ONE

FUNCTIONS, LIMITS, AND CONTINUITY

1.1 FUNCTIONS AND THEIR GRAPHS

Real Number can be expressed as a decimal.
A *rational number* is a repeating or terminating decimal.

Inequalities $a < b$ (a is less than b) if $b - a$ is positive;
$a > b$ (a is greater than b) if $a - b$ is positive.
$a \le b$ (a is less than or equal to b) if $a < b$ or $a = b$; $a \ge b$ if $a > b$ or $a = b$.

Intervals $(a,b) = \{x \mid a < x < b\}$ $\quad [a,b] = \{x \mid a \le x \le b\}$ $\quad (a,b] = \{x \mid a < x \le b\}$
$[a,b) = \{x \mid a \le x < b\}$ $\quad (a,+\infty) = \{x \mid x > a\}$ $\quad (-\infty,b) = \{x \mid x < b\}$
$[a,+\infty) = \{x \mid x \ge a\}$ $\quad (-\infty,b] = \{x \mid x \le b\}$ $\quad (-\infty,+\infty) = \mathrm{R}$

Absolute Value $|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}, \; |a-b| = \begin{cases} a-b & \text{if } a \ge b \\ b-a & \text{if } a < b \end{cases}$

Sign, Step Function $\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}. \; \mathrm{U}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$

Triangle Inequality $|a+b| \le |a| + |b|$

Graphing *Sketch* mean by hand; *plot* mean using a graphics calculator. Our plots label the axes.

1.1.1 Definition A *function* is a set of ordered pairs of numbers (x, y) in which no two distinct ordered pairs have the same first number. The set of all admissible values of x is called the *domain* of the function and the set of resulting values of y is called the *range* of the function. The numbers x and y are *variables*. A function like $|x|$ is said to be defined *in pieces*.

1.1.2 Definition If f is a function, then the *graph* of f is the set of all points (x, y) in R^2 for which (x, y) is an ordered pair in f, that is, the graph of the equation $y = f(x)$.

A function determines a correspondence between the domain and the range. For each number x in the domain there corresponds one and only one number y in the range. When addition, subtraction, and multiplication are the only operations required to calculate y, as in a polynomial function, the domain is the set of all real numbers. If division is required to calculate y, then the domain does not contain any replacement for x that results in division by zero. Thus, for a *rational function*, which is the quotient of two polynomials, the domain is the set of all real numbers, except for those replacements of x that result in a value of zero for the denominator of the fraction. Because the range of a function is a set of *real* numbers, and the square root of a negative number is not real, if the formula that defines a function contains the square root sign, then x must satisfy the inequality obtained by making the expression under the radical sign greater than or equal to zero.

Greatest Integer $[\![x]\!] = n$ if $n \le x < n+1$ where n is an integer. Note: $-[\![-n]\!] = n+1$ if $n < x \le n+1$.

3. (a) $\{(x, y) \mid y = x^2\}$ is a function; the domain is $(-\infty, +\infty)$
(b) $\{(x, y) \mid x = y^2\}$ is not a function; it contains $(4, 2)$ and $(4, -2)$
(c) $\{(x, y) \mid y = x^3\}$ is a function; the domain is $(-\infty, +\infty)$
(d) $\{(x, y) \mid x = y^3\}$ is a function; the domain is $(-\infty, +\infty)$

4. (a) $\{(x, y) \mid y = (x-1)^2 + 2\}$ is a function; the domain is $(-\infty, +\infty)$
(b) $\{(x, y) \mid x = (y+1)^2 - 2\}$ is not a function; it contains $(2, 1)$ and $(2, -3)$
(c) $\{(x, y) \mid y = (x+2)^3 - 1\}$ is a function; the domain is $(-\infty, +\infty)$
(d) $\{(x, y) \mid x = (y+1)^3 - 2\}$ is a function; the domain is $(-\infty, +\infty)$

5. $f(x) = 2x - 1$
(a) $f(3) = 2(3) - 1 = 5$
(b) $f(-2) = 2(-2) - 1 = -5$
(c) $f(0) = 2(0) - 1 = -1$
(d) $f(a+1) = 2(a+1) - 1 = 2a + 1$
(e) $f(x+1) = 2(x+1) - 1 = 2x + 1$
(f) $f(2x) = 2(2x) - 1 = 4x - 1$
(g) $2f(x) = 2(2x - 1) = 4x - 2$
(h) $f(x+h) = 2(x+h) - 1 = 2x + 2h - 1$
(i) $f(x) + f(h) = 2x - 1 + 2h - 1 = 2x + 2h - 2$
(j) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{(2x + 2h - 1) - (2x - 1)}{h} = \dfrac{2h}{h} = 2$

6. $f(x) = \frac{3}{x}$
(a) $f(1) = \frac{3}{1} = 3$
(b) $f(-3) = \dfrac{3}{-3} = -1$
(c) $f(6) = \frac{3}{6} = \frac{1}{2}$
(d) $f(\frac{1}{3}) = \dfrac{3}{\frac{1}{3}} = 9$
(e) $f(3/a) = \dfrac{3}{3/a} = a$ if $a \neq 0$
(f) $f(3/x) = \dfrac{3}{3/x} = x$ if $x \neq 0$
(g) $\dfrac{f(3)}{f(x)} = \dfrac{3/3}{3/x} = \frac{x}{3}$ if $x \neq 0$
(h) $f(x - 3) = \dfrac{3}{x - 3}$
(i) $f(x) - f(3) = \frac{3}{x} - \frac{3}{3} = \frac{3}{x} - 1$
(j) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{3/(x+h) - 3/x}{h} = \dfrac{3[x - (x+h)]}{hx(x+h)} = \dfrac{-3}{x(x+h)}$

7. $f(x) = 2x^2 + 5x - 3$
(a) $f(-2) = 2(-2)^2 + 5(-2) - 3 = -5$
(b) $f(-1) = 2(-1)^2 + 5(-1) - 3 = -6$
(c) $f(0) = -3$
(d) $f(3) = 2(3)^2 + 5(3) - 3 = 30$
(e) $f(h+1) = 2(h+1)^2 + 5(h+1) - 3 = 2h^2 + 9h + 4$
(f) $f(2x^2) = 2(2x^2)^2 + 5(2x^2) - 3 = 8x^4 + 10x^2 - 3$
(g) $f(x^2 - 3) = 2(x^2 - 3)^2 + 5(x^2 - 3) - 3 = 2x^4 - 7x^2$
(h) $f(x+h) = 2(x+h)^2 + 5(x+h) - 3 = 2x^2 + 4xh + 2h^2 + 5x + 5h - 3$
(i) $f(x) + f(h) = 2x^2 + 5x - 3 + 2h^2 + 5h - 3 = 2x^2 + 2h^2 + 5x + 5h - 6$
(j) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{(2x^2 + 4xh + 2h^2 + 5x + 5h - 3) - (2x^2 + 5x - 3)}{h} = \dfrac{4xh + 2h^2 + 5h}{h} = 4x + 2h + 5$

8. $g(x) = 3x^2 - 4$
(a) $g(-4) = 3(-4)^2 - 4 = 44$
(b) $g(\frac{1}{2}) = 3(\frac{1}{2})^2 - 4 = -\frac{13}{4}$
(c) $g(x^2) = 3(x^2)^2 - 4 = 3x^4 - 4$
(d) $g(3x^2 - 4) = 3(3x^2 - 4)^2 - 4 = 3(9x^4 - 24x^2 + 16) - 4 = 27x^4 - 72x^2 + 44$
(e) $g(x - h) = 3(x - h)^2 - 4 = 3(x^2 - 2xh + h^2) - 4 = 3x^2 - 6xh + 3h^2 - 4$
(f) $g(x) - g(h) = (3x^2 - 4) - (3h^2 - 4) = 3x^2 - 3h^2$
(g) $\dfrac{g(x+h) - g(x)}{h} = \dfrac{[3(x+h)^2 - 4] - (3x^2 - 4)}{h} = \dfrac{3(x^2 + 2xh + h^2) - 3x^2}{h} = \dfrac{6xh + 3h^2}{h} = 6x + 3h$

9. $F(x) = \sqrt{x + 9}$
(a) $F(x + 9) = \sqrt{(x + 9) + 9} = \sqrt{x + 18}$
(b) $F(x^2 - 9) = \sqrt{(x^2 - 9) + 9} = \sqrt{x^2} = |x|$
(c) $F(x^4 - 9) = \sqrt{(x^4 - 9) + 9} = \sqrt{x^4} = x^2$
(d) $F(x^2 + 6x) = \sqrt{x^2 + 6x + 9} = |x + 3|$
(e) $F(x^4 - 6x^2) = \sqrt{x^4 - 6x^2 + 9} = |x^2 - 3|$
(f) $\dfrac{F(x+h) - F(x)}{h} = \dfrac{\sqrt{x+h+9} - \sqrt{x+9}}{h} \cdot \dfrac{\sqrt{x+h+9} + \sqrt{x+9}}{\sqrt{x+h+9} + \sqrt{x+9}} = \dfrac{(x+h+9) - (x+9)}{h(\sqrt{x+h+9} + \sqrt{x+9})} = \dfrac{1}{\sqrt{x+h+9} + \sqrt{x+9}}$

10. $G(x)=\sqrt{4-x}$

(a) $G(4-x)=\sqrt{4-(4-x)}=\sqrt{x}$

(b) $G(4-x^2)=\sqrt{4-(4-x^2)}=\sqrt{x^2}=|x|$

(c) $G(4-x^4)=\sqrt{4-(4-x^4)}=\sqrt{x^4}=x^2$

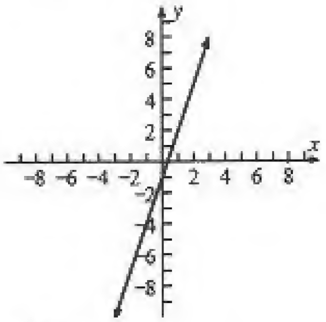
(d) $G(4x-x^2)=\sqrt{4-(4x-x^2)}=\sqrt{x^2-4x+4}=|x-2|$

(e) $G(-x^4-4x^2)=\sqrt{4-(-x^4-4x^2)}=\sqrt{x^4+4x^2+4}=x^2+2$

(f) $\dfrac{G(x+h)-G(h)}{h}=\dfrac{\sqrt{4-x-h}-\sqrt{4-x}}{h}\cdot\dfrac{\sqrt{4-x-h}+\sqrt{4-x}}{\sqrt{4-x-h}+\sqrt{4-x}}=\dfrac{(4-x-h)-(4-x)}{h(\sqrt{4-x-h}+\sqrt{4-x})}=\dfrac{-1}{\sqrt{4-x-h}+\sqrt{4-x}}$

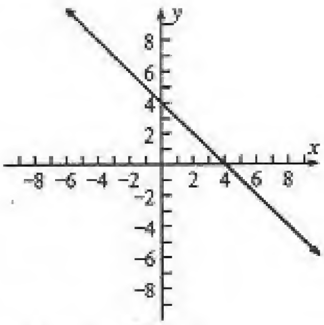
11. $f(x)=3x-1$

▷ D: $(-\infty,+\infty)$
R: $(-\infty,+\infty)$



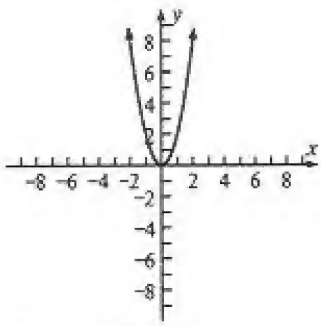
12. $g(x)=4-x$

▷ D: $(-\infty,+\infty)$
R: $(-\infty,+\infty)$



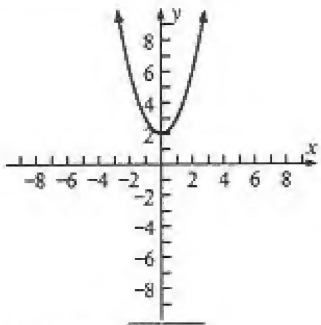
13. $2x^2$

▷ D: $(-\infty,+\infty)$
R: $[0,+\infty)$



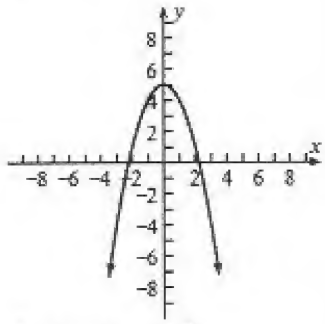
14. $G(x)=x^2+2$

▷ D: $(-\infty,+\infty)$
R: $[2,+\infty)$



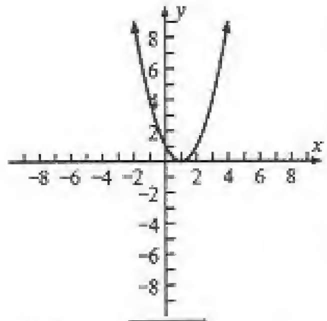
15. $g(x)=5-x^2$

▷ D: $(-\infty,+\infty)$
R: $(-\infty,5]$



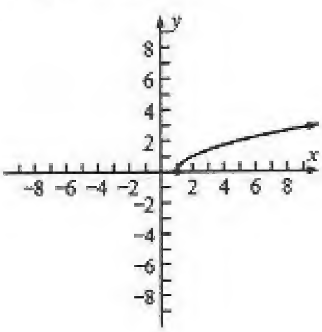
16. $f(x)=(x-1)^2$

▷ D: $(-\infty,+\infty)$
R: $[0,\infty)$



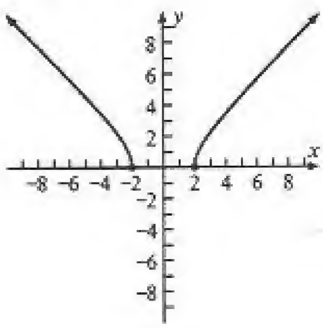
17. $G(x)=\sqrt{x-1}$

▷ D: $x-1\geq 0$,
$[1,+\infty)$
R: $[0,+\infty)$



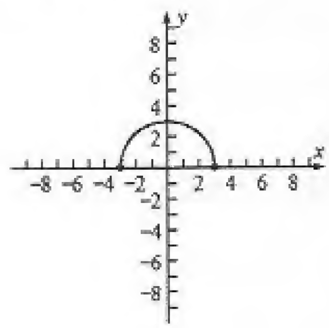
19. $f(x)=\sqrt{x^2-4}$

▷ D: $x^2-4\geq 0$,
$(-\infty,-2]\cup[2,+\infty)$
R: $[0,+\infty)$



21. $g(x)=\sqrt{9-x^2}$

▷ D: $9-x^2\geq 0$
$[-3,3]$
R: $[0,3]$

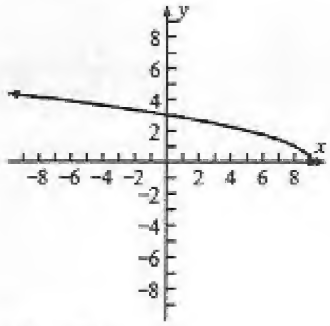


18. $F(x) = \sqrt{9-x}$

▹ D: $9 - x \geq 0$

$(-\infty, 9]$

R: $[0, +\infty)$

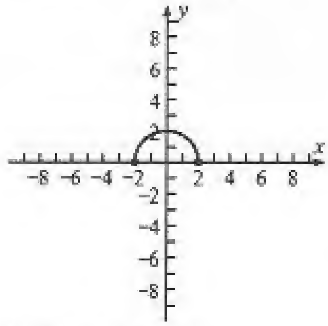


20. $g(x) = \sqrt{4-x^2}$

▹ D: $4 - x^2 \geq 0$

$[-2, 2]$

R: $[0, 2]$

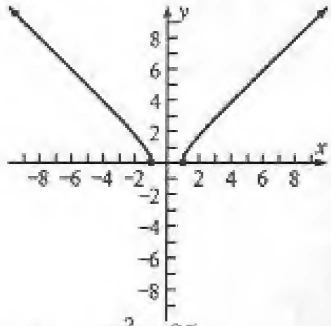


22. $f(x) = \sqrt{x^2-1}$

▹ D: $x^2 - 1 \geq 0$

$(-\infty, -1] \cup [1, +\infty)$

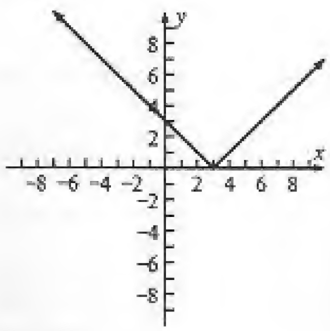
R: $[0, +\infty)$



23. $h(x) = |x-3|$

▹ D: $(-\infty, +\infty)$

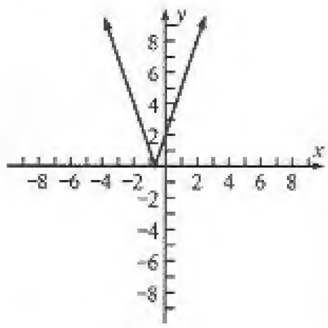
R: $[0, +\infty)$



25. $F(x) = |3x+2|$

▹ D: $(-\infty, +\infty)$

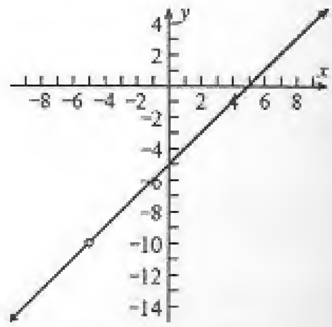
R: $[0, +\infty)$



27. $H(x) = \dfrac{x^2-25}{x+5}$

▹ $= \dfrac{(x-5)(x+5)}{x+5} = x - 5$

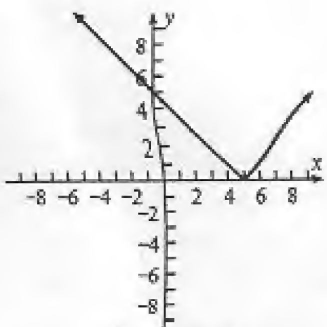
D: $x \neq -5$, R: $y \neq -10$



24. $H(x) = |5-x|$

▹ D: $(-\infty, +\infty)$

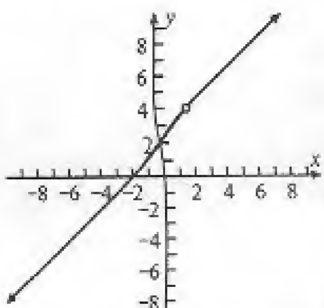
R: $[0, +\infty)$



26. $G(x) = \dfrac{x^2-4}{x-2}$

▹ $= \dfrac{(x-2)(x+2)}{x-2} = x + 2$

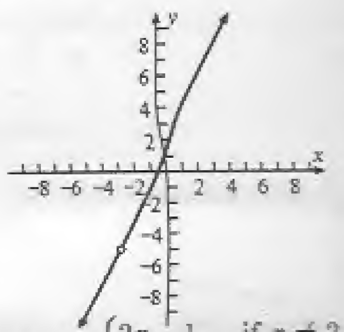
D: $x \neq 2$, R: $y \neq 4$



28. $f(x) = \dfrac{2x^2+7x+3}{x+3}$

▹ $= \dfrac{(2x+1)(x+3)}{x+3} = 2x + 1$

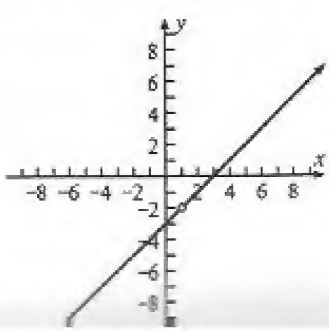
D: $x \neq -3$, R: $y \neq -5$



29. $f(x) = \dfrac{x^2-4x+3}{x-1}$

▹ $= \dfrac{(x-3)(x-1)}{x-1} = x - 3$

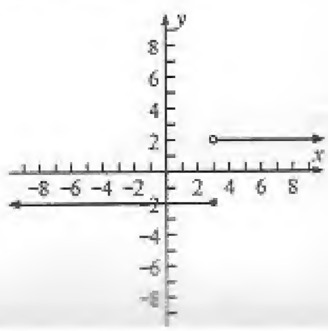
D: $x \neq 1$, R: $y \neq -2$



31. $f(x) = \begin{cases} -2 & \text{if } x \leq 3 \\ 2 & \text{if } 3 < x \end{cases}$

▹ D: $(-\infty, +\infty)$

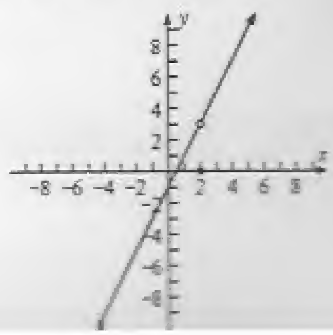
R: $\{-2, 2\}$



33. $g(x) = \begin{cases} 2x-1 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$

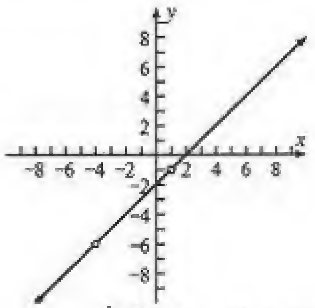
▹ D: $(-\infty, +\infty)$

R: $y \neq 3$



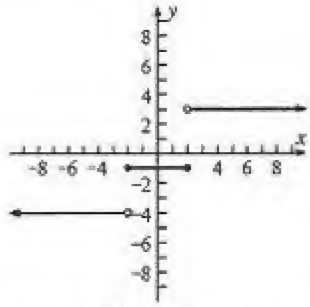
30. $g(x) = \dfrac{(x^2-4)(x-3)}{x^2-x-6}$

▹ $= \dfrac{(x-2)(x+2)(x-3)}{(x+2)(x-3)}$
$= x - 2$. D: $x \neq -2, 3$
R: $y \neq -4, 1$



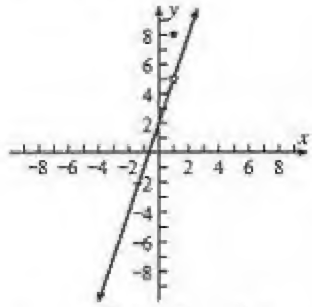
32. $g(x) = \begin{cases} -4 & \text{if } x < -2 \\ -1 & \text{if } -2 \leq x \leq 2 \\ 3 & \text{if } 2 < x \end{cases}$

▹ D: $(-\infty, +\infty)$
R: $\{-4, -1, 3\}$



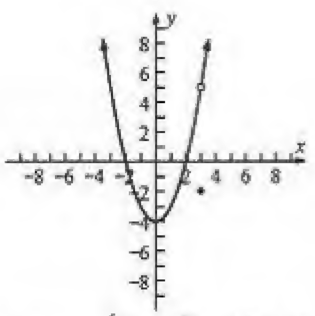
34. $f(x) = \begin{cases} 3x+2 & \text{if } x \neq 1 \\ 8 & \text{if } x = 1 \end{cases}$

▹ $f(x) = 3x + 2$
D: $(-\infty, +\infty)$
R: $y \neq 5$



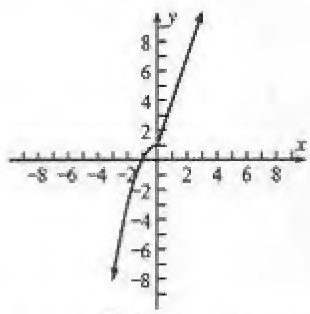
35. $F(x) = \begin{cases} x^2-4 & \text{if } x \neq 3 \\ -2 & \text{if } x = 3 \end{cases}$

▹ D: $(-\infty, +\infty)$, R: $[-4, +\infty)$
Note: $F(-3) = 5$



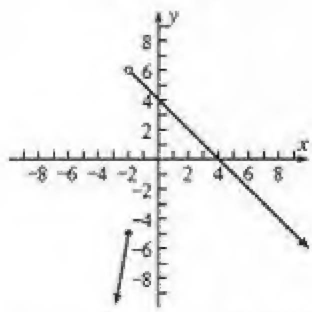
37. $G(x) = \begin{cases} 1-x^2 & \text{if } x < 0 \\ 3x+1 & \text{if } 0 \leq x \end{cases}$

▹ D: $(-\infty, +\infty)$
R: $(-\infty, 1) \cup [1, +\infty)$
$= (-\infty, +\infty)$



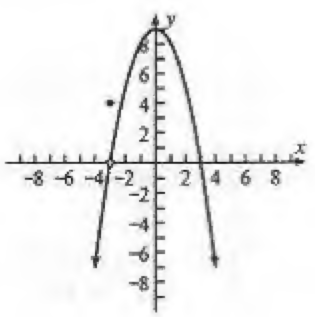
39. $g(x) = \begin{cases} 6x+7 & \text{if } x \leq -2 \\ 4-x & \text{if } -2 < x \end{cases}$

▹ D: $(-\infty, +\infty)$
R: $(-\infty, -5] \cup (-\infty, 6)$
$= (-\infty, 6)$



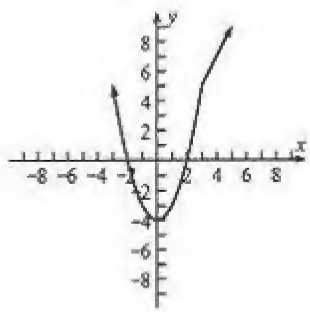
36. $G(x) = \begin{cases} 9-x^2 & \text{if } x \neq -3 \\ 4 & \text{if } x = -3 \end{cases}$

▹ D: $(-\infty, +\infty)$, R: $(-\infty, 9]$
Note: $G(3) = 0$



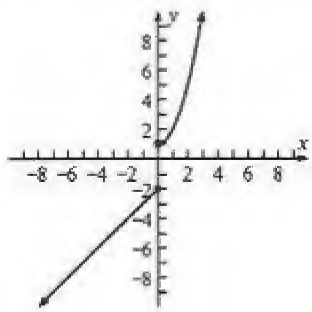
38. $F(x) = \begin{cases} x^2-4 & \text{if } x < 3 \\ 2x-1 & \text{if } 3 \leq x \end{cases}$

▹ D: $(-\infty, +\infty)$
R: $[-4, +\infty) \cup [5, +\infty)$
$= [-4, +\infty)$



40. $f(x) = \begin{cases} x-2 & \text{if } x \leq 0 \\ x^2+1 & \text{if } 0 < x \end{cases}$

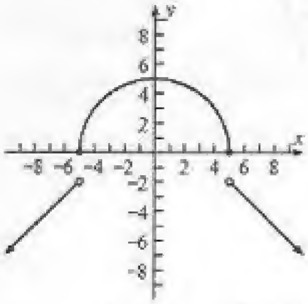
▹ D: $(-\infty, +\infty)$
R: $(-\infty, -2] \cup (1, +\infty)$



41. $h(x) = \begin{cases} x+3 & \text{if } x < -5 \\ \sqrt{25-x^2} & \text{if } -5 \le x \le 5 \\ 3-x & \text{if } 5 < x \end{cases}$

▷ D: $(-\infty, +\infty)$

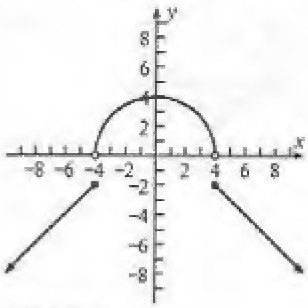
R: $(-\infty, -2) \cup [0, 5] \cup (-\infty, -2)$
$= (-\infty, -2) \cup [0, 5]$



42. $H(x) = \begin{cases} x+2 & \text{if } x \le -4 \\ \sqrt{16-x^2} & \text{if } -4 < x < 4 \\ 2-x & \text{if } 4 \le x \end{cases}$

▷ D: $(-\infty, +\infty)$

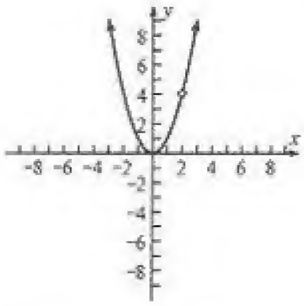
R: $(-\infty, -2] \cup (0, 4] \cup (-\infty, -2]$
$= (-\infty, -2] \cup (0, 4]$



43. $F(x) = \dfrac{x^3 - 2x^2}{x-2}$

▷ $= \dfrac{x^2(x-2)}{x-2} = x^2$

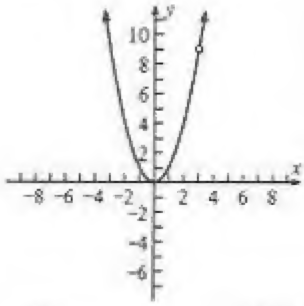
D: $x \ne 2$, R: $[0, +\infty)$
Note: $F(-2) = 4$



44. $G(x) = \dfrac{x^3 + 3x^2}{x+3}$

▷ $= \dfrac{x^2(x+3)}{x+3} = x^2$

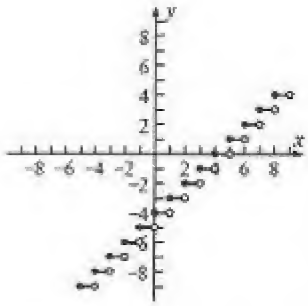
D: $x \ne -3$, R: $[0, +\infty)$
Note: $G(3) = 9$



45. $f(x) = [\![x-4]\!]$

▷ D: $(-\infty, +\infty)$

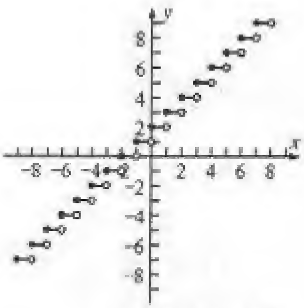
R: {integers}



46. $F(x) = [\![x+2]\!]$

▷ D: $(-\infty, +\infty)$

R: {integers}



47. (a) $U(x)$

▷ $= \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$

(b) $U(x-1)$

$= \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$

(c) $U(x) - 1$

$= \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$

(d) $U(x) - U(x-1)$

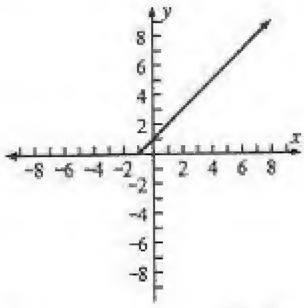
$= \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x < 1 \\ 0 & \text{if } x \ge 1 \end{cases}$

48. (a) $x\,\mathrm{U}(x)$

▹ $= \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

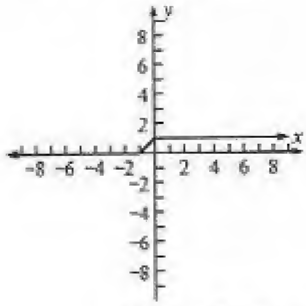
(b) $(x+1)\mathrm{U}(x+1)$

$= \begin{cases} 0 & \text{if } x < -1 \\ x+1 & \text{if } x \geq -1 \end{cases}$



(c) $(x+1)\mathrm{U}(x+1) - x\mathrm{U}(x)$

$= \begin{cases} 0 & \text{if } x < -1 \\ x & \text{if } -1 \leq x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$



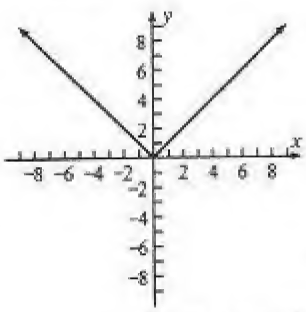
49. (a) $\operatorname{sgn} x$

▹ $= \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$

(b) $x \operatorname{sgn} x$

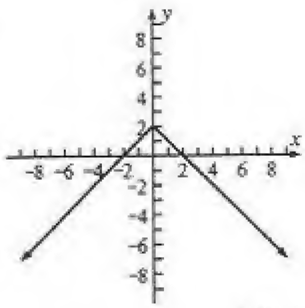
$= \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$

$= |x|$



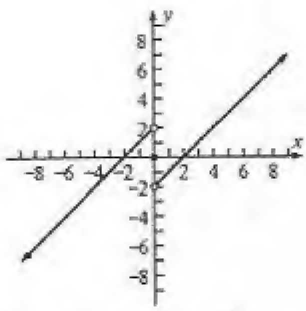
(c) $2 - x \operatorname{sgn} x$

$= \begin{cases} 2+x & \text{if } x < 0 \\ 2-x & \text{if } x \geq 0 \end{cases}$



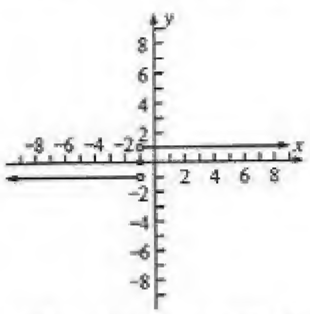
(d) $x - 2\operatorname{sgn} x$

$= \begin{cases} x+2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x-2 & \text{if } x > 0 \end{cases}$



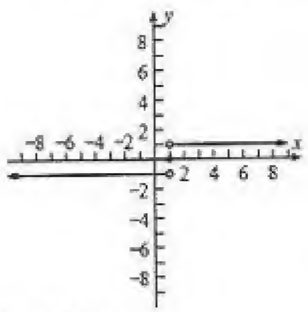
50. (a) $\operatorname{sgn}(x+1)$

▹ $= \begin{cases} -1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ 1 & \text{if } x > -1 \end{cases}$



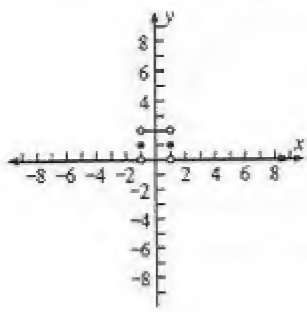
(b) $\operatorname{sgn}(x-1)$

$= \begin{cases} -1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$



(c) $\operatorname{sgn}(x+1) - \operatorname{sgn}(x-1)$

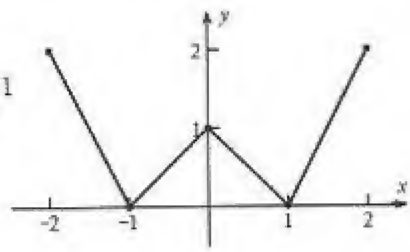
$= \begin{cases} 0 & \text{if } x < -1 \text{ or } x > 1 \\ 1 & \text{if } x = -1 \text{ or } x = 1 \\ 2 & \text{if } -1 < x < 1 \end{cases}$



51. Define $f(x)$ piecewise for the graph of the figure.

▹ Using the point-slope formula for each segment give

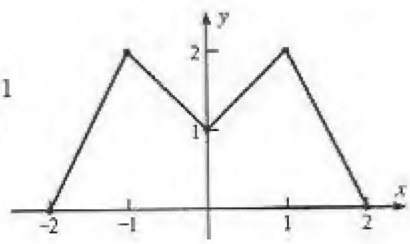
$$f(x) = \begin{cases} 2 - 2(x+2) & \text{if } -2 \leq x \leq -1 \\ 0 + 1(x+1) & \text{if } -1 < x \leq 0 \\ 1 - 1x & \text{if } 0 < x \leq 1 \\ 0 + 2(x-1) & \text{if } 1 < x \leq 2 \end{cases} = \begin{cases} -2x-2 & \text{if } -2 \leq x \leq -1 \\ x+1 & \text{if } -1 < x \leq 0 \\ 1-x & \text{if } 0 < x \leq 1 \\ 2x-2 & \text{if } 1 < x \leq 2 \end{cases}$$



52. Define $f(x)$ piecewise for the graph of the figure.

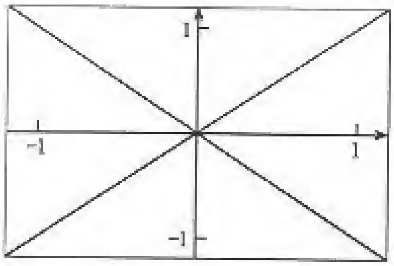
▹ Using the point-slope formula for each segment give

$$f(x) = \begin{cases} 0 + 2(x+2) & \text{if } -2 \leq x \leq -1 \\ 2 - 1(x+1) & \text{if } -1 < x \leq 0 \\ 1 + 1x & \text{if } 0 < x \leq 1 \\ 2 - 2(x-1) & \text{if } 1 < x \leq 2 \end{cases} = \begin{cases} 2x+4 & \text{if } -2 \leq x \leq -1 \\ 1-x & \text{if } -1 < x \leq 0 \\ 1+x & \text{if } 0 < x \leq 1 \\ 4-2x & \text{if } 1 < x \leq 2 \end{cases}$$



53. Define the graph of the figure as the union of the graphs of two functions f_1 and f_2.

▷ $f_1 = x$, $f_2 = -x$ or $f_1 = |x|$, $f_2 = -|x|$



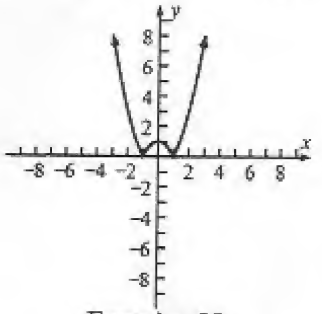
54. Define the graph of the letter Z as the union of f_1, f_2, f_3.

▷ Use relational operators to get segments, rather than lines.
$f_1 = .8(x > -.8)(x < .8)$
$f_2 = x(x > -.8)(x < .8)$
$f_3 = -.8(x > -.8)(x < .8)$

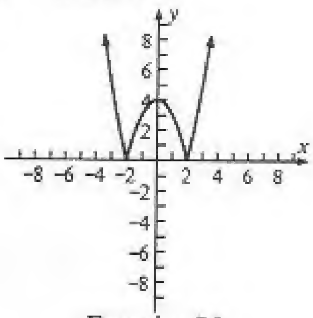
In Exercises 55–58, define the function piecewise and sketch the graph.

55. $f(x) = |x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } x \le -1 \\ 1 - x^2 & \text{if } -1 < x < 1 \\ x^2 - 1 & \text{if } x \ge 1 \end{cases}$

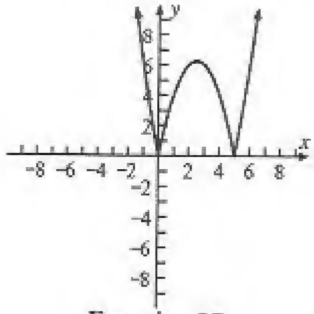
56. $g(x) = |4 - x^2| = \begin{cases} x^2 - 4 & \text{if } x \le -2 \\ 4 - x^2 & \text{if } -2 < x < 2 \\ x^2 - 4 & \text{if } x \ge 2 \end{cases}$



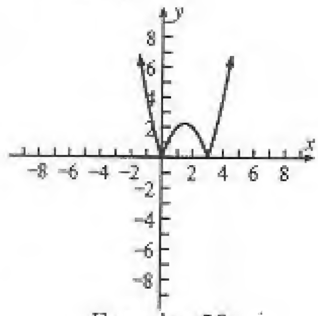
Exercise 55



Exercise 56



Exercise 57



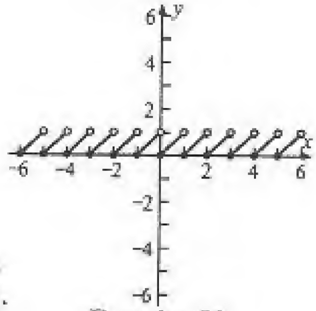
Exercise 58

57. $g(x) = |x| \cdot |5 - x| = |x(5 - x)|$
$= \begin{cases} x^2 - 5x & \text{if } x < 0 \\ 5x - x^2 & \text{if } 0 \le x \le 5 \\ x^2 - 5x & \text{if } x > 5 \end{cases}$

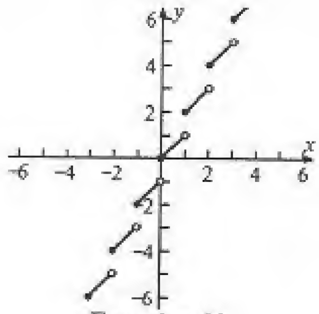
58. $g(x) = |x| \cdot |x - 3| = |x(x - 3)|$
$= \begin{cases} x^2 - 3x & \text{if } x < 0 \\ 3x - x^2 & \text{if } 0 \le x \le 3 \\ x^2 - 3x & \text{if } x > 3 \end{cases}$

In Exercises 59 and 60, sketch the graph of the function and determine its domain and range. Check by plotting.

59. $h(x) = x - [\![x]\!]$. D: $(-\infty, +\infty)$; R: $[0, 1)$



Exercise 59



Exercise 60

60. $F(x) = x + [\![x]\!]$. D: $(-\infty, +\infty)$; R: $[2k, 2k + 1)$, where k is any integer.

61. Define two other functions whose graphs resemble two different letters of the alphabet.

▷ U: $f(x) = 1 - \sqrt{1 - x^2}$, V: $f(x) = |x|$

1.2 OPERATIONS ON FUNCTIONS AND TYPES OF FUNCTIONS

1.2.1 Definition Given the two functions f and g:

(i) their *sum*, denoted by $f + g$, is the function defined by
$$(f + g)(x) = f(x) + g(x)$$

(ii) their *difference*, denoted by $f - g$, is the function defined by
$$(f - g)(x) = f(x) - g(x)$$

(iii) their *product*, denoted by $f \cdot g$, is the function defined by
$$(f \cdot g)(x) = f(x) \cdot g(x)$$

(iv) their *quotient*, denoted by f/g, is the function defined by
$$(f/g)(x) = f(x)/g(x)$$

In each case the *domain* of the resulting function consists of those values of x common to the domains of f and g, with the additional requirement in case (iv) that the values of x for which $g(x) = 0$ are excluded.

1.2.2 Definition Given two functions f and g, the *composite function*, denoted by $f \circ g$ and read f of g, is defined by $(f \circ g)(x) = f(g(x))$ and the *domain* of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

The composition of functions is associative, that is $(f \circ g) \circ h = f \circ (g \circ h)$.

1.2.3 Definition (i) A function is said to be an *even* function if for every x in the domain of f, $f(-x) = f(x)$.
(ii) A function is said to be an *odd* function if for every x in the domain of f, $f(-x) = -f(x)$.

In both parts (i) and (ii) it is understood that $-x$ is in the domain of f whenever x is.

Even powers of x are even functions; odd powers and odd roots of x are odd functions.

Combinations even $\pm$ even = even, odd $\pm$ odd = odd, even $\times$ even = even, odd $\times$ odd = even, even $\times$ odd = odd even/even = even, odd/odd = even, even/odd = odd, odd/even = odd. See Exercise 52.

Exercises 1.2

In *Exercises 1-10*, define the following functions and determine their domain D:
(a) $f+g$ (b) $f-g$ (c) $f \cdot g$ (d) f/g (e) g/f

▹ We adopt the common practice of omitting "(x)". Values not in the implicit domain are bolded.

1. $f = x-5$, $g = x^2-1$ (a) $f+g = x^2+x-6$, D: $(-\infty,\infty)$ (b) $f-g = -x^2+x-4$, D: $(-\infty,\infty)$
(c) $f \cdot g = (x-5)(x^2-1) = x^3-5x^2-x+5$, D: $(-\infty,\infty)$
(d) $f/g = \dfrac{x-5}{x^2-1}$, D: $x \neq \pm 1$ (e) $g/f = \dfrac{x^2-1}{x-5}$, D: $x \neq 5$

2. $f = \sqrt{x}$, $g = x^2+1$ (a) $f+g = \sqrt{x}+x^2+1$, D: $[0,+\infty)$ (b) $f-g = \sqrt{x}-x^2-1$, D: $[0,+\infty)$
(c) $f \cdot g = \sqrt{x}(x^2+1)$, D: $[0,+\infty)$ (d) $f/g = \dfrac{\sqrt{x}}{x^2+1}$, D: $[0,+\infty)$ (e) $g/f = \dfrac{x^2+1}{\sqrt{x}}$, D: $(0,+\infty)$

3. $f = \dfrac{x+1}{x-1}$, $g = \dfrac{1}{x}$ (a) $f+g = \dfrac{x+1}{x-1}\cdot\dfrac{x}{x}+\dfrac{1}{x}\cdot\dfrac{x-1}{x-1} = \dfrac{x^2+2x-1}{x(x-1)}$, D: $x \neq 0, 1$
(b) $f-g = \dfrac{x+1}{x-1}\cdot\dfrac{x}{x}-\dfrac{1}{x}\cdot\dfrac{x-1}{x-1} = \dfrac{x^2+1}{x(x-1)}$, D: $x \neq 0, 1$ (c) $f \cdot g = \dfrac{x+1}{x-1}\cdot\dfrac{1}{x} = \dfrac{x+1}{x(x-1)}$, D: $x \neq 0, 1$
(d) $\dfrac{f}{g} = \dfrac{x+1}{x-1} \div \dfrac{1}{x} = \dfrac{x+1}{x-1}\cdot x = \dfrac{x^2+x}{x-1}$, $x \neq 0, 1$ (e) $\dfrac{g}{f} = \dfrac{1}{x} \div \dfrac{x+1}{x-1} = \dfrac{1}{x}\cdot\dfrac{x-1}{x+1} = \dfrac{x-1}{x(x+1)}$, D: $x \neq -1, 0, 1$

4. $f = \sqrt{x}$, $g = 4-x^2$ (a) $f+g = \sqrt{x}+4-x^2$, D: $[0,+\infty)$ (b) $f-g = \sqrt{x}-4+x^2$, D: $[0,+\infty)$
(c) $f \cdot g = \sqrt{x}(4-x^2)$, D: $[0,+\infty)$ (d) $f/g = \dfrac{\sqrt{x}}{4-x^2}$, D: $[0,2) \cup (2,+\infty)$ (e) $g/f = \dfrac{4-x^2}{\sqrt{x}}$, D: $(0,+\infty)$

5. $f = \sqrt{x}$, $g = x^2-1$ (a) $f+g = \sqrt{x}+x^2-1$, D: $[0,+\infty)$ (b) $f-g = \sqrt{x}-x^2+1$, D: $[0,+\infty)$
(c) $f \cdot g = \sqrt{x}(x^2-1)$, D: $[0,+\infty)$ (d) $f/g = \dfrac{\sqrt{x}}{x^2-1}$, D: $[0,1) \cup (1,+\infty)$ (e) $g/f = \dfrac{x^2-1}{\sqrt{x}}$, D: $(0,+\infty)$

6. $f = |x|$, $g = |x-3|$ (a) $f+g = |x|+|x-3|$, D: $(-\infty,+\infty)$ (b) $f-g = |x|-|x-3|$, D: $(-\infty,+\infty)$
(c) $f \cdot g = |x||x-3|$, D: $(-\infty,+\infty)$ (d) $f/g = \dfrac{|x|}{|x-3|}$, D: $x \neq 3$ (e) $g/f = \dfrac{|x-3|}{|x|}$, D: $x \neq 0$

7. $f = x^2+1$, $g = 3x-2$ (a) $f+g = x^2+3x-1$, D: $(-\infty,+\infty)$ (b) $f-g = x^2-3x+3$, D: $(-\infty,+\infty)$
(c) $f \cdot g = (x^2+1)(3x-2) = 3x^3-2x^2+3x-2$, D: $(-\infty,+\infty)$ (d) $f/g = \dfrac{x^2+1}{3x-2}$, D: $x \neq \frac{2}{3}$
(e) $g/f = \dfrac{3x-2}{x^2+1}$, D: $(-\infty,+\infty)$

8. $f = \sqrt{x+4}$, $g = x^2-4$ (a) $f+g = \sqrt{x+4}+x^2-4$, D: $[-4,+\infty)$ (b) $f-g = \sqrt{x+4}-x^2+4$,
D: $[-4,+\infty)$ (c) $f \cdot g = \sqrt{x+4}(x^2-4)$, D: $[-4,+\infty)$ (d) $f/g = \dfrac{\sqrt{x+4}}{x^2-4}$, D: $[-4,-2) \cup (-2,2) \cup (2,+\infty)$
(e) $g/f = \dfrac{x^2-4}{\sqrt{x+4}}$, D: $(-4,+\infty)$

9. $f = \frac{1}{x+1}$, $g = \frac{x}{x-2}$ (a) $f + g = \frac{1}{x+1}\cdot\frac{x-2}{x-2} + \frac{x}{x-2}\cdot\frac{x+1}{x+1} = \frac{x^2+2x-2}{(x-2)(x+1)}$, d: $x \neq -1, 2$

(b) $f - g = \frac{1}{x+1}\cdot\frac{x-2}{x-2} - \frac{x}{x-2}\cdot\frac{x+1}{x+1} = \frac{-x^2-2}{(x+1)(x-2)}$, D: $x \neq -1, 2$

(c) $f \cdot g = \frac{x}{(x+1)(x-2)}$, D: $x \neq -1, 2$ (d) $\frac{f}{g} = \frac{1}{x+1} \div \frac{x}{x-2} = \frac{1}{x+1}\cdot\frac{x-2}{x} = \frac{x-2}{x(x+1)}$, D: $x \neq -1, 0, 2$

(e) $g/f = \frac{x}{x-2} \div \frac{1}{x+1} = \frac{x}{x-2}\cdot\frac{x+1}{1} = \frac{x^2+x}{x-2}$, D: $x \neq -1, 2$

10. $f = x^2$, $g = \frac{1}{\sqrt{x}}$ (a) $f + g = x^2 + \frac{1}{\sqrt{x}}$, D: $(0, +\infty)$ (b) $f - g = x^2 - \frac{1}{\sqrt{x}}$, D: $(0, +\infty)$

(c) $f \cdot g = \frac{x^2}{\sqrt{x}} = x^{3/2}$, $(0, +\infty)$ (d) $\frac{f}{g} = x^2 \div \frac{1}{\sqrt{x}} = x^{5/2}$, D: $(0, +\infty)$ (e) $\frac{g}{f} = \frac{1}{\sqrt{x}} \div x^2 = x^{-5/2}$, D: $(0, +\infty)$

In *Exercises 11–14*, compute $(f \circ g)(c)$ by two methods: (a) Find $g(c)$ first; (b) Find $(f \circ g)(x)$ first.

11. $f(x) = 3x^2 - 4x$, $g(x) = 2x - 5$, $c = 4$ ▹ (a) $g(4) = 2\cdot 4 - 5 = 3$, $f(3) = 3\cdot 3^2 - 4\cdot 3 = 15$

(b) $f(g(x)) = 3(2x-5)^2 - 4(2x-5) = 3(4x^2 - 20x + 25) - 8x + 20 = 12x^2 - 68x + 95$

$f(g(4)) = 12\cdot 4^2 - 68\cdot 4 + 95 = 15$

12. $f(x) = \sqrt{x^2 - 36}$, $g(x) = x^2 - 3x$, $c = 5$ ▹ (a) $g(5) = 5^2 - 3\cdot 5 = 10$, $f(10) = \sqrt{10^2 - 36} = 8$

(b) $f(g(x)) = \sqrt{(x^2-3x)^2 - 36} = \sqrt{x^4 - 6x^3 + 9x^2 - 36}$, $f(g(5)) = \sqrt{5^4 - 6\cdot 5^3 + 9\cdot 5^2 - 36} = 8$

13. $f(x) = \frac{1}{x-1}$, $g(x) = \frac{2}{x^2+1}$, $c = \frac{1}{2}$ ▹ (a) $g(\frac{1}{2}) = \frac{2}{(\frac{1}{2})^2 + 1} = \frac{8}{5}$, $f(\frac{8}{5}) = \frac{1}{\frac{8}{5} - 1} = \frac{5}{3}$

(b) $f(g(x)) = \frac{1}{\frac{2}{x^2+1} - 1} = \frac{x^2+1}{2 - (x^2+1)} = \frac{x^2+1}{1-x^2}$, $f(g(\frac{1}{2})) = \frac{(\frac{1}{2})^2 + 1}{1 - (\frac{1}{2})^2} = \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{3}$

14. $f(x) = \frac{2\sqrt{x+3}}{x}$, $g(x) = \frac{2x+5}{x^4}$, $c = -2$ ▹ (a) $g(-2) = \frac{2(-2)+5}{(-2)^4} = \frac{1}{16}$,

$f(\frac{1}{16}) = \frac{2\sqrt{\frac{1}{16}+3}}{\frac{1}{16}} = 32\sqrt{\frac{49}{16}} = 32\cdot\frac{7}{4} = 56$ (b) $f(g(x)) = \frac{2\sqrt{\frac{2x+5}{x^4} + 3}}{\frac{2x+5}{x^4}} = \frac{x^4}{2x+5}\cdot 2\sqrt{\frac{3x^4+2x+5}{x^4}}$

$= \frac{2x^2}{2x+5}\sqrt{3x^4+2x+5}$, $f(g(-2)) = \frac{2(-2)^2}{2(-2)+5}\sqrt{3(-2)^4 + 2(-2) + 5} = 8\sqrt{49} = 56$

In *Exercises 15–24*, find (a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$ and give their domains D.

15. $f(x) = x - 2$, $g(x) = x + 7$

▹ (a) $f(g(x)) = (x+7) - 2 = x + 5$, D: $(-\infty, +\infty)$ (b) $g(f(x)) = (x-2) + 7 = x + 5$, D: $(-\infty, +\infty)$

(c) $f(f(x)) = (x-2) - 2 = x - 4$ D: $(-\infty, +\infty)$ (d) $g(g(x)) = (x+7) + 7 = x + 14$, D: $(-\infty, +\infty)$

16. $f(x) = 3 - 2x$, $g(x) = 6 - 3x$

▹ (a) $f(g(x)) = 3 - 2(6 - 3x) = 6x - 9$, D: $(-\infty, +\infty)$ (b) $g(f(x)) = 6 - 3(3 - 2x) = 6x - 3$, D: $(-\infty, +\infty)$

(c) $f(f(x)) = 3 - 2(3 - 2x) = 4x - 3$, D: $(-\infty, +\infty)$ (d) $g(g(x)) = 6 - 3(6 - 3x) = 9x - 12$, D: $(-\infty, +\infty)$

17. $f(x) = x - 5$, $g(x) = x^2 - 1$ ▹ (a) $f(g(x)) = (x^2 - 1) - 5 = x^2 - 6$, D: $(-\infty, +\infty)$

(b) $g(f(x)) = (x-5)^2 - 1 = x^2 - 10x + 24$, D: $(-\infty, +\infty)$

(c) $f(f(x)) = (x-5) - 5 = x - 10$, D: $(-\infty, +\infty)$ (d) $g(g(x)) = (x^2-1)^2 - 1 = x^4 - 2x^2$, D: $(-\infty, +\infty)$

18. $f(x) = \sqrt{x}$, $g(x) = x^2 + 1$

▹ (a) $f(g(x)) = \sqrt{x^2+1}$, D: $(-\infty, +\infty)$ (b) $g((x)) = (\sqrt{x})^2 + 1 = x + 1$, D: $[0, +\infty)$

(c) $f(f(x)) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$, D: $[0, +\infty)$ (d) $g(g(x)) = (x^2+1)^2 + 1 = x^4 + 2x^2 + 2$, D: $(-\infty, +\infty)$

19. $f(x) = \sqrt{x-2}$, $g(x) = x^2 - 2$ ▹ (a) $f(g(x)) = \sqrt{(x^2-2) - 2} = \sqrt{x^2 - 4}$, D: $(-\infty, -2] \cup [2, +\infty)$

(b) $g(f(x)) = (\sqrt{x-2})^2 - 2 = x - 2 - 2 = x - 4$, D: $[2, +\infty)$

(c) $f(f(x)) = \sqrt{\sqrt{x-2} - 2}$, D: $x - 2 \geq 4$, $[6, +\infty)$ (d) $g(g(x)) = (x^2-2)^2 - 2 = x^4 - 4x^2 + 2$, D: $(-\infty, +\infty)$

20. $f(x) = x^2 - 1$, $g(x) = \frac{1}{x}$

▷ (a) $f(g(x)) = \left(\frac{1}{x}\right)^2 - 1 = \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2}$, D: $x \neq 0$ (b) $g(f(x)) = \frac{1}{x^2-1}$, D: $x \neq \pm 1$

(c) $f(f(x)) = (x^2-1)^2 - 1 = x^4 - 2x^2$, D: $(-\infty, +\infty)$ (d) $g(g(x)) = \frac{1}{1/x} = x$, D: $x \neq 0$

21. $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x}$

▷ (a) $f(g(x)) = \frac{1}{\sqrt{x}}$, D: $(0, +\infty)$ (b) $g(f(x)) = \sqrt{\frac{1}{x}} = \frac{1}{\sqrt{x}}$, D: $(0, +\infty)$

(c) $f(f(x)) = \frac{1}{1/x} = x$, D: $x \neq 0$ (d) $g(g(x)) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$, D: $[0, +\infty)$

22. $f(x) = \sqrt{x}$, $g(x) = -\frac{1}{x}$ ▷ (a) $f(g(x)) = \sqrt{-\frac{1}{x}}$, D: $(-\infty, 0)$ (b) $g(f(x)) = -\frac{1}{\sqrt{x}}$, D: $(0, +\infty)$

(c) $f(f(x)) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$, D: $[0, +\infty)$ (d) $g(g(x)) = -\frac{1}{-1/x} = x$, D: $x \neq 0$

23. $f(x) = |x|$, $g(x) = |x+2|$

▷ (a) $f(g(x)) = ||x+2|| = |x+2|$, D: $(-\infty, +\infty)$ (b) $g(f(x)) = ||x|+2| = |x|+2$, D: $(-\infty, +\infty)$

(c) $f(f(x)) = ||x|| = |x|$, D: $(-\infty, +\infty)$ (d) $g(g(x)) = ||x+2|+2| = |x+2|+2$, D: $(-\infty, +\infty)$

24. $f(x) = \sqrt{x^2-1}$, $g(x) = \sqrt{x-1}$

▷ (a) $f(g(x)) = \sqrt{(\sqrt{x-1})^2 - 1} = \sqrt{x-1-1} = \sqrt{x-2}$, D: $[2, +\infty)$

(b) $g(f(x)) = \sqrt{\sqrt{x^2-1} - 1}$, D: $\sqrt{x^2-1} \geq 1$, $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty)$

(c) $f(f(x)) = \sqrt{(\sqrt{x^2-1})^2 - 1} = \sqrt{x^2-1-1} = \sqrt{x^2-2}$, D: $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty)$

(d) $g(g(x)) = \sqrt{\sqrt{x-1} - 1}$, D: $x - 1 > 1$, $[2, +\infty)$

25. $f(x) = \sqrt{x}$ (a) $f(x^2) = \sqrt{x^2} = |x|$, D: $(-\infty, +\infty)$ (b) $[f(x)]^2 = (\sqrt{x})^2 = x$, D: $[0, +\infty)$

(c) $f(f(x)) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$, D: $[0, +\infty)$ (d) $f(f(-x)) = \sqrt[4]{-x}$, D: $(-\infty, 0]$

26. $f(x) = \frac{1}{1-x}$ (a) $f(x^2) = \frac{1}{1-x^2}$, D: $x \neq \pm 1$ (b) $[f(x)]^2 = \frac{1}{(1-x)^2}$, D: $x \neq 1$

(c) $f(f(x)) = \frac{1}{1 - 1/(1-x)} = \frac{1-x}{-x} = \frac{x-1}{x}$, D: $x \neq 0, 1$ (d) $f(f(-x)) = \frac{x+1}{x}$, D: $x \neq 0, -1$

In *Exercises 27–32*, express the function as $f(g(x))$ in two ways.

▷ Express the function using *of* twice; break at the first or the second.

27. $\sqrt{x^2-4}$ ▷ square root of the difference of $\cdots$, $f = \sqrt{x}$, $g = x^2 - 4$ or $f = \sqrt{x-4}$, $g = x^2$

28. $(9+x^2)^{-2}$ ▷ -2 power of the sum of $\cdots$, $f = x^{-2}$, $g = 9 + x^2$ or $f = (9+x)^{-2}$, $g = x^2$

29. $\left(\frac{1}{x-2}\right)^3$ ▷ 3 power of the reciprocal of $\cdots$, $f = x^3$, $g = \frac{1}{x-2}$ or $f = \left(\frac{1}{x}\right)^3$, $g = x - 2$

30. $\frac{4}{\sqrt[3]{x^3+3}}$ ▷ quotient of 4 and cube root of $\cdots$, $f = \frac{4}{x}$, $g = \sqrt[3]{x^3+3}$ or $f = \frac{1}{\sqrt[3]{x}}$, $g = x^3 + 3$

31. $(x^2+4x-5)^4$ ▷ 4 power of difference of $\cdots$, $f = x^4$, $g = x^2 + 4x - 5$ or $f = (x-5)^4$, $g = x^2 + 4x$

32. $\sqrt{|x|+4}$ ▷ square root of sum of $\cdots$, $f = \sqrt{x}$, $g = |x| + 4$ or $f = \sqrt{x+4}$, $g = |x|$

In *Exercises 33–38*, plot the function and determine if it is even, odd, or neither.

33. (a) $2x^4 - 3x^2 + 1$ is even (sum of even powers) (b) $5x^5 + 1$ is neither (odd + even)

34. (a) $x^2 + 2x + 2$ is neither (even + odd) (b) $x^6 - 1$ is even (even − even)

35. (a) $5x^3 - 7x$ is odd (odd + odd) (b) $g(x) = |x|$, $g(-x) = |-x| = |x| = g(x)$ even

36. (a) $4x^5 + 3x^3$ is odd (odd + odd) (b) $x^3 + 1$ is neither (odd + even)

37. (a) $\sqrt[3]{x}$ is odd (odd root) (b) $5x^4 - 4$ is even (even − even)

38. (a) $|x|/x$ is odd (even ÷ odd) (b) $2|x| + 3$ is even (even + even)

In *Exercises 39 and 40*, determine if the function is even, odd, or neither.

39. (a) $\frac{y^3-y}{y^2+1}$ is odd (odd ÷ even) (b) $\frac{r^2-1}{r^2+1}$ is even (even ÷ even) (c) $\frac{|x|}{x^2+1}$ is even (even ÷ even)

40. (a) $\frac{x^2-5}{2x^3+x}$ is odd (even ÷ odd) (b) $g(x)=\frac{x-1}{x+1}$, $g(-x)=\frac{-x-1}{-x+1}=\frac{x+1}{x-1}$ neither

(c) $f(x)=\begin{cases}-1 & \text{if } x<0\\ 1 & \text{if } x\geq 0\end{cases}$, $f(-x)=\begin{cases}-1 & \text{if } x>0\\ 1 & \text{if } x\leq 0\end{cases}$ and $-f(x)=\begin{cases}1 & \text{if } x<0\\ -1 & \text{if } x\geq 0\end{cases}$ neither

In *Exercises 41–44*, define the function in intervals, plot it, and determine if it is even, odd, or neither.

41. $\frac{|x|}{x}=\begin{cases}x/x=1 & \text{if } x>0\\ -x/x=-1 & \text{if } x<0\end{cases}$ is odd (even ÷ odd) **42.** $x|x|=\begin{cases}x\cdot x=x^2 & \text{if } x\geq 0\\ x(-x)=-x^2 & \text{if } x<0\end{cases}$ is even (odd · even)

43. $|x-2|-|x+2|=\begin{cases}-(x-2)+(x+2)=4 & \text{if } x<-2\\ -(x-2)-(x+2)=-2x & \text{if } -2\leq x<2\\ (x-2)-(x+2)=-4 & \text{if } x\geq 2\end{cases}$ is odd

44. $\frac{|x+1|-|x-1|}{x}=\begin{cases}[-(x+1)+(x-1)]/x=-2/x & \text{if } x<-1\\ [(x+1)-(x-1)]/x=2x/x=2 & \text{if } x\in[-1,0)\cup(0,1]\\ [(x+1)-(x-1)]/x=2/x & \text{if } x>1\end{cases}$ is even

45. $f\circ g\neq g\circ f$. See Exercises 16–24, (a) versus (b).

In *Exercises 46–50*, show that f and g are inverse functions by showing that $f(g(x))=x$ and $g(f(x))=x$.

46. $f=2x-3$, $g=\frac{1}{2}(x+3)$. $f\circ g=2[\frac{1}{2}(x+3)]-3=(x+3)-3=x$, $g\circ f=\frac{1}{2}[(2x-3)+2]=\frac{1}{2}(2x)=x$

47. $f=\frac{1}{x+1}$, $g=\frac{1-x}{x}$. $f\circ g=\frac{1}{(1-x)/x+1}\cdot\frac{x}{x}=\frac{x}{1-x+x}=x$, $g\circ f=\frac{1-1/(x+1)}{1/(x+1)}\cdot\frac{x+1}{x+1}=\frac{x+1-1}{1}=x$

48. $f=x^2$, $x\geq 0$; $g=\sqrt{x}$. $f\circ g=(\sqrt{x})^2=x$, $g\circ f=\sqrt{x^2}=x$

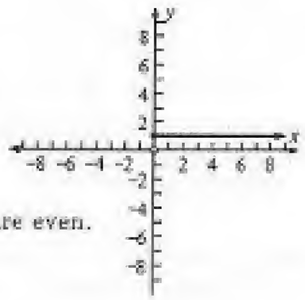
49. $f=x^2$, $x\leq 0$, $g=-\sqrt{x}$. $f\circ g=(-\sqrt{x})^2=x$, $g\circ f=-\sqrt{x^2}=-(-x)=x$

50. $f=(x-1)^3$, $g=1+\sqrt[3]{x}$. $f\circ g=[(1+\sqrt[3]{x})-1]^3=(\sqrt[3]{x})^3=x$, $g\circ f=1+\sqrt[3]{(x-1)^3}=1+(x-1)=x$

51. Find formulas for sgn(U(x)) and U(sgn(x)) and sketch their graph.

▷ $\text{sgn}(\text{U}(x))=\begin{cases}-1 & \text{if U}(x)<0\\ 0 & \text{if U}(x)=0\\ 1 & \text{if U}(x)=1\end{cases}=\left.\begin{cases}-1 & \text{never}\\ 0 & \text{if } x<0\\ 1 & \text{if } x\geq 0\end{cases}\right\}=\text{U}(x).$

$\text{U}(\text{sgn}(x))=\begin{cases}1 & \text{if sgn } x\geq 0\\ 0 & \text{if sgn } x<0\end{cases}=\begin{cases}1 & \text{if } x\geq 0\\ 0 & \text{if } x<0\end{cases}=\text{U}(x)$



52. If f and g are odd functions, prove that $f+g$ and $f-g$ are odd and $f\cdot g$ and f/g are even.

▷ $f(-x)=-f(x)$, $g(-x)=g(x)$.

$(f+g)(-x)=f(-x)+g(-x)=-f(x)-g(x)=-[f(x)+g(x)]=-(f+g)(x)$

$(f-g)(-x)=f(-x)-g(-x)=-f(x)+g(x)=-[f(x)-g(x)]=-(f-g)(x)$

$(f\cdot g)(-x)=f(-x)g(-x)=[-f(x)][-g(x)]=f(x)g(x)=(f\cdot g)(x)$

$(f/g)(-x)=f(-x)/g(-x)=-f(x)/-g(x)=f(x)/g(x)=(f/g)(x)$

53. If f is any function and g is even, $f(g(-x))=f(g(x))$ so $f\circ g$ is even.
If f and g are both odd, $f(g(-x))=f(-(g(x))=-f(g(x))$ so $f\circ g$ is odd.
If f is even and g is odd, $f(g(-x))=f(-g(x))=f(g(x))$ so $f\circ g$ is even.

54. Find formulas for $(f\circ g)(x)$. Sketch the graphs of f, g, and $f\circ g$.

▷ $f(g(x))=\begin{cases}2g(x) & \text{if } 0\leq g(x)\leq 1\\ 0 & \text{otherwise}\end{cases}=\begin{cases}x & \text{if } 0\leq x\leq 1\\ 2 & \text{otherwise}\end{cases}$

f g $f\circ g$ $g\circ f$

55. Find formulas for $(g \circ f)(x)$ and sketch its graph.

▷ $g(f(x)) = \begin{cases} 0 & \text{if } f(x) = 0 \\ \frac{1}{2}f(x) & \text{if } 0 < f(x) \le 1 \\ 1 & \text{if otherwise} \end{cases} = \begin{cases} 0 & \text{if } x \le 0 \text{ or } x > 1 \\ x & \text{if } 0 < x \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

56. If $f(x) = x^2 + 2x + 2$, find two functions g for which $f(g(x)) = x^2 - 4x + 5$.

▷ Assume g has the form $ax + b$. Then $f(g) = (ax+b)^2 + 2(ax+b) + 2 = a^2x^2 + (2ab + 2a)x + (b^2 + 2b + 2)$. $a^2 = 1$, $a = \pm 1$. If $a = 1$, $2b + 2 = -4$, $b = -3$. If $a = -1$, $-2b - 2 = -4$, $b = 1$. Both satisfy $b^2 + 2b + 2 = 5$. $g(x) = x - 3$ or $g(x) = 1 - x$.

57. If $f(x) = x^2$, find two functions g for which $f(g(x)) = 4x^2 - 12x + 9 = (2x-3)^2$. $g = 2x - 3$ or $-2x + 3$.

58. Prove that if f and g are linear functions, then so is $f \circ g$.

▷ Let $f = ax + b$, $g = cx + d$. $f(g) = a(cx + d) + b = acx + (ad + b) = \mathrm{A}x + \mathrm{B}$ with $\mathrm{A} = ac$, $\mathrm{B} = ad + b$.

59. Find the function $f(x)$ which is odd and even.

▷ $f(-x) = -f(x)$ and $f(-x) = f(x)$. Then $-f(x) = f(x)$ so $f(x) \equiv 0$.

60. If $f(x) = \frac{1}{x}$, $g(x) = -\frac{1}{x}$, $h(x) = -x$, is $(f \circ g)(x) = h(x)$?

▷ $(f \circ g)(x) = (g \circ f)(x) = -x$ if $x \neq 0$ while $h(x) = -x$ for all x.

61. Explain why $\mathrm{F}(x) = \sqrt{x+1}/\sqrt{x-4}$ and $\mathrm{G}(x) = \sqrt{(x+1)/(x-4)}$ are not the same.

▷ Dom $\mathrm{F} = \{x + 1 \ge 0\} \cap \{x - 4 > 0\} = (4, +\infty)$ but dom $\mathrm{G} = (-\infty, -1] \cup (4, +\infty)$

1.3 FUNCTIONS AS MATHEMATICAL MODELS

Variables should be described using units: x feet is the length.

y is Proportional to x Directly: $y = kx$. Inversely: $y = k/x$. Jointly proportional to x and z: $y = kxz$.

Extremum of $ax^2 + bx + c$ occurs when $x = -b/2a$; this can be obtained by completing the square.

Maximum Product of a set of positive numbers of constant sum is when factors are equal. See Ex. 28.

Minimum Sum of a set of positive numbers of constant product is when terms are equal. See Ex. 22.

Exercises 1.3

In each exercise, obtain a function as a mathematical model of the situation. Be sure to write a conclusion.

1. A payroll of p dollars is directly proportional to the number w of workers, and a crew of 12 workers earns \$810. (a) Find $p(w)$. (b) What is the payroll for a crew of 15 workers?

▷ $\mathrm{p} = \mathrm{kw}$. $810 = \mathrm{k}(12)$, $k = 67.5$ (a) $p(w) = 67.5w$ (b) $p(15) = 67.5 \cdot 15 = 1012.50$

2. A person's brain weight b lb is directly proportional to his body weight w lb, and a person weighing 150 lb has a 4 lb brain (a) $b(w)$. (b) Find the brain weight of a 176 lb person.

▷ $b = kw$. $4 = k(150)$, $k = \frac{2}{75}$. (a) $b(w) = \frac{2}{75}w$ (b) $b(176) = \frac{2}{75} \cdot 176 \approx 4.69$

3. The period p sec of a pendulum is directly proportional to the square root of the number x of feet in its length, and an 8 ft pendulum has a 2 sec period. (a) Find $p(x)$. (b) Find the period of a 2 ft pendulum.

▷ $p = k\sqrt{x}$. $2 = k\sqrt{8}$, $k = 1/\sqrt{2}$ (a) $p(x) = \sqrt{x/2}$ (b) $p(2) = \sqrt{1} = 1$. The period is 1 sec.

4. The frequency f per sec of a vibration is directly proportional to the square root of the tension t kg, and is 864/sec when the tension is 24 kg. (a) Find $f(t)$. (b) Find the frequency under a tension of 6 kg.

▷ $f = k\sqrt{t}$. $864 = k\sqrt{24} = 12k/\sqrt{6}$, $k = 72\sqrt{6}$ (a) $f(t) = 72\sqrt{6t}$ (b) $f(6) = 72 \cdot 6 = 432$

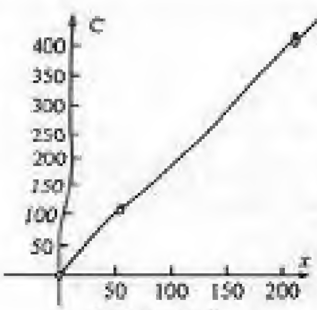
5. $\mathrm{C}(x)$ dollars is the cost of shipping x lb. $\mathrm{C}(x) = \begin{cases} 2.2x & \text{if } 0 < x \le 50 \\ 2.1x & \text{if } 50 < x \le 200 \\ 2.05x & \text{if } x > 200 \end{cases}$. $\mathrm{C}(50) = 2.2 \times 50 = 110$,

$\mathrm{C}(51) = 2.1 \times 51 = 107.10$, $\mathrm{C}(52) = 2.1 \times 52 = 109.20$, $\mathrm{C}(53) = 2.1 \times 53 = 111.30$, $\mathrm{C}(200) = 2.1 \times 200 = 420$
$\mathrm{C}(202) = 2.05 \times 202 = 414.10$, $\mathrm{C}(204) = 2.05 \times 204 = 418.20$, $\mathrm{C}(206) = 2.05 \times 206 = 422.30$

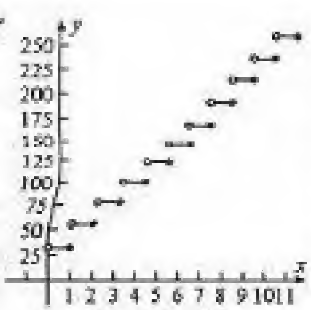
6. $y(x)$ cents is the cost of mailing x ounces. $y = 9 - 23[\![-x]\!]$. $y(1.6) = 9 - 23[\![-1.6]\!] = 9 - 23(-2) = 55$

$y(2) = 9 - 23[\![-2]\!] = 9 - 23(-2) = 55$ $\quad$ $y(2.1) = 9 - 23[\![-2.1]\!] = 9 - 23(-3) = 78$

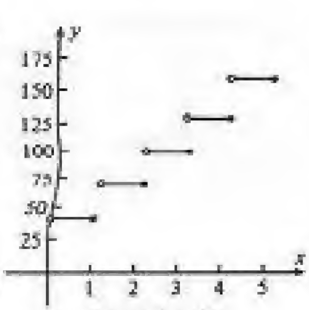
$y(8.4) = 9 - 23[\![-8.4]\!] = 9 - 23(-9) = 216$ $\quad$ $y(11) = 9 - 23[\![-11]\!] = 9 - 23(-11) = 262$



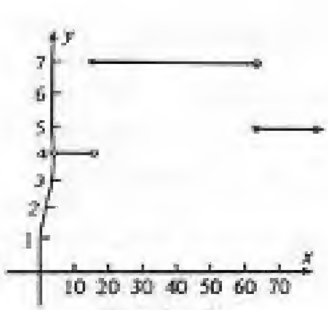
Exercise 5



Exercise 6



Exercise 7



Exercise 8

7. $y(x)$ cents is the cost of an x minute call. $y(x) = 10 - 30[\![-x]\!]$. $y(0.5) = 10 - 30[\![-0.5]\!] = 10 - 30(-1) = 40$
$y(2) = 10 - 30[\![-2]\!] = 10 - 30(-2) = 70$ $\quad y(2.5) = 10 - 30[\![-2.5]\!] = 10 - 30(-3) = 100$
$y(3) = 10 - 30[\![-3]\!] = 10 - 30(-3) = 100$ $\quad y(3.5) = 10 - 30[\![-3.5]\!] = 10 - 30(-4) = 130$
$y(5) = 10 - 30[\![-5]\!] = 10 - 30(-5) = 160$

8. The adult admission price is \$7, while the price for cildren under 12 years is \$4 and the price for seniors at least 60 yers is \$5. (a) Find a model of price as a function of age. (b) Sketch the graph of your function in (a).

▹ *$y(x)$ dollars is the admission for age x years.* $y(x) = \begin{cases} 4 & \text{if } 0 < x < 12 \\ 7 & \text{if } 12 \le x < 60 \\ 5 & \text{if } 60 \le x \end{cases}$.

9. The demand for a toy is $f(p) = \dfrac{5000}{p^2}$ where p dollars is the price. After t months the price is $g(t) = \frac{1}{20}t^2 + \frac{7}{20}t + 5$. (a) The demand after t months is $f(g(t)) = \dfrac{5000}{(\frac{1}{20}t^2 + \frac{7}{20}t + 5)^2} = \dfrac{2{,}000{,}000}{(t^2 + 7t + 100)^2}$.
(b) The demand after 5 months is $\dfrac{2{,}000{,}000}{(5^2 + 7\cdot 5 + 100)^2} = \dfrac{2000000}{160^2} = 78.125 \approx 78$

10. The number of large fish is $f(x) = \sqrt{20x} + 150$ where x is number of medium fish. The number if medium fish is $g(w) = \sqrt{w} + 5000$ where w is the number of small fish. (a) As a function of w the number of large fish is $f(g(w)) = \sqrt{20(\sqrt{w} + 5000)} + 150$. (b) There are 9 million small fish.
$f(g(9000000)) = \sqrt{20(3000 + 5000)} + 150 = 550$. There are 550 large fish.

11. If r cm is the radius of a sphere and A cm^2 is the surface area, then $A(r) = 4\pi r^2$. The radius is increasing at 3 cm/sec, so $r = f(t) = 3t$. (a) After t sec the area is $A(f(t)) = 4\pi(3t)^2 = 36\pi t^2$ cm^2
(b) After 4 sec the area is $36\pi \cdot 4^2 = 576\pi \approx 1809.6$ cm^2

12. If r ft is the radius of a sphere and V ft^3 is the volume, then $V(r) = \frac{4}{3}\pi r^3$. The radius is 2 ft and decreases at 4.5 in/min $= \frac{3}{8}$ ft/min so $r = f(t) = 2 - \frac{3}{8}t$. (a) After t minutes the volume is $V(f(t)) = \frac{4}{3}\pi(2 - \frac{3}{8}t)^3$. (b) After 3 minutes the volume is $\frac{4}{3}\pi(2 - \frac{3}{8}\cdot 3)^3 = 2.81$ ft^3.

13. A field of length x m is enclosed with 240 m of fence. Its width is $\frac{1}{2}(240 - 2x) = 120 - x$ m and its area is $a(x) = (120 - x)x = 120x - x^2$, $0 \le x \le 120$. $a(x) = 3600 - (x - 60)^2$ is greatest when $x = 60$ and the field is 60 m × 60 m.

14. A garden of length x ft is enclosed with 100 ft of fence. Its width is $\frac{1}{2}(100 - 2x) = 50 - x$ ft and its area is $a(x) = (50 - x)x = 50x - x^2$, $0 \le x \le 50$. $a(x) = 625 - (x - 25)^2$ is greatest when $x = 25$ and the garden is 25 ft × 25 ft

15. A field of length x m parallel to a river is enclosed with 240 m of fence. Its width is $\frac{1}{2}(240 - x)$ m and its area is $a(x) = \frac{1}{2}(240 - x)x = 120x - \frac{1}{2}x^2$, $0 \le x \le 240$. $a(x) = -\frac{1}{2}(x^2 - 240x + 120^2 + \frac{1}{2}\cdot 14400 = -\frac{1}{2}(x - 120)^2 + 7200$ is greatest when $x = 120$ and the field is 120 m × 60 m.

16. A garden of length x ft parallel to a house is enclosed with 100 ft of fence. Its width is $\frac{1}{2}(100 - x)$ ft and its area is $a(x) = \frac{1}{2}(100 - x)x = 50x - \frac{1}{2}x^2$, $0 \le x \le 100$. $a(x) = -\frac{1}{2}(x^2 - 100x + 50^2) + 1250 = -\frac{1}{2}(x - 50)^2 + 1250$ is greatest when $x = 50$ and the garden is 50 ft × 25 ft.

17. x in. squares are cut from the corners of 8 in. by 15 in. sheet and the sides are turned up. (a) Find the volume $V(x)$ in^3. (b) Find dom(V). (c) Maximize the volume graphically.

▹ $V = \ell wh = (8 - 2x)(15 - 2x)x$, $0 \le x \le 4$. The maximum volume is 90.74 in^3 when $x = \frac{5}{3} \approx 1.7$ (in.)

18. x cm squares are cut from the corners of 12 cm square and the sides are turned up. (a) Find the volume $V(x)$ cm^3. (b) Find dom(V). (c) Maximize the volume graphically.

▹ $V = \ell wh = (12-2x)^2 x = \frac{1}{4}\cdot 4x(12-2x)(12-2x)$ (sum = 24) $0 \le x \le 6$. The maximum volume is 128 cm^3 when $12 - 2x = 4x$, $x = 2$. Cut 2 cm squares.

19. x in. squares are cut from the corners of 12 in. by 15 in. sheet and the sides are turned up. (a) Find the volume $V(x)$ in^3. (b) Find dom(V). (c) Maximize the volume graphically.

▹ $V = \ell wh = (12-2x)(15-2x)x$, $0 \le x \le 6$. The maximum volume is 177 in^3 when $x = 2.21$ (in.)

20. x cm squares are cut from the corners of 40 cm by 50 cm sheet and the sides are turned up. (a) Find the volume $V(x)$ cm^3. (b) Find dom(V). (c) Maximize the volume graphically.

▹ $V = \ell wh = (40-2x)(50-2x)x$, $0 \le x \le 20$. The maximum volume is 6564.2 cm^3 when $x = 7.36$ (cm)

21. 60 in^3 is the volume of a cylinder, r in. is the radius, $60/\pi r^2$ in. is the height, $120/r$ in^2 is the lateral area, $2\pi r^2$ in^2 is the area of the ends. The cost is $\$k/in^2$ for the sides and $\$2k/in^2$ for the ends. The total cost is $C(r) = k\cdot 120/r + 2k\cdot 2\pi r^2 = k(120/r + 4\pi r^2)$, $r > 0$. $C = k(60/r + 60/r + 4\pi r^2)$ (product $= 14400\pi$) is minimum when $60/r = 4\pi r^2$, $r^3 = 15/\pi$, $r \approx 1.68$ in.

22. 60 in^3 is the volume of a cylinder, r in. is the radius, $60/\pi r^2$ in. is the height, $120/r$ in^2 is the lateral area, πr^2 in^2 is the area of the bottom. The total surface is $S(r) = 120/r + \pi r^2$, $r > 0$. $S = 60/r + 60/r + \pi r^2$ (product $= 3600\pi$) is minimum when $60/r = \pi r^2$, $r^3 = 60/\pi$, $r \approx 2.67$ in.

23. A page with margins of 1.5 in. at the top and bottom and 1 in. at the sides is to contain 24 in^2 of print. (a) Find the total area of the page, $A(x)$ in^2, when the width of the printed region is x in. (b) What is the domain D of A? (c) Approximate to the nearest hundredth of an inch the size of the smallest page.

▹ (a) The length of the printed region is $\frac{24}{x}$ in. $A(x) = (x+2)\left(\frac{24}{x}+3\right) = 30 + \left(3x + \frac{48}{x}\right)$ (product = 144)

(b) D: $x > 0$ (c) $A_{min} = 54$ when $3x = \frac{48}{x}$, $x = 4$. $x + 2 = 4 + 2 = 6$, $\frac{24}{x} + 3 = \frac{24}{4} + 3 = 9$. The smallest page is 6 in. wide and 9 in. long.

24. A lot with walkways 22 ft wide at the front and back and 15 ft at the sides is to contain a 13,200 ft^2 building. (a) Find the total area of the lot, $A(x)$ ft^2, when the width of the front is x ft. (b) What is the domain D of A? (c) Approximate to the nearest hundredth of a foot the size of the smallest lot.

▹ (a) The length of the building is $\frac{13{,}200}{x}$ ft. $A(x) = (x+30)\left(\frac{13200}{x} + 44\right) = 14520 + (44x + 396000/x)$

(b) D: $x > 0$ (c) Because the product of the variable terms is 1,742,000, the smallest lot has area 22,868.4 ft^2 when $44x = 396{,}000/x$, $x = \sqrt{9000} \approx 94.87$. The field is 124.87 ft by 183.14 ft.

25. A box of length x in. with square cross section has 100 in. as the sum of its length and girth. (a) Find the volume V in^3 as a function of x. (b) What is the domain D of V? (c) Approximate to the nearest inch the dimensions of the largest box.

▹ (a) The width of the box is $\frac{1}{4}(100-x)$ in. $V = x[\frac{1}{4}(100-x)]^2 = \frac{1}{32}\cdot 2x(100-x)(100-x)$ (sum = 200)

(b) D: $20 \le x \le 100$ (length ≥ width) (c) $V_{max} = \frac{1}{32}\left(\frac{200}{3}\right)^3 \approx 9259$ when $100 - x = 2x$, $x = \frac{100}{3}$.
The largest box is about 33 by 17 by 17 in.

26. The growth rate f bacteria/min of a colony is jointly proportional to the number x of bacteria and the number $1{,}000{,}000 - x$ of capacity. $f(x) = kx(1{,}000{,}000 - x)$, $f(1000) = 60 = 1000k\cdot 999{,}000$, $k = \frac{60}{999{,}000{,}000}$
$= \frac{1}{16{,}650{,}000}$. $f = x(1{,}000{,}000 - x)/16{,}650{,}000$, $0 \le x \le 1{,}000{,}000$. $f(100{,}000) = 100{,}000\cdot 900{,}000/16{,}650{,}000$
≈ 5405 bacteria/min. $f(x) = k(-x^2 + 1{,}000{,}000x)$ is maximum when $x = -1{,}000{,}000/-2 = 500{,}000$.

27. The growth rate f infected/day of an epidemic is jointly proportional to the number x of infected and the number $5{,}000 - x$ of capacity. $f(x) = kx(5{,}000 - x)$, $f(100) = 9 = 100k\cdot 4900$, $k = 9/490{,}000$.
$f(x) = 9x(5{,}000 - x)/490{,}000$, $0 \le x \le 5{,}000$. $f(200) = 9\cdot 200\cdot 4800/490{,}000 \approx 17.6 \approx 18$ people/day.
$f(x) = k(-x^2 + 5{,}000x)$ is maximum when $x = -5{,}000/-2 = 2500$.

28. The base of a pyramidal tent is $2x$ m square and a triangular side has height $2.5 - x$ m. The height h of the tent satisfies $h^2 + x^2 = (2.5 - x)^2$, $h = \sqrt{6.25 - 5x}$. The volume is $V = \frac{1}{3}(2x)^2\sqrt{6.25 - 5x}$, $0 \le x \le 1.25$.
$V(0.8) = \frac{1}{3}(2\cdot 0.8)^2\sqrt{6.25 - 5\cdot 0.8} = \frac{1}{3}\cdot 1.6^2\sqrt{2.25} = 1.28$ m^3. $V^2 = \frac{25}{9}x^4(1 - \frac{4}{5}x) = \frac{1}{9}5^6(\frac{1}{5}x)(\frac{1}{5}x)(\frac{1}{5}x)(\frac{1}{5}x)(1 - \frac{4}{5}x)$
Because the sum of the variable factors is 1, V is maximum when $\frac{1}{5}x = 1 - \frac{4}{5}x$, $x = 1$

1.4 GRAPHICAL INTRODUCTION TO LIMITS OF FUNCTIONS

The formal definition of limit is given in the next section:

1.4.1 Definition Let f be a function that is defined at every number in some open interval containing a, except possibly at the number a itself. The *limit of $f(x)$ as x approaches a is L*, written as

$$\lim_{x \to a} f(x) = \mathrm{L}$$

if the following statement is true:
Given any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - \mathrm{L}| < \epsilon$$

Note that $0 < |x - a|$ implies that $x \neq a$. This hypothesis must be used whenever $f(a)$ is not defined as in Exercises 29 and 30.

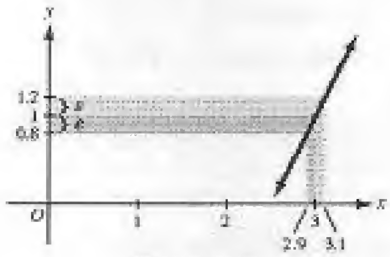
To find δ we usually factor the expression $|f(x) - \mathrm{L}|$ into the form $|x - a| \cdot |g(x)|$. If $f(x)$ is a first degree polynomial, then $|g(x)|$ is a constant, and we find δ as in Exercise 28. However, if $f(x)$ is not a first degree polynomial, then $|g(x)|$ is not a constant, and we must find an upper bound for $|g(x)|$, as in Exercises 36. We may often find this upper bound if we assume that $\delta \leq 1$.

Finally, we note that the definition does not tell us how to find the limit L. In Section 1.5 we have theorems that can be used to find L.

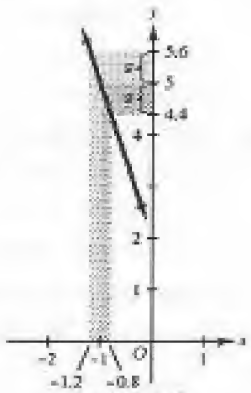
Exercises 1.4

1. $f(x) = 2x - 5$, $a = 3$, $\mathrm{L} = 1$, $\epsilon = 0.2$.
▹ From the figure, $\delta = .1$

2. $f(x) = 2 - 3x$, $a = -1$, $\mathrm{L} = 5$, $\epsilon = 0.6$.
▹ From the figure, $\delta = .4$



Exercise 1



Exercise 2

3 and 25. $f(x) = x - 1$, $a = 4$, $\mathrm{L} = 3$, $\epsilon = 0.03$.
▹ $x_1 - 1 = 3 - 0.03$, $x_1 - 4 = -0.03$. $x_2 - 1 = 3 + 0.03$, $x_2 - 4 = 0.03$. Choose $\delta = 0.03$.
$|f(x) - \mathrm{L}| = |(x-1) - 3| = |x - 4| < \epsilon$ when $|x - 4| < \epsilon = 0.03 = \delta$.

4 and 26. $f(x) = x + 2$, $a = 3$, $\mathrm{L} = 5$, $\epsilon = 0.02$
▹ $x_1 + 2 = 5 - 0.02$, $x_1 - 3 = -0.02$. $x_2 + 2 = 5 + 0.02$, $x_2 - 3 = 0.02$. Choose $\delta = 0.02$.
$|f(x) - \mathrm{L}| = |(x+2) - 5| = |x - 3| < \epsilon$ when $|x - 3| < \epsilon = 0.02 = \delta$

5. $f(x) = 2x + 4$, $a = 3$, $\mathrm{L} = 10$, $\epsilon = 0.01$
▹ $2x_1 + 4 = 10 - 0.01, 2x_1 - 6 = -.01$, $x_1 - 3 = -.005$. $2x_2 + 4 = 10 + .01$, $2x_2 - 6 = 0.01$, $x_2 - 3 = .005$. $\delta = .005$

6. $f(x) = 3x - 1$, $a = 2$, $\mathrm{L} = 5$, $\epsilon = 0.1$
▹ $3x_1 - 1 = 5 - 0.1$, $3x_1 - 6 = -0.1$, $x_1 - 2 = -\frac{1}{30}$. $3x_2 - 1 = 5 + 0.1$, $3x_2 - 6 = 0.1$, $x_2 - 2 = \frac{1}{30}$. Choose $\delta = \frac{1}{30}$.

7 and 27. $f(x) = 5x - 3$, $a = 1$, $\mathrm{L} = 2$, $\epsilon = 0.05$
▹ $5x_1 - 3 = 2 + 0.05$, $5x_1 - 5 = 0.05$, $x_1 - 1 = 0.01$. $5x_2 - 3 = 2 + 0.05$, $5x_2 - 5 = 0.05$, $x_2 - 1 = 0.01$. $\delta = 0.01$.
$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < \epsilon$ when $|x - 1| = \frac{1}{5}\epsilon = .01 = \delta$

8 and 28. $f(x) = 4x - 5$, $a = 2$, $\mathrm{L} = 3$, $\epsilon = 0.001$
▹ $4x_1 - 5 = 3 - .001$, $4x_1 - 8 = -.001$, $x_1 - 2 = -.00025$. $4x_2 - 5 = 3 + .001$, $4x_2 - 8 = .001$, $x_2 - 2 = .00025 = \delta$.
$|(4x - 5) - 3| = |4x - 8| = 4|x - 2| < \epsilon$ when $|x - 2| < \frac{1}{4}\epsilon = 0.00025 = \delta$

9. $f(x) = 3 - 4x$, $a = -1$, $\mathrm{L} = 7$, $\epsilon = 0.02$. f is decreasing.
▹ $3 - 4x_1 = 7 + .02$, $-4 - 4x_1 = .02$, $x_1 + 1 = -.005$. $3 - 4x_2 = 7 - .02$, $-4 - 4x_2 = -.02$, $x_2 + 1 = .005$. $\delta = .005$

10. $f(x) = 2 + 5x$, $a = -2$, $\mathrm{L} = -8$, $\epsilon = 0.002$
▹ $2 + 5x_1 = -8 - .002, 5x_1 + 10 = -.002, x_1 + 2 = -.0004$. $2 + 5x_2 = -8 + .002, 5x_2 + 10 = .002, x_2 + 2 = .0004 = \delta$

11. $f(x) = \dfrac{x^2-4}{x+2}$, $a = -2$, $\mathrm{L} = -4$, $\epsilon = 0.01$. ▷ $f(x) = x - 2$ if $x \neq -2$

▷ $x_1 - 2 = -4 - .01$, $x_1 - 2 = -4 - .01$, $x_1 + 2 = -.01$. $x_2 - 2 = -4 + .01$, $x_2 - 2 = -4 + .01$, $x_2 + 2 = .01 = \delta$

12. $f(x) = \dfrac{9x^2-1}{3x-1}$, $a = \frac{1}{3}$, $\mathrm{L} = 2$, $\epsilon = 0.01$ ▷ $f(x) = 3x + 1$ if $x \neq \frac{1}{3}$

$3x_1 + 1 = 2 - .01$, $3x_1 - 1 = -.01$, $x_1 - \frac{1}{3} = -\frac{1}{300}$. $3x_2 + 1 = 2 + .01$, $3x_2 - 1 = .01$, $x_2 - \frac{1}{3} = \frac{1}{300}$. $\delta = \frac{1}{300}$

13 and 29. $f(x) = \dfrac{4x^2-4x-3}{2x+1}$, $a = -\frac{1}{2}$, $\mathrm{L} = -4$, $\epsilon = 0.03$ ▷ $f(x) = 2x - 3$ if $x \neq -\frac{1}{2}$

$2x_1 - 3 = -4 - .03$, $2x_1 + 1 = -.03$, $x_1 + \frac{1}{2} = -.015$. $2x_1 - 3 = -4 + .03$, $2x_1 + 1 = .03$, $x_1 + \frac{1}{2} = .015$. $\delta = .015$

29. $\left|(2x-3) - (-4)\right| = |2x+1| = 2\left|x + \frac{1}{2}\right| < \epsilon$ when $0 < \left|x + \frac{1}{2}\right| < \frac{1}{2}\epsilon = .015 = \delta$

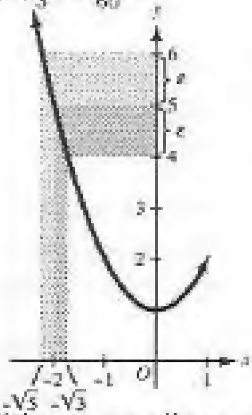
14 and 30. $f(x) = \dfrac{3x^2-8x-3}{x-3}$, $a = 3$, $\mathrm{L} = 10$, $\epsilon = .05$ ▷ $f(x) = 3x + 1$ if $x \neq 3$

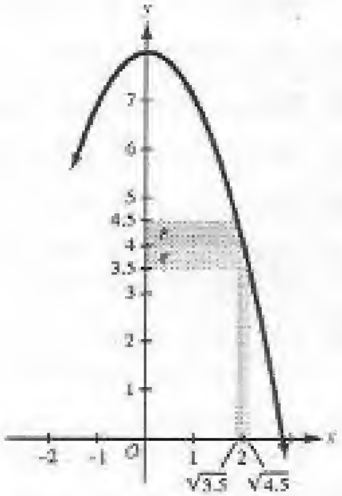
$3x_1 + 1 = 10 - .05$, $3x_1 - 9 = -.05$, $x_1 - 3 = -\frac{1}{60}$. $3x_2 + 1 = 10 + .05$, $3x_2 - 9 = .05$, $x_2 - 3 = \frac{1}{60}$. $\delta = \frac{1}{60}$

30. $\left|(3x+1) - 10\right| = |3x-9| = 3|x-3| < \epsilon$ when $0 < |x-3| < \frac{1}{3}\epsilon = \frac{1}{60} = \delta$

15. $f(x) = x^2 + 1$, $a = -2$, $\mathrm{L} = 5$, $\epsilon = 1$
▷ From the figure, $\delta = \min(-2 + \sqrt{6}, -\sqrt{3} + 2)$
$= \min(0.449, 0.268) = 0.268$

16. $f(x) = 9 - x^2$, $a = 2$, $\mathrm{L} = 5$, $\epsilon = 0.5$
▷ From the figure, $\delta = \min(2 - \sqrt{7/2}, \sqrt{9/2} - 2)$
$= \min(0.129, 0.121) = 0.121$





Note. For nonlinear f, the values of δ obtained from inequalities are usually smaller than those obtained directly. The factoring is really done first to determine what bounds we need.

17 and 31. $f(x) = x^2$, $a = 3$, $\mathrm{L} = 9$, $\epsilon = 0.5$ ▷ $a > 0 \Rightarrow x_1, x_2 > 0$. ${x_1}^2 = 9 - 0.5 = 8.5$,
$x_1 = \sqrt{8.5} = 2.915$, $3 - x_1 = .085$. ${x_2}^2 = 9 + 0.5 = 9.5$, $x_2 = \sqrt{9.5} = 3.082$. $x_2 - 3 = .082$. $\delta = .082$

31. Choose $\delta < 1$ so $-1 < x - 3 < 1$, $-5 < x + 3 < 7$. Then $\left|x^2 - 9\right| = |x+3||x-3| < 7|x-3| < \epsilon$ when $|x-3| < \frac{1}{7}\epsilon = \frac{1}{14} = \delta$

18 and 32. $f(x) = x^2$, $a = 0.5$, $\mathrm{L} = 0.25$, $\epsilon = 0.1$ ▷ $a > 0 \Rightarrow x_1, x_2 > 0$. ${x_1}^2 = .25 - .1 = .15$,
$x_1 = \sqrt{.15} = 0.387$, $.5 - x_1 = .113$. ${x_2}^2 = .25 + .1 = .35$, $x_2 = \sqrt{.35} = 0.592$, $x_2 - .5 = .092$. $\delta = .092$

32. Choose $\delta < 1$ so $-1 < x - .5 < 1$, $0 < x + .5 < 2$. Then $\left|x^2 - .25\right| = |x + .25||x - .25| < 2|x - .5| < \epsilon$ if $|x - .5| < \frac{1}{2}\epsilon = .05 = \delta$

19. $f(x) = x^2$, $a = -1$, $\mathrm{L} = 1$, $\epsilon = .2$ ▷ $a < 0 \Rightarrow x_1, x_2 < 0$. f is decreasing. ${x_1}^2 = 1 + .2 = 1.2$
$x_1 = -\sqrt{1.2} = -1.095$, $-1 - x_1 = .095$. ${x_2}^2 = 1 - .2 = -.8$, $x_2 = -\sqrt{.8} = -.894$, $x_1 - (-1) = .106$. $\delta = .095$

20. $f(x) = x^2 - 5$, $a = 1$, $\mathrm{L} = -4$, $\epsilon = 0.15$ ▷ $a > 0 \Rightarrow x_1, x_2 > 0$. ${x_1}^2 - 5 = -4 - .15$, ${x_1}^2 = .85$
$x_1 = \sqrt{.85} = .922$, $1 - x_1 = .078$. ${x_2}^2 - 5 = -4 + .15$, ${x_2}^2 = 1.15$, $x_2 = \sqrt{1.15} = 1.072$, $x_2 - 1 = .072$. $\delta = .072$

21 and 33. $f(x) = x^2 - 2x + 1$, $a = 2$, $\mathrm{L} = 1$, $\epsilon = 0.4$ ▷ $x_1, x_2 > 1$. ${x_1}^2 - 2x_1 + 1 = 1 - .4$, $(x_1 - 1)^2 = .6$
$x_1 - 1 = \sqrt{.6}$, $x_2 - 2 = \sqrt{.6} - 1 = -.225$. ${x_2}^2 - 2x_2 + 1 = 1 + .4$, $(x_2 - 1)^2 = 1.4$, $x_1 - 1 = \sqrt{1.4}$,
$x_2 - 2 = \sqrt{1.4} - 1 = .183$. $\delta = .183$

33. Choose $\delta < 1$ so $-1 < x - 2 < 1$, $1 < x < 3$. $\left|(x^2 - 2x + 1) - 1\right| = \left|x^2 - 2x\right| = |x||x-2| < 3|x-2| < \epsilon$ when $|x-2| < \frac{1}{3}\epsilon = \frac{2}{15} = \delta$.

22 and 34. $f(x) = x^2 + 4x + 4$, $a = -1$, $\mathrm{L} = 1$, $\epsilon = .08$ ▷ $x_1, x_2 > -2$. ${x_1}^2 + 4x_1 + 4 = 1 - .08$, $(x_1 + 2)^2 = .92$
$x_1 + 2 = \sqrt{.92}$, $x_1 + 1 = \sqrt{.92} - 1 = -.041$. ${x_2}^2 + 4x_2 + 4 = 1 + .08$, $(x_2 + 2)^2 = 1.08$, $x_2 + 2 = \sqrt{1.08}$.
$x_2 + 1 = \sqrt{1.08} - 1 = .0392$

34. Choose $\delta < 1$ so $-1 < x + 1 < 1, 1 < x + 3 < 3$. $\left|(x^2 + 4x + 4) - 1\right| = \left|x^2 + 4x + 3\right| = |x+3||x+1| < 3|x+1| < \epsilon$ when $|x+1| < \frac{1}{3}\epsilon = \frac{2}{75} = \delta$

23 and 35. $f(x) = 2x^2 + 5x + 3$, $a = -3$, $L = 6$, $\epsilon = .6$ ▷ $x_1, x_2 < -2.5$. f is decreasing. $2x_1{}^2 + 5x_1 + 3 = 6 + .6$, $2x_1{}^2 + 5x_1 - 3.6 = 0$, $x_1 = (-5 - \sqrt{5^2 + 4 \cdot 2 \cdot 3.6})/4 = -3.084$. $-3 - x_1 = .084$. $2x_2 + 5x_2 + 3 = 6 - .6$, $2x_2 + 5x_2 - 2.4 = 0$, $x_2 = (-5 - \sqrt{5^2 + 4 \cdot 2 \cdot 2.4})/4 = -2.912$. $x_2 + 3 = .088$. $2x_2 + 5x_2$. $\delta = .084$

35. Choose $\delta < 1$ so $-1 < x + 3 < 1$, $-2 < 2x + 6 < 2$, $-9 < 2x - 1 < -5$. $\left|(2x^2 + 5x + 3) - 6\right| = \left|2x^2 + 5x - 3\right| = |2x - 1||x + 3| < 9|x + 3| < \epsilon$ when $|x + 3| < \frac{1}{9}\epsilon = \frac{1}{15} = \delta$

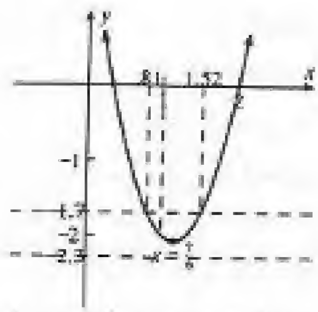
24 and 36. $f(x) = 3x^2 - 7x + 2$, $a = 1$, $L = -2$, $\epsilon = .3$

▷ $x_1, x_2 < \frac{7}{6}$. f is decreasing. $3x_1{}^2 - 7x_1 + 2 = -2 + .3$, $3x_1{}^2 - 7x_1 + 3.7 = 0$,

$x_1 = (7 - \sqrt{7^2 - 4 \cdot 3 \cdot 3.7})/6 = .809$, $1 - x_1 = .191$. Because

$f(x) \geq f(\frac{7}{6}) > -2 - .3$, we may take $x_2 = (7 + \sqrt{7^2 - 4 \cdot 3 \cdot 3.7})/6 = 1.524$,

$x_2 - 1 = .524$. $\delta = .191$.



36. Choose $\delta < 1$ so $-1 < x - 1 < 1$, $-3 < 3x - 3 < 3$, $-4 < 3x - 4 < 2$. $\left|(3x^2 - 7x + 2) - (-2)\right| = \left|3x^2 - 7x + 4\right| = |3x - 4||x - 1| < 4|x - 1| < \epsilon$ when $|x - 1| < \frac{1}{4}\epsilon = \frac{3}{40} = \delta$.

37. $15x$ dollars are earned working x hours. $|15x - 120| = 15|x - 8| < .25$ when $|x - 8| < \frac{1}{60}$. Within 1 min of 8 hr.

38. When the temperature is x degrees the volume is $\frac{4}{7}x$ cm^3. $79.95 < \frac{4}{7}x < 80.05$ when $139.9125 < x < 140.0875$.

39. When the side is x ft, the fence is $4x$ ft. $39.96 < 4x < 40.04$ $-.04 < 4x - 40 < .04$, $|x - 10| < .01$. Within .01 ft.

40. When the radius is r ft, the circumference is $2\pi r$ ft. $|2\pi r - 6\pi| < .1$, $|r - 3| < 1/20\pi$. Within .0159 ft.

41. When the side is x ft, the area is x^2 ft^2. Assume $-1 < x - 10 < 1$, $19 < x + 10 < 21$. $\left|x^2 - 100\right| = |x + 10||x - 10| < 21|x - 10| < .5$ when $|x - 10| < \frac{1}{42}$. Within $\frac{2}{7}$ in.

42. When the radius is r ft. the area is πr^2 ft^2. Assume $-1 < r - 3 < 1$, $5 < r + 3 < 7$. $\left|\pi r^2 - 9\pi\right| = \pi|r + 3||r - 3| < 7\pi|r - 3| < 0.2$ when $|r - 3| < 1/35\pi \approx 0.0091$. Within 0.0091 ft.

43. A body falls s ft in t sec, $s = kt^2$. $t = 2$: $64 = k2^2$, $k = 16$. $s = 16t^2$. $398 < 16t^2 < 402$, $t_1 = \sqrt{398/16} = 4.9875$, $5 - t_1 = .0125$. $t_2 = \sqrt{402/16} = 5.0125$, $t_2 - 5 = .0125$. Within .0125 sec.

44. A wind has pressure f psi when its velocity is v mph, $f = kv^2$. $v = 20$: $2 = k20^2$, $k = \frac{1}{200}$. $f = \frac{1}{200}v^2$.

$4.45 < \frac{1}{200}v^2 < 4.55$, $890 < v^2 < 910$, $29.833 < v < 30.166$, $-.167 < v - 30 < .166$. Within .166 mph.

1.5 DEFINITION OF THE LIMIT OF A FUNCTION AND LIMIT THEOREMS

The definition of limit was given in Section 1.4. The following theorems may often be used to find the limit of a function. "L.T." means "Limit Theorem".

L.T.1: Linear Function If m and b are any constants, $\lim\limits_{x \to a} (mx + b) = ma + b$.

L.T.2: Constant If c is a constant, then for any number a, $\lim\limits_{x \to a} c = c$.

L.T.3: Identity Function $\lim\limits_{x \to a} x = a$.

L.T.4: Sum/Difference If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} [f(x) \pm g(x)] = L \pm M$

$$\Leftrightarrow \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

L.T.5: Sum/Difference If $\lim\limits_{x \to a} f_1(x) = L_1$, $\lim\limits_{x \to a} f_2(x) = L_2$, …, and $\lim\limits_{x \to a} f_n(x) = L_n$, then

$$\lim_{x \to a} [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)] = L_1 \pm L \pm \cdots \pm L_n$$

$$\Leftrightarrow \lim_{x \to a} [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)] = \lim_{x \to a} f_1(x) \pm \lim_{x \to a} f_2(x) \pm \cdots \pm \lim_{x \to a} f_n(x)$$

L.T.6: Product If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} [f(x) \cdot g(x)] = L \cdot M$

$$\Leftrightarrow \lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

It follows from L.T.2 that if c is a constant, $\lim\limits_{x \to a} cf(x) = c \lim\limits_{x \to a} f(x)$.

L.T.7: Product If $\lim\limits_{x \to a} f_1(x) = L_1$, $\lim\limits_{x \to a} f_2(x) = L_2$, …, and $\lim\limits_{x \to a} f_n(x) = L_n$, then

$$\lim_{x \to a} [f_1(x) f_2(x) \cdots f_n(x)] = L_1 L \cdots L_n$$

$$\Leftrightarrow \lim_{x \to a} [f_1(x) f_2(x) \cdots f_n(x)] = \lim_{x \to a} f_1(x) \cdot \lim_{x \to a} f_2(x) \cdot \cdots \cdot \lim_{x \to a} f_n(x)$$

L.T.8: Power If $\lim_{x\to a} f(x) = \text{L}$ and n is any positive integer, then $\lim_{x\to a} [f(x)]^n = \text{L}^n$

$\Leftrightarrow \lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$

L.T.9: Quotient If $\lim_{x\to a} f(x) = \text{L}$ and $\lim_{x\to a} g(x) = \text{M}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\text{L}}{\text{M}}$ if $\text{M} \neq 0$

$\Leftrightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

L.T.10: Root If n is a positive integer, and $\lim_{x\to a} f(x) = \text{L}$, then $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\text{L}}$

$\Leftrightarrow \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

with the restriction that if n is even, $\text{L} > 0$.

Substitution Limit Theorems 1–10 imply that the limit as x approaches a of any formula (that is, excluding functions defined in pieces) can be found by substitution, if substitution is meaningful.

Fractions Note that we cannot use Limit Theorem 9 to find the limit of a fraction if the limit of the denominator is zero. However, when the limit of both the numerator and denominator is zero at $x = a$ we may be able to apply one of the following methods:

(i) If the numerator and denominator are both polynomials, we apply the *factor theorem* which states that if $x = a$ is a zero of a polynomial, then $x - a$ is a factor. The common factor $x - a$ may then be canceled. This is illustrated in Exercise 32.

(ii) If the denominator is a polynomial and a term of the numerator is a radical, we may multiply by its conjugate, which is not zero at a, and apply method (i) to the product. This is shown in Exercises 20 and 32.

In so doing, we use the following theorem, a consequence of Definition 1.5.1.

Theorem A If $\lim_{x\to a} g(x) = \text{L}$ and $f(x) = g(x)$ for all $x \neq a$, then $\lim_{x\to a} f(x) = \text{L}$.

We cannot use Limit Theorems 1–10 to find the limit of a fraction for which the limit of the denominator is zero and for which the limit of the numerator is not zero. In Section 1.7 we have a theorem for finding the limit of such a fraction.

Exercises 1.5

1. We wish to determine a $\delta > 0$ such that
   if $0 < |x-2| < \delta$ then $|7-7| < \epsilon$
   $\Leftrightarrow$ if $0 < |x-2| < \delta$ then $0 < \epsilon$
   Therefore for any $\epsilon > 0$, any choice of $\delta > 0$ will suffice; in particular take $\delta = \epsilon$.

2. We wish to determine a $\delta > 0$ such that
   if $0 < |x-5| < \delta$ then $|(-4)-(-4)| < \epsilon$
   $\Leftrightarrow$ if $0 < |x-5| < \delta$ then $0 < \epsilon$
   Therefore for any $\epsilon > 0$, any choice of $\delta > 0$ will suffice; in particular take $\delta = \epsilon$.

3. We wish to determine a $\delta > 0$ such that
   if $0 < |x-4| < \delta$ then $|(2x+1)-9| < \epsilon$
   $\Leftrightarrow$ if $0 < |x-4| < \delta$ then $2|x-4| < \epsilon$
   $\Leftrightarrow$ if $0 < |x-4| < \delta$ then $|x-4| < \frac{1}{2}\epsilon$
   Hence take $\delta = \frac{1}{2}\epsilon$; then $0 < |x-4| < \delta \Rightarrow 2|x-4| < 2(\frac{1}{2}\epsilon) \Rightarrow |(2x+1)-9| < \epsilon$.

4. We wish to determine a $\delta > 0$ such that
   if $0 < |x-1| < \delta$ then $|(4x+3)-7| < \epsilon$
   $\Leftrightarrow$ if $0 < |x-1| < \delta$ then $4|x-1| < \epsilon$
   $\Leftrightarrow$ if $0 < |x-1| < \delta$ then $|x-1| < \frac{1}{4}\epsilon$
   Hence take $\delta = \frac{1}{4}\epsilon$; then $0 < |x-1| < \delta \Rightarrow 4|x-1| < 4(\frac{1}{4}\epsilon) \Rightarrow |(4x+3)-7| < \epsilon$.

5. We wish to determine a $\delta > 0$ such that
if $0 < |x-3| < \delta$ then $|(7-3x)-(-2)| < \epsilon$
$\Leftrightarrow$ if $0 < |x-3| < \delta$ then $3|x-3| < \epsilon$
$\Leftrightarrow$ if $0 < |x-3| < \delta$ then $|x-3| < \frac{1}{3}\epsilon$
Hence, take $\delta = \frac{1}{3}\epsilon$; then $0 < |x-3| < \delta \Rightarrow 3|x-3| < 3(\frac{1}{3}\epsilon) \Rightarrow |(7-3x)-(-2)| < \epsilon$.

6. We wish to determine a $\delta > 0$ such that
if $0 < |x+4| < \delta$ then $|(2x+7)+1| < \epsilon$
$\Leftrightarrow$ if $0 < |x+4| < \delta$ then $2|x+4| < \epsilon$
$\Leftrightarrow$ if $0 < |x+4| < \delta$ then $|x+4| < \frac{1}{2}\epsilon$
Hence take $\delta = \frac{1}{2}\epsilon$; then $0 < |x+4| < \delta \Rightarrow 2|x+4| < 2(\frac{1}{2}\epsilon) \Rightarrow |(2x+7)-1| < \epsilon$.

7. We wish to determine a $\delta > 0$ such that
if $0 < |x+2| < \delta$ then $|(1+3x)-(-5)| < \epsilon$
$\Leftrightarrow$ if $0 < |x+2| < \delta$ then $3|x+2| < \epsilon$
$\Leftrightarrow$ if $0 < |x+2| < \delta$ then $|x+2| < \frac{1}{3}\epsilon$
Hence, take $\delta = \frac{1}{3}\epsilon$; then $0 < |x+2| < \delta \Rightarrow 3|x+2| < 3(\frac{1}{3}\epsilon) \Rightarrow |(1+3x)-(-5)| < \epsilon$.

8. We wish to determine a $\delta > 0$ such that
if $0 < |x+2| < \delta$ then $|(1+3x)-(-5)| < \epsilon$
$\Leftrightarrow$ if $0 < |x+2| < \delta$ then $3|x+2| < \epsilon$
$\Leftrightarrow$ if $0 < |x+2| < \delta$ then $|x+2| < \frac{1}{3}\epsilon$
Hence, take $\delta = \frac{1}{3}\epsilon$; then $0 < |x+2| < \delta \Rightarrow 3|x+2| < 3(\frac{1}{3}\epsilon) \Rightarrow |(1+3x)-(-5)| < \epsilon$.

9. We wish to determine a $\delta > 0$ such that
if $0 < |x+1| < \delta$ then $\left|\frac{x^2-1}{x+1}-(-2)\right| < \epsilon$
$\Leftrightarrow$ if $0 < |x+1| < \delta$ then $|x+1| < \epsilon$
Hence, take $\delta = \epsilon$; then $0 < |x+1| < \delta \Rightarrow 0 < |x+1| < \epsilon \Rightarrow \left|\frac{x^2-1}{x+1}-(-2)\right| < \epsilon$.

10. We wish to determine a $\delta > 0$ such that
if $0 < |x-3| < \delta$ then $\left|\frac{x^2-9}{x-3}-6\right| < \epsilon$
$\Leftrightarrow$ if $0 < |x-3| < \delta$ then $|x-3| < \epsilon$
Hence, take $\delta = \epsilon$; then $0 < |x-3| < \delta \Rightarrow 0 < |x-3| < \epsilon \Rightarrow \left|\frac{x^2-9}{x-3}-6\right| < \epsilon$.

11. $\lim\limits_{x\to 5}(3x-7) \overset{\text{L.T.1}}{=} 3(5)-7 = 8$

12. $\lim\limits_{x\to -4}(5x+2) \overset{\text{L.T.1}}{=} 5(-4)+2 = -18$

13. $\lim\limits_{x\to 2}(x^2+2x-1) \overset{\text{L.T.4}}{=} \lim\limits_{x\to 2} x^2 + \lim\limits_{x\to 2}(2x-5) \overset{\substack{\text{L.T.8}\\ \text{L.T.1}}}{=} (\lim\limits_{x\to 2} x)^2 + [2(2)-1] \overset{\text{L.T.3}}{=} (2)^2+3 = 7$

14. $\lim\limits_{x\to 3}(2x^2-4x+5) \overset{\text{L.T.4}}{=} \lim\limits_{x\to 3}2x^2 + \lim\limits_{x\to 3}(-4x+5) \overset{\substack{\text{L.T.1}\\ \text{L.T.7}}}{=} \lim\limits_{x\to 3}2\cdot\lim\limits_{x\to 3}x\cdot\lim\limits_{x\to 3}x + (-4)3+5 \overset{\text{L.T.1}}{=} 2\cdot 3\cdot 3-7 = 11$

15. $\lim\limits_{x\to -2}(x^3+8) \overset{\text{L.T.4}}{=} \lim\limits_{x\to -2} x^3 + \lim\limits_{x\to -2} 8 \overset{\substack{\text{L.T.8}\\ \text{L.T.3}}}{=} \left(\lim\limits_{x\to -2} x\right)^3 + 8 \overset{\text{L.T.2}}{=} (-2)^3+8 = 0$

16. $\lim\limits_{y\to -1}(y^3-2y^2+3y-4) \overset{\text{L.T.1}}{=} \lim\limits_{y\to -1} y^3 - \lim\limits_{y\to -1} 2y^2 + \lim\limits_{y\to -1}(3y-4)$
$\overset{\text{L.T.7}}{=} \lim\limits_{y\to -1} y\cdot\lim\limits_{y\to -1} y\cdot\lim\limits_{y\to -1} y - \lim\limits_{y\to -1} 2\cdot\lim\limits_{y\to -1} y\cdot\lim\limits_{y\to -1} y + \lim\limits_{y\to -1}(3y-4) \overset{\text{L.T.1}}{=} (-1)^3-2(-1)^2+3(-1)-4 = -10$

17. $\lim\limits_{x\to 3}\frac{4x-5}{5x-1} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to 3}(4x-5)}{\lim\limits_{x\to 3}(5x-1)} \overset{\text{L.T.1}}{=} \frac{4(3)-5}{5(3)-1} = \frac{1}{2}$

18. $\lim\limits_{x\to 2}\frac{3x+4}{8x-1} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to 2}(3x+4)}{\lim\limits_{x\to 2}(8x-1)} \overset{\text{L.T.1}}{=} \frac{3(2)+4}{8(2)-1} = \frac{10}{15} = \frac{2}{3}$

19. $\displaystyle\lim_{t\to 2}\frac{t^2-5}{2t^3+6} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{t\to 2}(t^2-5)}{\lim\limits_{t\to 20}(2t^3+6)} \overset{\text{L.T.4}}{=} \frac{\lim\limits_{t\to 2}t^2-\lim\limits_{t\to 2}5}{\lim\limits_{t\to 2}(2t^3)+\lim\limits_{t\to 2}(6)} \overset{\substack{\text{L.T.2}\\ \text{L.T.6}}}{=} \frac{\lim\limits_{t\to 2}t^2-5}{(\lim\limits_{t\to 2}2)(\lim\limits_{t\to 2}t^3)+6} \overset{\substack{\text{L.T.2}\\ \text{L.T.8}}}{=} \frac{(\lim\limits_{t\to 2}t)^2-5}{2(\lim\limits_{t\to 2}t)^3+6} =$

$\displaystyle\overset{\text{L.T.3}}{=} \frac{(2)^2-5}{2(2)^3+6} = -\frac{1}{22}$

20. $\displaystyle\lim_{x\to -1}\frac{2x+1}{x^2-3x+4} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to -1}(2x+1)}{\lim\limits_{x\to -1}(x^2-3x+4)} \overset{\text{L.T.4}}{=} \frac{\lim\limits_{x\to -1}(2x+1)}{\lim\limits_{x\to -1}x^2+\lim\limits_{x\to -1}(-3x+4)} \overset{\text{L.T.8}}{=} \frac{\lim\limits_{x\to -1}(2x+1)}{\left(\lim\limits_{x\to -1}x\right)^2+\lim\limits_{x\to -1}(-3x+4)}$

$\displaystyle\overset{\text{L.T.1}}{=} \frac{2(-1)+1}{(-1)^2+(-3)(-1)+4} = \frac{-1}{8}$

21. $\displaystyle\lim_{r\to 1}\sqrt{\frac{8r+1}{r+3}} \overset{\text{L.T.10}}{=} \sqrt{\lim_{r\to 1}\frac{8r+1}{r+3}} \overset{\text{L.T.9}}{=} \sqrt{\frac{\lim\limits_{r\to 1}(8r+1)}{\lim\limits_{r\to 2}(r+3)}} \overset{\text{L.T.1}}{=} \sqrt{\frac{8(1)+1}{1+3}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$

22. $\displaystyle\lim_{x\to 2}\sqrt{\frac{x^2+3x+4}{x^3+1}} \overset{\text{L.T.10}}{=} \sqrt{\lim_{x\to 2}\frac{x^2+3x+4}{x^3+1}} \overset{\text{L.T.9}}{=} \sqrt{\frac{\lim\limits_{x\to 2}(x^2+3x+4)}{\lim\limits_{x\to 2}x^3+1}} \overset{\text{L.T.4}}{=} \sqrt{\frac{\lim\limits_{x\to 2}x^2+\lim\limits_{x\to 2}(3x+4)}{\lim\limits_{x\to 2}x^3+\lim\limits_{x\to 2}1}}$

$\displaystyle\overset{\text{L.T.8}}{=} \sqrt{\frac{(\lim\limits_{x\to 2}x)^2+\lim\limits_{x\to 2}(3x+4)}{(\lim\limits_{x\to 2}x)^3+\lim\limits_{x\to 2}1}} \overset{\text{L.T.1}}{=} \sqrt{\frac{2^2+3\cdot 2+4}{2^3+1}} = \sqrt{\frac{14}{9}} = \frac{1}{3}\sqrt{14}$

23. $\displaystyle\lim_{x\to 4}\sqrt[3]{\frac{x^2-3x+4}{2x^2-x-1}} \overset{\text{L.T.10}}{=} \sqrt[3]{\lim_{x\to 4}\frac{x^2-3x+4}{2x^2-x-1}} \overset{\text{L.T.9}}{=} \sqrt[3]{\frac{\lim\limits_{x\to 4}(x^2-3x+4)}{\lim\limits_{x\to 4}(2x^2-x-1)}} \overset{\text{L.T.4}}{=} \sqrt[3]{\frac{\lim\limits_{x\to 4}x^2+\lim\limits_{x\to 4}(-3x+4)}{\lim\limits_{x\to 4}2x^2+\lim\limits_{x\to 4}(-x-1)}}$

$\displaystyle\overset{\substack{\text{L.T.7}\\ \text{L.T.1}\\ \text{L.T.8}}}{=} \sqrt[3]{\frac{(\lim\limits_{x\to 4}x)^2+(-8)}{\lim\limits_{x\to 4}2(\lim\limits_{x\to 4}x)^2+(-5)}} \overset{\substack{\text{L.T.3}\\ \text{L.T.2}}}{=} \sqrt[3]{\frac{(4)^2-8}{2(4)^2-5}} = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$

24. $\displaystyle\lim_{x\to -3}\sqrt[3]{\frac{5+2x}{5-x}} \overset{\text{L.T.10}}{=} \sqrt[3]{\lim_{x\to -3}\frac{5+2x}{5-x}} \overset{\text{L.T.9}}{=} \sqrt[3]{\frac{\lim\limits_{x\to -3}(5+2x)}{\lim\limits_{x\to -3}(5-x)}} \overset{\text{L.T.1}}{=} \sqrt[3]{\frac{5+2(-3)}{5-(-3)}} = \sqrt[3]{\frac{-1}{8}} = -\frac{1}{2}$

25. $f(x)=\dfrac{x-2}{x^2-4}$ (a)

x	1	1.5	1.9	1.99	1.999	3	2.5	2.1	2.01	2.001
f	.3333	.2857	.2564	.2506	.2501	.2000	.2222	.2439	.2494	.2499

(b) $\displaystyle\lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2}\frac{1}{x+2} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to 2}1}{\lim\limits_{x\to 2}(x+2)} \overset{\text{L.T.1}}{=} \frac{1}{4}$

26. $f(x)=\dfrac{2x^2+3x-2}{x^2-6x-16}$ (a)

x	-3	-2.5	-2.1	-2.01	-2.001	-1	-1.5	-1.9	-1.99	-1.999
f	.6364	.5714	.5149	.5015	.5002	.3333	.4211	.4848	.4985	.4998

(b) $\displaystyle\lim_{x\to -2}\frac{2x^2+3x-2}{x^2-6x-16} = \lim_{x\to -2}\frac{(x+2)(2x-1)}{(x+2)(x-8)} = \lim_{x\to -2}\frac{2x-1}{x-8} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to -2}(2x-1)}{\lim\limits_{x\to -2}(x-8)} \overset{\text{L.T.1}}{=} \frac{2(-2)-1}{-2-8} = \frac{-2}{-10} = \frac{1}{2}$

27. $f(x)=\dfrac{x^2+5x+6}{x^2-x-12}$ (a)

x	-4	-3.5	-3.1	-3.01	-3.001	-3.0001	-2	-2.5	-2.9	-2.99	-2.999	-2.9999
f	.2500	.2000	.1549	.1441	.1430	.1429	.0000	.0769	.1304	.1416	.1427	.1428

(b) $\displaystyle\lim_{x\to -3}\frac{x^2+5x+6}{x^2-x-12} = \lim_{x\to -3}\frac{(x+2)(x+3)}{(x-4)(x+3)} = \lim_{x\to -3}\frac{x+2}{x-4} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to -3}(x+2)}{\lim\limits_{x\to -3}(x-4)} \overset{\text{L.T.1}}{=} \frac{-3+2}{-3-4} = \frac{-1}{-7} = \frac{1}{7}$

28. $f(x)=\dfrac{2x-3}{4x^2-9}$ (a)

x	1	1.4	1.49	1.499	1.4999	2	1.6	1.51	1.501	1.5001
f	.2	.1724	.1672	.1667	.1667	.1429	.1613	.1661	.1666	.1667

(b) $\displaystyle\lim_{x\to 1.5}\frac{2x-3}{4x^2-9} = \lim_{x\to 1.5}\frac{2x-3}{(2x-3)(2x+3)} = \lim_{x\to 1.5}\frac{1}{2x+3} \overset{\text{L.T.9}}{=} \frac{\lim\limits_{x\to 1.5}1}{\lim\limits_{x\to 1.5}(2x+3)} \overset{\text{L.T.1}}{=} \frac{1}{2\cdot 1.5+3} = \frac{1}{6}$

29. $f(x)=\dfrac{3-\sqrt{x}}{9-x}$ (a)

x	8	8.5	8.9	8.99	8.999	10	9.5	9.1	9.01	9.001
f	.1716	.1690	.1671	.1667	.1667	.1623	.1644	.1662	.1666	.1667

(b) $\displaystyle\lim_{x\to 9}\frac{3-\sqrt{x}}{9-x}=\lim_{x\to 9}\frac{(3-\sqrt{x})(3+\sqrt{x})}{(9-x)(3+\sqrt{x})}=\lim_{x\to 9}\frac{9-x}{(9-x)(3+\sqrt{x})}=\lim_{x\to 9}\frac{1}{3+\sqrt{x}}\overset{\text{L.T.9}}{=}\frac{\lim_{x\to 9}1}{\lim_{x\to 9}(3+\sqrt{x})}$

$\displaystyle\overset{\substack{\text{L.T.4}\\\text{L.T.2}}}{=}\frac{1}{\lim_{x\to 9}3+\lim_{x\to 9}\sqrt{x}}\overset{\substack{\text{L.T.10}\\\text{L.T.2}}}{=}\frac{1}{3+\sqrt{\lim_{x\to 9}x}}\overset{\text{L.T.3}}{=}\frac{1}{3+\sqrt{9}}=\frac{1}{6}$

30. $f(x)=\dfrac{2-\sqrt{4-x}}{x}$ (a)

x	-1	-0.5	-0.1	-0.01	-.001	1	0.5	0.1	0.01	0.001
f	.2361	.2426	.2485	.2498	.2500	.2679	.2583	.2516	.2502	.2500

$\displaystyle\lim_{x\to 0}\frac{2-\sqrt{4-x}}{x}\cdot\frac{2+\sqrt{4-x}}{2+\sqrt{4-x}}=\lim_{x\to 0}\frac{4-(4-x)}{x(2+\sqrt{4-x})}=\lim_{x\to 0}\frac{1}{2+\sqrt{4-x}}\overset{\text{L.T.9}}{=}\frac{\lim_{x\to 0}1}{\lim_{x\to 0}(2+\sqrt{4-x})}$

$\displaystyle\overset{\text{L.T.4}}{=}\frac{\lim_{x\to 0}1}{\lim_{x\to 0}2+\lim_{x\to 0}\sqrt{4-x}}\overset{\text{L.T.10}}{=}\frac{\lim_{x\to 0}1}{\lim_{x\to 0}2+\sqrt{\lim_{x\to 0}(4-x)}}\overset{\text{L.T.1}}{=}\frac{1}{2+\sqrt{4}}=\frac{1}{4}$

31. $\displaystyle\lim_{x\to 7}\frac{x^2-49}{x-7}=\lim_{x\to 7}\frac{(x-7)(x+7)}{x-7}=\lim_{x\to 7}x+7\overset{\text{L.T.1}}{=}7+7=14$

32. $\displaystyle\lim_{z\to -5}\frac{z^2-25}{z+5}=\lim_{z\to -5}\frac{(z-5)(z+5)}{z+5}=\lim_{z\to -5}z-5\overset{\text{L.T.1}}{=}-5-5=-10$

33. $\displaystyle\lim_{x\to -3/2}\frac{4x^2-9}{2x+3}=\lim_{x\to -3/2}\frac{(2x+3)(2x-3)}{2x+3}=\lim_{x\to -3/2}2x-3\overset{\text{L.T.1}}{=}2(-\tfrac{3}{2})-3=-6$

34. $\displaystyle\lim_{x\to 1/3}\frac{3x-1}{9x^2-1}=\lim_{x\to 1/3}\frac{3x-1}{(3x-1)(3x+1)}=\lim_{x\to 1/3}\frac{1}{3x+1}\overset{\text{L.T.1}}{=}\frac{1}{3(1/3)+1}=\frac{1}{2}$

35. $\displaystyle\lim_{s\to 4}\frac{3s^2-8s-16}{2s^2-9s+4}=\lim_{s\to 4}\frac{(3s+4)(s-4)}{(2s-1)(s-4)}=\lim_{s\to 4}\frac{3s+4}{2s-1}\overset{\text{L.T.9}}{=}\frac{\lim_{s\to 4}(3s+4)}{\lim_{s\to 4}(2s-1)}\overset{\text{L.T.1}}{=}\frac{3(4)+4}{2(4)-1}=\frac{16}{7}$

36. $\displaystyle\lim_{x\to 4}\frac{3x^2-17x+20}{4x^2-25x+36}=\lim_{x\to 4}\frac{(x-4)(3x-5)}{(x-4)(4x-9)}=\lim_{x\to 4}\frac{3x-5}{4x-9}\overset{\text{L.T.9}}{=}\frac{\lim_{x\to 4}(3x-5)}{\lim_{x\to 4}(4x-9)}\overset{\text{L.T.1}}{=}\frac{3\cdot 4-5}{4\cdot 4-9}=\frac{7}{7}=1$

37. $\displaystyle\lim_{y\to -2}\frac{y^3+8}{y+2}=\lim_{y\to -2}\frac{(y+2)(y^2-2y+4)}{y+2}=\lim_{y\to -2}(y^2-2y+4)\overset{\text{L.T.4}}{=}\lim_{y\to -2}y^2+\lim_{y\to -2}(-2y+4)$

$\displaystyle\overset{\substack{\text{L.T.8}\\\text{L.T.1}}}{=}(\lim_{y\to -2}y)^2+[(-2)(-2)+4]\overset{\text{L.T.3}}{=}(-2)^2+8=12$

38. $\displaystyle\lim_{s\to 1}\frac{s^3-1}{s-1}=\lim_{s\to 1}\frac{(s-1)(s^2+s+1)}{s-1}=\lim_{s\to 1}(s^2+s+1)\overset{\text{L.T.4}}{=}\lim_{s\to 1}s^2+\lim_{s\to 1}(s+1)\overset{\text{L.T.8}}{=}(\lim_{s\to 1}s)^2+\lim_{s\to 1}(s+1)$

$\overset{\text{L.T.1}}{=}1^2+1+1=3$

39. $\displaystyle\lim_{y\to -3}\sqrt{\frac{y^2-9}{2y^2+7y+3}}=\lim_{y\to -3}\sqrt{\frac{(y-3)(y+3)}{(2y+1)(y+3)2}}=\lim_{y\to -3}\sqrt{\frac{y-3}{2y+1}}\overset{\text{L.T.10}}{=}\sqrt{\lim_{y\to -3}\frac{y-3}{2y+1}}$

$\displaystyle\overset{\text{L.T.9}}{=}\sqrt{\frac{\lim_{y\to -3}(y-3)}{\lim_{y\to -3}(2y+1)}}\overset{\text{L.T.1}}{=}\sqrt{\frac{-6}{-5}}=\sqrt{\frac{6}{5}}=\frac{1}{5}\sqrt{30}$

40. $\displaystyle\lim_{t\to 3/2}\sqrt{\frac{8t^3-27}{4t^2-9}}=\lim_{t\to 3/2}\sqrt{\frac{(2t-3)(4t^2+6t+9)}{(2t-3)(2t+3)}}=\lim_{t\to 3/2}\sqrt{\frac{4t^2+6t+9}{2t+3}}\overset{\substack{\text{L.T.10}\\\text{L.T.9}}}{=}\sqrt{\frac{\lim_{t\to 3/2}(4t^2+6t+9)}{\lim_{t\to 3/2}(2t+1)}}$

$\displaystyle\overset{\substack{\text{L.T.4}\\\text{L.T.7}}}{=}\sqrt{\frac{\lim_{t\to 3/2}4\cdot\lim_{t\to 3/2}t\cdot\lim_{t\to 3/2}t+\lim_{t\to 3/2}(6t+9)}{\lim_{t\to 3/2}(2t+1)}}\overset{\text{L.T.1}}{=}\sqrt{\frac{4(\frac{3}{2})^2+6\cdot\frac{3}{2}+9}{2\cdot\frac{3}{2}+1}}=\sqrt{\frac{27}{4}}=\frac{3}{2}\sqrt{3}$

41. $\lim\limits_{x\to1}\dfrac{\sqrt{x}-1}{x-1}=\lim\limits_{x\to1}\dfrac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)}=\lim\limits_{x\to1}\dfrac{1}{\sqrt{x}+1}\overset{\text{L.T.9}}{=}\dfrac{\lim\limits_{x\to1}1}{\lim\limits_{x\to1}(\sqrt{x}+1)}\overset{\substack{\text{L.T.2}\\\text{L.T.4}}}{=}\dfrac{1}{\lim\limits_{x\to1}\sqrt{x}+\lim\limits_{x\to1}1}$

$\overset{\substack{\text{L.T.10}\\\text{L.T.2}}}{=}\dfrac{1}{\sqrt{\lim\limits_{x\to1}x}+1}\overset{\text{L.T.3}}{=}\dfrac{1}{\sqrt{1}+1}=\frac{1}{2}$

42. $\lim\limits_{x\to-1}\dfrac{\sqrt{x+5}-2}{x+1}\cdot\dfrac{\sqrt{x+5}+2}{\sqrt{x+5}+2}=\lim\limits_{x\to-1}\dfrac{(x+5)-4}{(x+1)(\sqrt{x+5}+2)}=\lim\limits_{x\to-1}\dfrac{1}{\sqrt{x+5}+2}\overset{\text{L.T.9}}{=}\dfrac{\lim\limits_{x\to-1}1}{\lim\limits_{x\to-1}(\sqrt{x+5}+2)}$

$\overset{\text{L.T.4}}{=}\dfrac{\lim\limits_{x\to-1}1}{\lim\limits_{x\to-1}\sqrt{x+5}+\lim\limits_{x\to-1}2}\overset{\text{L.T.10}}{=}\dfrac{\lim\limits_{x\to-1}1}{\sqrt{\lim\limits_{x\to-1}(x+5)}+\lim\limits_{x\to-1}2}\overset{\text{L.T.1}}{=}\dfrac{1}{\sqrt{-1+5}+2}=\frac{1}{4}$

43. $\lim\limits_{h\to0}\dfrac{\sqrt{h+2}-\sqrt{2}}{h}=\lim\limits_{h\to0}\dfrac{(\sqrt{h+2}-\sqrt{2})(\sqrt{h+2}+\sqrt{2})}{h(\sqrt{h+2}+\sqrt{2})}=\lim\limits_{h\to0}\dfrac{(h+2)-2}{h(\sqrt{h+2}+\sqrt{2})}=\lim\limits_{h\to0}\dfrac{1}{\sqrt{h+2}+\sqrt{2}}$

$\overset{\text{L.T.9}}{=}\dfrac{\lim\limits_{h\to0}1}{\lim\limits_{h\to0}(\sqrt{h+2}+\sqrt{2})}\overset{\substack{\text{L.T.2}\\\text{L.T.4}}}{=}\dfrac{1}{\lim\limits_{h\to0}\sqrt{h+2}+\lim\limits_{h\to0}\sqrt{2}}\overset{\substack{\text{L.T.10}\\\text{L.T.2}}}{=}\dfrac{1}{\sqrt{\lim\limits_{h\to0}(h+2)}+\sqrt{2}}\overset{\text{L.T.1}}{=}\dfrac{1}{\sqrt{2}+\sqrt{2}}=\dfrac{1}{2\sqrt{2}}=\frac{1}{4}\sqrt{2}$

44. $\lim\limits_{x\to1}\dfrac{\sqrt[3]{x}-1}{x-1}=\lim\limits_{x\to1}\dfrac{\sqrt[3]{x}-1}{(\sqrt[3]{x})^3-1}=\lim\limits_{x\to1}\dfrac{\sqrt[3]{x}-1}{(\sqrt[3]{x}-1)[(\sqrt[3]{x})^2+\sqrt[3]{x}+1]}=\lim\limits_{x\to1}\dfrac{1}{(\sqrt[3]{x})^2+\sqrt[3]{x}+1}$

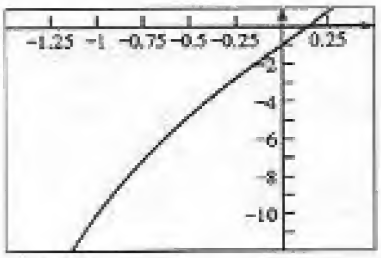
$\overset{\substack{\text{L.T.4}\\\text{L.T.9}}}{=}\dfrac{\lim\limits_{x\to1}1}{\lim\limits_{x\to1}(\sqrt[3]{x})^2+\lim\limits_{x\to1}\sqrt[3]{x}+\lim\limits_{x\to1}1}\overset{\substack{\text{L.T.8}\\\text{L.T.10}}}{=}\dfrac{\lim\limits_{x\to1}1}{(\sqrt[3]{\lim\limits_{x\to1}x})^2+\sqrt[3]{\lim\limits_{x\to1}x}+\lim\limits_{x\to1}1}\overset{\text{L.T.1}}{=}\dfrac{1}{1+1+1}=\frac{1}{3}$

45. $\lim\limits_{x\to-1}\dfrac{2x^2-x-3}{x^3+2x^2+6x+5}=\lim\limits_{x\to-1}\dfrac{(2x-3)(x+1)}{(x^2+x+5)(x+1)}=\lim\limits_{x\to-1}\dfrac{2x-3}{x^2+x+5}\overset{\text{L.T.9}}{=}\dfrac{\lim\limits_{x\to-1}(2x-3)}{\lim\limits_{x\to-1}(x^2+x+5)}$

$\overset{\substack{\text{L.T.4}\\\text{L.T.1}}}{=}\dfrac{-5}{\lim\limits_{x\to-1}x^2+\lim\limits_{x\to-1}(x+5)}\overset{\substack{\text{L.T.1}\\\text{L.T.8}}}{=}\dfrac{-5}{(\lim\limits_{x\to-1}x)^2+4}\overset{\text{L.T.3}}{=}\dfrac{-5}{(-1)^2+4}=-1$

46. $\lim\limits_{x\to-2}\dfrac{x^3-x^2-x+10}{x^2+3x+2}=\lim\limits_{x\to-2}\dfrac{(x+2)(x^2-3x+5)}{(x+2)(x+1)}=\lim\limits_{x\to-2}\dfrac{x^2-3x+5}{x+1}\overset{\substack{\text{L.T.9}\\\text{L.T.4}}}{=}\dfrac{\lim\limits_{x\to-2}x^2+\lim\limits_{x\to-2}(-3x+5)}{\lim\limits_{x\to-2}(x+1)}$

$\overset{\text{L.T.8}}{=}\dfrac{(\lim\limits_{x\to-2}x)^2+\lim\limits_{x\to-2}(-3x+5)}{\lim\limits_{x\to-2}(x+1)}\overset{\text{L.T.1}}{=}\dfrac{(-2)^2-3(-2)+5}{-2+1}=\dfrac{15}{-1}=-15$



47. $\lim\limits_{x\to2}f(x)=\lim\limits_{x\to2}(x^2+5x-3)\overset{\text{L.T.4}}{=}\lim\limits_{x\to2}x^2+\lim\limits_{x\to2}(5x-3)$

$\overset{\text{L.T.8}}{=}(\lim\limits_{x\to2}x)^2+\lim\limits_{x\to2}(5x-3)\overset{\text{L.T.1}}{=}2^2+5(2)-3=f(2)$

48. $\lim\limits_{x\to-1}F(x)=\lim\limits_{x\to-1}(2x^3+7x-1)\overset{\text{L.T.4}}{=}\lim\limits_{x\to-1}2x^3+\lim\limits_{x\to-1}(7x-1)$

$\overset{\text{L.T.7}}{=}\lim\limits_{x\to-1}2\cdot\lim\limits_{x\to-1}x\cdot\lim\limits_{x\to-1}x+\lim\limits_{x\to-1}(7x-1)\overset{\text{L.T.1}}{=}2(-1)^3+7(-1)-1=10=F(-1)$

49. $g(x)=\dfrac{x^2-1}{x-1}$. $g(1)$ does not exist because $0/0$ is not defined. $\lim\limits_{x\to1}g(x)=\lim\limits_{x\to1}(x+1)=1+1=2$

50. $G(x)=\dfrac{x-1}{x^2-1}$. $G(1)$ does not exist because $0/0$ is not defined. $\lim\limits_{x\to1}G(x)=\lim\limits_{x\to1}\dfrac{1}{x+1}=\dfrac{1}{1+1}=\dfrac{1}{2}$

51. $h(x)=\dfrac{\sqrt{x+9}-3}{x}$. $h(0)$ does not exist because $0/0$ is not defined.

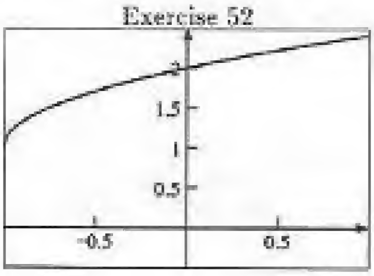
$\lim\limits_{x\to0}h(x)=\dfrac{(\sqrt{x+9}-3)(\sqrt{x+9}+3)}{x(\sqrt{x+9}+3)}=\lim\limits_{x\to0}\dfrac{x+9-9}{x(\sqrt{x+9}+3)}=\lim\limits_{x\to0}\dfrac{1}{\sqrt{x+9}+3}=\dfrac{1}{\sqrt{9}+3}=\dfrac{1}{6}$

52. $H(x)=\dfrac{x}{\sqrt{x+1}-1}$. $H(0)$ does not exist because $0/0$ is not defined.

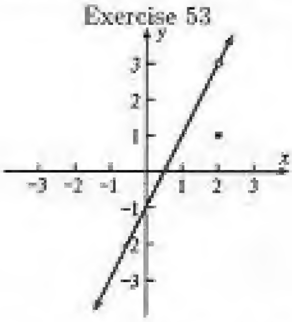
$\lim\limits_{x\to0}H(x)=\dfrac{x}{\sqrt{x+1}-1}\cdot\dfrac{\sqrt{x+1}+1}{\sqrt{x+1}+1}=\lim\limits_{x\to0}\dfrac{x(\sqrt{x+1}+1)}{(x+1)-1}=\lim\limits_{x\to0}(\sqrt{x+1}+1)=1+1=2$

53. $f(x)=\begin{cases}2x-1 & \text{if } x\neq 2\\ 1 & \text{if } x=2\end{cases}$. $\lim\limits_{x\to2} f(x)=\lim\limits_{x\to 2}(2x-1)=2\cdot 2-1=3\neq 1=f(2)$

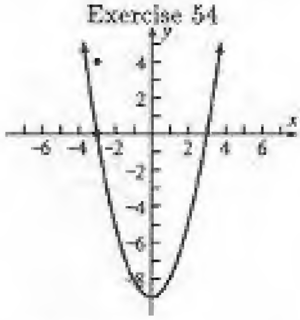
Exercise 52



Exercise 53



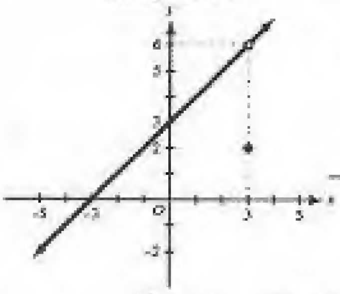
Exercise 54



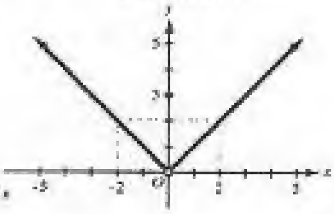
54. $f(x)=\begin{cases}x^2-9 & \text{if } x\neq -3\\ 4 & \text{if } x=-3\end{cases}$. $\lim\limits_{x\to-3} f(x)=\lim\limits_{x\to -3}(x^2-9)=(-3)^2-9\neq 4=f(-3)$

In Exercises 55–58, answer parts (a)–(c) from the graph of f sketched in the accompanying figure.

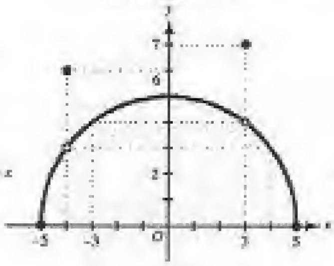
Exercise 55



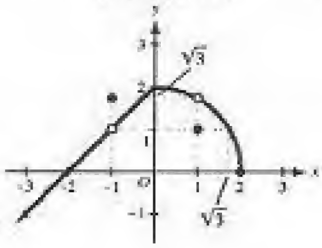
Exercise 56



Exercise 57



Exercise 58



55. $f(x)=\begin{cases}x+3 & \text{if } x\neq 3\\ 2 & \text{if } x=3\end{cases}$. $f(-3)=0$, $f(0)=3$, $f(3)=2$. $\lim\limits_{x\to-3} f(x)=0$. $\lim\limits_{x\to0} f(x)=3$. $\lim\limits_{x\to3} f(x)=6$

56. $f(x)=\begin{cases}|x| & \text{if } x\neq 0\\ 3 & \text{if } x=0\end{cases}$. $f(-2)=2$, $f(0)=3$, $f(2)=2$. $\lim\limits_{x\to-2} f(x)=2$. $\lim\limits_{x\to0} f(x)=0$. $\lim\limits_{x\to2} f(x)=2$

57. $f(x)=\begin{cases}5 & \text{if } x=-4\\ 6 & \text{if } x=3\\ \sqrt{25-x^2} & \text{if } x\in[-5,-4)\cup(-4,3)\cup(3,5]\end{cases}$. $f(-4)=5$, $f(-3)=4$, $f(3)=6$, $f(4)=3$

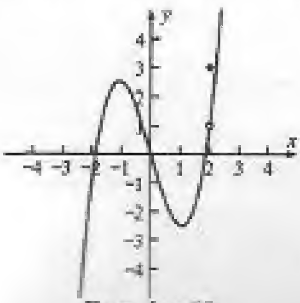
$\lim\limits_{x\to-4} f(x)=3$, $\lim\limits_{x\to-3} f(x)=4$, $\lim\limits_{x\to3} f(x)=4$, $\lim\limits_{x\to4} f(x)=3$

58. $f(x)=\begin{cases}\sqrt{3} & \text{if } x=-1\\ 1 & \text{if } x=1\\ x+2 & \text{if } x\in(-\infty,-1)\cup(-1,0)\\ \sqrt{4-x^2} & \text{if } x\in[0,1)\cup(1,2]\end{cases}$. $f(-1)=\sqrt{3}$, $f(0)=2$, $f(1)=1$, $f(\sqrt{3})=1$.

$\lim\limits_{x\to-1} f(x)=1$, $\lim\limits_{x\to0} f(x)=2$, $\lim\limits_{x\to1} f(x)=\sqrt{3}$, $\lim\limits_{x\to\sqrt{3}} f(x)=1$

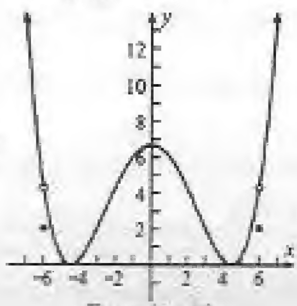
In Exercises 59–62, sketch the graph of a function f with domain $(-\infty,+\infty)$ satisfying the given properties.

60. $f(-3)=4$; $f(3)=-5$; $\lim\limits_{x\to-3} f(x)=-5$; $\lim\limits_{x\to3} f(x)=4$; $\lim\limits_{x\to a} f(x)=f(a)$ if $a\neq\pm3$; range$(f)=(-\infty,+\infty)$

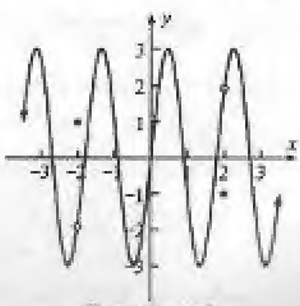


Exercise 59

Exercise 60



Exercise 61



Exercise 62

63. $\lim_{x\to a} f(x) = \mathrm{L} \Leftrightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - \mathrm{L}| < \epsilon$ when $0 < |x - a| < \delta$

$\Leftrightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $|[f(x) - \mathrm{L}] - 0| < \epsilon$ when $0 < |x - a| < \delta$

$\Leftrightarrow \lim_{x\to a}[f(x) - \mathrm{L}] = 0$

64. Let $t = x - a$, $x = t + a$.
$\lim_{x\to a} f(x) = \mathrm{L} \Leftrightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - \mathrm{L}| < \epsilon$ when $0 < |x - a| < \delta$

$\Leftrightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(t + a) - \mathrm{L}| < \epsilon$ when $0 < |t - 0| < \delta$

$\Leftrightarrow \lim_{t\to 0} f(t + a) = \mathrm{L}$

65. $\lim_{x\to a} \mathrm{P}(x) = \lim_{x\to a}(c_n x^n + \cdots + c_0) \overset{\text{L.T.5}}{=} \lim_{x\to a} c_n x^n + \cdots + \lim_{x\to a} a_0 \overset{\text{L.T.7}}{=} \lim_{x\to a} c_n \cdot \lim_{x\to a} x \cdots \lim_{x\to a} x + \cdots + \lim_{x\to a} a_0$
$\overset{\text{L.T.1}}{=} c_n a^n + \cdots + a_0 = \mathrm{P}(a)$. Let $\mathrm{R}(x) = \mathrm{P}(x)/\mathrm{Q}(x)$. $\lim_{x\to a} \mathrm{R}(x) = \mathrm{R}(a)$ provided $\lim_{x\to a} \mathrm{Q}(x) \neq 0$.

66. We are given $\lim_{x\to a} f(x)$ exists. If $\lim_{x\to a} g(x)$ did exist, then by L.T.4, so would $\lim_{x\to a}[f(x) + g(x)]$, contradicting the other hypothesis.

1.5 SUPPLEMENT

1.5.12 Theorem If a is any real number except 0, $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$.

1.5.13 Theorem If $a > 0$ and n is a positive integer, or $a \leq 0$ and n is an odd positive integer, $\lim_{x\to a} \sqrt[n]{x} = \lim_{x\to a} \sqrt[n]{a}$.

1.5.16 Theorem If $\lim_{x\to a} f(x) = \mathrm{L}_1$ and $\lim_{x\to a} f(x) = \mathrm{L}_2$, then $\mathrm{L}_1 = \mathrm{L}_2$.

Supplementary Exercises 1.5

▷ The key to a short solution is to factor in the second line before completing the first line.

1. $\lim_{x\to 1} x^2 = 1$ ▷ Choose $\delta \leq 1$ so $-1 < x - 1 < 1 \Rightarrow 1 < x + 1 < 3$.
Then $|x^2 - 1| = |x + 1||x - 1| < 3|x - 1| < \epsilon$ when $|x - 1| < \frac{1}{3}\epsilon$. $\delta = \min(1, \frac{1}{3}\epsilon)$.

2. $\lim_{x\to -3} x^2 = 9$ ▷ Choose $\delta \leq 1$ so $-1 < x + 3 < 1 \Rightarrow -7 < x - 3 < -5$.
Then $|x^2 - 9| = |x - 3||x + 3| < 7|x + 3| < \epsilon$ when $|x + 3| < \frac{1}{7}\epsilon$. $\delta = \min(1, \frac{1}{7}\epsilon)$.

3. $\lim_{x\to 5}(x^2 - 3x) = 10$ ▷ Choose $\delta \leq 1$ so $-1 < x - 5 < 1 \Rightarrow 6 < x + 2 < 8$.
Then $|x^2 - 3x - 10| = |x + 2||x - 5| < 8|x - 5| < \epsilon$ when $|x - 5| < \frac{1}{8}\epsilon$. $\delta = \min(1, \frac{1}{8}\epsilon)$.

4. $\lim_{x\to 2}(x^2 + 2x - 1) = 7$ ▷ Choose $\delta \leq 1$ so $-1 < x - 2 < 1 \Rightarrow -5 < x + 4 < 7$.
Then $|x^2 + 2x - 1 - 7| = |x + 4||x - 2| < 7|x - 2| < \epsilon$ when $|x - 2| < \frac{1}{7}\epsilon$. $\delta = \min(1, \frac{1}{7}\epsilon)$.

5. $\lim_{x\to -3}(5 - x - x^2) = -1$ ▷ Choose $\delta \leq 1$ so $-1 < x + 3 < 1 \Rightarrow -6 < x - 2 < -4$.
Then $|5 - x - x^2 + 1| = |x^2 + x - 6| = |x - 2||x + 3| < 6|x + 3| < \epsilon$ when $|x + 3| < \frac{1}{6}\epsilon$. $\delta = \min(1, \frac{1}{6}\epsilon)$.

6. $\lim_{x\to -1}(3 + 2x - x^2) = 0$ ▷ Choose $\delta \leq 1$ so $-1 < x + 1 < 1 \Rightarrow -5 < x - 3 < -3$.
Then $|3 + 2x - x^2 - 0| = |x^2 - 2x - 3| = |x - 3||x + 1| < 5|x + 1| < \epsilon$ when $|x + 1| < \frac{1}{5}\epsilon$. $\delta = \min(1, \frac{1}{5})$.

7. $\lim_{x\to 2}(6x^2 - 13x + 5) = 3$ ▷ Choose $\delta \leq 1$ so $-1 < x - 2 < 1 \Rightarrow -6 < 6x - 12 < 6 \Rightarrow 5 < 6x - 1 < 17$.
Then $|6x^2 - 13x + 5 - 3| = |6x - 1||x - 2| < 17|x - 2| < \epsilon$ when $|x - 2| < \frac{1}{17}\epsilon$. $\delta = \min(1, \frac{1}{17}\epsilon)$.

8. $\lim_{x\to 1}(4x^2 - 13x + 12) = 3$ ▷ Choose $\delta \leq 1$ so $-1 < x - 1 < 1 \Rightarrow -4 < 4x - 4 < 4 \Rightarrow -9 < 4x - 9 < -1$
Then $|4x^2 - 13x + 12 - 3| = |4x - 9||x - 1| < 9|x - 1| < \epsilon$ when $|x - 1| < \frac{1}{9}\epsilon$. $\delta = \min(1, \frac{1}{9}\epsilon)$.

10. We wish to prove by mathematical induction that if $\lim_{x\to a} f_1(x) = \mathrm{L}_1$, $\lim_{x\to a} f_2(x) = \mathrm{L}_2, \ldots,$ and $\lim_{x\to a} f_n(x) = \mathrm{L}_n$, then $\lim_{x\to a}[f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)] = \mathrm{L}_1 \pm \mathrm{L}_2 \pm \cdots \pm \mathrm{L}_n$ (1)

Proof: By L.T.4, $\lim_{x\to a}[f_1(x) \pm f_2(x)] = \mathrm{L}_1 \pm \mathrm{L}_2$. Therefore, Equation (1) holds when $n = 2$.

Assume Equation (1) holds when $n = k$; that is
$\lim_{x\to a}[f_1(x) \pm f_2(x) \pm \cdots \pm f_k(x)] = \mathrm{L}_1 \pm \mathrm{L}_2 \pm \cdots \pm \mathrm{L}_k$ (2)

We wish to prove that Equation (1) holds when $n = k + 1$. From L.T.4

$\lim_{x \to a} \{[f_1(x) \pm f_2(x) \pm \cdots \pm f_k(x)] \pm f_{k+1}(x)\} = \lim_{x \to a} [f_1(x) \pm f_2(x) \pm \cdots \pm f_k(x)] \pm \lim_{x \to a} [f_{k+1}(x)]$

From Equation (2) and because by hypothesis $\lim_{x \to a} f_{k+1}(x) = L_{k+1}$ the right-hand side of the above equation is $(L_1 \pm L_2 \pm \cdots \pm L_k) \pm L_{k+1}$. Therefore, Equation (1) holds for $n = k + 1$, and hence for every integer n.

11. We are given $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = 0$. In order to prove that $\lim_{x \to a} [f(x) \cdot g(x)] = 0$ we must show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

if $0 < |x - a| < \delta$ then $|f(x) \cdot g(x)| < \epsilon$ (1)

Because $\lim_{x \to a} f(x) = L$ it follows from Definition 2.1.1 that there is a $\delta_1 > 0$ such that

if $0 < |x - a| < \delta_1$ then $|f(x) - L| < 1$ (2)

Because $|a| - |b| \le |a - b|$ it follows that $|f(x)| - |L| \le |f(x) - L|$. Hence, from statement (2) there is a $\delta_1 > 0$ such that

if $0 < |x - a| < \delta_1$ then $|f(x)| < 1 + |L|$

Because $\lim_{x \to a} g(x) = 0$ it follows from Definition 2.1.1 that there is a $\delta_2 > 0$ such that

if $0 < |x - a| < \delta_2$ then $|g(x)| < \frac{\epsilon}{1 + |L|}$

Let $\delta = \min(\delta_1, \delta_2)$; it follows that

if $0 < |x - a| < \delta$ then $|f(x) \cdot g(x)| = |f(x)||g(x)| < (1 + |L|)\frac{\epsilon}{1 + |L|} = \epsilon$

Therefore statement (1) is proved and so $\lim_{x \to a} f(x)g(x) = 0$.

12. We are given $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M \Rightarrow \lim_{x \to a} [f(x) - L] = 0$ and $\lim_{x \to a} [g(x) - M] = 0$. Hence

$\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} \{[f(x) - L]g(x) + L[g(x) - M] + L \cdot M\}$

$\overset{\text{L.T.1}}{=} \lim_{x \to a} \{[f(x) - L]g(x)\} + \lim_{x \to a} \{L[g(x) - M]\} + \lim_{x \to a} L \cdot M \overset{\text{Ex. 11}}{=} 0 + 0 + L \cdot M$

13. We wish to prove by mathematical induction that if $\lim_{x \to a} f_1(x) = L_1$, $\lim_{x \to a} f_2(x) = L_2, \ldots,$ and $\lim_{x \to a} f_n(x) = L_n$, then $\lim_{x \to a} [f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_n(x)] = L_1 \cdot L_2 \cdot \cdots \cdot L_n$ (1)

Proof: By L.T.6, $\lim_{x \to a} [f_1(x) \cdot f_2(x)] = L_1 \cdot L_2$. Therefore, Equation (1) holds when $n = 2$.

Assume Equation (1) holds when $n = k$; that is

$\lim_{x \to a} [f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_k(x)] = L_1 \cdot L_2 \cdot \cdots \cdot L_k$ (2)

We wish to prove that Equation (1) holds when $n = k + 1$. From L.T.6

$\lim_{x \to a} \{[f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_k(x)] \cdot f_{k+1}(x)\} = \lim_{x \to a} [f_1(x) \cdot f_2(x) \cdot \cdots \pm f_k(x)] \cdot \lim_{x \to a} [f_{k+1}(x)]$

From Equation (2) and because by hypothesis $\lim_{x \to a} f_{k+1}(x) = L_{k+1}$ the right-hand side of the above equation is $(L_1 \cdot L_2 \cdot \cdots \cdot L_k) \cdot L_{k+1}$. Therefore, Equation (1) holds for $n = k + 1$, and hence for every integer n.

14. Suppose $a < 0$ and so $-a > 0$. By what was proved, $\lim_{y \to -a} \frac{1}{y} = \frac{1}{-a} \Rightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $\left|\frac{1}{y} - \frac{1}{-a}\right| < \epsilon$ when $0 < |y - (-a)| < \delta$. Replacing y by $-x$, we get the equivalent conclusion

$\left|\frac{1}{-x} - \frac{1}{-a}\right| < \epsilon$ when $0 < |-x - (-a)| < \delta \Leftrightarrow \left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon$ when $0 < |x - a| < \delta \Leftrightarrow \lim_{x \to a} \frac{1}{x} = \frac{1}{a}$.

15. Suppose $a < 0$ so $-a > 0$. By what was proved, $\lim_{y \to -a} \sqrt[n]{y} = \sqrt[n]{-a} \Rightarrow$ for any $\epsilon > 0$ there is a $\delta > 0$ such that $|\sqrt[n]{y} - \sqrt[n]{-a}| < \epsilon$ when $0 < |y - (-a)| < \delta$. Replacing y by $-x$, we get the equivalent conclusion

$|\sqrt[n]{-x} - \sqrt[n]{-a}| < \epsilon$ when $0 < |-x - (-a)| < \delta \overset{n \text{ is odd}}{\Leftrightarrow} |\sqrt[n]{x} - \sqrt[n]{a}| < \epsilon$ when $0 < |x - a| < \delta \Leftrightarrow \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

The case $a = 0$ follows from $a < 0$ and $a > 0$ and one;sided limits, discussed in the next section.

1.6 ONE-SIDED LIMITS

1.6.1 Definition Let f be a function that is defined at every number in some open interval (a, c). Then *limit of $f(x)$, as x approaches a from the right, is L*, written

$$\lim_{x\to a^+} f(x) = \mathrm{L}$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

$$\text{if } 0 < x - a < \delta \text{ then } \left|f(x) - \mathrm{L}\right| < \epsilon$$

1.6.2 Definition Let f be a function that is defined at every number in some open interval (c, a). Then *limit of $f(x)$, as x approaches a from the left, is L*, written

$$\lim_{x\to a^-} f(x) = \mathrm{L}$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

$$\text{if } 0 < a - x < \delta \text{ then } \left|f(x) - \mathrm{L}\right| < \epsilon$$

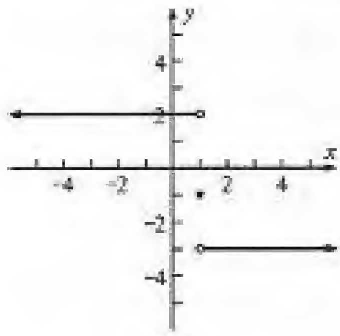
Limit Theorems 1–10 hold if "$x \to a$" is replaced by "$x \to a^+$" or "$x \to a^-$".

1.6.3 Theorem $\lim\limits_{x\to a} f(x)$ exists and equals L if and only if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist and equal L.

Exercises 1.6

In Exercises 1–22, sketch the graph. Then find the indicated limit or state why it does not exist.

1. $f(x) = \begin{cases} 2 & \text{if } x < 1 \\ -1 & \text{if } x = 1 \\ -3 & \text{if } 1 < x \end{cases}$

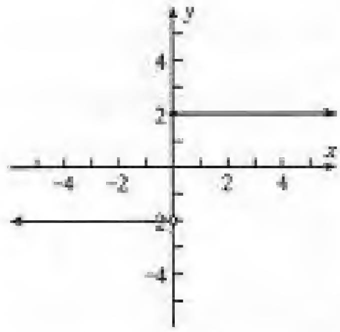


▷ (a) $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (-3) = -3$

(b) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 2 = 2$

(c) $\lim\limits_{x\to 1} f(x)$ does not exist because $\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$.

2. $f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$

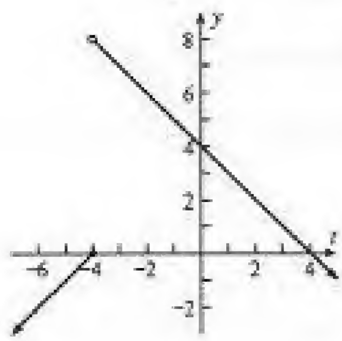


▷ (a) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2 = 2$

(b) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} -2 = -2$

(c) $\lim\limits_{x\to 0} f(x)$ does not exist because $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$.

3. $f(t) = \begin{cases} t + 4 & \text{if } t \leq -4 \\ 4 - t & \text{if } -4 < t \end{cases}$



▷ (a) $\lim\limits_{t\to -4^+} f(t) = \lim\limits_{t\to -4^+} (4 - t) = 8$

(b) $\lim\limits_{t\to -4^-} f(t) = \lim\limits_{t\to -4^-} (t + 4) = 0$

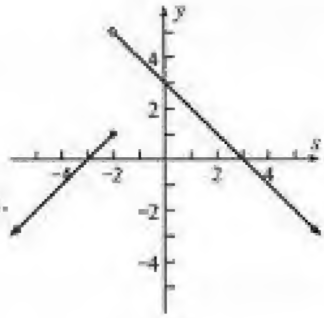
(c) $\lim\limits_{t\to -4} f(t)$ does not exist because $\lim\limits_{t\to -4^-} f(t) \neq \lim\limits_{t\to 4^+} f(t)$.

4. $g(s) = \begin{cases} s+3 & \text{if } s \le -2 \\ 3-s & \text{if } -2 < s \end{cases}$

▷ (a) Because $g(s) = 3 - s$ if $s > -2$ then $\lim\limits_{s \to -2^+} g(s) = \lim\limits_{s \to -2^+} (3 - s) = 5$.

(b) Because $g(s) = s + 3$ if $s < -2$, then $\lim\limits_{s \to -2^-} g(s) = \lim\limits_{s \to -2^-} (s + 3) = 1$.

(c) Because $\lim\limits_{s \to -2^+} g(s) \ne \lim\limits_{s \to -2^-} g(s)$, by Theorem 1.6.3 $\lim\limits_{s \to -2} g(s)$ does not exist.

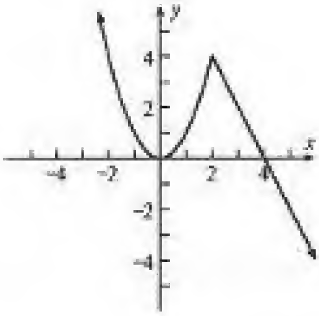


5. $F(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 8-2x & \text{if } 2 < x \end{cases}$

▷ (a) $\lim\limits_{x \to 2^+} F(x) = \lim\limits_{x \to 2^+} (8 - 2x) = 4$

(b) $\lim\limits_{x \to 2^-} F(x) = \lim\limits_{x \to 2^-} x^2 = 4$

(c) $\lim\limits_{x \to 2} F(x) = 4$ by Theorem 1.6.3.

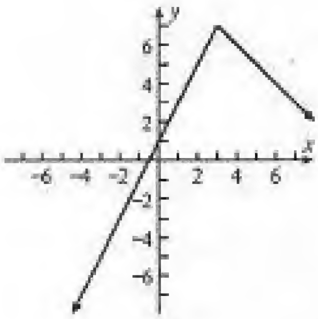


6. $h(x) = \begin{cases} 2x+1 & \text{if } x < 3 \\ 10-x & \text{if } x \ge 3 \end{cases}$

▷ (a) $\lim\limits_{x \to 3^+} h(x) = \lim\limits_{x \to 3^+} (10 - x) = 10 - 3 = 7$

(b) $\lim\limits_{x \to 3^-} h(x) = \lim\limits_{x \to 3^-} (2x + 1) = 2(3) + 1 = 7$

(c) $\lim\limits_{x \to 3} F(x) = 7$ by Theorem 1.6.3.

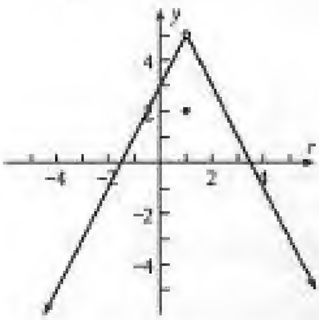


7. $g(r) = \begin{cases} 2r+3 & \text{if } r < 1 \\ 2 & \text{if } r = 1 \\ 7-2r & \text{if } 1 < r \end{cases}$

▷ (a) $\lim\limits_{r \to 1^+} g(r) = \lim\limits_{r \to 1^+} (7 - 2r) = 5$

(b) $\lim\limits_{r \to 1^-} g(r) = \lim\limits_{r \to 1^-} (2r + 3) = 5$

(c) $\lim\limits_{r \to 1} g(r) = 5$ by Theorem 1.6.3.

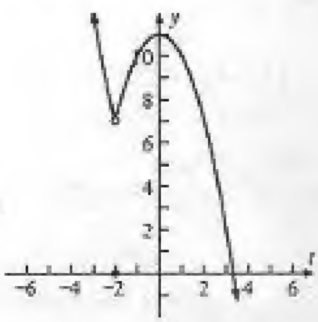


8. $g(t) = \begin{cases} 3+t^2 & \text{if } t < -2 \\ 0 & \text{if } t = -2 \\ 11-t^2 & \text{if } -2 < t \end{cases}$

▷ (a) Because $g(t) = 11 - t^2$ if $t > -2$, then $\lim\limits_{t \to -2^+} g(t) = \lim\limits_{t \to -2^+} (11 - t^2) = 7$.

(b) Because $g(t) = 3 + t^2$ if $t < -2$, then $\lim\limits_{t \to -2^-} g(t) = \lim\limits_{t \to -2^-} (3 + t^2) = 7$.

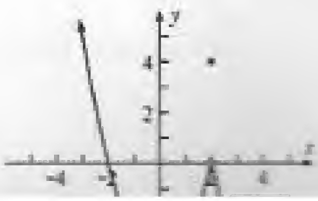
(c) Because $\lim\limits_{t \to -2^+} g(t) = \lim\limits_{t \to -2^-} g(t) = 7$, then by Theorem 1.6.3 $\lim\limits_{t \to -2} g(t) = 7$.



9. $f(x) = \begin{cases} x^2-4 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ 4-x^2 & \text{if } 2 < x \end{cases}$

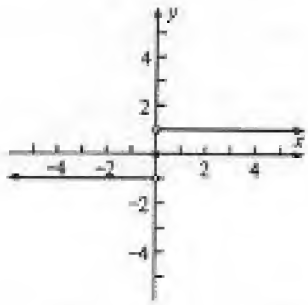
▷ (a) $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4 - x^2) = 0$

(b) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^2 - 4) = 0$



16. $S(x) = |\operatorname{sgn} x|$, where $\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$

▷ $$S(x) = |\operatorname{sgn} x| = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

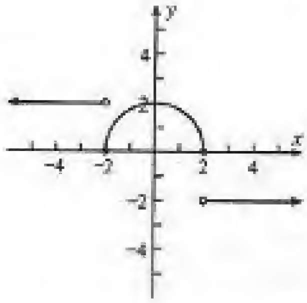


(a) Because $\operatorname{sgn} x = 1$ if $x > 0$, then $|\operatorname{sgn} x| = 1$ if $x > 0$, and
$\lim\limits_{x\to 0^+} S(x) = \lim\limits_{x\to 0^+} |\operatorname{sgn} x| = \lim\limits_{x\to 0^+} 1 = 1$

(b) Because $\operatorname{sgn} x = -1$ if $x < 0$, then $|\operatorname{sgn} x| = 1$ if $x < 0$, and
$\lim\limits_{x\to 0^-} S(x) = \lim\limits_{x\to 0^-} |\operatorname{sgn} x| = \lim\limits_{x\to 0^-} 1 = 1$

(c) Because $\lim\limits_{x\to 0^+} S(x) = \lim\limits_{x\to 0^-} S(x) = 1$, by Theorem 1.6.3 $\lim\limits_{x\to 0} S(x) = 1$.

17. $f(x) = \begin{cases} 2 & \text{if } x < -2 \\ \sqrt{4 - x^2} & \text{if } -2 \le x \le 2 \\ -2 & \text{if } 2 < x \end{cases}$



▷ (a) $\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-} 2 = 2$

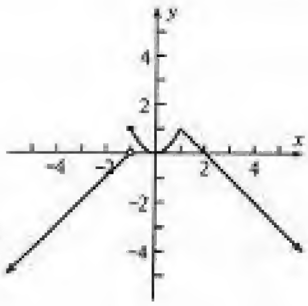
(b) $\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+} \sqrt{4 - x^2} = 0$

(c) $\lim\limits_{x\to -2} f(x)$ does not exist because $\lim\limits_{x\to -2^-} f(x) \ne \lim\limits_{x\to -2^+} f(x)$.

(d) $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \sqrt{4 - x^2} = 0$ (e) $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (-2) = -2$

(f) $\lim\limits_{x\to 2^+} f(x)$ does not exist because $\lim\limits_{x\to 2^-} f(x) \ne \lim\limits_{x\to 2^+} f(x)$.

18. $f(x) = \begin{cases} x + 1 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$



▷ (a) $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} (x + 1) = -1 + 1 = 0$

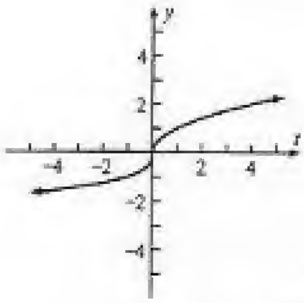
(b) $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} x^2 = (-1)^2 = 1$

(c) $\lim\limits_{x\to -1} f(x)$ does not exist because $\lim\limits_{x\to -1^-} f(x) \ne \lim\limits_{x\to -1^+} f(x)$.

(d) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x^2 = 1^2 = 1$ (e) $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (2 - x) = 2 - 1 = 1$

(f) $\lim\limits_{x\to 1} f(x) = 1$ by Theorem 1.6.3.

19. $f(t) = \begin{cases} \sqrt[3]{t} & \text{if } t < 0 \\ \sqrt{t} & \text{if } 0 \le t \end{cases}$



▷ (a) $\lim\limits_{t\to 0^+} f(t) = \lim\limits_{t\to 0^+} \sqrt{t} = 0$

(b) $\lim\limits_{t\to 0^-} f(t) = \lim\limits_{t\to 0^-} \sqrt[3]{t} = 0$

(c) $\lim\limits_{t\to 0} f(t) = 0$ by Theorem 1.6.3

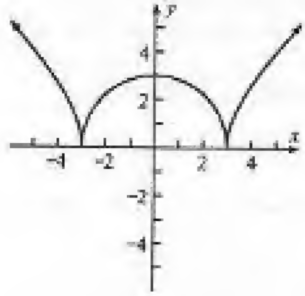
20. $g(x) = \begin{cases} \sqrt[3]{-x} & \text{if } x \le 0 \\ \sqrt[3]{x} & \text{if } 0 < x \end{cases}$

▷ (a) $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} \sqrt[3]{x} = 0$

(b) $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} \sqrt[3]{-x} = 0$

(c) Because $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^-} g(x) = 0$, then by Theorem 1.6.3 $\lim\limits_{x\to 0} g(x) = 0$.

21. $F(x) = \begin{cases} \sqrt{x^2-9} & \text{if } x \le -3 \text{ or } x \ge 3 \\ \sqrt{9-x^2} & \text{if } -3 < x < 3 \end{cases}$



(a) $\lim\limits_{x\to -3^-} F(x) = \lim\limits_{x\to -3^-} \sqrt{x^2-9} = 0$

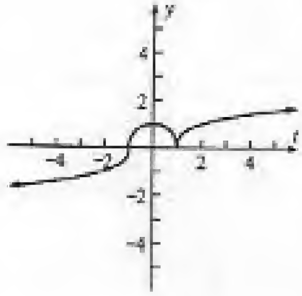
(b) $\lim\limits_{x\to -3^+} F(x) = \lim\limits_{x\to -3^+} \sqrt{9-x^2} = 0$

(c) $\lim\limits_{x\to -3} F(x) = 0$ by Theorem 1.6.3.

(d) $\lim\limits_{x\to 3^-} F(x) = \lim\limits_{x\to 3^-} \sqrt{9-x^2} = 0$ (e) $\lim\limits_{x\to 3^+} F(x) = \lim\limits_{x\to 3^+} \sqrt{x^2-9} = 0$

(f) $\lim\limits_{x\to 3} F(x) = 0$ by Theorem 1.6.3.

22. $G(t) = \begin{cases} \sqrt[3]{t+1} & \text{if } t \le -1 \\ \sqrt{1-t^2} & \text{if } -1 < t < 1 \\ \sqrt[3]{t-1} & \text{if } t \ge 1 \end{cases}$



(a) $\lim\limits_{t\to -1^-} G(t) = \lim\limits_{t\to -1^-} \sqrt[3]{t+1} = \sqrt[3]{-1+1} = 0$

(b) $\lim\limits_{t\to -1^+} G(t) = \lim\limits_{t\to -1^+} \sqrt{1-t^2} = \sqrt{1-(-1)^2} = 0$

(c) $\lim\limits_{t\to -1} G(t) = 0$ by Theorem 1.6.3.

(d) $\lim\limits_{t\to 1^-} G(t) = \lim\limits_{t\to 1^-} \sqrt{1-t^2} = \sqrt{1-1^2} = 0$

(e) $\lim\limits_{x\to 1^+} G(t) = \lim\limits_{t\to 1^+} \sqrt[3]{t-1} = \sqrt[3]{1-1} = 0$

(f) $\lim\limits_{t\to 1} G(t) = 0$ by Theorem 1.6.3.

23. $F(x) = x - 2\,\text{sgn}\,x = \begin{cases} x+2 & \text{if } x < 0 \\ x & \text{if } x = 0 \\ x-2 & \text{if } 0 < x \end{cases}$

(a) $\lim\limits_{x\to 0^+} F(x) = \lim\limits_{x\to 0^+} (x-2) = -2$ (b) $\lim\limits_{x\to 0^-} F(x) = \lim\limits_{x\to 0^-} (x+2) = 2$

(c) $\lim\limits_{x\to 0} F(x)$ does not exist because $\lim\limits_{x\to 0^-} F(x) \ne \lim\limits_{x\to 0^+} F(x)$.

24. Let h be defined by $h(x) = \text{sgn}\,x - U(x)$, where $\text{sgn}\,x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$ and $U(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$

$h(x) = \begin{cases} -1-0=-1 & \text{if } x < 0 \\ 0-1=-1 & \text{if } x = 0 \\ 1-1=\ \ 0 & \text{if } x > 0 \end{cases}$

(a) $h(x) = 0$ if $x > 0$, and $\lim\limits_{x\to 0^+} h(x) = 0$. (b) $h(x) = -1$ if $x < 0$, and $\lim\limits_{x\to 0^-} h(x) = -1$.

(c) Because $\lim\limits_{x\to 0^+} h(x) \ne \lim\limits_{x\to 0^-} h(x)$, then $\lim\limits_{x\to 0} h(x)$ does not exist.

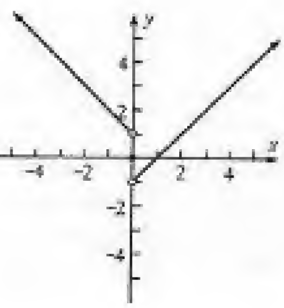
25. (a) $\lim\limits_{x\to 2^+} [\![x]\!] = 2$ (b) $\lim\limits_{x\to 2^-} [\![x]\!] = 1$

(c) $\lim\limits_{x\to 2} [\![x]\!]$ does not exist because $\lim\limits_{x\to 2^-} [\![x]\!] \ne \lim\limits_{x\to 2^+} [\![x]\!]$.

26. (a) $\lim\limits_{x\to 4^+} [\![x-3]\!] = [\![1^+]\!] = 1$ (b) $\lim\limits_{x\to 4^-} [\![x-3]\!] = [\![1^-]\!] = 0$

(c) $\lim\limits_{x\to 4} [\![x]\!]$ does not exist because $\lim\limits_{x\to 4^-} [\![x]\!] \ne \lim\limits_{x\to 4^+} [\![x]\!]$.

27. $h(x) = (x-1)\,\text{sgn}\,x = \begin{cases} 1-x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x-1 & \text{if } 0 < x \end{cases}$



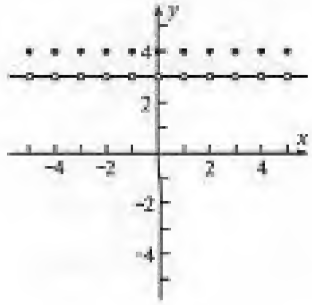
(a) $\lim\limits_{x\to 0^+} h(x) = \lim\limits_{x\to 0^+} (x-1) = -1$

(b) $\lim\limits_{x\to 0^-} h(x) = \lim\limits_{x\to 0^-} (1-x) = 1$

(c) $\lim\limits_{x\to 0} h(x)$ does not exist because $\lim\limits_{x\to 0^-} h(x) \ne \lim\limits_{x\to 0^+} h(x)$.

28. Let $G(x) = [\![x]\!] + [\![4 - x]\!]$.

▷ Let n be any integer. If $x = n$, then $G(x) = n + (4 - n) = 4$.
If $n < x < n + 1$ then $4 - n > 4 - x > 4 - (n + 1) = 3 - n$
so $G(x) = n + (3 - n) = 3$.



(a) $\lim\limits_{x\to 3^+} G(x) = \lim\limits_{x\to 3^+} 3 = 3$ (b) $\lim\limits_{x\to 3^-} G(x) = \lim\limits_{x\to 3^-} 3 = 3$

(c) Because $\lim\limits_{x\to 3^+} G(x) = \lim\limits_{x\to 3^-} G(x) = 3$, by Theorem 1.6.3 $\lim\limits_{x\to 3} G(x) = 3$.

29. $f(x) = \begin{cases} 3x + 2 & \text{if } x < 4 \\ 5x + k & \text{if } 4 \le x \end{cases}$

▷ $\lim\limits_{x\to 4} f(x)$ exists if and only if $\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^+} f(x)$.
$\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^-} (3x + 2) = 14$ $\lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^+} (5x + k) = 20 + k$
Therefore $\lim\limits_{x\to 4} f(x)$ exists if and only if $20 + k = 14$; hence $k = -6$.

30. $f(x) = \begin{cases} kx - 3 & \text{if } x \le -1 \\ x^2 + k & \text{if } x > -1 \end{cases}$

▷ $\lim\limits_{x\to -1} f(x)$ exists if and only if $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^+} f(x)$.
$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} (kx - 3) = -k - 3$ $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (x^2 + k) = 1 + k$
Therefore $\lim\limits_{x\to -1} f(x)$ exists if and only if $-k - 3 = 1 + k$; hence $k = -2$.

31. $f(x) = \begin{cases} x^2 & \text{if } x \le -2 \\ ax + b & \text{if } -2 < x < 2 \\ 2x - 6 & \text{if } 2 \le x \end{cases}$

▷ If $\lim\limits_{x\to -2} f(x)$ exists then $\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^+} f(x)$; that is, $4 = -2a + b$.
If $\lim\limits_{x\to 2} f(x)$ exists then $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$; that is, $2a + b = -2$.
Solving these equations simultaneously, we get $a = -\frac{3}{2}$, $b = 1$.

32. Given $f(x) = \begin{cases} 2x - a & \text{if } x < -3 \\ ax + 2b & \text{if } -3 \le x \le 3 \\ b - 5x & \text{if } 3 < x \end{cases}$

Find the values of a and b so that $\lim\limits_{x\to -3} f(x)$ and $\lim\limits_{x\to 3} f(x)$ both exist.

▷ Because of the two one-sided limits

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} (2x - a) = -6 - a$$
$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} (ax + 2b) = -3a + 2b$$

if $\lim\limits_{x\to -3} f(x)$ exists, by Theorem 1.6.3 we must have

$$-6 - a = -3a + 2b$$
$$a - b = 3$$

Because of the two one-sided limits

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax + 2b) = 3a + 2b$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (b - 5x) = b - 15$$

if $\lim\limits_{x\to 3} f(x)$ exists, by Theorem 1.6.3 we must have

$$3a + 2b = b - 15$$
$$3a + b = -15$$

Solving these equations simultaneously, we obtain $a = -3$ and $b = -6$.

33. $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } 0 < x \end{cases}$; $|f(x)| = 1$ if $x \ne 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (-1) = -1$ $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (1) = 1$
Because $\lim\limits_{x\to 0^-} f(x) \ne \lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0} f(x)$ does not exist. But $\lim\limits_{x\to 0} |f(x)| = \lim 1 = 1$.

34. If $\delta = \min(\delta_1, \delta_2)$ then the following statements are equivalent.
$\lim\limits_{x\to a} f(x) = L$
$\Leftrightarrow$ For every $\epsilon \ge 0$ there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ when $0 < |x - a| < \delta$
$\Leftrightarrow$ For every $\epsilon \ge 0$ there is a $\delta_1 > 0$ such that $|f(x) - L| < \epsilon$ when $0 < x - a < \delta$ and there is a $\delta_2 > 0$ such that $|f(x) - L| < \epsilon$ when $0 < -(x - a) < \delta$
$\Leftrightarrow$ $\lim\limits_{x\to a^+} f(x) = L$ and $\lim\limits_{x\to a^-} f(x) = L$.

In Exercises 35 and 36, evaluate the limits from the graph of function f sketched in the accompanying figure.

35. (a) $\lim_{x\to -1^+} f(x) = 0$ (b) $\lim_{x\to 0^-} f(x) = 3$ (c) $\lim_{x\to 0^+} f(x) = 0$ (d) $\lim_{x\to 0} f(x)$ does not exist (e) $\lim_{x\to 2^-} f(x) = 5$

(f) $\lim_{x\to 2^+} f(x) = 5$ (g) $\lim_{x\to 2} f(x) = 5$ (h) $\lim_{x\to 3^-} f(x) = 2$ (i) $\lim_{x\to 3^+} f(x) = 2$ (j) $\lim_{x\to 3} f(x) = 2$ (k) $\lim_{x\to 5^-} f(x) = 0$

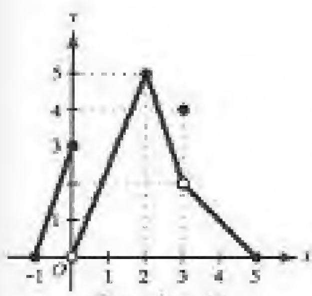
36. (a) $\lim_{x\to 0^+} f(x) = 4$ (b) $\lim_{x\to 1^-} f(x) = 2$ (c) $\lim_{x\to 1^+} f(x) = 2$

(d) Because $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$, by Theorem 1.6.3, $\lim_{x\to 1} f(x) = 3$

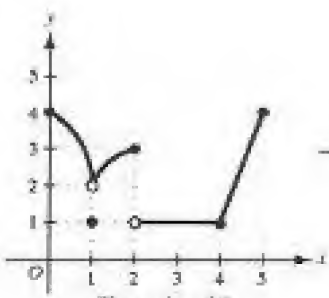
(e) $\lim_{x\to 2^-} f(x) = 3$ (f) $\lim_{x\to 2^+} f(x) = 1$. (g) Because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, $\lim_{x\to 2} f(x)$ does not exist.

(h) $\lim_{x\to 4^-} f(x) = 1$ (i) $\lim_{x\to 4^+} f(x) = 1$ (j) Because $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = 1$, by Theorem 1.6.3, $\lim_{x\to 4} f(x) = 1$

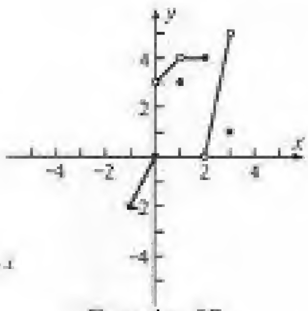
(k) $\lim_{x\to 5^-} f(x) = 4$



Exercise 35



Exercise 36



Exercise 37

Exercise 38

39. $f(x)$ dollars is the cost of shipping x lb. $f(x) = \begin{cases} 2.2x & \text{if } 0 < x \le 50 \\ 2.1x & \text{if } 50 < x \le 200 \\ 2.05x & \text{if } x > 200 \end{cases}$.

▹ (a) $\lim_{x\to 50^-} f(x) = \lim_{x\to 50^-} 2.2x = 2.2 \cdot 50 = 110$ (b) $\lim_{x\to 50^+} f(x) = \lim_{x\to 50^+} 2.1x = 2.1 \cdot 50 = 105$

(c) $\lim_{x\to 200^-} f(x) = \lim_{x\to 200^-} 2.1x = 2.1 \cdot 200 = 420$ (d) $\lim_{x\to 200^+} f(x) = \lim_{x\to 200^+} 2.05x = 2.05 \cdot 200 = 410$

40. $F(x)$ cents is the cost of mailing x ounces. $F(x) = 9 - 23[\![-x]\!]$.

▹ (a and b) If $0 < x \le 1$ then $F(x) = 9 - 23(-1) = 32$ and so $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} 32 = 32$ and $\lim_{x\to 1^-} F(x) = 32$

(c and d) If $1 < x \le 2$ then $F(x) = 9 - 23(-2) = 55$ and so $\lim_{x\to 1^+} F(x) = \lim_{x\to 1^+} 55 = 55$ and $\lim_{x\to 2^-} F(x) = 55$

(e) If $9 < x \le 10$ then $F(x) = 9 - 23(-10) = 239$ and so $\lim_{x\to 10^-} F(x) = \lim_{x\to 10^-} 239 = 239$

(f and g) If $10 < x \le 11$ then $F(x) = 9 - 23(-11) = 262$ and so $\lim_{x\to 10^+} F(x) = \lim_{x\to 10^+} 262 = 262$ and $\lim_{x\to 11^-} F(x) = 262$

41. $g(x)$ cents is the cost of an x minute call. $g(x) = 10 - 30[\![-x]\!]$.

▹ (a) $\lim_{x\to 1^-} g(x) = 10 - 30[\![-(1^-)]\!] = 10 - 30(-1) = 40$ (b) $\lim_{x\to 1^+} g(x) = 10 - 30[\![-(1^+)]\!] = 10 - 30(-2) = 70$

(c) $\lim_{x\to 2^-} g(x) = 10 - 30[\![-(2^-)]\!] = 10 - 30(-2) = 70$ (d) $\lim_{x\to 5^-} g(x) = 10 - 30[\![-(5^-)]\!] = 10 - 30(-5) = 160$

42. $G(x)$ dollars is the admission for age x years. $G(x) = \begin{cases} 4 & \text{if } 0 < x < 12 \\ 7 & \text{if } 12 \le x < 60 \\ 5 & \text{if } 60 \le x \end{cases}$.

▹ (a) $\lim_{x\to 12^-} G(x) = \lim_{x\to 12^-} 4 = 4$ (b) $\lim_{x\to 12^+} G(x) = \lim_{x\to 12^+} 7 = 7$

(c) $\lim_{x\to 60^-} G(x) = \lim_{x\to 60^-} 7 = 7$ (d) $\lim_{x\to 60^+} G(x) = \lim_{x\to 60^+} 5 = 5$

43. $f(x) = \begin{cases} x^2 + 3 & \text{if } x \le 1 \\ x + 1 & \text{if } 1 < x \end{cases}$ and $g(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2 & \text{if } 1 < x \end{cases}$

▸ (a) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 3) = 1^2 + 3 = 4$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 1 + 1 = 2$

(b) $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1$ and $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} 2 = 2$

(c) $f(x) \cdot g(x) = \begin{cases} (x^2 + 3)x^2 & \text{if } x \le 1 \\ (x + 1)(2) & \text{if } x > 1 \end{cases}$

(d) $\lim_{x \to 1^-} [f(x) \cdot g(x)] = (1^2 + 3)(1^2) = 4$ and $\lim_{x \to 1^+} [f(x) \cdot g(x)] = (1 + 1)(2) = 4$

Therefore, $\lim_{x \to 1} [f(x) \cdot g(x)]$ exists and is 4.

44. $f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ x - 1 & \text{if } x \ge 1 \end{cases}$ and $g(x) = \begin{cases} 1 - x & \text{if } x < 1 \\ 1 + x & \text{if } x \ge 1 \end{cases}$

(a) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 1) = 1 + 1 = 2$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 1) = 1 - 1 = 0$

Because $\lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$, then $\lim_{x \to 1} f(x)$ does not exist.

$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (1 - x) = 1 - 1 = 0$ and $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (1 + x) = 1 + 1 = 2$

Because $\lim_{x \to 1^-} g(x) \ne \lim_{x \to 1^+} g(x)$, then $\lim_{x \to 1} g(x)$ does not exist.

(b) $(f + g)(x) = \begin{cases} (x + 1) + (1 - x) & \text{if } x < 1 \\ (x - 1) + (1 + x) & \text{if } x \ge 1 \end{cases} = \begin{cases} 2 & \text{if } x < 1 \\ 2x & \text{if } x \ge 1 \end{cases}$

(c) $\lim_{x \to 1^-} [f(x) + g(x)] = \lim_{x \to 1^-} 2 = 2$ and $\lim_{x \to 1^+} [f(x) + g(x)] = \lim_{x \to 1^+} 2x = 2 \cdot 1 = 2$

Because $\lim_{x \to 1^-} [f(x) + g(x)] = \lim_{x \to 1^+} [f(x) + g(x)] = 2$, then $\lim_{x \to 1} [f(x) + g(x)] = 2$

(d) Because $\lim_{x \to 1} f(x)$ and $\lim_{x \to 1} g(x)$ do not exist, the hypotheses of Limit Theorem 4 are not satisfied.

1.7 INFINITE LIMITS

1.7.1 Definition Let f be a function that is defined at every number in some open interval containing a, except possibly at the number a itself. *As x approaches a, $f(x)$ increases without bound,* which is written

$$\lim_{x \to a} f(x) = +\infty$$

if for any number $N > 0$ there exists a $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } f(x) > N$$

1.7.2 Definition Let f be a function that is defined at every number in some open interval containing a, except possibly at the number a itself. *As x approaches a, $f(x)$ decreases without bound,* which is written

$$\lim_{x \to a} f(x) = -\infty$$

if for any number $N < 0$ there exists a $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ then } f(x) < N$$

We may also define one-sided limits that are infinite. The following theorems involve such definitions.

Limit Theorem 11 If r is any positive integer, then

$$\text{(i) } \lim_{x \to 0^+} \frac{1}{x^r} = +\infty$$

$$\text{(ii) } \lim_{x \to 0^-} \frac{1}{x^r} = \begin{cases} -\infty & \text{if } r \text{ is odd} \\ +\infty & \text{if } r \text{ is even} \end{cases}$$

Limit Theorem 12 If a is any real number, and if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = c$, where c is a constant not equal to 0, then

(i) if $c > 0$ and if $f(x) \to 0$ through positive values of $f(x)$, $\lim_{x\to a} \dfrac{g(x)}{f(x)} = +\infty$

(ii) if $c > 0$ and if $f(x) \to 0$ through negative values of $f(x)$, $\lim_{x\to a} \dfrac{g(x)}{f(x)} = -\infty$

(iii) if $c < 0$ and if $f(x) \to 0$ through positive values of $f(x)$, $\lim_{x\to a} \dfrac{g(x)}{f(x)} = -\infty$

(iv) if $c < 0$ and if $f(x) \to 0$ through negative values of $f(x)$, $\lim_{x\to a} \dfrac{g(x)}{f(x)} = +\infty$

The theorem is also valid if "$x \to a$" is replaced by "$x \to a^+$" or "$x \to a^-$".

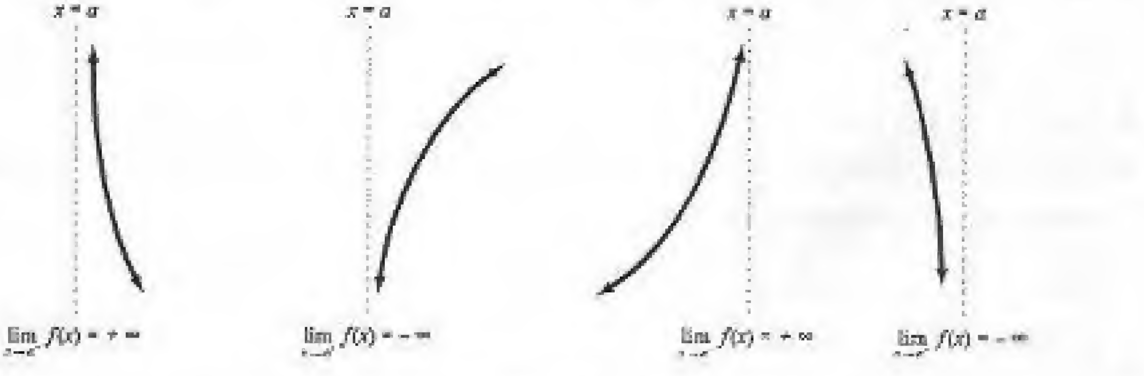
We can now usually find the limit of a fraction if either the numerator or the denominator has limit zero. If the numerator has limit zero and the denominator has a limit that is not zero, then by Limit Theorem 9 the limit of the fraction is zero. If the denominator has limit zero and the numerator has a limit that is not zero, then one of the cases of Limit Theorem 12 is usually satisfied and the limit of the fraction is $+\infty$ or $-\infty$, depending on which case. If both the numerator and denominator have limit zero, then one of the methods of Section 1.5 often works:

(i) Factor the numerator and denominator and cancel the common factors.
(ii) Rationalize either the numerator or denominator and proceed as in (i).

We will learn additional methods as we proceed through the book, but it is important to realize that some limits simply don't exist.

1.7.8 Definition The line $x = a$ is said to be a *vertical asymptote* of the graph of the function f if at least one of the following statements is true. Each statement is illustrated by the graph below it.

(i) $\lim_{x\to a^+} f(x) = +\infty$ (ii) $\lim_{x\to a^+} f(x) = -\infty$ (iii) $\lim_{x\to a^-} f(x) = +\infty$ (iv) $\lim_{x\to a^-} f(x) = -\infty$



Exercises 1.7

In Exercises 1–12, do the following: (a) Use a calculator to tabulate values of $f(x)$ for the specified values of x, and from these values make a statement regarding the apparent behavior of $f(x)$. (b) Support your answer in part (a) by plotting the graph of f. (c) Confirm your answer in part (a) analytically by computing the indicated limit.

▸ Let 0^+ and 0^- denote quantities that approach 0 through positive and negative values, respectively. If p and q are positive and negative numbers, then Limit Theorem 12 can be restated as below. See Solutions 9 and 10.
(i) $p/0^+ = +\infty$ (ii) $p/0^- = -\infty$ (iii) $q/0^+ = -\infty$ (iv) $q/0^- = +\infty$

1. (a) $f(x) = \dfrac{1}{x-5}$.

x	6	5.5	5.1	5.01	5.001	5.0001
$f(x)$	1	2	10	100	1000	10,000

(b) $\lim_{x\to 5^+} 1 = 1$ and $\lim_{x\to 5^+} (x-5) = 0$ through positive values. Hence, $\lim_{x\to 5^+} \dfrac{1}{x-5} = +\infty$.

2. (a) $f(x) = \dfrac{1}{x-5}$.

x	4	4.5	4.9	4.99	4.999	4.9999
$f(x)$	−1	−2	−10	−100	−1000	−10000

(b) $\lim_{x\to 5^-} 1 = 1$ and $\lim_{x\to 5^-} (x-5) = 0$ through negative values. Hence, $\lim_{x\to 5^-} \dfrac{1}{x-5} = -\infty$.

3. (a) $f(x) = \dfrac{1}{(x-5)^2}$.

x	6	5.5	5.1	5.01	5.001	5.0001
$f(x)$	1	4	100	10,000	1,000,000	100,000,000

x	4	4.5	4.9	4.99	4.999	4.9999
$f(x)$	1	4	100	10,000	1,000,000	100,000,000

(b) $\lim\limits_{x\to 5} 1 = 1$ and $\lim\limits_{x\to 5}(x-5)^2 = 0$ through positive values. Hence, $\lim\limits_{x\to 5}\dfrac{1}{(x-5)^2} = +\infty$.

4. (a) $f(x) = \dfrac{x+2}{1-x}$; x is 0, 0.5, 0.9, 0.99, 0.999, 0.9999; (b) $\lim\limits_{x\to 1^-}\dfrac{x+2}{1-x}$

▶ (a) From the values of $f(x)$ for the specified values of x given in the table, $f(x)$ appears to be increasing without bound as x approaches 1 from the left.

x	0	0.5	0.9	0.99	0.999	0.9999
$f(x) = \dfrac{x+2}{1-x}$	2	5	29	299	2999	29999

(b) $\lim\limits_{x\to 1^-}(x+2) = 3$ and $\lim\limits_{x\to 1^-}(1-x) = 0$. Because $x \to 1^-$, then $x < 1$ and $1 - x > 0$. Thus, $1-x$ approaches 0 through positive values. By Limit Theorem 12(i), we have $\lim\limits_{x\to 1^-}\dfrac{x+2}{1-x} = +\infty$

5. (a) $f(x) = \dfrac{x+2}{1-x}$.

x	2	1.5	1.1	1.01	1.001	1.0001
$f(x)$	−4	−7	−31	−301	−3001	−30,001

(b) $\lim\limits_{x\to 1^+}(x+2) = 3$ and $\lim\limits_{x\to 1^+}(1-x) = 0$ through negative values. Hence, $\lim\limits_{x\to 1^+}\dfrac{x+2}{1-x} = -\infty$.

6. (a) $f(x) = \dfrac{x+2}{(x-1)^2}$.

x	0	0.5	0.9	0.99	0.999	0.9999
$f(x)$	2	10	290	29,900	2,999,000	299,900,000

x	2	1.5	1.1	1.01	1.001	1.0001
$f(x)$	4	14	310	30,100	3,001,000	300,010,000

(b) $\lim\limits_{x\to 5} 1 = 1$ and $\lim\limits_{x\to 5}(x-5)^2 = 0$ through positive values. Hence, $\lim\limits_{x\to 5}\dfrac{1}{(x-5)^2} = +\infty$.

7. (a) $f(x) = \dfrac{x-2}{x+1}$.

x	0	−0.5	−0.9	−0.99	−0.999	−0.9999
$f(x)$	−2	−5	−29	−299	−2999	−29,999

(b) $\lim\limits_{x\to -1^+}(x-2) = -3$ and $\lim\limits_{x\to -1^+}(x+1) = 0$ through positive values. Hence, $\lim\limits_{x\to -1^+}\dfrac{x-2}{x+1} = -\infty$.

8. (a) $f(x) = \dfrac{x-2}{x+1}$; x is −2, −1.5, −1.1, −1.01, −1.001, −1.0001; (b) $\lim\limits_{x\to -1^-}\dfrac{x-2}{x+1}$

▶ (a) From the values of $f(x)$ for the specified values of x given in the table, $f(x)$ appears to be increasing without bound as x approaches −1 from the left.

x	−2	−1.5	−1.1	−1.01	−1.001	−1.0001
$f(x) = \dfrac{x-2}{x+1}$	4	7	31	301	3001	30001

(b) $\lim\limits_{x\to -1^-}(x-2) = -3$ and $\lim\limits_{x\to -1^-}(x+1) = 0$. Because $x \to -1^-$, then $x < -1$ and $x + 1 < 0$. Thus, $x+1$ approaches 0 through negative values. By Limit Theorem 12(iv), we have $\lim\limits_{x\to -1^-}\dfrac{x-2}{x+1} = +\infty$

9. (a) $f(x) = \dfrac{x}{x+4}$.

x	−5	−4.5	−4.1	−4.01	−4.001	−4.0001
$f(x)$	5	9	41	401	4001	40,001

(b) $\lim\limits_{x\to -4^-} x = -4$ and $\lim\limits_{x\to -4^-}(x+4) = 0$ through negative values. Hence, $\lim\limits_{x\to -4^-}\dfrac{x}{x+4} = +\infty$.
Alternatively, we could write the entire solution as: $\lim\limits_{x\to -4^-}\dfrac{x}{x+4} = \dfrac{-4}{0^-} = +\infty$.

10. (a) $f(x) = \dfrac{x}{x-4}$.

x	5	4.5	4.1	4.01	4.001	4.0001
$f(x)$	5	9	41	401	4001	40,001

(b) $\lim\limits_{x\to 4^+}\dfrac{x}{x-4} = \dfrac{4}{0^+} = +\infty$

11. (a) $f(x) = \dfrac{4x}{9-x^2}$.

x	−4	−3.5	−3.1	−3.01	−3.001	−3.0001
$f(x)$	2.3	4.3	20.3	200.3	2000.5	20,037

(b) $\lim\limits_{x\to -3^-} 4x = -12$ and $\lim\limits_{x\to -3^-}(9-x^2) = 0$ through negative values. Hence, $\lim\limits_{x\to -3^-}\dfrac{4x}{9-x^2} = +\infty$.

12. (a) $f(x) = \dfrac{4x^2}{9-x^2}$; x is 4, 3.5, 3.1, 3.01, 3.001, 3.0001; (b) $\lim\limits_{x\to3^+} \dfrac{4x^2}{9-x^2}$

» (a) The table gives the values of $f(x)$ for the specified values of x. From the table, $f(x)$ appears to be decreasing without bound as x approaches 3 from the right.

x	4	3.5	3.1	3.01	3.001	3.0001
$f(x) = \dfrac{4x^2}{9-x^2}$	-9.14	-15.1	-63.0	-603	-6003	-60003

(b) $\lim\limits_{x\to3^+}(4x^2) = 36$ and $\lim\limits_{x\to3^+}(9-x^2) = 0$. Furthermore, since $x \to 3^+$, then $x > 3$ and $9 - x^2 < 0$. Thus, $9 - x^2$ approaches 0 through negative values. By Limit Theorem 12(ii), $\lim\limits_{x\to3^+}\dfrac{4x^2}{9-x^2} = -\infty$

In Exercises 13–32, find the limit and support your answer by plotting the graph of the function.

13. $\lim\limits_{t\to2^+}\dfrac{t+2}{t^2-4} = \lim\limits_{t\to2^+}\dfrac{t+2}{(t+2)(t-2)} = \lim\limits_{t\to2^+}\dfrac{1}{t-2} = \dfrac{1}{0^+} = +\infty.$

14. $\lim\limits_{t\to2^-}\dfrac{-t+2}{(t-2)^3} = \lim\limits_{t\to2^-}\dfrac{-(t-2)}{(t-2)^3} = \lim\limits_{t\to2^-}\dfrac{-1}{(t-2)^2} = \dfrac{-1}{0^+} = -\infty$

15. $\lim\limits_{t\to2^-}\dfrac{t+2}{t^2-4} = \lim\limits_{t\to2^-}\dfrac{t+2}{(t+2)(t-2)} = \lim\limits_{t\to2^-}\dfrac{1}{t-2} = \dfrac{1}{0^-} = -\infty.$

16. $\lim\limits_{x\to0^+}\dfrac{\sqrt{3+x^2}}{x} = +\infty$

» $\lim\limits_{x\to0^+}\sqrt{3+x^2} = \sqrt{3}$ and $\lim\limits_{x\to0^+} x = 0$. Moreover, since $x \to 0^+$, x approaches 0 through positive values. Thus by Limit Theorem 12(i), $\lim\limits_{x\to0^+}\dfrac{\sqrt{3+x^2}}{x} = +\infty$

17. $\lim\limits_{x\to0^-}\sqrt{3+x^2} = \sqrt{3}$ and $\lim\limits_{x\to0^-} x = 0$ through negative values. Therefore, $\lim\limits_{x\to0^-}\dfrac{\sqrt{3+x^2}}{x} = -\infty.$

18. $\lim\limits_{x\to0}\dfrac{\sqrt{3+x^2}}{x^2} = \dfrac{\sqrt{3}}{0^+} = +\infty$

19. $\lim\limits_{x\to3^+}\dfrac{\sqrt{x^2-9}}{x-3} = \lim\limits_{x\to3^+}\dfrac{\sqrt{x-3}\sqrt{x+3}}{\sqrt{x-3}\sqrt{x-3}} = \lim\limits_{x\to3^+}\dfrac{\sqrt{x+3}}{\sqrt{x-3}}$; $\lim\limits_{x\to3^+}\sqrt{x+3} = \sqrt{6}$ and $\lim\limits_{x\to3^+}\sqrt{x-3} = 0$ through positive values. Therefore, $\lim\limits_{x\to3^+}\dfrac{\sqrt{x+3}}{\sqrt{x-3}} = +\infty.$

20. $\lim\limits_{x\to4^-}\dfrac{\sqrt{16-x^2}}{x-4}$

» Because both the numerator and denominator have limit 0, we cannot use Limit Theorem 12. We must factor. Because $x \to 4^-$, then $x < 4$ or, equivalently, $4 - x > 0$. Thus,

$$x - 4 = -(4-x) = -\sqrt{(4-x)^2}$$

Therefore

$$\begin{aligned}\lim_{x\to4^-}\frac{\sqrt{16-x^2}}{x-4} &= \lim_{x\to4^-}\frac{\sqrt{(4-x)(4+x)}}{-\sqrt{(4-x)^2}}\\ &= \lim_{x\to4^-}\frac{\sqrt{4-x}\cdot\sqrt{4+x}}{-\sqrt{4-x}\cdot\sqrt{4-x}}\\ &= \lim_{x\to4^-}\frac{\sqrt{4+x}}{-\sqrt{4-x}} \qquad (1)\end{aligned}$$

Because $\lim\limits_{x\to4^-}\sqrt{4+x} = \sqrt{8}$ and $\lim\limits_{x\to4^-} -\sqrt{4-x} = 0$ and $-\sqrt{4-x}$ approaches 0 through negative values, by Limit Theorem 12(ii) and Eq. (1) we have $\lim\limits_{x\to4^-}\dfrac{\sqrt{16-x^2}}{x-4} = -\infty$

21. $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=\lim_{x\to 0^+}\frac{x-1}{x^2}$; $\lim_{x\to 0^+}(x-1)=-1$ and $\lim_{x\to 0^+}x^2=0$ through positive values.

Therefore, $\lim_{x\to 0^+}\frac{x-1}{x^2}=-\infty$.

22. $\lim_{x\to 0^+}\frac{x^2-3}{x^3+x^2}=\lim_{x\to 0^+}\frac{x^2-3}{x^2(x+1)}=\frac{-3}{0^+}=-\infty$

23. $\lim_{x\to 0^-}(2-4x^3)=2$ and $\lim_{x\to 0^-}(5x^2+3x^3)=0$ through positive values. Hence, $\lim_{x\to 0^-}\frac{2-4x^3}{5x^2+3x^3}=+\infty$.

24. $\lim_{s\to 2^-}\left(\frac{1}{s-2}-\frac{3}{s^2-4}\right)$

▷
$$\begin{aligned}\lim_{s\to 2^-}\left(\frac{1}{s-2}-\frac{3}{s^2-4}\right)&=\lim_{s\to 2^-}\left(\frac{s+2}{(s-2)(s+2)}-\frac{3}{(s-2)(s+2)}\right)\\&=\lim_{s\to 2^-}\frac{s-1}{(s-2)(s+2)}\end{aligned}$$

Moreover, $\lim_{s\to 2^-}(s-1)=1$ and $\lim_{s\to 2^-}(s-2)(s+2)=0$. Because $s\to 2^-$, then $s-2<0$; thus $(s-2)(s+2)$ approaches 0 through negative values. Therefore, by Limit Theorem 12(ii), $\lim_{s\to 2^-}\left(\frac{1}{s-2}-\frac{3}{s^2-4}\right)=-\infty$

25. $\lim_{t\to -4^-}\left(\frac{2}{t^2+3t-4}-\frac{3}{t+4}\right)=\lim_{t\to -4^-}\left(\frac{2}{(t+4)(t-1)}-\frac{3(t-1)}{(t+4)(t-1)}\right)=\lim_{t\to -4^-}\frac{5-3t}{(t+4)(t-1)}$.

$\lim_{t\to -4^-}(5-3t)=17$ and $\lim_{t\to -4^-}(t+4)(t-1)=0$ through positive values. Therefore, $\lim_{t\to -4^-}\frac{5-3t}{(t+4)(t-1)}=+\infty$.

26. $\lim_{x\to 1^-}\frac{2x^3-5x^2}{x^2-1}=\frac{-3}{0^-}=+\infty$

27. $\lim_{x\to 3^-}([\![x]\!]-x)=\lim_{x\to 3^-}[\![x]\!]-3=\lim_{x\to 3^-}(2)-3=-1$ and $\lim_{x\to 3^-}(3-x)=0$ through positive values.

Therefore, $\lim_{x\to 3^-}\frac{[\![x]\!]-x}{3-x}=-\infty$.

28. $\lim_{x\to 1^-}\frac{[\![x^2]\!]-1}{x^2-1}$

▷ Since $x\to 1^-$, then $x<1$. If $0<x<1$, then $[\![x^2]\!]=0$. Thus

$$\lim_{x\to 1^-}[\![x^2]\!]-1=-1$$

Because $x<1$ and $x^2-1<0$ if $-1<x<1$, then

$$\lim_{x\to 1^-}(x^2-1)=0$$

and x^2-1 approaches 0 through negative values. Therefore, by Limit Theorem 12(iv),

$$\lim_{x\to 1^-}\frac{[\![x^2]\!]-1}{x^2-1}=+\infty$$

29. $\lim_{x\to 3^-}\frac{x^3+9x^2+20x}{x^2+x-12}=\lim_{x\to 3^-}\frac{x(x+5)(x+4)}{(x-3)(x+4)}=\lim_{x\to 3^-}\frac{x(x+5)}{x-3}$

$\lim_{x\to 3^-}x(x+5)=24$ and $\lim_{x\to 3^-}(x-3)=0$ through negative values. Hence, $\lim_{x\to 3^-}\frac{x(x+5)}{x-3}=-\infty$.

30. $\lim_{x\to -2^+}\frac{6x^2+x-2}{2x^2+3x-2}=\lim_{x\to -2^+}\frac{6x^2+x-2}{(x+2)(2x-1)}=\frac{20}{0^+(-5)}=-\infty$

31. $\lim\limits_{x\to1^+}\dfrac{x-1}{\sqrt{2x-x^2}-1}=\lim\limits_{x\to1^+}\dfrac{(x-1)(\sqrt{2x-x^2}+1)}{(\sqrt{2x-x^2}-1)(\sqrt{2x-x^2}+1)}=\lim\limits_{x\to1^+}\dfrac{(x-1)(\sqrt{2x-x^2}+1)}{2x-x^2-1}$

$=\lim\limits_{x\to1^+}\dfrac{(x-1)(\sqrt{2x-x^2}+1)}{-(x-1)^2}=\lim\limits_{x\to1^+}\dfrac{\sqrt{2x-x^2}+1}{1-x}$

$\lim\limits_{x\to1^+}(\sqrt{2x-x^2}+1)=2$ and $\lim\limits_{x\to1^+}(1-x)=0$ through negative values. Hence, $\lim\limits_{x\to1^+}\dfrac{\sqrt{2x-x^2}+1}{1-x}=-\infty$.

32. $\lim\limits_{x\to2^-}\dfrac{x-2}{2-\sqrt{4x-x^2}}$

▸ Because the numerator and denominator both have limit 0, Limit Theorem 12 does not apply. We rationalize the denominator.

$$\begin{aligned}\lim_{x\to2^-}\frac{x-2}{2-\sqrt{4x-x^2}}&=\lim_{x\to2^-}\frac{x-2}{2-\sqrt{4x-x^2}}\cdot\frac{2+\sqrt{4x-x^2}}{2+\sqrt{4x-x^2}}\\&=\lim_{x\to2^-}\frac{(x-2)(2+\sqrt{4x-x^2})}{4-(4x-x^2)}\\&=\lim_{x\to2^-}\frac{(x-2)(2+\sqrt{4x-x^2})}{(x-2)^2}\\&=\lim_{x\to2^-}\frac{2+\sqrt{4x-x^2}}{x-2}\qquad(1)\end{aligned}$$

Now $\lim\limits_{x\to2^-}(2+\sqrt{4x-x^2})=4$ and $\lim\limits_{x\to2^-}(x-2)=0$. Furthermore, $x-2$ approaches 0 through negative values. Thus, by Limit Theorem 12(ii) and Eq. (1) we conclude that $\lim\limits_{x\to2^-}\dfrac{x-2}{2-\sqrt{4x-x^2}}=-\infty$

33. $f(x)=\dfrac{x^2+x-6}{x^2-6x+8}=\dfrac{(x-2)(x+3)}{(x-2)(x-4)}=\dfrac{x+3}{x-4}$ if $x\neq2$.

$\lim\limits_{x\to2^-}f(x)=\lim\limits_{x\to2^+}f(x)=-\frac{5}{2}$. $\lim\limits_{x\to4^-}f(x)=\frac{7}{0^-}=-\infty$. $\lim\limits_{x\to4^+}f(x)=\frac{7}{0^+}=+\infty$

34. $f(x)=\dfrac{x^2+2x-3}{x^2+x-6}=\dfrac{(x+3)(x-1)}{(x+3)(x-2)}=\dfrac{x-1}{x-2}$ if $x\neq-3$.

$\lim\limits_{x\to-3^-}f(x)=\lim\limits_{x\to-3^+}f(x)=\frac{2}{1}=2$. $\lim\limits_{x\to2^-}f(x)=\frac{2}{0^-}=-\infty$. $\lim\limits_{x\to2^+}f(x)=\frac{2}{0^+}=+\infty$

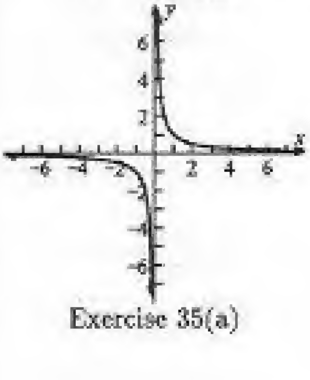
In Exercises 35–44, find the vertical asymptote(s) of the graph of the function, and sketch the graph.

35. (a) Because $\lim\limits_{x\to0^-}\frac{1}{x}=-\infty$ or because $\lim\limits_{x\to0^+}\frac{1}{x}=+\infty$, $x=0$ is a vertical asymptote.

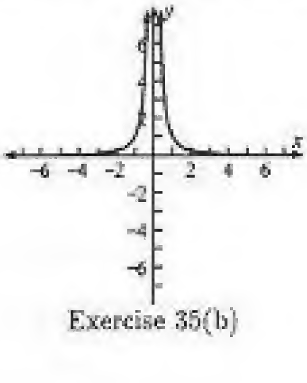
(b) Because $\lim\limits_{x\to0}\frac{1}{x^2}=+\infty$, $x=0$ is a vertical asymptote.

(c) Because $\lim\limits_{x\to0^-}\frac{1}{x^3}=-\infty$ or because $\lim\limits_{x\to0^+}\frac{1}{x^3}=+\infty$, $x=0$ is a vertical asymptote.

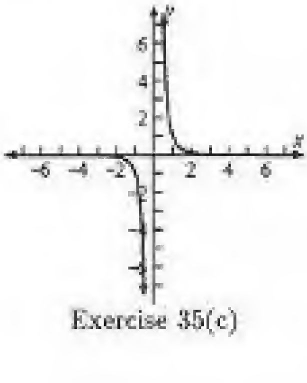
(d) Because $\lim\limits_{x\to0}\frac{1}{x^4}=+\infty$, $x=0$ is a vertical asymptote.



Exercise 35(a)



Exercise 35(b)



Exercise 35(c)

Exercise 35(d)

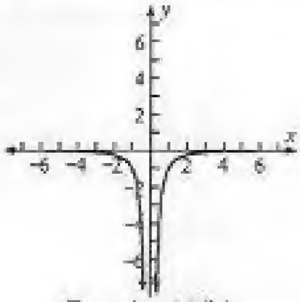
36. (a) Because $\lim\limits_{x\to 0^-} -\frac{1}{x} = +\infty$ by Limit Theorem 12(iv) or because $\lim\limits_{x\to 0^+} -\frac{1}{x} = -\infty$ by Limit Theorem 12(iii), $x = 0$ is a vertical asymptote.

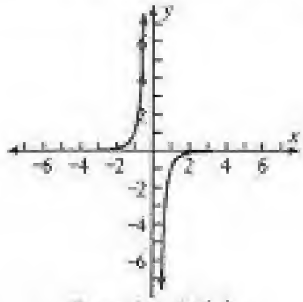
(b) Because $\lim\limits_{x\to 0} -\frac{1}{x^2} = -\infty$ by Limit Theorem 12(iii), $x = 0$ is a vertical asymptote.

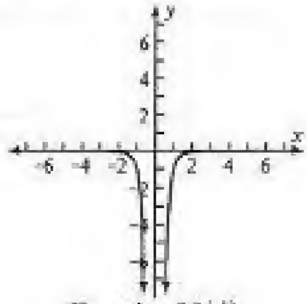
(c) Because $\lim\limits_{x\to 0^-} -\frac{1}{x^3} = +\infty$ by Limit Theorem 12(iv) or because $\lim\limits_{x\to 0^+} -\frac{1}{x^3} = -\infty$ by Limit Theorem 12(iii), $x = 0$ is a vertical asymptote.

(d) Because $\lim\limits_{x\to 0} -\frac{1}{x^4} = -\infty$ by Limit Theorem 12(iii), $x = 0$ is a vertical asymptote.


Exercise 36(a)


Exercise 36(b)


Exercise 36(c)


Exercise 36(d)

37. Because $\lim\limits_{x\to 4^-} \frac{2}{x-4} = -\infty$ or because $\lim\limits_{x\to 4^+} \frac{2}{x-4} = +\infty$, $x = 4$ is a vertical asymptote.

38. Because $\lim\limits_{x\to -1^-} \frac{3}{x+1} = \frac{3}{0^-} = -\infty$ or because $\lim\limits_{x\to -1^+} \frac{3}{x+1} = \frac{3}{0^+} = +\infty$, $x = -1$ is a vertical asymptote.

39. Because $\lim\limits_{x\to -3^-} \frac{-2}{x+3} = +\infty$ or because $\lim\limits_{x\to -3^+} \frac{-2}{x+3} = -\infty$, $x = -3$ is a vertical asymptote.

40. $f(x) = \frac{-4}{x-5}$

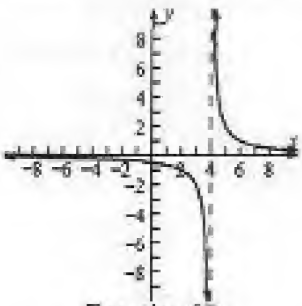
▶ If $x = 5$, then $x - 5 = 0$. Furthermore, because $x - 5 > 0$ if $x > 5$, and $-4 < 0$, by Limit Theorem 12(iii)

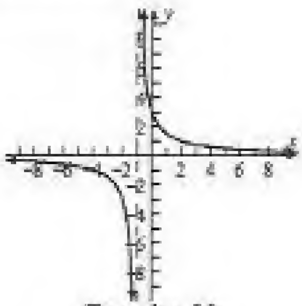
$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{-4}{x-5} = -\infty$$

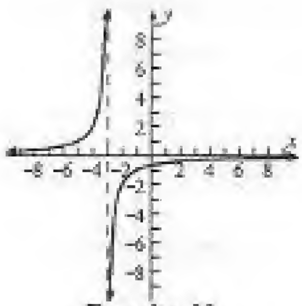
Or because $x - 5 < 0$ if $x < 5$, by Limit Theorem 12(iv)

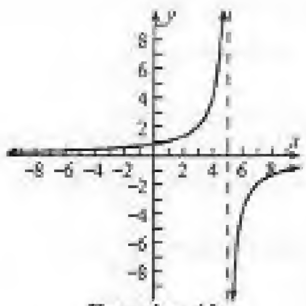
$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \frac{-4}{x-5} = -\infty$$

Thus, $x = 5$ is a vertical asymptote.


Exercise 37


Exercise 38


Exercise 39


Exercise 40

41. Because $\lim\limits_{x\to -3} \frac{-2}{(x+3)^2} = \frac{-2}{0^+} = -\infty$, $x = -3$ is a vertical asymptote.

42. Because $\lim\limits_{x\to 5} \frac{4}{(x-5)^2} = \frac{4}{0^+} = +\infty$, $x = 5$ is a vertical asymptote.

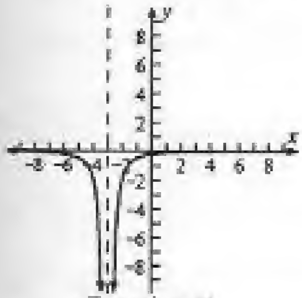
43. $f(x) = \frac{5}{x^2 + 8x + 15} = \frac{5}{(x+3)(x+5)}$

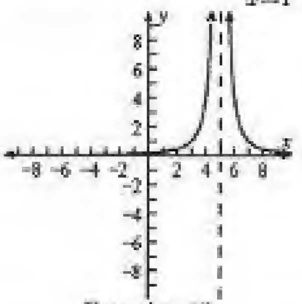
Because $\lim\limits_{x\to -5^-} f(x) = +\infty$ or because $\lim\limits_{x\to -5^+} f(x) = -\infty$, $x = -5$ is a vertical asymptote.

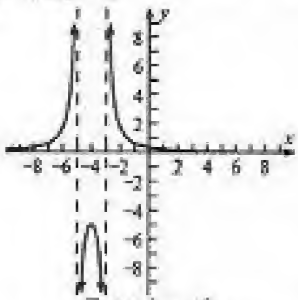
Because $\lim\limits_{x\to -3^-} f(x) = -\infty$ or because $\lim\limits_{x\to -3^+} f(x) = +\infty$, $x = -3$ is a vertical asymptote.

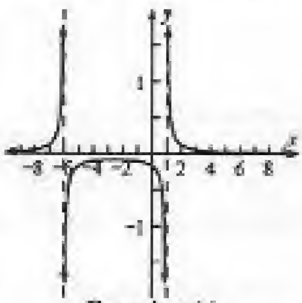
44. $f(x)=\dfrac{1}{x^2+5x-6}$

▶ $f(x)=\dfrac{1}{(x+6)(x-1)}$.
$\lim_{x\to-6^-}(x+6)=0$ and $\lim_{x\to-6^-}(x-1)=-7$. Because $x\to 6^-$, then $x+6<0$; thus $(x+6)(x-1)$ approaches 0 through negative values. Therefore, by Limit Theorem 12(ii) $\lim_{x\to-6^-}\dfrac{1}{x^2+5x-6}=-\infty$. Thus $x=-6$ is a vertical asymptote.
$\lim_{x\to1^+}(x+6)=7$ and $\lim_{x\to1^+}(x-1)=0$. Because $x\to1^+$, then $x-1>0$; thus $(x+6)(x-1)$ approaches 0 thru positive values. Hence, by Limit Theorem 12(i) $\lim_{x\to1^+}\dfrac{1}{x^2+5x-6}=+\infty$. Thus $x=1$ is a vertical asymptote.


Exercise 41


Exercise 42


Exercise 43


Exercise 44

In Exercises 45 and 46, evaluate the limits from the graph of function f sketched in the accompanying figure.

45. (a) $\lim_{x\to-2^+}f(x)=0$ (b) $\lim_{x\to-1^-}f(x)=-\infty$ (c) $\lim_{x\to-1^+}f(x)=+\infty$ (d) $\lim_{x\to0}f(x)=0$ (e) $\lim_{x\to1^-}f(x)=+\infty$
(f) $\lim_{x\to1^+}f(x)=+\infty$ (g) $\lim_{x\to1}f(x)=+\infty$ (h) $\lim_{x\to2^-}f(x)=1$ (i) $\lim_{x\to2^+}f(x)=-\infty$ (j) $\lim_{x\to3^-}f(x)=0$

46. (a) $\lim_{x\to-4^+}f(x)=+\infty$ (b) $\lim_{x\to-2^-}f(x)=2$ (c) $\lim_{x\to-2^+}f(x)=+\infty$ (d) $\lim_{x\to0}f(x)=0$ (e) $\lim_{x\to2^-}f(x)=-\infty$
(f) $\lim_{x\to2^+}f(x)=0$ (g) $\lim_{x\to3^-}f(x)=-\infty$ (h) $\lim_{x\to3^+}f(x)=-\infty$ (i) $\lim_{x\to3}f(x)=-\infty$ (j) $\lim_{x\to4^+}f(x)=0$

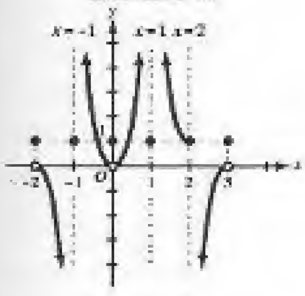
In Exercises 47 and 48, sketch the graph of a function f satisfying the given properties.

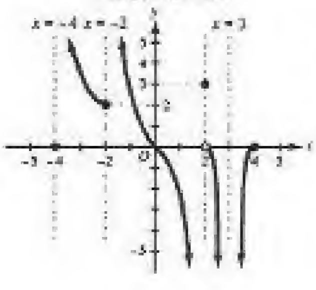
47. The domain of f is $[-5,5]$. $f(-5)=0$; $f(-3)=2$; $f(-1)=0$; $f(0)=0$; $f(1)=0$; $f(3)=-2$; $f(5)=-4$;
$\lim_{x\to-5^+}f(x)=+\infty$; $\lim_{x\to-3}f(x)=0$; $\lim_{x\to-1^-}f(x)=+\infty$; $\lim_{x\to-1^-}f(x)=-\infty$; $\lim_{x\to0}f(x)=0$; $\lim_{x\to1}=+\infty$;
$\lim_{x\to3}f(x)=0$; $\lim_{x\to5^-}f(x)=-\infty$

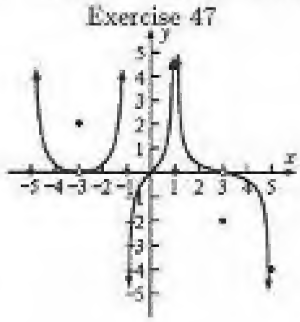
▶ The figure shows the graph of $\begin{cases}-(x+3)^2/[(x+5)(x+1)] & \text{if } -5<x<-1\\ x/(1-x^2) & \text{if } -1<x<1\\ (x-3)/[(x-1)(x-5)] & \text{if } 1<x<5\end{cases}$

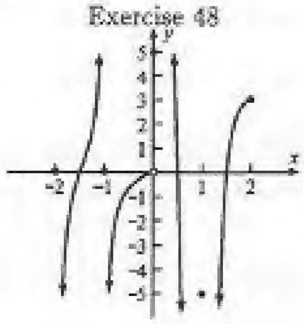
48. The domain of f is $[-2,2]$. $f(-2)=0$; $f(-1)=0$; $f(0)=5$; $f(1)=-5$; $f(2)=3$; $\lim_{x\to-2^+}f(x)=-\infty$;
$\lim_{x\to-1^-}f(x)=+\infty$; $\lim_{x\to-1^+}f(x)=-\infty$; $\lim_{x\to0^-}f(x)=0$; $\lim_{x\to0^+}f(x)=+\infty$; $\lim_{x\to1}f(x)=-\infty$; $\lim_{x\to2^-}f(x)=3$

▶ The figure shows the graph of $\begin{cases}-x(x+1.5)/[(x+2)(x+1)] & \text{if } x\in(-2,-1)\cup(-1,0)\\ 8(x-.5)(x-1.5)/[x(x-1)^2] & \text{if } x\in(0,1)\cup(1,2]\end{cases}$

Exercise 45


Exercise 46


Exercise 47


Exercise 48


49. $\lim_{t\to0^+}n\left(\dfrac{r}{t}+\dfrac{epk}{1000}\right)=+\infty$ because r and n are positive.

50. $f(x)=\dfrac{1}{x-2}$ and $g(x)=\dfrac{1}{2-x}$. (a) $\lim_{x\to2^-}\dfrac{1}{x-2}=\dfrac{1}{0^-}=-\infty$ so $\lim_{x\to2}f(x)$ does not exist. $\lim_{x\to2^-}\dfrac{1}{2-x}=\dfrac{1}{0^+}=+\infty$ so $\lim_{x\to2}g(x)$ does not exist. (b) $f(x)+g(x)=0$ if $x\neq2$. (c) $\lim_{x\to2}[f(x)+g(x)]=0$

(d) Because $\lim\limits_{x\to 2} f(x)$ and $\lim\limits_{x\to 2} g(x)$ do not exist, the hypotheses of Limit Theorem 4 are not satisfied.

51. $\lim\limits_{v\to c^-} \dfrac{m_0}{\sqrt{1-(v/c)^2}} = \dfrac{m_0}{0^+} = +\infty$. $\lim\limits_{v\to c^+} \dfrac{m_0}{\sqrt{1-(v/c)^2}}$ does not exist because $1-(v/c)^2 < 0$ if $v > c$. The limit as $v \to c$ does not exist because neither 1-sided limit exists.

1.7 SUPPLEMENT

Theorem B $\lim\limits_{x\to a} f(x) = +\infty$ if and only if $\lim\limits_{x\to a} -f(x) = -\infty$.

Proof Let $M = -N$. $\lim\limits_{x\to a} f(x) = +\infty$

$\Leftrightarrow$ for every $N > 0$ there is a $\delta > 0$ such that $f(x) > N$ when $0 < |x-a| < \delta$

$\Leftrightarrow$ for every $M < 0$ there is a $\delta > 0$ such that $-f(x) < M$ when $0 < |x-a| < \delta$

$\Leftrightarrow \lim\limits_{x\to a} -f(x) = -\infty$

Supplementary Exercises 1.7

1. Prove that $\lim\limits_{x\to 2} \dfrac{3}{(x-2)^2} = +\infty$ by using Definition 1.7.1.

▹ We must show that for any $N > 0$ there exists a $\delta > 0$ such that if $0 < |x-2| < \delta$ then $\dfrac{3}{(x-2)^2} > N$

or, equivalently, because $(x-4)^2 > 0$ and $N > 0$ if $0 < |x-2| < \delta$ then $(x-2)^2 < \dfrac{3}{N}$

or, equivalently, if $0 < |x-2| < \delta$ then $|x-3| < \sqrt{\dfrac{3}{N}}$

The above statement holds if $\delta = \sqrt{\dfrac{3}{N}}$.

2. Prove that $\lim\limits_{x\to 4} \dfrac{-2}{(x-4)^2} = -\infty$ by using Definition 1.7.2.

▹ We must show that for any $N < 0$ there exists a $\delta > 0$ such that if $0 < |x-4| < \delta$ then $\dfrac{-2}{(x-4)^2} < N$

or, equivalently, because $(x-4)^2 > 0$ and $N < 0$ if $0 < |x-4| < \delta$ then $(x-4)^2 < -\dfrac{2}{N}$

or, equivalently, if $0 < |x-4| < \delta$ then $|x-4| < \sqrt{-\dfrac{2}{N}}$

The above statement holds if $\delta = \sqrt{-\dfrac{2}{N}}$.

3. Prove Limit Theorem 11(ii): If r is any positive integer, then $\lim\limits_{x\to 0^-} \dfrac{1}{x^r} = \begin{cases} -\infty & \text{if } r \text{ is odd} \\ +\infty & \text{if } r \text{ is even} \end{cases}$.

If r is an odd positive integer we must show that for any $N < 0$ there is a $\delta > 0$ such that if $-\delta < x < 0$ then $\dfrac{1}{x^r} < N$

$\Leftrightarrow$ if $0 < -x < \delta$ then $\dfrac{1}{x^r} > -N$

$\Leftrightarrow$ if $0 < -x < \delta$ then $\dfrac{1}{(-x)^r} > -N$ (because r is an odd integer)

$\Leftrightarrow$ if $0 < -x < \delta$ then $(-x)^r < \dfrac{1}{-N}$ (because $-x > 0$ and $-N > 0$)

$\Leftrightarrow$ if $0 < -x < \delta$ then $-x < \left(-\dfrac{1}{N}\right)^{1/r}$ (because r is positive)

The last statement holds if $\delta = \left(-\dfrac{1}{N}\right)^{1/r}$.

If r is an even positive integer we must show that for any $N > 0$ there is a $\delta > 0$ such that if $-\delta < x < 0$ then $\dfrac{1}{x^r} > N$

$\Leftrightarrow$ if $0 < -x < \delta$ then $\dfrac{1}{(-x)^r} > N$ (because r is an even integer)

$\Leftrightarrow$ if $0 < -x < \delta$ then $(-x)^r < \dfrac{1}{N}$ (because $-x > 0$ and $N > 0$)

$\Leftrightarrow$ if $0 < -x < \delta$ then $-x < \left(\dfrac{1}{N}\right)^{1/r}$ (because r is positive)

The last statement holds if $\delta = \left(\dfrac{1}{N}\right)^{1/r}$.

4. Prove Theorem 1.7.4(ii).

▷ We want to prove that if $\lim\limits_{x\to a} f(x) = 0$ with $f(x) \to 0$ through negative values of $f(x)$, and $\lim\limits_{x\to a} g(x) = c$ with $c > 0$, then $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)} = -\infty$. Now $\lim\limits_{x\to a} -f(x) = 0$ with $-f(x) \to 0$ through positive values, and so, by case (i) $\lim\limits_{x\to a} \dfrac{g(x)}{-f(x)} = +\infty$. Hence by Theorem B, $\lim\limits_{x\to a} -\left(\dfrac{g(x)}{-f(x)}\right) = -\infty$, or equivalently, $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)} = -\infty$.

5. Prove Theorem 1.7.4(iii).

▷ We want to prove that if $\lim\limits_{x\to a} f(x) = 0$ with $f(x) \to 0$ through positive values of $f(x)$, and $\lim\limits_{x\to a} g(x) = c$ with $c < 0$, then $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)} = -\infty$. Now $\lim\limits_{x\to a} -g(x) = -c$ with $-c > 0$, and so, by case (i), $\lim\limits_{x\to a} \dfrac{-g(x)}{f(x)} = +\infty$. Hence by Theorem B, $\lim\limits_{x\to a} -\left(\dfrac{-g(x)}{f(x)}\right) = -\infty$, or equivalently, $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)} = -\infty$.

6. Prove Theorem 1.7.4(iv).

▷ We want to prove that if $\lim\limits_{x\to a} f(x) = 0$ with $f(x) \to 0$ through negative values of $f(x)$, and $\lim\limits_{x\to a} g(x) = c$ with $c < 0$, then $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)} = \infty$. Now $\lim\limits_{x\to a} -f(x) = 0$ with $-f(x) \to 0$ through positive values, and $\lim\limits_{x\to a} -g(x) = -c$ with $-c > 0$, and so, by case (i), $\lim\limits_{x\to a} \dfrac{-g(x)}{-f(x)} = +\infty$, or equivalently, $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)} = \infty$.

7. Prove Theorem 1.7.5: If c is any constant and
(i) if $\lim\limits_{x\to a} f(x) = +\infty$ and $\lim\limits_{x\to a} f(x) = c$ then $\lim\limits_{x\to a} [f(x) + g(x)] = +\infty$
(ii) if $\lim\limits_{x\to a} f(x) = -\infty$ and $\lim\limits_{x\to a} g(x) = c$ then $\lim\limits_{x\to a} [f(x) + g(x)] = -\infty$

Because $\lim\limits_{x\to a} g(x) = c$, then for any $\epsilon > 0$ there is a $\delta > 0$ such that
 if $0 < |x - a| < \delta$ then $|g(x) - c| < \epsilon$
Now $|g(x)| - |c| \le |g(x) - c|$. Hence, if $\epsilon = |c| + 1$ there is a δ_1 such that
 if $0 < |x - a| < \delta_1$ then $|g(x) - c| < |c| + 1$
$\Leftrightarrow$ if $0 < |x - a| < \delta_1$ then $|g(x)| < 2|c| + 1$
$\Leftrightarrow$ if $0 < |x - a| < \delta_1$ then $-2|c| - 1 < g(x) < 2|c| + 1$ (1)

To prove part (i) we must show that for any $N > 0$ there is a $\delta > 0$ such that
 if $0 < |x - a| < \delta$ then $f(x) + g(x) > N$
Because $\lim\limits_{x\to a} f(x) = +\infty$, then for $N + 2|c| + 1 > 0$ there is a $\delta_2 > 0$ such that
 if $0 < |x - a| < \delta_2$ then $f(x) > N + 2|c| + 1$
If $\delta = \min(\delta_1, \delta_2)$, by statement (1) we get
 if $0 < |x - a| < \delta$
  then $f(x) + g(x) > N + 2|c| + 1 - 2|c| - 1$
$\Leftrightarrow$ if $0 < |x - a| < \delta$ then $f(x) + g(x) > N$
Hence part (i) is proved.

To prove part (ii) we must show that for any $N < 0$ there is a $\delta > 0$ such that
 if $0 < |x - a| < \delta$ then $f(x) + g(x) < N$
Because $\lim\limits_{x\to a} f(x) = -\infty$, then for $N - 2|c| - 1 < 0$ there is a $\delta_3 > 0$ such that
 if $0 < |x - a| < \delta_3$ then $f(x) < N - 2|c| - 1$
If $\delta = \min(\delta_1, \delta_2)$, by statement (1) we get
 if $0 < |x - a| < \delta$
  then $f(x) + g(x) < N - 2|c| - 1 + 2|c| + 1$
$\Leftrightarrow$ if $0 < |x - a| < \delta$ then $f(x) + g(x) < N$
Hence part (ii) is proved.

8. Prove: Theorem 1.7.6:
If $\lim\limits_{x\to a} f(x) = +\infty$ and $\lim\limits_{x\to a} g(x) = c$, where c is any constant except 0, then
(i) if $c > 0$, $\lim\limits_{x\to a} f(x) \cdot g(x) = +\infty$ (ii) if $c < 0$, $\lim\limits_{x\to a} f(x) \cdot g(x) = -\infty$

▷ By Theorem B, $\lim\limits_{x\to a} -f(x) = -\infty$. By Theorem 1.7.7 proved below
if $c > 0$, $\lim\limits_{x\to a} -f(x) \cdot g(x) = -\infty$ if $c < 0$, $\lim\limits_{x\to a} -f(x) \cdot g(x) = +\infty$
Applying Theorem B again to the limit of the products, we obtain (i) and (ii).

9. Prove: Theorem 1.7.7:
If $\lim_{x\to a} f(x) = -\infty$ and $\lim_{x\to a} g(x) = c$, where c is any constant except 0, then
(i) if $c > 0$, $\lim_{x\to a} f(x)\cdot g(x) = -\infty$
(ii) if $c < 0$, $\lim_{x\to a} f(x)\cdot g(x) = +\infty$

▷ To prove part (i) we must show that for any $N < 0$ there is a $\delta > 0$ such that
if $0 < |x-a| < \delta$ then $f(x)\cdot g(x) < N$ (1)
Because $\lim_{x\to a} g(x) = c$ and $c > 0$ then for $\epsilon = \frac{1}{2}c > 0$ there is a $\delta_1 > 0$ such that
if $0 < |x-a| < \delta_1$ then $|g(x) - c| < \frac{1}{2}c$
$\Rightarrow$ if $0 < |x-a| < \delta_1$ then $g(x) - c > -\frac{1}{2}c$
$\Leftrightarrow$ if $0 < |x-a| < \delta_1$ then $g(x) > \frac{1}{2}c > 0$ (2)
Because $\lim_{x\to a} f(x) = -\infty$ then for $\frac{2N}{c} < 0$ there is a $\delta_2 > 0$ such that
if $0 < |x-a| < \delta_2$ then $-f(x) > -\frac{2N}{c} > 0$ (3)
Let $\delta = \min(\delta_1, \delta_2)$. It follows from statements (2) and (3) that
if $0 < |x-a| < \delta$ then $-f(x)\cdot g(x) > -\frac{2N}{c}\cdot\frac{c}{2} = -N$
This is statement (1) proving part (i).

To prove part (ii) we must show that for any $N > 0$ there is a $\delta > 0$ such that
if $0 < |x-a| < \delta$ then $f(x)\cdot g(x) > N$ (4)
Because $\lim_{x\to a} g(x) = c$ and $c < 0$ then for $\epsilon = -\frac{1}{2}c > 0$ there is a $\delta_3 > 0$ such that
if $0 < |x-a| < \delta_3$ then $|g(x) - c| < -\frac{1}{2}c$
$\Rightarrow$ if $0 < |x-a| < \delta_3$ then $-g(x) + c > \frac{1}{2}c$
$\Leftrightarrow$ if $0 < |x-a| < \delta_3$ then $-g(x) > -\frac{1}{2}c > 0$ (5)
Because $\lim_{x\to a} f(x) = -\infty$ then for $\frac{2N}{c} < 0$ there is a $\delta_4 > 0$ such that
if $0 < |x-a| < \delta_4$ then $-f(x) > -\frac{2N}{c} > 0$ (6)
Let $\delta = \min(\delta_3, \delta_4)$. It follows from statements (5) and (6) that
if $0 < |x-a| < \delta$ then $f(x)\cdot g(x) > -\frac{2N}{c}\cdot -\frac{c}{2} = N$
This is statement (4) proving part (ii).

10. Use Definition 1.7.1 to prove that $\lim_{x\to -3}\left|\frac{5-x}{3+x}\right| = +\infty$.

▷ Choose $\delta \le 1$ so that $|x+3| < \delta \Rightarrow -1 < x+3 < 1 \Rightarrow -9 < x-5 < -7 \Rightarrow 7 < |5-x| < 9$. Then $\left|\frac{5-x}{3+x}\right| > \frac{7}{\delta} > N$ whenever $\delta < \frac{7}{N}$. Choose $\delta = \min(1, 7/N)$.

1.8 CONTINUITY OF A FUNCTION AT A NUMBER

There are many theorems having a hypothesis that includes the condition that a function be continuous at a number a. Hence we must be able to determine if a function is continuous or discontinuous at a. If there is a break in the graph of f at the point where $x = a$, then f is discontinuous at a. This situation is included in the following analytic definition.

1.8.1 Definition The function f is said to be *continuous* at the number a if and only if the following three conditions are satisfied.

(i) $f(a)$ exists
(ii) $\lim_{x\to a} f(x)$ exists
(iii) $\lim_{x\to a} f(x) = f(a)$

If one or more of these three conditions fails to hold at a, the function f is said to be *discontinuous* at a.

We note that we may have to consider one-sided limits and use Theorem 1.6.3 to determine whether or not condition (ii) in Definition 1.8.1 is satisfied. This is illustrated in Exercise 7. If f is a function that is discontinuous at the number a, but for which $\lim_{x\to a} f(x)$ exists and is the real number L, then either $f(a) \ne L$ or else $f(a)$ does not exist. Such a discontinuity is called a *removable discontinuity* because if we define $f(a) = L$ the new function is continuous at a. A removable discontinuity will not show as a break when the graph is plotted on a graphics calculator unless the x-range is carefully chosen. If $\lim_{x\to a} f(x)$ is not a real number, then f has an *essential discontinuity* at a, and the discontinuity cannot be removed.

The following theorems concerning the continuity of a function follow from Definition 1.8.1. Often they can be used to determine if a function is continuous at a number.

1.8.2 Theorem If f and g are two functions that are continuous at the number a, then

(i) $f + g$ is continuous at a
(ii) $f - g$ is continuous at a
(iii) $f \cdot g$ is continuous at a
(iv) f/g is continuous at a, provided that $g(a) \neq 0$

1.8.3 Theorem A polynomial function is continuous at every number.

1.8.4 Theorem A rational function is continuous at every number in its domain.

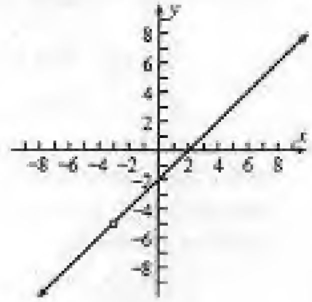
1.8.5 Theorem If n is a positive integer and $f(x) = \sqrt[n]{x}$, then
(i) if n is odd, f is continuous at every number
(ii) if n is even, f is continuous at every positive number

Theorem The absolute-value function is continuous at every number.

Exercises 1.8

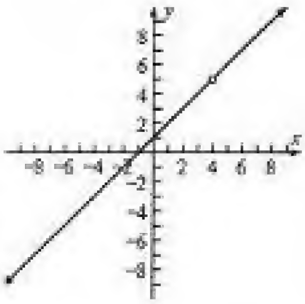
In Exercises 1–14, sketch the graph of the function. By observing where there is a break in the graph, determine the number at which the function is discontinuous; and show why Definition 1.8.1 is not satisfied at this number.

1. $f(x) = \dfrac{x^2 + x - 6}{x + 3} = \dfrac{(x+3)(x-2)}{x+3} = x - 2$ if $x \neq -3$.



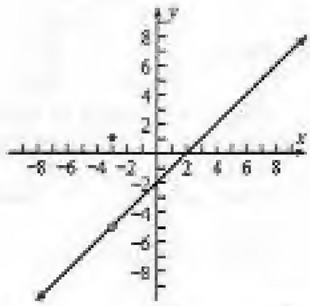
▸ There is a break in the graph at -3.
$f(-3)$ does not exist. Hence condition (i) of Definition 1.8.1 fails at -3.

2. $F(x) = \dfrac{x^2 - 3x - 4}{x - 4} = \dfrac{(x-4)(x+1)}{x-4} = x + 1$ if $x \neq 4$.



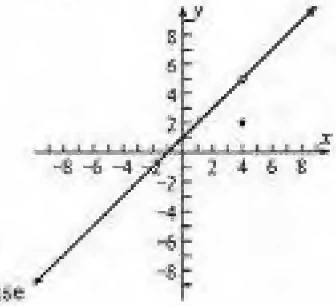
▸ There is a break in the graph at 4.
$f(4)$ does not exist. Hence condition (i) of Definition 1.8.1 fails at 4.

3. $g(x) = \begin{cases} \dfrac{x^2 + x - 6}{x + 3} & \text{if } x \neq -3 \\ 1 & \text{if } x = -3 \end{cases} = \begin{cases} x - 2 & \text{if } x \neq -3 \\ 1 & \text{if } x = -3 \end{cases}$



▸ There is a break in the graph at -3.
(i) $g(-3) = 1$; (ii) $\lim\limits_{x \to -3} g(x) = -5$;
(iii) $\lim\limits_{x \to -3} g(x) \neq g(-3)$.
Thus condition (iii) of Definition 1.8.1 fails at -3.
Hence, g is discontinuous at -3.

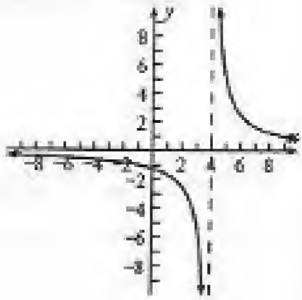
4. $G(x) = \begin{cases} \dfrac{x^2 - 3x - 4}{x - 4} & \text{if } x \neq 4 \\ 2 & \text{if } x = 4 \end{cases}$



▸ Because
$$\frac{x^2 - 3x - 4}{x - 4} = \frac{(x-4)(x+1)}{x-4} = x + 1 \quad \text{if } x \neq 4$$
then
$$G(x) = \begin{cases} x + 1 & \text{if } x \neq 4 \\ 2 & \text{if } x = 4 \end{cases}$$
There is a "hole" in the line at the point $(4, 5)$ because $G(x) \neq x + 1$ if $x = 4$, and thus G is discontinuous at 4. We show how Definition 1.8.1 is not satisfied. Because
$$\lim_{x \to 4} G(x) = \lim_{x \to 4} (x + 1) = 5 \quad \text{and} \quad G(4) = 2$$
then
$$\lim_{x \to 4} G(x) \neq G(4)$$
and thus condition (iii) of Definition 1.8.1 is not satisfied.

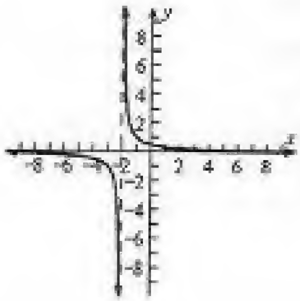
5. $h(x) = \dfrac{5}{x-4}$

▷ There is a break in the graph at 4.
$h(4)$ does not exist. Thus, condition (i) of Definition 1.8.1 fails at 4.
Therefore, f is discontinuous at 4.



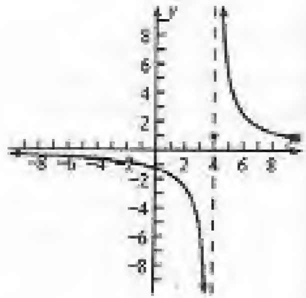
6. $\text{H}(x) = \dfrac{1}{x+2}$

▷ There is a break in the graph at -2.
$h(-2)$ does not exist. Thus, condition (i) of Definition 1.8.1 fails at -2.
Therefore, f is discontinuous at -2.



7. $f(x) = \begin{cases} \dfrac{5}{x-4} & \text{if } x \neq 4 \\ 1 & \text{if } x = 4 \end{cases}$

There is a break in the graph at 4.
$\lim\limits_{x\to 4^+} f(x) = +\infty$ and $\lim\limits_{x\to 4^-} f(x) = -\infty$. Hence, condition (ii) of Definition 1.8.1 fails at 4.
Therefore, f is discontinuous at 4.



8. $g(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 0 & \text{if } x = -2 \end{cases}$

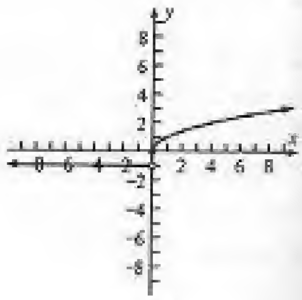
▷ There is a break in the graph at $x = -2$;
so we use Definition 1.8.1 at $a = -2$ to show there is a discontinuity. Because $g(-2) = 0$, condition (i) is satisfied. However

$$\lim_{x\to -2^+} g(x) = +\infty \quad \text{and} \quad \lim_{x\to -2^-} g(x) = -\infty$$

so condition (ii) is not satisfied. Therefore, g is discontinuous at -2.

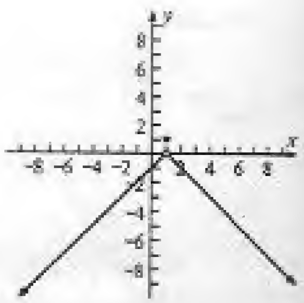
9. $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sqrt{x} & \text{if } 0 < x \end{cases}$

There is a break in the graph at 0.
$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (-1) = -1$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \sqrt{x} = 0$; therefore $\lim\limits_{x\to 0} f(x)$ does not exist. Thus condition (ii) of Definition 1.8.1 fails at 0. Hence f is discontinuous at 0.

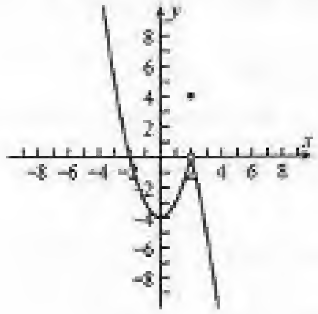


10. $f(x) = \begin{cases} x-1 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ 1-x & \text{if } x > 1 \end{cases}$

There is a break in the graph at 1.
(i) $f(1) = 1$; (ii) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x-1) = 0$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (1-x) = 0$. Thus $\lim\limits_{x\to 1} f(x) = 0$. (iii) $\lim\limits_{x\to 1} f(x) \neq f(1)$
Thus condition (iii) fails at 1. Hence f is discontinuous at 1.

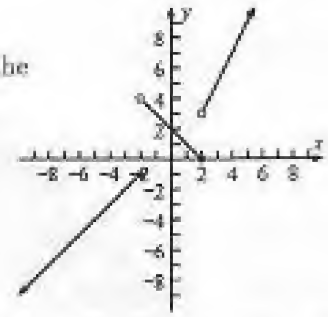


11. $g(t) = \begin{cases} t^2 - 4 & \text{if } t < 2 \\ 4 & \text{if } t = 2 \\ 4 - t^2 & \text{if } 2 < t \end{cases}$



▸ There is a break in the graph at 2.
(i) $g(2) = 4$; (ii) $\lim\limits_{t \to 2^-} g(t) = \lim\limits_{t \to 2^-} (t^2 - 4) = 0$ and
$\lim\limits_{t \to 2^+} g(t) = \lim\limits_{t \to 2^+} (4 - t^2) = 0$.
Therefore $\lim\limits_{t \to 0} g(t) = 0$. (iii) $\lim\limits_{t \to 2} g(t) \neq g(2)$.
Thus condition (iii) of Definition 1.8.1 fails at 2.
Hence g is discontinuous at 2.

12. $H(x) = \begin{cases} 1 + x & \text{if } x \leq -2 \\ 2 - x & \text{if } -2 < x \leq 2 \\ 2x - 1 & \text{if } 2 < x \end{cases}$



▸ There are breaks in the graph at the points where $x = -2$ and $x = 2$. We consider the points separately. Since $H(-2) = 1 + 2 = -1$, condition (i) of Definition 1.8.1 is satisfied when $a = -2$. Because

$$\lim_{x \to -2^+} H(x) = \lim_{x \to -2^+} (2 - x) = 4$$

and

$$\lim_{x \to -2^-} H(x) = \lim_{x \to -2^-} (1 + x) = -1$$

$\lim\limits_{x \to -2} H(x)$ does not exist. Therefore condition (ii) of Definition 1.8.1 is not satisfied when $a = -2$, and thus H is discontinuous at -2.
Since $H(2) = 2 - 2 = 0$, condition (i) of Definition 1.8.1 is satisfied when $a = 2$. Because

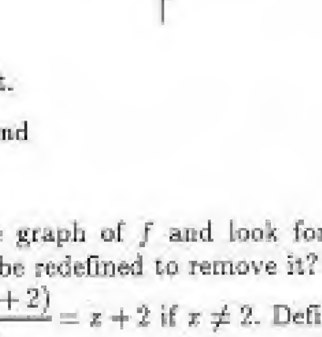
$$\lim_{x \to 2^+} H(x) = \lim_{x \to 2^+} (2x - 1) = 3$$

and

$$\lim_{x \to 2^-} H(x) = \lim_{x \to 2^-} (2 - x) = 0$$

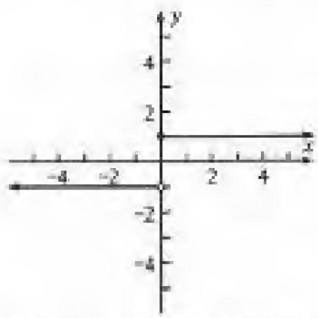
$\lim\limits_{x \to 2} H(x)$ does not exist. Thus condition (ii) in Definition 1.8.1 is not satisfied when $a = 2$, and thus H is discontinuous at 2.

13. $f(x) = -\dfrac{|x|}{x}$, $x \neq 0$.



▸ There is a break in the graph at 0.
$f(0)$ does not exist. Hence condition (i) of Definition 1.8.1 fails at 0.
Therefore, f is discontinuous at 0.

14. $g(x) = \begin{cases} |x|/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

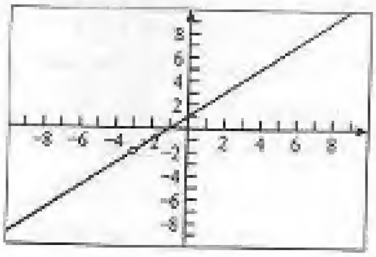


▸ There is a break in the graph at 0. (i) $f(0) = 1$.
(ii) $\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} -x/x = -1$ and
$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} x/x = 1$. Thus $\lim\limits_{x \to 0} g(x)$ does not exist.
Condition (ii) in Definition 1.8.1 is not satisfied at 0, and so g is discontinuous at 0.

In Exercises 15–28, f is discontinuous at a. (a) Plot the graph of f and look for a break at $x = a$. Does the discontinuity appear to be removable? If so, how should f be redefined to remove it? (b) Confirm analytically.

15. $f(x) = \dfrac{x^2 - 4}{x - 2}$; $a = 2$. ▸ $f(x) = \dfrac{(x-2)(x+2)}{x-2} = x + 2$ if $x \neq 2$. Define $f(2) = 2 + 2 = 4$

16. $f(x) = \dfrac{x^2+4x+3}{x+3}$; $a = -3$



▷ $f(x) = \dfrac{(x+3)(x+1)}{x+3} = x+1$ if $x \neq -3$.

To make f continuous, we should define $f(-3) = -3+1 = -2$.

17. $f(x) = \dfrac{x-9}{\sqrt{x}-3}$; $a = 9$

▷ $f(x) = \dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} = \sqrt{x}+3$ if $x \neq 9$. Define $f(9) = \sqrt{9}+3 = 6$

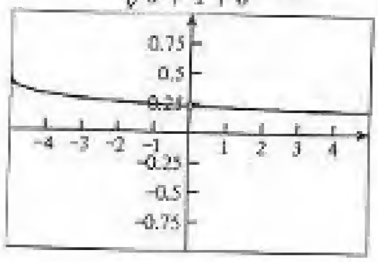
18. $f(x) = \dfrac{x-5}{\sqrt{x-1}-2}$; $a = 5$

▷ $f(x) = \dfrac{(x-1)-4}{\sqrt{x-1}-2} = \dfrac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{\sqrt{x-1}-2} = \sqrt{x-1}+2$ if $x \neq 5$. Define $f(5) = \sqrt{5-1}+2 = 4$.

19. $f(x) = \dfrac{\sqrt{x+4}-3}{x-5}$; $a = 5$

▷ $f(x) = \dfrac{\sqrt{x+4}-3}{(x+4)-9} = \dfrac{\sqrt{x+4}-3}{(\sqrt{x+4}-3)(\sqrt{x+4}+3)} = \dfrac{1}{\sqrt{x+4}+3}$ if $x \neq 5$. Define $f(5) = \dfrac{1}{\sqrt{5+4}+3} = \dfrac{1}{6}$.

20. $f(x) = \dfrac{\sqrt{x+5}-\sqrt{5}}{x}$; $a = 0$



▷ $f(x) = \dfrac{\sqrt{x+5}-\sqrt{5}}{(x+5)-5} = \dfrac{\sqrt{x+5}-\sqrt{5}}{(\sqrt{x+5}-\sqrt{5})(\sqrt{x+5}+\sqrt{5})} = \dfrac{1}{\sqrt{x+5}+\sqrt{5}}$

if $x \neq 0$. To make f continuous, we should define

$f(0) = \dfrac{1}{\sqrt{0+5}+\sqrt{5}} = \dfrac{1}{2\sqrt{5}}$.

21. $f(x) = \dfrac{\sqrt{2}-\sqrt{x+2}}{x}$; $a = 0$ ▷ $f(x) = \dfrac{\sqrt{2}-\sqrt{x+2}}{(x+2)-2} = \dfrac{\sqrt{2}-\sqrt{x+2}}{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})} = \dfrac{-1}{\sqrt{x+2}+\sqrt{2}}$

if $x \neq 0$. Define $f(0) = \dfrac{-1}{\sqrt{0+2}+\sqrt{2}} = \dfrac{-1}{2\sqrt{2}} = -\frac{1}{4}\sqrt{2}$

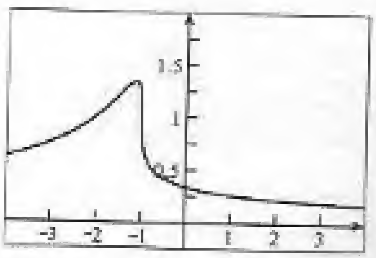
22. $f(x) = \dfrac{2-\sqrt{x+1}}{x-3}$; $a = 3$ ▷ $f(x) = \dfrac{2-\sqrt{x+1}}{(x+1)-4} = \dfrac{2-\sqrt{x+1}}{(\sqrt{x+1}-2)(\sqrt{x+1}+2)} = \dfrac{-1}{\sqrt{x+1}+2}$ if $x \neq 3$.

Define $f(3) = \dfrac{-1}{\sqrt{3+1}+2} = -\dfrac{1}{4}$.

23. $f(x) = \dfrac{\sqrt[3]{x}-2}{x-8}$; $a = 8$ ▷ $f(x) = \dfrac{x^{1/3}-2}{(x^{1/3})^3-2^3} = \dfrac{x^{1/3}-2}{(x^{1/3}-2)(x^{2/3}+2x^{1/3}+4)} = \dfrac{1}{x^{2/3}+2x^{1/3}+4}$

if $x \neq 8$. Define $f(8) = \dfrac{1}{8^{2/3}+2\cdot 8^{1/3}+4} = \dfrac{1}{12}$.

24. $f(x) = \dfrac{\sqrt[3]{x+1}-1}{x}$; $a = 0$



▷ $f(x) = \dfrac{(x+1)^{1/3}-1}{[(x+1)^{1/3}]^3-1^3} = \dfrac{(x+1)^{1/3}-1}{[(x+1)^{1/3}-1][(x+1)^{2/3}+(x+1)^{1/3}+1]}$

$= \dfrac{1}{(x+1)^{2/3}+(x+1)^{1/3}+1}$ if $x \neq 0$. To make f continuous,

we should define $f(0) = \dfrac{1}{(0+1)^{2/3}+(0+1)^{1/3}+1} = \dfrac{1}{3}$.

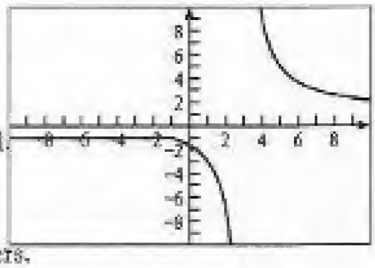
25. $f(x) = \dfrac{x+3}{3-|x|}$; $a = -3$

▷ $f(x) = \dfrac{x+3}{3-|x|}\cdot\dfrac{3+|x|}{3+|x|} = \dfrac{(x+3)(3+|x|)}{(3-x)(3+x)} = \dfrac{3+|x|}{3-x}$ if $x \neq -3$. Define $f(-3) = \dfrac{3+3}{3+3} = 1$.

26. $f(x)=\dfrac{x+5}{|x+1|-4}$; $a=-5$ ▶ $f(x)=\dfrac{x+5}{|x+1|-4}\cdot\dfrac{|x+1|+4}{|x+1|+4}=\dfrac{(x+5)(|x+1|+4)}{[(x+1)-4][(x+1)+4]}=\dfrac{|x+1|+4}{x-3}$ if $x\neq-5$. Define $f(-5)=\dfrac{|-5+1|+4}{-5-3}=-1$

27. $f(x)=\dfrac{x+3}{3-|x|}$; $a=3$ ▶ $\lim\limits_{x\to3^-}f(x)=\lim\limits_{x\to3^-}\dfrac{x+3}{3-x}=\dfrac{6}{0^+}=+\infty$. Essential discontinuity.

28. $f(x)=\dfrac{x+5}{|x+1|-4}$; $a=3$



▶ $\lim\limits_{x\to3}f(x)=\lim\limits_{x\to3}\dfrac{x+5}{(x+1)+4}$ does not exist because the denominator is approaching 0 and the numerator is not. Hence the discontinuity is essential.

In Exercises 29–40, determine the numbers at which the function is continuous and state the reason.

29. $f(x)=x^2(x+3)^2$ is a polynomial. Hence, f is continuous for all real numbers.

30. $f(x)=(x-5)^3(x^2+4)^5$ is a polynomial. Hence, f is continuous for all real numbers.

31. $g(x)=\dfrac{x}{x-3}$ is a rational function. Hence f is continuous on its domain: all real numbers except 3.

32. $h(x)=\dfrac{x+1}{2x+5}$

▶ Because h is a rational function, by Theorem 1.8.4, h is continuous at every number in its domain. Therefore, h is continuous at every number for which $2x+5\neq0$ or equivalently, $x\neq-\frac{5}{2}$.

33. $F(x)=\dfrac{x^3+7}{x^2-4}$ is a rational function. Hence F is continuous on its domain: all real numbers except 2 and -2.

34. $G(x)=\dfrac{x-2}{x^2+2x-8}=\dfrac{x-2}{(x+4)(x-2)}$ is a rational function. Hence G is continuous on its domain: all real numbers except 2 and -4. The discontinuity at 2 is removable.

35. $f(x)=\begin{cases}3x-1 & \text{if } x<2\\ 4-x^2 & \text{if } 2\leq x\end{cases}$. If $x\neq2$, $f(x)$ is a polynomial. Hence f is continuous at numbers other than 2.

$\lim\limits_{x\to2^-}f(x)=\lim\limits_{x\to2^-}(3x-1)=5$ and $\lim\limits_{x\to2^+}f(x)=\lim\limits_{x\to2^+}(4-x^2)=0$; thus $\lim\limits_{x\to2}f(x)$ does not exist. Therefore, f is discontinuous at 2.

36. $f(x)=\begin{cases}(x+2)^2 & \text{if } x\leq0\\ x^2+2 & \text{if } 0<x\end{cases}$

▶ Because f is a polynomial function for $x<0$ and for $x>0$, by Theorem 1.8.3, f is continuous at every $x\neq0$. Furthermore,

$$\lim_{x\to0^-}f(x)=\lim_{x\to0^-}(x+2)^2=4 \qquad \lim_{x\to0^+}f(x)=\lim_{x\to0^+}(x^2+2)=2$$

Therefore, $\lim\limits_{x\to0}f(x)$ does not exist, so f is discontinuous at 0.

37. $f(x)=\begin{cases}1/(x+1) & \text{if } x\leq1\\ 1/(3-x) & \text{if } x>1\end{cases}$. If $x\neq1$, $f(x)$ is a rational function. Hence f is continuous except at -1 and 3. $\lim\limits_{x\to1^-}f(x)=\lim\limits_{x\to1^-}1/(x+1)=\frac{1}{2}=f(1)$ and $\lim\limits_{x\to1^+}f(x)=\lim\limits_{x\to1^+}1/(3-x)=\frac{1}{2}$. Hence f is continuous at 1.

38. $f(x)=\begin{cases}1/x & \text{if } x<3\\ 2/(9-x) & \text{if } x\geq3\end{cases}$. If $x\neq3$, $f(x)$ is a rational function. Hence f is continuous if $x\neq0$ or 9. $\lim\limits_{x\to3^-}f(x)=\lim\limits_{x\to3^-}(1/x)=\frac{1}{3}$ and $\lim\limits_{x\to3^+}f(x)=\lim\limits_{x\to3^+}2/(9-x)=\frac{1}{3}=f(3)$. Hence f is continuous at 3.

39. $h(x)=\begin{cases}x+\sqrt[3]{x} & \text{if } x<0\\ x-\sqrt{x} & \text{if } x\geq0\end{cases}$. If $x\neq0$, f is continuous by Theorem 1.8.5.

$\lim\limits_{x\to0^-}h(x)=\lim\limits_{x\to0}(x+\sqrt[3]{x})=0$ and $\lim\limits_{x\to0^+}h(x)=\lim\limits_{x\to0^+}(x+\sqrt{x})=0$. Hence h is continuous for all real numbers.

40. $g(x) = \begin{cases} 2x - \sqrt[3]{x} & \text{if } x \le 1 \\ x\sqrt{x} & \text{if } x > 1 \end{cases}$

▷ The radicals are continuous by Theorem 1.8.5. $2x$ and x are polynomials and so are continuous. Hence if $x \ne 1$, g is continuous by Theorem 1.8.2 (ii) and (iii). Furthermore,

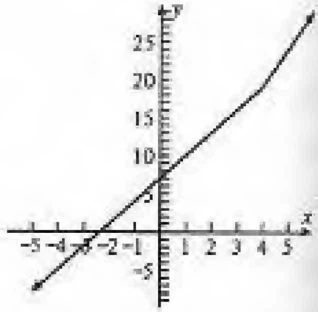
$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-}(2x - \sqrt[3]{x}) = 2 - 1 = 1 = g(1) \text{ and } \lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} x\sqrt{x} = 1 \cdot 1 = 1$$

Therefore g is continuous for all real numbers.

In Exercises 41–44: (a) Find c and k that make f continuous everywhere. (b) Sketch the graph of f.

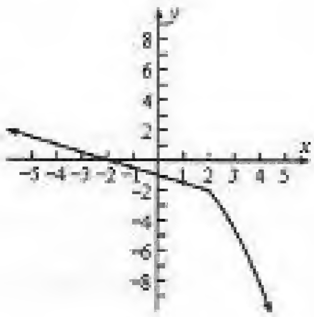
41. $f(x) = \begin{cases} 3x + 7 & \text{if } x \le 4 \\ kx - 1 & \text{if } 4 < x \end{cases}$

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (3x + 7) = 19 = f(4)$;
$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (kx - 1) = 4k - 1$
Hence f is continuous at 4 if and only if
$4k - 1 = 19$; $k = 5$.



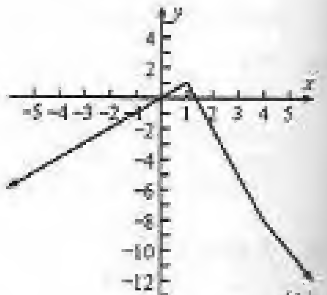
42. $f(x) = \begin{cases} kx - 1 & \text{if } x < 2 \\ kx^2 & \text{if } x \ge 2 \end{cases}$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} kx^2 = 4k = f(2)$
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} kx - 1 = 2k - 1$
Hence f is continuous at 2 if and only if
$4k = 2k - 1$; $2k = -1$; $k = -\frac{1}{2}$



43. $f(x) = \begin{cases} x & \text{if } x \le 1 \\ cx + k & \text{if } 1 < x < 4 \\ -2x & \text{if } 4 \le x \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1 = f(1)$; $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (cx + k) = c + k$
Therefore f is continuous at 1 if and only if $c + k = 1$.
$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (cx + k) = 4c + k$; $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (-2x) = -8 = f(4)$
Therefore f is continuous at 4 if and only if $4c + k = -8$.
Solving $c + k = 1$ and $4c + k = -8$ simultaneously, we get $c = -3$ and $k = 4$.



44. $f(x) = \begin{cases} x + 2c & \text{if } x < -2 \\ 3cx + k & \text{if } -2 \le x \le 1 \\ 3x - 2k & \text{if } 1 < x \end{cases}$ (1)

▷ For all values of c and k the function f is continuous at all x, except possibly at $x = -2$ and $x = 1$.

If f is continuous at -2, then

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x)$$
$$\lim_{x\to -2^-} (x + 2c) = \lim_{x\to -2^+} (3cx + k)$$
$$-2 + 2c = -6c + k$$

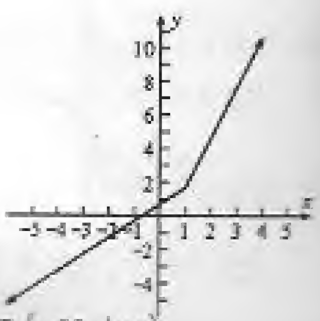
If f is continuous at 1, then

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$
$$\lim_{x\to 1^-} (3cx + k) = \lim_{x\to 1^+} (3x - 2k)$$
$$3c + k = 3 - 2k$$

Solving these equations simultaneously, we get $c = \frac{1}{3}$ and $k = \frac{2}{3}$.

Substituting these values for c and k into Eq. (1), we have

$f(x) = \begin{cases} x + \frac{2}{3} & \text{if } x < -2 \\ x + \frac{2}{3} & \text{if } -2 \le x \le 1 \\ 3x - \frac{4}{3} & \text{if } 1 < x \end{cases}$ or, equivalently, $f(x) = \begin{cases} x + \frac{2}{3} & \text{if } x \le 1 \\ 3x - \frac{4}{3} & \text{if } x > 1 \end{cases}$



Now, $f(1) = 1 + \frac{2}{3} = \frac{5}{3}$. Furthermore,

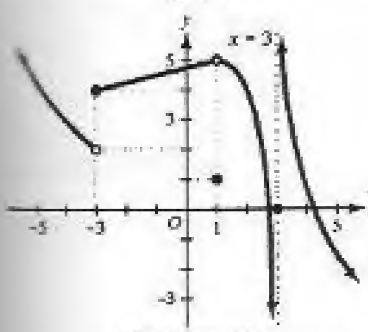
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x + \frac{2}{3}) = \frac{5}{3}$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3x - \frac{4}{3}) = \frac{5}{3}$

Thus, $\lim_{x\to 1} f(x) = \frac{5}{3} = f(1)$. Therefore, f is continuous at 1. Hence, f is continuous on $(-\infty, +\infty)$.

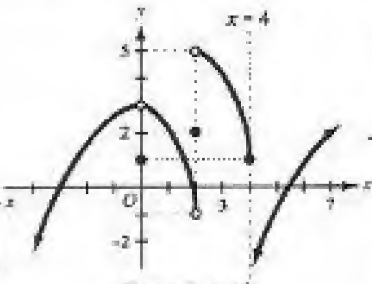
In Exercises 45–46, in (a)–(c) show that f in the figure is discontinuous by showing how Definition 1.8.1 fails.

45. (a) $x = -3$: $\lim_{x\to -3^-} f(x) \neq \lim_{x\to -3^+} f(x)$; essential (b) $x = 1$: $\lim_{x\to 1} f(x) = 5 \neq f(1)$; removable: define $f(1) = 5$
(c) $x = 3$: $\lim_{x\to 3^-} f(x)$ does not exist; essential

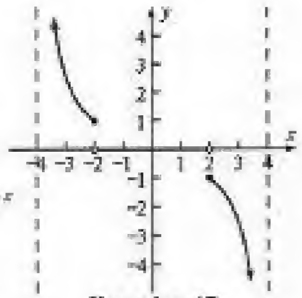
46. (a) $x = 0$: $\lim_{x\to 0} f(x) = 3 \neq f(0)$; removable: define $f(0) = 3$ (b) $x = 2$: $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$; essential
(c) $x = 4$: $\lim_{x\to 4^+} f(x)$ does not exist; essential



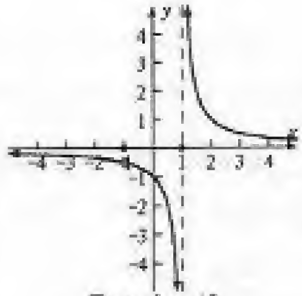
Exercise 45



Exercise 46



Exercise 47



Exercise 48

In Exercises 47 and 48, sketch a graph of some function f satisfying the given properties.

47. f is continuous in $(-4,-2)$, $(-2,2)$ and $(2,4)$; f is discontinuous at -2 and 2; $f(-2) = 0$ and $f(2) = 0$;

$$\lim_{x\to -4^+} f(x) = +\infty,\ \lim_{x\to -2^-} f(x) = 0;\ \lim_{x\to 2^+} f(x) = 0;\ \lim_{x\to 4^-} f(x) = -\infty$$

48. f is continuous on $(-\infty,-1)$, $(-1,1)$, and $(1,+\infty)$; f is discontinuous at -1 and 1; $f(-1) = 0$ and $f(1) = 0$; $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$ both exist but neither is 0; neither $\lim_{x\to 1^-} f(x)$ nor $\lim_{x\to 1^+} f(x)$ exists.

In Exercises 49–52, state where the function is discontinuous by showing how Definition 1.8.1 fails.

49. $f(x)$ dollars is the cost of shipping x lb. $f(x) = \begin{cases} 2.2x & \text{if } 0 < x \le 50 \\ 2.1x & \text{if } 50 < x \le 200 \\ 2.05x & \text{if } x > 200 \end{cases}$.

▶ $x = 50$: $\lim_{x\to 50^-} f(x) = 110 \neq 105 = \lim_{x\to 50^+} f(x)$. $x = 200$: $\lim_{x\to 200^-} f(x) = 420 \neq 410 = \lim_{x\to 200^+} f(x)$

50. $\mathrm{F}(x)$ cents is the cost of mailing x ounces. $\mathrm{F}(x) = 6 - 23[\![-x]\!]$.

▶ If n is any positive integer, $\lim_{x\to n^-} \mathrm{F}(x) \neq \lim_{x\to n^+} \mathrm{F}(x)$; condition (ii) is not satisfied.

51. $g(x)$ cents is the cost of an x minute call. $g(x) = 10 - 30[\![-x]\!]$.

▶ If n is any positive integer, $\lim_{x\to n^-} g(x) \neq \lim_{x\to n^+} g(x)$; condition (ii) is not satisfied.

52. $\mathrm{G}(x)$ dollars is the admission for age x years. $\mathrm{G}(x) = \begin{cases} 4 & \text{if } 0 < x < 12 \\ 7 & \text{if } 12 \le x < 60 \\ 5 & \text{if } 60 \le x \end{cases}$.

▶ $x = 12$: $\lim_{x\to 12^-} \mathrm{G}(x) = 4 \neq 7 = \lim_{x\to 12^+} \mathrm{G}(x)$ $\qquad x = 60$: $\lim_{x\to 60^-} \mathrm{G}(x) = 7 \neq 5 = \lim_{x\to 60^+} \mathrm{G}(x)$
Condition (ii) of Definition 1.8.1 is not satisfied at 12 and 60.

53. At t minutes, $r(t)$ meters is the radius. $r(t) = \begin{cases} 4t^2 + 20 & \text{if } 0 \le t \le 2 \\ 16t + 4 & \text{if } t > 2 \end{cases}$

▶ $\lim_{t\to 2^-} r(t) = \lim_{t\to 2^-} (4t^2 + 20) = 4(2)^2 + 20 = 36 = r(2)$ and $\lim_{t\to 2^+} r(t) = \lim_{t\to 2^+} (16t + 4) = 36$

54. At t minutes, $\mathrm{A}(t)$ m^2 is the area. $\mathrm{A}(t) = \pi r^2 = \begin{cases} \pi(4t^2 + 20)^2 & \text{if } 0 \le t \le 2 \\ \pi(16t + 4)^2 & \text{if } t > 2 \end{cases}$

▶ $\lim_{t\to 2^-} r(t) = \lim_{t\to 2^-} \pi(4t^2 + 20)^2 = \pi(36)^2 = r(2)$ and $\lim_{t\to 2^+} r(t) = \lim_{t\to 2^+} \pi(16t + 4)^2 = \pi(36)^2$

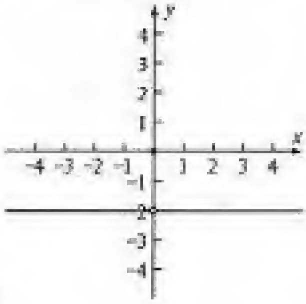
55. If n is a positive integer, $\lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{x^n - 1}{x - 1} = \lim_{x\to 1} \dfrac{(x-1)(x^{n-1} + x^{n-2} + \cdots + 1)}{x - 1}$
$= \lim_{x\to 1} (x^{n-1} + x^{n-2} + \cdots + 1) = 1 + 1 + \cdots + 1 = n$. The discontinuity is removed by defining $f(1) = n$.

56. The function f is defined by $f(x) = \lim_{n \to 0} \dfrac{2nx}{n^2 - nx}$.

Sketch the graph of f. At what values of x is f discontinuous?

▷ We divide the numerator and denominator by n to obtain the limit. Thus,

$$\lim_{n \to 0} \frac{2nx}{n^2 - nx} = \lim_{n \to 0} \frac{2x}{n - x} = \frac{2x}{-x} = -2 \quad \text{if } x \neq 0$$



Thus,

$$f(x) = -2 \quad \text{if} \quad x \neq 0$$

Furthermore, by replacing x with 0 in the definition of f,

$$f(0) = \lim_{n \to 0} \frac{0}{n^2} = 0$$

Because $\lim_{x \to 0} f(x) = \lim_{x \to 0} -2 = -2$ and $f(0) = 0$, f is discontinuous at $x = 0$.

57. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ and $g(x) = \begin{cases} 1 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$; $(f \cdot g)(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} = |x|$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$; hence $\lim f(x)$ does not exist.
Therefore, f is discontinuous at 0.

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 1 = 1$ and $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} x = 0$; thus $\lim g(x)$ does not exist.
Therefore, g is discontinuous at 0.

$(f \cdot g)(x) = |x|$ so $f \cdot g$ is continuous everywhere; in particular, $f \cdot g$ is continuous at 0.

58. Let $f(x) = x$ and $g(x) = \operatorname{sgn} x$. Then f is continuous at 0, g is discontinuous at 0 and $f(x)g(x) = |x|$ is continuous at 0.

59. Let $f(x) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } a \leq x \end{cases}$ and $g(x) = \begin{cases} 1 & \text{if } x < a \\ 0 & \text{if } a \leq x \end{cases}$.
Then f and g are discontinuous at a because condition (ii) of Definition 1.8.1 fails at a for both functions. But $(f + g)(x) = 1$ for every x; therefore $f + g$ is continuous everywhere; in particular, $f + g$ is continuous at a.

1.9 CONTINUITY OF A COMPOSITE FUNCTION AND CONTINUITY ON AN INTERVAL

1.9.1 Theorem If $\lim_{x \to a} g(x) = b$ and if the function f is continuous at b,

$$\lim_{x \to a} (f \circ g)(x) = f(b)$$

or, equivalently,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

1.9.2 Theorem If the function g is continuous at a and the function f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

1.9.3 Definition A function is said to be *continuous on an open interval* if and only if it is continuous at every number in the open interval.

1.9.4 Definition The function f is said to be *continuous from the right at the number a* if and only if the following three conditions are satisfied:

(i) $f(a)$ exists
(ii) $\lim_{x \to a^+} f(x)$ exists
(iii) $\lim_{x \to a^+} f(x) = f(a)$

1.9.5 Definition The function f is said to be *continuous from the left at the number a* if and only if the following three conditions are satisfied:

(i) $f(a)$ exists
(ii) $\lim_{x \to a^-} f(x)$ exists
(iii) $\lim_{x \to a^-} f(x) = f(a)$

A function f is continuous at a if and only if the function f is both continuous from the right at a and continuous from the left at a.

1.9.6 Definition A function whose domain includes the closed interval $[a,b]$ is said to be *continuous on* $[a,b]$ if and only if it is continuous on the open interval (a,b), as well as continuous from the right at a and continuous from the left at b.

A function f may be continuous on the closed interval $[a,b]$ with out being continuous at either a or b. For example, let f be the function defined as follows:

$$f(x)=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } 0\le x\le 1\\ 1 & \text{if } x>1\end{cases}$$

Then f is continuous on the closed interval $[0,1]$, but f is discontinuous at both $x=0$ and $x=1$.

1.9.7 Definition

(i) A function whose domain includes the interval half-open on the right $[a,b)$ is said to be *continuous on* $[a,b)$ if and only if it is continuous on the open interval (a,b) and continuous from the right at a.

(ii) A function whose domain includes the interval half-open on the left $(a,b]$ is said to be *continuous on* $(a,b]$ if and only if it is continuous on the open interval (a,b) and continuous from the left at b.

If a function f is continuous on any interval I, then f is continuous on any subset of the interval I.

1.9.8 Intermediate-Value Theorem If the function f is continuous on the closed interval $[a,b]$ and if $f(a)\neq f(b)$, then for any number k (strictly) between $f(a)$ and $f(b)$ there exists a number c in the open interval (a,b) such that $f(c)=k$.

Exercises 1.9

In Exercises 1-6, define $f\circ g$ and determine the numbers at which $f\circ g$ is continuous.

1 and 17. (a) $f(x)=\sqrt{x}$; $g(x)=9-x^2$ ▷ $(f\circ g)(x)=\sqrt{9-x^2}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.3, g is continuous for all x and $g(x)>0$ for $-3<x<3$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in $(-3,3)$. Because
$\lim\limits_{x\to -3^+}(f\circ g)(x)=0=(f\circ g)(-3)$ and $\lim\limits_{x\to 3^-}(f\circ g)(x)=0=(f\circ g)(3)$, $f\circ g$ is continuous on $[-3,3]$.

(b) $f(x)=\sqrt{x}$; $g(x)=x^2-16$ ▷ $(f\circ g)(x)=\sqrt{x^2-16}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.3, g is continuous for all x and $g(x)>0$ for $x<-4$ or $x>4$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in $(-\infty,-4)\cup(4,+\infty)$.
Because $\lim\limits_{x\to -4^-}(f\circ g)(x)=0=(f\circ g)(-4)$ and $\lim\limits_{x\to 4^+}(f\circ g)(4)=0=(f\circ g)(4)$,
$f\circ g$ is continuous on $(-\infty,-4]\cup[4,+\infty)$.

2 and 18. (a) $f(x)=\sqrt{x}$; $g(x)=16-x^2$ ▷ $(f\circ g)(x)=\sqrt{16-x^2}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.3, g is continuous for all x and $g(x)>0$ for $-4<x<4$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in $(-4,4)$. Because
$\lim\limits_{x\to -4^+}(f\circ g)(x)=0=(f\circ g)(-4)$ and $\lim\limits_{x\to 4^-}(f\circ g)(x)=0=(f\circ g)(4)$, $f\circ g$ is continuous on $[-4,4]$.

(b) $f(x)=\sqrt{x}$; $g(x)=x^2+4$ ▷ $(f\circ g)(x)=\sqrt{x^2+4}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.3, g is continuous for all x and $g(x)>0$ all x.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x, that is, on $(-\infty,\infty)$.

3 and 19. (a) $f(x)=\sqrt{x}$; $g(x)=\dfrac{1}{x-2}$ ▷ $(f\circ g)(x)=\sqrt{\dfrac{1}{x-2}}=\dfrac{1}{\sqrt{x-2}}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.4, g is continuous except at 2 and $g(x)>0$ for $x>2$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous in $(2,+\infty)$.
Because $\lim\limits_{x\to 2^+}(f\circ g)(x)$ does not exist, $f\circ g$ is continuous only on $(2,+\infty)$.

(b) $f(x) = \frac{1}{x-2}$; $g(x) = \sqrt{x}$ ▷ $(f \circ g)(x) = \frac{1}{\sqrt{x}-2}$

By Theorem 1.8.4, $f \circ g$ is continuous for $g(x) \neq 2$.
By Theorem 1.8.5(ii), g is continuous for $x > 0$ and $g(x) = 2$ for $x = 4$.
Hence by Theorem 1.9.2, $f \circ g$ is continuous for all positive numbers except 4.
Because $\lim_{x \to 0^+} (f \circ g)(x) = -\frac{1}{2} = (f \circ g)(0)$, $f \circ g$ is continuous on $[0, 4) \cup (4, +\infty)$.

4 and 20. (a) $f(x) = \sqrt[3]{x}$; $g(x) = \sqrt{x+1}$ ▷ $(f \circ g)(x) = \sqrt[6]{x+1}$

By Theorem 1.8.5(i), $f \circ g$ is continuous for all $g(x)$.
By Theorem 1.8.5(ii), g is continuous for $x > -1$.
Hence by Theorem 1.9.2, $f \circ g$ is continuous in $(-1, +\infty)$.
Because $\lim_{x \to -1^+} (f \circ g)(x) = 0 = (f \circ g)(-1)$, $f \circ g$ is continuous on $[-1, +\infty)$.

(b) $f(x) = \sqrt{x+1}$; $g(x) = \sqrt[3]{x}$ ▷ $(f \circ g)(x) = \sqrt{\sqrt[3]{x}+1}$

By Theorem 1.8.5(ii), $f \circ g$ is continuous for $g(x) > -1$.
By Theorem 1.8.5(i), g is continuous for all x and $g(x) > -1$ for $x > -1$.
Hence by Theorem 1.9.2, $f \circ g$ is continuous on $(-1, +\infty)$.
Because $\lim_{x \to -1^+} (f \circ g)(x) = 0 = (f \circ g)(-1)$, $f \circ g$ is continuous on $[-0, +\infty)$.

5 and 21. $f(x) = \frac{\sqrt{4-x^2}}{\sqrt{x-1}}$; $g(x) = |x|$ ▷ $(f \circ g)(x) = \frac{\sqrt{4-x^2}}{\sqrt{|x|-1}}$

$\sqrt{4-x^2}$ is continuous for x in $(-2, 2)$; $\sqrt{|x|-1}$ is continuous for x in $(-\infty, -1) \cup (1, +\infty)$.
By Theorem 1.8.2(iv), $f \circ g$ is continuous for x in $(-2, -1) \cup (1, 2)$.
Because $\lim_{x \to -2^+} (f \circ g)(x) = 0 = (f \circ g)(-2)$ and $\lim_{x \to 2^-} (x) = 0 = (f \circ g)(2)$,
$f \circ g$ is continuous on $[-2, -1) \cup (1, 2]$.

6 and 22. $f(x) = \frac{\sqrt{x^2-1}}{\sqrt{4-x}}$; $g(x) = |x|$ ▷ $(f \circ g)(x) = \frac{\sqrt{x^2-1}}{\sqrt{4-|x|}}$

$\sqrt{x^2-1}$ is continuous for x in $(-\infty, -1) \cup (1, +\infty)$; $\sqrt{4-|x|}$ is continuous for x in $(-4, 4)$.
By Theorem 1.8.2(iv), $f \circ g$ is continuous for x in $(-4, -1) \cup (1, 4)$.
Because $(f \circ g)(-4)$ and $(f \circ g)(4)$ do not exist, $f \circ g$ is continuous only on $(-4, -1) \cup (1, 4)$.

In Exercises 7–16, find the domain of the function. Determine for each interval if it is continuous on that interval.

7. $f(x) = \frac{2}{x+5}$; the domain of f is all real numbers except -5.

f is continuous on$(3, 7), (-5, +\infty), [-10, -5)$; f is discontinuous on$[-6, 4], (-\infty, 0), [-5, +\infty)$.

8. $g(x) = \frac{x}{x-2}$; $(-\infty, 0]$, $[0, +\infty)$, $(0, 2)$, $(0, 2]$, $[2, +\infty)$, $(2, +\infty)$

▷ The domain is the set of all real numbers, except 2. Because a rational function is continuous on its domain, and g is a rational function, g is continuous on any interval that does not include 2. Therefore, g is continuous on $(-\infty, 0]$, $(0, 2)$, and $(2, +\infty)$. And g is discontinuous on $[0, +\infty)$, $(0, 2]$, and $[2, +\infty)$.

9. $f(t) = \frac{1}{t^2-1}$; the domain of f is all real numbers except 1 and -1.

f is continuous on $(0, 1), (-1, 1), (-1, 0], (1, +\infty)$; f is discontinuous on $[0, 1]$, $(-\infty, -1]$.

10. $f(r) = \frac{r+3}{r^2-4}$; the domain of f is all real numbers except 2 and -2.

f is continuous on $(-2, 2)$, $(2, +\infty)$ and discontinuous on $(0, 4]$, $(-\infty, -2]$, $[-4, 4]$, $(-2, 2]$.

11. $g(x) = \sqrt{x^2-9}$; the domain of g is $\{x \mid x^2 - 9 \geq 0\} = (-\infty, -3] \cup [3, +\infty)$.
g is continuous on $(-\infty, -3)$, $(3, \infty)$; g is discontinuous on $(-3, 3)$.

Because $\lim_{x \to -3^-} g(x) = \lim_{x \to -3^-} \sqrt{x^2-9} = 0 = g(-3)$ and $\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} \sqrt{x^2-9} = 0 = g(3)$,
g is also continuous on $(-\infty, -3]$ and $[3, +\infty)$.

12. $f(x) = [\![x]\!]$; $(-\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{1}{2})$, $(1, 2)$, $[1, 2)$, $(1, 2]$

► The greatest integer function is defined for every real number, so the domain of f is the set of all real numbers. If n is any integer, then

$$[\![x]\!] = n - 1 \text{ if } n - 1 \le x < n \text{ so that } \lim_{x \to n^-} f(x) = n - 1$$

and

$$[\![x]\!] = n \text{ if } n \le x < n + 1 \text{ so that } \lim_{x \to n^+} f(x) = n$$

Therefore $\lim_{x \to n} f(x)$ does not exist and so f is discontinuous at each integer n. However, $f(n) = n$ so that f is continuous from the right at n because $\lim_{x \to n^+} f(x) = f(n)$ and discontinuous from the left at n because $\lim_{x \to n^-} f(x) \ne f(n)$. Therefore, f is continuous on the intervals $(\frac{1}{4}, \frac{1}{2})$, $(1, 2)$, and $[1, 2)$, while f is discontinuous on the intervals $(-\frac{1}{2}, \frac{1}{2})$ and $(1, 2]$.

13. $f(t) = \dfrac{|t - 1|}{t - 1}$; the domain of f is all real numbers except 1.

f is continuous on $(-\infty, 1)$ and $(1, \infty)$; f is discontinuous on $(-\infty, 1]$, $[-1, 1]$, $(-1, \infty)$.

14. $h(x) = \begin{cases} 2x - 3 & \text{if } x < -2 \\ x - 5 & \text{if } -2 \le x \le 1 \\ 3 - x & \text{if } x > 1 \end{cases}$; the domain of h is all real numbers.

$\lim_{x \to -2^-} h(x) = \lim_{x \to -2^-} (2x - 3) = -7$; $\lim_{x \to -2^+} h(x) = \lim_{x \to -2^+} (x - 5) = -7 = h(-2)$ and so h is continuous at -2.
$\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} (x - 5) = -4 = h(1)$; $\lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} (3 - x) = 2$ and so h is continuous from the left at 1.
f is continuous on $(-\infty, 1)$, $(-2, 1)$, $[-2, 1)$, $[-2, 1]$; f is discontinuous on $(-2, +\infty)$.

15. $f(x) = \sqrt{4 - x^2}$; the domain of f is $\{x \mid 4 - x^2 \ge 0\} = [-2, 2]$.
f is continuous on $(-2, 2)$; f is discontinuous on $(-\infty, -2]$ and $(2, +\infty)$.
Because $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \sqrt{4 - x^2} = 0 = f(-2)$ and $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sqrt{4 - x^2} = 0 = f(2)$,
f is also continuous on $[-2, 2]$, $(-2, 2]$, and $[-2, 2)$.

16. $F(y) = \dfrac{1}{3 + 2y - y^2}$; $(-1, 3)$, $[-1, 3]$, $[-1, 3)$, $(-1, 3]$

►
$$\frac{1}{3 + 2y - y^2} = \frac{1}{(3 - y)(1 + y)}$$

The domain of F is the set of all real numbers except 3 and -1. Because F is a rational function, F is continuous on its domain. Thus, F is continuous on $(-1, 3)$, and F is discontinuous on $[-1, 3]$, $[-1, 3)$, and $(-1, 3]$.

17. See Ex.1 18. See Ex.2 19. See Ex.3 20. See Ex.4 21. See Ex.5 22. See Ex.6

23. $f(x) = \begin{cases} 2 & \text{if } x < -2 \\ \sqrt{4 - x^2} & \text{if } -2 \le x \le 2 \\ -2 & \text{if } 2 < x \end{cases}$ Continuous on $(-\infty, -2) \cup [-2, 2] \cup (2, +\infty)$. See Ex.1.6.17 for fig. and limits.

24. Determine the largest interval (or union of intervals) on which the function is continuous:

$$f(x) = \begin{cases} x + 5 & \text{if } x < -3 \\ \sqrt{9 - x^2} & \text{if } -3 \le x \le 3 \\ 3 - x & \text{if } 3 < x \end{cases}$$

► Because f is a polynomial on the intervals $(-\infty, -3)$ and $(3, +\infty)$, f is continuous on these intervals. Because $9 - x^2$ is continuous for all x and positive on $(-3, 3)$, f is continuous on $(-3, 3)$ by Theorems 1.8.5 and 1.9.2. Because

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = 0 = f(-3) \quad \text{and} \quad \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (x + 5) = 2 \ne f(-3)$$

f is continuous from the right at -3 but not from the left. Because

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = 0 = f(3) \quad \text{and} \quad \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 3 - x = 0 = f(3)$$

f is continuous from the left and the right at 3 and hence is continuous there. Combining these facts, we see that f is continuous on $(-\infty, -3) \cup [-3, +\infty)$.

In Exercises 25–28, sketch the graph of a function f that satisfies the given properties.

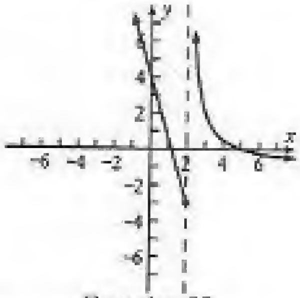
25. f is continuous on $(-\infty, 2]$ and $(2, +\infty)$; $\lim_{x\to 0} f(x) = 4$; $\lim_{x\to 2^-} f(x) = -3$; $\lim_{x\to 2^+} f(x) = +\infty$; $\lim_{x\to 5} f(x) = 4$

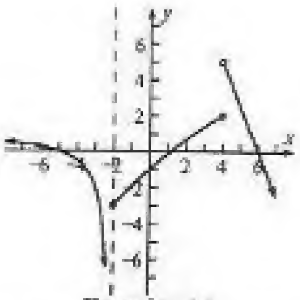
26. f is continuous on $(-\infty, -2)$, $[-2, 4]$, and $(4, +\infty)$; $\lim_{x\to -5} f(x) = 0$; $\lim_{x\to -2^-} f(x) = -\infty$; $\lim_{x\to -2^+} f(x) = -3$; $\lim_{x\to 0} f(x) = -1$; $\lim_{x\to 4^-} f(x) = 2$; $\lim_{x\to 4^+} f(x) = 5$; $\lim_{x\to 6} f(x) = 0$

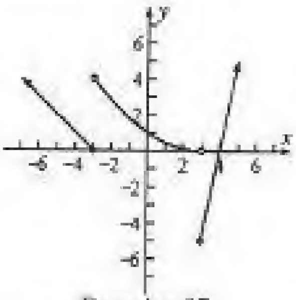
27. f is continuous on $(-\infty, -3]$, $(-3, 3)$, and $[3, +\infty)$; $\lim_{x\to -5} f(x) = 2$; $\lim_{x\to -3^-} f(x) = 0$; $\lim_{x\to -3^+} f(x) = 4$; $\lim_{x\to 0} f(x) = 1$; $\lim_{x\to 3^-} f(x) = 0$; $\lim_{x\to 3^+} f(x) = -5$; $\lim_{x\to 4} f(x) = 0$

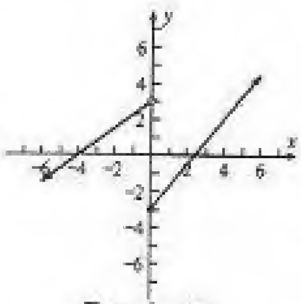
28. f is continuous on $(-\infty, 0)$ and $[0, +\infty)$; $\lim_{x\to -4} f(x) = 0$; $\lim_{x\to 0^-} f(x) = 3$; $\lim_{x\to 0^+} f(x) = -3$; $\lim_{x\to 4} f(x) = 2$.

▷ Because f is continuous at -4, then $f(-4) = \lim_{x\to -4} f(x) = 0$, so the graph contains the point $(-4, 0)$.
Because f is continuous from the right at 0, then $f(0) = \lim_{x\to 0^+} f(x) = -3$, and the graph contains $(0, 3)$.
Because f is continuous at 4, and $\lim_{x\to 4} f(x) = 2$, then $f(4) = 2$.
Because $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, there is a break in the graph at the point where $x = 0$.
Because $\lim_{x\to 0^-} f(x) = 3$, from the left the graph approaches the point $(0, 3)$, but this point is not on the graph.


Exercise 25


Exercise 26


Exercise 27


Exercise 28

In Exercises 29–34, prove that the function obtained in Exercises 1.3 is continuous on its domain.

29. (a) Ex. 13. $a(x) = (120 - x)x = 120x - x^2$, $0 \le x \le 120$, is a polynomial.
(b) Ex. 15. $a(x) = \frac{1}{2}(240 - x)x = 120x - \frac{1}{2}x^2$, $0 \le x \le 240$, is a polynomial.

30. (a) Ex. 14. $a(x) = (50 - x)x = 50x - x^2$, $0 \le x \le 50$, is a polynomial.
(b) Ex. 16. $a(x) = \frac{1}{2}(100 - x)x = 50x - \frac{1}{2}x^2$, $0 \le x \le 100$, is a polynomial.

31. (a) Ex. 17. $V(x) = (8 - 2x)(15 - 2x)x = 120x - 46x^2 + 4x^3$, $0 \le x \le 4$, is a polynomial.
(b) Ex. 19. $V(x) = (12 - 2x)(15 - 2x)x = 4x^3 - 54x^2 + 180x$, $0 \le x \le 6$, is a polynomial.

32. (a) Exercise 18. $V(x) = (12 - 2x)^2 x = 4x^3 - 48x^2 + 144x$, $0 \le x \le 6$.
(b) Exercise 20. $V(x) = (40 - 2x)(50 - 2x)x = 4x^3 - 180x^2 + 200x$, $0 \le x \le 20$.
Both of these are polynomials, continuous at every number by Theorem 1.8.3.

33. (a) Ex. 21. $C(r) = k(120/r + 4\pi r^2)$, $r > 0$, is a rational function, continuous on its domain by Th. 1.8.4.
(b) Ex. 23. $A(x) = (x + 2)(\frac{24}{x} + 3) = \dfrac{3x^2 + 30x + 48}{x}$, $x > 0$, is also a rational function.

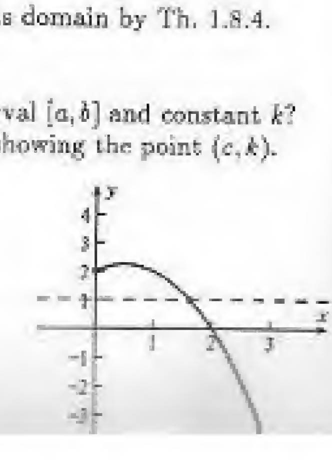
34. (a) Ex. 22. $C(r) = k(120/r + 2\pi r^2)$, $r > 0$, is a rational function, continuous on its domain by Th. 1.8.4.
(b) Ex. 24. $A(x) = (x + 30)\left(\dfrac{13200}{x} + 44\right)$, $x > 0$, is also a rational function.

In Exercises 35–42, does the intermediate-value theorem hold for the function f, interval $[a, b]$ and constant k? If so, solve $f(c) = k$ graphically and analytically to 4 decimals and sketch the graph showing the point (c, k).

35. $f(x) = 2 + x - x^2$; $[a, b] = [0, 3]$; $k = 1$

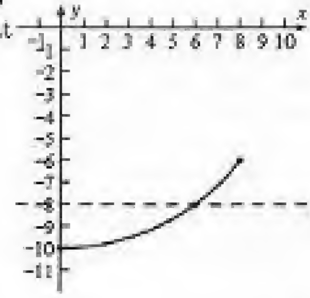
▷ 1 is between $f(0) = 2$ and $f(3) = -3$
and f is continuous on $[0, 3]$. Therefore,
the intermediate-value theorem holds and
there exists a number c between 0 and 3 such that $f(c) = 1$:

$2 + c - c^2 = 1$; $c^2 - c - 1 = 0$; $c = \frac{1}{2}(1 \pm \sqrt{5})$; and $\frac{1}{2}(1 + \sqrt{5}) \approx 1.6180$ is in $(0, 3)$.



36. $f(x) = -\sqrt{100 - x^2}$; $[a, b] = [0, 8]$; $k = -8$

▷ Because $f(a) = f(0) = -10$ and $f(b) = f(8) = -6$, then $f(a) \neq f(b)$, and k is between $f(a)$ and $f(b)$. Furthermore, f is continuous on $[0, 8]$. Thus, the hypothesis of the intermediate-value theorem is satisfied, and we can find a number c such that

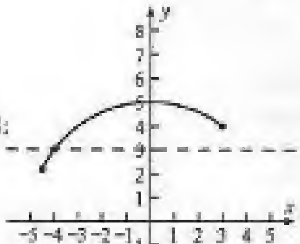


$$\begin{aligned} f(c) &= -8 \\ -\sqrt{100 - c^2} &= -8 \\ 100 - c^2 &= 64 \\ c &= \pm 6 \end{aligned}$$

Because -6 is not between 0 and 8, the only suitable choice of c is 6. The figure shows the graph of f and the line $y = -8$.

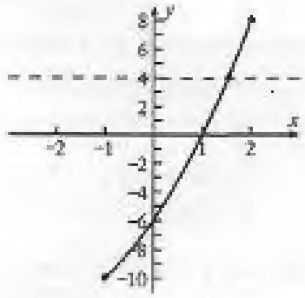
37. $f(x) = \sqrt{25 - x^2}$; [a, b] = $[-4.5, 3]$;

▷ 3 is between $f(-4.5) = \frac{1}{2}\sqrt{19} \approx 2.18$ and $f(3) = 4$ and f is continuous on $[-4.5, 3]$. Therefore, the intermediate-value theorem holds and there exists a number c between -4.5 and 3 such that $f(c) = 3$: $\sqrt{25 - c^2} = 3$; $25 - c^2 = 9$; $c^2 = 16$; $c = \pm 4$, and -4 is in $(-4.5, 3)$.



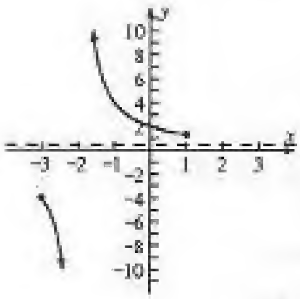
38. $f(x) = x^2 + 5x - 6$; [a, b] = $[-1, 2]$;

▷ 4 is between $f(-1) = -10$ and $f(2) = 8$ and f is continuous on $[-1, 2]$. Therefore, the intermediate-value theorem holds and there exists a number c between -1 and 2 such that $f(c) = 4$: $c^2 + 5c - 6 = 4$; $c^2 + 5c - 10 = 0$; $c = \frac{1}{2}(-5 \pm \sqrt{65})$ and $\frac{1}{2}(-5 + \sqrt{65}) \approx 1.5311$ is in $(-1, 2)$.



39. $f(x) = \dfrac{4}{x + 2}$; [a, b] = $[-3, 1]$;

▷ $\frac{1}{2}$ is between $f(-3) = -4$ and $f(1) = \frac{4}{3}$ but f is discontinuous at -2, and -2 is in $[-3, 1]$. Therefore the intermediate-value theorem does not hold and there may not be a number c between -3 and 1 such that $f(c) = \frac{1}{2}$: $\dfrac{4}{c + 2} = \dfrac{1}{2}$; $8 = c + 2$; $c = 6$. But 6 is not in $(-3, 1)$.

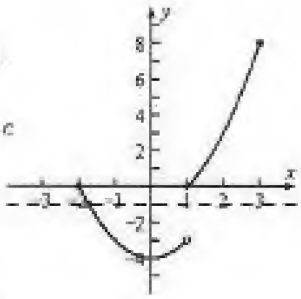


40. $f(x) = -\dfrac{5}{2x - 1}$; $[a, b] = [0, 1]$; $k = 2$

▷ Because f is discontinuous at $\frac{1}{2}$, then f is not continuous on the closed interval $[0, 1]$. Thus the hypothesis of the intermediate-value theorem is not satisfied and the theorem does not hold. The figure shows the graph of the function on $[0, 1]$ and the line $y = 2$. Because the line does not intersect the curve, there is no number c that satisfies the conclusion of the intermediate-value theorem.

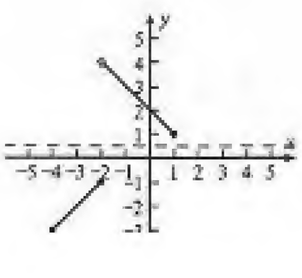
41. $f(x) = \begin{cases} x^2 - 4 & \text{if } -2 \leq x < 1 \\ x^2 - 1 & \text{if } 1 \leq x \leq 3 \end{cases}$; [a, b] = $[-2, 3]$; $k = -1$

▷ -1 is not between $f(-2) = 0$ and $f(3) = 8$ and f is not continuous on $[-2, 3]$. Hence the intermediate-value theorem does not hold and there may not be a number c between -2 and 3 such that $f(c) = -1$: $c^2 - 4 = -1$; $c^2 = 3$; $c = \pm\sqrt{3}$; and $-\sqrt{3}$ is in $(-2, 3)$.



42. $f(x) = \begin{cases} 1 + x & \text{if } -4 \leq x \leq -2 \\ 2 - x & \text{if } -2 < x \leq 1 \end{cases}$; $[a, b] = [-4, 1]$

▷ Because f is discontinuous at -2, and -2 is in $[-4, 1]$. Therefore the intermediate-value theorem does not hold and there may not be a number c between -4 and 1 such that $f(c) = \frac{1}{2}$. The graph shows there is no such number.

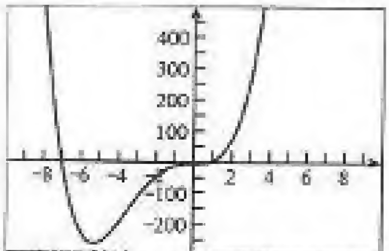


In Exercises 43–48, (a) apply the intermediate-zero theorem to show that f has the indicated number of zeros between a and b. (b) Approximate them to 2 decimal places.

43. $f(x) = x^3 - 6x + 3$. $f(-3) = -6$, $f(-2) = 7$; $f(0) = 3$, $f(1) = -2$; $f(2) = -1$, $f(3) = 12$. Thus there is a zero in each of the intervals: $(-3, -2)$, $(0, 1)$, $(1, 2)$. The zeros are $-2.669 \approx -2.67$, $0.524 \approx 0.52$, $2.1451 \approx 2.15$.

44. $f(x) = x^4 + 7x^3 + x - 8$; two zeros; $a = -10$; $b = 5$

▷ $f(-8) = 496$, $f(-7) = -15$; $f(0) = -8$, $f(1) = 1$
By the intermediate-zero theorem, there is a zero in each of the intervals $(-8, -7)$, $(0, 1)$. To two decimals, the zeros are -7.04 and 0.96.



45. $f(x) = 4x^4 - 3x^3 + 2x - 5$. $f(-1) = 0$; $f(1) = -2$, $f(2) = 39$. -1 is zero. There is a zero in the interval $(1, 2)$. The zero is $1.168 \approx 1.17$

46. $f(x) = 3x^4 - 21x^3 + 36x^2 + 2x - 8$. $f(-1) = 50$, $f(0) = -8$, $f(1) = 12$; $f(2) = 20$, $f(3) = -2$; $f(4) = 0$. 4 is a zero. There is one in each interval: $(-1, 0)$, $(0, 1)$, $(2, 3)$. They are $-0.440 \approx -0.44$, $0.518 \approx 0.52$, $2.922 \approx 2.92$.

47. $f(x) = x^3 - 4x^2 + x + 3$. $f(1) = 1$, $f(2) = -3$. To 3 decimals the root is 1.239.

48. Show that the intermediate-value theorem guarantees that $x^3 + x + 3 = 0$ has a root between -2 and 2 and use your graphics calculator to approximate the root to two decimal places.

▷ Let $f(x) = x^3 + x + 3$. $f(-2) = -7$ and $f(2) = 13$. Because 0 is between -7 and 13, there is a number c between -2 and 2 with $f(c) = 0$. Two 2 decimal places the root is -1.21.

49. $m(v) = \dfrac{m_0}{\sqrt{1 - (v/c)^2}}$ is continuous on $[0, c)$.

50. Let $x = a - t$. f is continuous at a

$\Rightarrow \lim\limits_{x \to a} f(x) = f(a)$

$\Rightarrow$ for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ when $0 < |x - a| < \delta$

$\Rightarrow$ for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(a - t) - f(a)| < \epsilon$ when $0 < |t| < \delta$

$\Rightarrow \lim\limits_{t \to a} f(a - t) = f(a)$

51. $\lim\limits_{x \to a} f(x) \overset{f(x) \geq 0}{=} \lim\limits_{x \to a} \sqrt{[f(x)]^2} \overset{\text{L.T.10}}{=} \sqrt{\lim\limits_{x \to a} [f(x)]^2}$, viewed as a one-sided limit if $\lim\limits_{x \to a} f(x) = 0$.

52. Prove that if $\lim\limits_{x \to a} f(x) = \mathrm{L}$, then $\lim\limits_{x \to a} |f(x)| = |\mathrm{L}|$.

▷ By the triangle inequality, $|f(x) - \mathrm{L}| \geq \big||f(x)| - |\mathrm{L}|\big|$. Therefore
$\lim\limits_{x \to a} f(x) = \mathrm{L}$

$\Rightarrow$ for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - \mathrm{L}| < \epsilon$ when $0 < |x - a| < \delta$

$\Rightarrow$ for every $\epsilon > 0$ there is a $\delta > 0$ such that $\big||f(x)| - |\mathrm{L}|\big| < \epsilon$ when $0 < |x - a| < \delta$

$\Rightarrow \lim\limits_{x \to a} |f(x)| = |\mathrm{L}|$.

Note that the converse is not true: $\lim\limits_{x \to a} |\operatorname{sgn} x| = 1$ but $\lim\limits_{x \to a} \operatorname{sgn} x$ does not exist.

53. Suppose f is a function for which $0 \leq f(x) \leq 1$ if $0 \leq x \leq 1$. Prove that if f is continuous on $[0, 1]$, there is at least one number c in $[0, 1]$ such that $f(c) = c$.

▷ If $f(0) = 0$, then $c = 0$ satisfies the conclusion. If $f(1) = 1$, then $c = 1$ satisfies the conclusion. Suppose that $f(0) \neq 0$ and $f(1) \neq 1$. Then because $0 \leq f(x) \leq 1$ if $0 \leq x \leq 1$, we have $f(0) > 0$ and $f(1) < 1$. Let g be the function defined by

$$g(x) = f(x) - x \tag{1}$$

Then $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. Because g is continuous on $[0, 1]$, and 0 is between $g(0)$ and $g(1)$, by the intermediate-value theorem there exists a number c between 0 and 1 such that

$$g(c) = 0$$

By Eq. (1), we have

$$f(c) - c = 0$$
$$f(c) = c$$

54. $f(x) = [\![x^2 - 2]\!]$. We seek the largest value of k for which f is continuous on the interval $[3, 3+k)$.
$f(3) = [\![9-2]\!] = [\![7]\!]$. If $x > 3$, then $f(x) \le 7$ if and only if $x^2 - 2 < 8$, that is $x < \sqrt{10}$.
Thus the largest value of k satisfies $3 + k = \sqrt{10}$; $k = \sqrt{10} - 3$.

55. Not equivalent: f continuous on the closed interval implies the existence of 1-sided limits at the endpoints; f continuous at every number in the closed interval implies the existence of 2-sided limits at the endpoints.

1.10 CONTINUITY OF THE TRIGONOMETRIC FUNCTIONS AND THE SQUEEZE THEOREM

1.10.1 Squeeze Theorem Suppose that the functions f, g, and h are defined on some open interval I containing a, except possibly at a itself, and that $f(x) \le g(x) \le h(x)$ for all x in I for which $x \ne a$. Also suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$ both exist and are equal to L. Then $\lim_{x\to a} g(x)$ also exists and is equal to L.

The squeeze theorem is used to prove the following results.

1.10.2 Theorem $\displaystyle\lim_{t\to 0} \frac{\sin t}{t} = 1$ Corollary: $\displaystyle\lim_{t\to 0} \frac{t}{\sin t} = 1$

1.10.5 Theorem $\displaystyle\lim_{t\to 0} \frac{1-\cos t}{t} = 0$

To verify Theorem 1.10.2 on a calculator, you must switch to radian mode. Note also that angle measure is based on arc length, which is not defined until Chapter 6. The above theorems are used to prove:

1.10.6 Theorem The sine and cosine functions are continuous at every real number.

1.10.7 Theorem The tangent, cotangent, secant, and cosecant functions are continuous on their domains.

Therefore, the tangent and secant functions are continuous at every real number, except $\frac{1}{2}\pi + k\pi$, where k is any integer. The cotangent and cosecant functions are continuous at every real number except $k\pi$, where k is any integer.

Useful Identities $1 - \cos^2 x = \sin^2 x$ $\qquad 1 - \cos x = 2\sin^2 \frac{1}{2}x$

Exercises 1.10

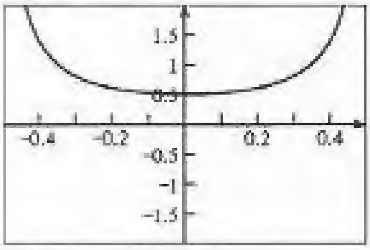
In Exercises 1–26, (a) Estimate the limit by plotting the graph of the function. (b) Find the limit analytically.

1. $\displaystyle\lim_{x\to 0}\frac{\sin 4x}{x} = 4\lim_{4x\to 0}\frac{\sin 4x}{4x} = 4\cdot 1 = 4$

2. $\displaystyle\lim_{x\to 0}\frac{2x}{\sin 3x} = \frac{2}{3}\cdot\lim_{3x\to 0}\frac{3x}{\sin 3x} = \frac{2}{3}\cdot 1 = \frac{2}{3}$

3. $\displaystyle\lim_{x\to 0}\frac{\sin 9x}{\sin 7x} = \frac{9}{7}\cdot\frac{\lim\limits_{9x\to 0}\frac{\sin 9x}{9x}}{\lim\limits_{7x\to 0}\frac{\sin 7x}{7x}} = \frac{9}{7}\cdot\frac{1}{1} = \frac{9}{7}$

4. $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{\sin 6x}$ ▷ The limit appears to be $\frac{1}{2}$.



To apply Theorem 1.10.2 we divide the numerator and denominator by $6x$.

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 6x} = \lim_{x\to 0}\frac{\frac{1}{2}\left(\frac{\sin 3x}{3x}\right)}{\frac{\sin 6x}{6x}} = \lim_{x\to 0}\frac{\frac{1}{2}\lim\limits_{x\to 0}\frac{\sin 3x}{3x}}{\lim\limits_{x\to 0}\frac{\sin 6x}{6x}} \qquad (1)$$

Furthermore, because both $3x$ and $6x$ approach 0 when x approaches 0, then by Theorem 1.10.2

$$\lim_{x\to 0}\frac{\sin 3x}{3x} = \lim_{3x\to 0}\frac{\sin 3x}{3x} = 1 \text{ and } \lim_{x\to 0}\frac{\sin 6x}{6x} = \lim_{6x\to 0}\frac{\sin 6x}{6x} = 1$$

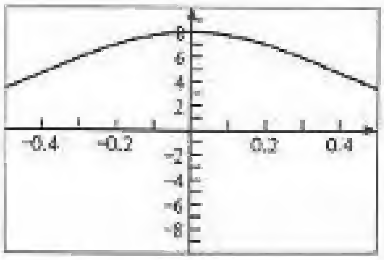
Substituting these into Eq. (1) we have $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{\sin 6x} = \frac{\frac{1}{2}(1)}{1} = \frac{1}{2}$

5. $\displaystyle\lim_{x\to 0}\frac{3x}{\sin 5x} = \frac{3}{5}\cdot\frac{1}{\lim\limits_{5x\to 0}\frac{\sin 5x}{5x}} = \frac{3}{5}\cdot\frac{1}{1} = \frac{3}{5}$

6. $\displaystyle\lim_{x\to 0}\frac{\sin^3 x}{x^2} = \lim_{x\to 0}\sin x\cdot\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 0\cdot 1 = 0$

7. $\displaystyle\lim_{x\to 0}\frac{x^2}{\sin^2 3x} = \left(\lim_{x\to 0}\frac{x}{\sin 3x}\right)^2 = \left(\frac{1}{3}\cdot\lim_{3x\to 0}\frac{3x}{\sin 3x}\right)^2 = \left(\frac{1}{3}\cdot 1\right)^2 = \frac{1}{9}$

8. $\lim\limits_{x\to 0} \dfrac{\sin^5 2x}{4x^5}$ ▹ The limit appears to be 8.



We apply Theorem 1.10.2. Because $2x \to 0$ when $x \to 0$, we have

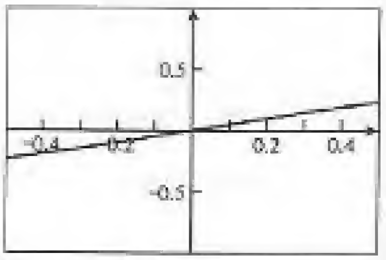
$$\lim_{x\to 0} \frac{\sin^5 2x}{4x^5} = \lim_{x\to 0} 8\left(\frac{\sin 2x}{2x}\right)^5 = 8\left(\lim_{2x\to 0} \frac{\sin 2x}{2x}\right)^5 = 8\cdot 1^5 = 8$$

9. $\lim\limits_{x\to 0} \dfrac{x}{\cos x} = \dfrac{\lim\limits_{x\to 0} x}{\lim\limits_{x\to 0} \cos x} = \dfrac{0}{1} = 0$

10. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{1+\sin x} = \dfrac{1-1}{1+0} = 0$

11. $\lim\limits_{x\to 0} \dfrac{1-\cos 4x}{x} = 4 \lim\limits_{4x\to 0} \dfrac{1-\cos 4x}{4x} = 4\cdot 0 = 0$

12. $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{4x}$ ▹ The limit appears to be 0.



Applying Theorem 1.10.5, we have

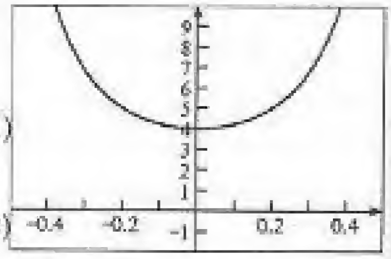
$$\lim_{x\to 0} \frac{1-\cos 2x}{4x} = \lim_{x\to 0} \frac{1}{2}\left(\frac{1-\cos 2x}{2x}\right) = \frac{1}{2} \lim_{2x\to 0} \frac{1-\cos 2x}{2x} = \frac{1}{2}\cdot 0 = 0$$

13. $\lim\limits_{x\to 0} \dfrac{3x^2}{1-\cos^2 \frac{1}{2}x} = \lim\limits_{x\to 0} \dfrac{3x^2}{\sin^2 \frac{1}{2}x} = \lim\limits_{x\to 0} \dfrac{12(\frac{1}{2}x)^2}{\sin^2 \frac{1}{2}x} = 12\left(\lim\limits_{x/2\to 0} \dfrac{\frac{1}{2}x}{\sin \frac{1}{2}x}\right)^2 = 12(1)^2 = 12$

14. $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{2x^2} = \lim\limits_{x\to 0} \dfrac{\sin^2 x}{2x^2} = \dfrac{1}{2}\left(\lim\limits_{x\to 0} \dfrac{\sin x}{x}\right)^2 = \dfrac{1}{2}(1)^2 = \dfrac{1}{2}$

15. $\lim\limits_{x\to 0} \dfrac{\tan x}{2x} = \lim\limits_{x\to 0} \dfrac{\frac{\sin x}{\cos x}}{2x} = \dfrac{1}{2}\cdot \lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x\to 0} \dfrac{1}{\cos x} = \dfrac{1}{2}\cdot 1\cdot 1 = \dfrac{1}{2}$

16. $\lim\limits_{x\to 0} \dfrac{\tan^4 2x}{4x^4}$ ▹ The limit appears to be 4.



$$\lim_{x\to 0} \frac{\tan^4 2x}{4x^4} = \lim_{x\to 0} \frac{\sin^4 2x}{4x^4 \cos^4 2x} = \lim_{x\to 0} \frac{4}{\cos^4 2x}\left(\frac{\sin 2x}{2x}\right)^4$$

$$= \lim_{x\to 0} \frac{4}{\cos^4 2x} \cdot \lim_{2x\to 0} \left(\frac{\sin 2x}{2x}\right)^4 \qquad (1)$$

Because the cosine function is continuous at 0,

$$\lim_{x\to 0} \frac{4}{\cos^4 2x} = \frac{4}{\cos^4 0} = \frac{4}{1^4} = 4 \qquad (2)$$

By Theorem 1.10.2

$$\lim_{2x\to 0} \left(\frac{\sin 2x}{2x}\right)^4 = 1^4 = 1 \qquad (3)$$

Substituting from Equations (2) and (3) into Eq. (1), we obtain

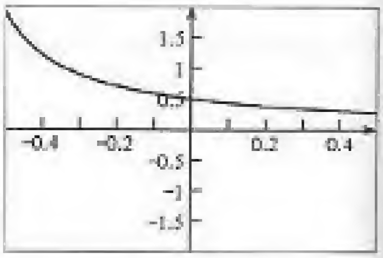
$$\lim_{x\to 0} \frac{\tan^4 2x}{4x^4} = 4\cdot 1 = 4$$

17. $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{\sin 3x} = \lim\limits_{x\to 0} \dfrac{1-\cos 2x}{x}\cdot \dfrac{x}{\sin 3x} = 2 \lim\limits_{2x\to 0} \dfrac{1-\cos 2x}{2x}\cdot \dfrac{1}{3}\cdot \dfrac{3x}{\sin 3x} = 2\cdot 0\cdot \dfrac{1}{3}\cdot 1 = 0$

18. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \dfrac{2\sin^2 \frac{1}{2}x}{x^2} = \lim\limits_{x\to 0} \dfrac{1}{2}\left(\dfrac{\sin \frac{1}{2}x}{\frac{1}{2}x}\right)^2 = \dfrac{1}{2}\left(\lim\limits_{x/2\to 0} \dfrac{\sin \frac{1}{2}x}{\frac{1}{2}x}\right)^2 = \dfrac{1}{2}(1)^2 = \dfrac{1}{2}$

19. $\lim\limits_{x\to 0} \dfrac{x^2+3x}{\sin x} = \lim\limits_{x\to 0} \dfrac{(x+3)x}{\sin x} = \lim\limits_{x\to 0} (x+3)\cdot \lim\limits_{x\to 0} \dfrac{x}{\sin x} = 3\cdot 1 = 3$

20. $\lim\limits_{x\to 0} \dfrac{\sin x}{3x^2+2x}$ ▹ The limit appears to be $\frac{1}{2}$.



Applying Theorem 1.10.2, we have

$$\lim_{x\to 0} \frac{\sin x}{3x^2+2x} = \lim_{x\to 0} \frac{\sin x}{x(3x+2)} = \lim_{x\to 0} \frac{\sin x}{x}\cdot \lim_{x\to 0} \frac{1}{3x+2} = 1\cdot \frac{1}{2} = \frac{1}{2}$$

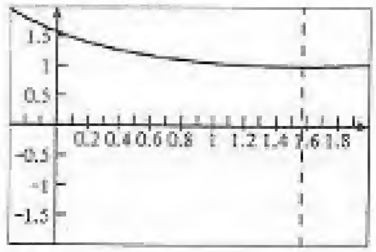
21. $\lim\limits_{t\to 0^+} \dfrac{\sin t}{t^2} = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t}\cdot \lim\limits_{t\to 0^+} \dfrac{1}{t} = +\infty$

22. $\lim\limits_{t\to 0^+} \dfrac{\sin 4t}{\cos 3t - 1} = \lim\limits_{t\to 0^+} -\dfrac{\sin 4t}{2\sin^2 \frac{3}{2}t} = \lim\limits_{t\to 0^+} -\dfrac{8}{9t}\cdot\dfrac{\sin 4t/4t}{(\sin \frac{3}{2}t/\frac{3}{2}t)^2} = -\infty$ by Limit Theorem 12

since $\lim\limits_{t\to 0^+} -\dfrac{8}{9t} = -\infty$ and $\lim\limits_{t\to 0^+} \dfrac{\sin 4t/4t}{(\sin \frac{3}{2}/\frac{3}{2}t)^2} = \dfrac{1}{(1)^2} = 1$

23. Let $t = \frac{1}{2}\pi - x$. Then $x = \frac{1}{2}\pi - t$. $\lim\limits_{x\to\pi/2} \dfrac{1-\sin x}{\frac{1}{2}\pi - x} = \lim\limits_{t\to 0} \dfrac{1-\sin(\frac{1}{2}\pi - t)}{t} = \lim\limits_{t\to 0} \dfrac{1-\cos t}{t} = 0$

24. $\lim\limits_{t\to\pi/2} \dfrac{\frac{1}{2}\pi - t}{\cos t}$ (*Hint:* Let $x = \frac{1}{2}\pi - t$.) ▷ The limit appears to be 1.



If $t = \frac{1}{2}\pi - x$, then $x = \frac{1}{2}\pi - t$. Thus

$$\lim_{t\to\pi/2} \frac{\frac{1}{2}\pi - t}{\cos t} = \lim_{x\to 0} \frac{x}{\cos(\frac{1}{2}\pi - x)} = \lim_{x\to 0} \frac{x}{\sin x} = 1$$

25. Let $t = x - \pi$. Then $x = \pi + t$.

$$\lim_{x\to\pi^+} \frac{\sin x}{x-\pi} = \lim_{t\to 0^+} \frac{\sin(\pi + t)}{t} = -\lim_{t\to 0^+} \frac{\sin t}{t} = -1$$

26. Let $t = x - \pi$. Then $x = t + \pi$. $\lim\limits_{x\to\pi^+} \dfrac{\tan x}{x-\pi} = \lim\limits_{t\to 0^+} \dfrac{\tan(t+\pi)}{t} = \lim\limits_{t\to 0^+} \dfrac{\tan t}{t} = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t}\cdot\lim\limits_{t\to 0^+} \dfrac{1}{\cos t} = 1\cdot 1 = 1$

27. Because $\sin x$ is continuous for all x and 2θ is continuous for all θ, the composition $\sin 2\theta$ is continuous for all θ, and so is the product $(V_0{}^2/g)\sin 2\theta$.

28. Let $F(\theta) = \dfrac{kW}{k\sin\theta + \cos\theta}$ where $0 < k < 1$. Prove that F is continuous on $[0, \frac{1}{2}\pi]$.

▷ Because $\sin\theta \geq 0$ and $\cos\theta \geq 0$ on $[0, \frac{1}{2}\pi]$ and both are not zero for the same θ, then $k\sin\theta + \cos\theta > 0$ on $[0, \frac{1}{2}\pi]$. Because $\sin\theta$, and hence $k\sin\theta$, and $\cos\theta$ are continuous for all θ, then by Theorem 1.8.2 (i) $k\sin\theta + \cos\theta$ is continuous for all θ. Hence by Theorem 1.8.2(iv) the quotient $F(\theta)$ is continuous on $[0, \frac{1}{2}\pi]$.

In Exercises 29–32, use the squeeze theorem to find the limit.

29. If $x \neq 0$ then $-1 \leq \cos\frac{1}{x} \leq 1$; so $-|x| \leq x\cos\frac{1}{x} \leq |x|$ (1)

Because $\lim\limits_{x\to 0}|x| = 0$, it follows from (1) and the squeeze theorem that $\lim\limits_{x\to 0} x\cos\frac{1}{x} = 0$.

30. Because $-1 \leq \sin x \leq 1$, then $-1 \leq \sin\dfrac{1}{\sqrt[3]{x}} \leq 1$ if $x \neq 0$.

Multiplying by x^2, we have $-x^2 \leq x^2\sin\dfrac{1}{\sqrt[3]{x}} \leq x^2$ if $x \neq 0$.

Because $\lim\limits_{x\to 0} x^2 = 0$ and $\lim\limits_{x\to 0}(-x^2) = 0$, we may apply the squeeze theorem to conclude that $\lim\limits_{x\to 0} x^2\sin\dfrac{1}{\sqrt[3]{x}} = 0$

31. Because $0 \leq |g(x) + 4| < 2(3-x)^4$, then $-2(3-x)^4 < g(x) + 4 < 2(3-x)^4$ (1)

Because $\lim\limits_{x\to 3} 2(3-x)^4 = 0$, it follows from (1) and the squeeze theorem that

$\lim\limits_{x\to 3} g(x) + 4 = 0$. Therefore $\lim\limits_{x\to 3} g(x) = -4$.

32. $\lim\limits_{x\to -2} g(x)$, if $|g(x) - 3| < 5(x+2)^2$ for all x.

▷ Because $0 \leq |g(x) - 3| < 5(x+2)^2$, then $-5(x+2)^2 < g(x) - 3 < 5(x+2)^2$ (1)

Because $\lim\limits_{x\to -2} 5(x+2)^2 = 0$, it follows from (1) and the squeeze theorem that

$\lim\limits_{x\to -2} g(x) - 3 = 0$. Therefore $\lim\limits_{x\to -2} g(x) = 3$.

In Exercises 33 and 34, find the limit if it exists.

33. As x approaches 0 so does $\sin x$. Hence

$$\lim_{x\to 0} \frac{\sin(\sin x)}{x} = \lim_{x\to 0} \frac{\sin(\sin x)}{\sin x}\cdot\frac{\sin x}{x} = \lim_{\sin x\to 0} \frac{\sin(\sin x)}{\sin x}\cdot\lim_{x\to 0} \frac{\sin x}{x} = 1\cdot 1 = 1$$

34. If $x \neq 0$ then $-1 \leq \sin(1/x) \leq 1$; so $-|\sin x| \leq \sin x\sin(1/x) \leq |\sin x|$ (1)

Because $\lim\limits_{x\to 0}|\sin x| = 0$, it follows from (1) and the squeeze theorem that $\lim\limits_{x\to 0} \sin x\cos(1/x) = 0$.

35. Because $1 - \cos^2 x \le f(x) \le x^2$ for all x in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, then

$$\sin^2 x \le f(x) \le x^2 \text{ for all } x \text{ in } (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi) \qquad (1)$$

Because $\lim_{x\to 0} \sin^2 x = 0$ and $\lim_{x\to 0} x^2 = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to 0} f(x) = 0$.

36. Given: $-\sin x \le f(x) \le 2 + \sin x$ for all x in the open interval $(-\pi, 0)$. Find $\lim_{x\to -\pi/2} f(x)$.

▶ Because $-\sin x \le f(x) \le 2 + \sin x$ for all x in $(-\pi, 0)$, then

$$-1 - \sin x \le f(x) - 1 \le 1 + \sin x \text{ for all } x \text{ in } (-\pi, 0) \qquad (1)$$

Because $\lim_{x\to -\pi/2} |1 + \sin x| = |1 - 1| = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to -\pi/2} f(x) - 1 = 0$ and so $\lim_{x\to -\pi/2} f(x) = 1$.

In Exercises 37–40, prove that the function is continuous on its domain.

37. Because $\tan x = \dfrac{\sin x}{\cos x}$, and both the sine and cosine functions are continuous,

the tangent function is continuous at all x for which $\cos x \neq 0$.
Therefore, the tangent function is continuous on its domain.

38. Because $\cot x = \dfrac{\cos x}{\sin x}$, and both the sine and cosine functions are continuous,

the cotangent function is continuous at all x for which $\sin x \neq 0$.
Therefore, the cotangent function is continuous on its domain.

39. Because $\sec x = \dfrac{1}{\cos x}$, and the cosine function is continuous,

the secant function is continuous at all values of x for which $\cos x \neq 0$.
Therefore, the secant function is continuous on its domain.

40. The cosecant function is defined by $\csc x = \dfrac{1}{\sin x}$.

The domain of the cosecant function is the set of all real numbers x for which $\sin x \neq 0$. Because a constant function and the sine function are continuous at every real number, and the cosecant function is the quotient of a constant function and the sine function, by Theorem 1.8.2(iv), the cosecant function is continuous at every real number x for which $\sin x \neq 0$. Thus the cosecant function is continuous on its domain.

41. If $|f(x)| \le M$ for all x, then $-M \le f(x) \le M$ and $-Mx^2 \le x^2 f(x) \le x^2 M$ (1)
Because $\lim_{x\to 0} Mx^2 = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to 0} x^2 f(x) = 0$.

42. If $|f(x)| \le M$ for all x, then $-M \le f(x) \le M$ and $-M|g(x)| \le f(x)g(x) \le M|g(x)|$ (1)
Because $\lim_{x\to a} M|g(x)| = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to a} f(x)g(x) = 0$.

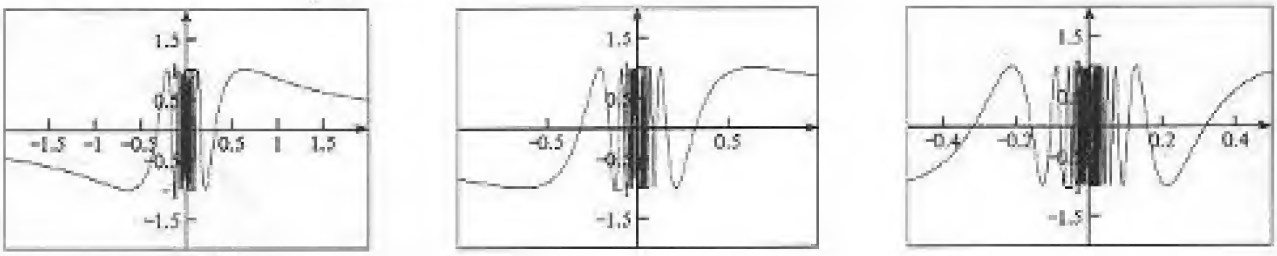
43. If $|f(x)| \le k|x - a|$ for all $x \neq a$, then $-k|x - a| \le f(x) \le k|x - a|$ (1)
Because $\lim_{x\to a} k|x - a| = 0$, it follows from (1) and the squeeze theorem that $\lim f(x) = 0$.

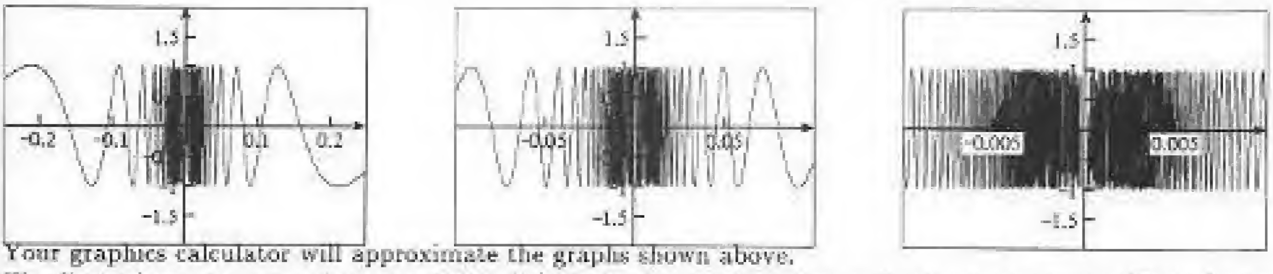
Remark: If $|f(x)| < g(x)$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} f(x) = 0$.

In fact $-g(x) < f(x) < g(x)$ and the remark follows from the squeeze theorem.
The results of Exercises 29, 31, 41 and 43 follow immediately from this remark.

44. Plot the graph of $f(x) = \sin(1/x)$ in each of the following windows: (a) $[-2, 2] \times [-2, 2]$ (b) $[-1, 1] \times [-2, 2]$ (c) $[-0.5, 0.5] \times [-2, 2]$ (d) $[-0.25, 0.25] \times [-2, 2]$ (e) $[-0.1, 0.1] \times [-2, 2]$ (f) $[-0.01, 0.01] \times [-2, 2]$ (g) Make and verify a conjecture about $\lim_{x\to 0} f(x)$.

▶





Your graphics calculator will approximate the graphs shown above.

The limit does not exist. Suppose $\lim_{x\to 0}\sin(1/x) = \mathrm{L}$. In the definition of limit, choose $\epsilon = 1$. Then for some $\delta > 0$ we must have $|\sin(1/x) - \mathrm{L}| < 1$ whenever $-\delta < x < \delta$. Let k be a positive integer $> 1/(2\pi\delta)$ and let $x_1 = 1/(2k+\frac{1}{2})\pi$ and $x_2 = -1/(2k+\frac{1}{2})\pi$. Then $-\delta < x_1 < \delta$ and $-\delta < x_2 < \delta$. By the triangle inequality we get the following contradiction:

$$1 + 1 > |\sin(1/x_1) - \mathrm{L}| + |\sin(1/x_2) - \mathrm{L}| \geq |[\sin(1/x_1) - \mathrm{L}] - [\sin(1/x_2) - \mathrm{L}]| = |\sin(1/x_1) - \sin(1/x_2)|$$
$$= |\sin(2k+\tfrac{1}{2})\pi - \sin[-(2k+\tfrac{1}{2})\pi]| = |1 - (-1)| = 2$$

Thus there is no such number L and the limit does not exist.

45. $\lim_{x\to 0}\cos(1/x)$ does not exist. See Ex.44.

46. $\lim_{x\to 0}\tan(1/x)$ does not exist. See Ex.44.

Miscellaneous Exercises for Chapter 1

1. $f(x) = 4 - x^2$ ▹ (a) $f(1) = 4 - 1^2 = 3$ (b) $f(-2) = 4 - (-2)^2 = 0$ (c) $f(3) = 4 - 3^2 = -5$
(d) $f(x-1) = 4 - (x-1)^2 = -x^2 + 2x + 3$ (e) $f(x^2) = 4 - (x^2)^2 = 4 - x^4$
(f) If $h \neq 0$, $\dfrac{f(x+h) - f(x)}{h} = \dfrac{[4 - (x+h)^2] - [4 - x^2]}{h} = \dfrac{-2xh - h^2}{h} = -2x - h$

2. $g(x) = \sqrt{1-x}$ ▹ (a) $g(1) = \sqrt{1-1} = 0$ (b) $g(-3) = \sqrt{1-(-3)} = \sqrt{4} = 2$
(c) $g(x+1) = \sqrt{1-(x+1)} = \sqrt{-x}$ (d) $g(1-x^2) = \sqrt{1-(1-x^2)} = \sqrt{x^2} = |x|$
(e) If $h \neq 0$, $\dfrac{g(x+h) - g(x)}{h} = \dfrac{\sqrt{1-(x+h)} - \sqrt{1-x}}{h} \cdot \dfrac{\sqrt{1-x-h} + \sqrt{1-x}}{\sqrt{1-x-h} + \sqrt{1-x}} = \dfrac{(1-x-h) - (1-x)}{h(\sqrt{1-x-h} + \sqrt{1-x})}$
$= \dfrac{-h}{h(\sqrt{1-x-h} + \sqrt{1-x})} = \dfrac{-1}{\sqrt{1-x-h} + \sqrt{1-x}}$

In *Exercises 3–6*, define the following functions and determine their domain D:
(a) $f+g$ (b) $f-g$ (c) $f \cdot g$ (d) f/g (e) g/f (f) $f \circ g$ (g) $g \circ f$

3. $f = \sqrt{x+2}$, $g = x^2 - 4$ (a) $f + g = \sqrt{x+2} + x^2 - 4$, D: $[-2, +\infty)$
(b) $f - g = \sqrt{x+2} - x^2 + 4$, D: $[-2, +\infty)$ (c) $f \cdot g = \sqrt{x+2}(x^2-4)$, D: $[-2, +\infty)$
(d) $f/g = \dfrac{\sqrt{x+2}}{x^2-4}$, D: $(-2, 2) \cup (2, +\infty)$ (e) $g/f = \dfrac{x^2-4}{\sqrt{x+2}}$, D: $(-2, +\infty)$
(f) $f(g(x)) = \sqrt{(x^2-4)+2} = \sqrt{x^2-2}$, D: $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, +\infty)$
(g) $g(f(x)) = (\sqrt{x+2})^2 - 4 = (x+2) - 4 = x - 2$, D: $[-2, +\infty)$

4. $f = x^2 - 9$, $g = \sqrt{x+5}$ (a) $f + g = x^2 - 9 + \sqrt{x+5}$, D: $[-5, +\infty)$
(b) $f - g = x^2 - 9 - \sqrt{x+5}$, D: $[-5, +\infty)$ (c) $f \cdot g = (x^2 - 9)\sqrt{x+5}$, D: $[-5, +\infty)$
(d) $f/g = \dfrac{x^2-9}{\sqrt{x+5}}$, D: $(-5, +\infty)$ (e) $g/f = \dfrac{\sqrt{x+5}}{x^2-9}$, D: $[-5, -3) \cup (-3, 3) \cup (3, +\infty)$
(f) $f(g(x)) = (\sqrt{x+5})^2 - 9 = (x+5) - 9 = x - 4$, D: $[-5, +\infty)$
(g) $g(f(x)) = \sqrt{(x^2-9)+5} = \sqrt{x^2-4}$, D: $(-\infty, -2] \cup [2, +\infty)$

5. $f = \frac{1}{x^2}$, $g = \sqrt{x}$ (a) $f + g = \frac{1}{x^2} + \sqrt{x}$, D: $(0, +\infty)$ (b) $f - g = \frac{1}{x^2} - \sqrt{x}$, D: $(0, +\infty)$

(c) $f \cdot g = \frac{1}{x^2} \cdot \sqrt{x} = x^{-3/2}$, D: $(0, +\infty)$ (d) $\frac{f}{g} = \frac{1/x^2}{\sqrt{x}} = x^{-5/2}$, D: $(0, +\infty)$ (e) $\frac{g}{f} = \frac{\sqrt{x}}{1/x^2} = x^{5/2}$, D: $(0, +\infty)$

(f) $f(g(x)) = \frac{1}{(\sqrt{x})^2} = \frac{1}{x}$, D: $(0, +\infty)$ (g) $g(f(x)) = \sqrt{\frac{1}{x^2}} = \frac{1}{|x|}$, D: $x \neq 0$

6. $f = \frac{x}{x-1}$, $g = \frac{1}{x+2}$ (a) $f + g = \frac{x}{x-1} \cdot \frac{x+2}{x+2} + \frac{1}{x+2} \cdot \frac{x-1}{x-1} = \frac{x^2 + 3x - 1}{(x-1)(x+2)}$, D: $x \neq -2, 1$

(b) $f - g = \frac{x}{x-1} \cdot \frac{x+2}{x+2} - \frac{1}{x+2} \cdot \frac{x-1}{x-1} = \frac{x^2 + x + 3}{(x-1)(x+2)}$, D: $x \neq -2, 1$

(c) $f \cdot g = \frac{x}{x-1} \cdot \frac{1}{x+2} = \frac{x}{(x-1)(x+2)}$, D: $x \neq -2, 1$ (d) $f/g = \frac{x/(x-1)}{1/(x+2)} = \frac{x^2 + 2x}{x-1}$, D: $x \neq -2, 1$

(e) $g/f = \frac{1/(x+2)}{x/(x-1)} = \frac{x-1}{x(x+2)}$, D: $x \neq -2, 0, 1$

(f) $f(g(x)) = \frac{1/(x+2)}{1/(x+2) - 1} = \frac{1}{1 - (x+2)} = -\frac{1}{x+1}$, D: $x \neq -2, -1$

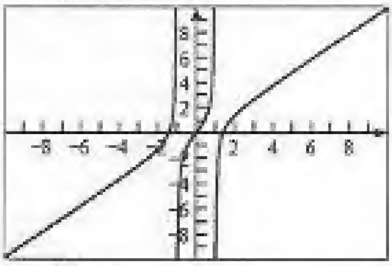
(g) $g(f(x)) = \frac{1}{x/(x-1) + 2} = \frac{x-1}{x + 2(x-1)} = \frac{x-1}{3x-2}$, D: $x \neq \frac{2}{3}, 1$

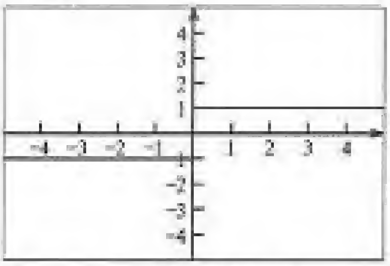
In *Exercises 7 and 8*, plot the graph and determine if the function is even, odd, or neither.

7. (a) $2x^3 - 3x$ is odd (odd − odd) (b) $5x^4 + 2x^2 - 1$ is even (sum of even)

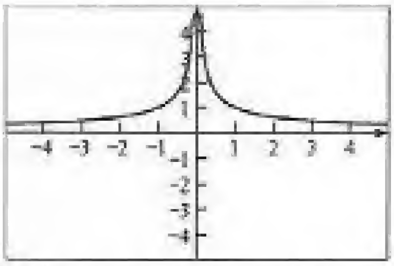
(c) $3x^5 - 2x^3 + x^2 - x$ is neither (odd + even) (d) $(x^2 + 1)/(x^3 - x)$ is odd (even ÷ odd)

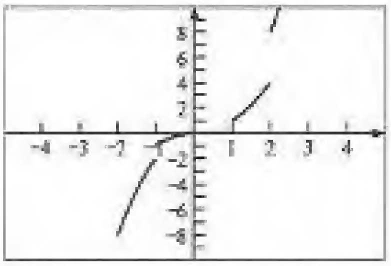
8. (a) $\frac{x^3 - 2x}{x^2 - 1}$ is odd (odd ÷ even) (b) $\frac{x}{|x|}$ is odd (odd ÷ even)





(c) $\sqrt[3]{x}/x$ is even (odd ÷ odd) (d) $F(x) = x^2 \cdot [\![x]\!]$ x is neither:
$F(0.5) = (0.5)^2[\![0.5]\!] = 0.25(0) = 0$ while $F(-0.5) = (-0.5)^2[\![-0.5]\!] = 0.25(1) = 0.25$





In *Exercises 9 and 10*, plot the graph of the function and determine its domain D and range R.

9. (a) $f(x) = 4 - 2x$
▷ D: $(-\infty, +\infty)$, R: $(-\infty, +\infty)$

(b) $g(x) = x^2 - 4$
D: $(-\infty, +\infty)$, R: $[-4, +\infty)$

(c) $h(x) = \sqrt{x^2 - 16}$
D: $(-\infty, -4] \cup [4, +\infty)$, R: $[0, +\infty)$

(d) $F(x) = \sqrt{16 - x^2}$
D: $[-4, 4]$, R: $[0, 4]$

(e) $f(x) = |5 - x|$
D: $(-\infty, +\infty)$, R: $[0, +\infty)$

(f) $g(x) = 5 - |x|$
D: $(-\infty, +\infty)$, R: $(-\infty, 5]$

10. (a) $g(x) = 3x + 2$
▷ D: $(-\infty, +\infty)$, R: $(-\infty, +\infty)$

(b) $f(x) = 9 - x^2$
D: $(-\infty, +\infty)$, R: $(-\infty, 9]$

(c) $H(x) = \sqrt{1 - x^2}$
D: $[-1, 1]$, R: $[0, 1]$

(d) $G(x) = \sqrt{x^2 - 1}$
D: $(-\infty, -1] \cup [1, +\infty)$, R: $[0, +\infty)$

(e) $g(x) = |x + 4|$
D: $(-\infty, +\infty)$, R: $[0, +\infty)$

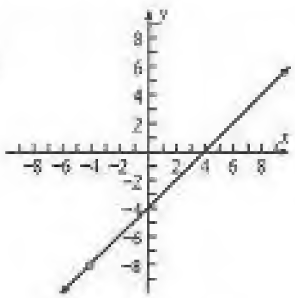
(f) $f(x) = |x| + 4$
D: $(-\infty, +\infty)$, R: $[4, +\infty)$

In Exercises 11–14, determine the domain D and range R of the function and sketch its graph.

11. (a) $g(x) = \dfrac{x^2-16}{x+4}$

▷ $= \dfrac{(x-4)(x+4)}{x+4} = x-4$

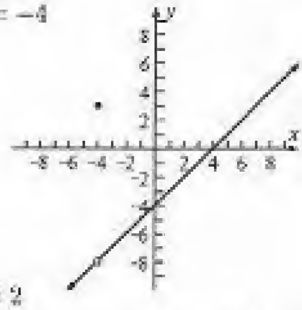
D: $x \neq -4$, R: $y \neq -8$



(b) $G(x) = \begin{cases} x-4 & \text{if } x \neq -4 \\ 3 & \text{if } x = -4 \end{cases}$

▷ D: $(-\infty, +\infty)$

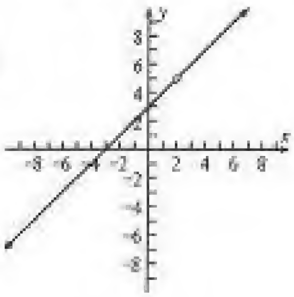
R: $y \neq -8$



12. (a) $f(x) = \dfrac{x^2+x-6}{x-2}$

▷ $= \dfrac{(x-2)(x+3)}{x-2} = x+3$

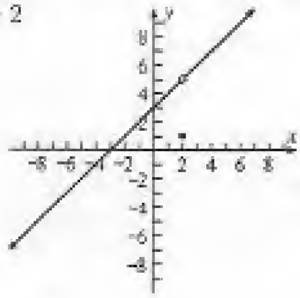
D: $x \neq 2$, R: $y \neq 5$



(b) $F(x) = \begin{cases} x+3 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $y \neq 5$

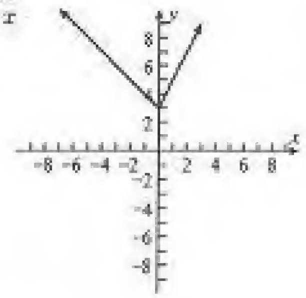


13. (a) $F(x) = \begin{cases} 3-x & \text{if } x < 0 \\ 3+2x & \text{if } 0 \leq x \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $(3, +\infty) \cup [3, +\infty)$

$= [3, +\infty)$

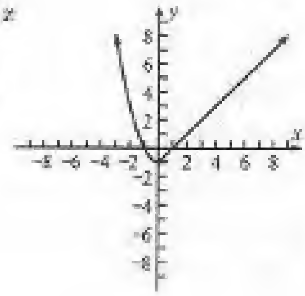


(b) $h(x) = \begin{cases} x^2-1 & \text{if } x \leq 0 \\ x-1 & \text{if } 0 < x \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $[-1, +\infty) \cup (-1, +\infty)$

$= [-1, +\infty)$

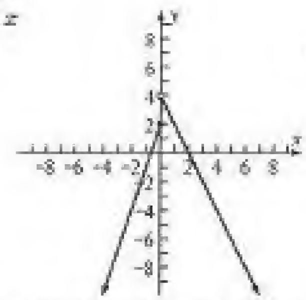


14. (a) $G(x) = \begin{cases} 3x+2 & \text{if } x \leq 0 \\ 4-2x & \text{if } 0 < x \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $(-\infty, 2] \cup (-\infty, 4)$

$= (-\infty, 4)$

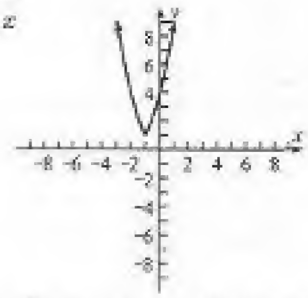


(b) $H(x) = \begin{cases} x^2 & \text{if } x < -1 \\ (x+2)^2 & \text{if } -1 \leq x \end{cases}$

▷ D: $(-\infty, +\infty)$

R: $(1, +\infty) \cup [1, +\infty)$

$= [1, +\infty)$



In Exercises 15–20, determine a $\delta > 0$ such that $|f(x) - L| < \epsilon$ when $0 < |x-a| < \delta$ by finding the smallest x_1 and largest x_2; and by using inequalities.

15. $f(x) = 2x-5$, $a = 3$, $L = 1$, $\epsilon = .05$

▷ $2x_1 - 5 = 1 - .05$, $6 - 2x_1 = .05$, $3 - x_1 = .025$; $2x_2 - 5 = 1 + .05$, $2x_2 - 6 = .05$, $x_2 - 3 = .025$. $\delta = .025$.

$|(2x-5)-1| = |2x-6| = 2|x-3| < .05$ when $|x-3| < .025 = \delta$

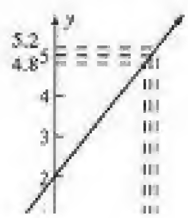
16. $f(x) = 3x+2$, $a = 1$, $L = 5$, $\epsilon = 0.2$

▷ Because f is increasing, x_1 corresponds to the smaller value of f.

$3x_1 + 2 = 5 - 0.2$, $3 - 3x_1 = 0.2$, $1 - x_1 = \frac{1}{15}$

$3x_2 + 2 = 5 + 0.2$, $3x_2 - 3 = 0.2$, $x_2 - 1 = \frac{1}{15}$. $\delta = \frac{1}{15}$.

$|(3x+2)-5| = |3x-3| = 3|x-1| < 0.2$ when $|x-1| < \frac{1}{15}$. $\delta = \frac{1}{15}$.



17. $f(x) = \dfrac{x^2 - 25}{x - 5}$, $a = 5$, $L = 10$, $\epsilon = 0.1$

▷ $f(x) = x + 5$ if $x \neq 5$. $x_1 + 5 = 10 - 0.1$, $5 - x_1 = 0.1$. $x_2 + 5 = 10 + 0.1$, $x_2 - 5 = 0.1$. $\delta = 0.1$

$\left|\dfrac{x^2 - 25}{x - 5} - 10\right| = \left|(x + 5) - 10\right| = |x - 5| < 0.1$ when $0 < |x - 5| < 0.1$. $\delta = 0.1$

18. $f(x) = \dfrac{2x^2 + 9x + 10}{x + 2}$, $a = -2$, $L = 1$, $\epsilon = .03$

▷ $f(x) = \dfrac{(2x + 5)(x + 2)}{x + 2} = 2x + 5$ if $x \neq -2$. $2x_1 + 5 = 1 - .03$, $-4 - 2x_1 = .03$, $-2 - x_1 = .015$

$2x_2 + 5 = 1 + .03$, $2x_2 + 4 = .03$, $x_2 - (-2) = .015$. $\delta = .015$

$\left|\dfrac{2x^2 + 9x + 10}{x + 2} - 1\right| = \left|(2x + 5) - 1\right| = |2x + 4| = 2|x + 2| < .03$ when $|x + 2| < .015$. $\delta = .015$

19. $f(x) = x^2 + 4$, $a = 2$, $L = 8$, $\epsilon = 0.3$ ▷ Because $a > 0$, then $x_1, x_2 > 0$.

${x_1}^2 + 4 = 8 - 0.3$, ${x_1}^2 = 3.7$, $x_1 = \sqrt{3.7} = 1.924$, $2 - x_1 = .076$. ${x_2}^2 + 4 = 8 + 0.3$, ${x_2}^2 = 4.3$,

$x_2 = \sqrt{4.3} = 2.074$, $x_2 - 2 = .074$. $\delta = .074$. Choose $\delta < 1$ so $-1 < x - 2 < 1$, $3 < x + 2 < 5$.

$\left|(x^2 + 4) - 8\right| = |x^2 - 4| = |x + 2||x - 2| < 5|x - 2| < 0.3$ when $|x - 2| < .06 = \delta$.

20. $f(x) = x^2 - 3x$, $a = 3$, $L = 0$, $\epsilon = .08$

▷ Because f is increasing at 3, x_1 corresponds to the smaller value of f.

Because $a > 0$, x_1 and $x_2 > 0$.

${x_1}^2 - 3x_1 = 0 - .08$, ${x_1}^2 - 3x_1 + .08 = 0$, $x_1 = \frac{1}{2}[3 + \sqrt{3^2 - 4(.08)}] = 2.973$.

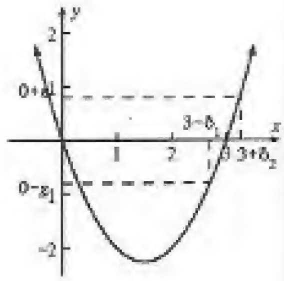
${x_2}^2 - 3x_2 = 0 + .08$, ${x_2}^2 - 3x_2 - .08 = 0$, $x_2 = \frac{1}{2}[3 + \sqrt{3^2 + 4(.08)}] = 3.026$

$3 - x_1 = .027$, $x_2 - 3 = .026$. $\delta = .026$

Choose $\delta < 1$ so $-1 < x - 3 < 1$, $2 < x < 4$.

$\left|(x^2 - 3x) - 0\right| = |x||x - 3| < 4|x - 3| < .08$ when $|x - 3| < .02 = \delta$

In the figure, ϵ and δ are exaggerated for clarity.



In Exercises 21–26, prove the limit by applying the definition (Definition 1.5.1).

21. $\lim\limits_{x \to 3}(2x - 5) = 1$ ▷ $\left|(2x - 5) - 1\right| = |2x - 6| = 2|x - 3| < \epsilon$ when $|x - 3| = \frac{1}{2}\epsilon = \delta$

22. $\lim\limits_{x \to -2}(8 - 3x) = 14$ ▷ $\left|(8 - 3x) - 14\right| = |-3x - 6| = 3|x + 2| < \epsilon$ when $|x + 2| < \frac{1}{3}\epsilon = \delta$

23. $\lim\limits_{x \to -1}(3x + 8) = 5$ ▷ $\left|(3x + 8) - 5\right| = |3x + 3| = 3|x + 1| < \epsilon$ when $|x + 1| < \frac{1}{3}\epsilon = \delta$

24. $\lim\limits_{x \to 5}(4x - 11) = 9$

▷ We wish to determine a $\delta > 0$ such that

if $0 < |x - 5| < \delta$ then $\left|(4x - 11) - 9\right| < \epsilon$

$\Leftrightarrow$ if $0 < |x - 5| < \delta$ then $4|x - 5| < \epsilon$

$\Leftrightarrow$ if $0 < |x - 5| < \delta$ then $|x - 5| < \frac{1}{4}\epsilon$

Hence take $\delta = \frac{1}{4}\epsilon$; then $0 < |x - 5| < \delta \Rightarrow 4|x - 5| < 4(\frac{1}{4}\epsilon) \Rightarrow \left|(4x - 11) - 9\right| < \epsilon$.

25. $\lim\limits_{x \to -3/4} \dfrac{16x^2 - 9}{4x + 3} = -6$

▷ If $x \neq -\frac{3}{4}$, then $\left|\dfrac{16x^2 - 9}{4x + 3} - (-6)\right| = \left|(4x - 3) + 6\right| = |4x + 3| = 4\left|x + \frac{3}{4}\right| < \epsilon$ when $0 < \left|x + \frac{3}{4}\right| < \frac{1}{4}\epsilon = \delta$

26. $\lim\limits_{x \to 1/3} \dfrac{1 - 9x^2}{1 - 3x} = 2$

▷ If $x \neq \frac{1}{3}$, then $\left|\dfrac{1 - 9x^2}{1 - 3x} - 2\right| = \left|(1 + 3x) - 2\right| = |3x - 1| = 3\left|x - \frac{1}{3}\right| < \epsilon$ when $0 < \left|x - \frac{1}{3}\right| < \frac{1}{3}\epsilon = \delta$

In Exercises 27–34, find the limit and, when appropriate, indicate the limit theorems used.

27. $\lim\limits_{x \to 2}(3x^2 - 4x + 5) \overset{\text{L.T.4}}{=} \lim\limits_{x \to 2} 3x^2 + \lim\limits_{x \to 2}(-4x + 5) \overset{\text{L.T.7,L.T.8}}{\overset{\text{L.T.1}}{=}} \lim\limits_{x \to 2} 3(\lim\limits_{x \to 2} x)^2 + (-3) \overset{\text{L.T.3}}{\overset{\text{L.T.2}}{=}} 3(2)^2 - 3 = 9$

28. $\lim\limits_{x\to-2} \dfrac{x^2-x-6}{x^2-5x-14}$

▷ $$\lim_{x\to-2} \frac{x^2-x-6}{x^2-5x-14} = \lim_{x\to-2} \frac{(x+2)(x-3)}{(x-2)(x-7)}$$
$$= \lim_{x\to-2} \frac{x-3}{x-7}$$
$$= \frac{\lim\limits_{x\to-2}(x-3)}{\lim\limits_{x\to-2}(x-7)} \qquad \text{(L.T. 9)}$$
$$= \frac{-5}{-9} = \frac{5}{9} \qquad \text{(L.T. 1)}$$

29. $\lim\limits_{z\to-3} \dfrac{z^2-9}{z+3} = \lim\limits_{z\to-3} \dfrac{(z-3)(z+3)}{z+3} = \lim\limits_{z\to-3}(z-3) \overset{\text{L.T.1}}{=} -6$

30. $\lim\limits_{h\to1} \dfrac{h^2-4}{3h^3+6} \overset{\text{L.T.10}}{=} \dfrac{\lim\limits_{h\to1}(h^2-4)}{\lim\limits_{h\to1}(3h^3+6)} \overset{\text{L.T.4}}{=} \dfrac{\lim\limits_{h\to1}h^2 - \lim\limits_{h\to1}4}{\lim\limits_{h\to1}3h^3 + \lim\limits_{h\to1}6} \overset{\substack{\text{L.T.8}\\ \text{L.T.7}}}{=} \dfrac{(\lim\limits_{h\to1}h)^2 - \lim\limits_{h\to1}4}{\lim\limits_{h\to1}3(\lim\limits_{h\to1}h)^3 + \lim\limits_{h\to1}6} \overset{\text{L.T.1}}{=} \dfrac{1^2-4}{3\cdot1^3+6} = \dfrac{-3}{9} = -\dfrac{1}{3}$

31. $\lim\limits_{x\to1/2} \sqrt[3]{\dfrac{4x^2+4x-3}{4x^2-1}} = \lim\limits_{x\to1/2} \sqrt[3]{\dfrac{(2x-1)(2x+3)}{(2x-1)(2x+1)}} = \lim\limits_{x\to1/2} \sqrt[3]{\dfrac{2x+3}{2x+1}} \overset{\text{L.T.10}}{=} \sqrt[3]{\lim\limits_{x\to1/2}\left(\dfrac{2x+3}{2x+1}\right)}$

$\overset{\text{L.T.9}}{=} \sqrt[3]{\dfrac{\lim\limits_{x\to1/2}(2x+3)}{\lim\limits_{x\to1/2}(2x+1)}} \overset{\text{L.T.1}}{=} \sqrt[3]{\dfrac{4}{2}} = \sqrt[3]{2}$

32. $\lim\limits_{t\to0} \dfrac{1-\sqrt{1+t}}{t}$

▷ $$\lim_{t\to0} \frac{1-\sqrt{1+t}}{t} = \lim_{t\to0} \frac{1-\sqrt{1+t}}{t} \cdot \frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}$$
$$= \lim_{t\to0} \frac{1-(1+t)}{t(1+\sqrt{1+t})} = \lim_{t\to0} \frac{-t}{t(1+\sqrt{1+t})}$$
$$= \lim_{t\to0} \frac{-1}{1+\sqrt{1+t}}$$
$$= \frac{-1}{\lim\limits_{t\to0}(1+\sqrt{1+t})} \qquad \text{(L.T. 9, 2)}$$
$$= \frac{-1}{1+\sqrt{1}} \qquad \text{(L.T. 5, 1, 10)}$$
$$= -\frac{1}{2}$$

33. $\lim\limits_{t\to0} \dfrac{\sqrt{9-t}-3}{t} = \lim\limits_{t\to0} \dfrac{(\sqrt{9-t}-3)(\sqrt{9-t}+3)}{t(\sqrt{9-t}+3)} = \lim\limits_{t\to0} \dfrac{-t}{t(\sqrt{9-t}+3)} = \lim\limits_{t\to0} \dfrac{-1}{\sqrt{9-t}+3} \overset{\text{L.T.9}}{=} \dfrac{\lim\limits_{t\to0}(-1)}{\lim\limits_{t\to0}(\sqrt{9-t}+3)}$

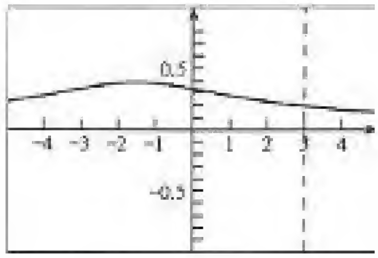
$\overset{\substack{\text{L.T.4}\\ \text{L.T.2}}}{=} \dfrac{-1}{\lim\limits_{t\to0}\sqrt{9-t} + \lim\limits_{t\to0}3} \overset{\substack{\text{L.T.10}\\ \text{L.T.2}}}{=} \dfrac{-1}{\sqrt{\lim\limits_{t\to0}(9-t)}+3} \overset{\text{L.T.1}}{=} \dfrac{-1}{\sqrt{9}+3} = -\dfrac{1}{6}$

34. $\lim\limits_{y\to-4} \sqrt{\dfrac{5y+4}{y-5}} \overset{\text{L.T.10}}{=} \sqrt{\lim\limits_{y\to-4} \dfrac{5y+4}{y-5}} \overset{\text{L.T.9}}{=} \sqrt{\dfrac{\lim\limits_{y\to-4}(5y+4)}{\lim\limits_{y\to-4}(y-5)}} \overset{\text{L.T.1}}{=} \sqrt{\dfrac{5(-4)+4}{-4-5}} = \sqrt{\dfrac{-16}{-9}} = \sqrt{\dfrac{16}{9}} = \dfrac{4}{3}$

In Exercises 35–42, find the limit if it exists and support your answer by plotting the graph.

35. $\displaystyle\lim_{x\to-1}\frac{2x^2-x-3}{3x^2+8x+5}=\lim_{x\to-1}\frac{(2x-3)(x+1)}{(3x+5)(x+1)}=\lim_{x\to-1}\frac{2x-3}{3x+5}=-\frac{5}{2}$

36. $\displaystyle\lim_{y\to3}\sqrt{\frac{y-3}{y^3-27}}$



▷ $$\begin{aligned}\lim_{y\to3}\sqrt{\frac{y-3}{y^3-27}}&=\lim_{y\to3}\sqrt{\frac{y-3}{(y-3)(y^2+3y+9)}}\\&=\lim_{y\to3}\sqrt{\frac{1}{y^2+3y+9}}\\&=\sqrt{\frac{1}{27}}=\tfrac{1}{9}\sqrt{3}\approx0.1925\end{aligned}$$

37. $\displaystyle\lim_{x\to9}\frac{2\sqrt{x}-6}{x-9}=2\lim_{x\to9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}=2\lim_{x\to9}\frac{x-9}{(x-9)(\sqrt{x}+3)}=2\lim_{x\to9}\frac{1}{\sqrt{x}+3}=2\cdot\frac{1}{6}=\frac{1}{3}$

38. Since $y\to5^-$, then $y<5$ and $5-y>0$. Thus,

$$\lim_{y\to5^-}\frac{\sqrt{25-y^2}}{y-5}=\lim_{y\to5^-}\frac{\sqrt{(5-y)(5+y)}}{-\sqrt{(5-y)^2}}=\lim_{y\to5^-}\frac{\sqrt{5-y}\cdot\sqrt{5+y}}{-\sqrt{5-y}\sqrt{5-y}}=\lim_{y\to5^-}\frac{\sqrt{5+y}}{-\sqrt{5-y}}=\frac{\sqrt{10}}{0^-}=-\infty$$

39. $\displaystyle\lim_{s\to7}\frac{5-\sqrt{4+3s}}{7-s}=\lim_{s\to7}\frac{5-\sqrt{4+3s}}{7-s}\cdot\frac{5+\sqrt{4+3s}}{5+\sqrt{4+3s}}=\lim_{s\to7}\frac{25-(4+3s)}{(7-s)(5+\sqrt{4+3s})}=\lim_{s\to7}\frac{3}{5+\sqrt{4+3s}}$
$\displaystyle=\frac{3}{5+\sqrt{4+3(7)}}=\frac{3}{10}$

40. $\displaystyle\lim_{x\to0^-}\frac{x^2-5}{2x^3-3x^2}$

▷ We apply Limit Theorem 12. Because $2x^3-3x^2=x^2(2x-3)$, then $2x^3-3x^2<0$ when $x<\frac{3}{2}$. Thus,

$$\lim_{x\to0^-}(2x^3-3x^2)=0$$

and $2x^3-3x^2$ is approaching 0 through negative values. Furthermore,

$$\lim_{x\to0^-}(x^2-5)=-5$$

Thus, by Limit Theorem 12(iv) we conclude that

$$\lim_{x\to0^-}\frac{x^2-5}{2x^3-3x^2}=+\infty$$

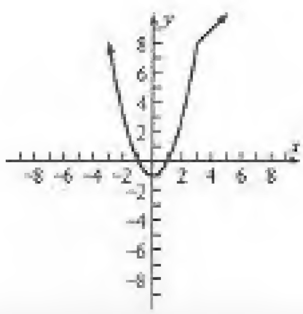
41. $\displaystyle\lim_{x\to2^+}([\![x]\!]-1)=1$ and $\displaystyle\lim_{x\to2^+}([\![x]\!]-x)=2-2=0$ through negative values. Therefore $\displaystyle\lim_{x\to0}\frac{[\![x]\!]-1}{[\![x]\!]-x}=-\infty$

42. $\displaystyle\lim_{x\to5}\frac{\sqrt{x-1}-2}{x-5}=\lim_{x\to5}\frac{\sqrt{x-1}-2}{x-5}\cdot\frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}=\lim_{x\to5}\frac{(x-1)-4}{(x-5)(\sqrt{x-1}+2)}=\lim_{x\to5}\frac{1}{\sqrt{x-1}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{4}$

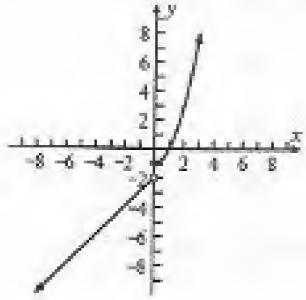
In Exercises 43–48, sketch the graph and find the limit or state why it does not exist.

43. $f(x)=\begin{cases}x^2-1 & \text{if } x\le3\\ x+5 & \text{if } 3<x\end{cases}$

▷ $\displaystyle\lim_{x\to3^-}f(x)=\lim_{x\to3^-}(x^2-1)=(3)^2-1=8;$
$\displaystyle\lim_{x\to3^+}f(x)=\lim_{x\to3^+}(x+5)=3+5=8.$
Hence, $\displaystyle\lim_{x\to3}f(x)=8.$



44. $g(x) = \begin{cases} x-2 & \text{if } x \le 0 \\ x^2-1 & \text{if } x > 0 \end{cases}$



▷ $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-}(x-2) = -2;$

$\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} x^2 - 1 = -1.$

Because $\lim\limits_{x\to 0^-} g(x) \ne \lim\limits_{x\to 0^+} g(x)$,

then $\lim\limits_{x\to 0} g(x)$ does not exist.

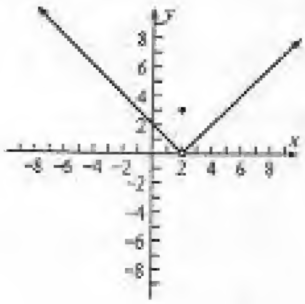
45. $h(t) = \dfrac{|t-1|}{t-1}$

▷ $\lim\limits_{t\to 1^-} h(t) = \lim\limits_{t\to 1^-} \dfrac{-(t-1)}{t-1} = \lim\limits_{t\to 1^-}(-1) = -1;$

$\lim\limits_{t\to 1^+} h(t) = \lim\limits_{t\to 1^+} \dfrac{t-1}{t-1} = \lim\limits_{t\to 1} 1 = 1.$

Hence, $\lim\limits_{t\to 1} h(t)$ does not exist.

46. $f(r) = \begin{cases} |r-2| & \text{if } r \ne 2 \\ 3 & \text{if } r = 2 \end{cases}$

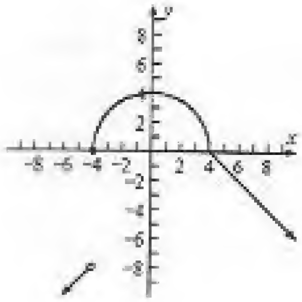


▷ $\lim\limits_{r\to 2^-} f(r) = \lim\limits_{r\to 2^-}(2-r) = 2-2 = 0;$

$\lim\limits_{r\to 2^+} f(r) = \lim\limits_{r\to 2^+}(r-2) = 2-2 = 0.$

Hence $\lim\limits_{r\to 2} f(r) = 0$

47. $g(x) = \begin{cases} x-4 & \text{if } x < -4 \\ \sqrt{16-x^2} & \text{if } -4 \le x \le 4 \\ 4-x & \text{if } x > 4 \end{cases}$



▷ $\lim\limits_{x\to -4^-} g(x) = \lim\limits_{x\to -4^-}(x-4) = -4-4 = -8$

$\lim\limits_{x\to -4^+} g(x) = \lim\limits_{x\to -4^+} \sqrt{16-x^2} = \sqrt{16-16} = 0$

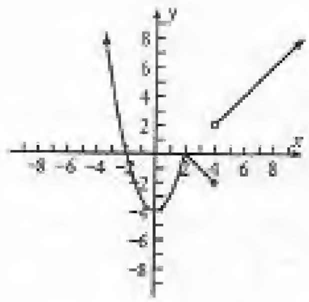
Hence $\lim\limits_{x\to -4} g(x)$ does not exist.

$\lim\limits_{x\to 4^-} g(x) = \lim\limits_{x\to 4^-} \sqrt{16-x^2} = \sqrt{16-16} = 0$

$\lim\limits_{x\to 4^+} g(x) = \lim\limits_{x\to 4^+}(4-x) = 4-4 = 0$

Hence $\lim\limits_{x\to 4} g(x) = 0.$

48. $h(x) = \begin{cases} x^2-4 & \text{if } x \le 2 \\ 2-x & \text{if } 2 < x \le 4 \\ x-2 & \text{if } x > 4 \end{cases}$



▷ (a) $\lim\limits_{x\to 2^-} h(x) = \lim\limits_{x\to 2^-}(x^2-4) = 4-4 = 0$

(b) $\lim\limits_{x\to 2^+} h(x) = \lim\limits_{x\to 2^+}(2-x) = 2-2 = 0$

(c) Because $\lim\limits_{x\to 2^-} h(x) = \lim\limits_{x\to 2^+} h(x) = 0$ then $\lim\limits_{x\to 2} h(x) = 0.$

(d) $\lim\limits_{x\to 4^-} h(x) = \lim\limits_{x\to 4^-}(2-x) = 2-4 = -2$

(e) $\lim\limits_{x\to 4^+} h(x) = \lim\limits_{x\to 4^+}(x-2) = 4-2 = 2$

(f) Because $\lim\limits_{x\to 4^-} h(x) \ne \lim\limits_{x\to 4^+} h(x)$, then $\lim\limits_{x\to 4} h(x)$ does not exist.

In Exercises 49–54, find the limit and support your answer graphically.

49. (a) $\lim\limits_{x\to -4^-} \dfrac{2x}{16-x^2} = \lim\limits_{x\to -4^-} \dfrac{2x}{(4-x)(4+x)} = \lim\limits_{x\to -4^-} \dfrac{2x/(4-x)}{4+x} = \dfrac{-1}{0^-} = +\infty$

(b) $\lim\limits_{x\to -4^+} \dfrac{2x}{16-x^2} = \lim\limits_{x\to -4^+} \dfrac{2x}{(4-x)(4+x)} = \lim\limits_{x\to -4^+} \dfrac{2x/(4-x)}{4+x} = \dfrac{-1}{0^+} = -\infty$

50. (a) $\lim\limits_{x\to-2^-}\dfrac{x-1}{x^2-4}=\lim\limits_{x\to-2^-}\dfrac{x-1}{(x-2)(x+2)}=\lim\limits_{x\to-2^-}\dfrac{(x-1)/(x-2)}{x+2}=\dfrac{\frac{3}{4}}{0^-}=-\infty$

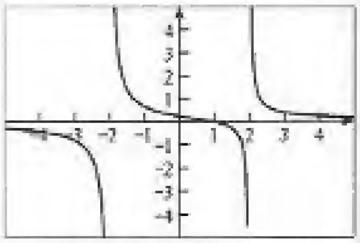
(b) $\lim\limits_{x\to-2^+}\dfrac{x-1}{x^2-4}=\lim\limits_{x\to-2^+}\dfrac{x-1}{(x-2)(x+2)}=\lim\limits_{x\to-2^+}\dfrac{(x-1)/(x-2)}{x+2}=\dfrac{\frac{3}{4}}{0^+}=+\infty$

51. (a) $\lim\limits_{x\to4^-}\dfrac{2x}{16-x^2}=\lim\limits_{x\to4^-}\dfrac{2x}{(4-x)(4+x)}=\lim\limits_{x\to4^-}\dfrac{2x/(4+x)}{4-x}=\dfrac{1}{0^+}=+\infty$

(b) $\lim\limits_{x\to4^+}\dfrac{2x}{16-x^2}=\lim\limits_{x\to4^+}\dfrac{2x}{(4-x)(4+x)}=\lim\limits_{x\to4^+}\dfrac{2x/(4+x)}{4-x}=\dfrac{1}{0^-}=-\infty$

52. (a) $\lim\limits_{x\to2^-}\dfrac{x-1}{x^2-4}=\lim\limits_{x\to2^-}\dfrac{x-1}{(x-2)(x+2)}=\lim\limits_{x\to2^-}\dfrac{(x-1)/(x+2)}{x-2}=\dfrac{\frac{1}{4}}{0^-}=-\infty$

(b) $\lim\limits_{x\to2^+}\dfrac{x-1}{x^2-4}=\lim\limits_{x\to2^+}\dfrac{x-1}{(x-2)(x+2)}=\lim\limits_{x\to2^+}\dfrac{(x-1)/(x+2)}{x-2}=\dfrac{\frac{1}{4}}{0^+}=+\infty$



53. (a) $\lim\limits_{t\to5}\dfrac{\sqrt{t-4}}{(t-5)^2}=\dfrac{1}{0^+}=+\infty$ (b) $\lim\limits_{t\to5^+}\dfrac{4-t}{\sqrt{t-5}}=\dfrac{-1}{0^+}=-\infty$

54. (a) $\lim\limits_{x\to-2}\dfrac{\sqrt{3-x}}{(x+2)^2}=\dfrac{\sqrt{5}}{0^+}=+\infty$ (b) $\lim\limits_{x\to-2^+}\dfrac{x-3}{\sqrt{x+2}}=\dfrac{-5}{0^+}=-\infty$

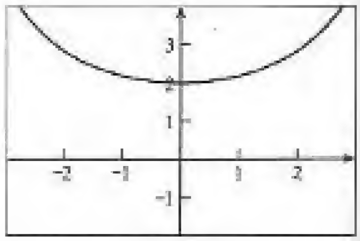
In Exercises 55–62, (a) plot the graph, conjecture the limit as $x\to0$ and (b) calculate it.

55. $\lim\limits_{x\to0}\dfrac{x}{\sin 3x}=\lim\limits_{3x\to0}\dfrac{1}{3}\cdot\dfrac{3x}{\sin 3x}=\dfrac{1}{3}$

56. $f(x)=\dfrac{x^2}{1-\cos x}$

▷ (a) See the figure. $f(x)$ appears to be approaching 2 as x approaches 0.

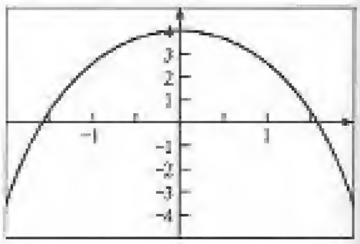
(b) $\lim\limits_{x\to0}\dfrac{x^2}{1-\cos x}=\lim\limits_{x\to0}\dfrac{x^2}{2\sin^2\frac{1}{2}x}=\lim\limits_{x/2\to0}2\Big(\dfrac{\frac{1}{2}x}{\sin\frac{1}{2}x}\Big)^2=2\Big(\lim\limits_{x/2\to0}\dfrac{\frac{1}{2}x}{\sin\frac{1}{2}x}\Big)^2$
$=2\cdot1=2$



57. $\lim\limits_{x\to0}\dfrac{\sin 5x}{\sin 2x}=\lim\limits_{x\to0}\dfrac{5}{2}\cdot\dfrac{\sin 5x/5x}{\sin 2x/2x}=\dfrac{5}{2}\cdot\dfrac{\lim\limits_{5x\to0}(\sin 5x/5x)}{\lim\limits_{2x\to0}(\sin 2x/2x)}=\dfrac{5}{2}\cdot\dfrac{1}{1}=\dfrac{5}{2}$

58. $\lim\limits_{x\to0}\dfrac{1-\cos 3x}{\sin 3x}=\lim\limits_{x\to0}\dfrac{2\sin^2\frac{3}{2}x}{2\sin\frac{3}{2}x\cos\frac{3}{2}x}=\lim\limits_{x\to0}\dfrac{\sin\frac{3}{2}x}{\cos\frac{3}{2}x}=\lim\limits_{x\to0}\tan\frac{3}{2}x=\tan 0=0$

59. $\lim\limits_{x\to0}\dfrac{1-\cos^2x}{x}=\lim\limits_{x\to0}\dfrac{\sin^2x}{x}=\lim\limits_{x\to0}\sin x\cdot\lim\limits_{x\to0}\dfrac{\sin x}{x}=0\cdot1=0$



60. $f(x)=\dfrac{4x}{\tan x}$

▷ (a) See the figure. $f(x)$ appears to be approaching 4 as x approaches 0.

(b) $\lim\limits_{x\to0}\dfrac{4x}{\tan x}=\lim\limits_{x\to0}\dfrac{4x}{\sin x/\cos x}=\lim\limits_{x\to0}4\cos x\cdot\lim\limits_{x\to0}\dfrac{x}{\sin x}=4\cos 0\cdot1=4$
since the cosine function is continuous.

61. $\lim\limits_{x\to0}\dfrac{\csc 3x}{\cot x}=\lim\limits_{x\to0}\dfrac{1/\sin 3x}{\cos x/\sin x}=\lim\limits_{x\to0}\dfrac{1}{3\cos x}\cdot\lim\limits_{3x\to0}\dfrac{3x}{\sin 3x}\cdot\lim\limits_{x\to0}\dfrac{\sin x}{x}=\dfrac{1}{3}\cdot1\cdot1=\dfrac{1}{3}$

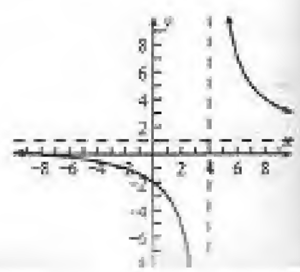
62. $\lim\limits_{x\to0}\dfrac{2x^2-3x}{2\sin x}=\lim\limits_{x\to0}\dfrac{x(2x-3)}{2\sin x}=\lim\limits_{x\to0}\dfrac{2x-3}{2}\cdot\lim\limits_{x\to0}\dfrac{x}{\sin x}=-\dfrac{3}{2}\cdot1=-\dfrac{3}{2}$

In Exercises 63–68, find the vertical asymptotes and use them to sketch the graph of the function.

▷ Note that if $f(x)-L$ is arbitrarily close to 0 for all large values of x, then the graph of $y=f(x)$ approaches the line $y=L$ at the extreme left and right, *a horizontal asymptote* (see Section 3.7).

63. $f(x)=\dfrac{x+8}{x-4}$

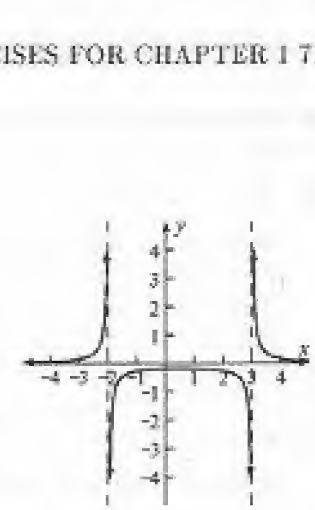
▷ Because $\lim\limits_{x\to4^-}f(x)=-\infty$ or $\lim\limits_{x\to4^+}f(x)=+\infty$,
$x=4$ is a vertical asymptote. Because
$f(x)-1=\dfrac{x+8}{x-4}-1=\dfrac{12}{x-4}$ is small when x is large,
$y=1$ is a horizontal asymptote.



64. $f(x) = \dfrac{-2}{x^2 - x - 6}$

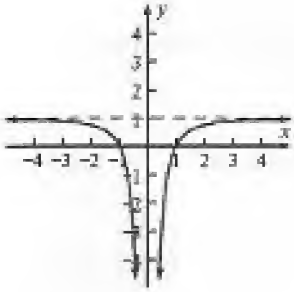
▷ To find the vertical asymptote, we factor the denominator.

$$f(x) = \frac{-2}{(x+2)(x-3)}$$



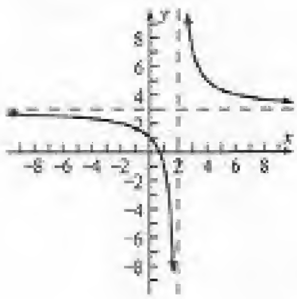
Because $(x+2)(x-3) = 0$ if $x = 3$ or $x = -2$, we find the following limits.
If $x > 3$, then $(x+2)(x-3) > 0$. Thus, by Limit Theorem 12(iii), we have $\lim\limits_{x\to 3^+} f(x) = -\infty$, and the line $x = 3$ is a vertical asymptote of the graph of f.
If $-2 < x < 3$, then $(x+2)(x-3) < 0$, so by Limit Theorem 12(iv) $\lim\limits_{x\to 3^-} f(x) = +\infty$. Therefore, the curve approaches the asymptote $x = 3$ as shown in the figure. Also, by Limit Theorem 12(iv) $\lim\limits_{x\to -2^+} f(x) = +\infty$, so the line $x = -2$ is a vertical asymptote for the graph of f. Because $f(x)$ is small when x is large, the line $y = 0$ is a horizontal asymptote.

65. $g(x) = 1 - \dfrac{1}{x^2}$



▷ Because $\lim\limits_{x\to 0^-} g(x) = -\infty$ or $\lim\limits_{x\to 0^+} g(x) = -\infty$, $x = 0$ is a vertical asymptote.
Because $g(x)$ is close to 1 when x is large, $y = 1$ is a horizontal asymptote.

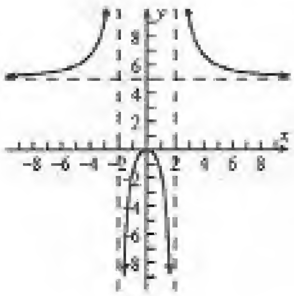
66. $f(x) = \dfrac{3x-2}{x-2}$



▷ Because $\lim\limits_{x\to 2^-} f(x) = -\infty$ or $\lim\limits_{x\to 2^+} f(x) = +\infty$, $x = 2$ is a vertical asymptote.

Because $f(x) - 3 = \dfrac{3x-2}{x-2} - 3 = \dfrac{4}{x-2}$ is small when x is large, $y = 3$ is a horizontal asymptote.

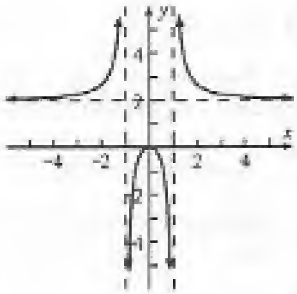
67. $f(x) = \dfrac{5x^2}{x^2-4}$



▷ Because $\lim\limits_{x\to 2^-} f(x) = -\infty$ or $\lim\limits_{x\to 2^+} f(x) = +\infty$, $x = 2$ is a vertical asymptote.
Because $\lim\limits_{x\to -2^-} f(x) = +\infty$ or $\lim\limits_{x\to -2^+} f(x) = -\infty$, $x = -2$ is a vertical asymptote.

Because $f(x) - 5 = \dfrac{5x^2}{x^2-4} - 5 = \dfrac{20}{x^2-4}$ is small when x is large, $y = 5$ is a horizontal asymptote.

68. $h(x) = \dfrac{2x^2}{x^2-1}$

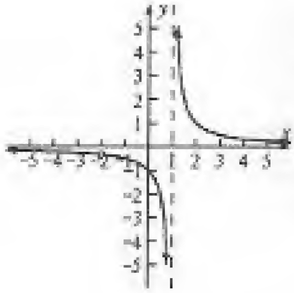


▷ Because $\lim\limits_{x\to 1^-} f(x) = -\infty$ or $\lim\limits_{x\to 1^+} f(x) = +\infty$, $x = 1$ is a vertical asymptote.
Because $\lim\limits_{x\to -1^-} f(x) = +\infty$ or $\lim\limits_{x\to -1^+} f(x) = -\infty$, $x = -1$ is a vertical asymptote.

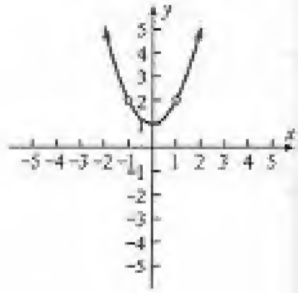
Because $f(x) - 2 = \dfrac{2x^2}{x^2-1} - 2 = \dfrac{2}{x^2-1}$ is small when x is large, $y = 2$ is a horizontal asymptote.

In Exercises 69–74, sketch the graph of the function; then by noting breaks in the graph, determine the values of at which the function is discontinuous, and show why Definition 1.8.1 is not satisfied at each discontinuity.

69. $f(x) = \dfrac{x+2}{x^2+x-2} = \dfrac{x+2}{(x+2)(x-1)} = \dfrac{1}{x-1}$ if $x \neq -2$
There are breaks in the graph at -2 and 1.
$f(-2)$ and $f(1)$ do not exist.
Hence condition (i) of Definition 2.6.1
fails at -2 and 1. Therefore,
f is discontinuous at -2 and 1.

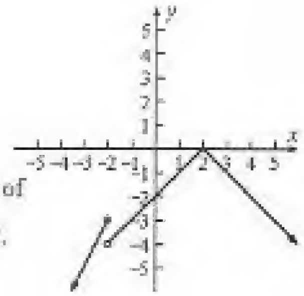


70. $g(x) = \dfrac{x^4-1}{x^2-1} = \dfrac{(x^2+1)(x+1)(x-1)}{(x+1)(x-1)} = x^2+1$ if
$x \neq \pm 1$. There are breaks in the graph at -1 and 1.
$f(-1)$ and $f(1)$ do not exist.
Hence condition (i) of Definition 1.8.1
fails at -1 and 1. Therefore,
f is discontinuous at -1 and 1.



71. $g(x) = \begin{cases} 2x+1 & \text{if } x \leq -2 \\ x-2 & \text{if } -2 < x \leq 2 \\ 2-x & \text{if } 2 < x \end{cases}$

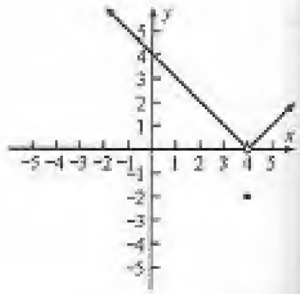
▷ There is a break in the graph at -2.
$\lim\limits_{x \to -2^-} g(x) = \lim\limits_{x \to -2^-} (2x+1) = -3$;
$\lim\limits_{x \to -2^+} g(x) = \lim\limits_{x \to -2^+} (x-2) = -4$;
therefore $\lim\limits_{x \to -2} g(x)$ does not exist. Thus condition (ii) of
Definition 1.8.1 fails at -2, so g is discontinuous at -2.
g is continuous at 2 because
$\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^+} g(x) = 0 = g(2)$.



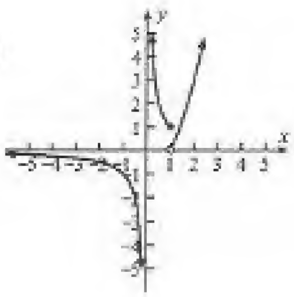
72. $F(x) = \begin{cases} |4-x| & \text{if } x \neq 4 \\ -2 & \text{if } x = 4 \end{cases}$

▷ There is a "hole" in the graph at the point $(4,0)$ because
$F(x) \neq |4-x|$ if $x = 4$, and thus F is discontinuous at 4.
We show how Definition 1.8.1 is not satisfied.
Because the absolute-value function is continuous,

$\lim\limits_{x \to 4} F(x) = \lim\limits_{x \to 4} |4-x| = |0| = 0$ and $F(4) = -2$.
Thus, $\lim\limits_{x \to 4} F(x) \neq F(4)$ and so condition (iii) of Definition 1.8.1 is not satisfied.

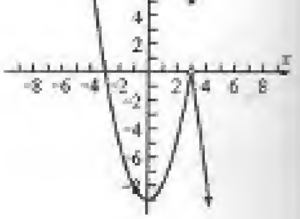


73. $h(x) = \begin{cases} 1/x & \text{if } x \leq 1 \\ x^2-1 & \text{if } 1 < x \end{cases}$
There are breaks in the graph at 0 and 1. $f(0)$ does not
exist. Hence condition (i) of Definition 2.6.1 fails at 0
and f is discontinuous at 0.
$\lim\limits_{x \to 1^+} h(x) = \lim\limits_{x \to 1^+} x^2 - 1 = 0$; $h(1) = \frac{1}{1} = 1$;
hence condition (iii) of Definition 1.8.1 fails at 1
and h is discontinuous at 1.



74. $f(x) = \begin{cases} x^2-9 & \text{if } x < 3 \\ 5 & \text{if } x = 3 \\ 9-x^2 & \text{if } x > 3 \end{cases}$

▷ There is a break in the graph at 3.
$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} (x^2-9) = 0$ but $f(3) = 5$
Thus condition (ii) or condition (iii) of Definition 1.8.1
is not satisfied.



In Exercises 75–78, prove that f is discontinuous at a. If the discontinuity is removable, redefine $f(a)$ to remove it.

75. $f(x) = \dfrac{x^2+2x-8}{x^2+3x-4} = \dfrac{(x+4)(x-2)}{(x+4)(x-1)} = \dfrac{x-2}{x-1}$ if $x \neq -4$.

$f(-4)$ does not exist. Therefore f is discontinuous at -4.

$\lim\limits_{x\to-4} f(x) = \lim\limits_{x\to-4} \dfrac{x-2}{x-1} = \dfrac{-6}{-5} = \dfrac{6}{5}$. The discontinuity is removable by redefining $f(-4) = \frac{6}{5}$.

76. $f(x) = \begin{cases} 4-x^2 & \text{if } x < 1 \\ 2x+3 & \text{if } 1 \leq x \end{cases}; \quad a = 1$

▷ $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} (4-x^2) = 3$ $\qquad$ $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} (2x+3) = 5$

Because $\lim\limits_{x\to1^-} f(x) \neq \lim\limits_{x\to1^+} f(x)$, then $\lim\limits_{x\to1} f(x)$ does not exist. Thus, there is an essential discontinuity at 1.

77. $f(x) = \begin{cases} 1/(x-2) & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

▷ $\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^-} \dfrac{1}{x-2} = \dfrac{1}{0^-} = -\infty$. Thus, there is an essential discontinuity at 2.

78. $f(x) = \dfrac{|2x-6|}{2x-6}$

$f(3)$ does not exist. Therefore, f is discontinuous at 3.

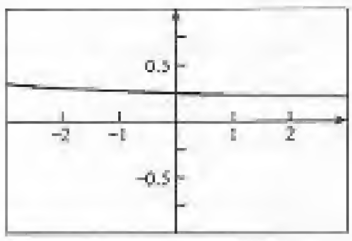
$\lim\limits_{x\to3^-} f(x) = \lim\limits_{x\to3^-} \dfrac{-(2x-6)}{2x-6} = \lim\limits_{x\to3^-} (-1) = -1$; $\lim\limits_{x\to3^+} f(x) = \lim\limits_{x\to3^+} \dfrac{2x-6}{2x-6} = \lim\limits_{x\to3^+} 1 = 1$

Because $\lim\limits_{x\to3^-} f(x) \neq \lim\limits_{x\to3^+} f(x)$, then $\lim\limits_{x\to3} f(x)$ does not exist. Therefore, the discontinuity is essential.

In Exercises 79–82, f is discontinuous at a. (a) Plot the graph of f and look for a break at $x = a$. Does the discontinuity appear to be removable? If so, how should f be redefined to remove it? (b) Confirm analytically.

79. $f(x) = \dfrac{|4-x|-3}{x-1}$; $a = 1$ $\qquad$ ▷ $\lim\limits_{x\to1} f(x) = \lim\limits_{x\to1} \dfrac{(4-x)-3}{x-1} = \lim\limits_{x\to1} \dfrac{1-x}{x-1} = \lim\limits_{x\to1} (-1) = -1$. Define $f(1) = -1$

80. $f(x) = \dfrac{2-\sqrt{x+4}}{x}$; $a = 0$ $\qquad$ ▷ We rationalize the numerator.



$f(x) = \dfrac{2-\sqrt{x+4}}{x} \cdot \dfrac{2+\sqrt{x+4}}{2+\sqrt{x+4}} = \dfrac{4-(x+4)}{x(2+\sqrt{x+4})} = \dfrac{-x}{x(2+\sqrt{x+4})}$

$= -\dfrac{1}{2+\sqrt{x+4}}$ if $x \neq 0$. To make f continuous, we should define

$f(0) = \dfrac{1}{2+\sqrt{0+4}} = \dfrac{1}{4}$.

81. $f(x) = \dfrac{x}{\sqrt{x+9}-3}$; $a = 0$ $\qquad$ ▷ $f(x) = \dfrac{x}{\sqrt{x+9}-3} \cdot \dfrac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$

$= \dfrac{x(\sqrt{x+9}+3)}{(x+9)-9} = \dfrac{x(\sqrt{x+9}+3)}{x} = \sqrt{x+9}+3$ if $x \neq 0$. Define $f(0) = \sqrt{9}+3 = 6$.

82. $f(x) = \dfrac{x-1}{\sqrt[3]{x}-1}$; $a = 1$

▷ $f(x) = \dfrac{(x^{1/3})^3-1}{x^{1/3}-1} = \dfrac{(x^{1/3}-1)(x^{2/3}+x^{1/3}+1)}{x^{1/3}-1} = x^{2/3}+x^{1/3}+1$ if $x \neq 1$. Define $f(1) = 1+1+1 = 3$.

In Exercises 83 and 84, (a) define $f \circ g$ and (b) find the numbers at which $f \circ g$ is continuous and state the reason.

83. (a) $f(x) = \sqrt{x}$; $g(x) = 25 - x^2$ $\qquad$ ▷ $(f \circ g)(x) = \sqrt{25-x^2}$

By Theorem 1.8.5(ii), $f \circ g$ is continuous for $g(x) > 0$.

By Theorem 1.8.3, g is continuous for all x and $g(x) > 0$ for $-5 < x < 5$.

Hence by Theorem 1.9.2, $f \circ g$ is continuous for all x in $(-5, 5)$.

(b) $f(x)=\dfrac{\sqrt{x^2-4}}{\sqrt{3-x}}$; $g(x)=|x|$ ▷ $(f\circ g)(x)=\dfrac{\sqrt{|x|^2-4}}{\sqrt{3-|x|}}=\dfrac{\sqrt{x^2-4}}{\sqrt{3-|x|}}$

By Theorems 1.8.2.4 and 1.8.5(ii), $f\circ g$ is continuous for $g(x)\in[(-\infty,-2)\cup(2,+\infty)]\cap(-\infty,3$ $=(-\infty,-2)\cup(2,3)=\text{I}$. The absolute-value function g is continuous for all x and $g(x)\in\text{I}$ fo $x\in(-3,-2)\cup(2,3)=\text{J}$. Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in J.

(c) $f(x)=\operatorname{sgn} x$; $g(x)=x^2-1$ ▷ $(f\circ g)(x)=\operatorname{sgn}(x^2-1)$

$f\circ g$ is continuous for $g(x)\neq 0$ and $g(x)\neq 0$ in $(-\infty,-1)\cup(-1,1)\cup(1,+\infty)=\text{I}$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in I.

84. (a) $f(x)=\sqrt{x}$; $g(x)=x^2-25$ ▷ $(f\circ g)(x)=\sqrt{x^2-25}$

By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)>0$.
By Theorem 1.8.3, g is continuous for all x and $g(x)>0$ for $(-\infty,-5)\cup(5,+\infty)=\text{I}$
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in I.

(b) $f(x)=\sqrt{x+1}$; $g(x)=\dfrac{1}{x-3}$ ▷ $(f\circ g)(x)=\sqrt{\dfrac{1}{x-3}+1}=\sqrt{\dfrac{x-2}{x-3}}$

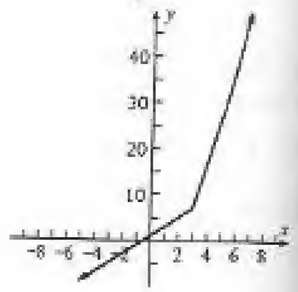
By Theorem 1.8.5(ii), $f\circ g$ is continuous for $g(x)+1>0$
By Theorem 1.8.4, g is continuous for all $x\neq 3$ and $g(x)+1>0$ when $\dfrac{x-2}{x-3}>0$; $x\in(-\infty,2)\cup(3,+\infty)$

(c) $f(x)=\operatorname{sgn} x$; $g(x)=x^2-x$ ▷ $(f\circ g)(x)=\operatorname{sgn}(x^2-x)$

$f\circ g$ is continuous for $g(x)\neq 0$. $g(x)=x(x-1)$ so $g(x)\neq 0$ in $(-\infty,0)\cup(0,1)\cup(1,+\infty)=\text{I}$.
Hence by Theorem 1.9.2, $f\circ g$ is continuous for all x in I.

In Exercises 85 and 86, find a and b that make f continuous at every number. Then sketch the graph of f.

85. $f(x)=\begin{cases}2x+1 & \text{if } x\le 3\\ ax+b & \text{if } 3<x<5\\ x^2+2 & \text{if } 5\le x\end{cases}$ ▷ $f(x)=\begin{cases}2x+1 & \text{if } x\le 3\\ 10x-23 & \text{if } 3<x<5\\ x^2+2 & \text{if } 5\le x\end{cases}$



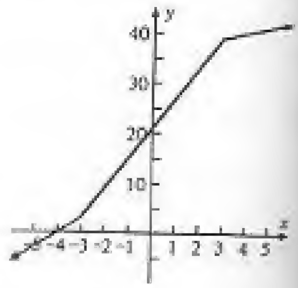
f is continuous on $(-\infty,3)$, $(3,5)$ and $(5,+\infty)$.

For f to be continuous at 3,
$$\lim_{x\to3^-}f(x)=\lim_{x\to3^+}f(x)$$
$$\lim_{x\to3^-}(2x+1)=\lim_{x\to3^+}(ax+b)$$
$$7=3a+b$$

For f to be continuous at 5,
$$\lim_{x\to5^-}f(x)=\lim_{x\to5^+}f(x)$$
$$\lim_{x\to5^-}(ax+b)=\lim_{x\to5^+}(x^2+2)$$
$$5a+b=27$$

Solving simultaneously, we get $a=10$ and $b=-23$.
The line $10x-23$ joins the parabola x^2+2 smoothly at $(5,27)$.

86. $f(x)=\begin{cases}3x+6a & \text{if } x<-3\\ 3ax-7b & \text{if } -3\le x\le 3\\ x-12b & \text{if } x>3\end{cases}$ ▷ $f(x)=\begin{cases}3x+12 & \text{if } x<-3\\ 6x+21 & \text{if } -3\le x\le 3\\ x+36 & \text{if } 6\end{cases}$



f is continuous on $(-\infty,-3)$, $(-3,3)$ and $(3,+\infty)$.

For f to be continuous at -3,
$$\lim_{x\to-3^-}f(x)=\lim_{x\to-3^+}f(x)$$
$$\lim_{x\to-3^-}(3x+6a)=\lim_{x\to-3^+}(3ax-7b)$$
$$-9+6a=-9a-7b$$
$$15a+7b=9$$

For f to be continuous at 3,
$$\lim_{x\to3^-}f(x)=\lim_{x\to3^+}f(x)$$
$$\lim_{x\to3^-}(3ax-7b)=\lim_{x\to3^+}(x-12b)$$
$$9a-7b=3-12b$$
$$9a+5b=3$$

Solving simultaneously, we get $a=2$ and $b=-3$.

87. $f(x)=\begin{cases}1 & \text{if } x \text{ is an integer}\\ 0 & \text{if } x \text{ is not an integer}\end{cases}$

(b) If a is an integer, then for all x in the open intervals $(a-1,a)$, $(a,a+1)$, $f(x)=0$. Thus $\lim\limits_{x\to a^-}f(x)=0$ and $\lim\limits_{x\to a^+}f(x)=0$.
Hence, $\lim\limits_{x\to a}f(x)=0$.

If a is not an integer, a is in the open interval $(k,k+1)$ for some integer k.
Then for all x in the open intervals (k,a) and $(a,k+1)$, $f(x)=0$.
Thus $\lim\limits_{x\to a^-}f(x)=0$ and $\lim\limits_{x\to a^+}f(x)=0$. Hence, $\lim\limits_{x\to a}f(x)=0$.
Therefore, $\lim\limits_{x\to a}f(x)$ exists for all values of a.

(c) f is discontinuous at every integer k because $\lim\limits_{x\to k}f(x)=0$, but $f(k)=1$.

However for any number a that is not an integer, $\lim_{x\to a} f(x) = 0$ and $f(a) = 0$.

Therefore f is continuous at all non-integers.

88. Give an example of a function for which $\lim_{x\to 0}|f(x)|$ exists but $\lim_{x\to 0} f(x)$ does not exist.

▷ Any f for which $\lim_{x\to 0^-} f(x) = -\lim_{x\to 0^+} f(x)$ will do, for then $\lim_{x\to 0^-}|f(x)| = \lim_{x\to 0^+}|f(x)|$.

A simple example is $f(x) = \operatorname{sgn} x$: $\lim_{x\to 0^-}\operatorname{sgn} x = \lim_{x\to 0^-} -1 = -1$ and $\lim_{x\to 0^+}\operatorname{sgn} x = \lim_{x\to 0^+} 1 = 1$.

In Exercises 89–92, find the largest interval or union of intervals on which the function is continuous.

89. (a) $f(x) = \sqrt{25 - x^2}$ ▷ The domain of f is $\{x \mid 25 - x^2 \geq 0\} = [-5, 5]$.

f is continuous on $(-5, 5)$. Because $\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} \sqrt{25 - x^2} = 0 = f(-5)$ and $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \sqrt{25 - x^2} = 0 = f(5)$, f is also continuous on $[-5, 5]$.

(b) $f(x) = \sqrt{x^2 - 25}$ ▷ The domain of f is $\{x \mid x^2 - 25 \geq 0\} = (-\infty, -5] \cup [5, +\infty) = \mathrm{I}$.

f is continuous on $(-\infty, -5) \cup (-5, +\infty)$. Because $\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} \sqrt{x^2 - 25} = 0 = f(-5)$ and $\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \sqrt{x^2 - 25} = 0 = f(5)$, f is also continuous on I.

90. (a) $f(x) = \dfrac{|x|+1}{|x|-1}$ ▷ The domain of f is $\{x \mid |x| - 1 \neq 0\} = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty) = \mathrm{I}$.

f is continuous on I. Since $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \dfrac{-x+1}{-x-1} = \dfrac{2}{0^+} = +\infty$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{x+1}{x-1} = \dfrac{2}{0^+} = +\infty$,
f is continuous only on I.

(b) $g(x) = \dfrac{\sqrt{9 - x^2}}{x - 2}$ ▷ The domain of g is $\{x \mid 9 - x^2 \geq 0\} - \{2\} = [-3, 2) \cup (2, 3] = \mathrm{I}$.

g is continuous on $(-3, 2) \cup (2, 3)$. Because $\lim_{x\to -3^+} g(x) = \lim_{x\to -3^+} \dfrac{\sqrt{9 - x^2}}{x - 2} = \dfrac{0}{-5} = 0 = g(-3)$,

$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \dfrac{\sqrt{9 - x^2}}{x - 2} = \dfrac{\sqrt{5}}{0^+} = +\infty$, and $\lim_{x\to 3^-} g(x) = \dfrac{\sqrt{9 - x^2}}{x - 2} = \dfrac{0}{1} = 0 = g(3)$, g is continuous on I.

91. (a) $f(x) = \dfrac{|x-2|}{x-2}$ ▷ The domain of f is $\{x \mid x - 2 \neq 0\} = (-\infty, 2) \cup (2, +\infty) = \mathrm{I}$.

Because $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{-(x-2)}{x-2} = \lim_{x\to 2^-}(-1) = -1$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \dfrac{x-2}{x-2} = \lim_{x\to 2^+} 1 = 1$,
f is continuous only on I.

(b) $g(x) = \dfrac{x}{x^2 - 4}$ ▷ The domain of g is $\{x \mid x^2 - 4 \neq 0\} = (-\infty, -2) \cup (-2, 2) \cup (2, +\infty) = \mathrm{I}$.

Because $\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+} \dfrac{x}{x^2 - 4} = \dfrac{-2}{0^-} = +\infty$ and $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \dfrac{x}{x^2 - 4} = \dfrac{2}{0^+} = +\infty$, g is continuous only on I.

92. $\mathrm{F}(x) = \begin{cases} x + 4 & \text{if } x < -4 \\ \sqrt{16 - x^2} & \text{if } -4 \leq x \leq 4 \\ 2 - x & \text{if } x > 4 \end{cases}$ ▷ The domain of F is all x. By Theorems 1.8.3 and 1.8.5(ii), F is continuous on $(-\infty, -4) \cup (-4, 4) \cup (4, +\infty)$. Because

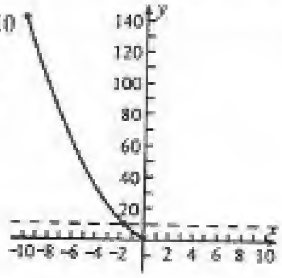
$\lim_{x\to -4^-} \mathrm{F}(x) = \lim_{x\to -4^-}(x + 4) = 0$ and $\lim_{x\to -4^+} \mathrm{F}(x) = \lim_{x\to -4^+} \sqrt{16 - x^2} = 0$
then F is continuous at -4. Because

$\lim_{x\to 4^-} \mathrm{F}(x) = \lim_{x\to 4^-} \sqrt{16 - x^2} = 0 = \mathrm{F}(4)$ and $\lim_{x\to 4^+} \mathrm{F}(x) = \lim_{x\to 4^+}(2 - x) = -2$
then F is continuous only from the left at 4. Therefore F is continuous on $(-\infty, 4] \cup (4, +\infty)$.

In Exercises 93–96, does the intermediate-value theorem hold for the function f, interval $[a,b]$ and constant k? If so, solve $f(c) = k$ graphically and analytically to 4 decimals and sketch the graph showing the point (c,k).

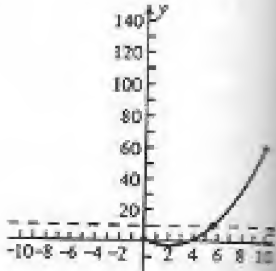
93. $f(x) = x^2 - 4x + 1$; $[a,b] = [-10,0]$; $k = 10$

▹ The interval from $f(-10) = 141$ to $f(0) = 1$ contains 10 and f is continuous on $[-10,0]$. Therefore, the intermediate-value theorem holds and there exists a number c between -10 and 0 such that $f(c) = 10$: $c^2 - 4c + 1 = 10$; $c^2 - 4c - 9 = 0$; $c = 2 \pm \sqrt{13}$ and $2 - \sqrt{13} \approx -1.6056$ is in $(-10,0)$.



94. $f(x) = x^2 - 4x + 1$; $[a,b] = [0,10]$; $k = 10$

▹ The interval from $f(0) = 1$ to $f(10) = 61$ contains 10 and f is continuous on $[0,10]$. Therefore, the intermediate-value theorem holds and there exists a number c between 0 and 10 such that $f(c) = 10$: $c^2 - 4c + 1 = 10$; $c^2 - 4c - 9 = 0$; $c = \frac{1}{2}(4 \pm \sqrt{52})$ and $\frac{1}{2}(4 + \sqrt{52}) \approx 5.6056$ is in $(0,10)$.

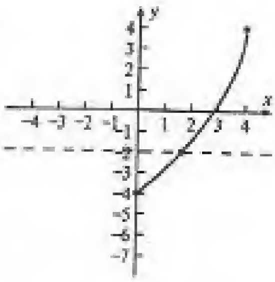


95. $f(x) = x - \sqrt{16 - x^2}$; $[a,b] = [0,4]$; $k = -2$

▹ The interval from $f(0) = -4$ to $f(4) = 4$ contains -2 and f is continuous on $[0,4]$. Therefore, the intermediate-value theorem holds and there exists a number c between 0 and 4 such that $f(c) = -2$:

$c - \sqrt{16 - c^2} = -2$; $c + 2 = \sqrt{16 - c^2}$;

$c^2 + 4c + 4 = 16 - c^2$; $2c^2 + 4c - 12 = 0$; $c = -1 \pm \sqrt{7}$ and $-1 + \sqrt{7} \approx 1.6458$ is in $(0,4)$.

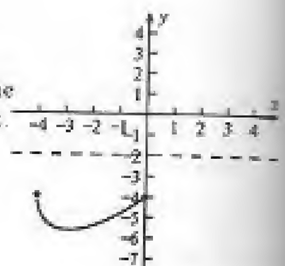


96. $f(x) = x - \sqrt{16 - x^2}$; $[a,b] = [-4,0]$; $k = -2$

▹ The interval from $f(-4) = -4$ to $f(0) = 4$ does not contain -2. Thus the hypothesis of the intermediate-value theorem is not satisfied and the theorem does not hold. The figure shows the graph of the function on $[-4,0]$ and the line $y = -2$. Because the line does not intersect the curve, there is no number c that satisfies the conclusion of the intermediate-value theorem. Try to solve $f(c) = -2$:

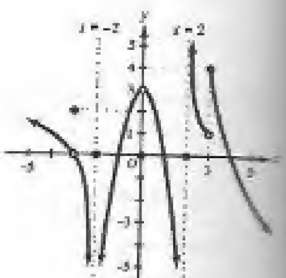
$c - \sqrt{16 - c^2} = -2$; $c + 2 = \sqrt{16 - c^2}$; $c^2 + 4c + 4 = 16 - c^2$; $2c^2 + 4c - 12 = 0$;

$c = -1 \pm \sqrt{7}$ and $-1 - \sqrt{7} \approx -3.6458$ is in $(-4,0)$. However, $c + 2$ cannot be negative, and so the solution is extraneous.

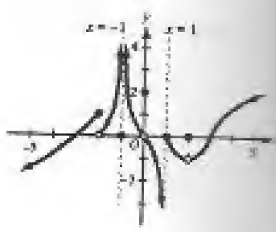


In Exercises 97 and 98, answer the questions from the graph of f in the figure.

97. (a) $\lim_{x\to -3} f(x) = 0$ (b) $\lim_{x\to -2} f(x) = -\infty$ (c) $\lim_{x\to 0} f(x) = 3$ (d) $\lim_{x\to 2^-} f(x) = -\infty$
(e) $\lim_{x\to 2^+} f(x) = +\infty$ (f) $\lim_{x\to 3^-} f(x) = 1$ (g) $\lim_{x\to 3^+} f(x) = 4$
(h) f is discontinuous at -3 (removable, define $f(-3) = 0$); -2 (essential); 0 (removable, define $f(0) = 3$); 2 (essential); 3 (essential)



98. (a) $\lim_{x\to -2^-} f(x) = 1$ (b) $\lim_{x\to -2^+} f(x) = 0$ (c) $\lim_{x\to -1} f(x) = +\infty$ (d) $\lim_{x\to 0} f(x) = 0$
(e) $\lim_{x\to 1^-} f(x) = -\infty$ (f) $\lim_{x\to 1^+} f(x) = 0$ (g) $\lim_{x\to 2} f(x) = -1$
(h) f is discontinuous at -2 (essential); -1 (essential); 0 (removable, define $f(0) = 0$); 1 (essential); 2 (removable, define $f(2) = -1$)

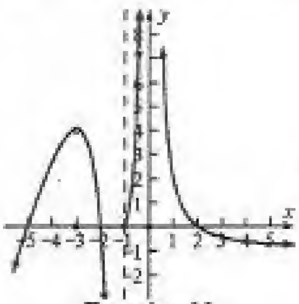


In Exercises 99–102, sketch the graph of a function satisfying the given conditions.

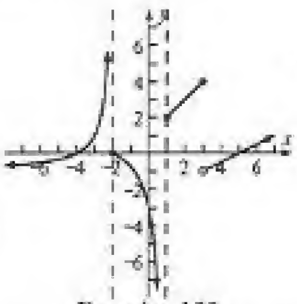
99. $-5, -3, -1, 0$ and 2 are the only zeros of f; $\lim_{x\to-3} f(x) = 4$; $\lim_{x\to-1^-} f(x) = +\infty$; $\lim_{x\to-1^+} f(x) = 0$; $\lim_{x\to0} f(x) = +\infty$; f is continuous on $(-\infty,-3)$, $(-3,-1)$, $(-1,0)$, $(0,+\infty)$

100. f is continuous on $(-\infty,-2)$, $[-2,1)$, $[1,3]$, and $(3,+\infty)$; $\lim_{x\to-4} f(x) = 0$; $\lim_{x\to-2^-} f(x) = +\infty$; $\lim_{x\to-2^+} f(x) = 0$; $\lim_{x\to0} f(x) = -3$; $\lim_{x\to1^-} f(x) = -\infty$; $\lim_{x\to1^+} f(x) = 2$; $\lim_{x\to3^-} f(x) = 4$; $\lim_{x\to3^+} f(x) = -1$; $\lim_{x\to5} f(x) = 0$.

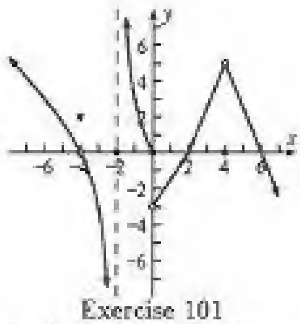
▷ Because f is continuous at -4 and $\lim_{x\to-4} f(x) = 0$ then $f(-4) = 0$, and the graph contains the point $(-4,0)$. Because $\lim_{x\to-2^-} f(x) = +\infty$, the line $x = -2$ is a vertical asymptote and the curve approaches the asymptote from the left in the upward direction. Because f is continuous from the right at -2 and $\lim_{x\to-2^+} f(x) = 0$, then $f(-2) = 0$, and the graph contains the point $(-2,0)$. The graph contains the point $(0,-3)$, because f is continuous at 0 and $\lim_{x\to0} f(x) = -3$. Because $\lim_{x\to1^-} f(x) = -\infty$, the line $x = 1$ is a vertical asymptote, and the curve approaches the asymptote from the left in the downward direction. Because f is continuous from the right at 1, and $\lim_{x\to1^+} f(x) = 2$, the graph contains the point $(1,2)$. Because f is continuous from the left at 3, and $\lim_{x\to3^-} f(x) = 4$, then the graph contains the point $(3,4)$. We are given that $\lim_{x\to3^+} f(x) = -1$. Because $\lim_{x\to3^-} f(x) = 4$, then f is discontinuous at 3, and there is a break in the graph at the point where $x = 3$. Because $\lim_{x\to3^+} f(x) = -1$, the point $(3,-1)$ is a limit point of the graph, but the point is not part of the graph; this is indicated by the open circle at the point $(3,-1)$. Finally, because $\lim_{x\to5} f(x) = 0$, and f is continuous at 5, the graph contains the point $(5,0)$.



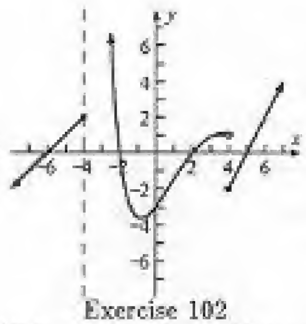
Exercise 99



Exercise 100



Exercise 101



Exercise 102

101. $f(-4) = 2$; $-2, 0, 2, 4$, and 5 are the only zeros of f; $\lim_{x\to-4} f(x) = 0$; $\lim_{x\to-2^-} f(x) = -\infty$; $\lim_{x\to-2^+} f(x) = +\infty$; $\lim_{x\to0^-} f(x) = 0$; $\lim_{x\to0^+} f(x) = -3$; $\lim_{x\to4} f(x) = 5$; f is continuous at all numbers except -4, -2, 0, and 4.

102. f is continuous on $(-\infty,-4]$, $(-4,4)$, and $[4,+\infty)$; $\lim_{x\to-6} f(x) = 0$; $\lim_{x\to-4^-} f(x) = 2$; $\lim_{x\to-4^+} f(x) = +\infty$; $\lim_{x\to-2} f(x) = 0$; $\lim_{x\to0} f(x) = -3$; $\lim_{x\to2} f(x) = 0$; $\lim_{x\to4^-} f(x) = 1$; $\lim_{x\to4^+} f(x) = -2$; $\lim_{x\to5} f(x) = 0$

103. x in. squares are cut from the corners of a 14 in × 18 in sheet and the sides turned up. (a) Find the volume $V(x)$ in^3. (b) Find dom(V). (c) Prove V is continuous on its domain. (d) Maximize the volume graphically.

▷ $V = \ell wh = (14-2x)(18-2x)x$, $0 \le x \le 7$. (c) V is a polynomial. (d) When $x \approx 2.6049$, $V_{max} = 292.86$ in^3.

104. An open box having a square base is to have a volume of 4000 in^3. (a) Find a mathematical model expressing the total surface area of the box as a function of the length of a side of the square base. (b) What is the domain of your function? (c) Prove that the function is continuous on its domain. (d) On your graphics calculator determine, to the nearest inch, the dimensions of the box that can be constructed with the least amount of material.

▷ (a) The base has side x in., total surface S in^2. Volume $= \ell wh = x^2h = 4000$, $h = \dfrac{4000}{x^2}$.

$S =$ area of bottom $+ 4\cdot$ area of a side $= x^2 + 4xh = x^2 + 4x\cdot\dfrac{4000}{x^2} = x^2 + \dfrac{16{,}000}{x}$ (b) Dom S is $(0,+\infty)$

(c) A rational function is continuous on its domain. (d) Because $S = x^2 + \dfrac{8{,}000}{x} + \dfrac{8{,}000}{x}$ has a constant product

$8{,}000^2$, the sum S is least when the terms are equal: $x^2 = \frac{8{,}000}{x}$, $x^3 = 8{,}000$, $x = 20$. The dimensions a 20 in × 20 in × 10 in.

105. A sign with margins of 4 m at the top and bottom and 2 m at the sides is to contain 50 m^2 of print. (a) Fin the total area of the sign, A(x) m^2, when the width of the printed region is x in. (b) Find the domain D of A (c) Prove that A is continuous on its domain (d) Determine to the nearest meter the size of the smallest sign.

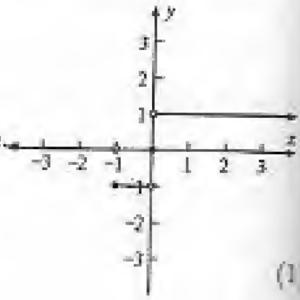
▷ (a) The length of the printed region is $\frac{50}{x}$ m. $A(x) = (x+4)(\frac{50}{x}+8) = 82 + (8x + \frac{200}{x})$ (product = 1600)

(b) D: $x > 0$ (c) A is a rational function (d) $A_{min} = 54$ when $8x = \frac{200}{x}$, $x^2 = 25$, $x = 5$. $x + 4 = 5 + 4 = 9$, $\frac{50}{x} + 8 = \frac{50}{5} + 8 = 18$. The smallest sign is 9 m wide and 18 m long.

106. The growth rate f fish/week is jointly proportional to the number x of fish and the number $10{,}000 - x$ o capacity. $f(x) = kx(10{,}000 - x)$, $f(1000) = 90 = 1000k \cdot 9000$, $k = 1/100{,}000$.
$f(x) = x(10{,}000 - x)/100{,}000$, $0 \le x \le 10{,}000$. A polynomial is continuous everywhere.
$f(x) = k(-x^2 + 10{,}000x)$ is maximum when $x = -10{,}000/-2 = 5000$.

107. $F(x) = \operatorname{sgn} x \cdot U(x+1)$

$$= \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \cdot \begin{cases} 0 & \text{if } x+1 < 0 \\ 1 & \text{if } x+1 \ge 0 \end{cases} = \begin{cases} -1 \cdot 0 = 0 & \text{if } x < -1 \\ -1 \cdot 1 = -1 & \text{if } -1 \le x < 0 \\ 0 \cdot 1 = 0 & \text{if } x = 0 \\ 1 \cdot 1 = 1 & \text{if } x > 0 \end{cases}$$



F is discontinuous at -1 and 0 because the left- and right-hand limits disagree there.

In Exercises 108 and 109, use the squeeze theorem to find the limit.

108. $\lim_{x\to 4} g(x)$, if $|g(x)+5| < 3(4-x)^2$ for all x.

▷ Because $0 \le |g(x)+5| < 3(4-x)^2$, then $-3(4-x)^2 < g(x) + 5 < 3(4-x)^2$ (1)

Because $\lim_{x\to 4} 3(4-x)^2 = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to 4} g(x) + 5 = 0$. Therefore $\lim_{x\to 4} g(x) = -5$.

109. If $x \ne 1$ then $-1 \le \sin(1/\sqrt[3]{x-1}) \le 1$; so $-(x-1)^2 \le (x-1)^2 \sin(1/x) \le (x-1)^2$ (1)

Because $\lim_{x\to 1}(x-1)^2 = 0$, it follows from (1) and the squeeze theorem that $\lim_{x\to 1} (x-1)^2 \sin(1/\sqrt[3]{x-1}) = 0$.

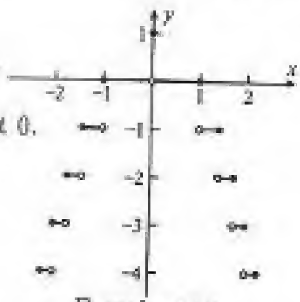
110. Sketch the graph of $f(x) = [\![1 - x^2]\!]$ for x in $[-2, 2]$.

▷ Let n be a negative integer. $[\![1-x^2]\!] = n$ when
$n \le 1 - x^2 < n+1$, $n - 1 \le -x^2 < n$, $-(n-1) \ge x^2 > -n$
$x \in [-\sqrt{-(n-1)}, -\sqrt{-n}) \cup (\sqrt{-n}, \sqrt{-(n-1)}]$
$\lim_{x\to 0} f(x) = \lim_{x\to 0} 0 = 0 \ne f(0) = 1$ so f is not continuous at 0.

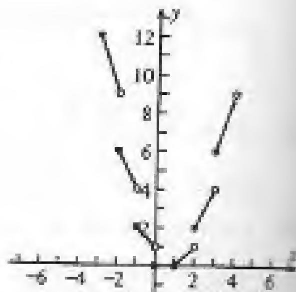
111. Sketch the graph $g(x) = (x-1)[\![x]\!]$ for x in $[-2, 2]$.

▷ $(x-1)[\![x]\!] = n(x-1)$ when $n \le x < n+1$.
$\lim_{x\to 1} g(x) = 0$ and g is continuous at 1.

$\lim_{x\to n} g(x)$ does not exist if n is any other integer.



Exercise 110



Exercise 111

112. Suppose $f(x) = g(x)$ for all values of x except a. (a) Prove that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if the limits exist.

(b) Prove that if $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} f(x)$ does not exist.

▷ (a) Suppose $\lim_{x\to a} f(x) = L$. Then

For every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ when $0 < |x - a| < \delta$.
Because $f(x) = g(x)$ when $x \ne a$, this is equivalent to
For every $\epsilon > 0$ there is a $\delta > 0$ such that $|g(x) - L| < \epsilon$ when $0 < |x - a| < \delta$.
$\Leftrightarrow \lim_{x\to a} g(x) = L$

(b) The contrapositive of (b) is "If $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} g(x)$" which is part of what was proved in (a).

113. (a) We are given that $\lim_{h\to 0} f(x+h) = f(x)$. Let $h = -k$. Then $\lim_{-k\to 0} f(x-k) = f(x)$.
But $-k\to 0$ is equivalent to $k\to 0$. Therefore, $\lim_{k\to 0} f(x-k) = f(x)$.
(b) Let $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
Then $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} 1 = 1$; $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} 1 = 1$; but $f(0) = 0$. Thus $\lim_{h\to 0} f(0+h) \neq f(0)$.

114. If t is any real number, then $f(t)$ exists by hypothesis. In order to prove that f is continuous at t we must show that $\lim_{x\to t} f(x) = f(t)$.
Because $f(a+b) = f(a) + f(b)$, it follows that $f(x) = f[t + (x-t)] = f(t) + f(x-t)$. Thus, if the limits exist,

$$\lim_{x\to t} f(x) = \lim_{x\to t} f(t) + \lim_{x\to t} f(x-t)$$

$$\lim_{x\to t} f(x) = f(t) + \lim_{x\to t} f(x-t) \tag{1}$$

Because f is continuous at 0, it follows from Theorem 1.9.1 that

$$\lim_{x\to t} f(x-t) = f[\lim_{x\to t}(x-t)] = f(0)$$

substituting into (1) we obtain

$$\lim_{x\to t} f(x) = f(t) + f(0) \tag{2}$$

To find $f(0)$ we apply $f(a+b) = f(a) + f(b)$ with $a = b = 0$. Then we have

$$f(0) = f(0) + f(0);\ f(0) = 0$$

Substituting into (2) we get $\lim_{x\to t} f(x) = f(t)$ which proves that f is continuous at t.

115. If the domain of f is the set of all real numbers, f is continuous at 0 and $f(a+b) = f(a)f(b)$ for all a and b, prove that f is continuous at every number.

▹ Because f is continuous at 0, $f(0) = \lim_{x\to 0} f(0+x) = \lim_{x\to 0} f(0)f(x) = f(0)f(0)$. Therefore, $f(0) = 0$ or $f(0) = 1$.
Suppose $f(0) = 0$. Then $f(x) = f(x+0) = f(x)f(0) = f(x)\cdot 0 = 0$. Thus f is a constant function and f is continuous.
Suppose $f(0) = 1$. Let a be any number. Then $\lim_{x\to 0} f(a+x) = \lim_{x\to 0}[f(a)f(x)] = f(a)f(0) = f(a)\cdot 1 = f(a)$.
Thus f is continuous at a.

116. Suppose the function f is defined on the open interval $(0,1)$ and $f(x) = \dfrac{\sin \pi x}{x(x-1)}$.
Define f at 0 and 1 so that f is continuous on the closed interval $[0,1]$.

▹ We must define $f(0)$ and $f(1)$ so that

$$\lim_{x\to 0^+} f(x) = f(0) \quad \text{and} \quad \lim_{x\to 0^-} f(x) = f(1)$$

Now

$$\begin{aligned}
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{\sin \pi x}{x} \cdot \frac{1}{x-1} \\
&= \pi \lim_{x\to 0^+} \frac{\sin \pi x}{\pi x} \cdot \lim_{x\to 0^+} \frac{1}{x-1} \\
&= \pi \lim_{\pi x\to 0} \frac{\sin \pi x}{\pi x}(-1) \\
&= -\pi && \text{(By Theorem 1.10.2)}
\end{aligned}$$

Thus, we define $f(0) = -\pi$. Next, we let $t = x - 1$. Then

$$\begin{aligned}
\lim_{x\to 1^-} f(t) &= \lim_{t\to 0^-} \frac{\sin(\pi t + \pi)}{(t+1)t} \\
&= \lim_{t\to 0^-} \frac{-\sin \pi t}{t} \cdot \lim_{t\to 0^-} \frac{1}{t+1} \\
&= -\pi \lim_{\pi t\to 0^-} \frac{\sin \pi t}{\pi t} \cdot 1 \\
&= -\pi \cdot 1
\end{aligned}$$

Thus we define $f(1) = -\pi$.

TWO

THE DERIVATIVE AND DIFFERENTIATION

2.1 THE TANGENT LINE AND THE DERIVATIVE

2.1.1 Definition Suppose the function f is continuous at x_1. The *tangent line* to the graph of f at the point $P(x_1, f(x_1))$ is

(i) the line through P having slope $m(x_1)$, given by

$$m(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{1}$$

if this limit exists

(ii) the line $x = x_1$ if

$$\lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x} \text{ is } +\infty \text{ or } -\infty \text{ and } \lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x} \text{ is } +\infty \text{ or } -\infty$$

If neither (i) nor (ii) of Definition 2.1.1 holds, then there is no tangent line to the graph of f at the point $P(x_1, f(x_1))$.

If we want to find the slope of the tangent line to the curve at more than one point, we first find the limit and then make the indicated replacements for x_1. However, if we are interested in finding the tangent line at only one point and if the coordinates of the point of tangency are known, it is easier to first make the indicated replacement and then find the limit.

2.1.2 Definition The *normal line* to a graph at a given point is the line perpendicular to the tangent line at that point.

Formula (1) for the slope of the tangent line is a special case of the formula for the derivative of a function. Following is one of the most important definitions in the calculus.

2.1.3 Definition The *derivative* of the function f is that function, denoted by f', such that its value at a number x in the domain of f is given by the equivalent formulas

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \qquad \text{(3) and (3')}$$

if this limit exists, where $z = x + \Delta x$. (f' is read "f prime," and $f'(x)$ is read "f prime of x.") We also use the symbols $\frac{d}{dx}f(x)$ and $D_x f(x)$ to represent $f'(x)$. If $y = f(x)$, then the symbols $D_x y$, y', dy/dx are sometimes used to represent $f'(x)$.

If Δy is defined by $\Delta y = f(x + \Delta x) - f(x)$ then $D_x y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

If x_1 is a particular number in the domain of f, then to find $f'(x_1)$ we may use either of the equivalent formulas,

$$f'(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \quad (4) \quad \text{or} \quad f'(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} \qquad (7)$$

To use formulas (3) and (4), note that
$(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$ and $(x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$

To use formulas (3') and (7), note that
$\frac{x^2 - x_1^2}{x - x_1} = \frac{(x - x_1)(x + x_1)}{x - x_1} = x + x_1$ and $\frac{x^3 - x_1^3}{x - x_1} = \frac{(x - x_1)(x^2 + xx_1 + x_1^2)}{x - x_1} = x^2 + xx_1 + x_1^2$

The slope of the tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$ is precisely the derivative of f evaluated at x_1. Thus formulas (3), (3'), (4), and (7) are interchangeable.

The equation of a (tangent) line with slope m and passing through (a, b) is $y = m(x - a) + b$.
A normal line has slope $-1/f'(x_1)$.

Exercises 2.1

In Exercises 1–6, find an equation of the tangent line at the point. Sketch the graph and a segment of the tangent.

1. $y = 9 - x^2$; $(2, 5)$ ▹ Let $f(x) = 9 - x^2$.

$$m(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{(9 - x^2) - (9 - 2^2)}{x - 2} = \lim_{x\to 2} \frac{-(x^2 - 2^2)}{x - 2} = \lim_{x\to 2} -(x + 2) = -4.$$

An equation of the tangent line is $y = -4(x - 2) + 5$; $y = -4x + 13$.

2. $y = x^2 + 4$; $(-1, 5)$ ▹ Let $f(x) = x^2 + 4$.

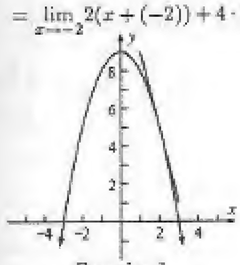
$$m(-1) = \lim_{x\to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x\to -1} \frac{x^2 + 4 - [(-1)^2 + 4]}{x - (-1)} = \lim_{x\to -1} \frac{x^2 - (-1)^2}{x - (-1)} = \lim_{x\to -1} (x + (-1)) = -2$$

An equation of the tangent line is $y = -2(x - (-1)) + 5$; $y = -2x + 3$.

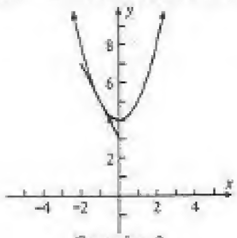
3. $y = 2x^2 + 4x$; $(-2, 0)$ ▹ Let $f(x) = 2x^2 + 4x$.

$$m(-2) = \lim_{x\to -2} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x\to -2} \frac{2x^2 + 4x - [2(-2)^2 + 4(-2)]}{x - (-2)} = \lim_{x\to -2} 2\frac{x^2 - (-2)^2}{x - (-2)} + 4\frac{x - (-2)}{x - (-2)}$$

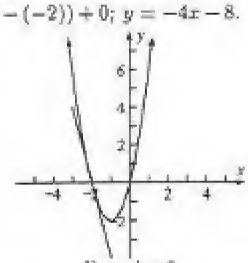
$= \lim_{x\to -2} 2(x + (-2)) + 4 \cdot 1 = 2(-4) + 4 = -4$. The tangent line is $y = -4(x - (-2)) + 0$; $y = -4x - 8$.



Exercise 1



Exercise 2



Exercise 3

4. $y = x^2 - 6x + 9$; $(3, 0)$ ▹ Let $f(x) = x^2 - 6x + 9$. Use formula (4) with $x_1 = 3$.

$$m(3) = \lim_{\Delta x\to 0} \frac{f(3 + \Delta x) - f(3)}{\Delta x} = \lim_{\Delta x\to 0} \frac{[(3 + \Delta x)^2 - 6(3 + \Delta x) + 9] - 0}{\Delta x} = \lim_{\Delta x\to 0} \frac{9 + 6\Delta x + (\Delta x)^2 - 18 - 6\Delta x + 9}{\Delta x}$$

$= \lim_{\Delta x\to 0} \frac{(\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0} \Delta x = 0$. An equation of the tangent line is $y = 0(x - 3) + 0$; $y = 0$.

5. $y = x^3 + 3$; $(1, 4)$ ▹ Let $f(x) = x^3 + 1$.

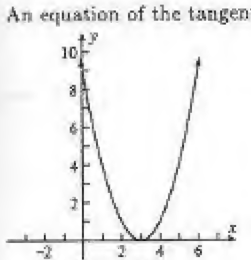
$$m(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{(x^3 + 3) - (1^3 + 3)}{x - 1} = \lim_{x\to 1} \frac{x^3 - 1^3}{x - 1} = \lim_{x\to 1} (x^2 + x + 1) = 3$$

An equation of the tangent line is $y = 3(x - 1) + 4$; $y = 3x + 1$.

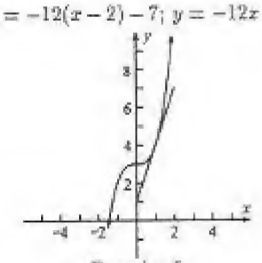
6. $y = 1 - x^3$; $(2, -7)$ ▹ Let $f(x) = 1 - x^3$.

$$m(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{1 - x^3 - (1 - 2^3)}{x - 2} = \lim_{x\to 2} \frac{-(x^3 - 2^3)}{x - 2} = \lim_{x\to 2} -(x^2 + 2x + 4) = -12$$

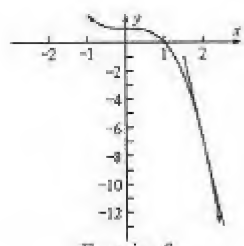
An equation of the tangent line is $y = -12(x - 2) - 7$; $y = -12x + 17$



Exercise 4



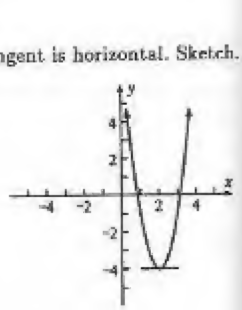
Exercise 5



Exercise 6

In Exercises 7–10, (a) find the slope of the tangent at $(x_1, f(x_1))$. (b) Find where the tangent is horizontal. Sketch.

7. $f(x) = 3x^2 - 12x + 8$.



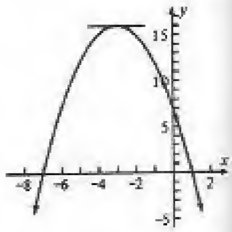
▷ (a) $m(x_1) = \lim\limits_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{[3(x_1 + \Delta x)^2 - 12(x_1 + \Delta x) + 8] - (3{x_1}^2 - 12x_1 + 8)}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{6x_1\Delta x + 3(\Delta x)^2 - 12\Delta x}{\Delta x} = 6x_1 - 12 = 6(x_1 - 2)$

(b) $m(x_1) = 0$ when $x_1 = 2$ and $f(2) = -4$;
so the graph has a horizontal tangent at $(2, -4)$.

8. $f(x) = 7 - 6x - x^2$



▷ (a) Applying formula (1) we have

$m(x_1) = \lim\limits_{\Delta x \to 0} \dfrac{[7 - 6(x_1 + \Delta x) - (x_1 + \Delta x)^2] - [7 - 6x_1 - {x_1}^2]}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{7 - 6x_1 - 6\Delta x - {x_1}^2 - 2x_1\Delta x - (\Delta x)^2 - 7 + 6x_1 + {x_1}^2}{\Delta x}$

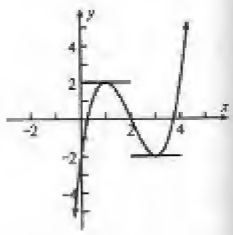
$= \lim\limits_{\Delta x \to 0} \dfrac{-6\Delta x - 2x_1\Delta x - (\Delta x)^2}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} (-6 - 2x_1 - \Delta x)$

$= -6 - 2x_1 = -2(x_1 + 3)$

(b) $m(x_1) = 0$ when $x_1 = -3$ and $f(-3) = 16$ so the graph has a horizontal tangent at $(-3, 16)$. Other points on the graph are $(-7, 0)$, $(-5, 12)$, $(-3, 16)$, $(-1, 12)$ and $(1, 0)$.

9. $f(x) = x^3 - 6x^2 + 9x - 2$



▷ (a) $m(x_1) = \lim\limits_{x \to x_1} \dfrac{f(x) - f(x_1)}{x - x_1} = \lim\limits_{x \to x_1} \dfrac{(x^3 - 6x^2 + 9x - 2) - ({x_1}^3 - 6{x_1}^2 + 9x_1 - 2)}{x - x_1}$

$= \lim\limits_{x \to x_1} \left(\dfrac{x^3 - {x_1}^3}{x - x_1} - 6\dfrac{x^2 - {x_1}^2}{x - x_1} + 9\dfrac{x - x_1}{x - x_1}\right) = \lim\limits_{x \to x_1} [(x^2 + x_1x + {x_1}^2) - 6(x + x_1) + 9]$

$= 3{x_1}^2 - 12x_1 + 9 = 3({x_1}^2 - 4x_1 + 3) = 3(x_1 - 1)(x_1 - 3)$

(b) $m(x_1) = 0$ when $x_1 = 1$, $f(1) = 2$ and $x_1 = 3$, $f(3) = -2$
so the graph has horizontal tangents at $(1, 2)$ and $(3, -2)$.

10. $f(x) = 2x^3 - 3x^2$

▷ $m(x_1) = \lim\limits_{x \to x_1} \dfrac{f(x) - f(x_1)}{x - x_1} = \lim\limits_{x \to x_1} \dfrac{(2x^3 - 3x^2) - (2{x_1}^3 - 3{x_1}^2)}{x - x_1}$

$= \lim\limits_{x \to x_1} \left(2\dfrac{x^3 - {x_1}^3}{x - x_1} - 3\dfrac{x^2 - {x_1}^2}{x - x_1}\right) = \lim\limits_{x \to x_1} [2(x^2 + x_1x + {x_1}^2) - 3(x + x_1)]$

$= 6{x_1}^2 - 6x_1 = 6x_1(x_1 - 1)$

(b) $m(x_1) = 0$ when $x_1 = 0$, $f(0) = 0$ and $x_1 = 1$, $f(1) = -1$
so the graph has horizontal tangents at $(0, 0)$ and $(1, -1)$.

In Exercises 11–16, find equations of the tangent and normal lines at the point. Plot the graph with these lines.

11. $y = \sqrt{x + 1}$; $(3, 2)$

▷ $m = \lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3} = \lim\limits_{x \to 3} \dfrac{\sqrt{x + 1} - 2}{(x + 1) - 4} = \lim\limits_{x \to 3} \dfrac{\sqrt{x + 1} - 2}{(\sqrt{x + 1})^2 - 2^2} = \lim\limits_{x \to 3} \dfrac{1}{\sqrt{x + 1} + 2} = \dfrac{1}{4}$

The tangent line has equation $y = \frac{1}{4}(x - 3) + 2 = \frac{1}{4}x + \frac{5}{4}$.

The normal line has slope -4 and equation $y = -4(x - 3) + 2 = -4x + 14$.

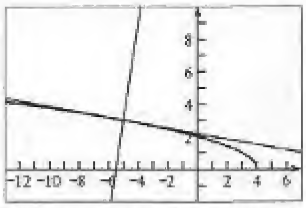
12. $y = \sqrt{4-x}$; $(-5, 3)$

▹ To find the slope of the tangent line to the curve at $(-5,3)$, use formula (7) with $f(x) = \sqrt{4-x}$ and $x_1 = -5$.

$$m(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1}$$

$$m(-5) = \lim_{x \to -5} \frac{\sqrt{4-x} - 3}{x - (-5)} = \lim_{x \to -5} - \frac{\sqrt{4-x} - 3}{(4-x) - 9} = \lim_{x \to -5} - \frac{\sqrt{4-x} - 3}{(\sqrt{4-x})^2 - 3^2}$$

$$= \lim_{x \to -5} - \frac{1}{\sqrt{4-x} + 3} = -\frac{1}{6}$$



Use the point-slope form with $m = -\frac{1}{6}$ to find an equation of the tangent line to the curve at the point $(-5,3)$.

$$y = -\tfrac{1}{6}(x - (-5)) + 3;\ y = -\tfrac{1}{6}x + \tfrac{13}{6}$$

Use the point-slope form with $m = 6$ to find an equation of the normal line to the curve at the point $(-5, 3)$.

$$y = 6(x - (-5)) + 3;\ y = 6x + 33$$

13. $y = 2x - x^3$; $(-2, 4)$ ▹ Let $f(x) = 2x - x^3$

$$m = \lim_{x \to -2} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2} \frac{2x - x^3 - (2(-2) - (-2)^3)}{x + 2} = \lim_{x \to -2}\left(2\frac{x+2}{x+2} - \frac{x^3 + 2^3}{x+2}\right)$$

$= \lim_{x \to -2}[2 - (x^2 - 2x + 4)] = -25$. The tangent line has equation $y = -10(x+2) + 4 = -10x + 16$

The normal line has slope $\frac{1}{10}$ and equation $y = \frac{1}{10}(x+2) + 4 = \frac{1}{10}x + \frac{21}{5}$.

14. $y = x^4 - 4x$; $(0, 0)$ ▹ Let $f(x) = x^4 - 4x$.

$$m(0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^4 - 4\Delta x}{\Delta x} = \lim_{\Delta x \to 0} [(\Delta x)^3 - 4] = -4.$$

The tangent line has equation $y = -4x$; $4x + y = 0$. The normal line has equation $y = \frac{1}{4}x$; $x - 4y = 0$.

15. $y = 4/x^2$; $(2, 1)$ ▹ Let $f(x) = 4/x^2$.

$$m(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{4}{x^2} - \frac{4}{2^2}}{x - 2} \cdot \frac{-2^2x^2}{-2^2x^2} = \lim_{x \to 2} \frac{4}{-2^2x^2} \cdot \frac{x^2 - 2^2}{x - 2} = \lim_{x \to 2} \frac{-1}{x^2} \cdot (x + 2) = -1$$

The tangent line has equation $y = -(x - 2) + 1 = -x + 3$.

The normal line has slope 1 and equation $y = 1(x - 2) + 1 = x - 1$.

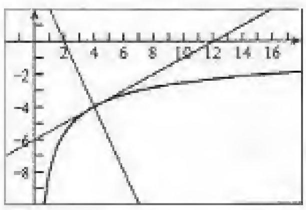
16. $y = -\dfrac{8}{\sqrt{x}}$; $(4, 4)$

▹ To find the slope of the tangent line to the curve at $(4, -4)$, use formula (7) with $f(x) = -8/\sqrt{x}$ and $x_1 = 4$.

$$m(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1}$$

$$m(4) = \lim_{x \to 4} \frac{-\frac{8}{\sqrt{x}} - (-4)}{x - 4} = \lim_{x \to 4} \frac{4\left(1 - \frac{2}{\sqrt{x}}\right)\sqrt{x}}{(x - 4)\sqrt{x}} = \lim_{x \to 4} \frac{4(\sqrt{x} - 2)}{(x - 4)\sqrt{x}} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2}$$

$$= \lim_{x \to 4} \frac{4(x - 4)}{(x - 4)\sqrt{x}(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{4}{\sqrt{x}(\sqrt{x} + 2)} = \frac{4}{\sqrt{4}(\sqrt{4} + 2)} = \frac{1}{2}$$



Use the point-slope form with $m = \frac{1}{2}$ to find an equation of the tangent line to the curve at the point $(4, -4)$.

$$y = \tfrac{1}{2}(x - 4) + (-4);\ y = \tfrac{1}{2}x - 6$$

Use the point-slope form with $m = -2$ to find an equation of the normal line to the curve at the point $(4, -4)$.

$$y = -2(x - 4) + (-4);\ y = -2x + 4$$

In Exercises 17–20, (a) Tabulate $[f(2+\Delta x)-f(2)]/\Delta x$ when $\Delta x = 0.10$ to 0.01 step -0.01 and -0.10 to -0.01 step 0.01 and guess the limit. (b) Find $f'(2)$ using formula (4). (c) Tabulate $[f(x)-f(2)]/(x-2)$ when $x = 2.10$ to 2.01 step -0.01 and 1.90 to 1.99 step 0.01 and guess the limit. (d) Find $f'(2)$ using formula (7).

▹ The symmetric difference quotient of §2.3 is included for comparison. It is exact for quadratic functions.

17. $f(x) = 3x^2 - 7x$

▹ Tables for (a) and (c). The limit appears to be 5.

Δx	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
$2+\Delta x$	2.10	2.09	2.08	2.07	2.06	2.05	2.04	2.03	2.02	2.01
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x} = \dfrac{f(x)-f(2)}{x-2}$	5.30	5.27	5.24	5.21	5.18	5.15	5.12	5.09	5.06	5.03
$[f(2+\Delta x)-f(2-\Delta x)]/2\Delta x$	5	5	5	5	5	5	5	5	5	5
Δx	−.10	−.09	−.08	−.07	−.06	−.05	−.04	−.03	−.02	−.01
$2+\Delta x$	1.90	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x} = \dfrac{f(x)-f(2)}{x-2}$	4.70	4.73	4.76	4.79	4.82	4.85	4.88	4.91	4.94	4.97
$[f(2+\Delta x)-f(2-\Delta x)]/2\Delta x$	5	5	5	5	5	5	5	5	5	5

(b) $\displaystyle\lim_{\Delta x\to 0}\frac{[3(2+\Delta x)^2-7(2+\Delta x)]-(3\cdot 2^2-7\cdot 2)}{\Delta x}$

$\displaystyle = \lim_{\Delta x\to 0}\frac{3(2^2+4\Delta x+\Delta x^2)-7(2+\Delta x)-(3\cdot 2^2-7\cdot 2)}{\Delta x}$

$\displaystyle = \lim_{\Delta x\to 0}\frac{12\Delta x+3\Delta x^2-7\Delta x}{\Delta x} = \lim_{\Delta x\to 0}(5-3\Delta x^2) = 5$

(d) $\displaystyle\lim_{x\to 2}\frac{(3x^2-7x)-(3\cdot 2^2-7\cdot 2)}{x-2}$

$\displaystyle = \lim_{x\to 2}\Big(3\frac{x^2-2^2}{x-2}-7\frac{x-2}{x-2}\Big)$

$\displaystyle = \lim_{x\to 2}[3(x+2)-7] = 5$

18. $f(x) = x^3$

▹ Tables for (a) and (c). The limit appears to be 12.

Δx	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
$2+\Delta x$	2.10	2.09	2.08	2.07	2.06	2.05	2.04	2.03	2.02	2.01
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x} = \dfrac{f(x)-f(2)}{x-2}$	12.61	12.55	12.49	12.45	12.36	12.30	12.24	12.18	12.12	12.06
$\dfrac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	12.0100	12.0081	12.0064	12.0049	12.0036	12.0025	12.0016	12.0009	12.0004	12.0001
Δx	−.10	−.09	−.08	−.07	−.06	−.05	−.04	−.03	−.02	−.01
$2+\Delta x$	1.90	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x} = \dfrac{f(x)-f(2)}{x-2}$	11.41	11.47	11.53	11.58	11.64	11.70	11.76	11.82	11.88	11.94
$\dfrac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	12.0100	12.0081	12.0064	12.0049	12.0036	12.0025	12.0016	12.0009	12.0004	12.0001

(b) $\displaystyle\lim_{\Delta x\to 0}\frac{(2+\Delta x)^3-2^3}{\Delta x}$

$\displaystyle = \lim_{\Delta x\to 0}\frac{2^3+3\cdot 2^2\Delta x+3\cdot 2\Delta x^2+\Delta x^3-2^2}{\Delta x}$

$\displaystyle = \lim_{\Delta x\to 0}\frac{12\Delta x+6\Delta x^2+\Delta x^3}{\Delta x} = \lim_{\Delta x\to 0}(12+6\Delta x+\Delta x^2) = 12$

(d) $\displaystyle\lim_{x\to 2}\frac{x^3-2^3}{x-2}$

$\displaystyle = \lim_{x\to 2}(x^2+2x+2^2) = 12$

19. $f(x) = \sqrt{6-x}$

▹ Tables for (a) and (c). The limit appears to be $-\frac{1}{4}$.

Δx	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
$2+\Delta x$	2.10	2.09	2.08	2.07	2.06	2.05	2.04	2.03	2.02	2.01
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x} = \dfrac{f(x)-f(2)}{x-2}$	−.2516	−.2514	−.2513	−.2511	−.2509	−.2508	−.2506	−.2505	−.2503	−.2502
$\dfrac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	−.250019	−.250016	−.250012	−.250009	−.250006	−.250005	−.250004	−.250002	−.250001	−.250000
Δx	−.10	−.09	−.08	−.07	−.06	−.05	−.04	−.03	−.02	−.01
$2+\Delta x$	1.90	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x} = \dfrac{f(x)-f(2)}{x-2}$	−.2485	−.2486	−.2488	−.2489	−.2491	−.2492	−.2494	−.2495	−.2497	−.2498
$\dfrac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	−.250019	−.250016	−.250012	−.250009	−.250006	−.250005	−.250004	−.250002	−.250001	−.250000

(b) $\lim\limits_{\Delta x \to 0} \dfrac{\sqrt{6-(2+\Delta x)} - \sqrt{6-2}}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{\sqrt{4-\Delta x} - 2}{\Delta x} \cdot \dfrac{\sqrt{4+\Delta x}+2}{\sqrt{4-\Delta x}+2}$

$= \lim\limits_{\Delta x \to 0} \dfrac{(4-\Delta x)-4}{\Delta x(\sqrt{4-\Delta x}+2)} = \lim\limits_{\Delta x \to 0} \dfrac{-1}{\sqrt{4-\Delta x}+2} = -\frac{1}{4}$

(d) $\lim\limits_{x \to 2} \dfrac{\sqrt{6-x} - \sqrt{6-2}}{x-2}$

$= \lim\limits_{x \to 2} - \dfrac{\sqrt{6-x} - \sqrt{4}}{(6-x)-4}$

$= \lim\limits_{x \to 2} \dfrac{-1}{\sqrt{6-x}+2} = -\frac{1}{4}$

20. $f(x) = 1/(4-x)$ ▷ Tables for (a) and (c). The limit appears to be $\frac{1}{4}$.

Δx	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
$2+\Delta x$	2.10	2.09	2.08	2.07	2.06	2.05	2.04	2.03	2.02	2.01
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x} = \dfrac{f(x)-f(2)}{x-2}$	.2632	.2618	.2604	.2591	.2577	.2564	.2551	.2538	.2525	.2513
$\dfrac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	.250627	.250507	.250401	.250307	.250225	.250156	.250100	.250056	.250025	.250006

Δx	−.10	−.09	−.08	−.07	−.06	−.05	−.04	−.03	−.02	−.01
$2+\Delta x$	1.90	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99
$\dfrac{f(2+\Delta x)-f(2)}{\Delta x} = \dfrac{f(x)-f(2)}{x-2}$	.2381	.2392	.2404	.2415	.2427	.2439	.2451	.2463	.2475	.2488
$\dfrac{f(2+\Delta x)-f(2-\Delta x)}{2\Delta x}$	.250627	.250507	.250401	.250307	.250225	.250156	.250100	.250056	.250025	.250006

(b) $\lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x}\left(\dfrac{1}{4-(2+\Delta x)} - \dfrac{1}{4-2}\right)$

$= \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x}\left(\dfrac{1}{2+\Delta x} - \dfrac{1}{2}\right)$

$= \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x} \cdot \dfrac{2-(2+\Delta x)}{(2+\Delta x)2} = \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x} \cdot \dfrac{\Delta x}{(2+\Delta x)2}$

$= \lim\limits_{\Delta x \to 0} \dfrac{1}{(2+\Delta x)2} = \frac{1}{4}$

(d) $\lim\limits_{x \to 2} \dfrac{1}{x-2}\left(\dfrac{1}{4-x} - \dfrac{1}{4-2}\right)$

$= \lim\limits_{x \to 2} \dfrac{1}{x-2}\left(\dfrac{1}{4-x} - \dfrac{1}{2}\right)$

$= \lim\limits_{x \to 2} \dfrac{1}{x-2} \cdot \dfrac{2-(4-x)}{(4-x)2} = \lim\limits_{x \to 2} \dfrac{1}{x-2} \cdot \dfrac{x-2}{(4-x)2}$

$= \lim\limits_{x \to 2} \dfrac{1}{(4-x)2} = \frac{1}{4}$

In Exercises 21–30, find $f'(x)$ (a) using formula 7 and (b) using formula (4).

21. $f(x) = \dfrac{8}{x-2}$; $x_1 = 6$

▷ (a) $f'(6) = \lim\limits_{x \to 6} \dfrac{1}{x-6}\left(\dfrac{8}{x-2} - \dfrac{8}{6-2}\right) = \lim\limits_{x \to 6} \dfrac{8}{x-6}\left(\dfrac{1}{x-2} - \dfrac{1}{4}\right) = \lim\limits_{x \to 6} \dfrac{8}{x-6} \cdot \dfrac{6-x}{(x-2)4} = \lim\limits_{x \to 6} \dfrac{-2}{x-2} = -\frac{1}{2}$

(b) $f'(6) = \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x}\left(\dfrac{8}{6+\Delta x-2} - \dfrac{8}{6-2}\right) = \lim\limits_{\Delta x \to 0} \dfrac{8}{\Delta x}\left(\dfrac{1}{4+\Delta x} - \dfrac{1}{4}\right) = \lim\limits_{\Delta x \to 0} \dfrac{8}{\Delta x} \cdot \dfrac{-\Delta x}{(4+\Delta x)4} = \lim\limits_{\Delta x \to 0} \dfrac{-2}{4+\Delta x} = -\frac{1}{2}$

22. $f(x) = \dfrac{2}{\sqrt{x}} - 1$; $x_1 = 4$ ▷ (a) $f'(4) = \lim\limits_{x \to 4} \dfrac{1}{x-4}\left[\dfrac{2}{\sqrt{x}} - 1 - \left(\dfrac{2}{\sqrt{4}} - 1\right)\right]$

$= \lim\limits_{x \to 4} \dfrac{2}{x-4}\left(\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{4}}\right) = \lim\limits_{x \to 4} \dfrac{-2}{x-4} \cdot \dfrac{\sqrt{x}-\sqrt{4}}{\sqrt{x}\sqrt{4}} = \lim\limits_{x \to 4} \dfrac{-2}{(\sqrt{x}+\sqrt{4})\sqrt{x}\sqrt{4}} = \dfrac{-2}{16} = -\frac{1}{8}$

(b) $f'(4) = \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x}\left[\dfrac{2}{\sqrt{4+\Delta x}} - 1 - \left(\dfrac{2}{\sqrt{4}} - 1\right)\right] = \lim\limits_{\Delta x \to 0} \dfrac{2}{\Delta x}\left(\dfrac{1}{\sqrt{4+\Delta x}} - \dfrac{1}{\sqrt{4}}\right) = \lim\limits_{\Delta x \to 0} \dfrac{2}{\Delta x} \cdot \dfrac{\sqrt{4}-\sqrt{4+\Delta x}}{\sqrt{x}\sqrt{4}}$

23. $f(x) = \sin x$; $x_1 = 0$ ▷ We use Theorem 1.10.2.

(a) $f'(0) = \lim\limits_{x \to 0} \dfrac{\sin x - \sin 0}{x-0} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ (b) $f'(0) = \lim\limits_{\Delta x \to 0} \dfrac{\sin(0+\Delta x) - \sin 0}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\sin \Delta x}{\Delta x} = 1$

24. $f(x) = \cos x$; $x_1 = 0$ ▷ We use Theorem 1.10.5.

(a) $f'(0) = \lim\limits_{x \to 0} \dfrac{\cos x - \cos 0}{x-0} = \lim\limits_{x \to 0} \dfrac{\cos x - 1}{x} = 0$ (b) $f'(0) = \lim\limits_{\Delta x \to 0} \dfrac{\cos(0+\Delta x) - \cos 0}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\cos \Delta x - 1}{\Delta x} = 0$

25. $f(x) = \sin x$; $x_1 = \frac{1}{2}\pi$ ▷ In (a), let $x - \frac{1}{2}x = \Delta x$ to get the same solution as (b).

$f'(\frac{1}{2}\pi) = \lim\limits_{x \to \pi/2} \dfrac{\sin x - \sin \frac{1}{2}\pi}{x - \frac{1}{2}\pi} = \lim\limits_{\Delta x \to 0} \dfrac{\sin(\frac{1}{2}\pi + \Delta x) - \sin \frac{1}{2}\pi}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\cos \Delta x - 1}{\Delta x} = 0$

26. $f(x) = \cos x$; $x_1 = \frac{1}{2}\pi$ ▷ In (a), let $x - \frac{1}{2}x = \Delta x$ to get the same solution as (b).

$f'(\frac{1}{2}x) = \lim\limits_{x \to \pi/2} \dfrac{\cos x - \cos \frac{1}{2}\pi}{x - \frac{1}{2}\pi} = \lim\limits_{\Delta x \to 0} \dfrac{\cos(\frac{1}{2}\pi + \Delta x) - \cos \frac{1}{2}\pi}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{-\sin \Delta x}{\Delta x} = -1$

27. $f(x) = \sec x$; $x_1 = 0$ ▷ We use Theorem 1.10.5.

(a) $f'(0) = \lim\limits_{x\to 0}\frac{\sec x - \sec 0}{x - 0} = \lim\limits_{x\to 0}\frac{1}{x}\left(\frac{1}{\cos x} - \frac{1}{\cos 0}\right) = \lim\limits_{x\to 0}\frac{1 - \cos x}{x\cos x} = \lim\limits_{x\to 0}\frac{1}{\cos x}\cdot\lim\limits_{x\to 0}\frac{1-\cos x}{x} = 1\cdot 0 = 0$

(b) $f'(0) = \lim\limits_{\Delta x\to 0}\frac{\sec(0+\Delta x) - \sec 0}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{1}{\cos(0+\Delta x)} - \frac{1}{\cos 0}\right) = \lim\limits_{\Delta x\to 0}\frac{1-\cos\Delta x}{\Delta x\cos\Delta x}$

$= \lim\limits_{\Delta x\to 0}\frac{1}{\cos\Delta x}\cdot\lim\limits_{\Delta x\to 0}\frac{1-\cos\Delta x}{\Delta x} = 1\cdot 0 = 0$

28. $f(x) = \tan x$; $x_1 = 0$ ▷ We use Theorem 1.10.2.

(a) $f'(0) = \lim\limits_{x\to 0}\frac{\tan x - \tan 0}{x - 0} = \lim\limits_{x\to 0}\frac{\sin x/\cos x}{x} = \lim\limits_{x\to 0}\frac{1}{\cos x}\cdot\lim\limits_{x\to 0}\frac{\sin x}{x} = 1\cdot 1 = 1$

(b) $f'(0) = \lim\limits_{\Delta x\to 0}\frac{\tan(0+\Delta x) - \tan 0}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{\sin\Delta x/\cos\Delta x}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{1}{\cos\Delta x}\cdot\lim\limits_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x} = 1\cdot 1 = 1$

29. $f(x) = \cot x$; $x_1 = \frac{1}{2}\pi$ ▷ In (a), let $x - \frac{1}{2}x = \Delta x$ to get the same solution as (b).

$f'(\frac{1}{2}\pi) = \lim\limits_{x\to\pi/2}\frac{\cot x - \cot\frac{1}{2}\pi}{x - \frac{1}{2}\pi} = \lim\limits_{\Delta x\to 0}\frac{\cot(\frac{1}{2}\pi + \Delta x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{-\tan\Delta x}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{-1}{\cos\Delta x}\cdot\lim\limits_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x} = -1\cdot 1 = -1$

30. $f(x) = \csc x$; $x_1 = \frac{1}{2}\pi$ ▷ In (a), let $x - \frac{1}{2}x = \Delta x$ to get the same solution as (b).

$f'(\frac{1}{2}\pi) = \lim\limits_{x\to\pi/2}\frac{\csc x - \csc\frac{1}{2}\pi}{x - \frac{1}{2}\pi} = \lim\limits_{\Delta x\to 0}\frac{\csc(\frac{1}{2}\pi + \Delta x) - \csc\frac{1}{2}\pi}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{1/\sin(\frac{1}{2}\pi + \Delta x) - 1}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{1/\cos\Delta x - 1}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\frac{1}{\cos\Delta x}\cdot\lim\limits_{\Delta x\to 0}\frac{1-\cos\Delta x}{\Delta x} = 1\cdot 0 = 0$

In Exercises 31–36, find $f'(x)$ by applying formula (3) or (3′).

31. $f(x) = -4$ ▷ $f'(x) = \lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{-4-(-4)}{\Delta x} = 0$

32. $f(x) = 10$ ▷ $f'(x) = \lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{10-10}{\Delta x} = 0$

33. $f(x) = 7x + 3$ ▷ $f'(x) = \lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{[7(x+\Delta x)+3] - (7x+3)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{7\Delta x}{\Delta x} = 7$

34. $f(x) = 8 - 5x$ ▷ $f'(x) = \lim\limits_{z\to x}\frac{f(z) - f(x)}{z - x} = \lim\limits_{z\to x}\frac{(8-5z)-(8-5x)}{z-x} = \lim\limits_{z\to x} -5\frac{z-x}{z-x} = -5$

35. $f(x) = 4 + 5x - 2x^2$ ▷ $f'(x) = \lim\limits_{z\to x}\frac{f(z) - f(x)}{z - x}$

$= \lim\limits_{z\to x}\frac{(4+5z-2z^2) - (4+5x-2x^2)}{z-x} = \lim\limits_{z\to x}\left(5\frac{z-x}{z-x} - 2\frac{z^2-x^2}{z-x}\right) = \lim\limits_{z\to x}[5 - 2(z+x)] = 5 - 4x$

36. $f(x) = 3x^2 - 2x + 1$ ▷ $f'(x) = \lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{[3(x+\Delta x)^2 - 2(x+\Delta x) + 1] - (3x^2 - 2x + 1)}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\frac{[3(x^2 + 2x\Delta x + \Delta x^2) - 2(x+\Delta x)] - (3x^2 - 2x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{\cancel{3x^2} + 6x\Delta x + 3\Delta x^2 - \cancel{2x} - 2\Delta x - \cancel{3x^2} + \cancel{2x}}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}(6x + 3\Delta x - 2) = 6x - 2$

In Exercises 37 – 40, find the derivative.

37. $\frac{d}{dx}(8 - x^3) = \lim\limits_{\Delta x\to 0}\frac{[8 - (x+\Delta x)^3] - (8 - x^3)}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{[8 - x^3 - 3x^2\Delta x - 3x(\Delta x)^2 - (\Delta x)^3] - (8 - x^3)}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\frac{-3x^2\Delta x - 3x(\Delta x)^2 - (\Delta x)^3}{\Delta x} = \lim\limits_{\Delta x\to 0}[-3x^2 - 3x\Delta x - (\Delta x)^2] = -3x^2$

38. $\frac{d}{dt}(t^3 + t) = \lim\limits_{z\to t}\frac{(z^3+z) - (t^3+t)}{z-t} = \lim\limits_{z\to t}\left(\frac{z^3 - t^3}{z-t} - \frac{z-t}{z-t}\right) = \lim\limits_{z\to t}(z^2 + zt + t^3 - 1) = 3t^2 - 1$

39. $D_r\left(\frac{2r+3}{3r-2}\right) = \lim\limits_{\Delta r\to 0}\frac{1}{\Delta r}\left[\frac{2(r+\Delta r)+3}{3(r+\Delta r)-2} - \frac{2r+3}{3r-2}\right] = \lim\limits_{\Delta r\to 0}\frac{[(2r+3) + 2\Delta \mathrm{D}r](3r-2) - (2r+3)[(3r-2) + 3\Delta r]}{\Delta r[3(r+\Delta r)-2](3r-2)}$

$= \lim\limits_{\Delta r\to 0}\frac{2\Delta r(3r-2) - 3\Delta r(2r+3)}{\Delta r[3(r+\Delta r)-2](3r-2)} = \lim\limits_{\Delta r\to 0}\frac{-13\Delta r}{\Delta r[3(r+\Delta r)-2](3r-2)} = \lim\limits_{\Delta r\to 0}\frac{-13}{[3(r+\Delta r)-2](3r-2)} = \frac{-13}{(3r-2)^2}$

40. $$D_x\left[\frac{1}{x^2}-x\right]=\lim_{\Delta x\to 0}\frac{\left[\frac{1}{(x+\Delta x)^2}-(x+\Delta x)\right]-\left(\frac{1}{x^2}-x\right)}{\Delta x}=\lim_{\Delta x\to 0}\left[-\Delta x+\frac{x^2-(x+\Delta x)^2}{(x+\Delta x)^2x^2}\right]\frac{1}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\left[-\Delta x+\frac{-2x\Delta x-(\Delta x)^2}{(x+\Delta x)^2x^2}\right]\frac{1}{\Delta x}=\lim_{\Delta x\to 0}\left[-1-\frac{2x+\Delta x}{(x+\Delta x)^2x^2}\right]=-1-\frac{2x}{x^4}=-1-\frac{2}{x^3}$$

In Exercises 41–44, find $\frac{dy}{dx}$.

41. $y=3x+\frac{6}{x^2}$ ▹ Let $f(x)=3x+\frac{6}{x^2}$.
$$\frac{dy}{dx}=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{1}{z-x}\left[\left(3z+\frac{6}{z^2}\right)-\left(3x+\frac{6}{x^2}\right)\right]=\lim_{z\to x}\left[3\frac{z-x}{z-x}+\frac{6}{z-x}\left(\frac{1}{z^2}-\frac{1}{x^2}\right)\right]$$
$$=\lim_{z\to x}\left[3+\frac{6}{z-x}\cdot\frac{-(z^2-x^2)}{z^2x^2}\right]=\lim_{z\to x}\left[3-\frac{6(z+x)}{z^2x^2}\right]=3-\frac{12x}{x^4}=3-\frac{12}{x^3}$$

42. $y=\sqrt[3]{x}$. ▹ Let $f(x)=x^{1/3}$.
$$\frac{dy}{dx}=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{z^{1/3}-x^{1/3}}{z-x}=\lim_{z\to x}\frac{z^{1/3}-x^{1/3}}{(z^{1/3})^3-(x^{1/3})^3}$$
$$=\lim_{z\to x}\frac{z^{1/3}-x^{1/3}}{(z^{1/3}-x^{1/3})[(z^{1/3})^2+(z^{1/3})(x^{1/3})+(x^{1/3})^2]}=\lim_{z\to x}\frac{1}{(z^{1/3})^2+(z^{1/3})(x^{1/3})+(x^{1/3})^2}$$
$=\frac{1}{x^{2/3}+x^{2/3}+x^{2/3}}=\frac{1}{3x^{2/3}}=\frac{3}{2}x^{-2/3}$. Note that while y is defined at 0, $\frac{dy}{dx}$ is not.

43. $y=\frac{1}{\sqrt{x-1}}$ ▹ Let $f(x)=\frac{1}{\sqrt{x-1}}$.
$$\frac{dy}{dx}=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{1}{z-x}\left(\frac{1}{\sqrt{z-1}}-\frac{1}{\sqrt{x-1}}\right)=\lim_{z\to x}\frac{-(\sqrt{z-1}-\sqrt{x-1})}{[(z-1)-(x-1)]\sqrt{z-1}\sqrt{x-1}}$$
$$=\lim_{z\to x}\frac{-1}{(\sqrt{z-1}+\sqrt{x-1})\sqrt{z-1}\sqrt{x-1}}=\frac{-1}{2(x-1)^{3/2}}=-\tfrac{1}{2}(x-1)^{-3/2}$$

44. $y=\frac{4}{2x-5}$ ▹ Let $f(x)=\frac{4}{2x-5}$.
$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{4}{2(x+\Delta x)-5}-\frac{4}{2x-5}}{\Delta x}\cdot\frac{(2x-5)[2(x+\Delta x)-5]}{(2x-5)[2(x+\Delta x)-5]}$$
$$=4\lim_{\Delta x\to 0}\frac{(2x-5)-[2(x+\Delta x)-5]}{(2x-5)[2(x+\Delta x)-5]\Delta x}=4\lim_{\Delta x\to 0}\frac{-2\Delta x}{(2x-5)[2(x+\Delta x)-5]\Delta x}$$
$$=4\lim_{\Delta x\to 0}\frac{-2}{(2x-5)[2(x+\Delta x)-5]}=4\left[\frac{-2}{(2x-5)(2x-5)}\right]=\frac{-8}{(2x-5)^2}$$

45. Find an equation of the tangent line to the curve $y=2x^2+3$ that is parallel to the line $8x-y+3=0$.

▹ $8x-y+3=0$, or $y=8x+3$, has slope 8. Let $f(x)=2x^2+3$. We wish to find an x_1 for which $m(x_1)=8$.
$$m(x_1)=\lim_{\Delta x\to 0}\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}=\lim_{\Delta x\to 0}\frac{[2(x_1+\Delta x)^2+3]-(2{x_1}^2+3)}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\frac{4x_1\Delta x+2(\Delta x)^2}{\Delta x}=\lim_{\Delta x\to 0}(4x_1+2\Delta x)=4x_1$$
Therefore, we have $4x_1=8$; $x_1=2$. So an equation of the tangent line at $(2,11)$ is $y=8(x-2)+11=8x-5$.

46. Find an equation of the tangent line to the curve $y=3x^2-4$ that is parallel to the line $3x+y=4$.

▹ $y=-3x+4$ has slope -3. Let $f(x)=3x^2-4$. We wish to find an x_1 for which $m(x_1)=-3$.
$$m(x_1)=\lim_{x\to x_1}\frac{f(x)-f(x_1)}{x-x_1}=\lim_{x\to x_1}\frac{(3x^2-4)-(3{x_1}^2-4)}{x-x_1}=\lim_{x\to x_1}3\frac{x^2-{x_1}^2}{x-x_1}=\lim_{x\to x_1}3(x+x_1)=6x_1$$
Hence $6x_1=-3$; $x_1=-\frac{1}{2}$ and $f(x_1)=3(-\frac{1}{2})^2-4=-\frac{13}{4}$. The tangent line is $y=-3(x+\frac{1}{2})-\frac{13}{4}=-3x-\frac{19}{4}$.

47. Find an equation of the normal line to the curve $y = 2 - \frac{1}{3}x^2$ that is parallel to the line $x - y = 0$.

▹ The line $x - y = 0$, or $y = x$, has slope 1 so a line perpendicular to it has slope -1.
Let $f(x) = 2 - \frac{1}{3}x^2$. We wish to find an x_1 for which $m(x_1) = -1$.

$$m(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[2 - \frac{1}{3}(x_1 + \Delta x)^2] - (2 - \frac{1}{3}{x_1}^2)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-2x_1 \Delta x - (\Delta x)^2}{3\Delta x} = \lim_{\Delta x \to 0} \frac{-2x_1 - \Delta x}{3} = -\frac{2}{3}x_1$$

We have $-\frac{2}{3}x_1 = -1$ when $x_1 = \frac{3}{2}$ and $f(\frac{3}{2}) = \frac{5}{4}$. Thus the required tangent line is at $(\frac{3}{2}, \frac{5}{4})$ and an equation is $y - \frac{5}{4} = -(x - \frac{3}{2})$, or $4x + 4y - 11 = 0$.

48. Find an equation of each normal line to the curve $y = x^3 - 3x$ that is parallel to the line $2x + 18y - 9 = 0$.

▹ We first find the slope of the tangent line to the given curve. Let $f(x) = x^3 - 3x$.

$$m(x_1) = \lim_{x \to x_1} \frac{(x^3 - 3x) - ({x_1}^3 - 3x_1)}{x - x_1} = \lim_{x \to x_1} \frac{(x - x_1)(x^2 + xx_1 + {x_1}^2) - 3(x - x_1)}{x - x_1} = \lim_{x \to x_1} (x^2 + xx_1 + {x_1}^2 - 3)$$
$$= 3{x_1}^2 - 3$$

Hence the slope of the normal line is $-\dfrac{1}{3{x_1}^2 - 3}$.

We find the slope of the given line by writing the given equation in slope-intercept form.

$$2x + 18y - 9 = 0; \qquad 18y = -2x + 9; \qquad y = -\tfrac{1}{9}x + \tfrac{1}{2}$$

Because $-\frac{1}{9}$ is the slope of the given line and the normal line is parallel to the given line, we have

$$-\frac{1}{9} = -\frac{1}{3{x_1}^2 - 3}; \qquad 3{x_1}^2 - 3 = 9; \qquad {x_1}^2 = 4; \qquad x_1 = \pm 2$$

We have found the x coordinate of each point on the given curve where the normal line is parallel to the given line. If we substitute $x = 2$ in the equation of the curve, we obtain $y = 2^3 - 3(2) = 2$. Thus, $(2, 2)$ is the point where the normal line intersects the given curve. Because the slope of the normal line is $-\frac{1}{9}$, we may write the point-slope form of the equation.

$$y - 2 = -\tfrac{1}{9}(x - 2); \qquad x + 9y - 20 = 0$$

Similarly, if $x = -2$, we have $y = (-2)^3 - 3(-2) = -2$, and

$$y + 2 = -\tfrac{1}{9}(x + 2); \qquad x + 9y + 20 = 0$$

49. Prove that there is no line through the point $(1,5)$ that is tangent to the the parabola $y = 4x^2$.

▹ At any point (x_1, y_1) on the curve we have $y_1 = 4{x_1}^2$. Since a tangent line at $(x_1, 4{x_1}^2)$ has slope $8x_1$ while the line through the points $(x_1, 4{x_1}^2)$ and $(1,5)$ has slope $\dfrac{4{x_1}^2 - 5}{x_1 - 1}$, we must have
$\dfrac{4{x_1}^2 - 5}{x_1 - 1} = 8x_1$; $4{x_1}^2 - 5 = 8{x_1}^2 - 8x_1$; $4{x_1}^2 - 8x_1 + 4 = -1$; $4(x_1 - 1)^2 = -1$
Since the last equation has no solution there can be no line through $(1,5)$ tangent to the curve $y = 4x^2$.

50. Prove that there is no line through the point $(1,2)$ that is tangent to the the parabola $y = 4 - x^2$.

▹ At any point (x_1, y_1) on the curve we have $y_1 = 4 - {x_1}^2$. Since a tangent line at $(x_1, 4 - {x_1}^2)$ has slope $-2x_1$ while the line through the points $(x_1, 4 - {x_1}^2)$ and $(1,2)$ has slope $\dfrac{4 - {x_1}^2 - 2}{x_1 - 1}$, we must have
$\dfrac{2 - {x_1}^2}{x_1 - 1} = -2x_1$; $2 - {x_1}^2 = -2{x_1}^2 + 2x_1$; ${x_1}^2 - 2x_1 + 1 = -1$; $(x_1 - 1)^2 = -1$
Since the last equation has no solution there can be no line through $(1,2)$ tangent to the curve $y = 4 - x^2$.

51. If g is continuous at a and $f(x) = (x - a)g(x)$, find $f'(a)$.

▹ $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a} \dfrac{(x - a)g(x) - (a - a)g(a)}{x - a} = \lim\limits_{x \to a} g(x) = g(a)$, since g is continuous at a.

52. If g is continuous at a and $f(x) = (x^2 - a^2)g(x)$, find $f'(a)$.

▹ $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a} \dfrac{(x^2 - a^2)g(x) - (a^2 - a^2)g(a)}{x - a} = \lim\limits_{x \to a} (x + a)g(x) = 2ag(a)$ (g is continuous at a)

In Exercises 53 and 54, use the formula $f''(x) = \lim\limits_{\Delta x \to 0} \dfrac{f'(x+\Delta x) - f'(x)}{\Delta x} = \lim\limits_{z \to x} \dfrac{f'(z) - f(z)}{z - x}$.

53. $f(x) = ax^2 + bx$. $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{[a(x+\Delta x)^2 + b(x+\Delta x)] - (ax^2 + bx)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{a(2x\Delta x + \Delta x^2) + b\Delta x}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} a(2x + \Delta x) + b = 2ax + b$. $f''(x) = \lim\limits_{\Delta x \to 0} \dfrac{[2a(x+\Delta x) + b] - (2ax + b)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{2a\Delta x}{\Delta x} = \lim\limits_{\Delta x \to 0} 2a = 2a$

54. $f(x) = \frac{a}{x}$. $f'(x) = \lim\limits_{z \to x} \dfrac{1}{z-x}\left(\dfrac{a}{z} - \dfrac{a}{x}\right) = \lim\limits_{z \to x} \dfrac{a}{z-x} \cdot \dfrac{-(z-x)}{zx} = \lim\limits_{z \to x} \dfrac{-a}{zx} = -\dfrac{a}{x^2}$

$f''(x) = \lim\limits_{z \to x} \dfrac{1}{z-x}\left(\dfrac{-a}{z^2} - \dfrac{-a}{x^2}\right) = \lim\limits_{z \to x} \dfrac{a}{z-x} \cdot \dfrac{z^2 - x^2}{z^2 x^2} = \lim\limits_{z \to x} \dfrac{a(z+x)}{z^2 x^2} = \dfrac{2ax}{x^4} = \dfrac{2a}{x^3}$

55. If $f'(a)$ exists, prove that $f'(a) = \lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a - \Delta x)}{2\Delta x}$.

▸ $\lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a-\Delta x)}{2\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a) + f(a) - f(a-\Delta x)}{2\Delta x}$

$= \lim\limits_{\Delta x \to 0} \left[\dfrac{f(a+\Delta x) - f(a)}{2} + \dfrac{f(a) - f(a-\Delta x)}{2}\right] = \lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a)}{2\Delta x} + \lim\limits_{\Delta x \to 0} \dfrac{f(a) - f(a-\Delta x)}{2\Delta x}$

$= \dfrac{1}{2} \lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a)}{2\Delta x} + \dfrac{1}{2} \lim\limits_{\Delta x \to 0} \dfrac{f(a-\Delta x) - f(a)}{2\Delta x} = -f'(a) + \dfrac{1}{2} \lim\limits_{(-\Delta x) \to 0} \dfrac{f(a + (-\Delta x)) - f(a)}{2}$

$= \frac{1}{2} f'(a) + \frac{1}{2} f'(a) = f'(a)$, as desired.

56. Let f be a function whose domain is R and (i) $f(a+b) = f(a) \cdot f(b)$ for all a and b. Furthermore, suppose that (ii) $f(0) = 1$ and (iii) $f'(0)$ exists. Prove that $f'(x)$ exist for all x and that $f'(x) = f'(0) \cdot f(x)$.

▸ Let x be any real number. $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} \overset{\text{i}}{=} \lim\limits_{\Delta x \to 0} \dfrac{f(x)f(\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} f(x) \cdot \dfrac{f(\Delta x) - 1}{\Delta x}$

$\overset{\text{L.T.6}}{=} f(x) \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x) - 1}{\Delta x} \overset{\text{ii}}{=} f(x) \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x) - f(0)}{\Delta x} \overset{\text{iii}}{=} f(x) f'(0)$

57. Plot the parabola $y = \frac{1}{4}x^2$ and its tangent line y_t at $(2,1)$ and explain what happens as you zoom in.

▸ $y'(2) = \lim\limits_{x \to 2} \dfrac{\frac{1}{4}x^2 - \frac{1}{4}2^2}{x - 2} = \lim\limits_{x \to 2} \dfrac{1}{4} \cdot \dfrac{x^2 - 2^2}{x - 2} = \lim\limits_{x \to 2} \dfrac{1}{4}(x+2) = 1$. $y_t = 1 + 1(x-2) = x - 1$

The tangent line and the curve become indistinguishable, a characteristic of the tangent line.

58. Plot the parabola $y = \sqrt{x}$ and its tangent line y_t at $(1,1)$ and explain what happens as you zoom in.

▸ $y'(1) = \lim\limits_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1} = \lim\limits_{x \to 1} \dfrac{1}{\sqrt{x} + 1} = \dfrac{1}{2}$. $y_t = 1 + \frac{1}{2}(x-1) = \frac{1}{2}x + \frac{1}{2}$. See Exercise 57.

2.2 DIFFERENTIABILITY AND CONTINUITY

Differentiable A function is said to be *differentiable* at x_1 if $f'(x_1)$ exists.

2.2.1 Theorem If a function is differentiable at x_1, then f is continuous at x_1.

The theorem implies that if $f'(x_1)$ exists, then there must be no break in the graph of f at the point where $x = x_1$. The converse of Theorem 2.2.1 is not true. That is, a function that is continuous at x_1 may not be differentiable at x_1. For example, the absolute-value function defined by $f(x) = |x|$ is not differentiable at $x = 0$ although f is continuous there. Note that the graph of $|x|$ has a corner at $(0,0)$. If a function is differentiable at a point, then the graph of the function must be smooth at that point. Furthermore, if a function is differentiable at a point, then the tangent line to the graph of the function at that point must not be vertical. That is, if the tangent line to the graph of f at x_1 is vertical, then f is not differentiable at x_1.

A function f defined on an open interval containing x_1 is differentiable if and only if both one-sided derivatives exist and are equal. One-sided derivatives are defined as follows.

2.2.2 Definition If the function f is defined at x_1, then the *derivative from the right* of f at x_1, denoted by $f'_+(x_1)$, is defined by

$$f'_+(x_1) = \lim_{\Delta x \to 0^+} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \quad \Leftrightarrow \quad f'_+(x_1) = \lim_{x \to x_1^+} \frac{f(x) - f(x_1)}{x - x_1}$$

if the limit exists.

2.2.3 Definition If the function f is defined at x_1, then the *derivative from the left* of f at x_1, denoted by $f'_-(x_1)$, is defined by

$$f'_-(x_1) = \lim_{\Delta x \to 0^-} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \quad \Leftrightarrow \quad f'_-(x_1) = \lim_{x \to x_1^-} \frac{f(x) - f(x_1)}{x - x_1}$$

if the limit exists.

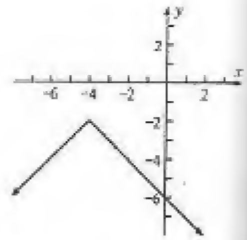
If f is continuous from the right and $\lim_{x \to x_1^+} f'(x)$ exists, then $\lim_{x \to x_1^+} f'(x) = f'_+(x_1)$.

If f is continuous from the left and $\lim_{x \to x_1^-} f'(x)$ exists, then $\lim_{x \to x_1^-} f'(x) = f'_-(x_1)$.

Exercises 2.2

In Exercises 1–20, do the following: (a) Sketch the graph of the function f. (b) Determine if f is continuous at x_1. (c) Find $f'_-(x_1)$ and $f'_+(x_1)$ if they exist. (d) Determine if f is differentiable at x_1.

1. $f(x) = \begin{cases} x+2 & \text{if } x \le -4 \\ -x-6 & \text{if } -4 < x \end{cases}$; $f(-4) = -4 + 2 = -2$.



▷ (b) $\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} (-x - 6) = -2$; $\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} (x + 2) = -2$

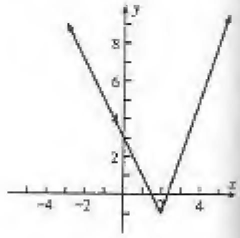
Therefore $\lim_{x \to -4} f(x) = -2 = f(-4)$. Thus, f is continuous at -4.

(c) $f'_-(-4) = \lim_{x \to -4^-} \dfrac{f(x) - f(-4)}{x - (-4)} = \lim_{x \to -4^-} \dfrac{(x+2) - (-2)}{x + 4} = 1$

$f'_+(-4) = \lim_{x \to -4^+} \dfrac{f(x) - f(-4)}{x - (-4)} = \lim_{x \to -4^+} \dfrac{(-x-6) - (-2)}{x + 4} = -1$

(d) Since $f'_-(-4) \ne f'_+(-4)$, $f'(-4)$ does not exist; so f is not differentiable at -4.

2. $f(x) = \begin{cases} 3 - 2x & \text{if } x < 2 \\ 3x - 7 & \text{if } x \ge 2 \end{cases}$; $f(2) = 3 \cdot 2 - 7 = -1$.



▷ (b) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - 2x) = -1$; $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 7) = -1$

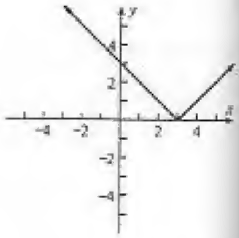
Therefore $\lim_{x \to 2} f(x) = -1 = f(2)$. Thus, f is continuous at 2.

(c) $f'_-(2) = \lim_{x \to 2^-} \dfrac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \dfrac{(3 - 2x) - (-1)}{x - 2} = \lim_{x \to 2^-} \dfrac{4 - 2x}{x - 2} = -2$

$f'_+(2) = \lim_{x \to 2^+} \dfrac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \dfrac{(3x - 7) - (-1)}{x - 2} = \lim_{x \to 2^+} \dfrac{3x - 6}{x - 2} = 3$

(d) Since $f'_-(2) \ne f'_+(2)$, $f'(2)$ does not exist; so f is not differentiable at 2.

3. $f(x) = |x - 3|$; $f(3) = 0$.



▷ (b) $\lim_{x \to 3^-} f(x) = \lim_{x \to 0} [-(x - 3)] = 0$; $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 3) = 0$

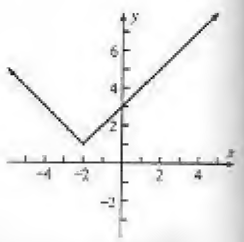
Therefore, $\lim_{x \to 3} f(x) = 0 = f(3)$. Thus, f is continuous at 3.

(c) $f'_-(3) = \lim_{x \to 3^-} \dfrac{f(x) - f(3)}{x - 3} = \lim_{x \to 3^-} \dfrac{-(x - 3) - 0}{x - 3} = \lim_{x \to 3^-} (-1) = -1$

$f'_+(3) = \lim_{x \to 3^+} \dfrac{f(x) - f(3)}{x - 3} = \lim_{x \to 3^+} \dfrac{(x - 3) - 0}{x - 3} = \lim_{x \to 3^+} 1 = 1$

(d) Since $f'_-(3) \ne f'_+(3)$, $f'(3)$ does not exist; so f is not differentiable at 3.

4. $f(x) = 1 + |x + 2|$; $x_1 = -2$



▷ $$f(x) = \begin{cases} 1 - (x + 2) & \text{if } x < -2 \\ 1 + (x + 2) & \text{if } x \ge -2 \end{cases} = \begin{cases} -x - 1 & \text{if } x < -2 \\ x + 3 & \text{if } x \ge -2 \end{cases}$$

(a) A sketch of the graph is shown at the right.

(b) Because $f(-2) = 1$, and

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (-x - 1) = 1$; $\quad \lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x + 3) = 1$

then f is continuous at -2.

(c) By Definition 2.2.3,

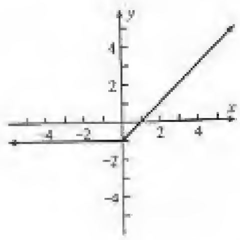
$$f'_-(-2) = \lim_{x\to-2^-} \frac{f(x)-f(-2)}{x-(-2)} = \lim_{x\to-2^-} \frac{(-x-1)-1}{x+2} = \lim_{x\to-2^-} \frac{-(x+2)}{x+2} = -1$$

and by Definition 2.2.3

$$f'_+(-2) = \lim_{x\to-2^+} \frac{f(x)-f(-2)}{x-(-2)} = \lim_{x\to-2^+} \frac{(x+3)-1}{x+2} = \lim_{x\to-2^+} \frac{x+2}{x+2} = 1$$

(d) Because $f'_-(-2) \neq f'_+(-2)$, then $f'(2)$ does not exist. Thus, f is not differentiable at -2. In the figure, note the corner at $x = -2$.

5. $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ x-1 & \text{if } 0 \le x \end{cases}$. $f(0) = 0 - 1 = -1$.



(b) $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} (-1) = -1$; $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} (x-1) = -1$

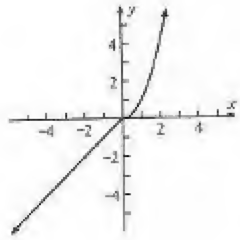
Therefore $\lim\limits_{x\to0} f(x) = -1 = f(0)$. Thus, f is continuous at 0.

(c) $f'_-(0) = \lim\limits_{x\to0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^-} \dfrac{-1-(-1)}{x} = \lim\limits_{x\to0^-} \dfrac{0}{x} = 0$

$f'_+(0) = \lim\limits_{x\to0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^+} \dfrac{(x-1)-(-1)}{x} = \lim\limits_{x\to0^+} \dfrac{x}{x} = 1$

(d) Since $f'_-(0) \neq f'_+(0)$, $f'(0)$ does not exist; so f is not differentiable at 0.

6. $f(x) = \begin{cases} x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$. $f(0) = 0$



(b) $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} x = 0$; $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} x^2 = 0$

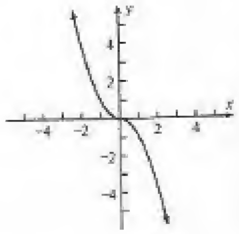
Therefore $\lim\limits_{x\to0} f(x) = 0 = f(0)$. Thus, f is continuous at 0.

(c) $f'_-(0) = \lim\limits_{x\to0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^-} \dfrac{x-0}{x-0} = 1$

$f'_+(0) = \lim\limits_{x\to0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^+} \dfrac{x^2-0}{x-0} = \lim\limits_{x\to0^+} x = 0$

(d) Since $f'_-(0) \neq f'_+(0)$, $f'(0)$ does not exist; so f is not differentiable at 0.

7. $f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ -x^2 & \text{if } 0 < x \end{cases}$. $f(0) = 0^2 = 0$



(b) $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} x^2 = 0$; $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} (-x^2) = 0$

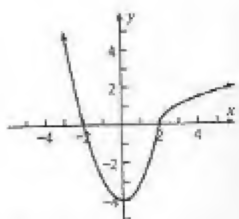
Therefore $\lim\limits_{x\to0} f(x) = 0 = f(0)$. Thus f is continuous at 0.

(c) $f'_-(0) = \lim\limits_{x\to0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^-} \dfrac{x^2-0}{x} = \lim\limits_{x\to0^-} x = 0$

$f'_+(0) = \lim\limits_{x\to0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to0^+} \dfrac{-x^2}{x} = \lim\limits_{x\to0^+} (-x) = 0$

(d) Since $f'_+(0) = f'_-(0) = 0$, $f'(0) = 0$; so f is differentiable at 0.

8. $f(x) = \begin{cases} x^2-4 & \text{if } x < 2 \\ \sqrt{x-2} & \text{if } x \ge 2 \end{cases}$; $x_1 = 2$



(a) A sketch of the graph is shown at the right.

(b) Because $f(2) = 0$ and

$\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^-} (x_2 - 4) = 0$; $\lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2^+} \sqrt{x-2} = 0$

then f is continuous at 2.

(c) By Definition 2.2.3,

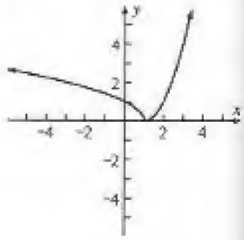
$$f'_-(2) = \lim_{x\to2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to2^-} \frac{(x^2-4)-0}{x-2} = \lim_{x\to2^-} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to2^-} (x+2) = 4$$

By Definition 2.2.2,

$$f'_+(2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{\sqrt{x-2}-0}{x-2} = \lim_{x\to 2^+} \frac{1}{\sqrt{x-2}} = +\infty$$

(d) *Because $f'_+(2) = +\infty$, $f'(2)$ does not exist, and f is not differentiable at 2. In the figure, note the* corner at $x = 2$.

9. $f(x) = \begin{cases} \sqrt{1-x} & \text{if } x < 1 \\ (1-x)^2 & \text{if } x \geq 1 \end{cases}$. $f(1) = (1-1)^2 = 0$.



▷ (b) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \sqrt{1-x} = 0$; $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (1-x)^2 = 0$

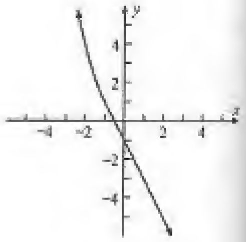
Therefore $\lim\limits_{x\to 1} f(x) = 0 = f(1)$. Thus f is continuous at 0.

(c) $f'_-(1) = \lim\limits_{x\to 1^-} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^-} \dfrac{\sqrt{1-x}-0}{-(1-x)} = \lim\limits_{x\to 1^-} \dfrac{-1}{\sqrt{1-x}} = -\infty$

$f'_+(1) = \lim\limits_{x\to 1^+} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^+} \dfrac{(1-x)^2-0}{x-1} = \lim\limits_{x\to 1^+} (x-1) = 0$

(d) Since $f'_-(1) \neq f'_+(1)$, $f'(1)$ does not exist; so f is not differentiable at 1.

10. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ -1-2x & \text{if } x \geq -1 \end{cases}$. $f(-1) = -1-2(-1) = 1$



▷ (b) $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} x^2 = 1$; $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (-1-2x) = 1$

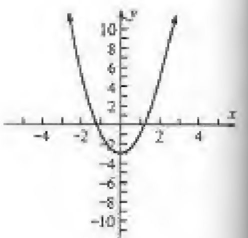
Therefore $\lim\limits_{x\to -1} f(x) = 1 = f(-1)$. Thus f is continuous at -1.

(c) $f'_-(-1) = \lim\limits_{x\to -1^-} \dfrac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\to -1^-} \dfrac{x^2-1}{x+1} = \lim\limits_{x\to -1^-} (x-1) = -2$

$f'_+(-1) = \lim\limits_{x\to -1^+} \dfrac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\to -1^+} \dfrac{(-1-2x)-1}{x+1} = \lim\limits_{x\to -1^+} \dfrac{-2x-2}{x+1} = -2$

(d) Since $f'_-(-1) = f'_+(-1) = -2$, $f'(-1) = -2$; so f is differentiable at -1.

11. $f(x) = \begin{cases} 2x^2-3 & \text{if } x \leq 2 \\ 8x-11 & \text{if } 2 < x \end{cases}$. $f(2) = 2(2)^2 - 3 = 5$.



(b) $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (2x^2-3) = 5$; $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (8x-1) = 5$

Therefore $\lim\limits_{x\to 2} f(x) = 5 = f(2)$. Thus f is continuous at 2.

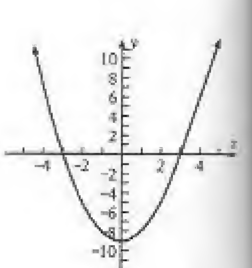
(c) $f'_-(2) = \lim\limits_{x\to 2^-} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2^-} \dfrac{(2x^2-3)-5}{x-2} = \lim\limits_{x\to 2^-} \dfrac{2(x-2)(x+2)}{x-2}$

$= \lim\limits_{x\to 2^-} 2(x+2) = 8$

$f'_+(2) = \lim\limits_{x\to 2^+} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2^+} \dfrac{(8x-11)-5}{x-2} = \lim\limits_{x\to 2^+} \dfrac{8(x-2)}{x-2} = 8$

(d) Since $f'_-(2) = f'_+(2) = 8$, $f'(2) = 8$; so f is differentiable at 2.

12. $f(x) = \begin{cases} x^2-9 & \text{if } x < 3 \\ 6x-18 & \text{if } 3 \leq x \end{cases}$; $x_1 = 3$



▷ (a) A sketch of the graph is shown at the right.

(b) Because $f(3) = 6(3) - 18 = 0$, and

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (x^2-9) = 0$; $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (6x-18) = 0$

then f is continuous at 3.

(c) By Definition 2.2.3 and by Definition 2.2.2

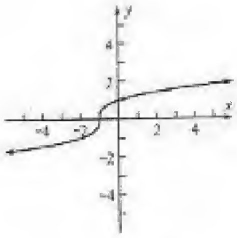
$$f'_-(3) = \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} \quad \text{and} \quad f'_+(3) = \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x \to 3^-} \frac{(x^2 - 9) - 0}{x - 3} \qquad\qquad = \lim_{x \to 3^+} \frac{(6x - 18) - 0}{x - 3}$$

$$= \lim_{x \to 3^-} (x + 3) = 6 \qquad\qquad = \lim_{x \to 3^+} 6 = 6$$

(d) Because $f'_-(3) = f'_+(3)$, then $f'(3)$ exists, and f is differentiable at 3. In the figure, note that the curve is smooth at $x = 3$.

13. $f(x) = (x+1)^{1/3}$



(b) $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} (x+1)^{1/3} = 0 = f(-1)$; so f is continuous at -1.

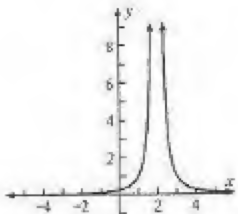
(c) $f'_-(-1) = f'_+(-1)$

$$= \lim_{x \to -1} \frac{f(x) - f(-1)}{x + 1} = \lim_{x \to -1} \frac{(x+1)^{1/3} - 0}{x + 1} = \lim_{x \to -1} \frac{1}{(x+1)^{2/3}} = +\infty$$

(d) $f'_-(-1)$, $f'_+(-1)$ and hence $f'(-1)$, do not exist;

so f is not differentiable at -1. Note that there is a vertical tangent at 0.

14. $f(x) = (x-2)^{-2}$



▹ f is not defined at 2 so f is not continuous, neither derivative exists and f is not differentiable at 2.

15. $f(x) = \begin{cases} 5 - 6x & \text{if } x \le 3 \\ -4 - x^2 & \text{if } 3 < x \end{cases}$. $f(3) = 5 - 6(3) = -13$.

(b) $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} (5 - 6x) = -13$; $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (-4 - x^2) = -13$

Therefore $\lim\limits_{x \to 3} f(x) = -13 = f(3)$. Thus, f is continuous at 3.

(c) $f'_-(3) = \lim\limits_{x \to 3^-} \dfrac{f(x) - f(3)}{x - 3} = \lim\limits_{x \to 3^-} \dfrac{(5 - 6x) - (-13)}{x - 3} = \lim\limits_{x \to 3^-} \dfrac{-6x + 18}{x - 3}$

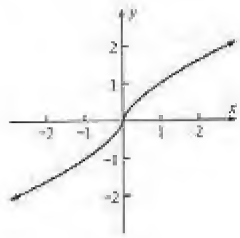
$$= \lim_{x \to 3^-} \frac{-6(x-3)}{x - 3} = \lim_{x \to 0} (-6) = -6$$

$$f'_+(3) = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3^+} \frac{(-4 - x^2) - (-13)}{x - 3} = \lim_{x \to 3^+} \frac{-x^2 + 9}{x - 3}$$

$$= \lim_{x \to 3^+} \frac{-(x-3)(x+3)}{x - 3} = \lim_{x \to 3^+} [-(x+3)] = -6$$

(d) Since $f'_-(3) = f'_+(3) = -6$, $f'(3) = -6$; so f is differentiable at 3.

16. $f(x) = \begin{cases} -x^{2/3} & \text{if } x \le 0 \\ x^{2/3} & \text{if } x > 0 \end{cases} \qquad x_1 = 0$



▹ (a) A sketch of the graph is shown at the right.

(b) Because $f(0) = 0$, and

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x^{2/3}) = 0; \qquad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^{2/3} = 0$$

then f is continuous at 0.

(c) By Definitions 2.2.3 and 2.2.2

$$f'_-(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} \quad \text{and} \quad f'_+(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

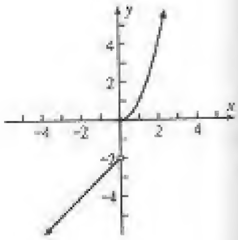
$$= \lim_{x \to 0^-} \frac{-x^{2/3}}{x} \qquad\qquad = \lim_{x \to 0^+} \frac{x^{2/3}}{x}$$

$$= \lim_{x \to 0^-} \frac{-1}{x^{1/3}} \qquad\qquad = \lim_{x \to 0^+} \frac{1}{x^{1/3}}$$

$$= +\infty \qquad\qquad = +\infty$$

(d) Because $f'_-(0) = +\infty$, then $f'(0)$ does not exist, and f is not differentiable at 0. Note that there is a vertical tangent at the point where $x = 0$.

17. $f(x) = \begin{cases} x-2 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \end{cases}$. $f(0) = 0^2 = 0$



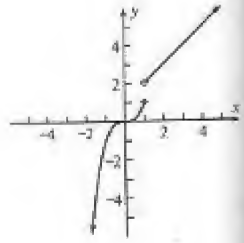
(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x-2) = -2$; $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$
Since $\lim_{x\to 0^-} f(x) \ne \lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ does not exist;
so f is not continuous at 0.

(c) $f'_-(0) = \lim_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \dfrac{(x-2)-0}{x} = -\infty$

$f'_+(0) = \lim_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \dfrac{x^2-0}{x} = \lim_{x\to 0^+} x = 0$

(d) f is not continuous at 0, so f is not differentiable at 0.

18. $f(x) = \begin{cases} x^3 & \text{if } x \le 1 \\ x+1 & \text{if } x > 1 \end{cases}$. $f(1) = 1^3 = 1$



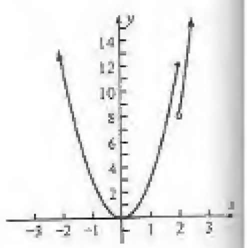
(b) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3 = 1$; $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x+1) = 2$
Since $\lim_{x\to 1^-} f(x) \ne \lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$ does not exist;
so f is not continuous at 1.

(c) $f'_-(1) = \lim_{x\to 1^-} \dfrac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \dfrac{x^3-1}{x-1} = \lim_{x\to 1}(x^2+x+1) = 3$

$f'_+(1) = \lim_{x\to 1^+} \dfrac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \dfrac{(x+1)-1}{x-1} = +\infty$

(d) f is not continuous at 1, so f is not differentiable at 1.

19. $f(x) = \begin{cases} 3x^2 & \text{if } x \le 2 \\ x^3 & \text{if } 2 < x \end{cases}$. $f(2) = 3(2)^2 = 12$



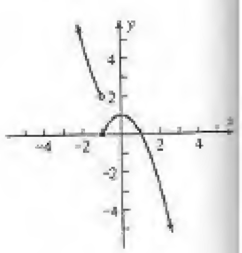
(b) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 3x^2 = 12$; $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^3 = 8$
Since $\lim_{x\to 2^-} f(x) \ne \lim_{x\to 2^+} f(x)$, f is not continuous at 2.

(c) $f'_-(2) = \lim_{x\to 2^-} \dfrac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \dfrac{3x^2-12}{x-2} = \lim_{x\to 2^-} \dfrac{3(x-2)(x+2)}{x-2}$
$= \lim_{x\to 2^-} 3(x+2) = 12$

$f'_+(2) = \lim_{x\to 2^+} \dfrac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \dfrac{x^3-12}{x-2} = -\infty$

(d) f is not continuous at 2, so f is not differentiable at 2.

20. $f(x) = \begin{cases} x^2+1 & \text{if } x < -1 \\ 1-x^2 & \text{if } -1 \le x \end{cases}$ $\qquad x_1 = -1$



(a) A sketch of the graph is shown at the right.

(b) Because

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}(x^2+1) = 2 \qquad \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(1-x^2) = 0$$

then $\lim_{x\to -1} f(x)$ does not exist, so f is discontinuous at -1.
Note that there is a break in the graph at the point where $x = -1$.

(c) Because $f(-1) = 1-(-1)^2 = 0$, by Definitions 2.2.3 and 2.2.2

$$f'_-(-1) = \lim_{x\to -1^-} \frac{f(x)-f(-1)}{x-(-1)} \quad \text{and} \quad f'_+(-1) = \lim_{x\to -1^+} \frac{f(x)-f(-1)}{x-(-1)}$$

$$= \lim_{x\to -1^-} \frac{(x^2+1)-0}{x+1} \qquad\qquad = \lim_{x\to -1^+} \frac{(1-x^2)-0}{x+1}$$

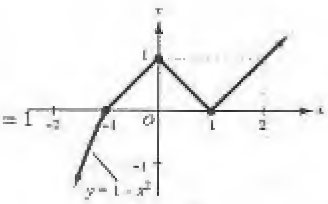
$$= -\infty \qquad\qquad = \lim_{x\to -1^+}(1-x) = 2$$

(d) Because f is discontinuous at -1 or because $f'(-1)$ does not exist, then f is not differentiable at -1.

In Exercises 21–26, (a) Define piecewise the continuous function shown in the figure; assume that each part of the graph that appears to be a line segment is a line segment. Find (b) $f'_-(-1)$, (c) $f'_+(-1)$, (d) $f'_-(0)$, (e) $f'_+(0)$, (f) $f'_-(1)$, and (g) $f'_+(1)$. (h) At what numbers is f not differentiable?

▸ The derivative at either end of a segment is its slope.

21. (a) $f(x) = \begin{cases} 1-x^2 & \text{if } x \le -1 \\ x+1 & \text{if } -1 < x \le 0 \\ -x+1 & \text{if } 0 < x \le 1 \\ x-1 & \text{if } x \ge 1 \end{cases}$

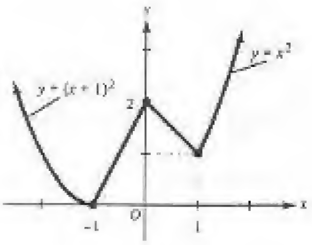


(b) $f'_-(-1) = \lim\limits_{x\to -1^-} \frac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\to -1^-} \frac{1-x^2-0}{x+1} = \lim\limits_{x\to -1^-}(1-x) = 2$

(c) and (d) $f'_+(-1) = f'_-(0) = 1$ (e) and (f) $f'_+(0) = f'_-(1) = -1$ (g) $f'_+(1) = 1$

(h) f is not differentiable at -1 because $f'_-(-1) \ne f'_+(-1)$; at 0 because $f'_-(0) \ne f'_+(0)$; at 1 because $f'_-(1) \ne f'_+(1)$

22. (a) $f(x) = \begin{cases} (x+1)^2 & \text{if } x \le -1 \\ 2x+2 & \text{if } -1 < x \le 0 \\ -x+2 & \text{if } 0 < x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$



(b) $f'_-(-1) = \lim\limits_{x\to -1^-} \frac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\to -1^-} \frac{(x+1)^2-0}{x+1} = \lim\limits_{x\to -1^-}(x+1) = 0$

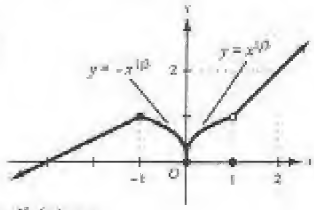
(c) and (d) $f'_+(-1) = f'_-(0) = 2$ (e) and (f) $f'_+(0) = f'_-(1) = -1$

(g) $f'_+(1) = \lim\limits_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^+} \frac{x^2-1}{x-1} = \lim\limits_{x\to 1^+}(x+1) = 2$

(h) f is not differentiable at -1 because $f'_-(-1) \ne f'_+(-1)$; at 0 because $f'_-(0) \ne f'_+(0)$; at 1 because $f'_-(1) \ne f'_+(1)$

23. (a) If $x \le -1$, $m = \frac{1-0}{-1-(-2)} = \frac{1}{2}$. $f(x) = \begin{cases} \frac{1}{2}(x+3) & \text{if } x \le -1 \\ -x^{1/3} & \text{if } -1 < x \le 0 \\ x^{1/3} & \text{if } 0 < x \le 1 \\ x & \text{if } x > 1 \end{cases}$ (b) $f'_-(-1) = \frac{1}{2}$ (c) $f'_+(-1)$

$= \lim\limits_{x\to -1^+} \frac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\to -1^+} \frac{-x^{1/3}-1}{x+1} = \lim\limits_{x\to -1^+} \frac{-1}{(x^{2/3}-x^{1/3}+1)} = -\frac{1}{3}$



(d) $f'_-(0) = \lim\limits_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \frac{-x^{1/3}-0}{x} = \lim\limits_{x\to 0^-} \frac{-1}{x^{2/3}} = -\infty$

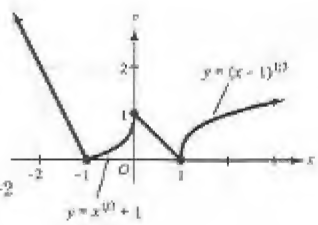
(e) $f'_+(0) = \lim\limits_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \frac{x^{1/3}-0}{x} = \lim\limits_{x\to 0^+} \frac{1}{x^{2/3}} = +\infty$

(f) $f'_-(1) = \lim\limits_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^-} \frac{x^{1/3}-1}{x-1} = \lim\limits_{x\to 1^-} \frac{1}{x^{2/3}+x^{1/3}+1} = \frac{1}{3}$ (g) $f'_+(1) = 1$

(h) f is not differentiable at -1 because $f'_-(-1) \ne f'_+(-1)$; at 0 because $f'_-(0)$ and $f'_+(0)$ do not exist (there is a vertical tangent); at 1 because $f'_-(1) \ne f'_+(1)$.

24. (a) For $x \le -1$, slope $= \frac{0-2}{-1-(-2)} = \frac{-2}{1} = -2$. Applying the point-slope formula at $(-1,0)$, $y = -2(x-(-1)) + 0 = -2x-2$.

For $0 < x \le 1$, slope $= \frac{0-1}{1-0} = -1$. Applying the point-slope formula at $(0,1)$, $y = -1(x-0)+1 = -x+1$. Therefore,



$f(x) = \begin{cases} -2x-2 & \text{if } x \le -1 \\ x^{1/3}+1 & \text{if } -1 < x \le 0 \\ -x+1 & \text{if } 0 < x \le 1 \\ (x-1)^{1/3} & \text{if } x > 1 \end{cases}$ (b) $f'_-(-1) = \text{slope}(-2x-2) = -2$

(c) $f'_+(-1) = \lim \frac{f(x)-f(-1)}{\ } = \lim \frac{x^{1/3}+1-0}{\ } = \lim \frac{1}{\ } = \frac{1}{\ }$

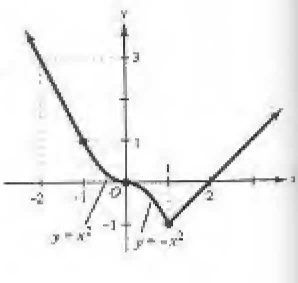
(d) $f'_-(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{(x^{1/3}+1)-1}{x} = \lim_{x\to 0^-} \frac{1}{x^{2/3}} = +\infty$

(e) and (f) $f'_-(0) = f'_+(1) = \text{slope}(-x+1) = -1$

(g) $f'_+(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{(x-1)^{1/3}-0}{x-1} = \lim_{x\to 1^+} \frac{1}{(x-1)^{2/3}} = +\infty$

(h) f is not differentiable at -1 because $f'_-(-1) \neq f'_+(-1)$;
at 0 because $f'_-(0)$ does not exist; at 1 because $f'_+(1)$ does not exist.

25. (a) $f(x) = \begin{cases} -2x-1 & \text{if } x \le -1 \\ x^2 & \text{if } -1 < x \le 0 \\ -x^2 & \text{if } 0 < x \le 1 \\ x-2 & \text{if } x > 1 \end{cases}$ (b) $f'_-(-1) = -2$



(c) $f'_+(-1) = \lim_{x\to -1^-} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1^+} \frac{x^2-1}{x+1} = \lim_{x\to -1^+} (x-1) = -2$

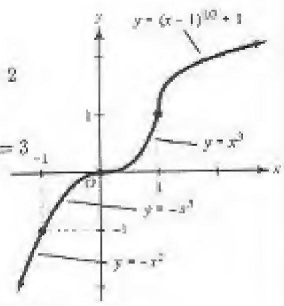
(d) $f'_-(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x^2-0}{x} = \lim_{x\to 0^-} x = 0$

(e) $f'_+(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{-x^2-0}{x} = \lim_{x\to 0^+} (-x) = 0$

(f) $f'_-(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{-x^2-(-1)}{x-1} = \lim_{x\to 1^-} -(x+1) = -2$

(g) $f'_+(1) = 1$ (h) f is not differentiable at 1 because $f'_-(1) \neq f'_+(1)$.

26. (a) $f(x) = \begin{cases} -x^2 & \text{if } x \le -1 \\ x^3 & \text{if } -1 < x \le 1 \\ (x-1)^{1/3}+1 & \text{if } x > 1 \end{cases}$



(b) $f'_+(-1) = \lim_{x\to -1^-} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1^+} \frac{-x^2-(-1)}{x+1} = \lim_{x\to -1^+} -(x-1) = 2$

(c) $f'_+(-1) = \lim_{x\to -1^-} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to -1^+} \frac{x^3-(-1)}{x+1} = \lim_{x\to -1^+} (x^2-x+1) = 3$

(d) and (e) $f'_-(0) = f'_+(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^3-0}{x} = \lim_{x\to 0} x^2 = 0$

(f) $f'_-(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{x^3-1}{x-1} = \lim_{x\to 1^-} (x^2+x+1) = 3$

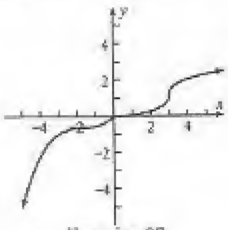
(g) $f'_+(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{[(x-1)^{1/3}+1]-1}{x-1} = \lim_{x\to 1^+} \frac{1}{(x-1)^{2/3}} = +\infty$

(h) f is not differentiable at -1 because $f'_-(-1) \neq f'_+(-1)$; at 1 because $f'_+(1)$ does not exist.

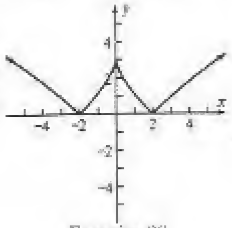
In Exercises 27–30, sketch the graph of a continuous function f defined on R and having the given properties.

27. The range of f is $(-\infty, +\infty)$, and f is differentiable at every number except 0 and 3; $f(-3) = -1$; $f(0) = 0$; $f(3) = 1$; $f'_-(0) = 1$; $f'_+(0) = 0$; $\lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = +\infty$

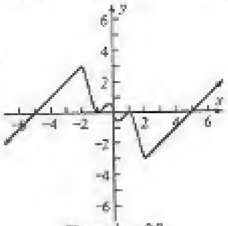
28. The range of f is $[0, +\infty)$; f is differentiable at every number except -2, 0, 2; $f(-2) = 0$; $f(0) = 3$; $f(2) = 0$; $f'_-(-2) = -1$; $f'_+(-2) = 1$; $f'_-(2) = -1$; $f'_+(2) = 1$; $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = +\infty$; $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = -\infty$



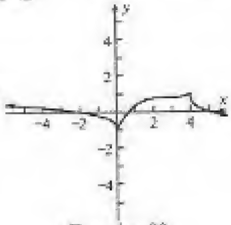
Exercise 27



Exercise 28



Exercise 29



Exercise 30

The range of f is R, f' exists except at -2, 0, 2; $f(-2)=0$; $f(-1)=0$; $f(0)=0$; $f(1)=0$; $f(2)=-3$; $f'_-(-2)=1$; $f'_+(-2)=-1$; $f'_-(2)=-1$; $f'_+(2)=1$; $\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x}=-\infty$; $\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x}=-\infty$

The range of f is $(-\infty,+\infty)$; f is differentiable at every number except 0 and 4; $f(-2)=0$; $f(0)=-1$; $f(4)=1$; $f(5)=0$; $f'_+(0)=2$; $f'_-(4)=\frac{1}{2}$; $\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x}=-\infty$; $\lim\limits_{x\to 4^+}\dfrac{f(x)-f(4)}{x-4}=-\infty$

Is the oil spill of Ex. 1.8.53 differentiable at 2? At t min, $r(t)$ m is the radius. $r(t)=\begin{cases}4t^2+20 & \text{if } 0\le t\le 2\\ 16t+4 & \text{if } t>2\end{cases}$

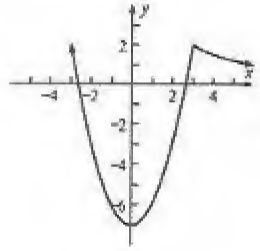
$r'_-(2)=\lim\limits_{t\to 2^-}\dfrac{f(t)-f(2)}{t-2}=\lim\limits_{t\to 2^-}\dfrac{(4t^2+20)-36}{t-2}=\lim\limits_{t\to 2^-}\dfrac{4t^2-16}{t-2}=\lim\limits_{t\to 2^-}4(t+2)=16$

$r'_+(2)=\lim\limits_{t\to 2^+}\dfrac{f(t)-f(2)}{t-2}=\lim\limits_{t\to 2^+}\dfrac{(16t+4)-36}{t-2}=\lim\limits_{t\to 2^+}16=16$. Since $r'_-(2)=r'_+(2)$, r is differentiable at 2.

$f(x)=x^2-1$ if $x<-1$ and $1-x^2$ if $-1\le x\le 1$. $\lim\limits_{x\to -1^-}f(x)=\lim\limits_{x\to -1}(x^2-1)=0=\lim\limits_{x\to -1^+}f(x)=\lim\limits_{x\to -1}1-x^2$ $=f(-1)\Rightarrow$continuous. $f'_-(-1)=\lim\limits_{x\to -1^-}\dfrac{(x^2-1)-0}{x+1}=\lim\limits_{x\to -1}(x-1)=-2$ but $f'_+(-1)=\lim\limits_{x\to -1^+}\dfrac{(1-x^2)-0}{x+1}=2$

$f(x)=\begin{cases}x^2-7 & \text{if } 0<x\le b\\ 6/x & \text{if } b<x\end{cases}$ (a) Find b so f is continuous at b. (b) Sketch. (c) Is f differentiable at b?

(a) $f(b)=b^2-7$; $\lim\limits_{x\to b^-}f(x)=\lim\limits_{x\to b^-}(x^2-7)=b^2-7$; $\lim\limits_{x\to b^+}f(x)=\lim\limits_{x\to b^+}\dfrac{6}{x}=\dfrac{6}{b}$



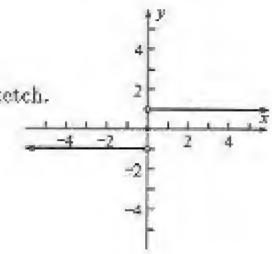
f will be continuous at b if $6/b=b^2-7$; $b^3-7b-6=0$; $(b-3)(b+1)(b+2)=0$. Thus f is continuous at b if $b=3$, -1, or -2; since $b>0$, only $b=3$ is admissible.

(c) $f'_-(3)=\lim\limits_{x\to 3^-}\dfrac{f(x)-f(3)}{x-3}=\lim\limits_{x\to 3^-}\dfrac{x^2-7-2}{x-3}=\lim\limits_{x\to 3^-}\dfrac{(x-3)(x+3)}{x-3}$

$=\lim\limits_{x\to 3^-}(x+3)=6$. $f'_+(3)=\lim\limits_{x\to 3^+}\dfrac{f(x)-f(3)}{x-3}=\lim\limits_{x\to 3^+}\dfrac{6/x-2}{x-3}=\lim\limits_{x\to 3^+}\dfrac{6-2x}{x(x-3)}$

$=\lim\limits_{x\to 3^+}\dfrac{-2}{x}=-\frac{2}{3}$. Hence $f'(3)$ does not exist, and f is not differentiable at 3.

$f(x)=\operatorname{sgn} x=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } x=0\\ +1 & \text{if } x>0\end{cases}$. Prove (a) $f'_-(0)$ and $f'_+(0)$ do not exist. (b) $\lim\limits_{x\to 0^-}f'(x)=0$ and $\lim\limits_{x\to 0^+}f'(x)=0$. (c) Sketch.

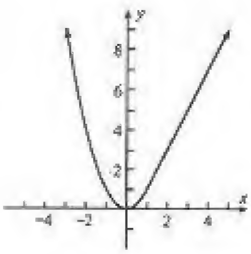


(a) $f'_-(0)=\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^-}\dfrac{-1-0}{x}=+\infty$;

$f'_+(0)=\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^+}\dfrac{1-0}{x}=+\infty$

(b) $\lim\limits_{x\to 0^-}f'(x)=\lim\limits_{x\to 0^-}\left[\dfrac{d}{dx}(-1)\right]=\lim\limits_{x\to 0^-}\left[\lim\limits_{\Delta x\to 0}\dfrac{(-1)-(-1)}{\Delta x}\right]=\lim\limits_{x\to 0^-}0=0$

$\lim\limits_{x\to 0^+}f'(x)=\lim\limits_{x\to 0^+}\left[\dfrac{d}{dx}(1)\right]=\lim\limits_{x\to 0^+}\left[\lim\limits_{\Delta x\to 0}\dfrac{1-1}{\Delta x}\right]=\lim\limits_{x\to 0^+}=0$



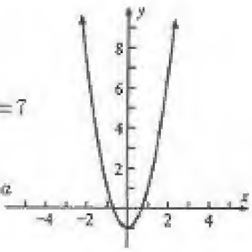
$f(x)=\begin{cases}x^2 & \text{if } x<1\\ ax+b & \text{if } x\ge 1\end{cases}$. Find a and b such that f is differentiable at 1. Sketch.

For continuity we need $1^2=a\cdot 1+b$.

$f'_-(1)=\lim\limits_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1^-}\dfrac{x^2-1}{x-1}=\lim\limits_{x\to 1^-}(x+1)=2$;

$f'_+(1)=\lim\limits_{x\to 1^+}\dfrac{f(x)-f(1)}{x-1}=\dfrac{(ax+b)-(a+b)}{x-1}=a$

f will be differentiable at 1 if $a=2$. Since $a+b=1$, then $b=-1$



$f(x)=\begin{cases}ax+b & \text{if } x<2\\ 2x^2-1 & \text{if } 2\le x\end{cases}$. Find a and b such that f is differentiable at 2. Sketch.

$f(2)=2(2)^2-1=7$; $\lim\limits_{x\to 2^-}f(x)=\lim\limits_{x\to 2^-}(ax+b)=2a+b$; $\lim\limits_{x\to 2^+}f(x)=\lim\limits_{x\to 2^+}(2x^2-1)=7$

For f to be differentiable at 2, f must be continuous at 2 so $2a+b=7$.

$f'_-(2)=\lim\limits_{x\to 2^-}\dfrac{f(x)}{x-2}=\lim\limits_{x\to 2^-}\dfrac{(ax+b)-7}{x-2}=\lim\limits_{x\to 2^-}\dfrac{(ax+b)-(2a+b)}{x-2}=\lim\limits_{x\to 2^-}\dfrac{a(x-2)}{x-2}=a$

$$f'_+(2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{(2x^2-1)-7}{x-2} = \lim_{x\to 2^+} \frac{2(x-2)(x+2)}{x-2} = \lim_{x\to 2^+} 2(x+2) = 8$$

f will be differentiable at 2 if $f'_-(2) = f'_+(2)$; so $a = 8$. Since $2a + b = 7$, $b = -9$.

In Exercises 37–40, treat the variable representing a nonnegative integer as if it was a nonnegative real number.

37. A trip is for up to 250 students. For up to 150 students the cost is \$15 per student and decreases by \$0.05 per student for each student over 150. (a) Express the income as function f of the number of students. (b) Prove that f is continuous on its domain. (c) If f differentiable at 150?

▷ (a) Let $f(x)$ dollars be the gross income if x students make the trip.

$$f(x) = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ [15 - .05(x-150)]x & \text{if } 150 < x \le 250 \end{cases} = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ 22.5x - .05x^2 & \text{if } 150 < x \le 250 \end{cases}$$

(b) $f(150) = 15(150) = 2250$; $\lim_{x\to 0} f(x) = \lim_{x\to 150^-} 15x = 15(150) = 2250$

$\lim_{x\to 150^+} f(x) = \lim_{x\to 150^+} [15 - 0.05(x-150)]x = 15(150) = 2250$

Therefore, $\lim_{x\to 150} f(x) = 2250 = f(150)$. Thus f is continuous at 150.

$$f'_-(150) = \lim_{x\to 150^-} \frac{f(x)-f(150)}{x-150} = \lim_{x\to 150^-} \frac{15x-2250}{x-150} = \lim_{x\to 150^-} \frac{15(x-150)}{x-150} = \lim_{x\to 150^-} 15 = 15$$

$$f'_+(150) = \lim_{x\to 150^+} \frac{f(x)-f(150)}{x-150} = \lim_{x\to 150^+} \frac{22.5x - .05x^2 - 2250}{x-150} = \lim_{x\to 150^+} \frac{-.05(x^2 - 450x + 45000)}{x-150}$$

$$= \lim_{x\to 150^+} \frac{-.05(x-150)(x-300)}{x-150} = \lim_{x\to 150^+} -.05(x-300) = 7.5$$

Since $f'_-(150) \ne f'_+(150)$, $f'(150)$ does not exist.

38. Do Exercise 37 if the reduction per student in excess of 150 is \$0.07.

▷ (a) Let $f(x)$ dollars be the gross income if x students make the trip.

$$f(x) = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ [15 - .07(x-150)]x & \text{if } 150 < x \le 250 \end{cases} = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ 25.5x - .07x^2 & \text{if } 150 < x \le 250 \end{cases}$$

(b) $f(150) = 15(150) = 2250$; $\lim_{x\to 150^-} f(x) = \lim_{x\to 150^-} 15x = 15(150) = 2250$

$\lim_{x\to 150^+} f(x) = \lim_{x\to 150^+} [15 - 0.05(x-150)]x = 15(150) = 2250$

Therefore, $\lim_{x\to 150} f(x) = 2250 = f(150)$. Thus f is continuous at 150.

$$f'_-(150) = \lim_{x\to 150^-} \frac{f(x)-f(150)}{x-150} = \lim_{x\to 150^-} \frac{15x-2250}{x-150} = \lim_{x\to 150^-} \frac{15(x-150)}{x-150} = \lim_{x\to 150^-} 15 = 15$$

$$f'_+(150) = \lim_{x\to 150^+} \frac{f(x)-f(150)}{x-150} = \lim_{x\to 150^+} \frac{25.5x - .07x^2 - 2250}{x-150} \overset{\text{long division}}{=} \lim_{x\to 150^+} (-.07x + 15) = 4.5$$

Since $f'_-(150) \ne f'_+(150)$, $f'(150)$ does not exist.

39. Each tree produces 600 oranges, at up to 20 trees per acre. For each additional tree per acre, yield per tree decreases by 15 oranges. (a) Express the number of oranges as a function $f(x)$ of the number x of trees per acre. (b) Prove f is continuous. (c) Is f differentiable at 20?

▷ (a) $f(x) = \begin{cases} 600x & \text{if } 0 \le x \le 20 \\ [600 - 15(x-20)]x & \text{if } x > 20 \end{cases} = \begin{cases} 600x & \text{if } 0 \le x \le 20 \\ 900x - 15x^2 & \text{if } 20 < x \le 60 \end{cases}$

(b) $f(20) = 600(20) = 12000$; $\lim_{x\to 20^-} f(x) = \lim_{x\to 20^-} 600x = 12000$; $\lim_{x\to 20^+} f(x) = \lim_{x\to 20^+} (900x - 15x^2) = 12000$

Therefore, $\lim_{x\to 20} f(x) = 12000 = f(20)$. Thus f is continuous at 100.

$$f'_-(20) = \lim_{x\to 20^-} \frac{f(x)-f(20)}{x-20} = \lim_{x\to 20^-} \frac{600x-12000}{x-20} = \lim_{x\to 20^-} \frac{600(x-20)}{x-20} = \lim_{x\to 20^-} 600 = 600$$

$$f'_+(20) = \lim_{x\to 20^+} \frac{f(x)-f(20)}{x-20} = \lim_{x\to 20^+} \frac{900x - 15x^2 - 12000}{x-20} \overset{\text{long division}}{=} \lim_{x\to 20^+} (-15x + 600) = 300$$

Since $f'_-(20) \ne f'_+(20)$, $f'(20)$ does not exist.

40. A club's annual dues is \$100 per member, less \$0.50 for each member over 600 and plus \$0.50 for each member less than 600. (a) Find a mathematical model expressing the club's revenue $f(x)$ as a function of the number x of members. (b) Prove that f is continuous on its domain. (c) Is f differentiable at 600?

(a) $f(x) = x[100 - \frac{1}{2}(x - 600)] = x(400 - \frac{1}{2}x) = 400x - \frac{1}{2}x^2,\ 0 \le x \le 800$

(b) and (c) Because f is a polynomial it is continuous and differentiable on $(0, 800)$.

Let $f(x) = |x|$. Find $f'(x)$, $x \neq 0$.

If $x \neq 0$, then since $|x| = \sqrt{x^2}$ we have $|x|^2 = x^2$ so

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{|x + \Delta x| - |x|}{\Delta x} = \lim_{\Delta x \to 0} \frac{(|x + \Delta x| - |x|)(|x + \Delta x| + |x|)}{\Delta x(|x + \Delta x| + |x|)}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x(|x + \Delta x| + |x|)} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x(|x + \Delta x| + |x|)} = \lim_{\Delta x \to 0} \frac{2x + \Delta x}{|x + \Delta x| + |x|} = \frac{2x}{2|x|} = \frac{|x|}{x}$$

Given $f(x) = [\![x]\!]$, find $f'(x_1)$ if x_1 is not an integer. Prove that $f'(x_1)$ does not exist if x_1 is an integer. If x_1 is an integer, what can you say about $f'_-(x_1)$ and $f'_+(x_1)$?

If x_1 is not an integer, then there is an integer N such that $N < x_1 < N + 1$. Hence, by definition, $f(x_1) = N$. Furthermore, for all x such that $N < x < N + 1$, $f(x) = N$. Thus, whenever $N < x < N + 1$, $f(x) - f(x_1) = 0$.

Therefore, $f'(x_1) = \lim\limits_{x \to x_1} \dfrac{f(x) - f(x_1)}{x - x_1} = \lim\limits_{x \to x_1} \dfrac{0}{x - x_1} = 0$

Thus, if x_1 is not an integer, $f'(x_1) = 0$. Furthermore, if x_1 is an integer, then

$$\lim_{x \to x_1^+} f(x) = x_1 \quad \text{and} \quad \lim_{x \to x_1^-} f(x) = x_1 - 1$$

Hence $\lim\limits_{x \to x_1} f(x)$ does not exist, and thus f is discontinuous at x_1. By Theorem 2.2.1, this proves that $f'(x_1)$ does not exist if x_1 is an integer. However, if x_1 is an integer, then

$$f'_-(x_1) = \lim_{x \to x_1^-} \frac{f(x) - f(x_1)}{x - x_1} = \lim_{x \to x_1^-} \frac{(x_1 - 1) - x_1}{x - x_1} = \lim_{x \to x_1^-} \frac{-1}{x - x_1} = +\infty$$

$$f'_+(x_1) = \lim_{x \to x_1^+} \frac{f(x) - f(x_1)}{x - x_1} = \lim_{x \to x_1^+} \frac{x_1 - x_1}{x - x_1} = \lim_{x \to x_1^+} \frac{0}{x - x_1} = 0$$

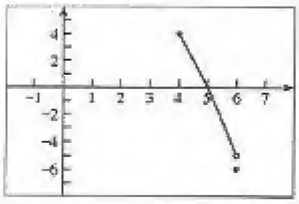
$f(x) = (x - 1)[\![x]\!]$

(a) $f'_-(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{(x - 1)[\![x]\!] - 0}{x - 1} = \lim\limits_{x \to 1^-} [\![x]\!] = 0$

(b) $f'_+(1) = \lim\limits_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^+} [\![x]\!] = 1$ (c) $f'_-(1) \neq f'_+(1)$; so $f'(1)$ does not exist.

$f(x) = (5 - x)[\![x]\!]$. Plot the graph of f for x in $[4, 6]$. Find, if they exist: (a) $f'_-(5)$; (b) $f'_+(5)$; (c) $f'(5)$.

$$f(x) = \begin{cases} (5 - x)4 & \text{if } 4 \le x < 5 \\ (5 - x)5 & \text{if } 5 \le x < 6 \\ -6 & \text{if } x = 6 \end{cases} \qquad f(5) = 0.$$



(a) $f'_-(5) = \lim\limits_{x \to 5^-} \dfrac{f(x) - f(5)}{x - 5} = \lim\limits_{x \to 5^-} \dfrac{(5 - x)4 - 0}{x - 5} = \lim\limits_{x \to 5^-} (-4) = -4$

(b) $f'_+(5) = \lim\limits_{x \to 5^+} \dfrac{f(x) - f(5)}{x - 5} = \lim\limits_{x \to 5^+} \dfrac{(5 - x)5 - 0}{x - 5} = \lim\limits_{x \to 5^+} (-5) = -5$

Because $f'_-(5) \neq f'_+(5)$, then $f'(5)$ does not exist.

$f(x) = (x - a)[\![x]\!]$; a is any integer.

$f'_-(a) = \lim\limits_{x \to a^-} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a^-} \dfrac{(x - a)[\![x]\!] - 0}{x - a} = \lim\limits_{x \to a^-} [\![x]\!] = a - 1$

$f'_+(a) = \lim\limits_{x \to a^+} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to a^+} [\![x]\!] = a$; so $f'_-(a) + 1 = f'_+(a)$.

Let the f be defined by $f(x) = \begin{cases} \dfrac{g(x) - g(a)}{x - a} & \text{if } x \neq a \text{ (i)} \\ g'(a) & \text{if } x = a \text{ (ii)} \end{cases}$. Prove that if $g'(a)$ exists (iii), f is continuous at a.

$\lim\limits_{x \to a} f(x) \overset{\text{i}}{=} \lim\limits_{x \to a} \dfrac{g(x) - g(a)}{x - a} \overset{\text{iii}}{=} g'(a) \overset{\text{ii}}{=} f(a)$. Therefore, by Definition 1.8.1, f is continuous at a.

(a) If $f(x) = |x|$ and $g(x) = -|x|$ then $(f + g)(x) = 0$ for all real numbers.

(b) If $F(x) = x^{-1}$ and $G(x) = -x^{-1}$ then $(F + G)(x) = 0$ for $x \neq 0$.

2.3 THE NUMERICAL DERIVATIVE

Symmetric Difference Quotient of f at a is $[f(a+\Delta x)-f(a-\Delta x)/(2\Delta x)$. As $\Delta x \to 0$, this approaches $f'(a)$, usually much faster than the ordinary difference quotient used in the definition, and sometimes approaches a limit when $f'(a)$ does not exist. The value of Δx is called the tolerance. With any tolerance, the result is exact for linear and quadratic functions. We shall used a fixed tolerance of 0.001. Thus

$$\mathbf{NDER}(f(x),x) = \frac{f(x+0.001)-f(x-0.001)}{0.002} \tag{4}$$

Note that some graphics calculators can plot the tangent line to a graph at any point and give the value of NDER in a single operation. If your graphics calculator can use rules of differentiation to get (a decimal expression for) the exact value DER1 (DER2 is the second derivative) of the derivative, you should use that capability (except in Exercises 27–30). The FRAC capability may be used to get the answer as a fraction.

Exercises 2.3

In Exercises 1–4, compare the symmetric difference quotient with standard one for the given Exercise of §2.1.

1. See Exercise 2.1.17 **2.** See Exercises 2.1.18 **3.** See Exercise 2.1.19 **4.** See Exercise 2.1.20

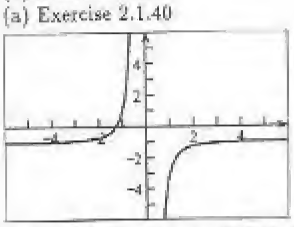
In Exercises 5–8, plot the graph of the numerical derivative at x to support the value found in the given Exercise.

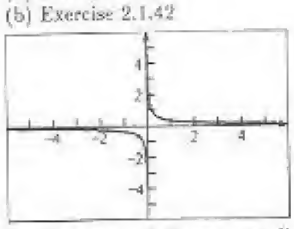
5. (a) Exercise 2.1.33 (b) Exercise 2.1.35 (c) Exercise 2.1.37

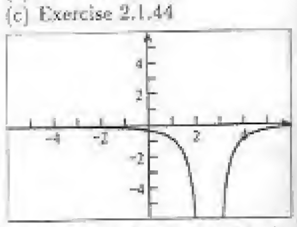
6. (a) Exercise 2.1.34 (b) Exercise 2.1.36 (c) Exercise 2.1.38

7. (a) Exercise 2.1.39 (b) Exercise 2.1.41 (c) Exercise 2.1.43

8. (a) Exercise 2.1.40 (b) Exercise 2.1.42 (c) Exercise 2.1.44







In Exercises 9–20, (a) use NDER to find the slope m of the tangent line to f at (x_1, y_1) where $y_1 = f(x_1)$; (b) write the equation of the tangent line T; (c) plot the tangent line and the graph of f.

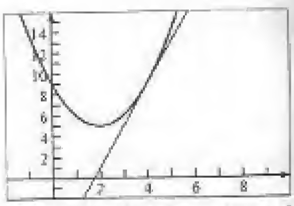
9. $f(x) = (x-1)^2$; $x_1 = 2$ ▷ $y_1 = f(2) = 1$. $m = f'(2) = 2$. T: $y = 2(x-2)+1 = 2x-3$

10. $f(x) = 2+2x-x^2$; $x_1 = -1$ ▷ $y_1 = f(-1) = -1$. $m = f'(-1) = 4$. T: $y = 4(x+1)-1 = 4x+3$

11. $f(x) = x^2-2x-4$; $x_1 = 3$ ▷ $y_1 = f(3) = -1$. $m = f'(3) = 4$. T: $y = 4(x-3)-1 = 4x-13$

12. $f(x) = (2-x)^2+5$; $x_1 = 4$

▷ $y_1 = f(4) = (2-4)^2+5 = 9$. Using NDER we find $m = f'(4) = 4$.
Therefore the equation of the tangent line is
$y = 4(x-4)+9$, $y = 4x-7$



13. $f(x) = \sqrt{x^2-16}$; $x_1 = -5$

▷ $y_1 = f(-5) = 3$. $m = f'(-5) = -\frac{5}{3} \approx -1.66\ldots$

T: $y = 3-\frac{5}{3}(x+5) = -\frac{5}{3}x - \frac{16}{3} \approx -1.66\ldots x - 5.33\ldots$

14. $f(x) = \sqrt{25-x^2}$; $x_1 = 3$ ▷ $y_1 = f(3) = 4$. $m = f'(3) = -.75$. T: $y = -.75(x-3)+4 = -.75x+6.25$

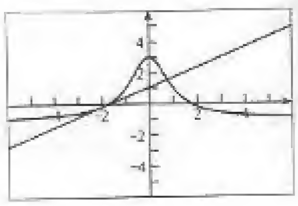
15. $f(x) = \dfrac{x^2-1}{x^2+4}$; $x_1 = 1$ ▷ $y_1 = f(1) = 0$. $m = f'(1) = 0.4$. T: $y = 0.4(x-1)+0 = 0.4x-0.4$

16. $f(x) = \dfrac{3-x^2}{1+x^2}$; $x_1 = -2$

▷ $y_1 = f(-2) = \dfrac{3-(-2)^2}{1+(-2)^2} = -\frac{1}{5} = -0.2$.

$m = f'(-2) = 0.64$ (using DER1) ≈ 0.6400001536 (using NDER)

Therefore the equation of the tangent line is
$y = 0.64(x+2)-0.2$, $y = 0.64x+1.08$



17. $f(x) = x \sin x$; $x_1 = 1$
▷ $y_1 = \sin 1 \approx 0.8415$. $m = f'(1) \approx 1.3818$. T: $y \approx 1.3818(x-1) + 0.8415 = 1.3818x - 0.5403$

18. $f(x) = x^2 \cos x$; $x_1 = 2$
▷ $y_1 = 4\cos 2 \approx -1.6646$. $m = f'(2) \approx -5.3018$. T: $y \approx -5.3018(x-2) - 1.6646 = -5.3018x + 8.9390$

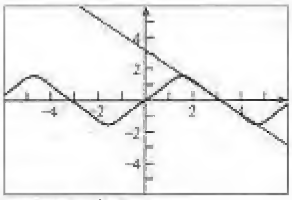
19. $f(x) = \sin(\cos x)$; $x_1 = 2$
▷ $y_1 = \sin(\cos 2) \approx -0.4042$. $m = f'(2) \approx -0.8317$. T: $y \approx -0.8317(x-2) - 0.4042 = -0.8317x + 1.2591$

20. $f(x) = \tan(\sin x)$; $x_1 = 3$
▷ $y_1 = \tan(\sin 3) \approx 0.1421$

$m = f'(3) = -1.009972795$ (using DER1) $= -1.009972957$ (using NDER)

Therefore the equation of the tangent line is approximately

$y = -1.0100(x-3) + 0.1421$, $y = -1.0100x + 3.1720$



21 and 22. Prove that if f is a linear or quadratic function then NDER($f(x)$,x) is exactly $f'(x)$.
▷ Let $f(x) = ax^2 + bx + c$ (a may be 0). NDER($f(x)$,x) $= \dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$

$$= \frac{[a(x+\Delta x)^2 + b(x+\Delta x) + c] - [a(x-\Delta x)^2 + b(x-\Delta x) + c]}{2\Delta x}$$

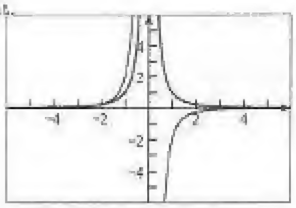
$$= \frac{[ax^2 + 2ax\Delta x + a\Delta x^2 + bx + b\Delta x] - [ax^2 - 2ax\Delta x + a\Delta x^2 + bx - b\Delta x]}{2\Delta x} = \frac{4ax\Delta x + 2b\Delta x}{2\Delta x} = 2ax + b = f'(x)$$

In Exercises 23–26, (a) plot f and NDER($f(x)$, x). For what values of of x is (b) NDER > 0 and (c) NDER < 0? For what values of x does $f(x)$ appear to be (d) increasing and (e) decreasing. (f) Compare your answers in parts (b) and (d) and in parts (c) and (e).
▷ (f) The answers are identical: a function increases if its derivative f' is positive and decreases if f' is negative.

23. $f(x) = x^2 + 2$ ▷ (b) and (d) $x > 0$; (c) and (e) $x < 0$

24. $f(x) = 1/x^2$ ▷ The figure at the right show f dark and f' light.
▷ (b) and (d): NDER > 0 and $f(x)$ is increasing if $x < 0$

(c) and (e): NDER < 0 and $f(x)$ is decreasing if $x > 0$



25. $f(x) = \sqrt{4-x^2}$ ▷ (b) and (d) $x < 0$; (c) and (e) $x > 0$

26. $f(x) = \sqrt{x^2-4}$ ▷ (b) and (d) $x > 2$; (c) and (e) $x < 2$

27. Let $f(x) = x^{1/3}$. Compute NDER($f(x)$,0) and explain why it exists even though $f'(0)$ does not exist.
▷ NDER $= (.001^{1/3} - (-.001)^{1/3})/.002 = (.1 + .1)/.002 = 100$. NDER always exists since we don't divide by 0.

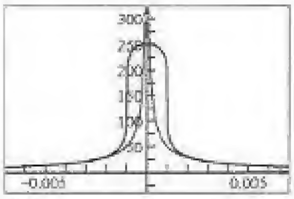
In Exercises 28 and 29, (a) show that $f'(0)$ does not exist; compute NDER($f(x)$,0) (b) by formula (4) and (c) by calculator. (d) Explain why NDER exists. (e) Plot the graph of NDER($f(x)$, x). (f) Compare its behavior at 0 with the result of (a).

28. $f(x) = x^{1/5}$
▷ (a) $f'(0) = \lim\limits_{x\to 0} \dfrac{x^{1/5} - 0}{x - 0} = \lim\limits_{x\to 0} \dfrac{1}{x^{4/5}} = +\infty$.

(b) and (c) NDER($x^{1/5}$,0) $= \dfrac{.001^{1/5} - (-.001)^{1/5}}{.002} = 251.19$.

(d) NDER always exists since we don't divide by 0

(f) The figure at the right shows NDER dark and f' light.



29. $f(x) = x^{2/3}$. $f'_-(0) = \lim\limits_{x\to 0^-} \dfrac{x^{2/3} - 0}{x - 0} = \lim\limits_{x\to 0^-} \dfrac{1}{x^{1/3}} = -\infty$; $f'_+(0) = \lim\limits_{x\to 0^+} \dfrac{1}{x^{1/3}} = +\infty$

Because f is an even function, NDER($f(x)$, 0) $= \dfrac{f(\Delta x) - f(-\Delta x)}{2\Delta x} = \dfrac{0}{2\Delta x} = 0$.

30. The inconsistency in Exercise 29 is because the graph has a cusp at the origin.

2.4 THEOREMS ON DIFFERENTIATION OF ALGEBRAIC FUNCTIONS AND HIGHER ORDER-DERIVATIVES

The differentiation formulas that are proved in this section may be used to differentiate any rational function and should be memorized. Algebraic functions are discussed in Section 2.9.

2.4.1 Theorem Constant Rule If c is a constant,

$$D_x(c) = 0$$

The derivative of a constant is zero.

2.4.3 Theorem Constant Multiple If c is a constant and if $D_x f(x)$ exists,

$$D_x[c \cdot f(x)] = c \cdot D_x f(x)$$

The derivative of a constant times a function is the constant times the derivative of the function, if this derivative exists.

2.4.4 Theorem Sum rule If $D_x f(x)$ and $D_x g(x)$ exist,

$$D_x[f(x) + g(x)] = D_x f(x) + D_x g(x)$$

The derivative of the sum of functions is the sum of their derivatives, if these derivatives exist. This rule holds for any number of terms.

2.4.6 Theorem Product rule If $D_x f(x)$ and $D_x g(x)$ exist,

$$D_x[f(x) \cdot g(x)] = f(x) \cdot D_x g(x) + g(x) \cdot D_x f(x) = D_x f(x) \cdot g(x) + f(x) \cdot D_x g(x)$$

The derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function, if these derivatives exist. Warning: In the first form, the product and quotient rules have their terms interchanged; in the second form the product and quotient rules are in the same order.

General Product Rule (Exercise 53.) If $f(x)$, $g(x)$, $h(x)$ are differentiable,
$[f(x)g(x)h(x)]' = f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$
In the first term, we differentiate the first factor, in the second term we differentiate the second factor, etc. This rule holds for any number of factors.

2.4.7 Theorem Quotient rule If $g(x) \neq 0$ and $D_x f(x)$ and $D_x g(x)$ exist, then

$$D_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot D_x f(x) - f(x) \cdot D_x g(x)}{[g(x)]^2} = \frac{D_x f(x) \cdot g(x) - f(x) \cdot D_x g(x)}{[g(x)]^2}$$

The derivative of the quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of denominator, all divided by the square of the denominator, if the derivatives exist.

2.4.2 Theorem Power rule If r is a positive integer or r is a negative integer and $x \neq 0$, then

$$D_x(cx^r) = crx^{r-1}$$

A power in the denominator is moved to the numerator, changing the sign of its exponent. The formula for the derivative of the power function was proved only for the case in which the exponent r is an integer. In Section 2.9 we show that the formula holds when r is any rational number, and in Section 5.5 we show that it is valid when r is any real number.

When convenient, we replace a given product by an equivalent sum before differentiating, because the formula for the derivative of a sum is easier to apply than the formula for the derivative of a product, as illustrated in Exercise 20. Also, if possible, we replace a given quotient by an equivalent sum before differentiating, as illustrated in Exercise 32.

Higher Order Derivatives The first derivative of the function f is denoted by f'; the second derivative of f is the first derivative of f' and is denoted by f''; the third derivative of f is the first derivative of f'' and is denoted by f'''; and so forth, provided these derivatives exist. If n is an integer greater than 1, the nth derivative of f is denoted by $f^{(n)}$, and $f^{(n)}$ is the first derivative of $f^{(n-1)}$.

Furthermore if $y = f(x)$, then $\frac{dy}{dx} = f'(x)$, $\frac{d^2y}{dx^2} = f''(x)$, $\frac{d^3y}{dx^3} = f'''(x)$, $\frac{d^4y}{dx^4} = f^{(4)}(x)$, and so on.

Leibniz Rule (Exercise 58.) If all derivatives exist, following the pattern of the binomial theorem,
$(fg)'' = f''g + 2f'g' + fg''$, $(fg)''' = f'''g + 3f''g' + 3f'g'' + fg''$, etc

Exercises 2.4

In Exercises 1–24, differentiate the function by applying the theorems of this section.

1. $f'(x) = \mathrm{D}_x(7x - 5) = \mathrm{D}_x(7x) - \mathrm{D}_x(5) = 7$

2. $g'(x) = \mathrm{D}_x(8 - 3x) = \mathrm{D}_x(8) - \mathrm{D}_x(3x) = -3$

3. $g'(x) = \mathrm{D}_x(1 - 2x - x^2) = \mathrm{D}_x(1) + \mathrm{D}_x(-2x) + \mathrm{D}_x(-x^2) = -2 - 2x$

4. $f(x) = 4x^2 + 4x + 1$
   - $f'(x) = \mathrm{D}_x(4x^2) + \mathrm{D}_x(4x) + \mathrm{D}_x(1) = 4(2x^{2-1}) + 4(1x^{1-1}) + 0 = 8x + 4$

5. $f'(x) = \mathrm{D}_x(x^3 - 3x^2 + 5x - 2) = \mathrm{D}_x(x^3) + \mathrm{D}_x(-3x^2) + \mathrm{D}_x(5x) + \mathrm{D}_x(-2) = 3x^2 - 6x + 5$

6. $f'(x) = \mathrm{D}_x(3x^4 - 5x^2 + 1) = \mathrm{D}_x(3x^4) - \mathrm{D}_x(5x^2) + \mathrm{D}_x(1) = 12x^3 - 10x$

7. $f'(x) = \mathrm{D}_x(\frac{1}{8}x^8 - x^4) = \mathrm{D}_x(\frac{1}{8}x^8) + \mathrm{D}_x(-x^4) = x^7 - 4x^3$

8. $g(x) = x^7 - 2x^5 + 5x^3 - 7x$
   - $g'(x) = 7x^{7-1} - 2(5x^{5-1}) + 5(3x^{3-1}) - 7(x^{1-1}) = 7x^6 - 10x^4 + 15x^2 - 7$

9. $\mathrm{F}'(t) = \mathrm{D}_t(\frac{1}{4}t^4 - \frac{1}{2}t^2) = \mathrm{D}_t(\frac{1}{4}t^4) + \mathrm{D}_t(-\frac{1}{2}t^2) = t^3 - t$

10. $\mathrm{H}'(x) = \mathrm{D}_x(\frac{1}{3}x^3 - x + 2) = \mathrm{D}_x(\frac{1}{3}x^3) - \mathrm{D}_x(x) + \mathrm{D}_x(2) = x^2 - 1$

11. $\mathrm{v}'(r) = \mathrm{D}_r(\frac{4}{3}\pi r^3) = 4\pi r^2$

12. $\mathrm{G}(y) = y^{10} + 7y^5 - y^3 + 1$
    - $\mathrm{G}'(y) = 10y^{10-1} + 7(5y^{5-1}) - 3y^{3-1} + 0 = 10y^9 + 35y^4 - 3y^2$

13. $\mathrm{F}'(x) = \mathrm{D}_x\left(x^2 + 3x + \frac{1}{x^2}\right) = \mathrm{D}_x(x^2) + \mathrm{D}_x(3x) + \mathrm{D}_x(x^{-2}) = 2x + 3 - 2x^{-3} = 2x + 3 - \frac{2}{x^3}$

14. $f'(x) = \mathrm{D}_x\left(\frac{x^3}{3} + \frac{3}{x^3}\right) = \mathrm{D}_x(\frac{1}{3}x^3) + \mathrm{D}_x(3x^{-3}) = x^2 - 9x^{-4} = x^2 - \frac{9}{x^4}$

15. $g'(x) = \mathrm{D}_x\left(4x^4 - \frac{1}{4x^4}\right) = \mathrm{D}_x(4x^4) + \mathrm{D}_x(-\frac{1}{4}x^{-4}) = 16x^3 + x^{-5} = 16x^3 + \frac{1}{x^5}$

16. $f(x) = x^4 - 5 + x^{-2} + 4x^{-4}$
    - $f'(x) = 4x^{4-1} - 0 + (-2)x^{-2-1} + 4(-4)x^{-4-1} = 4x^3 - 2x^{-3} - 16x^{-5}$

17. $g'(x) = \mathrm{D}_x\left(\frac{3}{x^2} + \frac{5}{x^4}\right) = \mathrm{D}_x(3x^{-2}) + \mathrm{D}_x(5x^{-4}) = -6x^{-3} - 20x^{-5} = -\frac{6}{x^3} - \frac{20}{x^5}$

18. $\mathrm{H}'(x) = \mathrm{D}_x\left(\frac{5}{6x^5}\right) = \mathrm{D}_x(\frac{5}{6}x^{-5}) = -\frac{25}{6}x^{-6} = -\frac{25}{6x^6}$

19. $f'(s) = \mathrm{D}_s[\sqrt{3}(s^3 - s^2)] = \sqrt{3}\mathrm{D}_s(s^3 - s^2) = \sqrt{3}(3s^2 - 2s)$

20. $g(x) = (2x^2 + 5)(4x - 1)$
    - We express $g(x)$ as a sum and use the sum rule.
      $g(x) = 8x^3 - 2x^2 + 20x - 5,$ $\qquad$ $g'(x) = 24x^2 - 4x + 20$

    ALTERNATE SOLUTION: We use the product rule.
    $$g'(x) = (2x^2 + 5)\mathrm{D}_x(4x - 1) + (4x - 1)\mathrm{D}_x(2x^2 + 5) = (2x^2 + 5)(4) + (4x - 1)(4x) = 8x^2 + 20 + 16x^2 - 4x$$
    $$= 24x^2 - 4x + 20$$

21. $f'(x) = \mathrm{D}_x[(2x^4 - 1)(5x^3 + 6x)] = (2x^4 - 1)\mathrm{D}_x(5x^3 + 6x) + (5x^3 + 6x)\mathrm{D}_x(2x^4 - 1)$
    $= (2x^4 - 1)(15x^2 + 6) + (5x^3 + 6x)(8x^3) = 30x^6 + 12x^4 - 15x^2 - 6 + 40x^6 + 48x^4 = 70x^6 + 60x^4 - 15x^2 - 6$

22. $f'(x) = \mathrm{D}_x[(4x^2 + 3)^2] = \mathrm{D}_x[(4x^2 + 3)(4x^2 + 3)] = \mathrm{D}_x(4x^2 + 3)\cdot(4x^2 + 3) + (4x^2 + 3)\mathrm{D}_x(4x^2 + 3)$
    $= 2(8x)(4x^2 + 3) = 64x^3 + 48x$

23. $G'(y) = D_y[(7-3y^3)^2] = D_y[(7-3y^3)(7-3y^3)] = (7-3y^3)D_y(7-3y^3) + (7-3y^3)D_y(7-3y^3)$
$= (7-3y^3)(-9y^2) + (7-3y^3)(-9y^2) = -18y^2(7-3y^3)$

24. $F(t) = (t^3-2t+1)(2t^2+3t)$ ▷ We use the product rule.
$F'(t) = (t^3-2t+1)D_t(2t^2+3t) + (2t^2+3t)D_t(t^3-2t+1) = (t^3-2t+1)(4t+3) + (2t^2+2t)(3t^2-2)$
$= 4t^4 - 8t^2 + 4t + 3t^3 - 6t + 3 + 6t^4 + 9t^3 - 4t^2 - 6t = 10t^4 + 12t^3 - 12t^2 - 8t + 3$

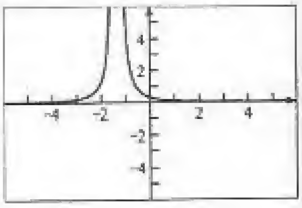
ALTERNATE SOLUTION: First express $F(t)$ as a sum and then differentiate.
$F(t) = 2t^5 + 3t^4 - 4t^3 - 4t^2 + 3t$ $\qquad F'(t) = 10t^4 + 12t^3 - 12t^2 - 8t + 3$

In Exercises 25–36, compute the derivative. In Exercises 25–30, check by plotting your answer and NDER.

25. $D_x[(x^2-3x+2)(2x^3+1)] = (x^2-3x+2)D_x(2x^3+1) + (2x^3+1)D_x(x^2-3x+2)$
$= (x^2-3x+2)(6x^2) + (2x^3+1)(2x-3) = 6x^4 - 18x^3 + 12x^2 + 4x^4 - 6x^3 + 2x - 3$
$= 10x^4 - 24x^3 + 12x^2 + 2x - 3$

26. $D_x\left(\dfrac{2x}{x+3}\right) = \dfrac{D_x(2x)\cdot(x+3) - 2xD_x(x+3)}{(x+3)^2} = \dfrac{2(x+3)-2x}{(x+3)^2} = \dfrac{3}{(x+3)^2}$

27. $D_x\left(\dfrac{x}{x-1}\right) = \dfrac{(x-1)D_x x - xD_x(x-1)}{(x-1)^2} = \dfrac{(x-1)(1)-x(1)}{(x-1)^2} = \dfrac{-1}{(x-1)^2}$



28. $D_y\left(\dfrac{2y+1}{3y+4}\right)$ ▷ We use the quotient rule.

$D_y\left(\dfrac{2y+1}{3y+4}\right) = \dfrac{(3y+4)\cdot D_y(2y+1) - (2y+1)\cdot D_y(3y+4)}{(3y+4)^2}$
$= \dfrac{(3y+4)(2)-(2y+1)(3)}{(3y+4)^2} = \dfrac{5}{(3y+4)^2}$

29. $\dfrac{d}{dx}\left(\dfrac{x^2+2x+1}{x^2-2x+1}\right) = \dfrac{(x^2-2x+1)D_x(x^2+2x+1) - (x^2+2x+1)D_x(x^2-2x+1)}{(x^2-2x+1)^2}$
$= \dfrac{(x^2-2x+1)(2x+2)-(x^2+2x+1)(2x-2)}{(x^2-2x+1)^2} = \dfrac{2(x-1)^2(x+1) - 2(x+1)^2(x-1)}{[(x-1)^2]^2}$
$= \dfrac{2(x-1)(x+1)[(x-1)-(x+1)]}{(x-1)^4} = \dfrac{2(x+1)(-2)}{(x-1)^3} = -\dfrac{4(x+1)}{(x-1)^3}$

30. $\dfrac{d}{dx}\left(\dfrac{4-3x-x^2}{x-2}\right) = \dfrac{D_x(4-3x-x^2)\cdot(x-2) - (4-3x-x^2)D_x(x-2)}{(x-2)^2} = \dfrac{(-3-2x)(x-2)-(4-3x-x^2)}{(x-2)^2}$
$= \dfrac{-x^2-4x-2}{(x-2)^2}$

31. $\dfrac{d}{dt}\left(\dfrac{5t}{1+2t^2}\right) = \dfrac{(1+2t^2)D_t(5t)-(5t)D_t(1+2t^2)}{(1+2t^2)^2} = \dfrac{(1+2t^2)(5)-(5t)(4t)}{(1+2t^2)^2} = \dfrac{5+10t^2-20t^2}{(1+2t^2)^2} = \dfrac{5-10t^2}{(1+2t^2)^2}$

32. $\dfrac{d}{dx}\left(\dfrac{x^4-2x^2+5x+1}{x^4}\right)$ ▷ We divide and use the formula for the derivative of a sum.

$\dfrac{d}{dx}\left(\dfrac{x^4-2x^2+5x+1}{x^4}\right) = \dfrac{d}{dx}(1-2x^{-2}+5x^{-3}+x^{-4}) = 4x^{-3} - 15x^{-4} - 4x^{-5}$
$= x^{-5}(4x^2 - 15x^1 - 4) = \dfrac{4x^2-15x-4}{x^5}$

33. $\dfrac{d}{dy}\left(\dfrac{y^3-8}{y^3+8}\right) = \dfrac{(y^3+8)(3y^2)-(y^3-8)(3y^2)}{(y^3+8)^2} = \dfrac{3y^2(y^3+8-y^3+8)}{(y^3+8)^2} = \dfrac{48y^2}{(y^3+8)^2}$

34. $\dfrac{d}{ds}\left(\dfrac{s^2-a^2}{s^2+a^2}\right) = \dfrac{D_s(s^2-a^2)\cdot(s^2+a^2)-(s^2-a^2)D_s(s^2+a^2)}{(s^2+a^2)^2} = \dfrac{2s(s^2+a^2)-(s^2-a^2)2s}{(s^2+a^2)^2} = \dfrac{4a^2s}{(s^2+a^2)^2}$

35. $D_x\left[\dfrac{2x+1}{x+5}(3x-1)\right] = \left(\dfrac{2x+1}{x+5}\right)D_x(3x-1) + (3x-1)D\left(\dfrac{2x+1}{x+5}\right)$
$= \left(\dfrac{2x+1}{x+5}\right)(3) + (3x-1)\dfrac{(x+5)2-(2x+1)1}{(x+5)^2} = \dfrac{6x+3}{x+5} + (3x-1)\dfrac{2x+10-2x-1}{(x+5)^2}$
$= \dfrac{(6x+3)(x+5)}{(x+5)^2} + \dfrac{(3x-1)9}{(x+5)^2} = \dfrac{6x^2+33x+15+27x-9}{(x+5)^2} = \dfrac{6x^2+60x+6}{(x+5)^2} = \dfrac{6(x^2+10x+1)}{(x+5)^2}$

36. $\mathrm{D}_x\left[\dfrac{(x^3+1)(x^2-2x^{-1}+1)}{x^2+3}\right] = \mathrm{D}_x\left[\dfrac{x^5+x^3-x^2-2x^{-1}+1}{x^2+3}\cdot\dfrac{x}{x}\right] = \mathrm{D}_x\left[\dfrac{x^6+x^4-x^3+x-2}{x^3+3x}\right]$

$= \dfrac{(x^3+3x)\mathrm{D}_x(x^6+x^4-x^3+x-2)-(x^6+x^4-x^3+x-2)\mathrm{D}_x(x^3+3x)}{(x+3x)^2}$

$= \dfrac{(x^3+3x)(6x^5+4x^3-3x^2+1)-(x^6+x^4-x^3+x-2)(3x^2+3)}{(x^3+3x)^2} = \dfrac{3x^8+16x^6+9x^4-8x^3+6x^2+6}{x^2(x^2+3)^2}$

In Exercises 37 and 38, find all the derivatives of f.

37. $f(x)=6x^5+3x^4-2x^3+5x^2-8x+9$, $f'(x)=30x^4+12x^3-6x^2+10x-8$, $f''(x)=120x^3+36x^2-12x+10$
$f'''(x)=360x^2+72x-12$, $f^{(4)}(x)=720x+72$, $f^{(5)}=720$, $f^{(n)}(x)=0$ if $n\ge 6$

38. $f(x)=2x^7-x^5+5x^3-8x+4$, $f'(x)=14x^6-5x^4+15x^2-8$, $f''(x)=84x^5-20x^3+30x$,
$f'''(x)=420x^4-60x^2+30$, $f^{(4)}(x)=1680x^3-120x$, $f^{(5)}(x)=5040x^2-120$, $f^{(6)}(x)=10080x$,
$f^{(7)}(x)=10080$, $f^{(n)}(x)=0$ if $n\ge 8$

39. $\mathrm{D}_t^{\,3}\left(\dfrac{1}{6t^3}\right)=\mathrm{D}_t^{\,2}(\mathrm{D}_t\tfrac{1}{6}t^{-3})=\mathrm{D}_t(\mathrm{D}_t-\tfrac{1}{2}t^{-4})=\mathrm{D}_t(2t^{-5})=-10t^{-6}$

40. Find $\dfrac{d^4}{dx^4}\left(x^5-\dfrac{1}{15x^5}\right)$ ▷ $=\mathrm{D}_x\mathrm{D}_x\mathrm{D}_x[\mathrm{D}_x(x^5-\tfrac{1}{15}x^{-5})]=\mathrm{D}_x\mathrm{D}_x\mathrm{D}_x(5x^4+\tfrac{1}{3}x^{-6})$
$=\mathrm{D}_x\mathrm{D}_x(20x^3-2x^{-7})=\mathrm{D}_x(60x^2+14x^{-8})=120x-112x^{-9}$

In Exercises 41 and 42, find $\dfrac{d^2y}{dx^2}$. Check by plotting your answer and your calculator's second derivative.

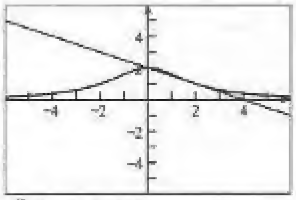
41. $y=\dfrac{x^4+1}{x^2}$ ▷ $=x^2+x^{-2}$, $y'=2x-2x^{-3}$, $y''=2+6x^{-4}$

42. $y=\dfrac{3}{x}-\dfrac{1}{3x^3}$ ▷ $=3x^{-1}-\tfrac{1}{3}x^{-3}$, $y'=-3x^{-2}+x^{-4}$, $y''=6x^{-3}-4x^{-5}$

In Exercises 43–46, find an equation of the tangent line or normal line, at the point. Check by plotting the line and the curve.

43. $y=x^3-4$, $(2,4)$, tangent line ▷ Let $f(x)=x^3-4$. Then $f'(x)=3x^2$, so $f'(2)=12$. Therefore the tangent line has slope 12; so an equation of the tangent line is $y=12(x-2)+4$, or $y=12x-20$.

44. $y=8/(x^2+4)$, $(2,1)$, tangent line.



▷ Because the slope of the tangent line is the value of the derivative,

$m(x)=\mathrm{D}_x\left(\dfrac{8}{x^2+4}\right)=\dfrac{(x^2+4)\cdot 0-8(2x)}{(x^2+4)^2}=\dfrac{-16x}{(x^2+4)^2}$

Therefore $\quad m(2)=-\dfrac{16(2)}{(2^2+4)^2}=-\tfrac{1}{2}$

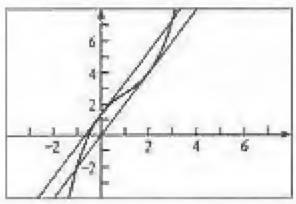
The line contains the point $(2,1)$ and has slope $-\tfrac{1}{2}$. An equation of the line is
$y-1=-\tfrac{1}{2}(x-2) \qquad x+2y-4=0$

45. $y=\dfrac{10}{14-x^2}$, $(4,-5)$, normal line ▷ Let $f(x)=\dfrac{10}{14-x^2}$. Then $f'(x)=\dfrac{(14-x^2)0-10(-2x)}{(14-x^2)^2}=\dfrac{20x}{(14-x^2)^2}$; so $f'(4)=\dfrac{80}{(-2)^2}$. The normal line at $(4,-5)$ has slope $-\frac{1}{20}$; so its equation is $y=-\frac{1}{20}(x-4)-5$ or $y=-\frac{1}{20}x-\frac{24}{5}$.

46. $y=4x^2-8x$, $(1,-4)$, normal line ▷ Let $f(x)=4x^2-8x$. Then $f'(x)=8x$; so $f'(1)=8$. The normal line at $(1,-4)$ has slope $-\frac{1}{8}$; so its equation is $y=-\frac{1}{8}(x-1)-4$ or $y=-\frac{1}{8}x-\frac{31}{8}$.

47. Find an equation of the line tangent to $y=3x^2-4x$ and parallel to the line $2x-y+3=0$.
▷ Let $f(x)=3x^2-4x$. Then $f'(x)=6x-4$. The given line has the equation $y=2x+3$ and has slope 2. $f'(x)=2$ when $6x-4=2$ so $x=1$. Since $f(1)=-1$, the desired tangent line passes through $(1,-1)$ and has slope 2; so its equation is $y=2(x-1)-1$ or $y=2x-3$.

48. Find an equation of each of the tangent lines to $3y=x^3-3x^2+6x+4$ that is parallel to $2x-y+3=0$. Check by plotting the curve and the lines.



▷ Let L_1 be one of the required lines and let (x_1,y_1) be the point at which L_1 is tangent to the curve. The slope of L_1 is given by $m(x_1)=y'(x_1)$.

Because $\quad y=\tfrac{1}{3}x^3-x^2+2x+\tfrac{4}{3} \qquad (1)$
then $\quad y'(x)=x^2-2x+2$
and $\quad m(x_1)={x_1}^2-2x_1+2 \qquad (2)$

Because the slope-intercept form of the equation of the given line is $y = 2x + 3$, the slope of this line is 2. Because L_1 is parallel to the given line, the slope of L_1 is also 2. Therefore, $m(x_1) = 2$, and by Eq. (2) we have

$$2 = x_1^2 - 2x_1 + 2, \quad x_1 = 0 \text{ or } x_1 = 2$$

If $x_1 = 0$, then by Eq. (1) $y = \frac{4}{3}$. Thus L_1 contains $(0, \frac{4}{3})$ and has slope 2. An equation of L_1 is

$$y = 2x + \tfrac{4}{3}$$

If $x_1 = 2$, then $y = \frac{1}{3}(2)^3 - (2)^2 + 2(2) + \frac{4}{3} = 4$. Therefore an equation of the second of the required lines is

$$y - 4 = 2(x - 2), \qquad y = 2x$$

49. Find an equation of each normal line to $y = x^3 - 4x$ that is parallel to the line $x + 8y - 8 = 0$.

▷ Let $f(x) = x^3 - 4x$. Then $f'(x) = 3x^2 - 4$. At the point $(x, f(x))$ on the curve, the slope of the normal line is $\dfrac{-1}{f'(x)} = \dfrac{-1}{3x^2 - 4}$. The given line has the equation $y = -\frac{1}{8}x + 1$ and hence has slope $-\frac{1}{8}$. Since the normal lines are parallel to the given line, we must have $\dfrac{-1}{3x^2 - 4} = -\dfrac{1}{8}$; $3x^2 - 4 = 8$; $x = \pm 2$

Now $f(2) = 0$ and $f(-2) = 0$; so the required normal lines are at $(2, 0)$ and $(-2, 0)$ and each line has slope $-\frac{1}{8}$. Hence, their equations are $y = -\frac{1}{8}(x - 2)$; $x + 8y - 2 = 0$ and $y = -\frac{1}{8}(x + 2)$; $x + 8y + 2 = 0$

50. Find an equation of the line tangent to $y = x^4 - 6x$ and perpendicular to the line $x - 2y + 6 = 0$.

▷ The given line has equation $y = \frac{1}{2}x + 3$ and slope $\frac{1}{2}$; a perpendicular has slope -2. Let $f(x) = x^4 - 6x$. Then $f'(x) = 4x^3 - 6$. At the point $(x, f(x))$ on the curve, the slope of the tangent line is $4x^3 - 6 = -2$ so $x = 1$. Because $f(1) = -5$, the required line is $y = -2(x - 1) - 5$ or $y = -2x - 3$.

51. Find an equation of each line through $(4, 13)$ that is tangent to the parabola $y = 2x^2 - 1$.

▷ Let $f(x) = 2x^2 - 1$. Then $f'(x) = 4x$. A line through the points $P(x_1, 2x_1^2 - 1)$ and $(4, 13)$ has slope $\dfrac{(2x_1^2 - 1) - 13}{x_1 - 4}$. This line will be tangent at P if its slope is $f'(x_1) = 4x_1$. Hence $\dfrac{(2x_1^2 - 1) - 13}{x_1 - 4} = 4x_1$; $2x_1^2 - 14 = 4x_1^2 - 16x_1$; $2x_1^2 - 16x_1 + 14 = 0$; $2(x_1 - 7)(x_1 - 1) = 0$ so $x_1 = 7$ or $x_1 = 1$. Now $f(7) = 97$ and $f'(7) = 28$ while $f(1) = 1$ and $f'(1) = 4$. Hence the equations of the tangent lines are $y - 97 = 28(x - 7)$; $28x - y - 99 = 0$ and $y - 1 = 4(x - 1)$; $4x - y - 3 = 0$

52. Given $f(x) = \frac{1}{3}x^3 + 2x^2 + 5x + 5$, show that $f'(x) \geq 0$ for all values of x.

▷ $f'(x) = x^2 + 4x + 5 = (x + 2)^2 + 1 \geq 1$ for all x.

53. Prove the general product rule for three factors. ▷ Applying the product rule twice, we have

$$\begin{aligned}\{[f(x) \cdot g(x)] \cdot h(x)\}' &= [f(x) \cdot g(x)]' \cdot h(x) + [f(x) \cdot g(x)] \cdot h'(x) \\ &= [f'(x) \cdot g(x) + f(x) \cdot g'(x)] \cdot h(x) + [f(x) \cdot g(x)] \cdot h'(x) \\ &= f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)\end{aligned}$$

In Exercises 54–57, use the general product rule to find the derivative.

54. $[(x^2 + 3)(2x - 5)(3x + 2)]' = (x^2 + 3)'(2x - 5)(3x + 2) + (x^2 + 3)(2x - 5)'(3x + 2) + (x^2 + 3)(2x - 5)(3x + 2)'$
$= 2x(2x - 5)(3x + 2) + (x^2 + 3)(2)(3x + 2) + (x^2 + 3)(2x - 5)(3)$
$= (12x^3 - 22x^2 - 20x) + (6x^3 + 4x^2 + 18x + 12) + (6x^3 - 15x^2 + 18x - 45) = 24x^3 - 33x^2 + 16x - 33$

55. $[(3x + 2)^2(x^2 - 1)]' = [(3x + 2)(3x + 2)(x^2 - 1)]'$
$= (3x + 2)'(3x + 2)(x^2 - 1) + (3x + 2)(3x + 2)'(x^2 - 1) + (3x + 2)(3x + 2)(x^2 - 1)'$
$= 3(3x + 2)(x^2 - 1) + (3x + 2)(3)(x^2 - 1) + (3x + 2)(3x + 2)(2x)$
$= (3x + 2)(3x^2 - 3 + 3x^2 - 3 + 6x^2 + 4x) = (3x + 2)(12x^2 + 4x - 6) = 2(3x + 2)(6x^2 + 2x - 3)$

56. $[(3x^3 + x^{-3})(x + 3)(x^2 - 5)]'$
$= (3x^3 + x^{-3})'(x + 3)(x^2 - 5) + (3x^3 + x^{-3})(x + 3)'(x^2 - 5) + (3x^3 + x^{-3})(x + 3)(x^2 - 5)'$
$= (9x^2 - 3x^{-4})(x + 3)(x^2 - 5) + (3x^3 + x^{-3})(1)(x^2 - 5) + (3x^3 + x^{-3})(x + 3)(2x)$
$= (9x^5 + 27x^4 - 45x^3 - 135x^2 - 3x^{-1} - 9x^{-2} + 15x^{-3} + 45x^{-4}) + (3x^5 + x^{-1} - 5x^{-3}) + (6x^5 + 18x^4 + 2x^{-1} + 6x^{-2})$
$= 18x^5 + 45x^4 - 60x^3 - 135x^2 - 3x^{-2} + 10x^{-3} + 45x^{-4}$

57. $[(2x^2 + x + 1)^3]' = D[(2x^2 + x + 1)(2x^2 + x + 1)(2x^2 + x + 1)]$
$= (2x^2 + x + 1)'(2x^2 + x + 1)(2x^2 + x + 1) + (2x^2 + x + 1)(2x^2 + x + 1)'(2x^2 + x + 1)$
$\qquad + (2x^2 + x + 1)(2x^2 + x + 1)(2x^2 + x + 1)'$
$= (4x + 1)(2x^2 + x + 1)(2x^2 + x + 1) + (2x^2 + x + 1)(4x + 1)(2x^2 + x + 1) + (2x^2 + x + 1)(2x^2 + x + 1)(4x + 1)$
$= 3(4x + 1)(2x^2 + x + 1)^2$

58. Prove the Leibniz rule for the second derivative.

➤ $(fg)'' = [(fg)']' = [f'g + fg']' = (f'g)' + (fg')' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''$

59. If $y = x^n$, prove by mathematical induction that $\dfrac{d^n y}{dx^n} = n!$

➤ If $n = 1$, $y = x$ and $\dfrac{dy}{dx} = 1 = 1!$. Now assume the formula is true for $n = k$.

If $y = x^{k+1}$, then $D^{k+1}y = D^{k+1}(x^{k+1}) = D[(k+1)x^k] = (k+1)D_x(x^k) = (k+1)k! = (k+1)!$.
Thus the formula holds true for $n = k + 1$ and hence for any integer n.

60. Use formula (7) of §2.1 to prove the power rule.

➤ $\dfrac{d}{dx}x^n = \lim\limits_{z \to x} \dfrac{z^n - x^n}{z - x} = \lim\limits_{z \to x} \dfrac{(z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}{z - x}$
$= \lim\limits_{z \to x}(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) = x^{n-1} + \cdots + x^{n-1}$ (n terms) $= nx^{n-1}$

61. Prove that if f and g are differentiable functions such that $f(0) = 0$ and $g(0) = 0$ then $f(x)g(x)$ cannot be x.

➤ Suppose $f(x)g(x) = x$. By the product rule: $f'(x)g(x) + f(x)g'(x) = 1$. At 0: $f'(0) \cdot 0 + 0 \cdot g'(0) = 0 \neq 1$.

62. Show how three theorem on differentiation enable us to differentiate any polynomial.

$(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0)' \overset{\text{sum rule}}{=} (c_n x^n)' + (c_{n-1}x^{n-1})' + \cdots + (c_1 x^1)' + (c_0 x^0)'$

$\overset{\text{constant multiple}}{=} c_n(x^n)' + c_{n-1}(x^{n-1})' + \cdots + c_1(x^1)' + c_0(x^0)' \overset{\text{power}}{=} nc_n x^{n-1} + (n-1)c_{n-1}x^{n-2} + \cdots + c_1$

2.5 RECTILINEAR MOTION

2.5.1 Definition If f is a function given by the equation $s = f(t)$ and a particle is moving along a straight line so that s is the number of units in the directed distance of the particle from a fixed point on the line at t units of time, then the *instantaneous velocity* of the particle at t units of time is v units of velocity, where

$$v = f'(t) \quad \Leftrightarrow \quad v = \frac{ds}{dt}$$

if it exists. The instantaneous rate of change of the velocity is called the *instantaneous acceleration*.

Exercises 2.5

In Exercises 1–8, a particle is moving along a horizontal line according to the given equation, where s meters is the directed distance of the particle from a point O at t seconds. Find the instantaneous velocity $v(t)$ meters per second at t seconds, and then find $v(t_1)$ for the particular value of t_1.

1. $s = 3t^2 + 1$; $v(t) = \frac{ds}{dt} = 6t$; $v(3) = 18$
2. $s = 8 - t^2$; $v(t) = \frac{ds}{dt} = -2t$; $v(5) = -10$
3. $s = \frac{1}{4}t^{-1}$; $v(t) = \frac{ds}{dt} = -\frac{1}{4}t^{-2}$; $v(\frac{1}{2}) = -1$
4. $s = \frac{3}{t^2}$; $t_1 = -2$

   ➤ $v(t) = D_t(3t^{-2}) = -6t^{-3}$; $v(-2) = -6(-2)^{-3} = \frac{3}{4}$
5. $s = 2t^3 - t^2 + 5$; $v(t) = \frac{ds}{dt} = 6t^2 - 2t$; $v(-1) = 8$
6. $s = 4t^3 + 2t - 1$; $v(t) = \frac{ds}{dt} = 12t^2 + 2$; $v(\frac{1}{2}) = 5$
7. $s = \frac{2t}{4+t}$; $v(t) = \frac{ds}{dt} = \frac{2(4+t) - 2t}{(4+t)^2} = \frac{8}{(4+t)^2}$; $v(0) = \frac{1}{2}$
8. $s = \frac{1}{t} + \frac{3}{t^2}$; $t_1 = 2$

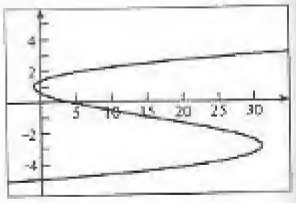
   ➤ $v(t) = D_t(t^{-1} + 3t^{-2}) = -t^{-2} - 6t^{-3}$; $v(2) = -2^{-2} - 6(2^{-3}) = -\frac{1}{4} - \frac{3}{4} = -1$

In Exercises 9–14, a particle is moving along a horizontal line according to the given equation where s meters is the directed distance of the particle from a point O at t seconds. The positive direction is to the right. Determine the intervals of time when the particle is moving to the right and when it is moving to the left. Also determine when the particle reverses its direction. Simulate the motion on your graphics calculator. In Exercises 15 and 16, plot the motion parametrically with $x = s$ and $y = t$.

9 and 15. $s = t^3 + 3t^2 - 9t + 4$

▷ $v(t) = \frac{ds}{dt} = 3t^2 + 6t - 9 = 3(t^2 + 2t - 3) = 3(t+3)(t-1)$.

$v(t) = 0$ when $t = -3$ and $t = 1$.

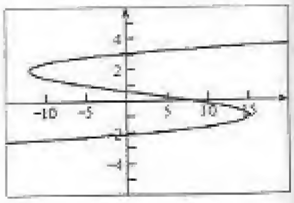


	$t+3$	$t-1$	v	particle is
$t < -3$	$-$	$-$	$+$	moving to the right
$t = -3$	0	$-$	0	changing direction from right to left
$-3 < t < 1$	$+$	$-$	$-$	moving to the left
$t = 1$	$+$	0	0	changing direction from left to right
$1 < t$	$+$	$+$	$+$	moving to the right

10 and 16. $s = 2t^3 - 3t^2 - 12t + 8$

▷ $v(t) = \frac{ds}{dt} = 6t^2 - 6t - 12 = 6(t^2 - t - 2) = 6(t+1)(t-2)$.

$v(t) = 0$ when $t = -1$ and $t = 2$



	$t+1$	$t-2$	v	particle is
$t < -1$	$-$	$-$	$+$	moving to the right
$t = -1$	0	$-$	0	changing direction from right to left
$-1 < t < 2$	$+$	$-$	$-$	moving to the left
$t = 2$	$+$	0	0	changing direction from left to right
$2 < t$	$+$	$+$	$+$	moving to the right

11. $s = \frac{2}{3}t^3 + \frac{3}{2}t^2 - 2t + 4$; $v(t) = \frac{ds}{dt} = 2t^2 + 3t - 2 = (t+2)(2t-1)$; $v(t) = 0$ when $t = -2$ and $t = \frac{1}{2}$

	$t+2$	$2t-1$	v	particle is
$t < -2$	$-$	$-$	$+$	moving to the right
$t = -2$	0	$-$	0	changing direction from right to left
$-2 < t < \frac{1}{2}$	$+$	$-$	$-$	moving to the left
$t = \frac{1}{2}$	$+$	0	0	changing direction from left to right
$\frac{1}{2} < t$	$+$	$+$	$+$	moving to the right

12. $s = \dfrac{t}{1+t^2}$

▷ The instantaneous velocity is given by

$$v(t) = \frac{d}{dt}\left(\frac{t}{1+t^2}\right) = \frac{(1+t^2)(1) - t(2t)}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}$$

$$= \frac{(1-t)(1+t)}{(1+t^2)^2} \tag{1}$$

Because $v(t) = 0$ when $t = 1$ and $t = -1$, the particle reverses direction at each of these times. Table 12a indicates the sign of each factor of v when t is in each interval. The sign of v is negative when v has one odd factor. The particle is moving to the right when $v > 0$ and is moving to the left when $v < 0$. The behavior of the motion is illustrated in the figure. Table 12b gives values of s and v for specific replacements of t, where we use the original equation of motion to calculate s and Eq. (1) to calculate v.

Table 12a

	$1-t$	$1+t$	$(1+t^2)^2$	v	particle is
$t < -1$	$+$	$-$	$+$	$-$	moving to the left
$t = -1$	$+$	0	$+$	0	reversing direction
$-1 < t < 1$	$+$	$+$	$+$	$+$	moving to the right
$t = 1$	0	$+$	$+$	0	reversing direction
$t > 1$	$-$	$+$	$+$	$-$	moving to the left

Table 12b

t	s	v
-2	$-\frac{2}{5}$	$-\frac{3}{25}$
-1	$-\frac{1}{2}$	0
0	0	1
1	$\frac{1}{2}$	0
2	$\frac{2}{5}$	$-\frac{3}{25}$

15. $s = \dfrac{1}{9+t^2}$; $v(t) = \dfrac{ds}{dt} = \dfrac{9+t^2-2t\cdot t}{(9+t^2)^2} = \dfrac{9-t^2}{(9+t^2)^2} = \dfrac{(3+t)(3-t)}{(9+t^2)^2}$; $v(t)=0$ when $t=-3$ and $t=3$

	$3+t$	$3-t$	$(9+t^2)^2$	v	particle is
$t<-3$	$-$	$+$	$+$	$-$	moving to the left
$t=-3$	0	$+$	$+$	0	changing direction from left to right
$-3<t<3$	$+$	$+$	$+$	$+$	moving to the right
$t=3$	$+$	0	$+$	0	changing direction from right to left
$3<t$	$+$	$-$	$+$	$-$	moving to the left

16. $s = \dfrac{t+1}{t^2+4}$; $v(t) = \dfrac{ds}{dt} = \dfrac{1\cdot(t^2+4)-(t+1)(2t)}{(t^2+4)^2} = \dfrac{-t^2-2t+4}{(t^2+4)^2}$; $v(t)=0$ when $t=-1\pm\sqrt{5}$

	$-t^2-2t+4$	$(t^2+4)^2$	v	particle is
$t<-1-\sqrt{5}$	$-$	$+$	$-$	moving to the left
$t=-1-\sqrt{5}$	0	$+$	0	changing direction from left to right
$-1-\sqrt{5}<t<-1+\sqrt{5}$	$+$	$+$	$+$	moving to the right
$t=-1+\sqrt{5}$	0	$+$	0	changing direction from right to left
$t>-1+\sqrt{5}$	$-$	$+$	$-$	moving to the left

In Exercises 17–21, an object moves in a vertical line such that $s=-16t^2+v_0t+s_0$, where s_0 feet and v_0 ft/sec are the initial height and velocity and s ft is the height after t sec and the positive direction is upward.

17. A stone is dropped from a height of 256 ft. (a) Write its equation of motion. (b) Find the velocity at 1 and 2 sec. (c) When does it reach the ground? (d) How fast is going then?

▷ (a) $v_0=0$, $s_0=256$. $s=-16t^2+256$ (b) $v(t)=\frac{ds}{dt}=-32t$. $v(1)=-32$. Therefore, 1 sec after the stone is dropped its velocity is -32ft/sec. $v(2)=-64$. Thus, 2 sec after the stone is dropped its velocity is -64ft/sec. (c) The stone reaches the ground when $s=0$. Then $-16t^2+256=0$; $t^2=16$; $t=4$. Thus the stone reaches the ground in 4 sec. (d) $v(4)=-128$. Hence when the stone reaches the ground its velocity is -128ft/sec.

18. A chandelier is dropped from a height of 160 ft. (a) Write its equation of motion. (b) Find the velocity at 1 and 1.5 sec. (c) When does it reach the ground? (d) How fast is going then?

▷ (a) $v_0=0$, $s_0=160$. $s=-16t^2+160$ (b) $v(t)=\frac{ds}{dt}=-32t$. $v(1)=-32$. Therefore, 1 sec after it is dropped its velocity is -32ft/sec. $v(1.5)=-48$. Thus, 1.5 sec after it is dropped its velocity is -48 ft/sec. (c) It reaches the ground when $s=0$. Then $-16t^2+160=0$; $t^2=10$; $t=\sqrt{10}$. Thus it reaches the ground in about 3.2 sec. (d) $v(\sqrt{10})=-32\sqrt{10}$. Hence when it reaches the ground its velocity is about -101.2 ft/sec.

19. A chandelier is thrown down from a height of 160 ft at 48 ft. (a) Write its equation of motion. (b) Find the velocity at 1 and 1.5 sec. (c) When does it reach the ground? (d) How fast is going then?

▷ (a) $v_0=-48$, $s_0=160$. $s=-16t^2-48t+160$ (b) $v(t)=\frac{ds}{dt}=-32t-48$. $v(1)=-80$. Hence, 1 sec after it is thrown its velocity is -80ft/sec. $v(1.5)=-96$. Thus, 1.5 sec after it is dropped its velocity is -96 ft/sec. (c) It reaches the ground when $s=0$. Then $-16t^2-48t+160=0$; $0=t^2+3t-10=(t+5)(t-2)$; $t=2$. Thus it reaches the ground in 2 sec. (d) $v(2)=-112$. Hence when it reaches the ground its velocity is -112 ft/sec.

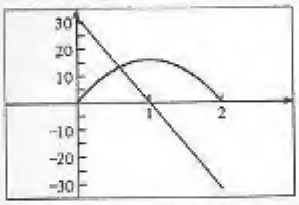
20. A ball is thrown up from the ground at 32 ft/sec. (a) Write its equation of motion. (b) Simulate its motion on your graphics calculator and estimate when and where its highest point will be. (c) Calculate the results. (d) Find its velocity and (e) its speed at 0.5 sec and 1.25 sec (f) Find its speed when it reaches the ground.

▷ (a) $v_0=32$, $s_0=0$. $s=-16t^2+32t$ (c) $v(t)=\frac{ds}{dt}=-32t+32$. The ball reaches its highest point when $v=0$; that is, when $t=1$ and $s(1)=16$. Therefore it takes the ball 1 sec to reach its maximum height of 16 ft. (d) $v(.75)=8$. Hence, .75 sec after it is thrown its velocity is 8ft/sec. $v(1.25)=-8$. Thus, 1.25 sec after it is thrown its velocity is -8 ft/sec. (e) In both cases, its speed is 8 ft/sec. (f) It reaches the ground when $s=0$. Then $-16t^2+32t=0$; $0=-16t(t-2)$; $t=2$. $v(2)=32$. Hence when it reaches the ground its speed is 32 ft/sec, the same as when it was thrown.

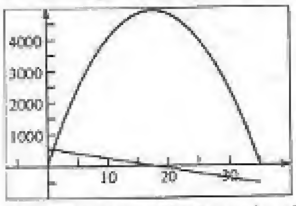
21. A rocket is fired up from the ground at 560 ft/sec. (a) Write its equation of motion. (b) Simulate its motion on your graphics calculator and estimate when and where its highest point will be. (c) Calculate the results. (d) Find its velocity and (e) its speed at 10 sec and 25 sec (f) Find its speed when it reaches the ground.

▷ (a) $v_0 = 560$, $s_0 = 0$. $s = -16t^2 + 560t$ (c) $v(t) = ds/dt = -32t + 560$. The rocket reaches its highest point when $v = 0$; that is, when $t = 17.5$ and $s(17.5) = 4900$. Therefore it takes the rocket 17.5 sec to reach its maximum height of 4900 ft. (d) $v(10) = 240$. Hence, 10 sec after it is fired its velocity is 240 ft/sec. $v(25) = -240$. Thus, 25 sec after it is fired its velocity is -240 ft/sec. (e) In both cases, its speed is 240 ft/sec. (f) It reaches the ground when $s = 0$. Then $-16t^2 + 560t = 0$; $0 = -16t(t - 35)$; $t = 35$ $v(35) = 560$. Hence when it reaches the ground its speed is 560 ft/sec, the same as when it was thrown.

22. Plot the path of the ball in Exercise 20 with $x = t$ and $y = s$, and also the velocity.



23. Plot the path of the rocket in Exercise 21 with $x = t$ and $y = s$, and also the velocity.



24. Simulate the motion of the particle of Ex. 3 on your graphics calculator. Explain why this supports the results.

▷ In Example 3 we have $s = 3t^2 - t^3$ and $v = \dfrac{ds}{dt} = 6t - 3t^2$. For visibility, we simulate the motion of the particle on the line $y = 2$. With our calculator in parametric mode, we let $x_1(t) = t^3 - 12t^2 + 36t - 24$ and $y_1(t) = 2$. In the window $[-1, 4] \times [-3, 5]$ we let $[t\text{Min}, t\text{Max}] = [0, 4]$ and t-step $= 0.05$. We now press the $\boxed{\text{TRACE}}$ key and then press the ◁ key and hold it down until the cursor is at $t = 0$. Notice the information at the bottom of the screen: $t = 0$, $x = 0$, $y = 2$. We press the ▷ key and hold it down. The cursor represents the particle moving along the line $y = 2$. Note that the particle is moving to the right until $t = 2$ and $x = 4$, when it stops and changes direction. The particle then moves to the left and disappears off the screen to the left.

In Exercises 25 and 26, a particle is moving along a line according the the equation, where s ft is the position at $t \geq 0$ sec. Find the time when the acceleration is 0 and the position and velocity at that time.

25. $s = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t + 1$ ▷ $v = t^2 - 3t + 2$, $a = \dfrac{dv}{dt} = 2t - 3 = 0$ when $t = \frac{3}{2}$. $s(\frac{3}{2}) = \frac{7}{4}$. $v(\frac{3}{2}) = -\frac{1}{4}$

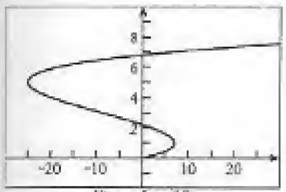
26. $s = 2t^3 - 6t^2 + 3t - 4$ ▷ $v = 6t^2 - 12t$, $a = \dfrac{dv}{dt} = 12t - 12 = 0$ when $t = 1$. $s(1) = -5$. $v(1) = -6$

In Exercises 27 and 28, make a table giving s, v, a. Include the intervals of time when the particle is moving to the left and right, and when the velocity or speed is increasing or decreasing. Plot with $x = s$ and $y = t$.

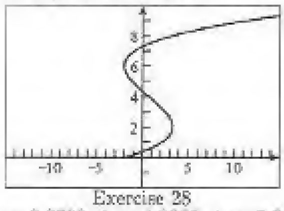
27. $s = t^3 - 9t^2 + 15t = t(t^2 - 9t + 15) = t(t - t_1)(t - t_2)$, where the roots of $t^2 - 9t + 15 = 0$ are
$t_1 = \frac{1}{2}(9 - \sqrt{81 - 4 \cdot 15}) = \frac{1}{2}(9 - \sqrt{21}) \approx 2.21$ and $t_2 = \frac{1}{2}(9 + \sqrt{21}) \approx 6.79$.
$v = D_t s = 3t^2 - 18t + 15 = 3(t^2 - 6t + 15) = 3(t - 1)(t - 5)$; $a = D_t s = 6t - 18 = 6(t - 3)$

	s	v	a	Conclusion
$t = 0$	0	+	−	Particle is at the origin and it is moving to the right The velocity is decreasing. The speed is decreasing.
$0 < t < 1$	+	+	−	Particle is right of the origin and it is moving to the right The velocity is decreasing. The speed is decreasing.
$t = 1$	7	0	−	Particle is 7 m right of the origin and it is changing its direction of motion from right to left. The velocity is decreasing. The speed is increasing.
$1 < t < t_1$	+	−	−	Particle is right of the origin, and it is moving to the left The velocity is decreasing. The speed is increasing.
$t = t_1$	0	−	−	Particle is at the origin, and it is moving to the left The velocity is decreasing. The speed is increasing.
$t_1 < t < 3$	−	−	−	Particle is left of the origin, and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = 3$	−	−	0	Particle is left of the origin, and it is moving to the left

				The velocity is not changing; so the speed is not changing.
$3 < t < 5$	$-$	$-$	$+$	Particle is left of the origin, and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = 5$	-25	0	$+$	Particle is 25 m left of the origin, and it is changing its direction of motion from left to right. The velocity is increasing. The speed is decreasing.
$5 < t < t_2$	$-$	$+$	$+$	Particle is left of the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t = t_2$	0	$+$	$+$	Particle is at the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t_2 < t$	$+$	$+$	$+$	Particle is right of the origin, and it is moving to the right The velocity is increasing. The speed is increasing.



Exercise 27



Exercise 28

29. $s = \frac{1}{6}t^3 - 2t^2 + 6t - 2 = \frac{1}{6}(t - t_1)(t - t_2)(t - t_3)$, where $t_1 \approx 0.3799$, $t_2 \approx 4.3365$, $t_3 \approx 7.2836$

$v = \frac{ds}{dt} = \frac{1}{2}t^2 - 4t + 6 = \frac{1}{2}(t^2 - 8t + 12) = \frac{1}{2}(t-2)(t-6)$; $a = \frac{ds}{dt} = t - 4$

	s	v	a	Conclusion
$0 < t < t_1$	$-$	$+$	$-$	Particle is left of the origin and it is moving to the right The velocity is decreasing. The speed is decreasing.
$t = t_1$	0	$+$	$-$	Particle is at the origin, and it is moving to the left
$t_1 < t < 2$	$+$	$+$	$-$	Particle is right of the origin and it is moving to the right The velocity is decreasing. The speed is decreasing.
$t = 2$	$\frac{10}{3}$	0	$-$	Particle is $\frac{10}{3}$ m right of the origin and it is changing its direction of motion from right to left. The velocity is decreasing. The speed is increasing.
$2 < t < 4$	$+$	$-$	$-$	Particle is right of the origin, and it is moving to the left The velocity is decreasing. The speed is increasing.
$t = 4$	$\frac{2}{3}$	$-$	0	Particle is right of the origin, and it is moving to the left The velocity is not changing; so the speed is not changing.
$4 < t < t_2$	$+$	$-$	$+$	Particle is right of the origin, and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = t_2$	0	$-$	$+$	Particle is at the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t_2 < t < 6$	$-$	$-$	$+$	Particle is left of the origin, and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = 6$	-2	0	$+$	Particle is 2 m left of the origin, and it is changing its direction of motion from left to right. The velocity is increasing. The speed is decreasing.
$6 < t < t_3$	$-$	$+$	$+$	Particle is left of the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t = t_3$	0	$+$	$+$	Particle is at the origin, and it is moving to the right The velocity is increasing. The speed is increasing.
$t_3 < t$	$+$	$+$	$+$	Particle is right of the origin, and it is moving to the right The velocity is increasing. The speed is increasing.

29. Simulate the motion of the particle of Exercise 27 on your graphics calculator.

30. Simulate the motion of the particle of Exercise 28 on your graphics calculator and explain why this supports your results.

▷ We have $s = \frac{1}{6}t^3 - 2t^2 + 6t - 2$ and $v = \frac{ds}{dt} = 6t - 3t^2$. For visibility, we simulate the motion of the particle on the line $y = 2$. With our calculator in parametric mode, we let $x_1(t) = \frac{1}{6}t^3 - 2t^2 + 6t - 2$ and $y_1(t) = 2$. In the window $[-13, 4] \times [-3, 5]$ we let $[t\text{Min}, t\text{Max}] = [0, 8]$ and t-step $= 0.05$. We now press the $\boxed{\text{TRACE}}$ key and then press the ◁ key and hold it down until the cursor is at $t = 0$. Notice the information at the bottom of the screen: $t = 0$, $x = -2$, $y = 2$. We press the ▷ key and hold it down. The cursor represents the particle moving along the line $y = 2$. Observe that the particle is moving to the right until $t = 2$ and $x = 4$, when it stops and changes direction. The particle then moves to the left until $t = 6$ and $x = -12$ when it stops and changes direction. Then it moves to the right and disappears off the screen to the right.

In Exercises 31 and 32, the equation of motion is $s = \frac{1}{2}at^2 + v_0 t + s_0$ where a the acceleration for the body. A stone dropped from a cliff hits the ground in T sec. (a) What is the height of the cliff? (b) With what velocity does it hit the ground? (c) What velocity is needed to throw it back to its original position?

31. On the moon, $a = -5.5$; $\text{T} = 4$. The stone starts at the origin. $s = -2.75t^2$. $s(4) = -44$; the cliff is 44 ft high. $v = -5.5t$, $v(4) = -22$. The stone hits the ground at 22 ft/sec and 22 ft/sec is the velocity needed to return it.

32. On Mars, $a = -12$; $\text{T} = 3$

▷ (a) The stone starts at the origin so $s_0 = 0$. The equation of motion is $s = -6t^2$.
Because $s(3) = -54$, the cliff is 54 ft high.
(b) $v(t) = \text{D}_t s = \text{D}_t(-6t^2) = -12t$ and $v(3) = -36$.
The stone hits the ground at 36 ft/sec and 36 ft/sec is the velocity needed to return it.

33. A sprinter is s meters from the finish t sec after the start, where $s = 100 - \frac{1}{4}(t^2 + 33t)$. Find his speed (a) at the start and (b) at the finish.

▷ $v = -\frac{ds}{dt} = \frac{1}{4}(2t + 33)$. (a) $v(0) = \frac{33}{4} = 8.25$. (b) $s = 0$ when $t^2 + 33t - 400 = 0$, $t = t_1 = \frac{1}{2}(-33 + \sqrt{2689}) \approx 9.42$
$v(t_1) = \frac{1}{4}\sqrt{2689} \approx 12.96$.

34. s ft is the distance of the ball from the starting point at t sec. $s = 24t + 10t^2$; $v(t) = \frac{ds}{dt} = 24 + 20t$

(a) $v(t_1) = 24 + 20t_1$. Therefore, at $t_1$sec the instantaneous velocity of the ball is $(24 + 20t_1)$ft/sec.
(b) $v(t) = 48$ when $24 + 20t = 48$; $20t = 24$; $t = \frac{6}{5}$. Hence it takes $\frac{6}{5}$ sec for the velocity to increase to 48 ft/sec.

35. s cm is the distance of the ball from its initial position at t sec. $s = 100t^2 + 100t$; $v(t) = \frac{ds}{dt} = 200t + 100$
The billiard ball hits the cushion when $s = 39$ so we have
$100t^2 + 100t = 39$; $100t^2 + 100t - 39 = 0$; $(10t - 3)(10t + 13) = 0$; $t = \frac{3}{10}$ or $t = -\frac{13}{10}$
We reject the negative value of t. Therefore, the billiard ball hits the cushion in $\frac{3}{10}$ sec and $v(\frac{3}{10}) = 160$. Thus the velocity of the billiard ball is 160cm/sec when it hits the cushion.

36. Two particles, A and B, move to the right along a horizontal line. They start at point O, s meters is the directed distance of the particles from O at t seconds, and the equations of motion are

$$s = 4t^2 + 5t \quad \text{(for particle A)} \qquad s = 7t^2 + 3t \quad \text{(for particle B)}$$

If $t = 0$ at the start, for what values of t will the velocity of particle A exceed the velocity of particle B?

▷ The velocity of A is given by $\quad v_A = \text{D}_t(4t^2 + 5t) = 8t + 5$

The velocity of B is given by $\quad v_B = \text{D}_t(7t^2 + 3t) = 14t + 3$

We want to find when $v_A > v_B$, or equivalently, when

$$\begin{aligned} 8t + 5 &> 14t + 3 \\ -6t &> -2 \\ t &< \tfrac{1}{3} \end{aligned}$$

Thus, the velocity of A exceeds the velocity of B when $0 \le t < \frac{1}{3}$.

2.6 THE DERIVATIVE AS A RATE OF CHANGE

2.6.1 Definition If $y = f(x)$, the *instantaneous rate of change of y per unit change in x* at x_1 is $f'(x_1)$ or, equivalently, the derivative of y with respect to x at x_1, if it exists. The *marginal cost* and *marginal revenue* are the derivatives of the cost and revenue functions.

Relative Rate of Change of y with respect to x at x_1 is given by $f'(x_1)/f(x_1) = (dy/dx)/y$ evaluated at $x = x_1$.

Exercises 2.6

1. A square of side x cm has area $A(x)$ cm^2. Find the average rate of change of A as x changes from (a) 4 to 4.6; (b) 4 to 4.3; (c) 4 to 4.1; (d) 4 to 4.05. (e) Find the instantaneous rate of change when $x = 4$.
▸ $A(x) = x^2$ (a) $\Delta A/\Delta x = (4.6^2 - 4^2)/(4.6 - 4) = 8.6$ (b) $\Delta A/\Delta x = (4.3^2 - 4^2)/(4.3 - 4) = 8.3$ (c) $\Delta A/\Delta x = (4.1^2 - 4^2)/(4.1 - 4) = 8.1$ (d) $(4.05^2 - 4^2)/(4.05 - .05) = 8.05$ (e) $A'(x) = 2x$, $A'(4) = 8$

2. A rectangle of width w in. and length $w + 4$ in. has area A(w) in^2. Find the average rate of change of A as w changes from (a) 3 to 3.2; (b) 3 to 3.1; (c) 3 to 3.01; (d) 3 to 3.001. (e) Find the instantaneous rate of change when $w = 3$.
▸ $A(w) = w(w + 4)$, $A(3) = 21$ (a) $\Delta A/\Delta w = (3.2 \cdot 7.2 - 21)/.2 = 10.2$ (b) $\Delta A/\Delta w = (3.1 \cdot 7.1 - 21)/.1 = 10.1$ (c) $\Delta A/\Delta w = (3.01 \cdot 7.01 - 21)/.01 = 10.01$ (d) $(3.001 \cdot 7.001 - 21)/.001 = 10.001 \approx 10.00$ (e) $A'(w) = 2w + 4$, $a'(3) = 10$

3. The measure of emission is $R = kT^4$, where T degrees is the Kelvin temperature. Find (a) the average rate of change as T increases from 200 to 300; (b) the instantaneous rate of change when $T = 200$.
▸ (a) $\Delta R/\Delta T = (k300^4 - k200^4)/(300 - 200) = 65{,}000{,}000k$ (b) $R'(T) = 4kT^3$, $R'(200) = 32{,}000{,}000k$

4. A circular cylinder of height 10 in. and base of r in. has volume V in^3. Find the average rate of change of V with respect to r as r changes from (a) 5.00 to 5.40; (b) 5.00 to 5.10; (c) 5.00 to 5.01. (d) Find the instantaneous rate of change of V with respect to r when r is 5.00.
▸ The volume of a circular cylinder is given by $V = \pi r^2 h = 10\pi r^2$. We simplify the calculation by factoring 10π.
(a) $\dfrac{\Delta V}{\Delta r} = \dfrac{10\pi \cdot 5.4^2 - 10\pi \cdot 5^2}{5.4 - 5} = \dfrac{10\pi(5.4^2 - 5^2)}{0.4} = 104\pi$ (b) $\dfrac{\Delta V}{\Delta r} = \dfrac{10\pi(5.1^2 - 5^2)}{5.1 - 5} = 101\pi$
(c) $\dfrac{\Delta V}{\Delta r} = \dfrac{10\pi(5.01^2 - 5^2)}{5.01 - 5} = 100.1\pi$ (d) $V'(r) = 20\pi r$, $V'(5) = 100\pi$

5. A circular plate of radius r in. has area $A(r)$ in^2 and circumference $C(r)$ in. Find the instantaneous rate of change of (a) $A(r)$ and (b) $C(r)$.
▸ (a) $A = \pi r^2$, $A' = 2\pi r$ (b) $C = 2\pi r$, $C' = 2\pi$ (c) Note that $A' = C$.

6. A right circular cylinder whose length is twice its radius r is capped by two hemispheres and has volume $V(r)$ cubic units. Find the instantaneous rate of change of $V(r)$ with respect to r.
▸ The two hemispheres combine to form a sphere of volume $\frac{4}{3}\pi r^3$. The cylinder has volume $\pi r^2 h = \pi r^2(2r) = 2\pi r^3$. Thus $V(r) = \frac{10}{3}\pi r^3$, $V'(r) = 10\pi r^2$.

7. Let x units be the total length of the solid of Exercise 6. Find the instantaneous rate of change of $V(x)$ with respect to x.
▸ $x = 4r$, $r = \frac{1}{4}x$. $V(x) = \frac{10}{3}\pi(\frac{1}{4}x)^3 = \frac{5}{96}\pi x^3$, $V'(x) = \frac{5}{32}\pi x^2$

8. Boyle's law for the expansion of a gas is $PV = C$, where P units is the pressure and V units is the volume, and C is a constant. (a) Show that V decreases at a rate proportional to the inverse square of P. (b) Find the instantaneous rate of change of V with respect to P when $P = 4$ and $V = 8$.
▸ (a) Solving for V, we find $V = CP^{-1}$ so that $V' = -CP^{-2} = -C/P^2$ which proves the assertion.
(b) When $P = 4$ and $V = 8$, we find $C = 4 \cdot 8 = 32$. The instantaneous rate of change is $V' = -32/4^2 = -2$.

9. t days after a sickness starts, the temperature is $f(t)$ °F, where $f(t) = 98.6 + 1.2t - 0.12t^2$, $0 \le t \le 10$. (a) Find the rate of change of f. What is the temperature and its rate of change after (b) 3 days and (c) 8 days. (d) When does the maximum temperature occur and what is it?
▸ (a) $f'(t) = 1.2 - 0.24t$ (b) $f(3) = 101.12$, $f'(3) = 0.48$ (c) $f(8) = 100.52$, $f'(8) = -0.72$ (d) The temperature is increasing as long as $f'(t) > 0$, that is $0.24t < 1.2$, $t < 5$. The maximum temperature is at 5 days; it is 101.6°.

10. Find the rate of change of the volume of a spherical tumor with respect to the radius when the radius is (a) 0.5 cm and (b) 1 cm.

▷ The volume is V cm^3 when the radius is r cm, where $V = \frac{4}{3}\pi r^3$. $V'(r) = 4\pi r^2$. (a) $V'(0.5) = 4\pi(0.5)^2 = \pi$ cm^3/cm (b) $V'(1) = 4\pi(1)^2 = 4\pi$ cm^3/cm.

11. Find the rate of change of the volume of a spherical cell with respect to the radius when the radius is (a) 1.5 μm and (b) 2 μm.

▷ The volume is V μm^3 when the radius is r μm, where $V = \frac{4}{3}\pi r^3$. $V'(r) = 4\pi r^2$. (a) $V'(1.5) = 4\pi(1.5)^2 = 9\pi$ $\mu\text{m}^3/\mu\text{m}$ (b) $V'(2) = 4\pi(2)^2 = 16\pi$ $\mu\text{m}^3/\mu\text{m}$.

12. Find the rate of change of the surface of a spherical tumor with respect to the radius when the radius is (a) 0.5 cm and (b) 1 cm.

▷ The surface is S cm^2 when the radius is r cm, where $S = 4\pi r^2$. $S'(r) = 8\pi r$.
(a) $S'(0.5) = 8\pi(0.5) = 4\pi$. The surface is increasing at the rate of 4π cm^2/cm.
(b) $S'(1) = 8\pi$. The surface is increasing at the rate of 8π cm^2/cm.

13. Find the rate of change of the surface of a spherical tumor with respect to the radius when the radius is (a) 1.5 μm and (b) 2 μm.

▷ The surface is S μm^2 when the radius is r μm, where $S = 4\pi r^2$. $S'(r) = 8\pi r$.
(a) $S'(1.5) = 8\pi(1.5) = 12\pi$. The surface is increasing at the rate of 9π $\mu\text{m}^2/\mu\text{m}$
(b) $S'(2) = 16\pi$. The surface is increasing at the rate of 16π $\mu\text{m}^2/\mu\text{m}$.

14. The height of a cone is twice the radius. Find the rate of change of the volume with respect to the radius when the height is (a) 4m and (b) 8m.

▷ When the radius is rm, the height is $h = 2r$m and the volume is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$ m^3. $V' = 2\pi r^2$.
(a) When $h = 4$, $r = 2$. $V'(2) = 8\pi$ m^3/m (b) When $h = 8$, $r = 4$. $V'(4) = 32\pi$ m^3/m.

15. T degrees is the temperature t hours after midnight. $T = 0.1(400 - 40t + t^2)$, $0 \le t \le 12$

▷ (a) The average rate of change of T with respect to t between 5 a.m. and 6 a.m. is

$$\frac{0.1[400 - 40(6) + (6)^2] - 0.1[400 - 40(5) + (5)^2]}{6 - 5} = \frac{0.1(-204 + 175)}{1} = -2.9$$

Therefore between 5 a.m. and 6 a.m. the average rate of change of the temperature with respect to time is a decrease of 2.9 degrees per hour. (b) $T'(t) = 0.1(-40 + 2t)$; $T'(5) = 0.1[-40 + 2(5)] = -3$. Therefore the instantaneous rate of change of the temperature at 5 a.m. is a decrease of 3 degrees per hour.

16. A worker can paint y frames x hours after starting work at 8 A.M., and $y = 3x + 8x^2 - x^3$, $0 \le x \le 4$. (a) Find the rate at which he is painting at 10 A.M. (b) Find the number of frames he paints between 10 and 11 A.M.

▷ (a) $y' = 3 + 16x - 3x^2$. At 10 A.M., $x = 2$. $y'(2) = 3 + 16(2) - 3(2)^2 = 23$. He is painting at the rate of 23 frames per hour. (b) At 11 A.M., $x = 3$. $y(3) - y(2) = (3\cdot 3 + 8\cdot 3^2 - 3^3) - (3\cdot 2 + 8\cdot 2^2 - 2^3) = 24$. He paints 24 frames between 10 A.M. and 11 A.M.

17. V liters is the volume of water in the pool t minutes after the draining starts, where $V = 250(1600 - 80t + t^2)$.

▷ (a) The number of liters per minute in the average rate at which the volume of water in the pool is changing during the first 5 min is $\dfrac{250[1600 - 80(5) + (5)^2] - 250[1600 - 80(0) + 0^2]}{5 - 0} = \dfrac{250(-375)}{5} = -18{,}750$

Therefore, the average rate at which the water leaves the pool during the first 5 min is 18,750 liters per minute. (b) $V'(t) = 250(-80 + 2t)$, $V'(5) = 250[-80 + 2(5)] = -17{,}500$. Therefore, 5 min after the draining starts the water is leaving the pool at the rate of 17,500 liters per minute.

18. Find the rate of change of the area of a circular ripple when it radius is (a) 4 cm and (b) 7 cm.

▷ When the radius is r cm, the area is $A = \pi r^2$ cm^2. $A' = 2\pi r$. $A'(4) = 8\pi$ and $A'(7) = 14\pi$. The area is increasing at the rate of (a) 8π cm^2/cm and (b) 14π cm^2/cm.

19. $C(x)$ dollars is the total cost of manufacturing x watches: $C(x) = 1500 + 3x + x^2$

▷ (a) C' is the marginal cost function: $C'(x) = 3 + 2x$ (b) $C'(40) = 83$
(c) The number of dollars in the actual cost of manufacturing the forty-first watch is
$C(41) - C(40) = [1500 + 3\cdot 41 + (41)^2] - [1500 + 3\cdot 40 + (40)^2] = 84$

20. The total revenue received from the sale of x desks is $R(x)$ dollars, and $R(x) = 200x - \frac{1}{3}x^2$. Find (a) the marginal revenue function; (b) the marginal revenue when $x = 30$; (c) the actual revenue from the sale of the thirty-first desk.

▸ (a) The marginal revenue function is given by $\quad R'(x) = 200 - \frac{2}{3}x$

(b) The marginal revenue when $x = 30$ is given by $\quad R'(30) = 200 - \frac{2}{3}(30) = 180$

The marginal revenue is \$180 when $x = 30$.

(c) The actual revenue from the sale of the thirty-first desk is given by

$R(31) - R(30) = [200(31) - \frac{1}{3}(31)^2] - [200(30) - \frac{1}{3}(30)^2] = 5879.67 - 5700 = 179.67$

The actual revenue from the sale of the thirty-first desk is \$179.67.

21. $R(x)$ dollars is the total revenue from the sale of x television sets: $R(x) = 600x - \frac{1}{20}x^3$

▸ (a) R' is the marginal revenue function: $R'(x) = 600 - \frac{3}{20}x^2$.

(b) $R'(20) = 600 - \frac{3}{20}(20)^2 = 600 - 60 = 540$. Therefore the marginal revenue is \$540 when $x = 20$.

(c) The number of dollars in the actual revenue from the sale of the twenty-first television set is $R(21) - R(20)$. $R(21) - R(20) = \left[600(21) - \frac{1}{20}(21)^3\right] - \left[600(20) - \frac{1}{20}(20)^3\right] = 536.95$

22. $C(x)$ dollars is the total cost of making x paperweights: $C(x) = 200 + \frac{50}{x} + \frac{x^2}{5}$.

▸ (a) C' is the marginal cost function: $C'(x) = -\frac{50}{x^2} + \frac{2x}{5}$. (b) $C'(10) = -\frac{50}{10^2} + \frac{2(10)}{5} = 3.5$. The marginal cost is \$3.50 when $x = 10$. (c) The number of dollars in the actual cost of making the 11th paperweight is

$C(11) - C(10) = \left[200 + \frac{50}{11} + \frac{11^2}{5}\right] - \left[200 + \frac{50}{10} + \frac{10^2}{5}\right] = 3.745 \approx 3.75$

23. p dollars is the annual gross earnings of the company t years after January 1, 1994. $p(t) = 0.4t^2 + 2t + 10$.

▸ $p'(t) = 0.8t + 2$. (a) On January 1, 1996, $t = 2$ and $p'(2) = 0.8(2) + 2 = 3.6$. Hence on January 1, 1996 the gross earnings are growing at a rate of 3.6 million dollars per year.

(b) The relative rate of growth of the gross earnings on January 1, 1996 is $p'(2)/p(2)$.

$p(2) = 0.4(2)^2 + 2(2) + 10 = 15.6$; $p'(2)/p(2) = 3.6/15.6 = 0.231 = 23.1\%$

(c) On January 1, 2000, $t = 6$ and $p'(6) = 0.8(6) + 2 = 6.8$. Hence on January 1, 2000 the gross earnings are growing at a rate of 6.8 million dollars per year.

(d) The relative rate of growth of the gross earnings on January 1, 2000 is $p'(6)/p(6)$.

$p(6) = 0.4(6)^2 + 2(6) + 10 = 36.4$; $p'(2)/p(2) = 6.8/36.4 = 0.187 = 18.7\%$

24. p dollars is the annual gross earnings of a company t years after April 1, 1993, where $p = 50{,}000 + 18{,}000t + 600t^2$. Find (a) the rate at which the gross earnings were growing on April 1, 1995; (b) the relative rate of growth of the gross earnings on April 1, 1995 to the nearest 0.1%; (c) the rate at which the gross earnings should be growing on April 1, 2003; (d) the anticipated relative rate of growth of the gross earnings on April 1, 2003 to the nearest 0.1%.

▸ $p' = 18{,}000 + 1200t$ (a) On April 1, 1995, $t = 2$ and $p'(2) = 18{,}000 + 1200(2) = 20{,}400$. Hence on April 1, 1995 the gross earnings are growing at a rate of \$20,400 per year.

(b) The relative rate of growth of the gross earnings on April 1, 1996 is $p'(2)/p(2)$.

$p(2) = 50{,}000 + 18{,}000(2) + 600(2)^2 = 88{,}400$; $p'(2)/p(2) = 20{,}400/88{,}400 = 0.2308 \approx 23.1\%$

(c) On April 1, 2003, $t = 10$ and $p'(10) = 18{,}000 + 1200(10)) = 30{,}000$. Hence on April 1, 2003 the gross earnings are growing at a rate of \$30,000 per year.

(d) The relative rate of growth of the gross earnings on January 1, 2003 is $p'(10)/p(10)$.

$p(10) = 50{,}000 + 18{,}000(10) + 600(10)^2 = 290{,}000$; $p'(10)/p(10) = 30{,}000/290{,}000 = 0.1034 \approx 10.3\%$

25. $P(t)$ is the number of people in a population t years after January 1, 1995, where $P(t) = 40t^2 + 200t + 10{,}000$.

▸ $P'(t) = 80t + 200$. (a) On January 1, 2004, $t = 9$ and $P'(9) = 80 \cdot 9 + 200 = 920$. Therefore on January 1, 2004 the population will be growing at the rate of 920 people per year.

(b) The relative rate of growth of the population on January 1, 2004 is $P'(9)/P(9)$.

$P(9) = 40(9)^2 + 200(9) + 10{,}000 = 15{,}040$; $P'(9)/P(9) = 920/15{,}040 = 0.061 = 6.1\%$

(c) On January 1, 2010, $t = 15$ and $P'(15) = 80 \cdot 15 + 200 = 1400$. Hence on January 1, 2010 the population will be growing at the rate of 1400 people per year.

(d) The relative rate of growth of the population on January 1, 2010 is $P'(15)/P(15)$.

$P(15) = 40(15)^2 + 200(15) + 10{,}000 = 22{,}000$; $P'(15)/P(15) = 1{,}400/22{,}000 = 0.064 = 6.4\%$

26. Let r be the reciprocal of n. Find the instantaneous rate of change and the relative rate of change of r with respect to n when n is (a) 4 and (b) 10.

▷ $r = \frac{1}{n}$, $r' = -\frac{1}{n^2}$ (a) $r'(4) = -\frac{1}{4^2} = -\frac{1}{16}$, $\frac{r'(4)}{r(4)} = \frac{-1/16}{1/4} = -\frac{1}{4}$ (b) $r'(10) = -\frac{1}{10^2} = -\frac{1}{100}$, $\frac{r'(10)}{r(10)} = \frac{-1/100}{1/10} = -\frac{1}{10}$

27. The profit of a store is $100y$ dollars when x dollars are spent daily on advertising and $y = 2500 + 36x - 0.2x^2$.

▷ $y'(x) = 36 - 0.4x$. (a) $y'(60) = 36 - 0.4(60) = 2$. Therefore the rate of change of y with respect to x is positive when $x = 60$. Thus it is profitable to increase the daily advertising budget when $x = 60$.
(b) $y'(300) = 36 - 0.4(300) = -84$. Because the rate of change of y with respect to x is negative when $x = 300$, it is not profitable to increase the daily advertising budget when $x = 300$.
(c) The maximum value for x below which it is profitable to increase the advertising budget occurs when $y'(x) = 0$. Thus $36 - 0.4 = 0$; $0.4x = 36$; $x = 90$.

28. The supply equation for a shirt is $x = 3p^2 + 2p$, where p dollars is the price per shirt when $1000x$ shirts are supplied. (a) Find the average rate of change of the supply when the price is increased for \$10 to \$11. (b) Find the marginal rate change of the supply when the price is \$10.

▷ (a) $x(11) - x(10) = [3(11)^2 + 2(11)] - [3(10)^2 + 2(10)] = 65$. The supply increases by 65,000 shirts when the price increases fro \$10 to \$11. (b) $x' = 6p + 2$. $x'(10) = 6(10) + 2 = 62$. The marginal rate of change is 62,000 shirts per dollar.

29. Find the slope of the tangent line at each point of the graph of $y = x^4 + x^3 - 3x^2$ where the rate of change of the slope is zero.

▷ slope $= y' = 4x^3 + 3x^2 - 6x$. $y'' = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x-1)(x+1) = 0$ when $x = -1, \frac{1}{2}$.
$y'(-1) = 4(-1)^3 + 3(-1)^2 - 6(-1) = 5$, $y'(\frac{1}{2}) = 4(\frac{1}{2})^3 + 3(\frac{1}{2})^2 - 6(\frac{1}{2}) = -\frac{7}{4}$

30. Find the instantaneous rate of change of the slope of the tangent line to $y = 2x^3 - 6x^2 - x + 1$ at $(3, -2)$.

▷ slope $= y' = 6x^2 - 12x - 1$. $y'' = 12x - 12$. $y''(3) = 12(3) - 12 = 24$

31. At t min $r(t)$ m is the radius of an oil spill. $r(t) = \begin{cases} 4t^2 + 20 & \text{if } 0 \le t \le 2 \\ 16t + 4 & \text{if } t > 2 \end{cases}$. In Ex. 2.2.31, it was shown that $r'(2) = 16$. Find the rate at which the the radius is changing at (a) 0.4 min; (b) 2 min; (c) 3.2 min.

▷ (a, b) If $r \le 2$, $r' = 8t$ so $r'(0.4) = 8(0.4) = 3.2$, $r'(2) = 8(2) = 16$ (c) If $r > 2$, $r' = 16$ so $r'(3.2) = 16$

32. Show that for any linear function f, the average rate of change of $f(x)$ as x changes from x_1 to $x_1 + k$ is the same as the instantaneous rate of change of $f(x)$ at x_1.

▷ Let $f(x) = mx + b$. The average rate of change is $\frac{f(x_1 + k) - f(x_1)}{(x_1 + k) - x_1} = \frac{m(x_1 + k) + b - (mx_1 + b)}{k} = \frac{mk}{k} = m$.
$f'(x) = m$ which the instantaneous rate of change at any point.

33. (a) The instantaneous rate of change of the area of a circle is $D_r(\pi r^2) = 2\pi r$, the circumference of the circle.
(b) The instantaneous rate of change of the volume of a sphere is $D_r(\frac{4}{3}\pi r^3) = 4\pi r^2$, the surface of the sphere.

2.7 DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

2.7.1 Theorem $D_x(\sin x) = \cos x$ **2.7.2 Theorem** $D_x(\cos x) = -\sin x$

2.7.3 Theorem $D_x(\tan x) = \sec^2 x$ **2.7.4 Theorem** $D_x(\cot x) = -\csc^2 x$

2.7.5 Theorem $D_x(\sec x) = \sec x \tan x$ **2.7.6 Theorem** $D_x(\csc x) = -\csc x \cot x$

Functions of rational multiples of π in the first quadrant requiring at most one radical (Ex. in Before Calculus):
$\tan \frac{1}{12}\pi = 2 - \sqrt{3}$ (9.2.5), $\sin \frac{1}{10}\pi = \cos \frac{2}{5}\pi = \frac{1}{4}(\sqrt{5} - 1)$ (10.6.22) $\tan \frac{1}{8}\pi = \sqrt{2} - 1$, $\sin \frac{1}{6}\pi = \cos \frac{1}{3}\pi = \frac{1}{2}$,
$\cos \frac{1}{6}\pi = \sin \frac{1}{3}\pi = \frac{1}{2}\sqrt{3}$, $\tan \frac{1}{6}\pi = \frac{1}{3}\sqrt{3}$, $\cos \frac{1}{5}\pi = \sin \frac{3}{10}\pi = \frac{1}{4}(\sqrt{5} + 1)$, $\sin \frac{1}{4}\pi = \cos \frac{1}{4}\pi = \frac{1}{2}\sqrt{2}$, $\tan \frac{1}{4}\pi = 1$,
$\tan \frac{1}{3}\pi = \sqrt{3}$, $\tan \frac{3}{8}\pi = \sqrt{2} + 1$ (9.3.28), $\tan \frac{5}{12}\pi = 2 + \sqrt{3}$ (9.2.6)

Exercises 2.7

1. $D_x(\cot x) = D_x\frac{\cos x}{\sin x} = \frac{(\cos x)'\sin x - \cos x(\sin x)'}{\sin^2 x} = \frac{(-\sin x)\sin x - \cos x(\cos x)}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$
$= -\csc x$

2. $D_x(\csc x) = D_x\frac{1}{\sin x} = \frac{(1)'\sin x - 1(\sin x)'}{\sin^2 x} = \frac{0 - \cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$

In Exercises 3–18, find the derivative of the function.

3. $f'(x) = D_x(3 \sin x) = 3 \cos x$

4. $g(x) = \sin x + \cos x$ ▷ $g'(x) = D_x(\sin x) + D_x(\cos x) = \cos x - \sin x$

5. $g'(x) = D_x(\tan x + \cot x) = \sec^2 x - \csc^2 x$

6. $f'(x) = D_x(4 \sec x - 2 \csc x) = 4 \sec x \tan x + 2 \csc x \cot x$

7. $f'(t) = D_t(2t \cos t) = 2 \cos t + 2t(-\sin t) = 2(\cos t - t \sin t)$

8. $f(x) = 4x^2 \cos x$
▷ We apply the product rule. $f'(x) = 4x^2 D_x(\cos x) + \cos x \cdot D_x(4x^2) = -4x^2 \sin x + 8x \cos x$

9. $g'(x) = D_x(x \sin x + \cos x) = (1 \sin x + x \cos x) - \sin x = x \cos x$

10. $g'(y) = D_y(3 \sin y - y \cos y) = 3 \cos y - [1 \cdot \cos y + y(-\sin y)] = 2 \cos y + y \sin y$

11. $h'(x) = D_x(4 \sin x \cos x) = 4[\sin x(-\sin x) + (\cos x)\cos x] = 4(-\sin^2 x + \cos^2 x) = 4 \cos 2x$

12. $f(x) = x^2 \sin x + 2x \cos x$
▷ $f'(x) = D_x(x^2 \sin x) + D_x(2x \cos x) = x^2 \cdot D_x(\sin x) + \sin x \cdot D_x(x^2) + 2x \cdot D_x(\cos x) + \cos(x) \cdot D_x(2x)$
$= x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x = x^2 \cos x + 2 \cos x$

13. $f'(x) = D_x(x^2 \cos x - 2x \sin x - 2 \cos x) = [x^2(-\sin x) + 2x \cos x] - (2x \cos x + 2 \sin x) + 2\sin x = -x^2 \sin x$

14. $h'(y) = D_y(y^3 - y^2 \cos y + 2y \sin y + 2 \cos y) = 3y^2 - [2y \cos y + y^2(-\sin y)] + (2 \sin y + 2y \cos y) - 2 \sin y$
$= 3y^2 + y^2 \sin y$

15. $f'(x) = D_x(3 \sec x \tan x) = 3[(\sec x \tan x)\tan x + \sec x(\sec^2 x)] = 3 \sec x(\tan^2 x + \sec^2 x)$

16. $f(t) = \sin t \tan t$
▷ $f'(t) = \sin t\, D_t(\tan t) + \tan t\, D_t(\sin t) = \sin t \sec^2 t + \tan t \cos t = \sin t \sec^2 t + \sin t$

17. $f'(y) = D_y(\cos y \cot y) = (\cos y)' \cot y + \cos y(\cot y)' = -\sin y \cot y - \cos y \csc^2 y$

18. $h'(x) = D_x(\cot x \csc x) = \cot x(-\csc x \cot x) + \csc x(-\csc^2 x) = -\csc x(\cot^2 x + \csc^2 x)$

In Exercises 19–30, find the derivative.

19. $D_z\left(\dfrac{2 \cos z}{z+1}\right) = \dfrac{(z+1)(-2 \sin z) - 2 \cos z \cdot 1}{(z+1)^2} = -2\dfrac{(z+1)\sin z + \cos z}{(z+1)^2}$

20. $D_t\left(\dfrac{\sin t}{t}\right)$ ▷ We apply the quotient rule.
$D_t\left(\dfrac{\sin t}{t}\right) = \dfrac{t \cdot D_t(\sin t) - \sin t \cdot D_t t}{t^2} = \dfrac{t \cos t - \sin t}{t^2}$

21. $\dfrac{d}{dx}\left(\dfrac{\sin x}{1 - \cos x}\right) = \dfrac{(1 - \cos x)\cos x - \sin x \cdot \sin x}{(1 - \cos x)^2} = \dfrac{\cos x - \cos^2 x - \sin^2 x}{(1 - \cos x)^2} = \dfrac{\cos x - 1}{(\cos x - 1)^2} = \dfrac{1}{\cos x - 1}$

22. $\dfrac{d}{dx}\left(\dfrac{x+4}{\cos x}\right) = \dfrac{1 \cdot \cos x - (x+4)(-\sin x)}{\cos^2 x} = \dfrac{\cos x + x \sin x + 4 \sin x}{\cos^2 x}$

23. $\dfrac{d}{dt}\left(\dfrac{\tan t}{\cos t - 4}\right) = \dfrac{(\cos t - 4)\sec^2 t - \tan t(-\sin t)}{(\cos t - 4)^2} = \dfrac{\sec t - 4\sec^2 t + \tan t \sin t}{(\cos t - 4)^2} = \dfrac{1 - 4 \sec t - \sin^2 t}{\cos t(\cos t - 4)^2}$

24. $\dfrac{d}{dy}\left(\dfrac{\cot y}{1 - \sin y}\right)$
▷ $\dfrac{d}{dy}\left(\dfrac{\cot y}{1 - \sin y}\right) = \dfrac{(1 - \sin y)D_y(\cot y) - \cot y\, D_y(1 - \sin y)}{(1 - \sin y)^2} = \dfrac{(1 - \sin y)(-\csc^2 y) - \cot y(-\cos y)}{(1 - \sin y)^2}$
$= \dfrac{-\csc^2 y + \csc y + \cot y \cos y}{(1 - \sin y)^2}$

25. $\dfrac{d}{dy}\left(\dfrac{1 + \sin y}{1 - \sin y}\right) = \dfrac{(1 - \sin y)(\cos y) - (1 + \sin y)(-\cos y)}{(1 - \sin y)^2} = \dfrac{\cos y - \sin y \cos y + \cos y + \sin y \cos y}{(1 - \sin y)^2} = \dfrac{2 \cos y}{(1 - \sin y)^2}$

26. $\frac{d}{dx}\left(\frac{\sin x-1}{\cos x+1}\right)=\frac{\cos x(\cos x+1)-(\sin x-1)(-\sin x)}{(\cos x+1)^2}=\frac{\cos^2 x+\cos x+\sin^2 x-\sin x}{(\cos x+1)^2}=\frac{1+\cos x-\sin x}{(\cos x+1)^2}$

27. $\mathrm{D}[(x-\sin x)(x+\cos x)]=(1-\cos x)(x+\cos x)+(x-\sin x)(1-\sin x)$

28. $\mathrm{D}_z[(z^2+\cos z)(2z-\sin z)]$

▷ Applying the product rule, we obtain

$$\begin{aligned}\mathrm{D}_z[(z^2+\cos z)(2z-\sin z)]&=(z^2+\cos z)\mathrm{D}_z(2z-\sin z)+(2z-\sin z)\mathrm{D}_z(z^2+\cos z)\\&=(z^2+\cos z)(2-\cos z)+(2z-\sin z)(2z-\sin z)\\&=(z^2+\cos z)(2-\cos z)+(2z-\sin z)^2\end{aligned}$$

29. $\mathrm{D}_t\left(\frac{2\csc t-1}{\csc t+2}\right)=\frac{(\csc t+2)(-2\csc t\cot t)-(2\csc t-1)(-\csc t\cot t)}{(\csc t+2)^2}$

$=\frac{-2\csc^2 t\cot t-4\csc t\cot t+2\csc^2 t\cot t-\csc t\cot t}{(\csc t+2)^2}=\frac{-5\csc t\cot t}{(\csc t+2)^2}$

Alternatively, $\mathrm{D}_t\left(\frac{2\csc t-1}{\csc t+2}\right)=\mathrm{D}_t\left(\frac{2-\sin t}{1+2\sin t}\right)=\frac{(1+2\sin t)(-\cos t)-(2-\sin t)(2\cos t)}{(1+2\sin t)^2}$

$=\frac{-\cos t-2\sin t\cos t-4\cos t+2\sin t\cos t}{(1+2\sin t)^2}=\frac{-5\cos t}{(1+2\sin t)^2}$

30. $\mathrm{D}_y\left(\frac{\tan y+1}{\tan y-1}\right)=\frac{\sec^2 y(\tan y-1)-(\tan y+1)\sec^2 y}{(\tan y-1)^2}=\frac{-2\sec^2 y}{(\tan y-1)^2}$

In Exercises 31–42, compute NDER$(f(x),a)$ on your calculator. Then compute the exact value of $f'(a)$.

31. $f(x)=x\cos x$; $f'(x)=\cos x-x\sin x$; so $f'(0)=\cos 0-0\sin 0=1$

32. $f(x)=x\sin x;\quad a=\frac{3}{2}\pi$

▷ NDER gives -0.9999995. Now we find $f'(x)$.

$$f'(x)=x\cdot\mathrm{D}_x\sin x+\sin x\cdot\mathrm{D}_x x=x\cos x+\sin x$$

Next, we replace x with $\frac{3}{2}\pi$. Thus,

$$f'(\tfrac{3}{2}\pi)=(\tfrac{3}{2}\pi)\cos(\tfrac{3}{2}\pi)+\sin(\tfrac{3}{2}\pi)=(\tfrac{3}{2}\pi)(0)+(-1)=-1$$

33. $f(x)=\frac{\cos x}{x}$; $f'(x)=\frac{x(-\sin x)-\cos x\cdot 1}{x^2}$; so $f'(\frac{1}{2}\pi)=\frac{\frac{1}{2}\pi(-1)-0}{(\frac{1}{2}\pi)^2}=\frac{-1}{\frac{1}{2}\pi}=-\frac{2}{\pi}$

34. $f(x)=\frac{\sec x}{x^2}$; $f'(x)=\frac{(\sec x\tan x)x^2-\sec x(2x)}{x^4}$; so $f'(\pi)=\frac{0+2\pi}{\pi^4}=\frac{2}{\pi^3}\approx 0.06450307$. NDER $=0.06450311$

35. $f(x)=x^2\tan x$; $f'(x)=2x\tan x+x^2\sec^2 x$; so $f'(\pi)=2\pi\tan\pi+\pi^2\sec^2\pi=2\pi(0)+\pi^2(-1)^2=\pi^2$

36. $f(x)=x^2\cos x-\sin x;\quad a=0$

▷ NDER gives -0.9999998

$$f'(x)=x^2\cdot\mathrm{D}_x\cos x+\cos x\cdot\mathrm{D}_x x^2-\mathrm{D}_x\sin x=-x^2\sin x+2x\cos x-\cos x$$

Thus, with $x=0$, we have

$$f'(0)=-0^2(\sin 0)+2(0)\cos 0-\cos 0=-1$$

37. $f(x)=\sin x(\cos x-1)$; $f'(x)=\cos x(\cos x-1)+\sin x(-\sin x)$; so $f'(\pi)=(-1)(-2)+0(0)=2$

38. $f(x)=(\cos x+1)(x\sin x-1)$; $f'(x)=-\sin x(x\sin x-1)+(\cos x+1)(\sin x+x\cos x)$;

$f'(\frac{1}{2}\pi)=-(\frac{1}{2}\pi-1)+1=2-\frac{1}{2}\pi\approx 0.4292037$. NDER $=0.4292041$

39. $f(x)=x\cos x+x\sin x$; $f'(x)=\cos x-x\sin x+\sin x+x\cos x$

so $f'(\frac{1}{4}\pi)=\frac{1}{2}\sqrt{2}-(\frac{1}{4}\pi)(\frac{1}{2}\sqrt{2})+\frac{1}{2}\sqrt{2}+(\frac{1}{4}\pi)(\frac{1}{2}\sqrt{2})=\sqrt{2}$

40. $f(x)=\tan x+\sec x;\quad a=\frac{1}{6}\pi$

▷ NDER gives 2.000002.

$$f'(x)=\sec^2 x+\sec x\tan x$$

$$f'(\tfrac{1}{6}\pi)=\left(\frac{2}{\sqrt{3}}\right)^2+\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}=\frac{4}{3}+\frac{2}{3}=2$$

41. $f(x)=2\cot x-\csc x$; $f'(x)=-2\csc^2 x+\csc x\cot x$; so $f'(\frac{2}{3}\pi)=-2\left(\frac{2}{\sqrt{3}}\right)^2+\frac{2}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{10}{3}$

42. $f(x) = \frac{1}{\cot x - 1}$; $f'(x) = \frac{0 - (-\csc^2 x)}{(\cot x - 1)^2}$; $f'(\frac{3}{4}\pi) = \frac{2}{4} = \frac{1}{2} = 0.5$. NDER $= 0.5000002$

43. (a)

h	1	0.5	0.1	0.01	0.001	-1	-0.5	-0.1	-0.01	$-.001$
$\frac{\sin(\frac{1}{3}\pi + h) - \sin \frac{1}{3}\pi}{h}$	.0226	.2674	.4559	.4956	.4995	.8188	.6915	.5424	.5043	.5006

The quotient appears to approach $\frac{1}{2}$. (b) $\lim_{h \to 0} \frac{\sin(\frac{1}{3}\pi + h) - \sin\frac{1}{3}\pi}{h} = \frac{d}{dx}\sin x\Big|_{x=\pi/3} = \cos \frac{1}{3}\pi = \frac{1}{2}$

44. (a) Use a calculator to tabulate to four decimal places values of $\frac{\cos(\frac{5}{6}\pi + h) - \cos \frac{5}{6}\pi}{h}$ when h is 1, 0.5, 0.1, 0.01, 0.001 and h is -1, -0.5, -0.1, -0.001. What does the quotient appear to be approaching as h approaches 0? (b) Find $\lim_{h \to 0} \frac{\cos(\frac{5}{6}\pi + h) - \cos \frac{5}{6}\pi}{h}$ by interpreting it as a derivative.

▶ (a) See the table. The quotient appears to be approaching -0.5 as h approaches 0.

h	1	0.5	0.1	0.01	0.001
$\frac{\cos(\frac{5}{6}\pi + h) - \cos \frac{5}{6}\pi}{h}$	-0.0226	-0.2674	-0.4559	-0.4957	-0.4996

h	-1	-0.5	-0.1	-0.01	-0.001
$\frac{\cos(\frac{5}{6}\pi + h) - \cos \frac{5}{6}\pi}{h}$	-0.8188	-0.6915	-0.5424	-0.5043	-0.5004

(b) By formula (4) of Section 2.1 and Theorem 2.7.2,

$$\lim_{h \to 0} \frac{\cos(\frac{5}{6}\pi + h) - \cos \frac{5}{6}\pi}{h} = D_x \cos x\,\Big|_{5\pi/6} = -\sin(\tfrac{5}{6}\pi) = -\tfrac{1}{2}$$

45.

h	0.1	0.01	0.001	0.0001	10^{-5}	-0.1	-0.01	-0.001	$-.0001$	-10^{-5}
$\frac{\tan(\frac{1}{4}\pi + h) - \tan \frac{1}{4}\pi}{h}$	2.2305	2.0203	2.0020	2.0002	2	1.8237	1.9803	1.9980	1.9998	2

The quotient appears to approach 2. (b) $\lim_{h \to 0} \frac{\tan(\frac{1}{4}\pi + h) - \tan \frac{1}{4}\pi}{h} = \frac{d}{dx}\tan x\Big|_{x=\pi/4} = \sec^2 \frac{1}{4}\pi = 2$

46.

h	0.1	0.01	0.001	0.0001	10^{-5}	-0.1	-0.01	-0.001	$-.0001$	-10^{-5}
$\frac{\sec(\frac{1}{6}\pi + h) - \sec \frac{1}{6}\pi}{h}$	0.7716	0.6764	0.6676	0.6668	0.6667	0.5775	0.6571	0.6657	0.6666	0.6667

Quotient appears to approach $\frac{2}{3}$. (b) $\lim_{h \to 0} \frac{\sec(\frac{1}{6}\pi + h) - \sec \frac{1}{6}\pi}{h} = \frac{d}{dx}\sec x\Big|_{x=\pi/6} = \sec \frac{1}{6}\pi \tan \frac{1}{6}\pi = \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{2}{3}$

47.

x	$\frac{3}{20}\pi$	$\frac{19}{120}\pi$	$\frac{33}{200}\pi$	$\frac{199}{1200}\pi$	$\frac{333}{2000}\pi$	$\frac{11}{60}\pi$	$\frac{7}{40}\pi$	$\frac{101}{600}\pi$	$\frac{67}{400}\pi$	$\frac{1001}{6000}\pi$
$x - \frac{1}{6}\pi$	$-\frac{1}{60}\pi$	$-\frac{1}{120}\pi$	$-\frac{1}{600}\pi$	$-\frac{1}{1200}\pi$	$-\frac{1}{6000}\pi$	$\frac{1}{60}\pi$	$\frac{1}{120}\pi$	$\frac{1}{600}\pi$	$\frac{1}{1200}\pi$	$\frac{1}{6000}\pi$
$\frac{\cos x - \cos \frac{1}{6}\pi}{x - \frac{1}{6}\pi}$	$-.4771$	$-.4886$	$-.4977$	$-.4989$	$-.4998$	$-.5224$	$-.5113$	$-.5023$	$-.5011$	$-.5002$

The quotient appears to approach $-\frac{1}{2}$. (b) $\lim_{x \to \pi/6} \frac{\cos x - \cos \frac{1}{6}\pi}{x - \frac{1}{6}\pi} = \frac{d}{dx}\cos x\Big|_{x=\pi/6} = \sin \frac{1}{6}\pi = -\frac{1}{2}$

48. (a) Use a calculator to tabulate to four decimal places values of $\frac{\sin x - \sin \frac{1}{3}\pi}{x - \frac{1}{3}\pi}$ when x is $\frac{3}{10}\pi$, $\frac{19}{60}\pi$, $\frac{33}{100}\pi$, $\frac{199}{600}\pi$, $\frac{333}{1000}\pi$ and x is $\frac{11}{30}\pi, \frac{7}{20}\pi, \frac{101}{300}\pi, \frac{67}{200}\pi, \frac{1001}{3000}\pi$. What does the quotient seem to be approaching as x approaches $\frac{1}{3}\pi$?

(b) Find $\lim_{x \to \pi/3} \frac{\sin x - \sin \frac{1}{3}\pi}{x - \frac{1}{3}\pi}$ by interpreting it as a derivative.

▷ (a) See the table. The quotient appears to be approaching 0.5 as x approaches $\frac{1}{3}\pi$.

x	$\frac{3}{10}\pi$	$\frac{19}{60}\pi$	$\frac{33}{100}\pi$	$\frac{199}{600}\pi$	$\frac{333}{1000}\pi$
$x-\frac{1}{3}\pi$	$-\frac{1}{30}\pi$	$-\frac{1}{60}\pi$	$-\frac{1}{300}\pi$	$-\frac{1}{600}\pi$	$-\frac{1}{3000}\pi$
$\dfrac{\sin x-\sin\frac{1}{3}\pi}{x-\frac{1}{3}\pi}$	0.5444	0.5224	0.5045	0.5023	0.5005

x	$\frac{11}{30}\pi$	$\frac{7}{20}\pi$	$\frac{101}{300}\pi$	$\frac{67}{200}\pi$	$\frac{1001}{3000}\pi$
$x-\frac{1}{3}\pi$	$\frac{1}{30}\pi$	$\frac{1}{60}\pi$	$\frac{1}{300}\pi$	$\frac{1}{600}\pi$	$\frac{1}{3000}\pi$
$\dfrac{\sin x-\sin\frac{1}{3}\pi}{x-\frac{1}{3}\pi}$	0.4538	0.4771	0.4955	0.4977	0.4994

(b) $\lim\limits_{x\to\pi/3}\dfrac{\sin x-\sin\frac{1}{3}\pi}{x-\frac{1}{3}\pi}=D_x\sin x\,|_{\pi/3}=\cos\frac{1}{3}\pi=\frac{1}{2}$

49.

x	$\frac{3}{5}\pi$	$\frac{19}{30}\pi$	$\frac{33}{50}\pi$	$\frac{199}{300}\pi$	$\frac{333}{500}\pi$	$\frac{11}{15}\pi$	$\frac{7}{10}\pi$	$\frac{101}{150}\pi$	$\frac{167}{100}\pi$	$\frac{1001}{1500}\pi$
$x-\frac{2}{3}\pi$	$-\frac{1}{15}\pi$	$-\frac{1}{30}\pi$	$-\frac{1}{150}\pi$	$-\frac{1}{300}\pi$	$-\frac{1}{1500}\pi$	$\frac{1}{15}\pi$	$\frac{1}{30}\pi$	$\frac{1}{150}\pi$	$\frac{1}{300}\pi$	$\frac{1}{1500}\pi$
$\dfrac{\csc x-\csc\frac{2}{3}\pi}{x-\frac{2}{3}\pi}$	0.4929	0.5736	0.6468	0.6567	0.6647	0.9116	0.7770	0.6872	0.6768	0.6687

Quotient seems to approach $\frac{2}{3}$. (b) $\lim\limits_{x\to 2\pi/3}\dfrac{\csc x-\csc\frac{2}{3}\pi}{x-\frac{2}{3}\pi}=\dfrac{d}{dx}\csc x\Big|_{x=2\pi/3}=-\csc\frac{2}{3}\pi\cot\frac{2}{3}\pi=-\dfrac{2}{\sqrt{3}}\left(-\dfrac{1}{\sqrt{3}}\right)=\frac{2}{3}$

50.

x	$\frac{29}{40}\pi$	$\frac{59}{80}\pi$	$\frac{299}{400}\pi$	$\frac{599}{800}\pi$	$\frac{2999}{4000}\pi$	$\frac{31}{40}\pi$	$\frac{61}{80}\pi$	$\frac{301}{400}\pi$	$\frac{601}{800}\pi$	$\frac{3001}{4000}\pi$
$x-\frac{3}{4}\pi$	$-\frac{1}{40}\pi$	$-\frac{1}{80}\pi$	$-\frac{1}{400}\pi$	$-\frac{1}{800}\pi$	$-\frac{1}{4000}\pi$	$\frac{1}{40}\pi$	$\frac{1}{80}\pi$	$\frac{1}{400}\pi$	$\frac{1}{800}\pi$	$\frac{1}{4000}\pi$
$\dfrac{\cot x-\cot\frac{3}{4}\pi}{x-\frac{3}{4}\pi}$	-1.8579	-1.9254	-1.9845	-1.9922	-1.9984	-2.1753	-2.0289	−2.0159	-2.0079	-2.0289

The quotient appears to approach -2. (b) $\lim\limits_{x\to 3\pi/4}\dfrac{\cot x-\cot\frac{3}{4}\pi}{x-\frac{3}{4}\pi}=\dfrac{d}{dx}\cot x\Big|_{x=3\pi/4}=-\csc^2\frac{3}{4}\pi=-(\sqrt{2})^2=-2$

51. Let $f(x)=\sin x$. Then $f'(x)=\cos x$.
(a) If $x=0$, $y=0$, and the slope of the tangent line is $f'(0)=\cos 0=1$; thus an equation of the tangent line at $(0,0)$ is $y=x$.
(b) If $x=\frac{1}{3}\pi, y=\frac{1}{2}\sqrt{3}$, and the slope of the tangent line is $f'(\frac{1}{3}\pi)=\cos(\frac{1}{3}\pi)=\frac{1}{2}$; thus an equation of the tangent line at $(\frac{1}{3}\pi,\frac{1}{2}\sqrt{3})$ is $y-\frac{1}{2}\sqrt{3}=\frac{1}{2}(x-\frac{1}{3}\pi)$; $3x-6y-\pi+3\sqrt{3}=0$.
(c) If $x=\pi$, $y=0$, and the slope of the tangent line is $f'(\pi)=-1$; thus an equation of the tangent line at $(\pi,0)$ is $y=-1(x-\pi)$; $x+y-\pi=0$.

52. Find an equation of the tangent line to the graph of the cosine function at the point where (a) $x=\frac{1}{2}\pi$; (b) $x=-\frac{1}{2}\pi$; (c) $x=\frac{1}{6}\pi$.

▷ Let f be the cosine function. That is $f(x)=\cos x$. Then $f'(x)=-\sin x$

(a) Because $f(\frac{1}{2}\pi)=\cos(\frac{1}{2}\pi)=0,\ f'(\frac{1}{2}\pi)=-\sin(\frac{1}{2}\pi)=-1$

then $y=0$ when $x=\frac{1}{2}\pi$ and the slope of the tangent line is -1. By the point-slope formula, an equation of the tangent line is $\quad y-0=-(x-\frac{1}{2}\pi),\qquad y=-x+\frac{1}{2}\pi$

(b) Because $\quad f(-\frac{1}{2}\pi)=\cos(-\frac{1}{2}\pi)=0,\quad f'(-\frac{1}{2}\pi)=-\sin(-\frac{1}{2}\pi)=1$

then $y=0$ when $x=-\frac{1}{2}\pi$ and the slope of the tangent line is 1. Thus, an equation of the tangent line is
$y=x+\frac{1}{2}\pi$

(c) Because $f(\frac{1}{6}\pi) = \cos(\frac{1}{6}\pi) = \frac{1}{2}\sqrt{3}, \quad f'(\frac{1}{6}\pi) = -\sin(\frac{1}{6}\pi) = -\frac{1}{2}$

then $y = \frac{1}{2}\sqrt{3}$ when $x = \frac{1}{6}\pi$ and the slope of the tangent line is $-\frac{1}{2}$. Thus, an equation of the tangent line is

$$y - \frac{1}{2}\sqrt{3} = -\frac{1}{2}(x - \frac{1}{6}\pi), \qquad y = -\frac{1}{2}x + \frac{1}{12}\pi + \frac{1}{2}\sqrt{3}$$

Let $f(x) = \tan x$. Then $f'(x) = \sec^2 x$.

(a) When $x = 0$, $y = 0$, and the slope of the tangent line is $f'(0) = \sec^2 0 = 1$; thus an equation of the tangent line at $(0, 0)$ is $y = x$.

(b) When $x = \frac{1}{4}\pi, y = 1$, and the slope of the tangent line is $f'(\frac{1}{4}\pi) = \sec^2(\frac{1}{4}\pi) = 2$; thus an equation of the tangent line at $(\frac{1}{4}\pi, 1)$ is $y - 1 = 2(x - \frac{1}{4}\pi)$; $4x - 2y + 2 - \pi = 0$.

(c) When $x = -\frac{1}{4}\pi$, $y = -1$, and the slope of the tangent line is $f'(-\frac{1}{4}\pi) = \sec^2(-\frac{1}{4}\pi) = 2$; thus an equation of the tangent line at $(-\frac{1}{4}\pi, -1)$ is $y + 1 = 2(x + \frac{1}{4}\pi)$; $4x - 2y - 2 + \pi = 0$.

Let $f(x) = \sec x$. Then $f'(x) = \sec x \tan x$.

(a) When $x = \frac{1}{4}\pi$, $y = \sqrt{2}$, and the slope of the tangent line is $f'(\frac{1}{4}) = \sec \frac{1}{4}\pi \tan \frac{1}{4}\pi = \sqrt{2}$; thus an equation of the tangent line at $(\frac{1}{4}\pi, \sqrt{2})$ is $y = \sqrt{2}(x - \frac{1}{4}\pi) + \sqrt{2}$.

(b) When $x = -\frac{1}{4}\pi, y = \sqrt{2}$, and the slope of the tangent line is $f'(-\frac{1}{4}\pi) = \sec(-\frac{1}{4}\pi)\tan(-\frac{1}{4}\pi) = -\sqrt{2}$; thus an equation of the tangent line at $(-\frac{1}{4}\pi, \sqrt{2})$ is $y = -\sqrt{2}(x + \frac{1}{4}\pi) + \sqrt{2}$.

(c) When $x = \frac{3}{4}\pi$, $y = -\sqrt{2}$, and the slope of the tangent line is $f'(\frac{3}{4}\pi) = \sec \frac{3}{4}\pi \tan \frac{3}{4}\pi = \sqrt{2}$; thus an equation of the tangent line at $(\frac{3}{4}\pi, -\sqrt{2})$ is $y = \sqrt{2}(x - \frac{3}{4}\pi) - \sqrt{2}$

In Exercises 55–58, a particle is moving along a straight line according to the equation where s cm is the directed distance from the origin at t seconds. (a) What is the instantaneous velocity and acceleration of the particle at t_1 seconds? (b) Find the instantaneous velocity and acceleration of the particle at t_1 seconds for each value of t_1.

$s = 4 \sin t$. (a) The instantaneous velocity and acceleration are given by $v(t) = \frac{ds}{dt} = 4 \cos t$ and $a(t) = \frac{dv}{dt} = -4 \sin t$ (b) $v(0) = 4 \cos 0 = 4$, $a(0) = -4 \sin 0 = 0$; $v(\frac{1}{3}\pi) = 4 \cos(\frac{1}{3}\pi) = 4(\frac{1}{2}) = 2$; $a(\frac{1}{3}\pi) = -4 \sin \frac{1}{3}\pi = -4 \cdot \frac{1}{2}\sqrt{3} = -2\sqrt{3}$; $v(\frac{1}{2}\pi) = 4 \cos(\frac{1}{2}\pi) = 4(0) = 0$, $a(\frac{1}{2}\pi) = -4 \sin \frac{1}{2}\pi = -4 \cdot 1 = -4$; $v(\frac{2}{3}\pi) = 4 \cos(\frac{2}{3}\pi) = 4(-\frac{1}{2}) = -2$; $a(\frac{2}{3}\pi) = -4 \sin \frac{2}{3}\pi = -4 \cdot \frac{1}{2}\sqrt{3} = -2\sqrt{3}$; $v(\pi) = 4 \cos \pi = 4(-1) = -4$, $a(\pi) = -4 \sin \pi = -4 \cdot 0 = 0$

$s = 6 \cos t$; t_1 is $0, \frac{1}{6}\pi, \frac{1}{2}\pi, \frac{5}{6}\pi$ and π

(a) Because $v(t) = D_t(6 \cos t) = -6 \sin t$ and $a(t) = D_t(-6 \sin t) = -6 \cos t$ the instantaneous velocity and acceleration of the particle at t sec is $-6 \sin t$ cm/sec and $-6 \cos t$ cm/sec^2.

(b) The instantaneous velocity and acceleration for each value of t_1 is given in the table below.

t_1	v	velocity	a	acceleration
0	$-6 \sin 0 = -6(0) = 0$	0	$-6 \cos 0 = -6$	-6 cm/sec^2
$\frac{1}{6}\pi$	$-6 \sin(\frac{1}{6}\pi) = -6(\frac{1}{2}) = -3$	-3 cm/sec	$-6 \cos \frac{1}{6}\pi = -6 \cdot \frac{1}{2}\sqrt{3} = -3\sqrt{3}$	$-3\sqrt{3}$ cm/sec^2
$\frac{1}{2}\pi$	$-6 \sin(\frac{1}{2}\pi) = -6(1) = -6$	-6 cm/sec	$-6 \cos \frac{1}{2}\pi = -6 \cdot 0 = 0$	0
$\frac{5}{6}\pi$	$-6 \sin(\frac{5}{6}\pi) = -6(\frac{1}{2}) = -3$	-3 cm/sec	$-6 \cos \frac{5}{6}\pi = -6(-\frac{1}{2}\sqrt{3}) = 3\sqrt{3}$	$3\sqrt{3}$ cm/sec^2
π	$-6 \sin \pi = -6(0) = 0$	0	$-6 \cos \pi = -6(-1) = 6$	6 cm/sec^2

$s = -3 \cos t$ (a) The instantaneous velocity and acceleration are given by $v(t) = \frac{ds}{dt} = 3 \sin t$ and $a(t) = \frac{dv}{dt} = 3 \cos t$. (b) $v(0) = 3(0) = 0$, $a(0) = 3 \cos 0 = 3 \cdot 1 = 3$; $v(\frac{1}{6}\pi) = 3 \cdot \frac{1}{2} = \frac{3}{2}$, $a(\frac{1}{6}\pi) = 3 \cos \frac{1}{6}\pi = 3 \cdot \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3}$; $v(\frac{1}{3}\pi) = 3 \cdot \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3}$, $a(\frac{1}{3}\pi) = 3 \cos \frac{1}{3}\pi = 3 \cdot \frac{1}{2} = \frac{3}{2}$; $v(\frac{1}{2}\pi) = 3 \cdot 1 = 3$, $v(\frac{2}{3}\pi) = 3 \cdot \frac{1}{2}\sqrt{3} = \frac{3}{2}\pi$, $a(\frac{2}{3}\pi) = 3 \cos \frac{2}{3}\pi = 3(-\frac{1}{2}) = -\frac{3}{2}$; $v(\frac{5}{6}\pi) = 3 \cdot \frac{1}{2} = \frac{3}{2}$, $a(\frac{5}{6}\pi) = 3 \cos \frac{5}{6}\pi = 3(\frac{1}{2}\sqrt{3}) = \frac{3}{2}\sqrt{3}$; $v(\pi) = 3 \cdot 0 = 0$, $a(\pi) = 3 \cos \pi = 3(-1) = -3$

$s = -\frac{1}{2} \sin t$ (a) The instantaneous velocity and acceleration are $v(t) = \frac{ds}{dt} = -\frac{1}{2} \cos t$ and $a(t) = \frac{1}{2} \sin t$.
(b) $v(0) = -\frac{1}{2} \cos 0 = -\frac{1}{2} \cdot 1 = -\frac{1}{2}$, $a(0) = \frac{1}{2} \sin 0 = \frac{1}{2} \cdot 0 = 0$; $v(\frac{1}{6}\pi) = -\frac{1}{2} \cos \frac{1}{6}\pi = -\frac{1}{2} \cdot \frac{1}{2}\sqrt{3} = -\frac{1}{4}\sqrt{3}$, $a(\frac{1}{6}\pi) = \frac{1}{2} \sin \frac{1}{6}\pi = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$; $v(\frac{1}{3}\pi) = -\frac{1}{2} \cos \frac{1}{3}\pi = -\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4}$, $a(\frac{1}{3}\pi) = \frac{1}{2} \sin \frac{1}{3}\pi = \frac{1}{2} \cdot \frac{1}{2}\sqrt{3} = \frac{1}{4}\sqrt{3}$;

$v(\frac{1}{2}\pi) = -\frac{1}{2}\cos\frac{1}{2}\pi = -\frac{1}{2}\cdot 0 = 0$, $a(\frac{1}{2}\pi) = \frac{1}{2}\sin\frac{1}{2}\pi = \frac{1}{2}\cdot 1 = \frac{1}{2}$; $v(\frac{2}{3}\pi) = -\frac{1}{2}\cos\frac{2}{3}\pi = -\frac{1}{2}(-\frac{1}{2}) = \frac{1}{4}$,
$a(\frac{2}{3}\pi) = \frac{1}{2}\sin\frac{2}{3}\pi = \frac{1}{2}\cdot\frac{1}{2}\sqrt{3} = \frac{1}{4}\sqrt{3}$; $v(\frac{5}{6}\pi) = -\frac{1}{2}\cos\frac{5}{6}\pi = -\frac{1}{2}(-\frac{1}{2}\sqrt{3}) = \frac{1}{4}\sqrt{3}$, $a(\frac{5}{6}\pi) = \frac{1}{2}\sin\frac{5}{6}\pi = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$;
$v(\pi) = -\frac{1}{2}\cos\pi = -\frac{1}{2}(-1) = \frac{1}{2}$, $a(\pi) = \frac{1}{2}\sin\pi = \frac{1}{2}\cdot 0 = 0$

59. $\mathrm{F} = \dfrac{k\mathrm{W}}{k\sin\theta + \cos\theta} = \dfrac{\frac{1}{2}\mathrm{W}}{\frac{1}{2}\sin\theta + \cos\theta}$

The instantaneous rate of change of F with respect to θ is $\mathrm{F}'(\theta) = \dfrac{\frac{1}{2}\mathrm{W}(\frac{1}{2}\cos\theta - \sin\theta)}{(\frac{1}{2}\sin\theta + \cos\theta)^2}$.

(a) $\mathrm{F}'(\frac{1}{4}\pi) = \dfrac{-\frac{1}{2}\mathrm{W}(\frac{1}{2}\cdot\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2})}{(\frac{1}{2}\cdot\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2})^2} = \dfrac{-\frac{1}{2}\mathrm{W}(-\frac{1}{4}\sqrt{2})}{(\frac{3}{4}\sqrt{2})^2} = \dfrac{\frac{1}{8}\sqrt{2}\mathrm{W}}{\frac{9}{8}} = \frac{1}{9}\sqrt{2}\mathrm{W}$ (b) $\mathrm{F}'(\frac{1}{2}\pi) = \dfrac{-\frac{1}{2}\mathrm{W}(\frac{1}{2}\cdot 0 - 1)}{(\frac{1}{2}\cdot 1 + 0)^2} = 2\mathrm{W}$

60. A projectile is shot from a gun at an angle of elevation having radian measure $\frac{1}{2}\alpha$ and an initial velocity of v_0 ft/sec. If R feet is the range of the projectile, then $\mathrm{R} = (v_0{}^2/g)\sin\alpha$, $0 \le \alpha \le \pi$ where g ft/sec^2 is the acceleration due to gravity. (a) If $v_0 = 480$, find the rate of change of R with respect to α when $\alpha = \frac{1}{2}\pi$ (That is, the angle of elevation has radian measure $\frac{1}{4}\pi$). Take $g = 32$. (b) Find the values of α for which $\mathrm{D}_\alpha\mathrm{R} > 0$.

▷ (a) Because $v_0 = 480$ and $g = 32$, we are given that $\quad \mathrm{R}(\alpha) = \dfrac{480^2}{32}\sin\alpha = 7200\sin\alpha$

Differentiating with respect to α, we have $\quad \mathrm{R}'(\alpha) = 7200\cos\alpha, \qquad \mathrm{R}'(\frac{1}{2}\pi) = 7200\cos(\frac{1}{2}\pi) = 0$

Hence, the rate of change of R with respect to α is 0 when $\alpha = \frac{1}{2}\pi$.
(b) If $\mathrm{R}'(\alpha) > 0$, we have $\quad 7200\cos\alpha > 0, \quad \cos\alpha > 0$

If $\cos\alpha > 0$ and $0 \le \alpha \le \pi$ then $0 \le \alpha < \frac{1}{2}\pi$. We conclude that $\mathrm{D}_\alpha\mathrm{R} > 0$ when $0 \le \alpha < \frac{1}{2}\pi$.

61. We wish to prove by mathematical induction that if k is any positive integer then

$$\mathrm{D}_x{}^n(\sin x) = \begin{cases} \sin x & \text{if } n = 4k \\ \cos x & \text{if } n = 4k+1 \\ -\sin x & \text{if } n = 4k+2 \\ -\cos x & \text{if } n = 4k+3 \end{cases} \quad (1)$$

We prove that formula (1) holds when $k = 1$. $\mathrm{D}_x(\sin x) = \cos x$;
$\mathrm{D}^2(\sin x) = -\sin x$; $\mathrm{D}^3(\sin x) = -\cos x$
Thus, by taking successive derivatives

$$\mathrm{D}_x{}^n(\sin x) = \begin{cases} \sin x & \text{if } n = 4 \\ \cos x & \text{if } n = 5 \\ -\sin x & \text{if } n = 6 \\ -\cos x & \text{if } n = 7 \end{cases} \quad (2)$$

Because (2) is (1) for $k = 1$, we have proved formula (1) for $k = 1$.

We now wish to show that the formula holds for $k = t + 1$ if it holds for $k = t$, that is, if

$$\mathrm{D}_x{}^n(\sin x) = \begin{cases} \sin x & \text{if } n = 4t \\ \cos x & \text{if } n = 4t+1 \\ -\sin x & \text{if } n = 4t+2 \\ -\cos x & \text{if } n = 4t+3 \end{cases} \quad (3)$$

When $k = t + 1$, then $4k = 4t + 4$, $4k + 1 = 4t + 5$, $4k + 2 = 4t + 6$, $4k + 3 = 4t + 7$. From (3) we have $\mathrm{D}^n(\sin x) = -\cos x$ if $n = 4t + 3$.
Thus, by taking successive derivatives

$$\mathrm{D}_x{}^n(\sin x) = \begin{cases} \sin x & \text{if } n = 4t+4 \\ \cos x & \text{if } n = 4t+5 \\ -\sin x & \text{if } n = 4t+6 \\ -\cos x & \text{if } n = 4t+7 \end{cases} \quad (4)$$

Because (4) is (1) when $k = t + 1$, we have proved that if (1) holds for $k = t$ it also holds for $k = t + 1$. Hence (1) holds for every positive integer k.

62. The formula of Ex. 61 can be expressed as $\mathrm{D}_x{}^n(\sin x) = \sin(x + \frac{1}{2}n\pi)$ for any positive integer n. We wish to prove by induction that if n is any positive integer, $\mathrm{D}_x{}^n(\cos x) = \cos(x + \frac{1}{2}n\pi)$. (1) We prove that formula (1) holds when $n = 1$: $\mathrm{D}_x(\cos x) = -\sin x$ and $\cos(x + \frac{1}{2}\pi) = -\sin x$. Suppose the formula is true for some integer k. Using this hypothesis and the formula for $\cos(\mathrm{A} + \mathrm{B})$, we have

$$\mathrm{D}_x{}^{k+1}(\cos x) = \mathrm{D}[\mathrm{D}_x{}^k(\cos x)] \overset{\text{hyp}}{=} \mathrm{D}[\cos(x + k\tfrac{1}{2}\pi)] \overset{\text{add}}{=} \mathrm{D}(\cos x\cos k\tfrac{1}{2}\pi - \sin x\sin k\tfrac{1}{2}\pi)$$
$$= -\sin x\cos k\tfrac{1}{2}\pi - \cos x\sin k\tfrac{1}{2}\pi = \cos(x + \tfrac{1}{2}\pi)\cos k\tfrac{1}{2}\pi - \sin(x + \tfrac{1}{2}\pi)\sin k\tfrac{1}{2}\pi \overset{\text{add}}{=} \cos[x + (k+1)\tfrac{1}{2}\pi]$$

We have proved that if (1) holds for $n = k$, it hold for $n = k + 1$. Hence (1) holds for every positive integer k.

THE DERIVATIVE OF A COMPOSITE FUNCTION AND THE CHAIN RULE

The Chain Rule If the function g is differentiable at x and the function f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The following differentiation formulas are special cases of the chain rule. In each formula we assume that u is a function differentiable at x. In the formulas involving the tangent, cotangent, secant, and cosecant functions we assume that the trigonometric function is defined at u.

$$D_x u^n = nu^{n-1} D_x u \quad \text{if } n \text{ is any integer}$$

$D_x \sin u = \cos u \, D_x u$	$D_x \cos u = -\sin u \, D_x u$
$D_x \tan u = \sec^2 u \, D_x u$	$D_x \cot u = -\csc^2 u \, D_x u$
$D_x \sec u = \sec u \tan u \, D_x u$	$D_x \csc u = -\csc u \cot u \, D_x u$

When computing derivatives by the chain rule we don't actually write the functions f and g, but we bear them in mind. If a function is described as "power of ...", "sine of ...", or "product of ...", then we first apply the power rule, sine rule, or product rule. In the final form of an answer, the simpler factors appear first.

Simple Harmonic Motion: An object moving on a line so that the measure of its acceleration is proportional to the measure of its displacement s from a fixed point on the line, and the acceleration and displacement are oppositely directed. It is a sum of terms of the form $A\cos(kx + a)$ and $B\sin(kx + b)$ of *amplitudes* $|A|$ and $|B|$, *period* $2\pi/|k|$ and *frequency* $|k|/2\pi$.

Exercises 2.8

Exercises 1–12, find the derivative of the function.

$f'(x) = D_x(2x+1)^3 = 3(2x+1)^2 D_x(2x+1) = 3(2x+1)^2(2) = 6(2x+1)^2$

$f'(x) = D_x(10-5x)^4 = 4(10-5x)^3 D_x(10-5x) = 4(10-5x)^3(-5) = -20(10-5x)^3$

$F'(x) = D_x(x^2+4x-5)^4 = 4(x^2+4x-5)^3 D_x(x^2+4x-5) = 4(x^2+4x-5)^3(2x+4) = 8(x+2)(x^2+4x-5)^3$

$g(r) = (2r^4 + 8r^2 + 1)^5$

Because $g(r)$ is the fifth power of $2r^4 + 8r^2 + 1$, we use the power rule first.

$g'(r) = 5(2r^4+8r^2+1)^4 \cdot D_r(2r^4+8r^2+1) = 5(2r^4+8r^2+1)^4(8r^3+16r) = 40r(r+2)(2r^4+8r^2+1)^4$

$f'(t) = D_t(2t^4-7t^3+2t-1)^2 = 2(2t^4-7t^3+2t-1)D_t(2t^4-7t^3+2t-1)$
$= 2(2t^4-7t^3+2t-1)(8t^3-21t^2+2)$

$H'(z) = D_z(z^3-3z^2+1)^{-3} = -3(z^3-3z^2+1)^{-4}D_z(z^3-3z^2+1) = -3(z^3-3z^2+1)^{-4}(3z^2-6z)$
$= -9z(z-2)(z^3-3z^2+1)^{-4}$

$f'(x) = D[(x^2+4)^{-2}] = -2(x^2+4)^{-3}D_x(x^2+4) = -2(x^2+4)^{-3}(2x) = -4x(x^2+4)^{-3}$

$g(x) = \sin x^2$

Because $g(x)$ is the sine of x^2, we use the sine rule first.

$g'(x) = \cos x^2 \, D_x(x^2) = \cos x^2 \, (2x) = 2x \cos x^2$

$f'(x) = D_x(4\cos 3x - 3\sin 4x) = -4(\sin 3x)(3) - 3(\cos 4x)(4) = -12(\sin 3x + \cos 4x)$

$G'(x) = D_x(\sec^2 x) = 2\sec x \, D_x(\sec x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$

$h'(x) = D_x(\frac{1}{3}\sec^3 2x - \sec 2x) = (\sec^2 2x - 1)D_x(\sec 2x) = (\sec^2 2x - 1)(\sec 2x \tan 2x)(2)$
$= (\tan^2 2x)(2\sec 2x \tan 2x) = 2\sec 2x \tan^3 2x$

12. $f(x) = \cos(3x^2 + 1)$

▷ Applying the chain rule to the cosine rule, we have

$$f'(x) = -\sin(3x^2 + 1)\mathrm{D}_x(3x^2 + 1) = -\sin(3x^2 + 1)(6x) = -6x \sin(3x^2 + 1)$$

In Exercises 13–16, compute the derivative.

13. $\frac{d}{dx}(\sec^2 x \tan^2 x) \overset{\text{prod}}{=} \mathrm{D}_x(\sec^2 x)\tan^2 x + \sec^2 x\, \mathrm{D}_x(\tan^2 x) = 2 \sec x(\sec x \tan x)\tan^2 x + \sec^2 x(2 \tan x \sec^2 x)$
$= 2 \sec^2 x \tan^3 x + 2 \sec^2 x \tan x = 2 \sec^2 x \tan x(\tan^2 x + \sec^2 x)$

14. $\frac{d}{dt}(2 \sin^3 t \cos^2 t) \overset{\text{prod}}{=} 2[\mathrm{D}_t(\sin^3 t) \cdot \cos^2 t + \sin^3 t \cdot \mathrm{D}_t(\cos^2 t)] = 2[3 \sin^2 t \cdot \mathrm{D}_t \sin t \cdot \cos^2 t + \sin^3 t \cdot 2 \cos t \cdot \mathrm{D}_t \cos t]$
$= 6 \sin^2 t \cos t - 4 \sin^4 t \cos t$

15. $\frac{d}{dt}(\cot^4 t - \csc^4 t) = 4 \cot^3 t(-\csc^2 t) - 4 \csc^3 t(-\csc t \cot t) = -4 \cot t \csc^2 t(\cot^2 t - \csc^2 t)$
$= -4 \cot t \csc^2 t(-1) = 4 \cot t \csc^2 t$. Alternatively,
$\frac{d}{dt}(\cot^4 t - \csc^4 t) = \frac{d}{dt}[(\cot^2 t - \csc^2 t)(\cot^2 t + \csc^2 t)] = \frac{d}{dt}[-(2 \cot^2 t - 1)] = 4 \cot t \csc^2 t$

16. $\frac{d}{dx}[(4x^2 + 7)^2(2x^3 + 1)^4]$

▷ Because the function is the product of $(4x^2 + 7)^2$ and $(2x^3 + 1)^4$, we use the product rule first, then the power rule. In the final step we remove the repeated factors. Note that we cannot describe the whole function as a power of something.

$$\begin{aligned}
&\frac{d}{dx}[(4x^2 + 7)^2(2x^3 + 1)^4] \\
&\quad = (4x^2 + 7)^2 \cdot \mathrm{D}_x(2x^3 + 1)^4 + (2x^3 + 1)^4 \cdot \mathrm{D}_x(4x^2 + 7)^2 \\
&\quad = (4x^2 + 7)^2 \cdot 4(2x^3 + 1)^3 \mathrm{D}_x(2x^3 + 1) + (2x^3 + 1)^4 \cdot 2(4x^2 + 7)\mathrm{D}_x(4x^2 + 7) \\
&\quad = (4x^2 + 7)^2 \cdot 4(2x^3 + 1)^3(6x^2) + (2x^3 + 1)^4 \cdot 2(4x^2 + 7)(8x) \\
&\quad = 24x^2(4x^2 + 7)^2(2x^3 + 1)^3 + 16x(4x^2 + 7)(2x^3 + 1)^4 \\
&\quad = 8x(4x^2 + 7)(2x^3 + 1)^3[3x(4x^2 + 7) + 2(2x^3 + 1)] \\
&\quad = 8x(4x^2 + 7)(2x^3 + 1)^3(16x^3 + 21x + 2)
\end{aligned}$$

In Exercises 17–24, find the derivative of the function and check by plotting your answer and NDER.

17. $\frac{d}{dx}\left(\frac{x-7}{x+2}\right)^2 = 2\left(\frac{x-7}{x+2}\right)^1 \frac{d}{dx}\left(\frac{x-7}{x+2}\right) = 2\left(\frac{x-7}{x+2}\right)\frac{(x+2)1 - (x-7)1}{(x+2)^2} = \frac{2(x-7)9}{(x+2)^2} = \frac{18(x-7)}{(x+2)^2}$

18. $\frac{d}{dt}\left[\left(\frac{2t^2+1}{3t^3+1}\right)^2\right] = 2\left(\frac{2t^2+1}{3t^3+1}\right) \cdot \mathrm{D}_t\left(\frac{2t^2+1}{3t^3+1}\right) = 2 \cdot \frac{2t^2+1}{3t^3+1} \cdot \frac{(3t^3+1)(4t) - (2t^2+1)(9t^2)}{(3t^3+1)^2} = 2 \cdot \frac{2t^2+1}{3t^3+1} \cdot \frac{-t^4 - 9t^2 + 4t}{(3t^3+1)^2}$
$= \frac{-2t(2t^2+1)(6t^3+9t-4)}{(3t^3+1)^3}$

19. $g'(t) = \mathrm{D}_t \sin^2(3t^2 - 1) = 2 \sin(3t^2 - 1)\mathrm{D}_t \sin(3t^2 - 1) = 2 \sin(3t^2 - 1)\cos(3t^2 - 1)(6t) = 6t \sin(6t^2 - 2)$

20. $g(x) = \tan^2 x^2$

▷ Because the function is the square of $\tan x^2$, we apply the power rule first.

$g'(x) = 2 \tan x^2 \cdot \mathrm{D}_x(\tan x^2) = 2 \tan x^2 \sec^2 x^2 \cdot \mathrm{D}_x(x^2) = 4x \tan x^2 \sec^2 x^2$

21. $f'(x) = \mathrm{D}_x(\tan^2 x - x^2)^3 = 3(\tan^2 x - x^2)^2 \mathrm{D}_x(\tan^2 x - x^2) = 3(\tan^2 x - x^2)^2[2 \tan x(\sec^2 x) - 2x]$
$= 6(\tan^2 x - x^2)^2(\tan x \sec^2 x - x)$

22. $G'(x) = \mathrm{D}_x(2 \sin x - 3 \cos x)^3 = 3(2 \sin x - 3 \cos x)^2 \mathrm{D}_x(2 \sin x - 3 \cos x)$
$= 3(2 \sin x - 3 \cos x)^2(2 \cos x + 3 \sin x)$

23. $F'(x) = \mathrm{D}[4 \cos(\sin 3x)] = -4 \sin(\sin 3x)\mathrm{D}_x(\sin 3x) = -4 \sin(\sin 3x)\cos 3x(3) = -12 \cos 3x \sin(\sin 3x)$

24. $f(x) = \sin^2(\cos 2x)$

▷ Because $f(x)$ is the square of a sine, we use the power rule first, then the sine rule.

$$\begin{aligned}
f'(x) &= 2 \sin(\cos 2x)\mathrm{D}_x \sin(\cos 2x) = 2 \sin(\cos 2x)\cos(\cos 2x)\mathrm{D}_x \cos 2x \\
&= 2 \sin(\cos 2x)\cos(\cos 2x)(-\sin 2x)\mathrm{D}_x(2x) = 2 \sin(\cos 2x)\cos(\cos 2x)(-2 \sin 2x) \\
&= -2 \sin 2x \sin(2 \cos 2x)
\end{aligned}$$

Exercises 25 and 26, find an equation of the tangent line at the point. Check by plotting the curve and tangent.

$y = (x^2 - 1)^2$ at $(2, 9)$

$y'(x) = 2(x^2 - 1)\mathrm{D}_x(x^2 - 1) = 2(x^2 - 1)(2x) = 4x(x^2 - 1)$. $y'(2) = 24$. $y = 24(x - 2) + 9$; $y = 24x - 39$

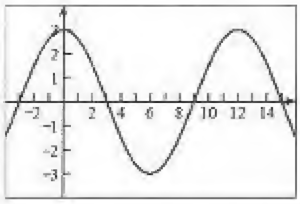
$y = 4 \tan 2x$ at $(\frac{1}{8}\pi, 4)$

$y = (x) = 4 \sec^2 2x\ \mathrm{D}_x(2x) = 8 \sec^2 2x$. $y'(\frac{1}{8}\pi) = 8 \sec^2 \frac{1}{4}\pi = 8(\sqrt{2})^2 = 16$. $y = 16(x - \frac{1}{8}\pi) + 4$

Exercises 27–30, s cm is the distance of a weight from its central position at t seconds, and the positive direction is upward. (a) Find the velocity and acceleration. (b) Show that the motion is simple harmonic. (c) Find the amplitude A, period p, and frequency f of the motion. (d) Simulate the motion on your calculator. (e) Plot the graph.

$s = 6 \sin \frac{1}{4}\pi t$ (a) $v = \frac{ds}{dt} = \frac{3}{2}\pi \cos \frac{1}{4}\pi t$, $a = \frac{dv}{dt} = -\frac{3}{8}\pi^2 \sin \frac{1}{4}\pi t$ (b) Because $a = -(\frac{1}{4}\pi)^2 s$ and $-(\frac{1}{4}\pi)^2$ is negative, the motion is simple harmonic. (c) $\mathrm{A} = 6$, $p = 2\pi/\frac{1}{4}\pi = 8$, $f = 1/p = \frac{1}{8}$

$s = 3 \cos \frac{1}{6}\pi t$ ▷ (a) The graph is at the right.



(a) $v = \frac{ds}{dt} = \mathrm{D}_t(3 \cos \frac{1}{6}\pi t) = -\frac{1}{2}\pi \sin \frac{1}{6}\pi t$,

$a = \frac{dv}{dt} = \mathrm{D}_t(-\frac{1}{2}\pi \sin \frac{1}{6}\pi t) = -\frac{1}{12}\pi^2 \cos \frac{1}{6}\pi t = -(\frac{1}{6}\pi)^2(3 \cos \frac{1}{6}\pi t)$

(b) Because $-(\frac{1}{6}\pi)^2$ is a constant, then a, the measure of the acceleration, is proportional to s, the measure of the displacement. Furthermore, because $-(-\frac{1}{6}\pi)^2$ is negative, then a and s are oppositely directed. Thus, the motion is simple harmonic. (c) The amplitude is 3. The period is $2\pi/\frac{1}{6}\pi = 12$ and the frequency is $\frac{1}{12}$. (d) To simulate the motion, in parametric mode let $x_1(t) = 2$, $y_1(t) = 3 \cos \frac{1}{6}\pi t$. In the window $[0, 4] \times [-4, 4]$, we let $[t\mathrm{Min}, t\mathrm{Max}] = [0, 12]$, t-step $= .05$. Press the $\boxed{\text{TRACE}}$ key, press the ◁ key until the cursor is at 0, the press the ▷ key to observe the motion.

$s = 4 \cos \pi(2t - \frac{1}{3})$ (a) $v = \frac{ds}{dt} = -8\pi \sin \pi(2t - \frac{1}{3})$, $a = \frac{dv}{dt} = -16\pi^2 \cos \pi(2t - \frac{1}{3})$ (b) Because $a = -(2\pi)^2 s$ and $-(2\pi)^2$ is negative, the motion is simple harmonic. (c) $\mathrm{A} = 4$, $p = 2\pi/2\pi = 1$, $f = 1/1 = 1$

$s = 8 \sin \pi(3t + \frac{1}{2})$ (a) $v = \frac{ds}{dt} = 24\pi \cos \pi(3t + \frac{1}{2})$, $a = \frac{dv}{dt} = -72\pi^2 \sin \pi(3t + \frac{1}{2})$ (b) Because $a = -(3\pi)^2 s$ and $-(3\pi)^2$ is negative, the motion is simple harmonic. (c) $\mathrm{A} = 8$, $p = 2\pi/3\pi = \frac{2}{3}$, $f = 1/p = \frac{3}{2}$

Exercises 31 and 32, s m is the distance of a particle from the origin at t seconds. Find (a) the velocity and (b) the acceleration. (c) Show that the motion is simple harmonic.

$s = b \cos(kt + c)$ (a) $\frac{ds}{dt} = -bk \sin(kt + c)$ (b) $\frac{d^2s}{dt^2} = -bk^2 \cos(kt + c)$ (c) Because $\frac{d^2s}{dt^2} = -k^2 s$, the measure of the acceleration is always proportional to the measure of the displacement and the acceleration and displacement are oppositely directed. Therefore, the motion is simple harmonic.

$s = \mathrm{A} \sin 2\pi kt + \mathrm{B} \cos 2\pi kt$, where A, B, and k are constants.

We find the velocity and then the acceleration of the particle by differentiating with respect to t. Thus,

$$v = \frac{ds}{dt} = 2\pi k\mathrm{A} \cos 2\pi kt - 2\pi k\mathrm{B} \sin 2\pi kt$$

$$a = \frac{dv}{dt} = -4\pi^2 k^2 \mathrm{A} \sin 2\pi kt - 4\pi^2 k^2 \mathrm{B} \cos 2\pi kt = -4\pi^2 k^2(\mathrm{A} \sin 2\pi kt + \mathrm{B} \cos 2\pi kt)$$

Substituting, we have $a = -4\pi^2 k^2 s$. Because $-4\pi^2 k^2$ is a constant, then a, the measure of the acceleration, is proportional to s, the measure of the displacement. Furthermore, because $-4\pi^2 k^2$ is negative, then a and s are oppositely directed. Thus, the motion is simple harmonic.

Exercises 33–36, s ft is the distance of a particle from the origin at t seconds. (a) Find v and a. (b) Show that the motion is simple harmonic. (c) Simulate the motion on your calculator.

$s = 5 \sin \pi t + 3 \cos \pi t$ (a) $v = \frac{ds}{dt} = 5\pi \cos \pi t - 3\pi \sin \pi t$, $a = \frac{dv}{dt} = -5\pi^2 \sin \pi t - 3\pi^2 \cos \pi t$

(b) Because $a = -\pi^2 s$ and $-\pi^2$ is negative, the motion is simple harmonic. The amplitude is $\sqrt{5^2 + 3^2} = \sqrt{34}$.

34. $s = \sin(6t - \frac{1}{3}\pi) + \sin(6t + \frac{1}{6}\pi)$; $v = \frac{ds}{dt} = 6\cos(6t - \frac{1}{3}\pi) + 6\cos(6t + \frac{1}{6}\pi)$

$a = \frac{dv}{dt} = -36\sin(6t - \frac{1}{3}\pi) - 36\sin(6t + \frac{1}{6}\pi) = -36[\sin(6t - \frac{1}{3}\pi) + \sin(6t + \frac{1}{6}\pi)] = -36s$

Because $a = -36s$, the measure of the acceleration is always proportional to the measure of the displacement and the acceleration and displacement are oppositely directed. Therefore, the motion is simple harmonic. The amplitude is $\sqrt{2}$.

35. $s = 5 - 10\sin^2 2t = 5 - 10 \cdot \frac{1}{2}[1 - \cos 2(2t)] = 5\cos 4t$; $v = \frac{ds}{dt} = -20\sin 4t$; $a = \frac{dv}{dt} = -80\cos 4t = -16s$
Because $a = -16s$, the measure of the acceleration is always proportional to the measure of the displacement and the acceleration and displacement are oppositely directed. Therefore, the motion is simple harmonic.

36. $s = 8\cos^2 6t - 4$

▷ Because $s = 4(2\cos^2 6t - 1)$ we may apply the identity $\cos 2x = 2\cos^2 x - 1$ with x replaced by $6t$ to simplify the given equation. Thus, $s = 4\cos 12t$. We differentiate to find v and a. Thus,

(a) $v = \frac{ds}{dt} = 4(-\sin 12t)(12) = -48\sin 12t$

$a = \frac{dv}{dt} = -48(\cos 12t)(12) = -144(4\cos 12t) = -144s$

(b) Because the acceleration is proportional to the displacement, and a and s are oppositely directed, the motion is simple harmonic. (c) To simulate the motion, in parametric mode let $x_1(t) = 2$, $y_1(t) = 4\cos 12t$. In the window $[0,4]\times[-5,5]$, we let $[t\text{Min}, t\text{Max}] = [0,2]$, t-step = .05. Press the $\boxed{\text{TRACE}}$ key, press the ◁ key until the cursor is at 0, the press the ▷ key to observe the motion.

37. (a) $h(\theta) = 10 - 10\cos\theta = 10(1 - \cos\theta) = 10 \cdot 2\sin^2 \frac{1}{2}\theta = 20\sin^2 \frac{1}{2}\theta$. $h'(\theta) = 2 \cdot 20 \sin\frac{1}{2}\theta \cos\frac{1}{2}\theta \cdot \frac{1}{2} = 10\sin\theta$

(b) $h'(\frac{1}{6}\pi) = 10\sin\frac{1}{6}\pi = 10 \cdot \frac{1}{2} = 5$ (c) $h'(\frac{1}{3}\pi) = 10\sin\frac{1}{3}\pi = 10 \cdot \frac{1}{2}\sqrt{3} = 5\sqrt{3}$ (d) $h'(\frac{1}{2}\pi) = 10\sin\frac{1}{2}\pi = 10 \cdot 1 = 10$

38. $K = 25\sin\alpha$. $K'(\alpha) = 25\cos\alpha$ (a) $K'(\frac{1}{6}\pi) = 25\cos\frac{1}{6}\pi = 25 \cdot \frac{1}{2}\sqrt{3} = \frac{25}{2}\sqrt{3}$ (b) $K'(\frac{1}{4}\pi) = 25\cos\frac{1}{4}\pi = 25 \cdot \frac{1}{2}\sqrt{2}$

(c) $K'(\frac{1}{3}\pi) = 25\cos\frac{1}{3}\pi = 25 \cdot \frac{1}{2} = \frac{25}{2}$

39. $P = \left(\frac{40+T}{140}\right)^5$, $T > 80$. Find the rate of change of P with respect to T when (a) $T = 100$; (b) $P = 32$.

▷ $P' = \frac{5}{140}\left(\frac{40+T}{140}\right)^4$ (a) $P'(100) = \frac{5}{140} \cdot 1^4 = \frac{1}{28}$

(b) $P = 32 = \left(\frac{40+T}{140}\right)^5$ when $2 = \frac{40+T}{140}$, $T = 240$. $P'(240) = \frac{5}{140}\left(\frac{280}{140}\right)^4 = \frac{5 \cdot 16}{140} = \frac{4}{7}$

40. If $q = -\frac{A}{\omega}\cos(\omega t + \phi)$, find i, the rate of change of q with respect to t.

▷ $i = \frac{dq}{dt} = \frac{A}{\omega}\sin(\omega t + \phi) \cdot \omega = A\sin(\omega t + \phi)$

41. If $\theta = 0.2\cos\pi(t - 0.5)$ at t seconds, find, to the nearest tenth, how fast the angle θ is changing at 3.1 sec.

▷ $\theta'(t) = -0.2\pi\sin\pi(t - 0.5)$, $\theta'(3.1) = -0.2\pi\sin 2.6\pi = -0.598 \approx -0.60$ rad/sec

42. $V = \frac{1}{16}x(100 - x)^2$, $V' = \frac{1}{16}[(100 - x)^2 + x \cdot 2(100 - x)(-1)] = \frac{1}{16}(100 - x)(100 - 3x)$. $V'(32) = \frac{1}{16} \cdot 68 \cdot 4 = 17$,

$V'(33) = \frac{1}{16} \cdot 67 \cdot 1 = 4.1875 \approx 4.2$, $V'(34) = \frac{1}{16} \cdot 66(-2) = -8.25 \approx -8.3$

43. $E(t) = 50\sin 120\pi t$; $E'(t) = 50\cos 120\pi t(120\pi) = 6{,}000\pi\cos 120\pi t$
(a) $E'(0.02) = 6{,}000\pi\cos 2.4\pi = 6000\pi\cos .4\pi \approx 5824.8$. Hence, the instantaneous rate of change of $E(t)$ with respect to t at $t = 0.02$ sec is 5824.8 volts/sec. (b) $E'(0.2) = 6000\pi\cos 24\pi = 6000\pi \approx 18{,}850$. Hence, the instantaneous rate of change of $E(t)$ with respect to t at $t = 0.2$sec is 18,850 volts/sec.

44. A wave produced by a simple sound has the equation $P(t) = 0.003\sin 1800\pi t$ where $P(t)$ dynes per square centimeter is the difference between the atmospheric pressure and the air pressure at the eardrum at t seconds. Find the instantaneous rate of change of $P(t)$ with respect to t at (a) $\frac{1}{9}$ sec; (b) $\frac{1}{8}$ sec; (c) $\frac{1}{7}$ sec.

▷ The instantaneous rate of change of $P(t)$ with respect to t is $P'(t)$.

$P'(t) = 0.003\cos 1800\pi t \cdot D_x(1800\pi t) = 0.003(1800\pi)\cos 1800\pi t = 5.4\pi\cos 1800\pi t$

$P'(\frac{1}{9}) = 5.4\pi\cos(1800\pi \cdot \frac{1}{9}) = 5.4\pi\cos 200\pi = 5.4\pi\cos 0 = 5.4\pi \approx 17.0$

$P'(\frac{1}{8}) = 5.4\pi\cos(1800\pi \cdot \frac{1}{8}) = 5.4\pi\cos 225\pi = 5.4\pi\cos\pi = -5.4\pi \approx -17.0$

$P'(\frac{1}{7}) = 5.4\pi\cos(\frac{1800}{7}\pi) \approx -15.3$

Thus the instantaneous rate of change of $P(t)$ with respect to t is

(a) 17.0 dynes/cm^2 per second when $t = \frac{1}{5}$; (b) -17.0 dynes/cm^2 per second when $t = \frac{1}{8}$;
(c) -15.3 dynes/cm^2 per second when $t = \frac{1}{7}$.

The demand equation for a particular toy is $p^2x = 5000$, where x toys are demanded per month when p dollars is the price per toy. It is expected that in t months, where $t \in [0, 6]$, the price of the toy will be p dollars, where $20p = t^2 + 7t + 100$. What is the anticipated rate of change of the demand with respect to time in 5 months? Do not express x in terms of t, but use the chain rule.

► We have $x = 5000p^{-2}$ and $p = \frac{1}{20}(t^2 + 7t + 100)$

By the chain rule $\quad \dfrac{dx}{dt} = \dfrac{dx}{dp} \cdot \dfrac{dp}{dt} = -10{,}000p^{-3} \cdot \frac{1}{20}(2t + 7) = -\dfrac{500(2t + 7)}{p^3} \qquad (1)$

When $t = 5$ then $\quad p = \frac{1}{20}[(5)^2 + 7(5) + 100] = 8$

Substituting for t and p in Eq. (1), we obtain $\quad \left.\dfrac{dx}{dt}\right|_{t=5} = -\dfrac{500(17)}{8^3} \approx -16.6$

that is, in 5 months the demand will be decreasing at the rate of 16.6 toys per month.

At t min, the area of an oil spill is $A(t) = \begin{cases} \pi(4t^2 + 20)^2 & \text{if } 0 \le t \le 2 \\ \pi(16t + 4)^2 & \text{if } t > 2 \end{cases}$ (a) Prove that A is differentiable at 2.
(b) Define $A'(t)$; find the rate at which the area of the spill is changing at (c) 0.4 min, (d) 2 min, (e) 3.2 min.

► (a) $f'_-(2) = 2\pi(4t^2 + 20)8t\Big|_{t=2} = 1152\pi$; $f'_+(2) = 2\pi(16t + 4)16\Big|_{t=2} = 1152\pi$. Thus A is differentiable at 2 and (b) $A'(t) = \begin{cases} 16\pi t(4t^2 + 20) & \text{if } 0 < t \le 2 \\ 32\pi(16t + 4) & \text{if } t > 2 \end{cases}$ (c) $A'(0.4) = 16\pi(0.4)[4(0.4)^2 + 20] = 414.99 \approx 415$ m^2/min

(d) $A'(2) = 1152\pi \approx 3619$ m^2/min (e) $A'(3.2) = 32\pi[16(3.2) + 4] \approx 9571$ m^2/min.

$f(x) = x^3$; $g(x) = f(x^2)$

► (a) $f'(x) = 3x^2$, so $f'(x^2) = 3(x^2)^2 = 3x^4$ (b) $g'(x) = D[f(x^2)] = f'(x^2)(2x) = 3x^4(2x) = 6x^5$

Given $f(u) = u^2 + 5u + 5$ and $g(x) = (x + 1)/(x - 1)$. Find the derivative of $f \circ g$ in two ways: (a) by first finding $(f \circ g)(x)$; (b) by using the chain rule.

► (a) $D_x(f \circ g)(x) = D_x\left[\left(\dfrac{x+1}{x-1}\right)^2 + 5\left(\dfrac{x+1}{x-1}\right) + 5\right] = 2\left(\dfrac{x+1}{x-1}\right)D_x\left(\dfrac{x+1}{x-1}\right) + 5D_x\left(\dfrac{x+1}{x-1}\right)$

$= \left[2\left(\dfrac{x+1}{x-1}\right) + 5\right]\dfrac{1(x-1) - (x+1)1}{(x-1)^2} = \left[2\left(\dfrac{x+1}{x-1}\right) + 5\right] \cdot \dfrac{-2}{(x-1)^2}$

(b) $f'(u) = 2u + 5$ and $g'(x) = D_x\left(\dfrac{x+1}{x-1}\right) = \dfrac{-2}{(x-1)^2}$. $(f \circ g)'(x) = f'(g(x))g'(x) = \left[2\left(\dfrac{x+1}{x-1}\right) + 5\right] \cdot \dfrac{-2}{(x-1)^2}$

$D_x(\cos x) = D[\sin(\frac{1}{2}\pi - x)] = \cos(\frac{1}{2}\pi - x)(-1) = -\cos(\frac{1}{2}\pi - x) = -\sin x$

Use the chain rule to prove that (a) the derivative of an even function is an odd function, and (b) the derivative of an odd function is an even function, provided that these derivatives exist.

► (a) If f is an even function, then $f(-x) = f(x)$.
Differentiating on both sides with respect to x and using the chain rule on the left side, we have
$f'(-x)D_x(-x) = f'(x)$; $\quad f'(-x)(-1) = f'(x)$; $\quad f'(-x) = -f'(x)$
Therefore f' is an odd function.
(b) If f is an odd function, then $f(-x) = -f(x)$.
Differentiating on both sides with respect to f we have
$f'(-x)D_x(-x) = -f'(x)$; $\quad f'(-x)(-1) = -f'(x)$; $\quad f'(-x) = f'(x)$
Therefore f' is an even function.

51. Use the result of Exercise 50(a) to prove that if f and g are differentiable, g is an even function and $h(x) = (f \circ g)(x)$, then $h'(0) = 0$.

► If g is an even function then by Ex.51(a), g' is an odd function so $g(-x) = g(x)$ and $g'(-x) = -g'(x)$ for every x in the domain of g'. If $h(x) = (f \circ g)(x)$, then $h'(x) = f'(g(x)) \cdot g'(x)$, so
$h'(-x) = f'(g(-x)) \cdot g'(-x) = f'(g(x))(-g'(x)) = -h'(x)$.
Substituting $x = 0$ gives $h'(0) = -h'(0)$; $2h'(0) = 0$; $h'(0) = 0$.

52. Suppose that f and g are functions such that $f'(x) = 1/x$ (1) and $(f \circ g)(x) = x$. Prove that if $g'(x)$ exists, then $g'(x) = g(x)$.

► $D_x(f \circ g)(x) = D_x x$. By the chain rule, $f'(g(x))g'(x) = 1$. By (1), $[1/g(x)]g'(x) = 1$; $g'(x) = g(x)$

53. Let $f(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

(a) Prove that f is continuous at 0. (b) Compute $f'(x)$. (c) Prove that f' is discontinuous at 0.

▷ (a) Because $|\sin x| \le 1$, $0 \le |x^2\sin(1/x)| \le x^2 \to 0$ as $x \to 0$. Hence by the squeeze theorem $\lim_{x\to 0} f(x) = 0 = f(0)$ and so f is continuous at 0. (b) $f'(x) = 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x}\left(-\frac{1}{x^2}\right) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ if $x \neq 0$ and $f'(0) = \lim_{x\to 0}\frac{x^2\sin(1/x) - 0}{x - 0} = x\sin\frac{1}{x} = 0$. (c) As $x \to 0$, the first term of $f'(x)$ approaches 0 but the second term oscillates between ± 1 and so $\lim_{x\to 0} f'(x)$ does not exist.

54. If f'' and g'' exist, find $(f\circ g)''$.

▷ $(f\circ g)'' = D_x(D_x(f\circ g)) \overset{\text{chain}}{=} D_x(f'(g)g') \overset{\text{product}}{=} D_x(f'(g))g' + f'(g)D_xg' \overset{\text{chain}}{=} f''(g)(g')^2 + f'(g)g''$

55. See Exercise 27. **56.** See Exercise 28.

57. Suppose that f and g are such that (i) $g'(x_1)$ and $f'(g(x_1))$ exist and (ii) for all $x \neq x_1$ in some open interval containing x_1, $g(x) - g(x_1) \neq 0$. Then $\dfrac{(f\circ g)(x) - (f\circ g)(x_1)}{x - x_1} = \dfrac{(f\circ g)(x) - (f\circ g)(x_1)}{g(x) - g(x_1)} \cdot \dfrac{g(x) - g(x_1)}{x - x_1}$.

(a) Prove that as $x \to x_1$, $g(x) \to (x_1)$ and hence that $(f\circ g)'(x_1) = f'(g(x_1))g'(x_1)$ thus simplifying the proof of the chain rule under the additional hypothesis (ii). (b) Explain why this proof applies at $x_1 = 0$ if $f(x) = x^2$ and $g(x) = x^3$, but not if $g(x) = \operatorname{sgn} x$.

▷ (a) Because $f'(g(x_1))$ exists, for every $\epsilon > 0$ there is a $\delta > 0$ such that $\left|\dfrac{f(g(x)) - f(g(x_1))}{g(x) - g(x_1)} - f'(g(x_1))\right| < \epsilon$ whenever $0 < |g(x) - g(x_1)| < \delta$. Because $g'(x_1)$ exists, $g(x)$ is continuous at x_1 and so $g(x) \to g(x_1)$ as $x \to x_1$. Hence there is a δ_1 such that $|g(x) - g(x_1)| < \delta$ whenever $0 < |g(x) - g(x_1)| < \delta_1$. By (ii), we have $0 < |g(x) - g(x_1)| < \delta$ whenever $0 < |x - x_1| < \delta_1$, which proves that the first factor has limit $f'(g(x_1))$. The second factor has limit $g'(x_1)$ by definition, which completes the proof. (b) The proof does not apply to $g(x) = \operatorname{sgn} x$ because $g'(0)$ does not exist.

2.9 THE DERIVATIVE OF THE POWER FUNCTION FOR RATIONAL EXPONENTS AND IMPLICIT DIFFERENTIATION

Two theorems proved in this section extend the formulas for the derivative of a power function and for the derivative of a composite power function to include those powers with rational number exponents. To apply them to radicals, rewrite using exponents.

2.9.1 Theorem If f is the power function defined by $f(x) = x^r$, where r is any rational number, then f is differentiable and

$$f'(x) = rx^{r-1}, \quad \text{where } r \ge 1 \text{ if } x = 0 \text{ and } r \text{ has an odd denominator if } x < 0.$$

2.9.2 Theorem If f and g are functions such that $f(x) = [g(x)]^r$, where r is any rational number, and if $g'(x)$ exists, then f is differentiable, and

$$f'(x) = r[g(x)]^{r-1}g'(x)$$

Algebraic Function A sum of terms of the form cx^ny^m is identically 0. We can find y' without solving for y.

Implicit Differentiation With implicit differentiation we must be careful to distinguish the independent variable from the dependent variable. If x is the independent variable, then we regard y as a function of x. Thus,

$$D_x x = 1$$

but $D_x y$ is not known. Similarly,

$$D_x(x^r) = rx^{r-1}$$

but to find $D_x(y^r)$ we must use the chain rule since y^r is a composite function of x. Thus,

$$D_x(y^r) = ry^{r-1}\frac{dy}{dx}$$

If we regard y as the independent variable and differentiate implicitly with respect to y, then the roles of x and y are interchanged. Thus,

$$D_y(x^r) = rx^{r-1}\frac{dx}{dy} \quad \text{and} \quad D_y(y^r) = ry^{r-1}$$

Exercises 2.9

In Exercises 1–12, find the derivative of the function.

1. $f'(x) = \mathrm{D}_x(4x^{1/2} + 5x^{-1/2}) = \frac{1}{2}\cdot 4x^{-1/2} - \frac{1}{2}\cdot 5x^{-3/2} = 2x^{-1/2} - \frac{5}{2}x^{-3/2}$

2. $f'(x) = \mathrm{D}_x(3x^{2/3} - 6x^{1/3} + x^{-1/3}) = \frac{2}{3}\cdot 3x^{-1/3} - \frac{1}{3}\cdot 6x^{-2/3} - \frac{1}{3}x^{-4/3} = 2x^{-1/3} - 2x^{-2/3} - \frac{1}{3}x^{-4/3}$

3. $g'(x) = \mathrm{D}_x(\sqrt{1 + 4x^2}) = \mathrm{D}_x(1 + 4x^2)^{1/2} = \frac{1}{2}(1 + 4x^2)^{-1/2}(8x) = 4x(1 + 4x^2)^{-1/2} = \dfrac{4x}{\sqrt{1 + 4x^2}}$

4. $f(s) = \sqrt{2 - 3s^2}$

■ We replace the radical sign by a fractional exponent and use the chain power rule, Theorem 2.9.2.

$$f(s) = (2 - 3s^2)^{1/2} \qquad f'(s) = \tfrac{1}{2}(2 - 3s^2)^{-1/2}\cdot(-6s) = \frac{-3s}{\sqrt{2 - 3s^2}}$$

5. $f'(x) = \mathrm{D}_x(5 - 3x)^{2/3} = \frac{2}{3}(5 - 3x)^{-1/3}(-3) = -2(5 - 3x)^{-1/3}$

6. $g'(x) = \mathrm{D}_x(\sqrt[3]{4x^2 - 1}) = \mathrm{D}_x(4x^2 - 1)^{1/3} = \frac{1}{3}(4x^2 - 1)^{-2/3}\cdot \mathrm{D}_x(4x^2 - 1) = \frac{8}{3}x(4x^2 - 1)^{-2/3}$

7. $g'(y) = \mathrm{D}_y \dfrac{1}{\sqrt{25 - y^2}} = \mathrm{D}_y(25 - y^2)^{-1/2} = -\frac{1}{2}(25 - y^2)^{-3/2}(-2y) = \dfrac{y}{(25 - y^2)^{3/2}}$

8. $f(x) = (5 - 2x^2)^{-1/3}$

■ We apply Theorem 2.9.2. Thus,

$$f'(x) = -\tfrac{1}{3}(5 - 2x^2)^{-(1/3)-1}\mathrm{D}_x(5 - 2x^2) = -\tfrac{1}{3}(5 - 2x^2)^{-4/3}(-4x) = \tfrac{4}{3}x(5 - 2x^2)^{-4/3} = \frac{4x}{3(5 - 2x^2)^{4/3}}$$

9. $h'(t) = \mathrm{D}_t(2\cos\sqrt{t}) = \mathrm{D}_t[2\cos(t^{1/2})] = -2\sin(t^{1/2})(\frac{1}{2}t^{-1/2}) = -t^{-1/2}\sin(t^{1/2}) = -\sin\sqrt{t}/\sqrt{t}$

10. $f'(x) = \mathrm{D}_x(4\sec\sqrt{x}) = 4\sec\sqrt{x}\tan\sqrt{x}\,\mathrm{D}_x(x^{1/2}) = 4\sec\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2}x^{-1/2} = 2\sec\sqrt{x}\tan\sqrt{x}/\sqrt{x}$

11. $g'(r) = \mathrm{D}_r(\cot\sqrt{3r}) = \mathrm{D}_r[\csc^2(\sqrt{3}r^{1/2})] = -\csc^2(\sqrt{3}r^{1/2})(\frac{1}{2}\sqrt{3}r^{-1/2}) = -\sqrt{3}\csc^2\sqrt{3r}/2\sqrt{r}$

12. $g(x) = \sqrt{3\sin x}$

■ We introduce a fractional exponent and then use Theorem 2.9.2.

$$g(x) = \sqrt{3}\sqrt{\sin x} = \sqrt{3}(\sin x)^{1/2}$$
$$g'(x) = \sqrt{3}(\tfrac{1}{2})(\sin x)^{-1/2}\mathrm{D}_x\sin x = \frac{\sqrt{3}\cos x}{2\sqrt{\sin x}}$$

In Exercise 13–16, compute the derivative and check by plotting NDER.

13. $\dfrac{d}{dt}\sqrt{\dfrac{\sin t}{1 - \sin t}} = \dfrac{d}{dt}\Big(\dfrac{\sin t}{1 - \sin t}\Big)^{1/2} = \dfrac{1}{2}\Big(\dfrac{\sin t}{1 - \sin t}\Big)^{-1/2}\cdot\dfrac{\cos t(1 - \sin t) - \sin t(-\cos t)}{(1 - \sin t)^2} = \dfrac{\cos t}{\sqrt{\sin t}(1 - \sin t)^{3/2}}$

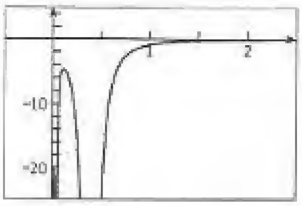
14. $\dfrac{d}{dx}\sqrt{\dfrac{\cos x - 1}{\sin x}} = \dfrac{d}{dx}\Big(\dfrac{\cos x - 1}{\sin x}\Big)^{1/2} = \dfrac{1}{2}\Big(\dfrac{\cos x - 1}{\sin x}\Big)^{-1/2}\dfrac{(-\sin x)\sin x - (\cos x - 1)\cos x}{\sin^2 x} = \dfrac{1}{2}\Big(\dfrac{\sin x}{\cos x - 1}\Big)^{1/2}\dfrac{\cos x - 1}{\sin^2 x}$

$= \dfrac{1}{2}\sqrt{\dfrac{\cos x - 1}{\sin^3 x}}$. Note that $\dfrac{\sqrt{\cos x - 1}}{2\sin^{3/2}x}$ is incorrect because $\sqrt{\cos x - 1}$ does not exist.

15. $\mathrm{D}_x(\sqrt{9 + \sqrt{9 - x}}) = \mathrm{D}_x\{[9 + (9 - x)^{1/2}]^{1/2}\} = \frac{1}{2}[9 + (9 - x)^{1/2}]^{-1/2}\mathrm{D}[9 + (9 - x)^{1/2}]$

$= \frac{1}{2}[9 + (9 - x)^{1/2}]^{-1/2}[\frac{1}{2}(9 - x)^{-1/2}(-1)] = \dfrac{-1}{4\sqrt{9 + \sqrt{9 - x}}\sqrt{9 - x}}$

16. $\mathrm{D}_x\Big(\sqrt{x}\tan\sqrt{\frac{1}{x}}\Big)$



■
$$= \mathrm{D}_x(x^{1/2}\tan x^{-1/2}) = x^{1/2}\,\mathrm{D}_x\tan x^{-1/2} + \tan x^{-1/2}\,\mathrm{D}_x(x^{1/2})$$
$$= x^{1/2}\sec^2 x^{-1/2}(-\tfrac{1}{2}x^{-3/2}) + \tan x^{-1/2}(\tfrac{1}{2}x^{-1/2})$$
$$= \tfrac{1}{2}x^{-1}(-\sec^2 x^{-1/2} + x^{1/2}\tan x^{-1/2})$$
$$= \frac{-\sec^2\sqrt{\frac{1}{x}} + \sqrt{x}\tan\sqrt{\frac{1}{x}}}{2x}$$

The figure shows a plot of NDER$(\sqrt{x}\tan\sqrt{1/x})$.

In Exercises 17–32, find dy/dx by implicit differentiation.

17. $x^2+y^2=16$; $2x+2y\frac{dy}{dx}=0$; $\frac{dy}{dx}=-\frac{x}{y}$

18. $4x^2-9y^2=1$; $8x-18y\frac{dy}{dx}=0$; $\frac{dy}{dx}=\frac{4x}{9y}$

19. $x^3+y^3=8xy$; $3x^2+3y^2\frac{dy}{dx}=8y+8x\frac{dy}{dx}$; $\frac{dy}{dx}(3y^2-8x)=8y-3x^2$; $\frac{dy}{dx}=\frac{8y-3x^2}{3y^2-8x}$

20. $x^2+y^2=7xy$

We differentiate with respect to x on both sides.

$$D_x x^2+D_x y^2=D_x(7xy) \tag{1}$$

Then $D_x x^2=2x$. Because y is a function of x, we use the chain rule to find $D_x y^2$.

$$D_x y^2=(2y)\frac{dy}{dx} \tag{2}$$

To find $D_x(7xy)$ we use the product rule

$$\begin{aligned} D_x(7xy)&=(7x)D_x y+y\cdot D_x(7x)\\ &=(7x)\frac{dy}{dx}+7y \end{aligned} \tag{3}$$

Substituting from (2) and (3) into (1), we have

$$2x+2y\frac{dy}{dx}=7x\frac{dy}{dx}+7y$$

Next, we solve algebraically for $\frac{dy}{dx}$.

$$2y\frac{dy}{dx}-7x\frac{dy}{dx}=7y-2x$$

$$(2y-7x)\frac{dy}{dx}=7y-2x$$

$$\frac{dy}{dx}=\frac{7y-2x}{2y-7x} \tag{4}$$

ALTERNATE SOLUTION: We divide on both sides by x^2. Thus

$$1+\left(\frac{y}{x}\right)^2=7\left(\frac{y}{x}\right)$$

Therefore

$$\frac{y}{x}=k \tag{5}$$

where the constant k is one of the two roots of the quadratic equation

$$1+k^2=7k$$

It follows from Eq. (5) that

$$y=kx \tag{6}$$

Thus the graph consists of two lines through the origin. Because y is not a function of x in any neighborhood of $(0,0)$, then dy/dx does not exist at $(0,0)$. Differentiating both sides of Eq. (6) with respect to x and using Eq. (5) we find

$$\frac{dy}{dx}=k=\frac{y}{x} \tag{7}$$

Using the given relation, it is easy to show that Eq. (7) is equivalent to Eq. (4).

21. $\frac{1}{x}+\frac{1}{y}=1$; $-x^{-2}-y^{-2}\frac{dy}{dx}=0$; $\frac{dy}{dx}=-\frac{x^{-2}}{y^{-2}}=-\frac{y^2}{x^2}$

22. $\frac{3}{x}-\frac{3}{y}=2x$; $3x^{-2}-3y^{-2}\frac{dy}{dx}=2$; $\frac{dy}{dx}=\frac{3x^{-2}-2}{3y^{-2}}=\frac{3y^2-2x^2y^2}{3x^2}$

23. $\sqrt{x}+\sqrt{y}=4$; $\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}\frac{dy}{dx}=0$; $\frac{dy}{dx}=\frac{x^{-1/2}}{y^{-1/2}}=-\frac{\sqrt{y}}{\sqrt{x}}$

24. $2x^3y+3xy^3=5$

▸ We differentiate with respect to x, and have

$$D_x(2x^3y)+D_x(3xy^3)=0$$

Because $2x^3y$ is the product of two functions of x, namely $2x^3$ and y, and because $3xy^3$ is the product of $3x$ and y^3, we use the product rule. Thus,

$$\left(2x^3\frac{dy}{dx}+y\cdot 6x^2\right)+\left(3x\cdot 3y^2\frac{dy}{dx}+y^3\cdot 3\right)=0$$
$$(2x^3+9xy^2)\frac{dy}{dx}=-6x^2y-3y^3$$
$$\frac{dy}{dx}=\frac{-6x^2y-3y^3}{2x^3+9xy^2}$$

$x^2y^2=x^2+y^2$; $2xy^2+2x^2y\frac{dy}{dx}=2x+2y\frac{dy}{dx}$; $\frac{dy}{dx}(2x^2y-2y)=2x-2xy^2$; $\frac{dy}{dx}=\frac{2x-2xy^2}{2x^2y-2y}=\frac{x-xy^2}{x^2y-y}$

$(2x+3)^4=3y^4$; $4(2x+3)^3=12y^3\frac{dy}{dx}$; $\frac{dy}{dx}=\frac{(2x+3)^3}{3y^3}$

$y=\cos(x-y)$; $\frac{dy}{dx}=-\sin(x-y)\left(1-\frac{dy}{dx}\right)=-\sin(x-y)+\sin(x-y)\frac{dy}{dx}$

$\frac{dy}{dx}[1-\sin(x-y)]=-\sin(x-y)$; $\frac{dy}{dx}=\frac{-\sin(x-y)}{1-\sin(x-y)}=\frac{\sin(x-y)}{\sin(x-y)-1}$

$x=\sin(x+y)$

Differentiating with respect to x, we have

$$1=\cos(x+y)\mathrm{D}_x(x+y)$$
$$1=\cos(x+y)\left(1+\frac{dy}{dx}\right)$$
$$1=\cos(x+y)+\cos(x+y)\cdot\frac{dy}{dx}$$
$$1-\cos(x+y)=\cos(x+y)\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{1-\cos(x+y)}{\cos(x+y)}$$

$\sec^2x+\csc^2y=4$; $2\sec x(\sec x\tan x)+2\csc y(-\csc y\cot y)\frac{dy}{dx}=0$; $\frac{dy}{dx}=\frac{\sec^2x\tan x}{\csc^2y\cot y}$

$\cot xy+xy=0$; $(-\csc^2xy+1)\left(y+x\frac{dy}{dx}\right)=0$; $\frac{dy}{dx}=-\frac{y}{x}$

Alternatively, $xy=k$, where k is one of the roots of $\cot k+k=0$; $y=\frac{k}{x}$, $\frac{dy}{dx}=-\frac{k}{x^2}=-\frac{y}{x}$

$x\sin y+y\cos x=1$; $\sin y+x\cos y\frac{dy}{dx}+\frac{dy}{dx}\cos x-y\sin x=0$; $\frac{dy}{dx}(x\cos y+\cos x)=y\sin x-\sin y$

$\frac{dy}{dx}=\frac{y\sin x-\sin y}{x\cos y+\cos x}$

$\cos(x+y)=y\sin x$

We differentiate on both sides with respect to x.

$$-\sin(x+y)\mathrm{D}_x(x+y)=y\,\mathrm{D}_x(\sin x)+\sin x\,\mathrm{D}_xy$$
$$-\sin(x+y)\left(1+\frac{dy}{dx}\right)=y\cos x+\sin x\frac{dy}{dx}$$
$$-\sin(x+y)-\sin(x+y)\frac{dy}{dx}=y\cos x+\sin x\frac{dy}{dx}$$
$$[\sin(x+y)+\sin x]\frac{dy}{dx}=y\cos x+\sin(x+y)$$
$$\frac{dy}{dx}=-\frac{y\cos x+\sin(x+y)}{\sin x+\sin(x+y)}$$

In Exercises 33–36, find the indicated line. check by plotting the line and the curve.

The tangent line to $y=\sqrt{x^2+9}$ at $(4,5)$.

$y'=\frac{x}{\sqrt{x^2+9}}$, $m=y'(4)=\frac{4}{\sqrt{4^2+9}}=\frac{4}{5}$. Tangent line: $y=\frac{4}{5}(x-4)+5=\frac{4}{5}x+\frac{9}{5}$

The normal line to $y=\sqrt{16+x^2}$ at the origin.

$y'=\frac{x}{\sqrt{16+x^2}}$. $y'(0)=0$. The tangent line is horizontal; the normal line is $x=0$.

The normal line to $9x^3-y^3=1$ at $(1,2)$.

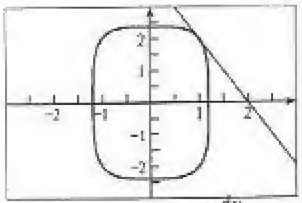
$27x^2-3y^2y'=0$; $y'=\frac{9x^2}{y^2}$. $m=-\frac{1}{y'(1)}=-\frac{2^2}{9\cdot 1^2}=-\frac{4}{9}$. Normal line: $y=-\frac{4}{9}(x-1)+2=-\frac{4}{9}x+\frac{22}{9}$

36. The tangent line to the curve $16x^4 + y^4 = 32$ at the point $(1,2)$.

▷ Differentiating implicitly, we obtain

$64x^3 + 4y^3\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{64x^3}{4y^3} = -\frac{16x^3}{y^3}$. At $(1,2)$: $\frac{dy}{dx} = -\frac{16}{8} = -2$

Equation of tangent line: $y - 2 = -2(x - 1)$; $2x + y - 4 = 0$



To plot the curve we solve for y. Then

$y^4 = 32 - 16x^4$; $y = \pm(32 - 16x^4)^{1/4}$

37. $xy = (1 - x - y)^2$; $y + x\frac{dy}{dx} = 2(1 - x - y)\left(-1 - \frac{dy}{dx}\right)$. The tangent line is parallel to the x axis when $\frac{dy}{dx} = 0$. Setting $\frac{dy}{dx} = 0$ gives $y = 2(x + y - 1)$; $y = 2 - 2x$. Substituting gives $x(2 - 2x) = (x - 1)^2$; $0 = 3x^2 - 4x + 1$ $= (3x - 1)(x - 1)$. At the points $(\frac{1}{3}, \frac{4}{3})$ and $(1, 0)$, the tangent line is parallel to the x axis. The graph is an ellipse in the first quadrant tangent to the y axis at $(0,1)$ and the x axis at $(1,0)$.

38. Find equation for the two lines through $(-1,3)$ that are tangent to the curve $x^2 + 4y^2 - 4x - 8y + 3 = 0$.

▷ Let ℓ_1 be one of the required lines and let (x_1, y_1) be the point at which ℓ_1 is tangent to the curve. To find m_1, the slope of ℓ_1, we use Eq. (1) and differentiate implicitly with respect to x. Thus,

$$2x + 8y\frac{dy}{dx} - 4 - 8\frac{dy}{dx} = 0; \quad \frac{dy}{dx} = \frac{-2x + 4}{8y - 8} = \frac{-x + 2}{4y - 4}. \text{ Hence, } m_1 = \frac{-x_1 + 2}{4y_1 - 4} \tag{2}$$

Because ℓ_1 contains $(-1,3)$ and (x_1, y_1), by definition of slope we have $\quad m_1 = \frac{y_1 - 3}{x_1 + 1}$ (3)

Eliminating m_1 between Eqs. (2) and (3), we obtain $\quad x_1^{\,2} + 4y_1^{\,2} - x_1 - 16y_1 + 10 = 0$ (4)

The curve contains (x_1, y_1) so this pair must satisfy Eq. (1). That is, $\quad x_1^{\,2} + 4y_1 - 4x_1 - 8y_1 + 3 = 0$ (5)

By subtracting terms of Eq. (5) from corresponding terms of Eq. (4), we get

$$3x_1 - 8y_1 + 7 = 0; \qquad x_1 = \frac{8y_1 - 7}{3} \tag{6}$$

Substituting from Eq. (6) into Eq. (4), we have

$$\frac{64y_1^{\,2} - 112y_1 + 49}{9} + 4y_1^{\,2} - \frac{8y_1 - 7}{3} - 16y_1 + 10 = 0$$

$$100y_1^{\,2} - 260y_1 + 160 = 0; \qquad 5y_1^{\,2} - 14y_1 + 8 = 0; \qquad (y_1 - 2)(5y_1 - 4) = 0;\ y_1 = 2 \text{ and } y_1 = \tfrac{4}{5}$$

If $y_1 = 2$ then $x_1 = \frac{8(2) - 7}{3} = 3$, $m_1 = \frac{-3 + 2}{4(2) - 4} = -\frac{1}{4}$, and an equation of ℓ_1 is

$y - 2 = -\frac{1}{4}(x - 3)$; $\quad x + 4y - 11 = 0$

If $y_1 = \frac{4}{5}$ then $x_1 = \frac{8(\frac{4}{5}) - 7}{3} = -\frac{1}{5}$, $m_1 = \frac{\frac{1}{5} + 2}{4(\frac{4}{5}) - 4} = -\frac{11}{4}$, and an equation of ℓ_1 is

$y - \frac{4}{5} = -\frac{11}{4}(x + \frac{1}{5})$; $\quad 11x + 4y - 1 = 0$

In Exercises 39–42: (a) Find two functions defined by the equation; (b) sketch the graph of each and (c) the equation. (d) Find the derivative of each function and state its domain. (e) Find dy/dx by implicit differentiation and check with (d). (f) Find an equation of each tangent line at x_1.

39. $y^2 = 4x - 8$; $x_1 = 3$

▷ (a) $y = \pm 2\sqrt{x - 2}$; $f_1(x) = 2\sqrt{x - 2}$, domain: $x \geq 2$; $f_2(x) = -2\sqrt{x - 2}$, domain $x \geq 2$

(d) $f_1{}'(x) = \frac{1}{\sqrt{x - 2}}$, domain: $x > 2$; $f_2{}'(x) = -\frac{1}{\sqrt{x - 2}}$, domain: $x > 2$

(e) $y^2 = 4x - 8$; $2y\frac{dy}{dx} = 4$; $\frac{dy}{dx} = \frac{2}{y}$

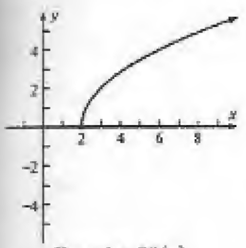
If $y = f_1(x) = 2\sqrt{x - 2}$, then $\frac{dy}{dx} = \frac{1}{y} = \frac{2}{2\sqrt{x - 2}} = \frac{1}{\sqrt{x - 2}} = f_1{}'(x)$

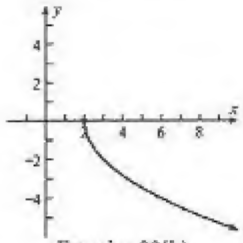
If $y = f_2(x) = -2\sqrt{x - 2}$, then $\frac{dy}{dx} = \frac{2}{y} = \frac{2}{-2\sqrt{x - 2}} = -\frac{1}{\sqrt{x - 2}} = f_2{}'(x)$

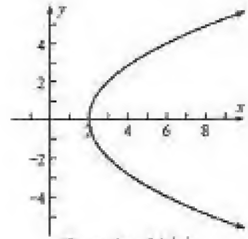
Therefore, the result in part (e) agrees with the results in part (d).

(f) $f_1(3) = 2$, $f_1{}'(3) = 1$. Tangent line to the graph of f_1: $y - 2 = 1(x - 3)$; $x - y - 1 = 0$

$f_2(3) = -2$, $f_2{}'(3) = -1$. Tangent line to the graph of f_2: $y + 2 = -1(x - 3)$; $x + y - 1 = 0$


Exercise 39(a)


Exercise 39(b)


Exercise 39(c)

40. $y^2 - x^2 = 16$; $x_1 = -3$

(a) Because $y = \pm\sqrt{x^2+16}$, the functions are given by $f_1(x) = \sqrt{x^2+16}$ and $f_2(x) = -\sqrt{x^2+16}$. The domain of each function is $(-\infty, +\infty)$.

(b) A sketch of the graph of f_1 is shown in fig. a and a sketch of the graph of f_2 is shown in fig. 3b.

(c) A sketch of the graph of the given equation is shown in fig. c.

(d) $f_1(x) = (x^2+16)^{1/2}$ $\qquad f_2(x) = -(x^2+16)^{1/2}$

$f_1{}'(x) = \frac{1}{2}(x^2+16)^{-1/2}(2x)$ $\qquad f_2{}'(x) = -\frac{1}{2}(x^2+16)^{-1/2}(2x)$

$f_1{}'(x) = \dfrac{x}{\sqrt{x^2+16}}$ $\qquad f_2{}'(x) = \dfrac{-x}{\sqrt{x^2+16}}$

The domain of each derivative is $(-\infty, +\infty)$.

(e) Differentiating the given equation implicitly with respect to x, we have

$2y\dfrac{dy}{dx} - 2x = 0$; $\qquad \dfrac{dy}{dx} = \dfrac{x}{y}$

For $y = f_1(x) = \sqrt{x^2+16}$ and $y = f_2(x) = -\sqrt{x^2+16}$, from part (d),

$\dfrac{dy}{dx} = f_1{}'(x) = \dfrac{x}{\sqrt{x^2+16}} = \dfrac{x}{y}$ and $\dfrac{dy}{dx} = f_2{}'(x) = \dfrac{-x}{\sqrt{x^2+16}} = \dfrac{x}{y}$

which agree with the result found by implicit differentiation.

(f) For $y = f_1(x)$, if $x_1 = -3$, then $y_1 = f_1(x_1) = \sqrt{(-3)^2+16} = 5$; $m(x_1) = f_1{}'(x_1) = \frac{-3}{5}$

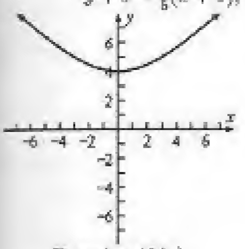
Thus an equation of the tangent line to the graph of f_1 at $(-3, 5)$ is

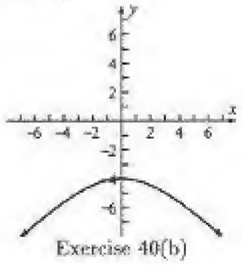
$$y - 5 = -\tfrac{3}{5}(x+3); \quad 3x + 5y - 16 = 0$$

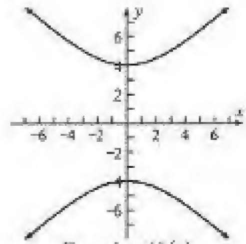
For $y = f_2(x)$, if $x_1 = -3$, then $y_1 = f_2(x_1) = -\sqrt{(-3)^2+16} = -5$; $m(x_1) = f_2{}'(x_1) = \frac{-3}{-5} = \frac{3}{5}$

Thus an equation of the tangent line to the graph of f_2 at $(-3, -5)$ is

$$y + 5 = \tfrac{3}{5}(x+3); \quad 3x - 5y + 16 = 0$$


Exercise 40(a)


Exercise 40(b)


Exercise 40(c)

41. $x^2 - y^2 = 9$; $x_1 = -5$

(a) $y^2 = x^2 - 9$; $y = \pm\sqrt{x^2-9}$, $f_1(x) = \sqrt{x^2-9}$, domain: $|x| \geq 3$; $f_2(x) = -\sqrt{x^2-9}$, domain: $|x| \geq 3$

(d) $f_1{}'(x) = x(x^2-9)^{-1/2}$, domain: $|x| > 3$; $f_2{}'(x) = -x(x^2-9)^{-1/2}$, domain: $|x| > 3$

(e) $x^2 - y^2 = 9$; $2x - 2y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = \dfrac{x}{y}$

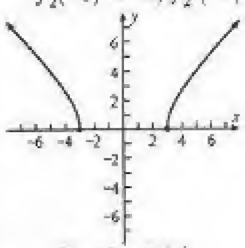
If $y = f_1(x) = \sqrt{x^2-9}$, then $\dfrac{dy}{dx} = \dfrac{x}{y} = \dfrac{x}{\sqrt{x^2-9}} = f_1{}'(x)$

If $y = f_2(x) = \sqrt{x^2-9}$, then $\dfrac{dy}{dx} = \dfrac{x}{y} = \dfrac{x}{-\sqrt{x^2-9}} = f_2{}'(x)$

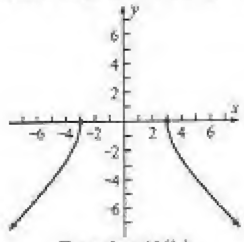
Therefore, the result in part (e) agrees with the results in part (d).

(f) $f_1(-5) = 4$, $f_1{}'(-5) = -\frac{5}{4}$. Tangent line to graph of f_1: $y - 4 = -\frac{5}{4}(x + 5)$; $5x + 4y + 9 = 0$

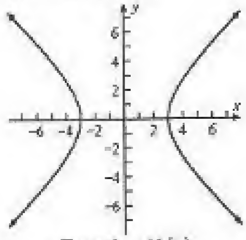
$f_2(-5) = -4$, $f_2{}'(-5) = \frac{5}{4}$. Tangent line to graph of f_2: $y + 4 = \frac{5}{4}(x + 5)$; $5x - 4y + 9 = 0$



Exercise 41(a)



Exercise 41(b)



Exercise 41(c)

42. $x^2 + y^2 = 25$, $x_1 = 4$

▷ (a) $y^2 = 25 - x^2$; $y = \pm\sqrt{25 - x^2}$, $f_1(x) = \sqrt{25 - x^2}$, domain $|x| \le 5$; $f_2(x) = -\sqrt{25 - x^2}$, domain $|x| \le 5$

(d) $f_1{}'(x) = -x(25 - x^2)^{-1/2}$, domain: $|x| < 5$; $f_2{}'(x) = x(25 - x^2)^{-1/2}$, domain: $|x| < 5$

(e) $x^2 + y^2 = 25$; $2x + 2y\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{x}{y}$

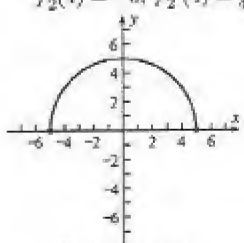
If $y = f_1(x) = \sqrt{25 - x^2}$, then $\frac{dy}{dx} = -\frac{x}{y} = \frac{-x}{\sqrt{25 - x^2}} = f_1{}'(x)$

If $y = f_2(x) = -\sqrt{25 - x^2}$, then $\frac{dy}{dx} = -\frac{x}{y} = \frac{x}{\sqrt{25 - x^2}} = f_2{}'(x)$

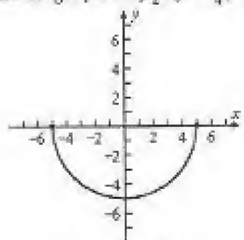
Therefore, the result in part (e) agrees with the results in part (d).

(f) $f_1(4) = 3$, $f_1{}'(4) = -\frac{3}{4}$. Tangent line to graph of f_1: $y = -\frac{3}{4}(x - 4) + 3 = -\frac{3}{4}x + 6$

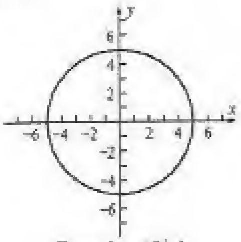
$f_2(4) = -3$, $f_2{}'(4) = \frac{3}{4}$. Tangent line to graph of f_2: $y = \frac{3}{4}(x - 4) - 3 = \frac{3}{4}x - 6$



Exercise 42(a)



Exercise 42(b)



Exercise 42(c)

43. $x^2 + y^2 = 1$; $2x + 2y\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{x}{y} = -xy^{-1}$

$\frac{d^2y}{dx^2} = -y^{-1} + xy^{-2}\frac{dy}{dx} = -\frac{1}{y} + \frac{x}{y^2}\left(-\frac{x}{y}\right) = -\frac{1}{y^3}(y^2 + x^2) = -\frac{1}{y^3}$, since $x^2 + y^2 = 1$.

44. Given $x^{1/2} + y^{1/2} = 2$, show that $\frac{d^2y}{dx^2} = \frac{1}{x^{3/2}}$.

▷ We first solve for y. Thus,

$y^{1/2} = 2 - x^{1/2}$; $\quad y = (2 - x^{1/2})^2 = 4 - 4x^{1/2} + x$

$\frac{dy}{dx} = -2x^{-1/2} + 1$; $\quad \frac{d^2y}{dx^2} = x^{-3/2} = \frac{1}{x^{3/2}}$

45. $x^3 + y^3 = 1$; $3x^2 + 3y^2\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{x^2}{y^2} = -x^2y^{-2}$

$\frac{d^2y}{dx^2} = -2xy^{-2} + 2x^2y^{-3}\frac{dy}{dx} = -\frac{2x}{y^2} + \frac{2x^2}{y^3}\left(-\frac{x^2}{y^2}\right) = -\frac{2x}{y^5}(y^3 + x^3) = -\frac{2x}{y^5}$, since $x^3 + y^3 = 1$.

46. $x^2 + 25y^2 = 100$; $2x + 50y\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{x}{25y} = -\frac{1}{25}xy^{-1}$

$\frac{d^2y}{dx^2} = -\frac{1}{25}\left(y^{-1} - xy^{-2}\frac{dy}{dx}\right) = -\frac{1}{25}\left[\frac{1}{y} + \frac{x}{y^2}\left(\frac{x}{25y}\right)\right] = -\frac{1}{25 \cdot 25y^3}(25y^2 + x^2) = -\frac{4}{25y^3}$, since $x^2 + 25y^2 = 100$.

$s = \sqrt{4t^2+3}$, $t \geq 0$; $v = \frac{ds}{dt} = \frac{1}{2}(4t^2+3)^{-1/2}(8t) = \frac{4t}{\sqrt{4t^2+3}}$

(a) $v = 0$ when $\frac{4t}{\sqrt{4t^2+3}} = 0$; $t = 0$

(b) $v = 1$ when $\frac{4t}{\sqrt{4t^2+3}} = 1$; $4t = \sqrt{4t^2+3}$; $16t^2 = 4t^2+3$; $12t^2 = 3$; $t^2 = \frac{1}{4}$; $t = \frac{1}{2}$

(c) $v = 2$ when $\frac{4t}{\sqrt{4t^2+3}} = 2$; $2t = \sqrt{4t^2+3}$; $4t^2 = 4t^2+3$. There is no such value of t.

An object is moving along a line according to the equation of motion $s = \sqrt{5+t^2}$. Find the value of t for which the measure of the velocity is (a) 0; (b) 1.

The velocity is given by $v = s'(t) = \frac{1}{2}(5+t^2)^{-1/2}(2t) = \frac{t}{\sqrt{5+t^2}}$

(a) We let $v = 0$. Thus, $\quad \frac{t}{\sqrt{5+t^2}} = 0; \quad t = 0$

(b) We let $v = 1$. Thus, $\quad \frac{t}{\sqrt{5+t^2}} = 1; \quad t = \sqrt{5+t^2}; \quad t^2 = 5+t^2; \quad 0 = 5$

We conclude that the instantaneous velocity is never 1.

$C(x)$ dollars is the total cost of producing x liters of a liquid: $C(x) = 6 + 4\sqrt{x}$.

(a) C' is the marginal cost function: $C'(x) = \frac{2}{\sqrt{x}}$; $C'(16) = \frac{2}{\sqrt{16}} = \frac{1}{2}$.

Thus the marginal cost when 16 liters are produced is 50 cents per liter.

(b) We wish to find x for which $C'(x) = 0.40$: $\frac{2}{\sqrt{x}} = 0.40$; $\sqrt{x} = 5$; $x = 25$

Therefore 25 liters are produced when the marginal cost is \$0.40 per liter.

$C(x)$ dollars is the total cost of producing x units of a commodity: $C(x) = 40 + 3x + 9\sqrt{2x}$

(a) C' is the marginal cost function: $C'(x) = 3 + 9/\sqrt{2x}$; $C'(50) = 3.9$

Thus the marginal cost when 50 units are produced is \$3.90 per unit.

(b) We wish to find x for which $C'(x) = 4.50$: $\frac{9}{\sqrt{2x}} = 4.50$; $\sqrt{2x} = 6$; $x = 18$

Therefore 18 units are produced when the marginal cost is \$4.50 per liter.

$R = px = (30\sqrt{300-2x}) = 30x(300-2x)^{1/2}$, $x \in [0, 150]$. $R'(x) = 30(300-2x)^{1/2} + 30x[\frac{1}{2}(300-2x)^{-1/2}(-2)]$
$= 30(300-2x)^{-1/2}[300-2x-x] = \frac{30(300-3x)}{\sqrt{300-2x}} = \frac{90(100-x)}{\sqrt{300-2x}}$

$R'(x) = 0$ when $x = 100$; therefore 100 apartments must be rented.

The daily production of a particular factory is $f(x)$ units when the capital investment is x thousands of dollars, and $f(x) = 200\sqrt{2x+1}$. If the current capitalization is \$760,000, use the derivative to estimate the change in the daily production if the capital investment is increased by \$1000.

$\frac{\Delta f}{\Delta x} \approx f'(x)$ and $\Delta x = 1$, then $\Delta f \approx f'(x)$.

$$f(x) = 200(2x+1)^{1/2}$$

$$f'(x) = 200(\tfrac{1}{2})(2)(2x+1)^{-1/2} = \frac{200}{\sqrt{2x+1}}$$

Because the current capitalization is \$760,000, then $x = 760$. Thus,

$$f'(760) = \frac{200}{\sqrt{2(760)+1}} = \frac{200}{39} \approx 5.13$$

The daily production will increase by approximately 5.1 units if the capital investment is increased by \$1000.

t minutes after the airplane was directly over the statue, the horizontal distance traveled is $\frac{9}{2}t$ km and s km is the line of sight distance between the plane and the statue. Then from the Pythagorean theorem

$s^2 = 4 + (\frac{9}{2}t)^2$; $s = (4 + \frac{81}{4}t^2)^{1/2} = \frac{1}{2}(16+81t^2)^{1/2}$; $\frac{ds}{dt} = \frac{1}{4}(16+81t^2)^{-1/2}(162t) = \frac{81t}{2\sqrt{16+81t^2}}$

We wish to find $\frac{ds}{dt}$ when $t = \frac{1}{3}$. $\left.\frac{ds}{dt}\right|_{t=1/3} = \frac{81 \cdot \frac{1}{3}}{2\sqrt{16+81(\frac{1}{3})^2}} = \frac{27}{2\sqrt{25}} = 2.7$

Therefore, the line of sight distance between the plane and the statue is changing at the rate of 2.7km/min 20 sec after the plane was directly over the statue.

54. At t hours after 8 AM the first ship is 24t miles north of P, the second ship is $32(t-2)$ miles east of P and the distance between them is s miles. Then $s^2 = (24t)^2 + [32(t-2)]^2$. Differentiating implicitly,
$2s\frac{ds}{dt} = 2(24t)24 + 2[32(t-2)]32$, $\frac{ds}{dt} = \frac{24^2t + 32^2(t-2)}{s}$

(a) At 9 AM, $t = 1$, $s^2 = 24^2 + (-32)^2 = 1600$, $s = 40$, $\frac{ds}{dt} = \frac{24^2(1) + 32^2(-1)}{40} = -11.2$.

Thus at 9AM the distance between the ships is decreasing at 11.2 knots.

(b) At 11 AM, $t = 3$, $s^2 = (24\cdot 3)^2 + (32\cdot 1)^2 = 6208$, $s = \sqrt{6208}$, $\frac{ds}{dt} = \frac{24^2\cdot 3 + 32^2\cdot 1}{\sqrt{6208}} = \frac{2752}{\sqrt{6208}} \approx 34.9$

Thus at 11AM the distance between the ships is increasing at 34.9 knots.

55. $D_x(|x|) = D_x(x^2)^{1/2} = \frac{1}{2}(x^2)^{-1/2}\cdot 2x = \frac{x}{(x^2)^{1/2}} = \frac{x}{|x|} = \frac{|x|}{x} = \operatorname{sgn}(x)$, $x \neq 0$.

56. Find $D_x{}^2(|x|)$ when it exists.

▷ We use the definition. $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$. $D_x|x| = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$, $D_x{}^2|x| = 0$ if $x \neq 0$.

57. $f'(x) = D_x|x^2 - 4| = [(x^2-4)^2]^{1/2} = \frac{1}{2}[(x^2-4)^2]^{-1/2}[2(x^2-4)(2x)] = \frac{2x(x^2-4)}{[(x^2-4)^2]^{1/2}} = \frac{2x(x^2-4)}{|x^2-4|}$

Alternatively, $D_x|x^2-4| = \operatorname{sgn}(x^2-4)D_x(x^2-4) = 2x\operatorname{sgn}(x^2-4)$, $x \neq \pm 2$.

58. Find the derivative of the function $g(x) = x|x|$.

▷ Because $|x|$ is not differentiable at 0, we cannot use the product rule.
If $x > 0$ then $|x| = x$ and $g(x) = x(x) = x^2$ so that $g'(x) = 2x = 2|x|$.
If $x < 0$ then $|x| = -x$ and $g(x) = -x(x) = -x^2$ so that $g'(x) = -2x = 2|x|$.
If $x = 0$ then $|x| = 0$ and $g(x) = 0$. Furthermore,

$$g'_-(0) = \lim_{x\to 0^-}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0^-}\frac{-x^2-0}{x} = \lim_{x\to 0^-} -x = 0$$

$$g'_+(0) = \lim_{x\to 0^+}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0^+}\frac{x^2-0}{x} = \lim_{x\to 0^+} x = 0$$

Therefore $g'(0) = 0 = 2|0|$. In each case we find $g'(x) = 2|x|$.

59. $f(x) = |x|^3 = x^3\operatorname{sgn} x$, $f'(x) = 3x^2\operatorname{sgn} x = 3x|x|$, $f''(x) = 6x\operatorname{sgn} x = 6|x|$

60. Given $g(x) = |f(x)|$. Prove that if $f'(x)$ and $g'(x)$ exist, then $|g'(x)| = |f'(x)|$.

▷ $g'(x) = D[f(x)^2]^{1/2} = \frac{1}{2}[f(x)^2]^{-1/2}D[f(x)^2] = |f(x)|^{-1}f(x)f'(x)$ and
$|g'(x)| = |f(x)|^{-1}|f(x)||f'(x)| = |f'(x)|$, provided $f(x) \neq 0$. Suppose $f(a) = 0$. Then
$g_+{}'(a) = \lim_{x\to a^+}\frac{|f(x)|-|f(a)|}{x-a} = \lim_{x\to a^+}\frac{|f(x)|}{x-a} \geq 0$ and $g_-{}'(x) = \lim_{x\to a^-}\frac{|f(x)|-|f(a)|}{x-a} = \lim_{x\to a^-}\frac{|f(x)|}{x-a} \leq 0$
Because $g'(a)$ exists, then $g'(a) = 0$ and $\lim_{x\to a}\frac{f(x)}{x-a} = 0$. Therefore
$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{f(x)}{x-a} = 0$ and so $|g'(a)| = |f'(a)|$.

61. $s(t) = h|\cos t|/(1+t)^2 = h(\cos^2 t)^{1/2}(1+t)^{-2}$; $s'(t) = h[D_t(\cos^2 t)^{1/2}\cdot(1+t)^{-2} + (\cos^2 t)^{1/2}\cdot D_t(1+t)^{-2}]$
$= h[-\sin t\cos t(\cos^2 t)^{-1/2}(1+t)^{-2} - 2(\cos^2 t)^{1/2}(1+t)^{-3}]$
$s'(1.4) = -0.1957h$ ft/sec, $s'(1.6) = 0.14454h$ ft/sec, $s'(1.8) = 0.1035h$ ft/sec, $s'(2.2) = 0.0430h$ ft/sec

62. Prove that the sum of the x and y intercepts of any tangent line to the parabola $x^{1/2} + y^{1/2} = k^{1/2}$ is k.

▷ Differentiating implicitly gives $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$; $y' = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{y^{1/2}}{x^{1/2}}$.

The tangent at (x_0, y_0) is $y - y_0 = -\frac{y_0{}^{1/2}}{x_0{}^{1/2}}(x - x_0)$; $\frac{x}{x_0{}^{1/2}} + \frac{y}{y_0{}^{1/2}} = \frac{x_0}{x_0{}^{1/2}} + \frac{y_0}{y_0{}^{1/2}} = x_0{}^{1/2} + y_0{}^{1/2} = k^{1/2}$

The sum of the x-intercept and y-intercept is $x_0{}^{1/2}k^{1/2} + y_0{}^{1/2}k^{1/2} = (x_0{}^{1/2} + y_0{}^{1/2})k^{1/2} = k^{1/2}k^{1/2} = k$.

63. $x^{2/3} + y^{2/3} = 1$. Differentiating implicitly, $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$; $\frac{2}{3}y^{-1/3}\frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$; $\frac{dy}{dx} = -\frac{y^{1/3}}{x^{2/3}}$. If $x = -\frac{1}{8}$ then $(-\frac{1}{8})^{2/3} + y^{2/3} = 1$; $y^{2/3} = 1 - \frac{1}{4} = \frac{3}{4}$. Therefore

(i) $y^{1/3} = \frac{1}{2}\sqrt{3}$ and $y = \frac{3}{8}\sqrt{3}$. $\frac{dy}{dx} = -\frac{\frac{1}{2}\sqrt{3}}{(-\frac{1}{8})^{1/3}} = -\frac{\frac{1}{2}\sqrt{3}}{-\frac{1}{2}} = \sqrt{3}$.

An equation of the tangent line is $y - \frac{3}{8}\sqrt{3} = \sqrt{3}(x + \frac{1}{8})$; $\sqrt{3}x - y + \frac{1}{2}\sqrt{3} = 0$ or

(ii) $y^{1/3} = -\frac{1}{2}\sqrt{3}$ and $y = -\frac{3}{8}\sqrt{3}$. $\frac{dy}{dx} = -\frac{-\frac{1}{2}\sqrt{3}}{(-\frac{1}{8})^{1/3}} = -\frac{-\frac{1}{2}\sqrt{3}}{-\frac{1}{2}} = -\sqrt{3}$.

An equation of the tangent line is $y + \frac{3}{8}\sqrt{3} = -\sqrt{3}(x + \frac{1}{8})$; $\sqrt{3}x + y + \frac{1}{2}\sqrt{3} = 0$.

64. Suppose that $g(x) = \sqrt{9 - x^2}$ and $h(x) = f(g(x))$, where f is differentiable at 3. Prove that $h'(0) = 0$.

▸ We have $h(x) = f((9 - x^2)^{1/2})$. By the chain rule,

$$h'(x) = f'((9-x^2)^{1/2})D_x(9 - x^2)^{1/2} = f'((9 - x^2)^{1/2}) \cdot \tfrac{1}{2}(9 - x^2)^{-1/2}(-2x) = f'((9 - x^2)^{1/2}) \cdot \frac{-x}{\sqrt{9 - x^2}}$$

Because $f'(3)$ exists, then $h'(0) = f'(3) \cdot 0 = 0$

65. Show that if $xy = 1$, then $\frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2} = 4$.

▸ Because $xy = 1$, then

$$\begin{aligned} y &= x^{-1} & x &= y^{-1} \\ \frac{dy}{dx} &= -x^{-2} & \frac{dx}{dy} &= -y^{-2} \\ \frac{d^2y}{dx^2} &= 2x^{-3} & \frac{d^2x}{dy^2} &= 2y^{-3} \end{aligned}$$

Multiplying the last two equations, we have

$$\frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2} = (2x^{-3})(2y^{-3}) = \frac{4}{(xy)^3} \tag{1}$$

Because $xy = 1$, Eq. (1) is equivalent to $\quad \frac{d^2y}{dx^2} \cdot \frac{d^2x}{dy^2} = 4$

66. $f(x) = x^r$. Let $r = \frac{p}{q}$, $q > 0$ with p and q integers, and set $y = f(x)$. Then $y = x^{p/q}$ becomes $y^q = x^p$. Differentiating implicitly gives $qy^{q-1}\frac{dy}{dx} = px^{p-1}$; $\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q} \cdot \frac{x^p x^{p/q}}{y^q x} = r \cdot \frac{x^p x^{p/q}}{x^p x} = rx^{p/q-1} = rx^{r-1}$

Therefore $f'(x) = rx^{r-1}$.

67. The proof given in Exercise 66 is not complete because it was assumed that f is differentiable.

68. Compute $(f \circ g)'(0)$ if $f(x) = x^6 + 7x^3$ and $g(x) = x^{1/3}$. Explain why the chain rule cannot be applied to perform this computation.

▸ $(f \circ g)(x) = f(g(x)) = (x^{1/3})^6 + 7(x^{1/3})^3 = x^2 + 7x$ and so $(f \circ g)'(x) = 2x + 7$; $(f \circ g)'(0) = 2(0) + 7 = 7$
However, $g'(x) = \frac{1}{3}x^{-2/3}$ and $g'(0)$ is not defined. Hence the chain does not apply at 0.

2.10 RELATED RATES

If x is a variable that is a function of time, which is represented by t, then the rate of change of x with respect to time is given by dx/dt. The following steps should be taken to solve problems involving the rate of change with respect to time for two or more related variables.

1. Select a letter to represent each variable.
2. Identify the constant rates of change that are given.
3. Identify the rate of change that must be found.
4. Find an equation that expresses the relation between the variables.
5. Differentiate with respect to t on both sides of the equation.
6. Replace the given rates of change with their constant values.
7. Replace the variables by their values at the particular time of interest.
8. Solve the resulting equation for the unknown rate of change.

Often it is helpful to draw a figure to find the equation in step 4. Be careful to distinguish the *variables*, which represent length, area, volume, etc., from the *rates of change* of these variables. Although a variable

that represents length may appear as a dimension in the figure, the rate of change of this variable does not appear in the figure. To find the equation you may use any formulas from geometry for length, area, and volume. You may use the fact that corresponding sides of similar triangles are proportional.

Following are some of the formulas from geometry.

1. The Pythagorean theorem for a right triangle: $a^2 + b^2 = c^2$
2. Circumference of a circle: $C = 2\pi r$
3. Area formulas for plane figures:
   a. Rectangle: $A = \ell w$
   b. Triangle: $A = \frac{1}{2}bh$
   c. Equilateral triangle of side s: $A = \frac{1}{4}\sqrt{3}\, s^2$
   d. Parallelogram: $A = bh$
   e. Trapezoid: $A = \frac{1}{2}(b_1 + b_2)h$
   f. Circle: $A = \pi r^2$
4. Surface area formulas for solids:
   a. Right circular cylinder:
      (i) Lateral area: $S = 2\pi rh$
      (ii) Total area: $S = 2\pi rh + 2\pi r^2$
   b. Right circular cone:
      (i) Lateral area: $S = \pi r\sqrt{r^2 + h^2}$
      (ii) Total area: $S = \pi r\sqrt{r^2 + h^2} + \pi r^2$
   c. Sphere: $S = 4\pi r^2$
5. Volume formulas for solids:
   a. Rectangular parallelepiped: $V = \ell wh$
   b. Circular cylinder: $V = \pi r^2 h$
   c. Circular cone: $V = \frac{1}{3}\pi r^2 h$
   d. Sphere: $V = \frac{4}{3}\pi r^3$
   e. Prism: $V = Bh$, where B is the area of the base.
   f. Pyramid: $V = \frac{1}{3}Bh$, where B is the area of the base.

Exercises 2.10

In Exercises 1–8, x and y are functions of a third variable t.

1. $2x + 3y = 8$; $2\frac{dx}{dt} + 3\frac{dy}{dt} = 0$; $\frac{dx}{dt} = -\frac{3}{2}\frac{dy}{dt}$. Because $\frac{dy}{dt} = 2$, then $\frac{dx}{dt} = -\frac{3}{2}\cdot 2 = -3$.

2. $\frac{x}{y} = 10$; $x = 10y$; $\frac{dx}{dt} = 10\frac{dy}{dt}$. Because $\frac{dx}{dt} = -5$, then $-5 = 10\frac{dy}{dt}$; $\frac{dy}{dt} = -\frac{1}{2}$

3. $xy = 20$; $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$; $\frac{dx}{dt} = -\frac{x}{y}\frac{dy}{dt}$. When $x = 2$, $y = 10$. Because $\frac{dy}{dt} = 10$, then $\left.\frac{dx}{dt}\right|_{x=2} = -\frac{2}{10}\cdot 10 = -2$

4. If $2 \sin x + 4 \cos y = 3$ and $\frac{dy}{dt} = 3$, find $\frac{dx}{dt}$ at $(\frac{1}{6}\pi, \frac{1}{3}\pi)$.

▷ We differentiate with respect to t on both sides of the given equation. Because x and y are functions of t, we must apply the chain rule. Thus

$$2 \cos x \cdot \frac{dx}{dt} - 4 \sin y \cdot \frac{dy}{dt} = 0$$

We let $\frac{dy}{dt} = 3$, $x = \frac{1}{6}\pi$ and $y = \frac{1}{3}\pi$ and solve for $\frac{dx}{dt}$.

$2(\cos \frac{1}{6}\pi)\frac{dx}{dt} - 4(\sin \frac{1}{3}\pi)(3) = 0$; $\quad 2(\frac{1}{2}\sqrt{3})\frac{dx}{dt} - 4(\frac{1}{2}\sqrt{3})(3) = 0$; $\quad \sqrt{3}\frac{dx}{dt} - 6\sqrt{3} = 0$; $\quad \frac{dx}{dt} = 6$

5. $\sin^2 x + \cos^2 y = \frac{5}{4}$; $2 \sin x \cos x\frac{dx}{dt} + 2 \cos y(-\sin y)\frac{dy}{dt} = 0$; $\sin 2x\frac{dx}{dt} - \sin 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{\sin 2x}{\sin 2y}\frac{dx}{dt}$. Let $\frac{dx}{dt} = -1$, $x = \frac{3}{4}\pi$, $y = -\pi$. Then $\left.\frac{dy}{dt}\right|_{y=3\pi/4} = \frac{\sin \frac{4}{3}\pi}{\sin \frac{3}{2}\pi}(-1) = \frac{-\frac{1}{2}\sqrt{3}}{-1} = -\frac{1}{2}\sqrt{3}$

6. $x^2 + y^2 = 25$ and $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $y = 4$. ▷ Because the value of x is not given, we solve for x.

$x = \pm\sqrt{25 - y^2}$; $\frac{dx}{dt} = \pm\frac{y}{\sqrt{25 - y^2}}\cdot\frac{dy}{dt}$; $5 = \pm\frac{4}{3}\cdot\frac{dy}{dt}$; $\frac{dy}{dt} = \pm\frac{15}{4}$

7. $\sqrt{x} + \sqrt{y} = 5$; $x^{1/2} + y^{1/2} = 5$; $\frac{1}{2}x^{-1/2}\frac{dx}{dt} + \frac{1}{2}y^{-1/2}\frac{dy}{dt} = 0$; - $\frac{1}{x^{1/2}} = -\frac{1}{y^{1/2}}\frac{dy}{dt}$; $\frac{dx}{dt} = -\frac{x^{1/2}}{y^{1/2}}\frac{dy}{dt}$

Because $\frac{dy}{dt} = 3$ and when $x = 1$, $y = 16$, then $\left.\frac{dx}{dt}\right|_{x=1} = -\frac{1^{1/2}}{16^{1/2}} \cdot 3 = -\frac{3}{4}$.

8. If $y(\tan x + 1) = 4$ and $\frac{dy}{dt} = -4$, find $\frac{dx}{dt}$ when $x = \pi$.

• Because the value of y is not given, we solve for y.

$$y = 4(\tan x + 1)^{-1}$$

Differentiating with respect to t on both sides, we obtain

$$\frac{dy}{dt} = -4(\tan x + 1)^{-2}\sec^2 x \cdot \frac{dx}{dt}$$

We let $\frac{dy}{dt} = -4$ and $x = \pi$ and solve for $\frac{dx}{dt}$.

$$-4 = -4(\tan \pi + 1)^{-2}\sec^2 \pi \cdot \frac{dx}{dt}$$

$$-4 = -4\frac{dx}{dt}; \quad \frac{dx}{dt} = 1$$

9. At t sec, let x ft be the horizontal distance from the child to the kite, $x > 0$, and let S ft be the length of the string. Then from the Pythagorean Theorem, $S^2 = 40^2 + x^2$; $2S\frac{dS}{dt} = 2x$; $\frac{dS}{dt} = \frac{x}{S}$.

When $S = 50$ we have $2500 = 1600 + x^2$; $x^2 = 900$; $x = 30$. Because $\frac{dx}{dt} = 3$, we have $\left.\frac{dS}{dt}\right|_{S=50} = \frac{30}{50} \cdot 3 = \frac{9}{5}$.

Therefore, when the length of the string released is 50 ft, the string is being paid out at the rate of $\frac{9}{5}$ ft/sec.

10. At t min, let r m be the radius and V ft^3 be the volume of the spherical balloon.

$V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. Because $\frac{dV}{dt} = 5$, $\frac{dr}{dt} = \frac{5}{4\pi r^2}$. Therefore $\left.\frac{dr}{dt}\right|_{r=6} = \frac{5}{4\pi \cdot 36} = \frac{5}{108\pi}$.

Hence when the balloon is 12m in diameter, the diameter is increasing at the rate of $\frac{5}{72\pi}$ m/min.

11. At t min, let r ft be the radius and V ft^3 be the volume of the spherical snowball.

$V = \frac{4}{3}\pi r^3$; $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. Because $\frac{dV}{dt} = 8$, $\frac{dr}{dt} = \frac{8}{4\pi r^2} = \frac{2}{\pi r^2}$. Therefore $\left.\frac{dr}{dt}\right|_{r=2} = \frac{2}{\pi \cdot 4} = \frac{1}{2\pi}$.

Hence when the snowball is 4ft in diameter, the radius is increasing at the rate of $\frac{1}{2\pi}$ ft/min.

12. A spherical snowball with diameter 6 ft starts to melt at the rate of $\frac{1}{4}$ ft^3/min. Find the rate at which the radius is changing when the radius is 2 ft.

• At t min, let r ft be the radius and V ft^3 be the volume of the spherical snowball.

Because the snowball is melting at the rate of $\frac{1}{4}$ ft^3/min, we have $\quad \frac{dV}{dt} = -\frac{1}{4}$

We must find $\frac{dr}{dt}$ when $r = 2$. The volume of a sphere is given by the formula $\quad V = \frac{4}{3}\pi r^3$

Differentiating with respect to t on both sides, we have $\quad \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

Substituting for $\frac{dV}{dt}$ and r, we obtain $\quad -\frac{1}{4} = 4\pi(2^2)\frac{dr}{dt}$; $\quad \frac{dr}{dt} = \frac{-1}{64\pi} \approx -0.0050$

We conclude that the radius is decreasing at the rate of 0.0050 ft/min when the radius is 2 ft.

13. At t min, let V cubic meters be the volume of the conical pile, let r meters be the radius of the base of the pile, and let h meters be the height of the pile.

Because $r = \frac{1}{2}h$, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{1}{2}h)^2 = \frac{1}{12}\pi h$; $\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$; $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$. Because $\frac{dV}{dt} = 10$, then $\left.\frac{dh}{dt}\right|_{h=8} = \frac{4 \cdot 10}{\pi \cdot 8^2} = \frac{5}{8\pi}$. Thus, when the pile is 8 m high, the height is increasing at the rate of $\frac{5}{8\pi}$ m/min.

In Exercises 14 and 15, a light hangs 15ft above a horizontal path and a man 6ft tall is walking away at 5 ft/sec.

14. How fast is the man's shadow lengthening?

• At t sec, let x ft be the man's distance from the light and s ft the length of his shadow. Then $dx/dt = 5$. From similar triangles, $\frac{\text{base}}{\text{height}} = \frac{s}{6} = \frac{x+s}{15}$; $5s = 2x + 2s$; $3s = 2x$; $s = \frac{2}{3}x$; $\frac{ds}{dt} = \frac{2}{3}\frac{dx}{dt} = \frac{2}{3} \cdot 5 = \frac{10}{3}$. The man's shadow is lengthening at the rate of $\frac{10}{3}$ ft/sec.

15. How fast is the tip of his shadow moving?

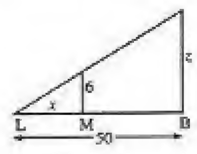
▹ At t sec, let x ft be the distance of the man from the light and let y ft be the horizontal distance of the tip of the man's shadow from the bottom of the post holding the light. From similar triangles,

$\frac{\text{base}}{\text{height}} = \frac{y-x}{6} = \frac{y}{15}$; $5y - 5x = 2y$; $3y = 5x$; $\frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt}$. Because $\frac{dx}{dt} = 5$, $\frac{dy}{dt} = \frac{5}{3} \cdot 5 = \frac{25}{3}$.

Therefore, the tip of the man's shadow is moving at the rate of $\frac{25}{3}$ ft/sec.

16. A man 6 ft tall is walking toward a building at the rate of 5 ft/sec. If there is a light on the ground 50 ft from the building, how fast is his shadow on the building growing shorter when he is 30 ft from the building?

▹ At t sec, let x ft be the distance from the man to the light and z ft the length of his shadow on the building. The figure shows the man at point M, between point L (the light), and point B (the base of the building). Because the man is walking at the rate of 5 ft/sec, we are given that $dx/dt = 5$. Because dz/dt is the rate of change of the length of the shadow, we want to find dz/dt when the man is 30 ft from the building, that is, when $x = 50 - 30 = 20$. By similar triangles we have



$$\frac{z}{50} = \frac{6}{x}; \qquad z = 300x^{-1}$$

Differentiating on both sides with respect to t, we obtain $\quad \frac{dz}{dt} = -300x^{-2}\frac{dx}{dt}$

Replacing x by 20 and $\frac{dx}{dt}$ by 5, we have $\quad \frac{dz}{dt} = \frac{-300}{(20)^2}(5) = -\frac{15}{4}$

Therefore, the shadow is growing shorter at the rate of $\frac{15}{4}$ ft/sec when the man is 30 ft from the building.

17. At t days, r cm is the radius and V cm^3 is the volume of the spherical tumor. When $r = 0.5$, $\frac{dr}{dt} = 0.001$. $V = \frac{4}{3}\pi r^3$; $\left.\frac{dV}{dt}\right|_{r=0.5} = 4\pi r^2 \left.\frac{dr}{dt}\right|_{r=0.5} = 4\pi(0.5)^2(0.001) = 0.001\pi$

When the radius of the tumor is 0.5 cm, its volume is increasing at the rate of $0.001\pi \approx 0.003$ cm^3 per day.

18. At t days, r μm is the radius and V μm^3 is the volume of the spherical cell. When $r = 1.5$, $\frac{dr}{dt} = 0.01$. $V = \frac{4}{3}\pi r^3$; $\left.\frac{dV}{dt}\right|_{r=1.5} = 4\pi r^2 \left.\frac{dr}{dt}\right|_{r=1.5} = 4\pi(1.5)^2(0.01) = 0.09\pi$

When the radius of the cell is 1.5 μm, its volume is increasing at the rate of $0.09\pi \approx 0.028$ μm^3 per day.

19. At t days, r cm is the radius and S cm^2 is the surface area of the spherical tumor.

When $r = 0.5$, $\frac{dr}{dt} = 0.001$. $S = 4\pi r^2$; $\left.\frac{dS}{dt}\right|_{r=0.5} = 8\pi r \left.\frac{dr}{dt}\right|_{r=0.5} = 8\pi(0.5)(0.001) = 0.004\pi$.

When the radius of the tumor is .5 cm, its surface area is increasing at the rate of $.004\pi \approx .012$ cm^2 per day.

20. A bacterial cell is spherical in shape. If the radius of the cell is increasing at the rate of 0.01 micrometers per day when it is 1.5 μm, what is the rate of increase of the surface area of the cell at that time?

▹ t days after the cell began to grow, let r μm be its radius and S μm^2 be its surface area. Because the cell is growing at the rate of 0.01 μm/day, we are given that

$$\frac{dr}{dt} = 0.01$$

We must find $\frac{dS}{dt}$ when $r = 1.5$. The area of a sphere is given by the formula $\quad S = 4\pi r^2$

Differentiating with respect to t on both sides, we have $\quad \frac{dS}{dt} = 8\pi r\frac{dr}{dt}$

Substituting for r and $\frac{dr}{dt}$, we obtain $\quad \frac{dS}{dt} = 8\pi(1.5)0.01 = 0.12\pi \approx 0.377$

Thus the surface area is increasing at the rate of 0.377 μm^2/day when the radius is 1.5 μm.

21. At t min, let V cubic meters be the volume of water in the tank, let r meters be the radius of the surface of the water, and let h meters be the depth of the water. From similar triangles, $\frac{h}{r} = \frac{24}{12}$; $r = \frac{1}{2}h$.

The volume of water in the tank can be expressed in terms of the volume of a cone.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{1}{2}h)^2 h = \frac{1}{12}\pi h^3$; $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$, $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$

Because $\frac{dV}{dt} = -6$, we have $\left.\frac{dh}{dt}\right|_{h=10} = \frac{4(-6)}{\pi \cdot 10^2} = -\frac{6}{25\pi}$.

Hence, when the water is 10m deep, the water level is lowering at the rate of $\frac{6}{25\pi}$ m/min.

22. At t min, let x ft be the depth of the water, b ft its width and V ft^3 its volume. We are given $dV/dt = 2$. From similar triangles $b = x$ and so $V = \frac{1}{2} \cdot 12bh = 6x^2$. $\frac{dV}{dt} = 12x\frac{dx}{dt}$. When $x = 1$, $2 = 12(1)\frac{dx}{dt}$; $\frac{dx}{dt} = \frac{1}{6}$. When the water is 1 ft deep it is rising at the rate of $\frac{1}{6}$ ft/min.

23. At t min, P pounds per square foot is the pressure and V cubic feet is the volume of gas. C is a constant. $PV = C$; $P\frac{dV}{dt} + V\frac{dP}{dt} = 0$; $\frac{dP}{dt} = -\frac{P}{V}\frac{dV}{dt}$. Because $\frac{dV}{dt} = 3$ and $P = 3000$ when $V = 5$, we have $\left.\frac{dP}{dt}\right|_{V=5} = -\frac{3000}{5} \cdot 3 = -1800$. Thus, when $V = 5$, the pressure is decreasing at the rate of 1800 lb/ft^2 per min.

24. The adiabatic law (no gain or loss of heat) for the expansion of air is $PV^{1.4} = C$, where P is the number of pounds per square unit of pressure, V is the number of cubic units of volume, and C is a constant. At a specific instant, the pressure is 40 lb/in. and is increasing at the rate of 8 lb/in.^{2} each second. If $C = \frac{5}{16}$, what is the rate of change of volume at this instant?

» Let t seconds be the time since the pressure began increasing. Then P and V are functions of t. When $P = 40$ we are given that $\frac{dP}{dt} = 8$ and we must find $\frac{dV}{dt}$. We are given that

$$PV^{1.4} = \frac{5}{16}; \quad V^{7/5} = \frac{5}{16}P^{-1}; \quad V = \left(\frac{5}{16}\right)^{5/7} P^{-5/7}$$

Differentiating with respect to t, we have

$$\frac{dV}{dt} = -\frac{5}{7}\left(\frac{5}{16}\right)^{5/7} P^{-12/7}\frac{dP}{dt}$$

When $P = 40 = 5 \cdot 2^3$ and $\frac{dP}{dt} = 8$ we have

$$\frac{dV}{dt} = -\frac{5}{7}\left(\frac{5}{2^4}\right)^{5/7}(5 \cdot 2^3)^{-12/7}(8) = -\frac{5}{7}\left(\frac{1}{5 \cdot 2^8}\right)(8) = -\frac{1}{224}$$

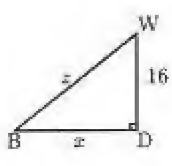
Thus, the volume is decreasing at the rate of $\frac{1}{224}$ cubic units per second at this instant.

25. At t sec, let r cm be the radius and let A cm^2 be the area of the disturbed region. $A = \pi r^2$; $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. Because $\frac{dr}{dt} = 16$, $\left.\frac{dA}{dt}\right|_{r=4} = 2\pi(4)16 = 128\pi$. Thus, when the radius is 4 cm, the area of the disturbed region is increasing at the rate of 128π cm^2/sec.

26. At t min, let r m be the radius of the oil, h m its depth, V m^3 its volume. We are given $\frac{dV}{dt} = 3\pi$. From similar triangles, $\frac{r}{h} = \frac{2.5}{10}$; $r = \frac{1}{4}h$. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{1}{4}h)^2 h = \frac{1}{48}\pi h^3$. $\frac{dV}{dt} = \frac{1}{16}\pi h^2\frac{dh}{dt}$. $3\pi = \frac{1}{16}\pi(8)^2\frac{dh}{dt}$; $\frac{dh}{dt} = \frac{3}{4}$. Thus, when the depth of the oil is 8m, it is increasing at the rate of $\frac{3}{4}$m/min.

27. At t sec after the truck leaves the intersection, let x ft be the distance traveled by the truck, let y feet be the distance traveled by the automobile, and let s ft be the distance between the automobile and the truck. Then $s = \sqrt{x^2 + (120 - y)^2}$; $D_t s = \frac{xD_t x + (120 - y)(-D_t y)}{\sqrt{x^2 + (120 - y)^2}}$. Because $D_t x = 40$, $D_t y = 30$, and when $t = 2$, $x = 80$ and $y = 60$, we have $\left.D_t s\right|_{t=2} = \frac{40 \cdot 80 + 60(-30)}{\sqrt{80^2 + 60^2}} = \frac{1400}{100} = 14$. Thus, 2 sec after the truck leaves the intersection the automobile and the truck are separating at the rate of 14 ft/sec.

28. A rope is attached to a boat at water level, and a woman on the dock is pulling on the rope at the rate of 50 ft/min. If her hands are 16 ft above the water level, how fast is the boat approaching the dock when the amount of rope out is 20 ft?



» Let x feet be the distance between the boat and the dock and z feet the amount of rope out t min after the woman began to pull in the boat. In the figure the boat is at point B, the dock enters the water at point D, and the woman's hands are at point W. Because the woman is pulling in the rope at the rate of 50 ft/min, then z is decreasing, and $\frac{dz}{dt} = -50$. Since $\frac{dx}{dt}$ is the rate of change of the distance between the boat and the dock, we want to find $\frac{dx}{dt}$ when $z = 20$. By the Pythagorean theorem we have

$$z^2 = x^2 + 16^2 \tag{1}$$

Differentiating on both sides with respect to t, we obtain

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} \qquad (2)$$

Substituting $z = 20$ in Eq. (1), we get

$20^2 = x^2 + 16^2; \qquad 144 = x^2; \qquad x = 12$

Substituting for z, $\frac{dz}{dt}$, and x in Eq. (2) we find

$2(20)(-50) = 2(12)\frac{dx}{dt}; \qquad \frac{dx}{dt} = -\frac{250}{3}$

The boat is approaching the dock at the rate of $\frac{250}{3}$ ft/min when 20 ft of rope is out.

29. C dollars is the cost of producing x units in t weeks. $C = 0.08x^3 - x^2 + 10x + 48$;
$D_tC = (.24x^2 - 2x + 10)D_tx$. Because $D_tx = 2$ and at the present time $x = 50$, we have

$D_tC\Big|_{x=50} = (.24(50)^2 - 2(50) + 10)(2) = 1020$

Thus the cost is increasing at the rate of \$1020 per week.

30. x thousand boxes are demanded if p cents is the price of a box, where $px + 50p = 16{,}000$; $x = 16{,}000p^{-1} - 50$.
$\frac{dx}{dt} = \frac{dx}{dp}\cdot\frac{dp}{dt} = -16{,}000p^{-2}\frac{dp}{dt} = -\frac{16{,}000}{160^2}\cdot 0.4 = -0.25$. Demand is decreasing at the rate of 250 boxes a week.

31. x units are supplied per month when p dollars is the price per unit. $x = 1000\sqrt{3p^2 + 20p}$;

$\frac{dx}{dp} = \frac{500(6p+20)}{\sqrt{3p^2+20p}}\frac{dp}{dt}$. At the present, $p = 20$ and $\frac{dp}{dt} = \frac{1}{2}$. Thus, $\frac{dx}{dt}\Big|_{p=20} = \frac{500(120+20)\frac{1}{2}}{\sqrt{3\cdot 20^2 + 20\cdot 20}} = \frac{35000}{40} = 875$.

Hence, the supply is increasing at the rate of 875 units per month.

32. Suppose that y workers are needed to produce x units of a certain commodity, and $x = 4y^2$. If production of the commodity this year is 250,000 units and the production is increasing at the rate of 18,000 units per year, what is the current rate at which the labor force should be increased?

▷ Let t years be the time, and both x and y are functions of t. When $x = 250{,}000$, we are are given that $\frac{dx}{dt} = 18{,}000$ and we must find $\frac{dy}{dt}$. Solving the given equation for y, we have

$x = 4y^2; \qquad y = \frac{1}{2}x^{1/2}$

Differentiating with respect to t, we get $\qquad \frac{dy}{dt} = \frac{1}{4}x^{-1/2}\frac{dx}{dt}$

Substituting for x and $\frac{dx}{dt}$, we obtain $\qquad \frac{dy}{dt} = \frac{1}{4}(250{,}000^{-1/2})(18{,}000) = 9$

Thus, the labor force should be increased at the rate of 9 workers per year at present.

33. $100x$ shirts are demanded per week when p dollars is the price of a shirt.

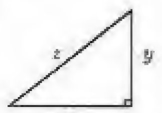
$2px + 65p - 4950 = 0$; $x = \frac{2475}{p} - 32.5$; $\frac{dx}{dt} = -\frac{2475}{p^2}\frac{dp}{dt}$

This week, $p = 30$ and $\frac{dp}{dt} = \frac{1}{5}$. Therefore, $\frac{dx}{dt}\Big|_{p=30} = -\frac{2475}{30^2}\frac{1}{5} = -\frac{11}{20}$.
Because $\frac{dx}{dt} = -\frac{11}{20}$, then $100\frac{dx}{dt} = -55$. Hence the demand is decreasing at the rate of 55 shirts per week.

34. The hypotenuse is 40 cm; α is such that $d\alpha/dt = \frac{1}{36}\pi$ rad/sec. The measures of sides are 40 sin α and 40 cos α.
A cm^2 is the area. $A = \frac{1}{2}(40\sin\alpha)(40\cos\alpha) = 400\sin 2\alpha$. When $\alpha = \frac{1}{6}\pi$,

$\frac{dA}{dt} = 800\cos 2\alpha\cdot\frac{d\alpha}{dt} = 800\cos\frac{1}{3}\pi\cdot\frac{1}{36}\pi = 800\cdot\frac{1}{2}\cdot\frac{1}{36}\pi = \frac{100}{9}\pi$. The area is increasing at about 34.9 cm^2/sec.

35. At t hr, let the distance from the intersection be x km for the first truck and y km for the second truck. See the figure. $z = \sqrt{x^2+y^2}$; $D_tz = \frac{xD_tx + yD_ty}{\sqrt{x^2+y^2}}$.
Since $D_tx = -k$ and $D_ty = -k$, when $x = m$ and $y = m$, we have



$D_tz = \frac{m(-k) + m(-k)}{\sqrt{m^2+m^2}} = \frac{-2mk}{\sqrt{2}m} = -k\sqrt{2}$. Thus, when the two trucks are each m km from the intersection, they are approaching each other at the rate of $k\sqrt{2}$ km/hr.

In Exercises 36 and 37, a horizontal trough is 16 meters long, and its ends are isosceles trapezoids with an altitude of 4 m, a lower base of 4 m, and and upper base of 6 m.

36. Water is being poured into the trough at the rate of 10 m^3/min. How fast is the water level rising when the water is 2 m deep?

37. If the water level is decreasing at the rate of 25 cm/min when the water is 3 m deep, at what rate is water being drawn from the trough?

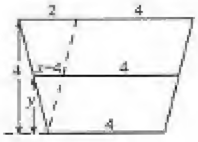
» The figure illustrates one end of the trough. Let y m be the depth of the water, x m the width of the surface of the water, and V m^3 its volume, t minutes after water began pouring into the trough

Because water is being poured into the trough at the rate of 10 m^3/min, we are given that $dV/dt = 10$. Since dy/dt is the rate at which the depth of the water is changing, we want to find dy/dt when $y = 2$. The water that is in the trough is in the shape of a prism with altitude 16 m and base a trapezoid. Because the volume of a prism is the area of the base times its altitude, and the area of a trapezoid is given by the formula $A = \frac{1}{2}(b_1 + b_2) \cdot h$, we have

$$V = \tfrac{1}{2}(4 + x) \cdot y \cdot 16; \; V = 8y(x + 4) \qquad (1)$$

We express x as a function of y. By similar triangles in the figure,

$$\frac{x-4}{y} = \frac{2}{4} = \frac{1}{2}; \qquad x = \frac{y}{2} + 4$$



Substituting the value for x into Eq. (1), we obtain

$$V = 8y\left(\frac{y}{2} + 8\right); \qquad V = 4y^2 + 64y; \qquad \frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt} = (8y + 64)\frac{dy}{dt}$$

In Exercise 36, we substitute $\frac{dV}{dt} = 10$ and $y = 2$. Then $\quad 10 = 80\frac{dy}{dt}; \quad \frac{dy}{dt} = \frac{1}{8}$

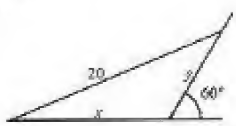
Thus, the water level is rising at the rate of $\frac{1}{8}$ m/min when the water is 2 m deep.

In Exercise 37, we substitute $\frac{dy}{dt} = -\frac{1}{4}$ and $y = 3$. Then $\quad \frac{dV}{dt} = (8 \cdot 3 + 64)\left(-\frac{1}{4}\right) = -22.$

Thus, water is being drawn from the trough at the rate of $22 m^3$/min.

38. Let the base of the ladder be x m from the the wall, and the top y m from the ground after t sec. We are given $\frac{dx}{dt} = -1.5$. Because the ladder is 7 m long, then $y = \sqrt{49 - x^2}$; $\frac{dy}{dt} = \frac{-x}{\sqrt{49 - x^2}}$. When $x = 2$, $\frac{dy}{dt} = \frac{-2}{\sqrt{49 - 2^2}}(-1.5) = \frac{3}{\sqrt{45}} = \frac{\sqrt{5}}{5}$. The ladder is sliding up at about 0.447 m/sec.

39. At t sec since the bottom of the ladder started to be moved toward the embankment, let the bottom of the embankment be x ft from the bottom of the ladder and y ft from the top of the ladder. See the figure. From the law of cosines we have $x^2 + y^2 - 2xy \cos 120^\circ = 400$; $x^2 + y^2 - 2xy(-\frac{1}{2}) = 400$; $y^2 + (x)y + (x^2 - 400) = 0$



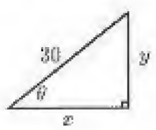
$$y = \frac{-x + \sqrt{x^2 - 4(x^2 - 400)}}{2} = \frac{\sqrt{1600 - 3x^2}}{2} - \frac{x}{2}; \; \frac{dy}{dt} = \left(\frac{-3x}{2\sqrt{1600 - 3x^2}} - \frac{1}{2}\right)\frac{dx}{dt}$$

Since $\frac{dx}{dt} = -1$, when $x = 4$ we get

$$\left.\frac{dy}{dt}\right|_{x=4} = \left(\frac{-12}{2\sqrt{1600 - 3 \cdot 4^2}} - \frac{1}{2}\right) \cdot (-1) = \frac{12}{8\sqrt{97}} - \frac{1}{2} = \frac{3\sqrt{97} + 97}{194} \approx 0.65$$

Therefore, the top of the ladder is moving at the rate of $\frac{3\sqrt{97} + 97}{194}$ ft/sec ≈ 0.65 ft/sec at the given instant.

40. If a ladder of length 30 ft that is leaning against a wall has its upper end sliding down the wall at the rate of $\frac{1}{2}$ft/sec, what is the rate of change of the measure of the acute angle made by the ladder with the ground when the upper end is 18 ft above the ground?

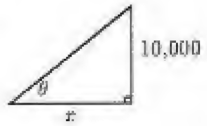


» At t sec, let the distance from the bottom of the wall to the bottom of the ladder be x ft and to the top of the ladder be y ft. Let θ radians be the measure of the acute angle made by the ladder with the ground at t sec.

See the figure. $\sin\theta = \frac{y}{30}$; $\cos\theta\frac{d\theta}{dt} = \frac{1}{30}\frac{dy}{dt}$; $\frac{d\theta}{dt} = \frac{1}{30\cos\theta}\frac{dy}{dt}$

When $y = 18$, $x = \sqrt{30^2 - 18^2} = \sqrt{576} = 24$; so $\cos\theta = \frac{24}{30}$; $30\cos\theta = 24$. Since $\frac{dy}{dt} = -\frac{1}{2}$, we have $\left.\frac{d\theta}{dt}\right|_{y=18} = \frac{1}{24}(-\frac{1}{2}) = -\frac{1}{48}$. Hence, the measure of the acute angle made by the ladder with the ground is decreasing at the rate of $\frac{1}{48}$ rad/sec at the given instant.

41. At t sec after takeoff, let x ft be the horizontal distance from the airplane to the point 10,000 ft directly above the observer, and let θ be the radian measure of the angle of elevation of the airplane. See the figure. $\theta = \pi/3$, $\frac{d\theta}{dt} = \frac{1}{60}$, and



$x = 10{,}000\cot\theta$, $\frac{dx}{dt} = -10{,}000\csc^2\theta\,\frac{d\theta}{dt}$, $\left.\frac{dx}{dt}\right|_{\theta=\pi/3} = -10{,}000\cdot\frac{4}{3}\cdot\frac{1}{60} = -\frac{2000}{9}$

The minus sign implies that x is decreasing. The speed of the plane is $\frac{2000}{9}$ ft/sec.

42. At t min, the beam makes an angle θ with a perpendicular to the shore and is x mi from its foot. We are given $\frac{d\theta}{dt} = \frac{32\text{ rev}}{\text{min}}\cdot\frac{2\pi\text{ rad}}{\text{rev}} = \frac{64\pi\text{ rad}}{\text{min}}$. $x = 4\tan\theta$. When $\theta = 45°$, $\frac{dx}{dt} = 4\sec^2\theta\,\frac{d\theta}{dt} = 4(2)64\pi = 512\pi$. The beam is moving at about 1608.5 mi/min.

43. At t sec after lift-off the rocket is x yd high and the radar dish makes an angle θ with the ground. After 10 sec the average velocity is 50 yd/sec so its height is 500 yd. $x = 1000\tan\theta$, $\frac{dx}{dt} = 1000\sec^2\theta\,\frac{d\theta}{dt}$;

$\frac{d\theta}{dt} = \frac{dx/dt}{1000(\tan^2\theta + 1)} = \frac{100}{1000[(500/1000)^2 + 1]} = \frac{2}{25}$. The dish revolves at $\frac{2}{25}$ rad/sec.

44. Water is poured at the rate of 8 ft^3/min into a tank in the form of a cone. The cone is 20 ft deep and 10 ft in diameter at the top. If there is a leak in the bottom and the water is rising at the rate of 1 in./min, when the water is 16 ft deep, how fast is the water leaking?

▷ After t min, V cubic feet is the volume of the water, h feet is its depth, and r feet is the radius of its surface. Because the water level is rising at the rate of 1 in./min when the water is 16 ft deep, and 1 in. is $\frac{1}{12}$ ft, we are given that

$$\frac{dh}{dt} = \frac{1}{12} \text{ when } h = 16$$

We find dV/dt at this moment. Applying the formula for the volume of a cone, we have

$$V = \frac{1}{3}r^2h \tag{1}$$

Because the altitude of the tank is 20 ft and the radius of the tank is 5 ft, by similar triangles in the figure, we have $r/h = 5/20$ or, equivalently, $r = \frac{1}{4}h$. Substituting the value of r into Eq. (1), we obtain

$$V = \frac{1}{3}\pi(\frac{1}{4}h)^2h = \frac{1}{48}\pi h^3$$

Differentiating with respect to t, we have

$$\frac{dV}{dt} = \frac{1}{16}\pi h^2\frac{dh}{dt}$$

We substitute $h = 16$ and $dh/dt = \frac{1}{12}$. Thus $\quad \frac{dV}{dt} = \frac{1}{16}\pi(16^2)\frac{1}{12} = \frac{4}{3}\pi$

Because water is being poured into the tank at the rate of 8 ft^3/min and the rate of change of volume of the water in the tank is $\frac{4}{3}\pi$ ft^3/min, we conclude that water is leaking out of the tank at the rate of $8 - \frac{4}{3}\pi$, or approximately 3.81 ft^3/min.

45. The volume of a balloon is decreasing at a rate proportional to its surface area. Show that the radius of the balloon shrinks at a constant rate.

▷ r units is the radius of the balloon at time t, V cubic units is its volume and S square units is its surface area. We are given that

$$\frac{dV}{dt} = kS \tag{1}$$

where k is a negative constant. We must show that $\frac{dr}{dt}$ is a negative constant. Now $\quad V = \frac{4}{3}\pi r^3$

Differentiating on both sides with respect to t, we have $\quad \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

Substituting from $S = 4\pi r^2$ into the above, we get $\quad \frac{dV}{dt} = S\frac{dr}{dt} \quad (2)$

Substituting from Eq. (1) into Eq.(2), we obtain $\quad kS = S\frac{dr}{dt}$; $\frac{dr}{dt} = k$

Because dr/dt is a negative constant k, we conclude that the radius shrinks at a constant rate.

Miscellaneous Exercises for Chapter 2

In Exercises 1–14, find the derivative of the function.

1. $f(x) = 5x^3 - 7x^2 + 2x - 3$ ▹ $f'(x) = 15x^2 - 14x + 2$

2. $g(x) = 5(x^4 + 3x^7)$ ▹ $g'(x) = 5(4x^3 + 21x^6)$

3. $g(x) = \frac{x^2}{4} + \frac{4}{x^2} = \frac{1}{4}x^2 + 4x^{-2}$ ▹ $g'(x) = \frac{1}{2}x - 8x^{-3} = \frac{x}{2} - \frac{8}{x^3}$

4. $f(x) = \frac{4}{x^2} - \frac{3}{x^4}$ ▹ We apply the power rule. Thus,

$$f'(x) = \mathrm{D}_x(4x^{-2} - 3x^{-4}) = -8x^{-3} + 12x^{-5} = -\frac{8}{x^3} + \frac{12}{x^5}$$

5. $\mathrm{F}(x) = 2x^{1/2} - \frac{1}{2}x^{-1/2}$ ▹ $\mathrm{F}'(x) = x^{-1/2} + \frac{1}{4}x^{-3/2}$

6. $\mathrm{G}(x) = \frac{x^2 - 4x + 4}{x - 1}$ ▹ $\mathrm{G}'(x) \overset{\text{quot}}{=} \frac{(2x - 4)(x - 1) - (x^2 - 4x + 4)1}{(x-1)^2} = \frac{2x^2 - 6x + 4 - x^2 + 4x - 4}{2} = \frac{x^2 - 2x}{(x-1)^2}$

7. $\mathrm{G}(t) = (3t^2 - 4)(4t^3 + t - 1)$ ▹ $\mathrm{G}'(t) = (3t^2 - 4)(12t^2 + 1) + (4t^3 + t - 1)(6t)$
$= 36t^4 - 45t^2 - 4 + 24t^4 + 6t^2 - 6t = 60t^4 - 39t^2 - 6t - 4$

8. $f(x) = (x^4 - 2x)(4x^2 + 2x + 5)$ ▹ We mulitiply and use the power rule.
$f(x) = 4x^6 + 2x^5 + 5x^4 - 8x^3 - 4x^2 - 10x$ $\qquad f'(x) = 24x^5 + 10x^4 + 20x^3 - 24x^2 - 8x - 10$

9. $g(x) = \frac{x^2 + 1}{x^3 - 1}$ ▹ $g'(x) = \frac{3x^2(x^3 - 1) - 3x^2(x^3 + 1)}{(x^3 - 1)^2} = \frac{-6x^2}{(x^3 - 1)^2}$

10. $h(y) = \frac{y^2}{y^3 + 8}$ ▹ $h'(y) \overset{\text{quot}}{=} \frac{2y(y^3 + 8) - y^2(3y^2)}{(y^3 + 8)^2} = \frac{2y^4 + 16y - 3y^4}{(y^3 + 8)^2} = \frac{16y - y^4}{(y^3 + 8)^2}$

11. $f(s) = (2s^3 - 2s + t)^4$ ▹ $f'(s) = 4(2s^3 - 3s + 7)(6s^2 - 3)$

12. $\mathrm{F}(x) = (4x^4 - 4x^2 + 1)^{-1/3}$ ▹ We factor and use the power chain rule.
$\mathrm{F}(x) = [(2x^2 - 1)^2]^{-1/3} = (2x^2 - 1)^{-2/3}$
$\mathrm{F}'(x) = -\frac{2}{3}(2x^2 - 1)^{-5/3}(4x) = -\frac{8}{3}x(2x^2 - 1)^{-5/3} = \frac{-8x}{3(2x^2 - 1)^{5/3}}$

13. $\mathrm{F}(x) = (x^2 - 1)^{3/2}(x^2 - 4)^{1/2}$ ▹ $\mathrm{F}'(x) = \frac{3}{2}(x^2 - 1)^{1/2}(2x)(x^2 - 4)^{1/2} + \frac{1}{2}(x^2 - 1)^{3/2}(x^2 - 4)^{-1/2}(2x)$
$= x(x^2 - 1)^{1/2}(x^2 - 4)^{-1/2}[3(x^2 - 4) + (x^2 - 1)] = x(x^2 - 1)^{1/2}(x^2 - 4)^{-1/2}(4x^2 - 13)$

14. $g(x) = (x^4 - x)^{-3}(5 - x^2)^{-1}$ ▹ $g'(x) = (x^4 - x)^{-3}\mathrm{D}_x(5 - x^2)^{-1} + (5 - x^2)^{-1}\mathrm{D}_x(x^4 - x)^{-3}$
$= (x^4 - x)^{-3}(-1)(5 - x^2)^{-2}(-2x) + (5 - x^2)^{-1}(-3)(x^4 - x)^{-4}(4x^3 - 1)$
$= (x^4 - x)^{-4}(5 - x^2)^{-2}[2x(x^4 - x) - 3(5 - x^2)(4x^3 - 1)]$
$= (x^4 - x)^{-4}(5 - x^2)^{-2}(14x^5 - 60x^3 - 5x^2 + 15) = \frac{14x^5 - 60x^3 - 5x^2 + 15}{x^4(x^3 - 1)^4(x^2 - 5)^2}$

In Exercises 15–20, compute the derivative.

15. $\mathrm{D}_x[(x + 1)\sin x - x\cos x] = \sin x + (x + 1)\cos x - \cos x + x\sin x = (x + 1)\sin x + x\cos x$

16. $\mathrm{D}_t(\sin^2 3t)$ ▹ We use the power rule first.
$= 2\sin 3t \cdot \mathrm{D}_t(\sin 3t) = 2\sin 3t \cdot 3\cos 3t = 6\sin 3t\cos 3t$

17. $\frac{d}{dt}(\sqrt{\tan 4t})$ ▹ $= \frac{d}{dt}(\tan 4t)^{1/2} = \frac{1}{2}(\tan 4t)^{-1/2}\sec^2 4t \cdot 4 = \frac{2\sec^2 4t}{\sqrt{\tan 4t}}$

18. $\frac{d}{dx}\left(x\cos\frac{1}{x}\right)$ ▹ $\cos\frac{1}{x} + x\left(-\sin\frac{1}{x} \cdot -\frac{1}{x^2}\right) = \cos\frac{1}{x} + \frac{1}{x}\sin\frac{1}{x}$

19. $\mathrm{D}_w[\sin(\cos 3w) - \sin w\cos 3w]$ ▹ $\cos(\cos 3w)(-3\sin 3w) - [\cos w\cos 3w + \sin w(-3\sin 3w)]$
$= -3\sin 3w\cos(\cos 3w) - \cos w\cos 3w + 3\sin w\sin 3w$

20. $\mathrm{D}_x[\tan 2x\sec x + \tan(2\sec x)$ ▹ We use the product rule in the first term; the chain rule in the second.
$= \mathrm{D}_x(\tan 2x) \cdot \sec x + \tan 2x \cdot \mathrm{D}_x(\sec x) + \sec^2(2\sec x) \cdot \mathrm{D}_x(2\sec x)$
$= 2\sec^2 2x\sec x + \tan 2x\tan x\sec x + \sec^2(2\sec x) \cdot 2\sec x\tan x$

In Exercises 21–24, compute the derivative and check by plotting your answer and NDER in the same window.

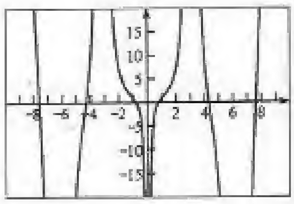
21. $f(x) = \left(\dfrac{2x}{x^2+1}\right)^2$ ▹ $f'(x) = 2\left(\dfrac{2x}{x^2+1}\right)\cdot\dfrac{2(x^2+1)-2x(2x)}{(x^2+1)^2} = \dfrac{8x(1-x^2)}{(x^2+1)^3}$

22. $g(x) = \sqrt{\dfrac{x}{4-x^2}} = \left(\dfrac{x}{4-x^2}\right)^{1/2}$; $g'(x) = \frac{1}{2}\left(\dfrac{x}{4-x^2}\right)^{-1/2}\dfrac{1(4-x^2)-x(-2x)}{(4-x^2)^2} = \frac{1}{2}\left(\dfrac{4-x^2}{x}\right)^{1/2}\dfrac{4+x^2}{(4-x^2)^2} = \dfrac{4+x^2}{2\sqrt{x(4-x^2)^3}}$

23. $g(x) = \dfrac{\tan x}{1+x}$ ▹ $g'(x) = \dfrac{\sec^2 x(1+x) - \tan x}{(1+x)^2}$

24. $f(x) = \dfrac{1+x^2}{\sin x}$

▹ $f'(x) = \dfrac{2x\sin x - (1+x^2)\cos x}{\sin^2 x}$



In Exercises 25–28, find $\dfrac{dy}{dx}$.

25. $4x^2 + 4y^2 - y^3 = 0$

▹ $8x + 8y\dfrac{dy}{dx} - 3y^2\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx}(8y - 3y^2) = -8x$; $\dfrac{dy}{dx} = \dfrac{-8x}{8y-3y^2}$

26. $xy^2 + 2y^3 = x - 2y$

▹ $x\left(2y\dfrac{dy}{dx}\right) + y^2(1) + 6y^2\dfrac{dy}{dx} = 1 - 2\dfrac{dy}{dx}$; $(2xy + 6y^2 + 2)\dfrac{dy}{dx} = 1 - y^2$; $\dfrac{dy}{dx} = \dfrac{1-y^2}{2xy+6y^2+2}$

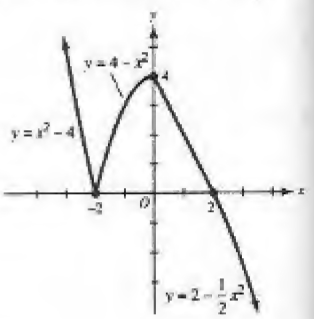
27. $\tan x + \tan y = xy$ ▹ $\sec^2 x + \sec^2 y\dfrac{dy}{dx} = y + x\dfrac{dy}{dx}$; $\dfrac{dy}{dx}(\sec^2 y - x) = y - \sec^2 x$; $\dfrac{dy}{dx} = \dfrac{y - \sec^2 x}{\sec^2 y - x}$

28. $\sin(x+y) + \sin(x-y) = 1$ ▹ We differentiate implicitly with respect to x on both sides.

$\cos(x+y)\left(1 + \dfrac{dy}{dx}\right) + \cos(x-y)\left(1 - \dfrac{dy}{dx}\right) = 0$; $[\cos(x+y) - \cos(x-y)]\dfrac{dy}{dx} = -\cos(x+y) - \cos(x-y)$;

$\dfrac{dy}{dx} = \dfrac{\cos(x-y) + \cos(x+y)}{\cos(x-y) - \cos(x+y)} = \dfrac{\cos x \cos y}{\sin x \sin y}$

In Exercises 29 and 30, assume that each part of the accompanying graph of the continuous function f that appears to be a line segment is a line segment. (a) Define f piecewise; find (b) $f'_-(-2)$, (c) $f'_+(-2)$ (d) $f'_-(0)$, (e) $f'_+(0)$, (f) $f'_-(2)$, $f'_+(2)$. (h) At what numbers is f not differentiable?



29. (a) $f(x) = \begin{cases} x^2 - 4 & \text{if } x \le -2 \\ 4 - x^2 & \text{if } -2 < x \le 0 \\ 4 - 2x & \text{if } 0 < x \le 2 \\ 2 - \frac{1}{2}x^2 & \text{if } x > 2 \end{cases}$

(b) $f'_-(-2) = \displaystyle\lim_{x\to-2^-}\frac{(x^2-4)-0}{x+2} = \lim_{x\to-2^-}(x-2) = -4$

(c) $f'_+(-2) = \displaystyle\lim_{x\to-2^-}\frac{(4-x^2)-0}{x+2} = \lim_{x\to-2^+}(2-x) = 4$

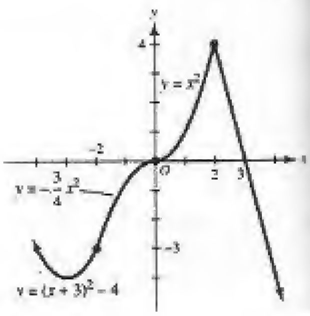
(d) $f'_-(0) = \displaystyle\lim_{x\to0^-}\frac{(4-x^2)-4}{x-0} = \lim_{x\to0^-}(-x) = 0$

(e) $f'_+(0) = \displaystyle\lim_{x\to0^+}\frac{(4-2x)-4}{x-0} = \lim_{x\to0^+}(-2) = -2$

(f) $f'_-(2) = \displaystyle\lim_{x\to2^-}\frac{(4-2x)-0}{x-2} = \lim_{x\to2^-}(-2) = -2$

(g) $f'_+(2) = \displaystyle\lim_{x\to2^+}\frac{(2-\frac{1}{2}x^2)-0}{x-2} = \lim_{x\to2^+}-\tfrac{1}{2}(x+2) = -2$

(h) f is not differentiable at -2 and 0.



30. (a) $\begin{cases} x^2 + 6x + 5 & \text{if } x \le -2 \\ -\frac{3}{4}x^2 & \text{if } -2 < x \le 0 \\ x^2 & \text{if } 0 < x \le 2 \\ 12 - 4x & \text{if } x > 2 \end{cases}$

(b) $f'_-(-2) = \displaystyle\lim_{x\to-2^-}\frac{(x^2+6x+5)+3}{x+2} = \lim_{x\to-2^-}(x+4) = 2$

(c) $f'_+(-2) = \lim\limits_{x \to -2^-} \dfrac{-\frac{3}{4}x^2 + 3}{x + 2} = \lim\limits_{x \to -2^+} -\frac{3}{4}(x - 2) = 3$

(d) $f'_-(0) = \lim\limits_{x \to 0^-} \dfrac{-\frac{3}{4}x^2 - 0}{x - 0} = \lim\limits_{x \to 0^-} (-\frac{3}{4}x) = 0$

(e) $f'_+(0) = \lim\limits_{x \to 0^+} \dfrac{x^2 - 0}{x - 0} = \lim\limits_{x \to 0^+} x = 0$

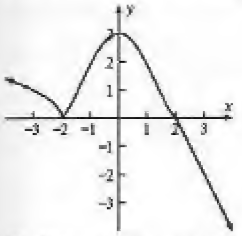
(f) $f'_-(2) = \lim\limits_{x \to 2^-} \dfrac{x^2 - 4}{x - 2} = \lim\limits_{x \to 2^-} (x + 2) = 4$

(g) $f'_+(2) = \lim\limits_{x \to 2^+} \dfrac{(12 - 4x) - 4}{x - 2} = \lim\limits_{x \to 2^+} -4 = -4$

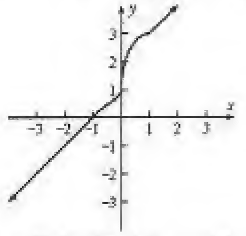
(h) f is not differentiable at -2 and 2.

In Exercises 31 and 32, sketch the graph of a continuous function f defined on R and having the given properties.

31. f is differentiable except at -2, 2; $f(x) > 0$ if $x < -2$; $f(-2) = 0$; $0 < f(x) \le 3$ if $-2 < x < 2$; $f(0) = 3$; $f(2) = 0$; $f(x) < 0$ if $x > 2$; $f'_+(-2) = 1$; $f'(0) = 0$; $f'_-(2) = -1$; $f'_+(2) = -2$; $\lim\limits_{x \to -2^-} \dfrac{f(x) - f(-2)}{x + 2} = -\infty$



32. f is differentiable except at -1, 0, and 1; the range of f is $(-\infty, +\infty)$; $f(-1) = 0$; $f(0) = 1$; $f(1) = 3$; $f'_-(-1) = 1$; $f'_+(-1) = 2$; $f'_-(1) = 0$; $f'_+(1) = 1$; $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x} = +\infty$



33. Find an equation of the tangent line to $y = x^3 - 3x - 1$ at $(2, 1)$. Check by plotting the curve and the tangent.

▸ $f'(x) = 3x^2 - 3$. $m = f'(2) = 3(2)^2 - 3 = 9$. Equation of tangent line: $y = 9(x - 2) + 1$; $y = 9x - 17$

34. Find an equation of the normal line to $y = \dfrac{8x}{x^2 + 3}$ at $(3, 2)$. Check by plotting the curve and the normal.

▸ $f'(x) = 8\dfrac{1(x^2 + 3) - x(2x)}{(x^2 + 3)^2} = 8\dfrac{3 - x^2}{(x^2 + 3)^2}$. $m = -\dfrac{1}{f'(3)} = -\dfrac{12^2}{8(-6)} = 3$. Normal line: $y = 3(x - 3) + 2$; $y = 3x - 7$

35. Find equations of the tangent lines to $y = 2x^3 + 4x^2 - x$ having slope $\frac{1}{2}$. Check by plotting the lines and curve.

▸ $y' = 6x^2 + 8x - 1 = \frac{1}{2}$ when $0 = 12x^2 + 16x - 3 = (6x - 1)(2x + 3)$; so $x = \frac{1}{6}, x = -\frac{3}{2}$.

At $x = \frac{1}{6}$, $y = 2(\frac{1}{6})^3 + 4(\frac{1}{6})^2 - \frac{1}{6} = -\frac{5}{108}$ Equation of tangent line: $y = -\frac{5}{108} + \frac{1}{2}(x - \frac{1}{6})\frac{1}{2}x - \frac{7}{54}$

At $x = -\frac{3}{2}$, $y = 2(-\frac{3}{2})^3 + 4(-\frac{3}{2})^2 - (-\frac{3}{2}) = \frac{15}{4}$ Equation of tangent line: $y = \frac{15}{4} + \frac{1}{2}(x + \frac{3}{2}) = \frac{1}{2}x + \frac{9}{2}$

36. Find an equation of the normal line to the curve $x - y = \sqrt{x + y}$ at the point $(3, 1)$.

▸ Differentiating implicitly with respect to x on both sides of the given equation, we get

$$1 - D_x y = \tfrac{1}{2}(x + y)^{-1/2}(1 + D_x y)$$

Replacing x by 3 and y by 1 and solving for $D_x y$, we have

$$1 - D_x y = \tfrac{1}{2}(4)^{-1/2}(1 + D_x y) = \tfrac{1}{4}(1 + D_x y)$$

$$4 - 4D_x y = 1 + D_x y; \qquad D_x y = \tfrac{3}{5}$$

Hence, the slope of the tangent to the curve at the point $(3, 1)$ is $\frac{3}{5}$ and so the slope of the normal line there is $-\frac{5}{3}$. Thus an equation for the normal line is

$$y - 1 = -\tfrac{5}{3}(x - 3); \quad 5x + 3y - 18 = 0$$

37. Find equations of the tangent and normal lines to the curve $2x^3 + 2y^3 - 9xy = 0$ at the point $(2, 1)$.

▸ $6x^2 + 6y^2\dfrac{dy}{dx} = 9y - 9x\dfrac{dy}{dx} = 0$; $(6y^2 - 9x)\dfrac{dy}{dx} = 9y - 6x^2$; $\dfrac{dy}{dx} = \dfrac{3y - 2x^2}{2y^2 - 3x}$; $\left.\dfrac{dy}{dx}\right|_{(2,1)} = \dfrac{3 \cdot 1 - 2 \cdot 2^2}{2 \cdot 1^2 - 3 \cdot 2} = \dfrac{5}{4}$

At $(2, 1)$ the tangent line has slope $\frac{5}{4}$. Its equation is $y - 1 = \frac{5}{4}(x - 2)$; $5x + 4y - 6 = 0$.
At $(2, 1)$ the normal line has slope $-\frac{4}{5}$.Its equation is $y - 1 = -\frac{4}{5}(x - 2)$; $4x + 5y - 13 = 0$.

38. Find equations of the tangent and normal to $y = 8\sin^3 2x$ at $(\frac{1}{12}\pi, 1)$. Check by plotting the lines and curve.

▷ $y' = 48\sin^2 2x\cos 2x$. $m_{tangent} = f'(\frac{1}{12}\pi) = 48\sin^2\frac{1}{6}\pi\cos\frac{1}{6}\pi = 6\sqrt{3}$. Tangent line: $y = 6\sqrt{3}(x - \frac{1}{12}\pi) + 1$
$m_{normal} = -\dfrac{1}{6\sqrt{3}} = -\frac{1}{18}\sqrt{3}$. Equation of normal line: $y = -\frac{1}{18}\sqrt{3}(x - \frac{1}{12}\pi) + 1$

39. Prove that the line tangent to $y = -x^4 + 2x^2 + x$ at $(1, 2)$ is tangent to the curve at another point and find it.

▷ $y'(x) = -4x^3 + 4x + 1$; $y'(1) = 1$. At $(1, 2)$ the tangent line has slope 1. Its equation is $y - 2 = 1(x - 1)$; $y = x + 1$. To determine the intersections of the line and the original curve we solve the first and last equations simultaneously.
$-x^4 + 2x^2 + x = x + 1$; $x^4 - 2x^2 + 1 = 0$; $(x^2 - 1)^2 = 0$; $(x - 1)^2(x + 1)^2 = 0$; $x = 1, 1, -1, -1$
When $x = -1$, $y = x + 1 = 0$. There are two ways to proceed.
Method 1: $y'(-1) = 1$. Hence the tangent line at $(-1, 0)$ has equation $y - 0 = 1(x + 1)$ or equivalently, $y = x + 1$, which is the tangent line at $(1, 2)$.
Method 2: Since the roots $1, -1$ are repeated, the line is tangent at $(1, 0)$ and $(-1, 0)$.

40. Prove that the tangent lines to the curves $4y^3 - x^2y - x + 5y = 0$ and $x^4 - 4y^3 + 5x + y = 0$ at the origin are perpendicular.

▷ We show that the product of the slopes of the tangent lines at $(0, 0)$ is -1. The first curve is

$$4y^3 - x^2y - x + 5y = 0$$

Differentiating implicitly with respect to x, using the product rule on the second term, we have

$$12y^2\frac{dy}{dx} - x^2\frac{dy}{dx} - 2xy - 1 + 5\frac{dy}{dx} = 0$$

We let $x = 0$ and $y = 0$ and solve for $\frac{dy}{dx}$.

$$-1 + 5\frac{dy}{dx} = 0; \qquad \frac{dy}{dx} = \frac{1}{5}$$

Thus, $m_1 = \frac{1}{5}$ is the slope of the tangent line to the first curve at the origin. The second curve is

$$x^4 - 4y^3 + 5x + y = 0$$

Differentiating implicitly with respect to x, we have

$$4x^3 - 12y^2\frac{dy}{dx} + 5 + \frac{dy}{dx} = 0$$

When $x = 0$ and $y = 0$, we have

$$5 + \frac{dy}{dx} = 0; \qquad \frac{dy}{dx} = -5$$

Thus, $m_2 = -5$. Because $m_1m_2 = \frac{1}{5}(-5) = -1$ then the tangent lines at the origin are perpendicular.

41. $y = \sqrt{3 - 2x} = (3 - 2x)^{1/2}$; $\frac{dy}{dx} = \frac{1}{2}(3 - 2x)^{-1/2}(-2) = -(3 - 2x)^{-1/2}$
$\frac{d^2y}{dx^2} = -(3 - 2x)^{-3/2}(-2) = -(3 - 2x)^{-3/2}$; $\frac{d^3y}{dx^3} = -(3 - 2x)^{-5/2}(-2) = -3(3 - 2x)^{-5/2}$

42. If $\frac{dy}{dx} = y^k$, find $\frac{d^3y}{dx^3}$. ▷ $\frac{d^2y}{dx^2} = ky^{k-1}\frac{dy}{dx} = ky^{k-1}y^k = ky^{2k-1}$
$\frac{d^3y}{dx^3} = k(2k - 1)y^{2k-2}\frac{dy}{dx} = k(2k - 1)y^{2k-2}y^k = k(2k - 1)y^{3k-2}$

43. $f(x) = \frac{1}{12}x^4 + \frac{2}{3}x^3 + \frac{3}{2}x^2 + 8x + 2$; $f'(x) = \frac{1}{3}x^3 + 2x^2 + 3x + 8$; $f''(x) = x^2 + 4x + 3$
We wish to find when $f''(x) > 0$, that is when $x^2 + 4x + 3 > 0$; $(x + 3)(x + 1) > 0$.
Case 1: $x + 3 > 0$ and $x + 1 > 0$; $x > -3$ and $x > -1$. Hence $x > -1$.
Case 2: $x + 3 < 0$ and $x + 1 < 0$; $x < -3$ and $x < -1$. Hence $x < -3$.
Therefore $f''(x) > 0$ when either $x < -3$ or $x > -1$.

44. Find the rate of change of y with respect to x at the point $(3, 2)$ if $7y^2 - xy^3 = 4$.

▷ Differentiating implicitly with respect to x, using the product rule on the second term, we have

$$14y\frac{dy}{dx} - y^3 - 3xy^3\frac{dy}{dx} = 0$$

We let $x = 3$ and $y = 2$ and solve for $\frac{dy}{dx}$. $\qquad -8 - 44\frac{dy}{dx} = 0; \qquad \frac{dy}{dx} = -\frac{2}{11}$

The rate of change of y with respect to x is $-\frac{2}{11}$.

In Exercise 45 and 46, a particle is moving along a horizontal line according to the given equation where s meters is the directed distance of the particle from a point O at t seconds. The positive direction is to the right. Determine the intervals of time when the particle is moving to the right and when it is moving to the left. Also determine when it reverses its direction.

45. $s = 2t^3 + 3t^2 - 12t - 5$

$v(t) = \frac{ds}{dt} = 6t^2 + 6t - 12 = 6(t^2 + t - 2) = 6(t+2)(t-1)$. $v(t) = 0$ when $t = -2$ and $t = 1$.

	$t+2$	$t-1$	Conclusion
$t < -2$	$-$	$-$	v is positive and the particle is moving to the right
$t = -2$	0	$-$	v is zero; particle is changing direction from right to left
$-2 < t < 1$	$+$	$-$	v is negative and the particle is moving to the left
$t = 1$	$+$	0	v is zero and the particle is changing direction from left to right
$1 < t$	$+$	$+$	v is positive and the particle is moving to the right

46. $s = \dfrac{t-1}{t^2 - 2t + 5}$

$v(t) = \dfrac{ds}{dt} = \dfrac{1(t^2 - 2t + 5) - (t-1)(2t-2)}{(t^2 - 2t + 5)^2} = \dfrac{-(t^2 - 2t - 3)}{(t^2 - 2t + 5)^2} = \dfrac{-(t+1)(t-3)}{(t^2 - 2t + 5)^2}$. $v(t) = 0$ when $t = -1$ and $t = 3$.

	$-(t+1)$	$t-3$	Conclusion
$t < -1$	$+$	$-$	v is negative and the particle is moving to the left
$t = -1$	$+$	0	v is zero and the particle is changing direction from left to right
$-1 < t < 3$	$-$	$-$	v is positive and the particle is moving to the right
$t = 3$	$-$	0	v is zero and the particle is changing direction from left to right
$3 < t$	$-$	$+$	v is negative and the particle is moving to the left

In Exercises 47 and 48, make a table giving s, v, a. Include the intervals of time when the particle is moving to the left and right, and when the velocity or speed is increasing or decreasing. Plot position as a function of t.

47. $s = 4 - 9t + 6t^2 - t^3 = -(t-1)^2(t-4)$; $v = \frac{ds}{dt} = -9 + 12t - 3t^2 = -3(t^2 - 4t + 3) = -3(t-1)(t-3)$;

$a = \frac{dv}{dt} = 12 - 6t = 6(2-t)$

	s	v	a	Conclusion
$0 \le t < 1$	$+$	$-$	$+$	Particle is right of the origin and it is moving to the left The velocity is increasing. The speed is decreasing.
$t = 1$	0	0	$+$	Particle is at the origin and it is changing its direction of motion from left to right. The velocity is increasing. The speed is increasing.
$1 < t < 2$	$+$	$+$	$+$	Particle is right of the origin, and it is moving to the right. The velocity is increasing. The speed is increasing.
$t = 2$	$+$	$+$	0	Particle is right of the origin, and it is moving to the right. The velocity is not changing; so the speed is not changing.
$2 < t < 3$	$+$	$+$	$-$	Particle is right of the origin, and it is moving to the right. The velocity is decreasing. The speed is decreasing.
$t = 3$	$+$	0	$-$	Particle is right of the origin, and it is changing its direction of motion from right to left. The velocity is decreasing. The speed is increasing.
$3 < t < 4$	$+$	$-$	$-$	Particle is right of the origin, and it is moving to the left. The velocity is decreasing. The speed is increasing.
$t = 4$	0	$-$	$-$	Particle is at the origin, and it is moving to the left. The velocity is decreasing. The speed is increasing.
$4 < t$	$-$	$-$	$-$	Particle is left of the origin, and it is moving to the left. The velocity is decreasing. The speed is increasing.

48. $s = t^3 - 3t^2 - 9t + 13 = (t - t_0)(t - t_1)(t - t_2)$, where $t_0 \approx -2.5446$, $t_1 \approx 1.1671$, $t_2 \approx 4.3776$

$v = \frac{ds}{dt} = 3t^2 - 6t - 9 = 3(t^2 - 2t - 3) = 3(t+1)(t-3)$; $a = \frac{ds}{dt} = 6t - 6 = 6(t-1)$

	s	v	a	Conclusion
$0 \le t < 1$	+	−	−	Particle is right of the origin and it is moving to the left. The velocity is decreasing. The speed is increasing.
$t = 1$	+	−	0	Particle is right of the origin and it is moving to the left. The velocity is not changing; so the speed is not changing.
$1 < t < t_1$	+	−	+	Particle is right of the origin, and it is moving to the left. The velocity is increasing. The speed is decreasing.
$t = t_1$	0	−	+	Particle is at the origin, and it is moving to the left. The velocity is increasing. The speed is decreasing.
$t_1 < t < 3$	−	−	+	Particle is left of the origin, and it is moving to the left. The velocity is increasing. The speed is decreasing.
$t = 3$	−	0	+	Particle is left of the origin, and it is changing its direction of motion from left to right. The velocity is increasing. The speed is increasing.
$3 < t < t_2$	−	+	+	Particle is left of the origin, and it is moving to the right. The velocity is increasing. The speed is increasing.
$t = t_2$	0	+	+	Particle is at the origin, and it is moving to the right. The velocity is increasing. The speed is increasing.
$t_2 < t$	+	+	+	Particle is right of the origin, and it is moving to the right. The velocity is increasing. The speed is increasing.

In Exercises 49 and 50, a particle is moving along a line where s feet is the directed distance at t seconds. Find th time when the instantaneous acceleration is zero and the position and velocity at this time.

49. $s = 9t^2 + 2\sqrt{2t} + 1 = 9t^2 + 2\sqrt{2}t^{1/2} + 1$; $v = ds/dt = 18t + \sqrt{2}t^{-1/2}$; $a = dv/dt = 18 - \frac{1}{2}\sqrt{2}t^{-3/2}$

$a = 0$ when $t^{3/2} = 2^{-3/2}3^{-2}$, $t = 2^{-1}3^{-4/3}$. Then $s = \frac{1}{4}3^{4/3} + 1$, $v = 3^{5/3}$

50. $s = \frac{4}{9}t^{3/2} + 2t^{1/2}$, $v = ds/dt = \frac{2}{3}t^{1/2} + t^{-1/2}$, $a = dv/dt = \frac{1}{3}t^{-1/2} - \frac{1}{2}t^{-3/2}$

$a = 0$ when $t = \frac{3}{2}$. Then $s = \frac{4}{3}\sqrt{6}$, $v = \frac{2}{3}\sqrt{6}$

51. A bag is dropped from 200 ft. (a) Write an equation of motion. (b) Find the velocity at 1 sec and 3 sec. (c) How long does it take to hit the ground? (d) What is its speed when it hits the ground?

▸ (a) $s = 200 - 16t^2$ (b) $v = ds/dt = -32t$. $v(1) = -32$ ft/sec. $v(3) = -32 \cdot 3 = -96$ ft/sec

(c) $s = 0$ when $t^2 = \frac{200}{16} = \frac{50}{4}$, $t = \frac{5}{2}\sqrt{2} \approx 3.54$ sec (d) $v(\frac{5}{2}\sqrt{2}) = -80\sqrt{2} \approx -113$. Its speed is 113 ft/sec.

52. A bag is thrown downward from an altitude of 200 ft with a velocity of 20 ft/sec. (a) Use Equation (10) in Exercises 2.5 to write an equation of motion and simulate the motion on your calculator. (b) Find the velocity at 1 sec and 3 sec. (c) How long does it take to hit the ground? (d) What is its speed when it hits the ground?

▸ (a) Equation (10) is $s = -16t^2 + v_0t + s_0$. Because $v_0 = -20$ and $s_0 = 200$, the equation of motion is $s = -16t^2 - 20t + 200$

(b) $v = D_ts = D_t(-16t^2 - 20t + 200) = -32t - 20$. $v(1) = -32 - 20 = -52$. $v(3) = -32 \cdot 3 - 20 = -116$

The velocity at 1 sec is −52 ft/sec; the velocity at 3 sec is −116 ft/sec (if it falls into a deep hole).

(c) When $s = 0$; $-16t_1^{\ 2} - 20t_1 + 200 = 0$; $t_1 = \frac{5}{8}(\sqrt{33} - 1)$. The bag hits the ground after about 2.97 sec.

(d) $v(t_1) = -32 \cdot \frac{5}{8}(\sqrt{33} - 1) - 20 = -20\sqrt{33}$. Its speed is about 114.9 ft/sec when it hits the ground.

53. A ball is thrown upward from a height of 112 ft with a velocity of 96 ft/sec. (a) Write an equation of motion and simulate the motion. (b) Estimate and (c) calculate how high the ball will go and when it gets there. (d) Estimate and (e) calculate how long it takes the ball to reach the ground. (f) Find the velocity at 2 sec and 4 sec. (g) Find the speed at 2 sec and 4 sec. (h) Find the velocity when it hits the ground.

▸ (a) $s = -16t^2 + 96t + 112$ (c) $v = -32t + 96 = 0$ when $t = 3$. $s(3) = -16 \cdot 3^2 + 96 \cdot 3 + 112 = 256$ ft

(d) $s = 0$ when $0 = -16(t^2 - 6t - 7) = -16(t - 7)(t + 1)$, $t = 7$ sec. (f) $v(2) = -32 \cdot 2 + 96 = 32$

$v(4) = -32 \cdot 4 + 96 = -32$ (g) The speed is 32 ft/sec at both times. (h) $v(7) = -32 \cdot 7 + 96 = -128$

In Exercises 54–56, a particle is moving along a line where at t sec, s cm is the directed distance form the origin, v cm/sec is the velocity and a cm/sec^2 is the acceleration. (a) Find v and a. (b) Show that the motion is simple harmonic. (c) Simulate the motion on your graphics calculator.

54. $s = 5 - 2\cos^2 t = 5 - (1 + \cos 2t) = 4 - \cos 2t$ (a) $v = ds/dt = 2\sin 2t$, $a = dv/dt = 4\cos 2t$. (b) The motion is simple harmonic because a is proportional to the distance from a fixed point ($s = 4$) and oppositely directed.

55. $s = \cos 2t + 2\sin 2t$ (a) $v = ds/dt = -2\sin 2t + 4\cos 2t$, $a = dv/dt = -4\cos 2t - 8\sin 2t$
(b) The motion is simple harmonic because $a = -4s$.

56. $s = \sin(4t + \frac{1}{3}\pi) + \sin(4t + \frac{1}{6}\pi)$

(a) $v = ds/dt = D_t[\sin(4t + \frac{1}{3}\pi) + \sin(4t + \frac{1}{6}\pi)] = 4[\cos(4t + \frac{1}{3}\pi) + \cos(4t + \frac{1}{6}\pi)]$
$a = dv/dt = D_t 4[\cos(4t + \frac{1}{3}\pi) + \cos(4t + \frac{1}{6}\pi)] = -16[\sin(4t + \frac{1}{3}\pi) + \sin(4t + \frac{1}{6}\pi)]$
(b) Because -16 is a constant, then a, the measure of the acceleration, is proportional to s, the measure of the displacement. Furthermore, because -16 is negative, then a and s are oppositely directed. Thus, the motion is simple harmonic.

57. Profit on an item is \$200 if not more than 800 are produced each week and decreases \$0.20 per item for each item over 800. (a) Express the profit as a function f of the number x of items sold. (b) Prove that f is continuous. (c) Determine if f is differentiable at 800.

(a) If $0 \le x \le 800$, then $f(x) = 200x$. Otherwise, $f(x) = [200 - .2(x - 800)]x = (360 - .2x)x = 360x - .2x^2$, $800 < x \le 1800$ (b) $\lim_{x \to 800^-} f(x) = \lim_{x \to 800^-} 200x = 200 \cdot 800 = f(200)$ and $\lim_{x \to 800^+} f(x) = \lim_{x \to 800^+} (360 - .2x)x = 200 \cdot 800$ so f is continuous at 800. (c) $\lim_{x \to 800^-} f'(x) = \lim_{x \to 800^-} 200 = 200$ and $\lim_{x \to 800^+} f'(x) = \lim_{x \to 800^+} (360 - .4x) = 40$. Hence f is not differentiable at 800.

58. If $R = kT^4$, find (a) the average rate of change of R as T increases from 200 to 300; (b) the instantaneous rate of change of R with respect to T when T is 200.

(a) $\dfrac{\Delta R}{\Delta T} = \dfrac{k300^4 - k200^4}{300 - 200} = 65{,}000{,}000k$ (b) $R'(T) = 4kT^3$; $R'(200) = 4k \cdot 200^3 = 32{,}000{,}000k$

59. A square units is the area of an isosceles right triangle for which the length of each leg is x units. Thus $A = \frac{1}{2}x^2$; $A'(x) = x$. (a) The average rate of change of A with respect to x as x changes from 8 to 8.01 is

$$\frac{\frac{1}{2} \cdot 8.01^2 - \frac{1}{2} \cdot 8^2}{8.01 - 8} = \frac{0.08005}{0.01} = 8.005$$

(b) The instantaneous rate of change of A with respect to x when $x = 8$ is $A'(8) = 8$.

60. If $y = x^{2/3}$, find the relative rate of change of y with respect to x when (a) $x = 8$, and (b) $x = c$, where c is a constant.

The relative rate of change of y with respect to x is $\dfrac{dy/dx}{y}$. Now, $\qquad \dfrac{dy}{dx} = \frac{2}{3}x^{-1/3}$

Thus, $\qquad \dfrac{dy/dx}{y} = \dfrac{\frac{2}{3}x^{-1/3}}{x^{2/3}} = \dfrac{2}{3x}$

(a) If $x = 8$, we have $\qquad \dfrac{dy/dx}{y} = \dfrac{2}{3(8)} = \dfrac{1}{12}$

The relative rate of change of y with respect to x is $\frac{1}{12}$ when $x = 8$.

(b) If $x = c$, we have $\qquad \dfrac{dy/dx}{y} = \dfrac{2}{3c}$

The relative rate of change of y with respect to x is $2/3c$ when $x = c$.

61. $100y$ calculators are supplied when m dollars is the price per calculator. $y = m^2 + \sqrt{m}$
(a) The average rate of change of y with respect to m when m increases from 16 to 17 is

$$\frac{(17^2 + \sqrt{17}) - (16^2 - \sqrt{16})}{17 - 16} \approx \frac{33.123}{1} = 33.123$$

Therefore the average rate of change of $100y$ with respect to m when m increases from 16 to 17 is 3,312.3. Thus, the average rate of change of the supply when the price increases from \$16 to \$17 is 3312.3 calculators per \$1 increase in price.

(b) $y'(m) = m + \frac{1}{2}m^{-1/2}$. The instantaneous rate of change of y with respect to m when m is 16 is $y'(16) = 2(16) + \frac{1}{2}(16)^{-1/2} = 32.125$. Therefore, the instantaneous rate of change of $100y$ when $m = 16$ is 3212.5. Hence, the instantaneous rate of change of the supply with respect to the price when the price is \$16 is 3212.5 calculators per \$1 increase in price.

62. The remainder theorem of algebra states that if $P(x)$ is a polynomial in x, and r is any real number, then there is a polynomial $Q(x)$ such that $P(x) = Q(x)(x-r) + P(r)$. What is $\lim_{x \to r} Q(x)$?

▷ Solving the given equation for $Q(x)$, we obtain $Q(x) = \dfrac{P(x) - P(r)}{x - r}$. Because P is a polynomial function, then P is differentiable at every real number. Thus we may apply the Definition 2.1.3′ to conclude that

$$\lim_{x \to r} Q(x) = \lim_{x \to r} \frac{P(x) - P(r)}{x - r} = P'(r)$$

63. $f(x) = \dfrac{3}{x+2}$; $f'(x) = \lim_{x \to -5} \dfrac{f(x) - f(x)}{x + 5} = \lim_{x \to -5} \dfrac{1}{x+5}\left(\dfrac{3}{x+2} - (-1)\right) = \lim_{x \to -5} \dfrac{x+5}{(x+2)(x+5)} = \lim_{x \to -5} \dfrac{1}{x+2} = -\dfrac{1}{3}$

64. Use the definition of a derivative to find $f'(x)$ if $f(x) = 3x^2 - 5x + 1$.

▷ $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[3(x + \Delta x)^2 - 5(x + \Delta x) + 1] - (3x^2 - 5x + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(2x\Delta x + \Delta x^2) - 5\Delta x}{\Delta x} = \lim_{\Delta x \to 0} [3(2x + \Delta x) - 5] = 6x - 5$$

65. $f(x) = \sqrt{4x - 3}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{4(x + \Delta x) - 3} - \sqrt{4x - 3}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\sqrt{4x + 4\Delta x - 3} - \sqrt{4x - 3})(\sqrt{4x + 4\Delta x - 3} + \sqrt{4x - 3})}{\Delta x(\sqrt{4x + 4\Delta x - 3} + \sqrt{4x - 3})}$$

$$= \lim_{\Delta x \to 0} \frac{4x + 4\Delta x - 3 - 4x + 3}{\Delta x(\sqrt{4x + 4\Delta x - 3} + \sqrt{4x - 3})} = \lim_{\Delta x \to 0} \frac{4}{\sqrt{4x + 4\Delta x - 3} + \sqrt{4x - 3}} = \frac{2}{\sqrt{4x - 3}}$$

66. Use the definition of a derivative to find $f'(5)$ if $f(x) = \sqrt{3x + 1}$.

▷ $$f'(5) = \lim_{x \to 5} \frac{\sqrt{3x + 1} - 4}{x - 5} = \lim_{x \to 5} \frac{(\sqrt{3x+1} - 4)(\sqrt{3x+1} + 4)}{(x - 5)(\sqrt{3x+1} + 4)} = \lim_{x \to 5} \frac{(3x + 1) - 16}{(x - 5)(\sqrt{3x+1} + 4)} = \lim_{x \to 5} \frac{3}{\sqrt{3x+1} + 4} = \frac{3}{8}$$

67. $f(x) = \sqrt{2 + \cos x} = (2 + \cos x)^{1/2}$; $f'(x) = \frac{1}{2}(2 + \cos x)^{-1/2}(-\sin x)$

$$f''(x) = \tfrac{1}{2}[-\tfrac{1}{2}(2 + \cos x)^{-3/2}(-\sin x)^2 + (2 + \cos x)^{-1/2}(-\cos x)]$$

$$f''(x) = \tfrac{1}{2}[-\tfrac{1}{2}(2 - 1)^{-3/2}(0) + (2 - 1)^{-1/2}(1)] = \tfrac{1}{2}$$

68. Find $f''(x)$ if $f(x) = 3\sin^2 x - 4\cos^2 x$.

▷ Because we will be taking two derivatives, it pays to simplify first.

$$f(x) = 3 \cdot \tfrac{1}{2}(1 - \cos 2x) - 4 \cdot \tfrac{1}{2}(1 + \cos 2x) = -\tfrac{1}{2} - \tfrac{7}{2}\cos 2x$$

$$f'(x) = 7 \sin 2x$$

$$f''(x) = 14 \cos 2x$$

69. $f(x) = (|x + 1| - |x|)^2$; $f'(x) = 2(|x + 1| - |x|)D_x(\sqrt{(x+1)^2} - \sqrt{x^2})$

$= 2(|x + 1| - |x|)\left(\dfrac{x+1}{\sqrt{(x+1)^2}} - \dfrac{x}{\sqrt{x^2}}\right) = 2(|x + 1| - |x|)\left(\dfrac{x+1}{|x+1|} - \dfrac{x}{|x|}\right)$. Alternatively,

$$f(x) = \begin{cases} (-x - 1 + x)^2 = 1 & \text{if } x < -1 \\ (x + 1 + x)^2 = (2x + 1)^2 & \text{if } -1 \le x \le 0; \\ (x + 1 - x)^2 = 1 & \text{if } x > 0 \end{cases} \quad f'(x) = \begin{cases} 0 & \text{if } x < -1 \\ 8x + 4 & \text{if } -1 < x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

70. Find $f'(-3)$ if $f(x) = (|x| - x)\sqrt[3]{9x}$.

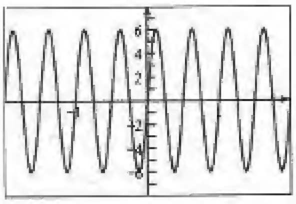
▷ Because x is negative, then $|x| = -x$. Therefore

$$f(x) = -2x(9x)^{1/3} = -2 \cdot 3^{2/3}x^{4/3}; \quad f'(x) = -8 \cdot 3^{-1/3}x^{1/3}; \quad f'(-3) = -8 \cdot 3^{-1/3}(-3)^{1/3} = 8$$

In Exercises 71 and 72, s cm is the distance of a weight from its central position at t seconds, and the positive direction is upward. (a) Find the velocity and acceleration. (b) Show that the motion is simple harmonic. (c) Find the amplitude A, period p, and frequency f of the motion. (d) Simulate the motion on your calculator. (e) Plot the graph.

71. $s = 5 \sin \frac{1}{6}\pi t$ (a) $v = \frac{ds}{dt} = \frac{5}{6}\pi \cos \frac{1}{6}\pi t$, $a = \frac{dv}{dt} = -\frac{5}{36}\pi^2 \sin \frac{1}{6}\pi t$ (b) Because $a = -(\frac{5}{36}\pi)^2 s$ and $-(\frac{5}{36}\pi)^2$ is negative, the motion is simple harmonic. (c) $A = 5$, $p = 2\pi/\frac{1}{6}\pi = 12$, $f = 1/p = \frac{1}{12}$

72. $s = 6 \cos \pi(4t - \frac{1}{2})$ ▹ The graph is at the right.



(a) $v = \frac{ds}{dt} = D_t[6 \cos \pi(4t - \frac{1}{2})] = -24\pi \sin \pi(4t - \frac{1}{2})$. $a = \frac{dv}{dt} =$
$D_t[-24\pi \sin \pi(4t - \frac{1}{2}t)] = -96\pi^2 \cos \pi(4t - \frac{1}{2}) = -(4\pi)^2[6 \cos \pi(4t - \frac{1}{2})]$
(b) Because $-(4\pi)^2$ is a constant, then a, the measure of the acceleration, is proportional to s, the measure of the displacement. Furthermore, because $-(4\pi)^2$ is negative, then a and s are oppositely directed. Thus, the motion is simple harmonic. (c) The amplitude is 6. The period is $2\pi/4\pi = \frac{1}{2}$ and the frequency is 2. (d) To simulate the motion, in parametric mode let $x_1(t) = 2$, $y_1(t) = 6 \cos \pi(4t - \frac{1}{2})$. In the window $[0,4] \times [-8,8]$, we let $[t\text{Min}, t\text{Max}] = [0,4]$, t-step $= .05$. Press the $\boxed{\text{TRACE}}$ key, press the ◃ key until the cursor is at 0, the press the ▹ key to observe the motion.

In Exercises 73 and 74, s ft is the directed distance of a particle from the origin at t seconds, v ft/sec is the velocity and a ft/sec^2 is the acceleration. (a) Find the v and a. (b) Show that the motion is simple harmonic. (c) Simulate the motion on your calculator.

73. $s = 2 \cos (3t + \frac{1}{3}\pi) + 4 \sin (3t - \frac{1}{6}\pi)$ (a) $v = ds/dt = -6 \sin(3t + \frac{1}{3}\pi) + 12 \cos(3t - \frac{1}{6}\pi)$, $a = dv/dt =$
$-18 \cos(3t + \frac{1}{3}\pi) - 36 \sin(3t - \frac{1}{6}\pi)$ (b) Because $a = -9s$ and -9 is negative, the motion is simple harmonic.

74. $s = 3 - 6 \sin^2 4t = 3 - 6 \cdot \frac{1}{2}(1 - \cos 8t) = 3 \cos 8t$ (a) $v = ds/dt = -24 \sin 8t$, $a = dv/dt = -192 \cos 8t$
(b) Because $a = -64s$ and -64 is negative, the motion is simple harmonic.

75. $s = \cos 2t + \cos t$; $v = \frac{ds}{dt} = -2 \sin 2t - \sin t$; $a = \frac{d^2 s}{dt^2} = -4 \cos 2t - \cos t$
Since the acceleration is not proportional to the displacement, the motion is not simple harmonic.

76. A particle is moving in a straight line according to the equation $s = \sqrt{A + Bt^2}$, where A and B are positive constants. Prove that the measure of the acceleration of the particle is inversely proportional to s^3 for any t.

▹ We are given that $s = (A + Bt^2)^{1/2}$ (1)

We differentiate to find first the velocity and then the acceleration.

$$v = \frac{ds}{dt} = \tfrac{1}{2}(A + Bt^2)^{-1/2}(2Bt) = Bt(A + Bt^2)^{-1/2}$$

$$a = \frac{dv}{dt} = B\left[t\frac{d}{dt}(A + Bt^2)^{-1/2} + (A + Bt^2)^{-1/2}\right]$$

$$= B[t(-\tfrac{1}{2})(A + Bt^2)^{-3/2}(2Bt) + (A + Bt^2)^{-1/2}]$$

$$= B(A + Bt^2)[-Bt^2 + (A + Bt^2)] = \frac{AB}{(A + Bt^2)^{3/2}} \qquad (2)$$

Substituting from Eq. (1) into Eq. (2), we obtain $a = \frac{AB}{s^3}$

Because we are given that A and B are constants, then we have proved that the measure of the acceleration is inversely proportional to s^3 for any t.

77. $C(x)$ dollars is the total cost of manufacturing x chairs and $C(x) = x^2 + 40x + 800$.
▹ (a) C' is the marginal cost function: $C'(x) = 2x + 40$ (b) $C'(20) = 2(20) + 40 = 80$
(c) The number of dollars in the actual cost of manufacturing the twenty-first chair is
$C(21) - C(20) = [(21)^2 + 40(2) + 800] - [(20)^2 + 40(20) + 800] = 81$

78. $R(x)$ dollars is the revenue from the sale of x lamps and $R(x) = 100x - \frac{1}{6}x^2$.

▸ (a) R' is the marginal revenue function : $R'(x) = 100 - \frac{1}{3}x$ (b) $R'(15) = 100 - \frac{1}{3}(15) = 95$
(c) The actual revenue from the sale of the sixteenth lamp is
$R(16) - R(15) = [100(16) - \frac{1}{6}(16)^2] - [100(15) - \frac{1}{6}(15)^2] = \$94.83$

79. After t weeks there are x prey and y predators where $y = \frac{1}{60{,}000}x^2 - \frac{1}{100}x + 40$ and $x = 300t + 375$.

▸ If $t = 10$, $x = 300 \cdot 10 + 375 = 3375$ and $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (\frac{x}{30{,}000} - \frac{1}{100})300 = (\frac{3375}{30{,}000} - \frac{1}{100})300 = 30.75$

▪ After 9 weeks the predator population is increasing at the rate of 648 fish per week.

80. The demand equation for a particular candy bar is $px + x + 20p = 3000$ where $1000x$ candy bars are demand ed per week when p cents is the price per bar. If the current price of the candy is 49 cents per bar and the price per bar is increasing at the rate of 0.2 cents per week, find the rate of change in the demand.

▸ The rate of change of the demand is $\frac{d}{dt}(1000x) = 1000\frac{dx}{dt}$. We solve the given equation for x and differentiate with respect to t using the chain rule.

$$px + x = 3000 - 20p$$

$$x = \frac{3000 - 20p}{p + 1}$$

$$\frac{dx}{dt} = \frac{dx}{dp} \cdot \frac{dp}{dt} = \frac{(p+1)(-20) - (3000 - 20p)(1)}{(p+1)^2} \cdot \frac{dp}{dt} = \frac{-3020}{(p+1)^2} \cdot \frac{dp}{dt}$$

We multiply on each side by 1000 and let $p = 49$ and $dp/dt = 0.2$. Thus,

$$1000\frac{dx}{dt} = 1000 \cdot \frac{-3020}{(50)^2}(0.2) = -241.6$$

▪ The demand is decreasing by about 242 bars/week at the current price.

81. At t hours after 6 P.M., let x nautical miles be the distance of the first ship from the port, let y nautical miles be the distance of the second ship from the port, and let z nautical miles be the distance between the two ships. By the law of cosines,

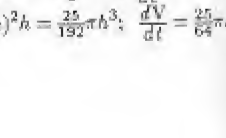
$$z^2 = x^2 + y^2 - 2xy \cos \tfrac{1}{4}\pi;\ z = [x^2 + y^2 - 2xy(\tfrac{1}{2}\sqrt{2})]^{1/2};\ D_t z = \frac{2xD_t x + 2yD_t y - \sqrt{2}yD_t x - \sqrt{2}xD_t y}{2(x^2 + y^2 - \sqrt{2}xy)^{1/2}}$$

We are given $D_t x = 20$ and $D_t y = 15$. At the instant the second ship has traveled 90 nautical miles, $t = 6$. For $t = 6$, $x = 20(6 + 6) = 240$ and $y = 90$. Hence

$$D_t z = \frac{2(240)20 + 2(90)15 - \sqrt{2}(90)20 - \sqrt{2}(240)15}{2[(240)^2 + (90)^2 - \sqrt{2}(240)(90)]^{1/2}} = 12.44$$

▪ When the second ship has traveled 90 nautical miles the ships are separating at the rate of 10.6 knots.

82. From the figure, the hypotenuse of a right triangle is 10 and one leg is 6. From the Pythagorean theorem we find the depth of the trough is 8 m. Let x m be the depth of the water. From similar triangles we find the width of the water's surface is $5 + 2 \cdot \frac{3}{4}x = 5 + \frac{3}{2}x$. The area of the trapezoidal water surface is A m^2 where $A = \frac{1}{2}x[5 + (5 + \frac{3}{2}x)] = 5x + \frac{3}{4}x^2$ and its volume is V m^3 where $V = 80A = 400x + 60x^2$. When $x = 5$ and $dx/dt = 0.1$,



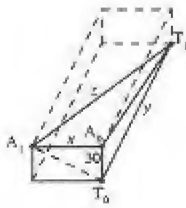
$$\frac{dV}{dt} = (400 + 120x)\frac{dx}{dt} = (400 + 120 \cdot 5)(0.1) = 100$$

▪ When the water is 5 m deep it is leaking out at 100 m^3/hr.

83. The water in the funnel forms a cone and the volume of this water is increasing at the rate of 8in^3/sec. At t sec, let r in. be the radius of this cone and h in. the height of this cone. Then $V = \frac{1}{3}\pi r^2 h$, $\frac{dV}{dt} = 8$ and we wish to find $\frac{dh}{dt}$ when $h = 5$. Now $\frac{r}{h} = \frac{5}{8}$ so $r = \frac{5}{8}h$. Hence $V = \frac{1}{3}\pi(\frac{5}{8}h)^2 h = \frac{25}{192}\pi h^3$; $\frac{dV}{dt} = \frac{25}{64}\pi h^2 \frac{dh}{dt}$;
$\frac{dh}{dt} = \frac{64}{25\pi h^2}\frac{dV}{dt}$. Because $\frac{dV}{dt} = 8$, we obtain $\left.\frac{dh}{dt}\right|_{h=5} = \frac{64 \cdot 8}{25\pi \cdot 5^2} = \frac{512}{625\pi} \approx 0.26$.

▪ The surface of the water is rising at the rate of 0.26in/sec when it is 5 in. deep.

34. As the last car of a train passes under a bridge, an automobile crosses the bridge on a roadway perpendicular to the track and 30 ft above it. The train is traveling at the rate of 80 ft/sec and the automobile is traveling at the rate of 40 ft/sec. How fast are the train and the automobile separating after 2 sec? Refer to the figure. The roadway is in the line A_0A_1 and the railway track is in the line T_0T_1 We are given that line A_0A_1 is perpendicular to line T_0T_1. The last car of the train is at point T_0 when the automobile is at point A_0. After t sec the last car of the train is at point T_1 and the automobile is at point A_1. The other variables are defined as follows.



x feet is the distance of the automobile from point A_0 at t sec

y feet is the distance of the last car of the train from point T_0 at t sec

z feet is the distance from the auto to the last car of the train at t sec

Because the train is traveling at 80 ft/sec, we are given that $y = 80t$. Because the automobile is traveling at 40 ft/sec, we are given that $x = 40$. Since dz/dt represents the rate at which the automobile and the train are separating, we want to find dz/dt when $t = 2$. Hence we must find an equation with variables x, y, and z.

Because triangle $A_1T_0T_1$ is a right triangle, we have $\qquad z^2 = \left|\overline{A_1T_0}\right|^2 + y^2 \qquad (1)$

Because triangle $A_1A_0T_0$ is a right triangle, we have $\qquad \left|\overline{A_1T_0}\right|^2 = x^2 + 30 \qquad (2)$

Substituting from Eq. (2) into Eq. (1) and solving for z, we obtain

$$z^2 = x^2 + (30)^2 + y^2 = (40t)^2 + (30)^2 + (80t)^2$$

$$z = (8000t^2 + 900)^{1/2}$$

Differentiating with respect to t, we get

$$\frac{dz}{dt} = \frac{1}{2}(8000t^2 + 900)^{-1/2}(16000t) = \frac{800t}{\sqrt{80t^2 + 9}}$$

When $t = 2$ we obtain $\qquad \dfrac{dz}{dt} = \dfrac{1600}{\sqrt{329}} \approx 88.2$

- After 2 sec the train and the automobile are separating at the rate of about 88.2 ft/sec.

At t sec, let y ft be the height of the man's shadow and x ft his distance from the building. We have by similar triangles

$$\frac{y}{6} = \frac{x}{40 - x} \text{ so } y = \frac{6x}{40 - x};\ \frac{dy}{dx} = \frac{6(40 - x) - 6x(-1)}{(40 - x)^2}\frac{dx}{dt} = \frac{240}{(40 - x)^2}\frac{dy}{dx}$$

When $x = 30$ and $\dfrac{dx}{dt} = -4$, then $\dfrac{dy}{dt} = \dfrac{240(-4)}{(40 - 30)^2} = -\dfrac{960}{100} = -9.6$.

- The man's shadow is growing shorter at the rate of 9.6 ft/sec.

At t days, r cm is the radius of the burn and A cm^2 is the area. $A = \pi r^2$, $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt} = 2\pi(1)(-0.05) = 0.1\pi$.

- When the radius of the burn is 1 cm, its area is decreasing at about 0.31 cm^2/day.

$$f(x) = \begin{cases} x^2 + 2 & \text{if } x \le 3 \\ 20 - x^2 & \text{if } 3 < x \end{cases}$$

(b) $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (x^2 + 2) = 11$, $\lim\limits_{x\to 3^+} f(x) = (20 - x^2) = 11$ so $\lim\limits_{x\to 3} f(x) = 11$.

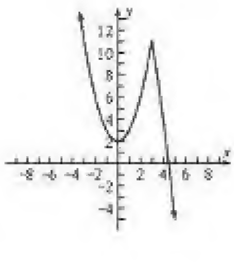
$f(3) = 11$. Because $\lim\limits_{x\to 3} f(x) = f(3)$, f is continuous at 3.

(c)
$$f'_-(3) = \lim_{x\to 3^-} \frac{f(x) - f(3)}{x - 3} = \lim_{x\to 3^-} \frac{(x^2 + 2) - 11}{x - 3} = \lim_{x\to 3^-} \frac{x^2 - 9}{x - 3}$$

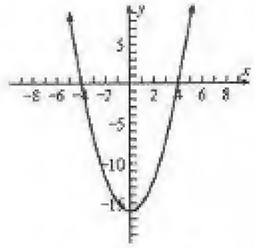
$$= \lim_{x\to 3^-} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x\to 3^-} (x + 3) = 6$$

$$f'_+(3) = \lim_{x\to 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{x\to 3^+} \frac{(20 - x^2) - 11}{x - 3} = \lim_{x\to 3^+} \frac{9 - x^2}{x - 3}$$

$$= \lim_{x\to 3^+} \frac{(3 - x)(3 + x)}{x - 3} = \lim_{x\to 3^+} -(x + 3) = -6$$

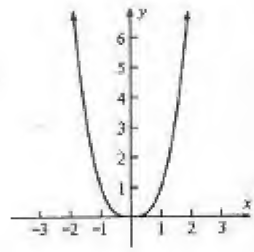


Because $f'_-(3) \ne f'_+(3)$, $f'(3)$ does not exist. Hence, f is not differentiable at 3.



88. Given $f(x) = \begin{cases} x^2 - 16 & \text{if } x < 4 \\ 8x - 32 & \text{if } x \ge 4 \end{cases}$. (a) Sketch the graph of f. (b) Determine if f is continuous at 4. (c) Determine if f is differentiable at 4.

▷ (a) The graph is sketched in the figure at the right.
(b) $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(x^2 - 16) = 0$ and $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}(8x - 32) = 0 = f(4)$.
Hence f is continuous at 4.
(c) Because f is continuous at 4, $f'_-(4) = \lim_{x\to 4^-} f'(x) = \lim_{x\to 4^-}(2x) = 8$ and
$f'_+(4) = \lim_{x\to 4^+} f'(x) = 8$. Because these are equal, f is differentiable at 4.



89. $f(x) = |x|^3$
(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x)^3 = 0$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^3 = 0$.
Therefore, $\lim_{x\to 0} f(x) = 0$.

(c) $f'_-(0) = \lim_{x\to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \dfrac{(-x)^3 - 0}{x} = \lim_{x\to 0^-} (-x^2) = 0$

$f'_+(0) = \lim_{x\to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \dfrac{x^3 - 0}{x} = \lim_{x\to 0^+} x^2 = 0$.

Therefore, $f'(0) = 0$.

90. $f(x) = x^2 \operatorname{sgn} x = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$. $f'(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x & \text{if } x > 0 \end{cases}$. Because $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = 0$, $f'(0) = 0$ and $f'(x) = |x|$. f is differentiable everywhere and f' is continuous everywhere.

91. $f(x) = \begin{cases} ax^2 + b & \text{if } x \le 1 \\ \frac{1}{|x|} & \text{if } x > 1 \end{cases}$ ▷ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (ax^2 + b) = a + b$, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x} = 1$.

Because $f'(1)$ exists, f is continuous at 1 and so $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$ and $a + b = 1$.
$f'_-(1) = \lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-}(2ax) = 2a$ and

$f'_+(1) = \lim_{x\to 1^+} \dfrac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \dfrac{(1/x) - 1}{x - 1} = \lim_{x\to 1^+} \dfrac{1 - x}{x(x - 1)} = \lim_{x\to 1^+} \dfrac{-1}{x} = -1$.

Because $f'(1)$ exists, $f'_-(1) = f'_+(1)$. Therefore $2a = -1$, $a = -\frac{1}{2}$.
Substituting $a = -\frac{1}{2}$ in $a + b = 1$ we obtain $-\frac{1}{2} + b = 1$, $b = \frac{3}{2}$.

92. Suppose $f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax^2 + bx + c & \text{if } x \ge 1 \end{cases}$
Find the values of a, b, and c so that $f''(1)$ exists.

▷ If $x \ne 1$, we may differentiate f as follows:
$f'(x) = \begin{cases} 3x^2 & \text{if } x < 1 \\ 2ax + b & \text{if } x > 1 \end{cases}$
Because $f''(1)$ exists, then $f'(1)$ exists and so f is continuous at 1. Therefore

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$$
$$\lim_{x\to 1^+} ax^2 + bx + c = \lim_{x\to 1^-} x^3$$
$$a + b + c = 1$$

and $f(1) = 1$. Because $f''(1)$ exists, then f' is continuous at 1. Therefore

$$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^-} f'(x)$$
$$\lim_{x\to 1^+} 2ax + b = \lim_{x\to 1^-} 3x^2$$
$$2a + b = 3$$

and $f'(1) = 3$. Because $f''(1)$ exists, then

$$f''_+(1) = f''_-(1)$$

$$\lim_{x\to1^+}\frac{f'(x)-f'(1)}{x-1}=\lim_{x\to1^-}\frac{f'(x)-f'(1)}{x-1}$$
$$\lim_{x\to1^+}\frac{2ax+b-3}{x-1}=\lim_{x\to1^-}\frac{3x^2-3}{x-1}$$

Substituting $b=3-2a$ from Eq. (4), we have

$$\lim_{x\to1^+}2a=\lim_{x\to1^-}3(x+1)$$
$$2a=6; \qquad a=3$$

Because $b=3-2a$ then $b=-3$. Substituting for a and b in Eq. (3) we get

$$3+(-3)+c=1; \qquad c=1$$

Although we are not given that f'' is continuous at 1, if the limits exist it is true that

$$\lim_{x\to1^+}f''(x)=\lim_{x\to1^-}f''(x)$$

Differentiating $x^2+y^2=r^2$ implicitly gives $2x+2y\frac{dy}{dx}=0$; $\frac{dy}{dx}=-\frac{x}{y}$. The line through (x,y) and the origin has slope $\frac{y}{x}$ which is the negative reciprocal of the slope of the tangent line. Hence they are perpendicular.

$f(u)=1/u^2$ and $g(x)=\sqrt{x}/\sqrt{2x^3-6x+1}$. Plotting $y=2x^3-6x+1$, we find dom $g\approx[0,.168)\cup(1.692,\infty)$.
(a) $(f\circ g)(x)=(2x^3-6x+1)/x=2x^2-6+x^{-1}$ and $(f\circ g)'(x)=4x-x^{-2}$, $x\in(0,.168)\cup(1.692,\infty)$.

(b) $f(g(x))'=f'(g(x))\cdot g'(x)$ where $f'(g(x))=-2(g(x))^{-3}=-2x^{-3/2}(2x^3-6x+1)^{3/2}$ and (1)

$g'(x)=D_x[x^{1/2}(2x^3-6x+1)^{-1/2}]=\frac{1}{2}x^{-1/2}(2x^3-6x+1)^{-1/2}-\frac{1}{2}(2x^3-6x+1)^{-1/2}(3x^2-6)$ (2)
Multiply (1) and (2) and simplify to get the result.

$f(x)=3x+|x|$ and $g(x)=\frac{3}{4}x-\frac{1}{4}|x|$.

$$f'_-(0)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{(3x-x)-0}{x}=\lim_{x\to0^-}2=2$$
$$f'_+(0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^+}\frac{(3x+x)-0}{x}=\lim_{x\to0^+}4=4$$

Because $f'_-(0)\neq f'_+(0)$, then $f'(0)$ does not exist.

$$g'_-(0)=\lim_{x\to0^-}\frac{g(x)-g(0)}{x-0}=\lim_{x\to0^-}\frac{(\frac{3}{4}x+\frac{1}{4}x)-0}{x}=\lim_{x\to0^-}1=1$$
$$g'_+(0)=\lim_{x\to0^+}\frac{g(x)-g(0)}{x-0}=\lim_{x\to0^+}\frac{(\frac{3}{4}x-\frac{1}{4}x)-0}{x}=\lim_{x\to0^+}\frac{1}{2}=\frac{1}{2}$$

Because $g'_-(0)\neq g'_+(0)$, then $g'(0)$ does not exist.

$(f\circ g)(x)=3(\frac{3}{4}x-\frac{1}{4}x)+\left|\frac{3}{4}x-\frac{1}{4}|x|\right|$

If $x<0$, $(f\circ g)(x)=3(\frac{3}{4}x+\frac{1}{4}x)+\left|\frac{3}{4}x+\frac{1}{4}x\right|=3x+|x|=3x-x=2x$.

If $x\geq0$, $(f\circ g)(x)=3(\frac{3}{4}x-\frac{1}{4}x)+\left|\frac{3}{4}x-\frac{1}{4}x\right|=\frac{3}{2}x+\left|\frac{1}{2}x\right|=\frac{3}{2}x+\frac{1}{2}x=2x$.

Thus $(f\circ g)(x)=2x$ for all x. Hence $(f\circ g)'(x)=2$. Therefore $(f\circ g)'(0)=2$.

Give an example of two functions f and g for which f is differentiable at $g(0)$, g is not differentiable at 0, and $f\circ g$ is differentiable at 0.

Let f and g be defined by $f(x)=x^2$ and $g(x)=x^{2/3}$. Then

$$g'(x)=\tfrac{2}{3}x^{-1/3}$$

Hence, $g'(0)$ does not exist, and thus g is not differentiable at 0. Moreover, because $f'(x)=2x$, then f is differentiable for all x, and in particular f is differentiable at 0. However, $f\circ g$ is defined by

$$(f\circ g)(x)=f(g(x))=(x^{2/3})^2=x^{4/3}$$

Thus $(f\circ g)'(x)=\frac{4}{3}x^{1/3}$. Therefore, $f\circ g$ is differentiable at all x, and in particular $f\circ g$ is differentiable at 0. $f(x)=\cos x$ gives another example.

97. Let $f(x) = |x|$ and $g(x) = x^2$. Then $(f \circ g)(x) = f(g(x)) = f(x^2) = |x^2| = x^2$.
$g(0) = 0$ and $f(x) = |x|$ is not differentiable at $g(0)$.
$g'(x) = 2x$. Therefore, $g'(0) = 0$. Thus, g is differentiable at 0.
$(f \circ g)(x) = x^2$. Therefore $f \circ g$ is differentiable at 0.

98. $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^n & \text{if } x \geq 0 \end{cases}$, $f'(x) = \begin{cases} 0 & \text{if } x < 0 \\ nx^{n-1} & \text{if } x > 0 \end{cases}$, where n is rational. (a) f is continuous if $n \geq 0$ (b) f is differentiable at 0 if $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} (nx^{n-1}) = 0$, that is, if $n > 1$. (c) f' is then continuous.

99. $\lim_{x \to x_1} \dfrac{xf(x_1) - x_1 f(x)}{x - x_1} = \lim_{x \to x_1} \dfrac{xf(x_1) - x_1 f(x_1) + x_1 f(x_1) - x_1 f(x)}{x - x_1}$

$= \lim_{x \to x_1} \dfrac{f(x_1)(x - x_1) - x_1[f(x) - f(x_1)]}{x - x_1} = f(x_1) \lim_{x \to x_1} 1 - x_1 \lim_{x \to x_1} \dfrac{f(x) - f(x_1)}{x - x_1} = f(x_1) - x_1 f'(x_1)$

100. Let f and g be two functions whose domains are the set of all real numbers. Furthermore, suppose that

(i) $g(x) = xf(x) + 1$; (ii) $g(a + b) = g(a) \cdot g(b)$ for all a and b; (iii) $\lim_{x \to 0} f(x) = 1$

Prove that $g'(x) = g(x)$.

▶ $g'(x) = \lim_{\Delta x \to 0} \dfrac{g(x + \Delta x) - g(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{g(x) \cdot g(\Delta x) - g(x)}{\Delta x}$ (Hypothesis ii)

$= g(x) \cdot \lim_{\Delta x \to 0} \dfrac{g(x) - 1}{\Delta x}$

$= g(x) \cdot \lim_{\Delta x \to 0} \dfrac{[\Delta x \cdot f(\Delta x) + 1] - 1}{\Delta x}$ (Hypothesis i)

$= g(x) \cdot \lim_{\Delta x \to 0} f(\Delta x)$

$= g(x) \cdot 1$ (Hypothesis iii)

Therefore, $g'(x) = g(x)$.

101. A counterexample. $f(x) = |x|$, $g(x) = -1$, and $x_1 = 1$. Here f and g are both differentiable at 1; but $(f \circ g)(x) = |x - 1|$; so $f \circ g$ is not differentiable at 1.

102. $g(x) = |f(x)| = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ f(x) & \text{if } f(x) \geq 0 \end{cases}$, $g^{(n)}(x) = \begin{cases} -f^{(n)}(x) & \text{if } f(x) < 0 \\ f^{(n)}(x) & \text{if } f(x) > 0 \end{cases} = \dfrac{f(x)}{|f(x)|} f^{(n)}(x)$ if $f \neq 0$

103. We wish to prove by mathematical induction that $D_x{}^n(\sin x) = \sin(x + \frac{1}{2}n\pi)$.
If $n = 1$ we have $D_x(\sin x) = \cos x = \sin(x + \frac{1}{2}\pi)$. Therefore the formula is true for $n = 1$.
We assume the formula is true for $n = k$, that is $D_x{}^k(\sin x) = \sin(x + \frac{1}{2}k\pi)$. Then
$D_x{}^{(k+1)}(\sin x) = D_x[D_x{}^k(\sin x)] = D_x[\sin(x + \frac{1}{2}k\pi)] = \cos(x + \frac{1}{2}k\pi) = \sin[(x + \frac{1}{2}k\pi) + \frac{1}{2}\pi] = \sin[x + \frac{1}{2}(k + 1)\pi]$
Therefore, the formula holds for $n = k + 1$ if it holds for $n = k$. Hence it holds for any positive integer n.

104. If $y = 1/(1 - 2x)$, prove by mathematical induction that

$$\frac{d^n y}{dx^n} = \frac{2^n n!}{(1 - 2x)^{n+1}} \tag{1}$$

▶ We must show that (a) Eq. (1) is true for $n = 1$, and (b) whenever Eq. (1) is true for $n = k$, then Eq. (1) is also true for $n = k + 1$.

(a) We are given that

$$y = (1 - 2x)^{-1}$$

Then $$\frac{dy}{dx} = -(1 - 2x)^{-2}(-2) = \frac{2}{(1 - 2x)^2} \tag{2}$$

Because Eq. (1) becomes Eq. (2) if $n = 1$, then Eq. (1) holds when $n = 1$.

(b) Suppose that Eq. (1) holds when $n = k$. That is, suppose that

$$\frac{d^k y}{dx^k} = \frac{2^k k!}{(1 - 2x)^{k+1}} = 2^k k!(1 - 2x)^{-k-1}$$

Differentiating with respect to x, we obtain

$$\begin{aligned}\frac{d^{k+1}y}{dx^{k+1}} &= 2^k k!(-k-1)(1-2x)^{(-k-1)-1}\cdot(-2)\\ &= 2^{k+1}k!(k+1)(1-2x)^{-k-2}\end{aligned} \tag{3}$$

Because $(k+1)\cdot k! = (k+1)!$, Eq. (3) is equivalent to

$$\frac{d^{k+1}y}{dx^{k+1}} = \frac{2^{k+1}(k+1)!}{(1-2x)^{k+2}} \tag{4}$$

If n is replaced by $k+1$ in Eq. (1), the result is Eq. (4). Therefore, we have shown that whenever Eq. (1) holds for $n=k$, it also holds for $n=k+1$.

THREE

BEHAVIOR OF FUNCTIONS AND THEIR GRAPHS, EXTREME FUNCTION VALUES, AND APPROXIMATIONS

3.1 MAXIMUM AND MINIMUM FUNCTION VALUES

3.1.1 Definition The function f is said to have a *relative maximum value* at c if there exists an open interv containing c, on which f is defined, such that $f(c) \geq f(x)$ for all x in this interval.

3.1.2 Definition The function f is said to have a *relative minimum value* at c if there exists an open interv containing c, on which f is defined, such that $f(c) \leq f(x)$ for all x in this interval.

If the function f has either a relative maximum or a relative minimum value at then f is said to have a *relative extremum* at c.

3.1.3 Theorem If $f(x)$ exists for all values of x in the open interval (a, b), and if f has a relative extremum c where $a < c < b$, then if $f'(c)$ exists, $f'(c) = 0$.

3.1.4 Definition If c is a number in the domain of the function f, and if either $f'(c) = 0$ or $f'(c)$ does exist, then c is called a *critical number* of f.

We conclude that a necessary (but not sufficient) condition for a function to have relative extremum at c is for c to be a critical number.

3.1.5 Definition The function f is said to have an *absolute maximum value on an interval* if there is som number c in the interval such that $f(c) \geq f(x)$ for all x in the interval. In such a case $f(c)$ the absolute maximum value of f on the interval.

3.1.6 Definition The function f is said to have an *absolute minimum value on an interval* if there is som number c in the interval such that $f(c) \leq f(x)$ for all x in the interval. In such a case $f(c)$ the absolute minimum value of f on the interval.

An *absolute extremum* of a function f on an interval I is either an absolute maxim value or an absolute minimum value of f on I. If f has an absolute extremum on I, then absolute extremum must occur either at a critical number of f or at an endpoint of I. If I not a closed interval or f is not continuous on I, then f may not have an absolute extrem on I. To show that f has no absolute maximum on I, we show that for some c in either $\lim_{x \to c} f(x) = +\infty$ or $\lim_{x \to c} f(x) = d$ but $f(x) < d$ in I, where the limits may be o sided. To show that f has no absolute minimum on I, we show that for some c in I eit $\lim_{x \to c} f(x) = -\infty$ or $\lim_{x \to c} f(x) = d$ but $f(x) > d$ on I.

3.1.7 Extreme-Value Theorem If the function f is continuous on the closed interval $[a, b]$, then f has an absol maximum value and an absolute minimum value on $[a, b]$.

The following steps can be used to find the absolute extrema of f on $[a, b]$ if f continuous on $[a, b]$.

1. Find each number c in (a, b) such that $f'(c) = 0$ or $f'(c)$ does not exist.
2. Find the function value $f(c)$ for each number c of step 1.
3. Find the function values $f(a)$ and $f(b)$.
4. The largest of the values from steps 2 and 3 is the absolute maximum value, and smallest of the values from steps 2 and 3 is the absolute minimum value of f on closed interval $[a, b]$.

Include in step 1 any points, such as the break points of a definition in pieces, where f' *might* not exist. It is easier to evaluate $f(c)$ than to *prove* that $f'(c)$ does not exist.

Exercises 3.1

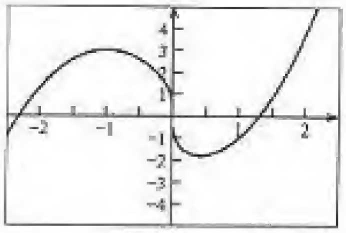
In Exercises 1–8, (a) plot the function and estimate the critical numbers. (b) Calculate the critical numbers.

1. $f(x) = x^3 + 7x^2 - 5x$; $f'(x) = 3x^2 + 14x - 5 = (3x-1)(x+5)$
$f'(x) = 0$ when $x = \frac{1}{3}$ and $x = -5$. Thus $\frac{1}{3}$ and -5 are the critical numbers of f.

2. $g(x) = 2x^3 - 2x^2 - 16x + 1$; $g'(x) = 6x^2 - 4x - 16 = 2(3x^2 - 2x - 8) = 2(3x+4)(x-2)$
$g'(x) = 0$ when $x = -\frac{4}{3}$ and $x = 2$. Thus $-\frac{4}{3}$, 2 are the critical numbers of g.

3. $g(x) = x^{6/5} - 12x^{1/5}$; $g'(x) = \frac{6}{5}x^{1/5} - \frac{12}{5}x^{-4/5} = \frac{6}{5}x^{-4/5}(x-2)$. $g'(0)$ does not exist and 0 is in the domain of g. $g'(x) = 0$ when $x = 2$. Thus 0 and 2 are the critical numbers of g.

4. $f(x) = x^{7/3} + x^{4/3} - 3x^{1/3}$ ▷ A plot of f is shown at the right.



The domain of f is $(-\infty, +\infty)$.

$$f'(x) = \tfrac{7}{3}x^{4/3} + \tfrac{4}{3}x^{1/3} - x^{-2/3}$$
$$= \tfrac{1}{3}x^{-2/3}(7x^2 + 4x - 3) = \frac{(7x-3)(x+1)}{3x^{2/3}}$$

Because $f'(0)$ does not exist and 0 is in the domain of f, by Definition 3.1.4, 0 is a critical number of f. If $f'(x) = 0$, then

$$0 = (7x-3)(x+1); \qquad x = \tfrac{3}{7} \quad \text{or} \quad x = -1$$

Because $f'(\frac{3}{7}) = 0$, $f'(-1) = 0$, and because $\frac{3}{7}$ and -1 are in the domain of f, by Definition 3.1.4, $\frac{3}{7}$ and -1 are also critical numbers of f.

5. $f(x) = \dfrac{x+1}{x^2 - 5x + 4} = \dfrac{x+1}{(x-4)(x-1)}$. Neither $f(4)$ nor $f(1)$ is defined. The domain of f is $\{x \mid x \neq 4, x \neq 1\}$.

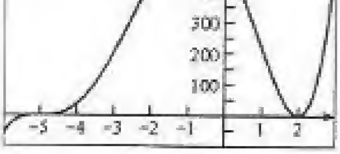
$$f'(x) = \frac{(x^2-5x+4)\cdot 1 - (x+1)(2x-5)}{(x^2-5x+4)^2} = \frac{x^2 - 5x + 4 - (2x^2 - 3x - 5)}{(x^2-5x+4)^2} = \frac{-x^2 - 2x + 9}{(x^2-5x+4)^2}$$

$f'(x) = 0$ when $-x^2 - 2x + 9 = 0$; $x^2 + 2x - 9 = 0$; $x = -1 \pm \sqrt{10}$
Therefore $-1 + \sqrt{10}$ and $-1 - \sqrt{10}$ are the critical numbers of f.
Note that $f'(x)$ is undefined when $x^2 - 5x + 4 = 0$. However, the only solutions of this equation are 1 and 4, numbers which are not in the domain of f. Therefore, neither 1 nor 4 is a critical number of f.

6. $f(x) = \dfrac{2x-9}{x^2-9}$. $x = \pm 3$ are not in the domain of f. Hence they will not be in the domain of f' but will not be critical numbers. $f'(x) = \dfrac{2(x^2-9) - (2x-9)2x}{(x^2-9)^2} = \dfrac{-2x^2 + 18x - 18}{(x^2-9)^2}$. $f'(x) = 0$ when $-2x^2 + 18x - 18 = 0$; $x^2 - 9x + 9 = 0$; $x = \frac{1}{2}(9 \pm 3\sqrt{5})$. Therefore, $\frac{1}{2}(9 + 3\sqrt{5})$ and $\frac{1}{2}(9 - 3\sqrt{5})$ are the critical numbers of f.

7. $G(x) = (x-2)^3(x+1)^2$; $G'(x) = 3(x-2)^2(x+1)^2 + 2(x-2)^3(x+1) = (x-2)^2(x+1)(5x-1)$
$G'(x) = 0$ when $x = 2$, $x = -1$, and $x = \frac{1}{5}$. Thus the critical numbers of G are 2, -1, and $\frac{1}{5}$.

8. $F(x) = (5+x)^3(2-x)^2$ ▷ A plot of F is shown at the right.



The domain of F is $(-\infty, +\infty)$. Because $(2-x)^2 = (x-2)^2$,

$$F(x) = (x+5)^3(x-2)^2$$

Differentiating, we have

$$F'(x) = (x+5)^3 D_x(x-2)^2 + (x-2)^2 D_x(x+5)^3$$
$$= (x+5)^3(2)(x-2) + (x-2)^2(3)(x+5)^2$$
$$= (x+5)^2(x-2)[(x+5)(2) + (x-2)(3)]$$
$$= (x+5)^2(x-2)(5x+4)$$

$F'(x)$ is defined at every real number. If $F'(0) = 0$, then either $x = -5$, $x = 2$, or $x = -\frac{4}{5}$. Because each of these numbers is in the domain of F, we conclude that -5, 2, and $-\frac{4}{5}$ are the critical numbers of F.

In Exercises 9–14, (a) Calculate the critical numbers. Check by plotting (b) f; and (c) NDER$(f(x), x)$.

9. $f(x) = x^4 + 11x^3 + 34x^2 + 15x - 2$. $f'(x) = 4x^3 + 33x^2 + 68x + 15 = (x+5)(4x+1)(x+3)$.
$f'(x) = 0$ when $x = -5$, $x = -\frac{1}{4}$, and $x = -3$. Thus -5, $-\frac{1}{4}$, -3 are the critical numbers of f.

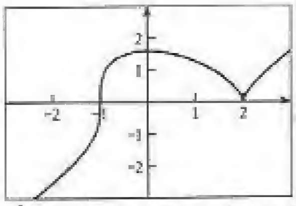
10. $f(x) = x^4 + 4x^3 - 2x^2 - 12x$; $f'(x) = 4x^3 + 12x^2 - 4x - 12 = 4[x^2(x+3) - (x+3)] = 4(x-1)(x+1)(x+3)$
$f'(x) = 0$ when $x = 1$, $x = -1$, and $x = -3$. Thus $1, -1, -3$ are the critical numbers of f.

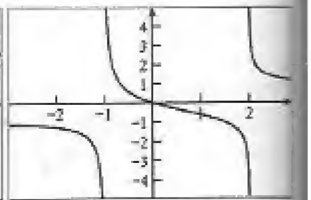
11. $f(t) = (t^2 - 4)^{2/3}$; $f'(t) = \frac{2}{3}(t^2-4)^{-1/3}(2t)$. $f'(-2)$ and $f'(2)$ do not exist and -2 and 2 are in the domain of f; $f'(t) = 0$ when $t = 0$. Thus $-2, 2, 0$ are the critical numbers of f.

12. $f(w) = (w^3 - 3w^2 + 4)^{1/3}$

▹ The domain of f is $(-\infty, +\infty)$. Plots of f and NDER(f) are shown below

$$f'(w) = \tfrac{1}{3}(w^3 - 3w^2 + 4)^{-2/3}(3w^2 - 6w) = \frac{w(w-2)}{(w^3-3w^2+4)^{2/3}} = \frac{w(w-2)}{[(w+1)(w-2)^2]^{2/3}} = \frac{w}{(w+1)^{2/3}(w-2)^{1/3}}$$





The factored form of the denominator $w^3 - 3w^2 + 4$ was found by trial and error using synthetic division. Because $f'(w)$ is not defined at -1 and at 2, both -1 and 2 are critical numbers of f. Because $f'(0) = 0$, then 0 is also a critical number of f.

13. $f(x) = \dfrac{x^2+4}{x-2}$; $f'(x) = \dfrac{2x(x-2) - (x^2+4)1}{(x-2)^2} = \dfrac{x^2 - 4x - 4}{(x-2)^2}$. f' is defined when f is defined and $f'(x) = 0$ when $x^2 - 4x - 4 = 0$; $x = 2 \pm 2\sqrt{2}$. Hence the critical numbers of f are $2 + 2\sqrt{2}$ and $2 - \sqrt{2}$.

14. $f(x) = \dfrac{x^2+2x+5}{x-1}$; $f'(x) = \dfrac{(2x+2)(x-1) - (x^2+2x+5)1}{(x-1)^2} = \dfrac{x^2-2x-7}{(x-1)^2}$. f' is defined when f is defined and $f'(x) = 0$ when $x^2 - 2x - 7 = 0$; $x = 1 \pm 2\sqrt{2}$. Hence the critical numbers of f are $1 + 2\sqrt{2}$ and $1 - \sqrt{2}$.

In Exercises 15–18, find the critical numbers of the function.

15. $f(x) = \sin 2x \cos 2x = \frac{1}{2}\sin 4x$; $f'(x) = 2\cos 4x$. $f'(x) = 0$ when $4x = \frac{1}{2}(2k+1)\pi$, where k is any integer.

▪ The critical numbers of f are all numbers $\frac{1}{8}(2k+1)\pi$, where k is any integer.

16. $f(x) = \sin 2x + \cos 2x$

▹ The domain of f is $(-\infty, +\infty)$. We have

$$f'(x) = 2\cos 2x - 2\sin 2x$$

Thus, $f'(x)$ is defined for all x. If $f'(x) = 0$, we have

$$2\cos 2x - 2\sin 2x = 0; \quad \sin 2x = \cos 2x$$

Dividing on both sides by $\cos 2x$, we obtain

$$\tan 2x = 1$$

Thus, $\quad 2x = \frac{1}{4}\pi + n\pi;\ x = \frac{1}{8}\pi + \frac{1}{2}n\pi$

▪ Each number of the form $\frac{1}{8}\pi + \frac{1}{2}n\pi$, where n is an integer, is a critical number.

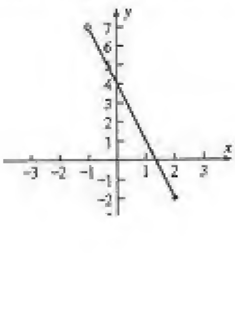
17. $F(x) = \sec^2 3x$; $F'(x) = 2\sec 3x \cdot \sec 3x \tan 3x(3) = 6\sec^2 3x \tan 3x$. $f'(x)$ does not exist when $3x = \frac{1}{2}(2k+1)\pi$ where k is any integer, but these values of x are not in the domain of f; $f'(x) = 0$ when $3x = k\pi$, where k is any integer. Thus the critical numbers of f are all numbers $\frac{1}{3}k\pi$, where k is any integer.

18. $G(x) = \tan^2 4x$; $G'(x) = 8\tan 4x \sec^2 4x$. $G'(x)$ does not exist when $4x = \frac{1}{2}(2k+1)\pi$ where k is any integer, but these values of x are not in the domain of G; $G'(x) = 0$ when $4x = k\pi$, where k is any integer.

▪ The critical numbers of G are all numbers $\frac{1}{4}k\pi$, where k is any integer.

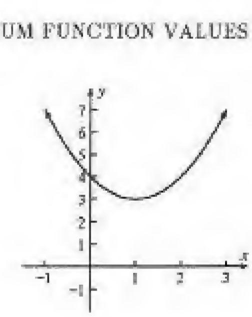
In Exercises 19–38, (a) sketch the graph of the function on the interval I. (b) Locate any extrema on the interval.

19. $f(x) = 4 - 3x$; $I = (-1, 2]$; $f'(x) = 3$
$f'(x)$ is never 0, so f has no relative extrema.
$f(2) = -2$ is an absolute minimum on I.
There is no absolute maximum on I because $\lim\limits_{x\to -1} f(x) = 7$ but $f(x) < 7$ on I.



$f(x) = x^2 - 2x + 4$; $I = (-\infty, +\infty)$

(a) The graph of the function is given at the right.



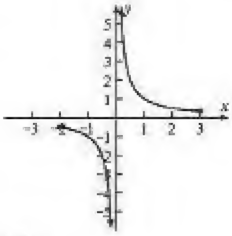
(b) $f'(x) = 2x - 2 = 2(x-1)$. $f'(1) = 0$ and 0 is in I.
Therefore 1 is the critical number of f in I.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to+\infty} f(x) = +\infty$

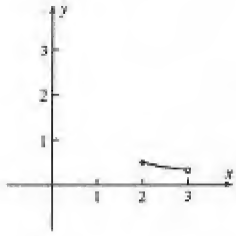
and so f has no absolute maximum on I.
$f(1) = 3$ is an absolute minimum on I.

$g(x) = \frac{1}{x}$, $I = [-2, 3]$

0 is in I and $\lim\limits_{x\to0^+} g(x) = +\infty$ and $\lim\limits_{x\to0^-} g(x) = -\infty$.
Hence g has neither an absolute maximum value nor an absolute minimum value on I.



$f(x) = \frac{1}{x}$; $I = [2, 3)$; $f'(x) = -\frac{1}{x^2}$

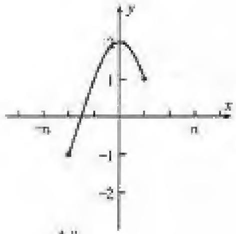


We see that $f'(x) \neq 0$ for all x in I. Furthermore, because $x^2 \neq 0$ for all x in I, then $f'(x)$ is defined for all x in I. We conclude that the function f does not have a critical number in I, and thus an absolute extremum, if it exists, must occur at an endpoint of I. Because $x \geq 2$ on I, and so $f(x) \leq \frac{1}{2}$ on I, and $f(2) = \frac{1}{2}$, then $\frac{1}{2}$ is the absolute maximum value of f on I. There is no absolute minimum value of f on I because $\lim\limits_{x\to3^-} f(x) = \frac{1}{3}$ but $f(x) < \frac{1}{3}$ on I.

$f(x) = 2\cos x$, $I = [-\frac{2}{3}\pi, \frac{1}{3}\pi]$; $f'(x) = 2\sin x$
$f'(0) = 0$ and 0 is in I. Therefore 0 is the critical number of f in I.

$f(-\frac{2}{3}\pi) = 2\cos(-\frac{2}{3}\pi) = 2(-\frac{1}{2}) = -1$;

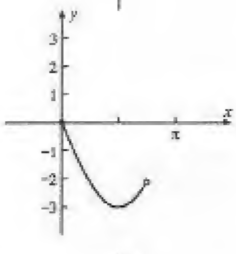
$f(0) = 2\cos 0 = 2 \cdot 1 = 2$; $f(\frac{1}{3}\pi) = 2\cos(\frac{1}{3}\pi) = 1$



The absolute minimum value of f on I is -1 and $f(-\frac{2}{3}\pi) = -1$.
The absolute maximum value of f on I is 2 and $f(0) = 2$.

$G(x) = -3\sin x$; $I = [0, \frac{3}{4}\pi)$

(a) The graph of the function is given at the right.

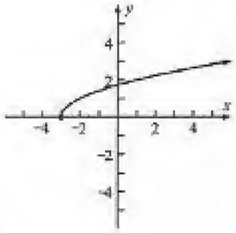


(b) $f'(x) = -3\cos x$. $f'(\frac{1}{2}\pi) = 0$ and $\frac{1}{2}\pi$ is in I.
Therefore $\frac{1}{2}\pi$ is the critical number of G in I.
$f(0) = -3\sin 0 = 0$; $f(\frac{1}{2}\pi) = -3\sin\frac{1}{2}\pi = -3$

and $\lim\limits_{x\to3/4\pi^-} G(x) = -3\sin\frac{3}{4}\pi = -\frac{3}{2}\sqrt{2} \approx -2.12$.

The absolute minimum value of f on I is -3 and $f(\frac{1}{2}\pi) = -3$.
The absolute maximum value of f on I is 0 and $f(0) = 0$.

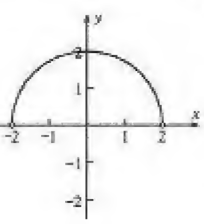
$f(x) = \sqrt{3+x}$, $I = [-3, +\infty)$; $f'(x) = \dfrac{1}{2\sqrt{3+x}}$
$f(-3) = 0$ and $f(x) > 0$ on I.
Thus the absolute minimum value of f on I is 0.
Because $\lim\limits_{x\to\infty} f(x) = +\infty$, there is no absolute maximum value of f on I.



$f(x) = \sqrt{4 - x^2}$; $I = (-2, 2)$

$f'(x) = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \dfrac{-x}{\sqrt{4 - x^2}}$

The only critical number of f in I is 0. Because $f(x) \leq 2$ on I and $f(0) = 2$, then the absolute maximum value of f on I is 2. There is no absolute minimum value of f on I because $\lim\limits_{x\to2^-} f(x) = 0$ but $f(x) > 0$ on I.

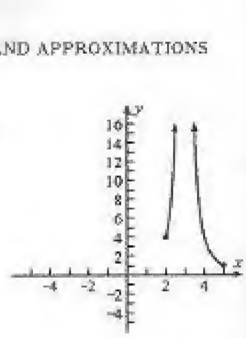


27. $h(x) = \dfrac{4}{(x-3)^2}$, $I = [2,5]$; $h'(x) = -\dfrac{8}{(x-3)^3}$

3 is in I and $\lim\limits_{x\to 3} h(x) = +\infty$. Therefore h has no absolute maximum value on I.

$h'(x)$ is never 0 so there are no critical numbers of h.

$h(2) = 4$ and $h(5) = 1$, so the absolute minimum value of h on I is 1.



28. $g(x) = \dfrac{3x}{9-x^2}$; $I = (-3,2)$

▹ (a) The graph of g is shown at the right.

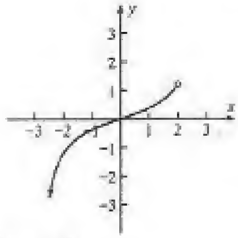
(b) $g'(x) = \dfrac{3(9-x^2)-3x(-2x)}{(9-x^2)^2} = \dfrac{3(9+x^2)}{(9-x^2)^2}$

-3 is an endpoint of I and $\lim\limits_{x\to -3^+} g(x) = -\infty$.

Therefore g has no absolute minimum value on I.

$g'(x)$ is never 0 so there are no critical numbers of g.

$\lim\limits_{x\to 2^-} g(x) = \frac{6}{5}$ and $g(x) < \frac{6}{5}$ on I. Therefore g has no absolute maximum value on I.



29. $F(x) = |x-4|+1$, $I = (0,6)$; $F'(x) = \dfrac{x-4}{|x-4|} = \operatorname{sgn}(x-4)$, $x \neq 4$.

$F'(4)$ does not exist and 4 is in I;

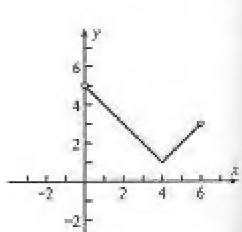
$F'(x)$ is never 0, so 4 is the critical number.

$F(0) = 5$, $F(4) = 1$, $F(6) = 3$

F has an absolute minimum value of 1 on I and $F(4) = 1$.

$\lim\limits_{x\to 0^+} F(x) = 5$ but $F(x) < 5$ for all x in I.

Hence F has no absolute maximum value on I.

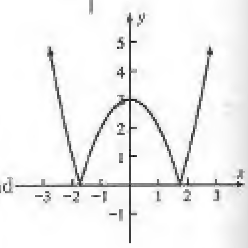


30. $f(x) = |4-x^2|$; $I = (-\infty,+\infty)$

Because $4-x^2 \geq 0$ if $-2 \leq x \leq 2$, and $4-x^2 < 0$ if $x \leq -2$ or $x > 2$, then

$$f(x) = \begin{cases} x^2-4 & \text{if } x < -2 \\ 4-x^2 & \text{if } -2 \leq x \leq 2 \\ x^2-4 & \text{if } x > 2 \end{cases} ; \quad f'(x) = \begin{cases} 2x & \text{if } x < -2 \\ -2x & \text{if } -2 < x < 2 \\ 2x & \text{if } x > 2 \end{cases}$$

Because $f'_-(-2) = -4$ and $f'_+(-2) = 4$, then $f'(-2)$ is not defined. Because $f'_-(2) = -4$ and $f'_+(2) = 4$, $f'(2)$ is not defined. Also, $f'(0) = 0$. Thus, the critical numbers of f are -2, 0, and 2. The absolute minimum value of f is 0 and it occurs at $x = 2$ and at $x = -2$. Because the values of $f(x)$ can be arbitrarily large, there is no absolute maximum value of f. There is a relative maximum value of 4 which occurs at the critical number $x = 0$.



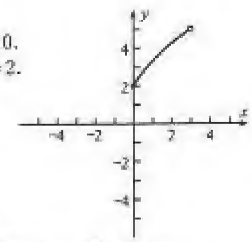
31. $g(x) = \sqrt{4+7x}$, $I = [0,3)$; $g'(x) = \dfrac{7}{2\sqrt{4+7x}}$

$g'(-\frac{4}{7})$ does not exist but $-\frac{4}{7}$ is not in I; $g'(x)$ is never 0.

The absolute minimum value of g on I is 2 and $g(0) = 2$.

Because $\lim\limits_{x\to 3^-} g(x) = 5$ and $g(x) < 5$ for all x in I,

there is no absolute maximum value of g on I.

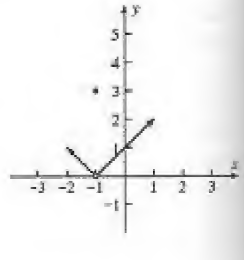


32. $\begin{cases} |x+1| & \text{if } x \neq -1 \\ 3 & \text{if } x = -1 \end{cases}$; $I = [-2,1]$

▹ Because $\quad \lim\limits_{x\to -1} f(x) = |0| = 0 \qquad (1)$

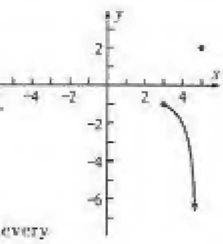
and $f(-1) = 3$, then f is not continuous at $x = -1$. Therefore $f'(-1)$ is not defined and thus -1 is a critical number of f. The graph of f on I is shown at the right. The absolute maximum value of f on I is 3 and it occurs at $x = -1$. Because Eq. (1) holds and $f(x) > 0$ on I,

there is no absolute minimum value of f on I.



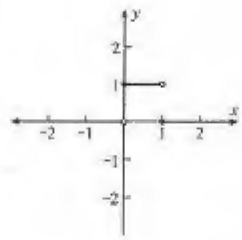
$f(x) = \begin{cases} \dfrac{2}{x-5} & \text{if } x \neq 5 \\ 2 & \text{if } x = 5 \end{cases}$, $I = [3, 5]$

Because $\lim\limits_{x \to 5^-} f(x) = -\infty$, there is no absolute minimum value of f on I.
The absolute maximum value of f on I is 2 and $f(5) = 2$.

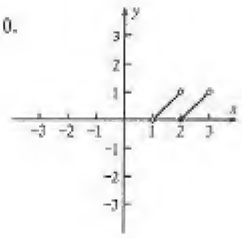


$F(x) = U(x) - U(x-1)$ where $U(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

$I = (-1, 1)$. $F(x) = (x \geq 0) - (x \geq 1)$
$= (0 \leq x < 1) + (x \geq 1) - (x \geq 1) = (0 \leq x < 1)$
$F'(x)$ is not defined at 0 and 1, and is 0 elsewhere. Thus every point of I is a critical point, which is useless. F has an absolute maximum value of 1 taken at every point of $[0, 1)$ and an absolute minimum value of 0 taken at every point of $(-1, 0)$.

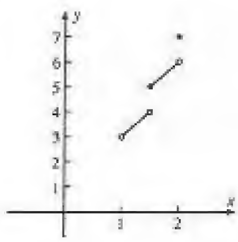


$f(x) = x - [\![x]\!]$, $I = (1, 3)$
The absolute minimum value of f on I is 0 and $f(2) = 0$.
There is no absolute maximum value of f on I because $\lim\limits_{x \to 2^-} f(x) = 1$ or because $\lim\limits_{x \to 3^-} f(x) = 1$ and $f(x) < 1$ for all x in I.

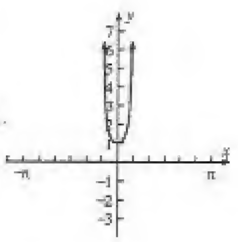


$h(x) = 2x + [\![2x - 1]\!]$; $I = (1, 2]$
(a) The graph of h is shown at the right.
(b) $h(x) = \begin{cases} 2x + 1 & \text{if } 1 < x < \frac{3}{2} \\ 2x + 2 & \text{if } \frac{3}{2} \leq x < 2 \\ 7 & \text{if } x = 2 \end{cases}$

$h'(x)$ is not defined at $\frac{3}{2}$ and is 2 elsewhere. Thus the critical number of h in I is $\frac{3}{2}$ and $h(\frac{3}{2}) = 5$. Because $\lim\limits_{x \to 1^+} h(x) = 3$ and $h(x) > 3$ in I, there is no absolute minimum value of h in I.
The absolute maximum value of h on I is 7 and $h(2) = 7$.



$g(x) = \sec 3x$, $I = [-\frac{1}{6}\pi, \frac{1}{6}\pi]$; $g'(x) = 3 \sec 3x \tan 3x$
$g'(0) = 0$ and 0 is in I. Thus 0 is the critical number.
Because $\lim\limits_{x \to \pi/6} g(x) = +\infty$, g has no absolute maximum value on I.
The absolute minimum value of g on I is 1 and $g(0) = 1$.

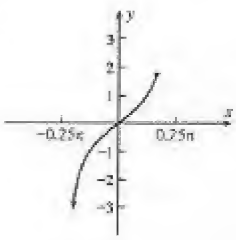


$f(x) = \tan 2x$; $I = [-\frac{1}{4}\pi, \frac{1}{6}\pi]$

Because $f(-\frac{1}{4}\pi) = \tan(-\frac{1}{2}\pi)$, which is not defined, f is discontinuous at $-\frac{1}{4}\pi$. Furthermore,

$$\lim_{x \to -\pi/4^+} f(x) = \lim_{x \to -\pi/4^-} \tan 2x = -\infty$$

Therefore f does not have an absolute minimum on I. $f'(x) = 2\sec^2 2x$
If $-\frac{1}{4}\pi < x < \frac{1}{6}\pi$ then $-\frac{1}{2}\pi < 2x < \frac{1}{3}\pi$ and $\sec 2x > 0$. Furthermore, although $\sec 2x$ is not defined if $x = -\frac{1}{4}\pi$, $-\frac{1}{4}\pi$ is not in the domain of f. Thus f has no critical numbers. Finally, $f(x) \leq f(\frac{1}{6}\pi) = \tan \frac{1}{3}\pi = \sqrt{3}$.
We conclude that the absolute maximum value of f is $\sqrt{3}$.



In Exercises 39–46, find the absolute extrema of the function on the closed interval and check by plotting.

$f(x) = x^4 - 8x^2 + 16$ (a) $I = [-4, 0]$; (b) $I = [-3, 2]$

$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2)$
Because f is continuous on each interval, f has an absolute maximum value and an absolute minimum value on each. $f'(x) = 0$ when $x = 0$, $x = -2$, $x = 2$.

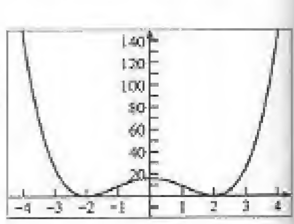
(a) The critical numbers of f on $[-4, 0]$ are -2 and 0.
$f(-4) = 144$, $f(-2) = 0$, $f(0) = 16$
The absolute minimum value is 0;
the absolute maximum value is 144.

(b) The critical numbers of f on $[-3, 2]$ are -2 and 0.
$f(-3) = 25$, $f(-2) = 0$, $f(0) = 16$, $f(2) = 0$
The absolute minimum value is 0;
the absolute maximum value is 25.

40. $f(x) = x^4 - 8x^2 + 16$ (a) $I = [0,4]$; (b) $I = [-1,4]$

▷ $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2)$

Because f is continuous on each interval, f has an absolute maximum value and an absolute minimum value on each. $f'(x) = 0$ when $x = 0$, $x = -2$, $x = 2$.



(a) The critical numbers of f on $[0,3]$ are 0 and 2.
$f(0) = 16$, $f(2) = 0$, $f(3) = 25$

▪ The absolute minimum value is 0; the absolute maximum value is 25.

(b) The critical numbers of f on $[-1,4]$ are 0 and 2
$f(-1) = 9$, $f(2) = 0$, $f(4) = 144$
The absolute minimum value is 0; the absolute maximum value is 144.

41. $f(t) = 2 \sin t$, $I = [-\pi,\pi]$; $f'(t) = 2 \cos t$

Because f is continuous on I, f has an absolute maximum value and an absolute minimum value on I.
$f'(-\frac{1}{2}\pi) = 0$ and $f'(\frac{1}{2}\pi) = 0$. Therefore, the critical numbers of f on I are $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.
$f(-\pi) = 0$, $f(-\frac{1}{2}\pi) = -2$, $f(\frac{1}{2}\pi) = 2$, $f(\pi) = 0$

▪ The absolute minimum value is -2; the absolute maximum value is 2.

42. $f(x) = \frac{1}{2} \csc 2x$, $I = [-\frac{1}{4}\pi, \frac{1}{6}\pi]$

Because $\lim\limits_{x \to 0^+} f(x) = +\infty$ and $\lim\limits_{x \to 0^-} f(x) = -\infty$, f does not have any absolute extreme value on I.

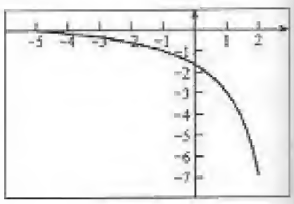
43. $g(w) = \dfrac{w}{w+2}$, $I = [-1,2]$; $g'(w) = \dfrac{1(w+2) - w}{(w+2)^2} = \dfrac{2}{(w+2)^2}$

Because g is continuous on I, g has an absolute maximum value and an absolute minimum value on I.
Because $g'(w)$ exists everywhere on I and $g'(w)$ is never 0, f has no critical numbers.
$g(-1) = -1$ and $g(2) = \frac{1}{2}$

▪ The absolute minimum value is -1; the absolute maximum value is $\frac{1}{2}$.

44. $f(x) = \dfrac{x+5}{x-3}$; $I = [-5,2]$ ▷ A plot is shown at the right.

$$f'(x) = \frac{(x-3)\cdot 1 - (x+5)\cdot 1}{(x-3)^2} = \frac{-8}{(x-3)^2}$$



Because $f'(x)$ is defined and $f'(x) \neq 0$ for all x in I, there are no critical numbers of f in I and so the absolute extrema of f must occur at the endpoints of $[-5,2]$. Because $f(-5) = 0$ and $f(2) = -7$, we conclude that

▪ 0 is the absolute maximum value of f on I and -7 is the absolute minimum value of f on I.

45. $f(x) = (x+1)^{2/3}$, $I = [-2,1]$; $f'(x) = \frac{2}{3}(x+1)^{-1/3}$

Because f is continuous on I, f has an absolute maximum value and an absolute minimum value on I.
$f'(-1)$ does not exist and -1 is in the domain of f; $f'(x)$ is never 0. Thus -1 is the only critical number.

$$f(-2) = 1,\ f(-1) = 0,\ f(1) = \sqrt[3]{4}$$

▪ The absolute minimum value is 0; the absolute maximum value is $\sqrt[3]{4}$.

46. $g(x) = 1 - (x-3)^{2/3}$, $I = [-5,4]$; $g'(x) = -\frac{2}{3}(x-3)^{-1/3}$

Because f is continuous on I, f has an absolute maximum value and an absolute minimum value on I.
$f'(3)$ does not exist and 3 is in the domain of f; $f'(x)$ is never 0. Thus 3 is the only critical number.

$$f(-5) = -3,\ f(3) = 3,\ f(4) = 0$$

▪ The absolute minimum value is -3; the absolute maximum value is 3.

In Exercises 47–52, plot to estimate the absolute extrema of the function on the closed interval, then use calculus

47. $f(x) = x^3 + 5x - 4$, $I = [-3,-1]$; $f'(x) = 3x^2 + 5$

Because f is continuous on I, f has an absolute maximum value and an absolute minimum value on I.
$f'(x)$ exists everywhere and $f'(x)$ is never 0. Therefore, f has no critical numbers.

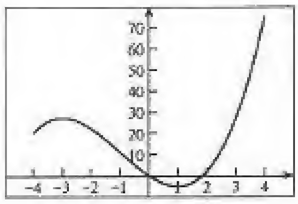
$$f(-3) = -46,\ f(-1) = -10$$

▪ The absolute minimum value is -46; the absolute maximum value is -10.

$g(x) = x^3 + 3x^2 - 9x$, $\mathrm{I} = [-4, 4]$ ▷ A plot is shown at the right.
Because g is continuous on I, g has an absolute maximum value and an absolute minimum value on I.
$g'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x+3)(x-1)$
$g'(x)$ exists everywhere. $g'(x) = 0$ when $x = -3$, $x = 1$.
The critical numbers of g on I are -3 and 1.
We evaluate f at the critical number and at each endpoint of I.

$$g(-4) = 20,\ g(-3) = 27,\ g(1) = -5,\ g(4) = 76$$

- The absolute minimum value is -5; the absolute maximum value is 76.



$g(t) = 2\sec\frac{1}{2}t$, $\mathrm{I} = [-\frac{1}{3}\pi, \frac{1}{2}\pi]$; $g'(t) = \sec\frac{1}{2}t\tan\frac{1}{2}t$
$g'(t)$ exists everywhere on I and $f'(0) = 0$. The critical number of g on I is 0.
$g(-\frac{1}{3}\pi) = 2\sec(-\frac{1}{6}\pi) = \frac{4}{3}\sqrt{3}$, $g(0) = 2$, $g(\frac{1}{2}\pi) = 2\sec\frac{1}{4}\pi = 2\sqrt{2}$
- The absolute minimum value is 2; the absolute maximum value is $2\sqrt{2}$.

$f(t) = 3\cos 2t$; $\mathrm{I} = [\frac{1}{6}\pi, \frac{3}{4}\pi]$; $f'(t) = -6\sin 2t$
$f'(t) = 0$ if $2t = n\pi$, that is, if $t = \frac{1}{2}n\pi$, where n is any integer. The only critical number of f in I is $\frac{1}{2}\pi$.
$f(\frac{1}{2}\pi) = 3\cos\pi = -3$ $f(\frac{1}{6}\pi) = 3\cos\frac{1}{3}\pi = \frac{3}{2}$ $f(\frac{3}{4}\pi) = 3\cos\frac{3}{2}\pi = 0$ Therefore, the absolute maximum value of f on I is $\frac{3}{2}$ which occurs at $\frac{1}{6}\pi$, and the absolute minimum value of f is -3 which occurs at $\frac{1}{2}\pi$.

$f(x) = (x-1)^{1/3} + 4$, $\mathrm{I} = [0, 2]$; $f'(x) = \frac{1}{3}(x-1)^{-2/3}$
$f'(1)$ is not defined and $f'(x)$ is never 0. The critical number of f on I is 1.

$$f(0) = 3,\ f(1) = 4,\ f(2) = 5$$

- The absolute minimum value is 3; the absolute maximum value is 5.

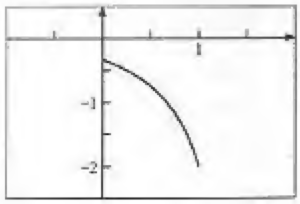
$f(x) = \dfrac{x+1}{2x-3}$, $\mathrm{I} = [0, 1]$ ▷ A plot is shown at the right.

$$f'(x) = \frac{1(2x-3) - 2(x+1)}{(2x-3)^2} = \frac{-5}{(2x-3)^2}$$

Because f is continuous on I, f has an absolute maximum value and an absolute minimum value on I. Because $f'(x)$ exists everywhere on I and $f'(x)$ is never 0, f has no critical numbers.

$$f(0) = -\tfrac{1}{3},\ f(1) = -2$$

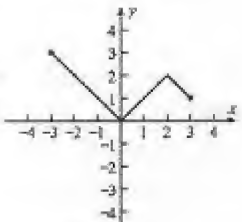
- The absolute minimum value is -2; the absolute maximum value is $-\frac{1}{3}$.



In Exercises 53–56, (a) sketch the graph on the closed interval I. (b) Determine the absolute extrema on I.

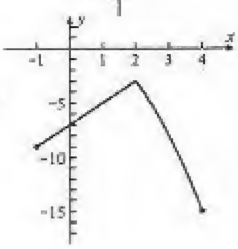
$f(x) = \begin{cases} |x| & \text{if } -3 \le x \le 2 \\ 4 - x & \text{if } 2 < x \le 3 \end{cases}$, $\mathrm{I} = [-3, 3]$; $f'(x) = \begin{cases} -1 & \text{if } x \in (-3, 0) \cup (2, 3) \\ 1 & \text{if } 0 < x < 3 \end{cases}$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} |x| = 2$; $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4 - x) = 2$
Because f is continuous on I, f has an absolute maximum value and an absolute minimum value on I. Because $f'(0)$ and $f'(2)$ do not exist and $f'(x)$ is never 0, 0 and 2 are the critical numbers of f.
$f(-3) = |3| = 3$, $f(0) = |0| = 0$, $f(2) = |2| = 2$, $f(3) = 4 - 3 = 1$
- The absolute minimum value is 0; the absolute maximum value is 3.

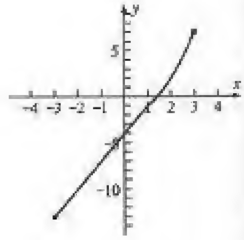


$f(x) = \begin{cases} 2x - 7 & \text{if } -1 \le x \le 2 \\ 1 - x^2 & \text{if } 2 < x \le 4 \end{cases}$, $\mathrm{I} = [-1, 4]$; $f'(x) = \begin{cases} 2 & \text{if } -1 < x < 2 \\ -2x & \text{if } 2 < x < 4 \end{cases}$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (2x - 7) = -3$; $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (1 - x^2) = -3$
Because f is continuous on I, f has an absolute maximum value and an absolute minimum value on I. Because $f'(2)$ does not exist and $f'(x)$ is never 0, 2 is the critical numbers of f.
$f(-1) = 2(-1) - 7 = -9$, $f(2) = 2(2) - 7 = -3$, $f(4) = 1 - 4^2 = -15$
- The absolute minimum value is -15; the absolute maximum value is -3.



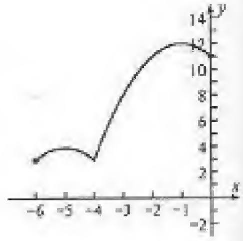
55. $F(x) = \begin{cases} 3x-4 & \text{if } -3 \le x < 1 \\ x^2-2 & \text{if } 1 \le x \le 3 \end{cases}$, $I = [-3, 3]$; $F'(x) = \begin{cases} 3 & \text{if } -3 < x < 1 \\ 2x & \text{if } 1 < x < 3 \end{cases}$



$\lim\limits_{x\to 1^-} F(x) = \lim\limits_{x\to 1^-} (3x-4) = -1$; $\lim\limits_{x\to 1^+} F(x) = \lim\limits_{x\to 1^+} (x^2-2) = -1$.
Thus F is continuous at 1. Because F is continuous on I, F has an absolute maximum value and an absolute minimum value on I.
$F'(x)$ is never 0 and $F'(x)$ exists except perhaps at 1; $F'_-(1) = 3$ and $F'_+(1) = 2$; therefore $F'(1)$ does not exist; thus 1 is a critical number of F.
$F(-3) = -13$, $F(1) = -1$, $F(3) = 7$

■ The absolute minimum value is -13; the absolute maximum value is 7.

56. $G(x) = \begin{cases} 4-(x+5)^2 & \text{if } -6 \le x \le -4 \\ 12-(x+1)^2 & \text{if } -4 < x \le 0 \end{cases}$; $I = [-6, 0]$



▷ $$G'(x) = \begin{cases} -2(x+5) & \text{if } -6 < x < -4 \\ -2(x+1) & \text{if } -4 < x < 0 \end{cases}$$

Because $G'(-5) = 0$ and $G'(-1) = 0$, then -5 and -1 are critical numbers of G. Furthermore,

$$G'_-(-4) = \lim_{x\to -4^-} G'(x) = \lim_{x\to -4^-} -2(x+5) = -2$$

and

$$G'_+(-4) = \lim_{x\to -4^+} G'(x) = \lim_{x\to -4^+} -2(x+1) = 6$$

Therefore, $G'(-4)$ is not defined, and so -4 is a critical number of G. We evaluate G at each endpoint of $[-6, 0]$ and at each critical number of G.

$$G(-6) = 3 \quad G(-5) = 4 \quad G(-4) = 3 \quad G(-1) = 12 \quad G(0) = 11$$

The absolute maximum value of G on I is 12 which occurs at -1, and the absolute minimum value of G on I is 3 which occurs at both -6 and -4. The graph is shown at the right.

57. If $\lim\limits_{x\to c} f(x) = L < 0$ then there is an an open interval I containing c such that $f(x) < 0$ for every $x \ne c$ in I.

▷ Choose $\epsilon = -\frac{1}{2}L$ in the definition of limit. The there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$ then $|f(x) - L| < -\frac{1}{2}L \Leftrightarrow \frac{1}{2}L < f(x) - L < -\frac{1}{2}L \Rightarrow f(x) < \frac{1}{2}L$.
Hence $f(x) < 0$ if $x \in I = (c-\delta, c+\delta)$ and $x \ne c$.

58. If the function f has a relative maximum value at c then $-f$ has a relative minimum value at c. By the first part of the proof, $-f'(c) = 0$; hence $f'(c) = 0$.

59. If f is differentiable everywhere and $f'(c) = 0$, can we conclude that f has a local extremum at c?

▷ No. If $f(x) = x^3$ then $f'(x) = 3x^2$ and $f'(0) = 0$ but f does not have a local extremum at 0.

60. If the function f has a local extremum at a number c, can we we conclude that $f'(c) = 0$?

▷ No. $f'(c)$ may not exist. If $f(x) = |x|$, then 0 is a local minimum of f but $f'(0)$ does not exist.

3.2 APPLICATIONS INVOLVING AN ABSOLUTE EXTREMUM ON A CLOSED INTERVAL

The procedure for solving the Exercises in this section is as follows.

1. Identify the variable for which you want to find an absolute extreme value. This is the dependent variable.
2. Express the dependent variable as a function of some other variable.
3. Verify that the function found in step 2 has a closed interval I for its domain and that the function is continuous on that closed interval.
4. Follow the steps given in Section 3.1 for finding the absolute extrema of a function on a closed interval.

Often it is helpful to first express the dependent variable as a function of two other variables. In this case, you must find another equation involving those two variables and then use the equation to eliminate one of the variables. Sometimes a figure is helpful, and sometimes the formulas from geometry given in Section 2.10 are helpful.

Exercises 3.2

1. Find a number in $I = [\frac{1}{3}, 2]$ such that the sum of the number and its reciprocal is (a) minimum; (b) maximum.
   - Let x be the number. We seek the extrema of $f(x) = x + \frac{1}{x}$; $f'(x) = 1 - \frac{1}{x^2}$. f is continuous and $f'(x)$ exists on any interval that does not contain 0. $f'(x) = 0$ when $1 = \frac{1}{x^2}$, $x^2 = 1$, $x = \pm 1$, but only $x = 1$ is in I.
   $f(\frac{1}{3}) = \frac{1}{3} + 3 = \frac{10}{3}$, $f(1) = 1 + 1 = 2$, $f(2) = 2 + \frac{1}{2} = \frac{5}{2}$. (a) $x = 1$ minimizes the sum (b) $x = \frac{1}{3}$ maximizes
2. Find the number in $I = [-1, 1]$ such that the number minus its square is (a) maximum; (b) minimum.
   - Let x be the number. We seek the extrema of $f(x) = x - x^2$; $f'(x) = 1 - 2x$. f is continuous and $f'(x)$ exists on any interval. $f'(x) = 0$ when $2x = 1$; $x = \frac{1}{2} \in I$. $f(-1) = -2$, $f(\frac{1}{2}) = \frac{1}{4}$, $f(1) = 0$ (a) $x = \frac{1}{2}$ maximizes the difference (b) $x = -1$ minimizes the difference.

Use calculus to solve the indicated exercise of Exercises 1.3.

3. Ex. 13. A field of length x m is enclosed with 240 m of fence. Its width is $\frac{1}{2}(240 - 2x) = 120 - x$ m and its area is $a(x) = (120 - x)x = 120x - x^2$, $0 \le x \le 120$. $a'(x) = 120 - 2x = 0$ when $x = 60$. $a(0) = 0$, $a(60) = 3600$, $a(120) = 0$. The area is greatest when $x = 60$ and the field is 60 m × 60 m.
4. Ex. 14. A garden of length x ft is enclosed with 100 ft of fence. Its width is $\frac{1}{2}(100 - 2x) = 50 - x$ ft and its area is $a(x) = (50 - x)x = 50x - x^2$, $0 \le x \le 50$. $a'(x) = 50 - 2x = 0$ when $x = 25$. $a(0) = 0$, $a(25) = 625$, $a(50) = 0$. The area is greatest when $x = 25$ and the garden is 25 ft × 25 ft.
5. Ex. 15. A field of length x m parallel to a river is enclosed with 240 m of fence. Its width is $\frac{1}{2}(240 - x)$ m and its area is $a(x) = \frac{1}{2}(240 - x)x = 120x - \frac{1}{2}x^2$, $0 \le x \le 240$. $a'(x) = 120 - x = 0$ when $x = 120$. $a(0) = 0$, $a(120) = 7200$, $a(240) = 0$. The area is greatest when $x = 120$ and the field is 120 m × 60 m.
6. Ex. 16. A garden of length x ft parallel to a house is enclosed with 100 ft of fence. Its width is $\frac{1}{2}(100 - x)$ ft and its area is $a(x) = \frac{1}{2}(100 - x)x = 50x - \frac{1}{2}x^2$, $0 \le x \le 100$. $a'(x) = 50 - x = 0$ when $x = 50$. $a(0) = 0$. $a(50) = 1250$, $a(100) = 0$. The area is greatest when $x = 50$ and the garden is 50 ft × 25 ft.
7. Ex. 17. When x in. squares are cut from the corners of 8 in. by 15 in. sheet and the sides are turned up, the volume is $V(x)$ in^3 where $V(x) = \ell wh = (8 - 2x)(15 - 2x)x = 4x^3 - 46x^2 + 120x$, $0 \le x \le 4$.
   $V'(x) = 12x^2 - 92x + 120 = 4(3x^2 - 23x + 30) = 4(3x - 5)(x - 6)$. $V'(x) = 0$ when $x = \frac{5}{3} \in I$ and $x = 6 \notin I$.
   $V(0) = 0$, $V(\frac{5}{3}) = \frac{4900}{27}$, $V(4) = 0$. The maximum volume is $\frac{4900}{27} \approx 90.74$ in^3 when $\frac{5}{3}$ in. squares are cut off.
8. Ex. 18. When x cm squares are cut from the corners of a 12 cm square and the sides are turned up, the volume is $V(x)$ cm^3 where $V(x) = \ell wh = (12 - 2x)^2 x = 4x^3 - 48x^2 + 144x$, $0 \le x \le 6$.
   $V'(x) = 12x^2 - 96x + 144 = 12(x^2 - 8x + 12) = 12(x - 2)(x - 6)$. $V'(x) = 0$ when $x = 2 \in I$ and $x = 6 \notin I$.
   $V(0) = 0$, $V(2) = 128$, $V(6) = 0$. The maximum volume is 128 cm^3 when 2 cm squares are cut off each corner.
9. Ex. 19. When x in. squares are cut from the corners of 12 in. by 15 in. sheet and the sides are turned up, the volume is $V(x)$ in^3 where $V(x) = \ell wh = (12 - 2x)(15 - 2x)x = 4x^3 - 54x^2 + 180x$, $0 \le x \le 6$. $V'(x) = 12x^2 - 108x + 180 = 12(x^2 - 9x + 15)$. $V'(x) = 0$ when $x = \frac{1}{2}(9 + \sqrt{21}) \approx 6.79 \notin I$ and $x = \frac{1}{2}(9 - \sqrt{21}) \approx 2.21 \in I$. $V(0) = 0$, $V(2.21) \approx 177$, $V(6) = 0$. The maximum volume is 177 in^3 when 2.21 in. squares are cut off.
10. Ex. 20. When x in. squares are cut from the corners of 40 cm by 50 cm sheet and the sides are turned up, the volume is $V(x)$ cm^3 where $V(x) = \ell wh = (40 - 2x)(50 - 2x)x = 4x^3 - 180x^2 + 2000x$, $0 \le x \le 20$. $V'(x) = 12x^2 - 360x + 2000 = 4(3x^2 - 90x + 500)$. $V'(x) = 0$ when $x = \frac{5}{3}(9 + \sqrt{21}) \approx 22.64 \notin I$ and $x = \frac{5}{3}(9 + \sqrt{21}) \approx 7.36 \in I$. $V(0) = 0$, $V(7.36) \approx 6564.2$, $V(20) = 0$. The maximum volume is 6564.2 cm^3; cut 7.36 cm squares.
11. Ex. 25. A box of length x in. with square cross section has 100 in. as the sum of its length and girth has volume $V(x)$ in^3. The width of the box is $\frac{1}{4}(100 - x)$ in. $V(x) = x[\frac{1}{4}(100 - x)]^2 = \frac{1}{16}(x^3 - 200x^2 + 10000x)$, $20 \le x \le 100$ (length ≥ width). $V'(x) = \frac{1}{16}(3x^2 - 400x + 10000) = \frac{1}{16}(3x - 100)(x - 100)$. $V'(x) = 0$ when $x = \frac{100}{3} \in I$ and $x = 100 \in I$. $V(20) = 8000$, $V(\frac{100}{3}) = \frac{1}{32}(\frac{200}{3})^2 \approx 9259$, $V(100) = 0$.
   - The largest box is about 33 by 17 by 17 in.

12. Ex. 26. The growth rate f bacteria/min of a colony is jointly proportional to the number x of bacteria and the number $1{,}000{,}000 - x$ of capacity. $f(x) = kx(1{,}000{,}000 - x) = k(1{,}000{,}000x - x^2)$, $0 \le x \le 1{,}000{,}000$. We do not need to determine k. $f'(x) = k(1{,}000{,}000 - 2x)$. $f'(x) = 0$ when $x = 500{,}000 \in \mathrm{I}$. $f(0) = 0$, $f(500{,}000) = 500{,}000^2k$, $f(1{,}000{,}000) = 0$. The growth rate is greatest when there are 500,000 bacteria.

13. Ex. 27. The growth rate f infected/day of an epidemic is jointly proportional to the number x of infected and the number $5{,}000 - x$ of capacity. $f(x) = kx(5{,}000 - x) = k(5000x - x^2)$, $0 \le x \le 5{,}000$. We do not need to determine k. $f'(x) = k(5000 - 2x)$. $f'(x) = 0$ when $x = 2500 \in \mathrm{I}$. $f(0) = 0$, $f(2500) = 2500^2k$, $f(5000) = 0$.

▪ The growth rate is greatest when 2,500 are infected.

14. Ex. 28. The base of a pyramidal tent is $2x$ m square and a triangular side has height $2.5 - x$ m. Because the slant height of the tent is $2.5 - x$, the height h satisfies $h^2 + x^2 = (2.5 - x)^2$, $h = \sqrt{6.25 - 5x}$. The volume is $V(x)$ m^3, where $V(x) = \frac{1}{3}(2x)^2\sqrt{6.25 - 5x} = \frac{4}{3}x^2(6.25 - 5x)^{1/2}$, $0 \le x \le 1.25$. We maximize $f(x) = V(x)^2 = \frac{16}{9}x^4(6.25 - 5x) = \frac{80}{9}x^4(1.25 - x)$. $f'(x) = \frac{80}{9}[4x^3(1.25 - x) + x^4(-1)] = \frac{80}{9}x^3(5 - 5x)$. $f'(x) = 0$ when $x = 0 \in \mathrm{I}$, $x = 1 \in \mathrm{I}$. $f(0) = 0$, $f(1) = \frac{20}{9}$, $f(1.25) = 0$. The volume is maximum when the base is 2m square.

15. In Exercise 2.2.37, a trip is for up to 250 students. For up to 150 students the cost is \$15 per student and decreases by \$0.05 per student for each student over 150. If $f(x)$ dollars is the gross income if x students make the trip, then $f(x) = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ 22.5x - .05x^2 & \text{if } 150 < x \le 250 \end{cases}$. $f'(x) = \begin{cases} 15 & \text{if } 0 < x < 150 \\ 22.5 - .1x & \text{if } 150 < x < 250 \end{cases}$.

$f'(150)$ might not exist and $f'(225) = 0$. $f(0) = 0$, $f(150) = 2250$, $f(225) = 2531.25$, $f(250) = 2500$

▪ The gross income is greatest when there are 225 students.

16. In Exercise 2.2.38, a trip is for up to 250 students. For up to 150 students the cost is \$15 per student and decreases by \$0.07 per student for each student over 150. How many students yield the largest income?

▹ We showed that if $f(x)$ dollars is the gross income if x students make the trip, then

$f(x) = \begin{cases} 15x & \text{if } 0 \le x \le 150 \\ 25.5x - .07x^2 & \text{if } 150 < x \le 250 \end{cases}$. $f'(x) = \begin{cases} 15 & \text{if } 0 < x < 150 \\ 25.5 - .14x & \text{if } 150 < x < 250 \end{cases}$

$f'(150)$ might not exist and $f'(182.1) = 0$. Because the number of students is an integer, we test 182 and 183.

$$f(0) = 0,\ f(150) = 2250,\ f(182) = 2322.32,\ f(183) = 2322.27\ f(250) = 2000$$

▪ The gross income is greatest when there are 182 students.

17. In Exercise 2.2.39, each tree produces 600 oranges, at up to 20 trees per acre. For each additional tree per acre, yield per tree decreases by 15 oranges. If $f(x)$ oranges are produced when there are number x of trees per acre, then $f(x) = \begin{cases} 600x & \text{if } 0 \le x \le 20 \\ 900x - 15x^2 & \text{if } 20 < x \le 60 \end{cases}$. $f'(x) = \begin{cases} 600 & \text{if } 0 < x < 20 \\ 900 - 30x & \text{if } 20 < x < 60 \end{cases}$

$f'(20)$ might not exist and $f'(30) = 0$. $f(0) = 0$, $f(20) = 12{,}000$, $f(30) = 13{,}500$, $f(60) = 0$

▪ The number of oranges is greatest when 30 trees per acre are planted.

18. In Exercise 2.2.40, a club's annual dues is \$100 per member, less \$0.50 for each member of 600 and plus \$0.50 for each member less than 600. If $f(x)$ dollars is the clubs revenue when there are x members, then $f(x) = 400x - \frac{1}{2}x^2$, $0 \le x \le 800$. $f'(x) = 400 - x = 0$ when $x = 400$. $f(0) = 0$, $f(400) = 80000$, $f(800) = 0$

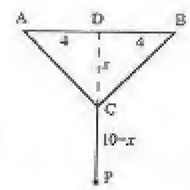
▪ The revenue is greatest when there are 400 members.

19. Find two nonnegative numbers whose sum is 12 such that (a) their product is an absolute maximum; (b) the sum of their squares is an absolute minimum.

▹ If x is the smaller number then $12 - x$ is the larger, $0 \le x \le 6$. (a) $f(x) = x(12 - x) = 12x - x^2$, $f'(x) = 12 - 2x = 0$ when $x = 6 \in \mathrm{I}$. $f(0) = 0$, $f(6) = 36$. The numbers are 6 and 6. (b) $g(x) = x^2 + (12 - x)^2$, $g'(x) = 2x - 2(12 - x) = 4x - 24 = 0$ when $x = 6$. $g(0) = 144$, $g(6) = 72$. The numbers are 6 and 6.

Suppose a weight is to be held 10 ft below a horizontal line AB by a wire in the shape of a Y. If the points A and B are 8 ft apart, what is the shortest total length of wire that can be used?

See the figure at the right. The weight is at point P and the line PD is perpendicular to AB. We are given that $|\overline{AB}| = 8$ and $|\overline{PD}| = 10$. Because triangle ABC is isosceles, then $|\overline{AD}| = |\overline{DB}| = 4$. Let



$x =$ the number of feet in $|\overline{CD}|$
$z =$ the number of feet in the total length of the wire

We want to find the absolute minimum value of z. We first express z as a function of x. We note that

$$z = |\overline{AC}| + |\overline{BC}| + |\overline{PC}| \qquad (1)$$

Because triangles ADC and BDC are right triangles, we have $|\overline{AC}| = \sqrt{x^2+16}$ and $|\overline{BC}| = \sqrt{x^2+16}$. And because $|\overline{PC}| = 10 - x$, from Eq. (1) we have

$$z(x) = 2\sqrt{x^2+16} + 10 - x$$

Because $|\overline{PC}| \geq 0$, $z(x)$ is defined and continuous on the closed interval $[0, 10]$.

$$z'(x) = \frac{2x}{\sqrt{x^2+16}} - 1$$

We note that $z'(x)$ is defined for all x. If $z'(x) = 0$, then

$$\frac{2x}{\sqrt{x^2+16}} = 1; \qquad 4x^2 = x^2 + 16; \qquad x^2 = \tfrac{16}{3}; \qquad x = \tfrac{4}{3}\sqrt{3}$$

We evaluate $z(x)$ at the critical number $\frac{4}{3}\sqrt{3}$ and at the endpoints 0 and 10.

$$z(0) = 2(4) + 10 = 18, \quad z(\tfrac{4}{3}\sqrt{3}) = 2\sqrt{\tfrac{64}{3}} + 10 - \tfrac{4}{3}\sqrt{3} = 10 + 4\sqrt{3} \approx 16.9, \quad z(10) = 2\sqrt{116} \approx 21.5$$

The shortest total length of wire that can be used is $(10 + 4\sqrt{3})$ ft.

A woman on an island at A, 4 km from the nearest point B on a beach, wishes to go in the least time to C, 6 km down the beach from B. She rows at 5 km/hr to point P between B and C, then walks at 8 km/hr to C.

Let x km be the distance from B to P. Then $\sqrt{x^2+16}$ km is the distance from P to A and $(6-x)$ km is the distance from P to C. The time of the trip equals the time of the part by water plus the time of the part by land. If the trip takes $f(x)$ hours, then

$$f(x) = \tfrac{1}{5}\sqrt{x^2+16} + \tfrac{1}{8}(6-x),\ 0 \leq x \leq 6; \ f'(x) = \frac{x}{5\sqrt{x^2+16}} - \frac{1}{8}$$

f is continuous on $[0, 6]$. Therefore f has an absolute minimum value on $[0, 6]$.

Set $f'(x) = 0$: $\dfrac{x}{5\sqrt{x^2+16}} - \dfrac{1}{8} = 0$; $8x = 5\sqrt{x^2+16}$; $64x^2 = 25x^2 + 400$; $39x^2 = 400$; $x^2 = \dfrac{400}{39}$; $x = \pm\dfrac{20}{\sqrt{39}}$

Because $f'(x)$ exists everywhere, $20/\sqrt{39} \approx 3.20$ is the only critical number of f in $[0, 6]$.

$$f(0) = \tfrac{31}{20} = 1.55, \ f(20/\sqrt{39}) = \tfrac{1}{10}\sqrt{39} + \tfrac{3}{4} \approx 1.37, \ f(6) = \tfrac{2}{5}\sqrt{13} \approx 1.44$$

The absolute minimum value of f is 1.37 when $x = 20/\sqrt{39}$. For the fastest route, P is 3.2 km from B.

A woman on an island at A, 4 km from the nearest point B on a beach, wishes to go in the least time to C, 3 km down the beach from B. She rows at 5 km/hr to point P between B and C, then walks at 8 km/hr to C.

Let x km be the distance from B to P. Then $\sqrt{x^2+16}$ km is the distance from P to A and $(3-x)$ km is the distance from P to C. The time of the trip equals the time of the part by water plus the time of the part by land. If the trip takes $f(x)$ hours, then

$$f(x) = \tfrac{1}{5}\sqrt{x^2+16} + \tfrac{1}{8}(3-x),\ 0 \leq x \leq 3; \ f'(x) = \frac{x}{5\sqrt{x^2+16}} - \frac{1}{8}$$

f is continuous on $[0, 3]$. Therefore f has an absolute minimum value on $[0, 3]$.

Set $f'(x) = 0$: $\dfrac{x}{5\sqrt{x^2+16}} - \dfrac{1}{8} = 0$; $8x = 5\sqrt{x^2+16}$; $64x^2 = 25x^2 + 400$; $39x^2 = 400$; $x^2 = \dfrac{400}{39}$; $x = \pm\dfrac{20}{\sqrt{39}}$

Because $f'(x)$ exists everywhere and $20/\sqrt{39} \approx 3.20$, there is no critical number of f in $[0, 3]$.

$$f(0) = \tfrac{47}{40} = 1.175, \ f(3) = 1$$

The absolute minimum value of f is 1 when $x = 3$. Hence, for the fastest route the point P should be C.

23. Find the dimensions of the right circular cylinder of greatest lateral surface area that can be inscribed in a sphere with a radius of 6 in.

▷ When r in. is the radius of the cylinder, let h in. be its height and $A(r)$ in^2 its lateral surface area. See the figure for Ex. 24. By the Pythagorean Theorem $r^2 + (\frac{1}{2}h)^2 = 36$; $h = 2\sqrt{36 - r^2}$.

Hence $A(r) = 2\pi rh = 4\pi r\sqrt{36 - r^2} = 4\pi\sqrt{36r^2 - r^4}$, $0 \le r \le 6$

$$A'(r) = \frac{4\pi(72r - 4r^3)}{2\sqrt{36r^2 - r^4}} = \frac{8\pi(18 - r^2)}{\sqrt{36 - r^2}}$$

$A'(-3\sqrt{2}) = 0$ and $A'(3\sqrt{2}) = 0$, and $A'(x)$ exists on $(0,6)$, so $3\sqrt{2}$ is the critical number. $A(0) = 0$, $A(3\sqrt{2}) = 72\pi$, $A(6) = 0$. Hence A has an absolute maximum value when $r = 3\sqrt{2}$ and

$h = 2\sqrt{36 - r^2} = 2\sqrt{36 - 18} = 2\sqrt{18} = 6\sqrt{2}$.

■ The right-circular cylinder of greatest lateral surface area has a radius of $3\sqrt{2}$ in. and a height of $6\sqrt{2}$ in.

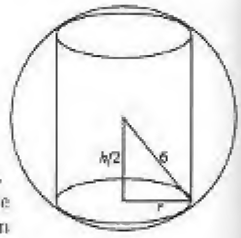
24. Find the dimensions of the right circular cylinder of greatest volume that can be inscribed in a sphere with a radius of 6 in.

▷ See the figure at the right. Let

r inches be the radius of the cylinder

h inches be the altitude of the cylinder

V cubic inches be the volume of the cylinder



We want to determine r and h so that V has an absolute maximum value.

The volume of a circular cylinder is given by

$$V = \pi r^2 h \qquad (1)$$

To express V in terms of one variable, we find another equation involving r and h. Because the center of the sphere bisects the altitude of the cylinder and the altitude of the cylinder is perpendicular to the base, we have from the Pythagorean theorem

$$(\tfrac{1}{2}h)^2 + r^2 = 6^2; \qquad r^2 = 36 - \tfrac{1}{4}h^2 \qquad (2)$$

Substituting from Eq. (2) into Eq. (1) we have

$$V(h) = \pi h(36 - \tfrac{1}{4}h^2) = \pi(36h - \tfrac{1}{4}h^3)$$

Because h cannot exceed the diameter of the sphere, V is defined on the closed interval $[0, 12]$. We have

$$V'(h) = \pi(36 - \tfrac{3}{4}h^2)$$

We note that $V'(h)$ is defined for all h. If $V'(h) = 0$, then

$$36 - \tfrac{3}{4}h^2 = 0; \qquad h^2 = 48; \qquad h = 4\sqrt{3}$$

We evaluate V at the critical number $4\sqrt{3}$ and at the endpoints 0 and 12.

$$V(0) = 0 \qquad V(4\sqrt{3}) = 4\sqrt{3}\pi \cdot 24 = 96\pi\sqrt{3} \qquad V(12) = 0$$

Therefore the greatest possible volume is $96\pi\sqrt{3}$ in^3. Furthermore, from Eq. (2), if $h^2 = 48$, then $r^2 = 24$ and so $r = 2\sqrt{6}$. Hence for the cylinder of greatest volume the altitude is $4\sqrt{3}$ in., and the radius is $2\sqrt{6}$ in.

25. The square of the distance from the point (x,y) on the circle $x^2 + y^2 = 9$ to $(4,5)$ is

$(x-4)^2 + (y-5)^2 = x^2 - 8x + 16 + y^2 - 10y + 25 = (x^2 + y^2 + 41) - 8x - 10y = 50 - 8x - 10y$

If $y \ge 0$, $y = \sqrt{9 - x^2}$. If $S_1(x)$ is the square of the distance from $(4,5)$ when $y \ge 0$, then

$S_1(x) = 50 - 8x - 10\sqrt{9 - x^2}$, $-3 \le x \le 3$; $S_1'(x) = -8 + \dfrac{10x}{\sqrt{9 - x^2}}$

If $y \le 0$, $y = -\sqrt{9 - x^2}$. If $S_2(x)$ is the square of the distance from $(4,5)$ when $y \le 0$, then

$S_2(x) = 50 - 8x + 10\sqrt{9 - x^2}$, $-3 \le x \le 3$; $S_2'(x) = -8 - \dfrac{10x}{\sqrt{9 - x^2}}$

Both S_1 and S_2 are continuous on $[-3, 3]$. Therefore the extreme-value theorem applies.

Set $S_1'(x) = 0$: $\dfrac{10x}{\sqrt{9 - x^2}} = 8 \Rightarrow x > 0$; $5x = 4\sqrt{9 - x^2}$; $25x^2 = 16(9 - x^2)$; $41x^2 = 144$, $x = \dfrac{12}{\sqrt{41}}$.

Set $S_2'(x) = 0$: $\dfrac{10x}{\sqrt{9 - x^2}} = -8 \Rightarrow x < 0$; $5x = -4\sqrt{9 - x^2}$; $25x^2 = 16(9 - x^2)$; $41x^2 = 144$, $x = -\dfrac{12}{\sqrt{41}}$.

When $x = \pm 3$, $\sqrt{9-x^2} = 0$. Thus $S_1(-3) = S_2(-3) = 50 - 8(-3) = 74$, $S_1(3) = S_2(3) = 50 - 8(3) = 26$.

When $x = \pm\frac{12}{\sqrt{41}}$, $\sqrt{9-x^2} = \sqrt{9-\frac{144}{41}} = \sqrt{\frac{225}{41}} = \frac{15}{\sqrt{41}}$. Thus

$S_1\left(\frac{12}{\sqrt{41}}\right) = 50 - 8\cdot\frac{12}{\sqrt{41}} - 10\cdot\frac{15}{\sqrt{41}} = 50 - \frac{246}{\sqrt{41}} = 50 - 6\sqrt{41}$ and

$S_2\left(-\frac{12}{\sqrt{41}}\right) = 50 - 8\left(-\frac{12}{\sqrt{41}}\right) + 10\cdot\frac{15}{\sqrt{41}} = 50 + \frac{246}{\sqrt{41}} = 50 + 6\sqrt{41}$.

Hence the absolute minimum distance is $\sqrt{50-6\sqrt{41}} = \sqrt{41 - 2\cdot 3\sqrt{41} + 9} = \sqrt{41} - 3$

and the absolute maximum distance is $\sqrt{50+6\sqrt{41}} = \sqrt{41} + 3$.

Alternatively, let $f(\theta)$ be the distance $|AP|$ when the measure of angle POA is θ. Then $|OA|^2 = 4^2 + 5^2 = 41$ and $|OP|^2 = 9$. By the law of cosines,

$|AP|^2 = |OP|^2 + |OA|^2 - 2|OP||OA|\cos\theta = 9 + 41 - 2\cdot 3\sqrt{41}\cos\theta = 50 - 6\sqrt{41}\cos\theta$

Hence $f(\theta) = \sqrt{50 - 6\sqrt{41}\cos\theta}$, $0 \le \theta \le \pi$. Because $-1 \le \cos\theta \le 1$,

(a) $f(\theta) \ge \sqrt{50 - 6\sqrt{41}(1)} = \sqrt{41} - 3 = a$ on $[0,\pi]$ and $f(0) = a$. a is the absolute minimum value.

(b) $f(\theta) \le \sqrt{50 - 6\sqrt{41}(-1)} = \sqrt{41} + 3 = b$ on $[0,\pi]$ and $f(\pi) = b$. b is the absolute minimum value.

Find the area of the largest rectangle with two vertices on the x-axis and two on the parabola $y = 9 - x^2$.

- Because the parabola is symmetric with respect to the y-axis, if one vertex is at $(x,0)$, $0 \le x \le 3$, another is at $(-x,0)$. The two on the parabola are $(\pm x, 9-x^2)$. We wish to maximize the area $A(x) = 2x(9-x^2)$ $= 18x - 2x^3$; $A'(x) = 18 - 6x^2$. $A'(x) = 0$ when $6x^2 = 18$, $x^2 = 3$, $x = \sqrt{3} \in I$ and $x = -\sqrt{3} \notin I$.

$$A(0) = 0,\ A(\sqrt{3}) = 2\sqrt{3}(9-3) = 12\sqrt{3},\ A(3) = 0$$

- The largest rectangle has area $12\sqrt{3} \approx 20.78$.

$f(x) = \frac{1}{2}x^2(k-x) = \frac{1}{2}kx^2 - \frac{1}{2}x^3$, $0 \le x \le k$; $f'(x) = kx - \frac{3}{2}x^2 = x(k - \frac{3}{2}x)$

f is continuous on $[0,k]$ so f has an absolute maximum value on $[0,k]$.

$f'(0) = 0$, $f'(\frac{2}{3}k) = 0$ and $f'(x)$ exists on $[0,k]$. Hence $\frac{2}{3}k$ is the only critical number.

$$f(0) = 0,\ f(\tfrac{2}{3}k) = \tfrac{2}{27}k^3,\ f(k) = 0$$

Thus f has an absolute maximum value at $\frac{2}{3}k$.

- The greatest decrease in blood pressure occurs when $\frac{2}{3}k$ mg of the drug is taken.

If $V(r)$ cm/sec is the velocity of air during a cough when r cm is the radius of the trachea, then $V(r) = kr^2(R-r)$ where R is the normal radius, k is a positive constant, and r is in $[\frac{1}{2}R, R]$. Determine the radius of the trachea which maximizes the velocity.

- $V(r) = k(Rr^2 - r^3)$. $V'(r) = k(2Rr - 3r^2) = kr(2R - 3r)$

V is continuous on $[\frac{1}{2}R, R]$ so f has an abeolute maximum value on $[\frac{1}{2}R, R]$.

$V'(0) = 0$, $f'(\frac{2}{3}R) = 0$ and $V'(x)$ exists on $[\frac{1}{2}R, R]$. Hence $\frac{2}{3}R$ is the only critical number.

$$V(\tfrac{1}{2}R) = k(R\cdot\tfrac{1}{4}R^2 - \tfrac{1}{8}R^3) = \tfrac{1}{8}kR^3,\ V(\tfrac{2}{3}R) = k(R\cdot\tfrac{4}{9}R^2 - \tfrac{8}{27}R^3) = \tfrac{4}{27}kR^3,\ V(R) = 0$$

Because $\frac{4}{27} > \frac{1}{8}$, V has an absolute maximum value at $\frac{2}{3}R$.

- The greatest velocity occurs when the radius is $\frac{2}{3}$ the normal radius.

Let $S(x)$ be the strength of the beam when its breadth is x cm. Because a diagonal of the beam is a diameter of the log, its depth is $\sqrt{144^2 - x^2}$ cm, and for some positive constant k,

$S(x) = kx(144^2 - x^2) = k(144^2x - x^3)$, $0 \le x \le 144$; $S'(x) = k(144^2 - 3x^2) = 3k(3\cdot 48^2 - x^2)$

S is continuous on $[0,144]$. Therefore S has an absolute maximum value on $[0,144]$.

$S'(x) = 0$ when $x = \pm 48\sqrt{3}$ and $S'(x)$ exists on $[0,144]$, so $48\sqrt{3}$ is the critical number.

$$S(0) = 0,\ S(48\sqrt{3}) = 663{,}552\sqrt{3}\ k,\ S(144) = 0$$

Thus S has an absolute maximum value when $x = 48\sqrt{3}$ and $\sqrt{144^2 - x^2} = 48\sqrt{6}$.

- The dimensions of the strongest beam are $48\sqrt{3} \approx 83.14$ cm by $48\sqrt{6} \approx 117.58$ cm.

Let $S(x)$ be the stiffness of the beam when its breadth is x cm. Because a diagonal of the beam is a diameter of the log, its depth is $\sqrt{4a^2 - x^2}$ cm, and for some positive constant k, $S(x) = kx(4a^2 - x^2)^{3/2}$, $0 \le x \le 2a$;

$S'(x) = k[(4a^2 - x^2)^{3/2} + \frac{3}{2}x(4a^2 - x^2)^{1/2}(-2x)] = k\sqrt{4a^2 - x^2}[(4a^2 - x^2) - x^2] = 4k\sqrt{4a^2 - x^2}(a^2 - x^2)$

S is continuous on $[0, 2a]$. Therefore S has an absolute maximum value on $[0, 2a]$.
$S'(x) = 0$ when $x = \pm a$ and $S'(x)$ exists on $[0, 2a]$, so a is the critical number.

$$S(0) = 0,\ S(a) = ka(3a^2)^{3/2} = 3\sqrt{3}ka^4,\ S(2a) = 0$$

Thus S has an absolute maximum value when $x = 2a$ and $\sqrt{4a^2 - x^2} = \sqrt{3}a$.

■ The dimensions of the stiffest beam are $2a$ cm by $\sqrt{3}a$.

31. Let x ft be the length of the piece of wire bent into the shape of a circle of radius $x/2\pi$. Then $(10 - x)$ ft is the length of the piece of wire bent into the shape of a square of side length $\frac{1}{4}(10 - x)$ft. Let $A(x)$ ft^2 be the combined area of the two figures.

$A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{10 - x}{4}\right)^2,\ 0 \le x \le 10;\ A'(x) = \frac{1}{2\pi}x - \frac{1}{8}(10 - x)$

A is continuous on $[0, 10]$ so A has an absolute maximum and an absolute minimum value.

Set $A'(x) = 0$: $\frac{1}{2\pi}x - \frac{5}{4} + \frac{1}{8}x = 0$; $4x + \pi x = 10\pi$; $x = \frac{10\pi}{\pi + 4}$

$A'\left(\frac{10\pi}{\pi + 4}\right) = 0$ and $A'(x)$ exists on $[0, 10]$ so the only critical number of A is $\frac{10\pi}{\pi + 4}$.

$A(0) = \frac{25}{4} = 6.25$, $A\left(\frac{10\pi}{\pi + 4}\right) = \frac{25}{\pi + 4} \approx 3.5$, $A(10) = \frac{25}{\pi} \approx 8.0$

Thus A has an absolute minimum value when $x = \frac{10\pi}{\pi + 4}$ and an absolute maximum when $x = 10$.

(a) The combined area is smallest when $x = \frac{10\pi}{\pi + 4}$; that is, when the radius of the circle is $\frac{5}{\pi + 4}$ ft and the length of a side of the square is $\frac{10}{\pi + 4}$ ft.

(b) The combined area is largest when $x = 10$; that is, when the radius of the circle is $\frac{5}{\pi}$ ft and there is no square.

32. A piece of wire 10 ft long is cut into two pieces. One piece is bent into the shape of an equilateral triangle and the other piece is bent into the shape of a square. How should the wire be cut so that (a) the combined area of the two figures is as small as possible; (b) the combined area of the two figures is as large as possible?

▷ If each side of the equilateral triangle is x ft long, then $(10 - 3x)$ ft is left for the square. Hence each side of the square has length $\frac{1}{4}(10 - 3x)$ ft. The number of square feet in the area of the equilateral triangle is $\frac{1}{4}\sqrt{3}\,x^2$ and the number of square feet in the area of the square is $[\frac{1}{4}(10 - 3x)]^2$. Hence, if A ft^2 is the combined area of the triangle and the square, then

$$A(x) = \tfrac{1}{4}\sqrt{3}\,x^2 + \tfrac{1}{16}(10 - 3x)^2$$

Because $x \ge 0$ and $10 - 3x \ge 0$, we find the absolute maximum and minimum values of A on the closed interval $[0, \frac{10}{3}]$. Differentiating, we have

$$A'(x) = \tfrac{1}{2}\sqrt{3}\,x + \tfrac{1}{8}(10 - 3x)(-3)$$

We note that $A'(x)$ is defined for all x. If $A'(x) = 0$, we have

$$\begin{aligned}
0 &= \tfrac{1}{2}\sqrt{3}\,x - \tfrac{1}{8}(10 - 3x)(-3)\\
0 &= 4\sqrt{3}\,x + (10 - 3x)(-3)\\
0 &= 4\sqrt{3}\,x - 30 + 9x\\
30 &= (4\sqrt{3} + 9)x\\
x &= \frac{30}{9 + 4\sqrt{3}} \cdot \frac{9 - 4\sqrt{3}}{9 - 4\sqrt{3}} = \frac{30(9 - 4\sqrt{3})}{81 - 48} = \tfrac{10}{11}(9 - 4\sqrt{3}) \approx 1.88
\end{aligned}$$

We evaluate A at the critical number 1.88 and at the endpoints 0 and $\frac{10}{3}$.

$A(0) = \frac{25}{4} = 6.25$ $\qquad A(1.88) \approx 2.72$ $\qquad A(\frac{10}{3}) = \frac{25}{9}\sqrt{3} \approx 4.81$

The absolute minimum and maximum of A are 2.72 and 6.25 respectively. Thus the combined area of the two figures is as small as possible if the side of the triangle is exactly $\frac{10}{11}(9 - 4\sqrt{3})$ ft long. Hence, the wire should be cut so that the piece that is bent into an equilateral triangle is $\frac{30}{11}(9 - 4\sqrt{3})$ ft ≈ 5.65 ft long. And to make the combined area as large as possible, all of the wire should be used for the square.

$R(\theta) = \dfrac{v_0{}^2 \sin 2\theta}{g}$, $0 \le \theta \le \frac{1}{2}\pi$, $0 \le 2\theta \le \pi$. The absolute maximum value of the sine function on $[0, \pi]$ is 1 when the independent variable is $\frac{1}{2}\pi$. Thus the absolute maximum value of $R(\theta)$ is $\dfrac{v_0{}^2}{g}$ when $\theta = \frac{1}{4}\pi$.

If a body of weight W pounds is dragged along a horizontal floor at a constant velocity by means of a force of magnitude F pounds and directed at an angle of θ radians with the plane of the floor, then F is given by the equation $F = \dfrac{kW}{k \sin\theta + \cos\theta}$ where k is a constant called the coefficient of friction and $0 < k < 1$. If $0 \le \theta \le \frac{1}{2}\pi$, find $\cos\theta$ when F is least.

Because $0 \le \theta \le \frac{1}{2}\pi$, both $\sin\theta$ and $\cos\theta$ are positive. Furthermore, k and W are positive. Therefore F is least when $g(\theta) = k \sin\theta + \cos\theta$ is greatest.

$$g'(\theta) = k\cos\theta - \sin\theta$$

Note that $g'(\theta)$ is defined for all θ. If $g'(\theta) = 0$, then

$$k\cos\theta = \sin\theta; \quad k^2\cos^2\theta = \sin^2\theta; \quad k^2\cos^2\theta = 1 - \cos^2\theta; \qquad (k^2+1)\cos^2\theta = 1$$

$$\cos^2\theta = \frac{1}{k^2+1}; \qquad \cos\theta = \frac{1}{\sqrt{k^2+1}}$$

The only critical number of F occurs when $\cos\theta = (k^2+1)^{-1/2}$. We evaluate $g(\theta)$ at the endpoints of the interval $[0, \frac{1}{2}\pi]$ and at the critical number.

$$g(0) = k\sin 0 + \cos 0 = 1$$

$$g(\tfrac{1}{2}\pi) = k\sin\tfrac{1}{2}\pi + \cos\tfrac{1}{2}\pi = k$$

Because $\sin\theta = k\cos\theta$ at the critical number, then

$$g(\theta) = k\sin\theta + \cos\theta = k(k\cos\theta) + \cos\theta = (k^2+1)\cos\theta = \frac{k^2+1}{\sqrt{k^2+1}} = \sqrt{k^2+1}$$

Because $\sqrt{k^2+1}$ is greater than both k and 1, we conclude that the maximum value of g and the minimum value of F occur when $\cos\theta = (k^2+1)^{-1/2}$.

C dollars is the cost when x machines are used to produce A and y machines are used to produce B, where $C = 3x^2 + 42y$. If 15 machines are working, how many should be used to produce each product for the cost to be least?

Because $y = 15 - x$, $C(x) = 3x^2 + 42(15 - x)$, $0 \le x \le 15$. $C'(x) = 6x - 42 = 0$ when $x = 7 \in I$. $C(0) = 630$, $C(7) = 483$, $C(15) = 675$. The cost is least when 7 machines produce A and 8 produce B.

(a) In Example 1, $C(x) = 12{,}500\sqrt{9 + x^2} + 10{,}000(k - x)$, where $k \ge 0$ and $C'(x) = 0$ when $x = \pm 4$. For what values of k will the absolute minimum value of C occur at a number in the open interval $(0, k)$?
(b) More generally, Let $f(x) = u\sqrt{a^2 + x^2} + v(b - x)$, $x \in [0, b]$, $u > v > 0$. If the absolute minimum value of f occur at a number in the open interval $(0, b)$, show that $av < b\sqrt{u^2 - v^2}$.

(a) If the absolute minimum value of C occurs in the interval $(0, k)$ then the critical number 4 must be in that interval, that is $k > 4$.
(b) To locate the critical number, we find the derivative, set it to zero and solve for x.

$f'(x) = \dfrac{ux}{\sqrt{a^2 + x^2}} - v = 0$ when $\dfrac{ux}{\sqrt{a^2 + x^2}} = v$; $ux = v\sqrt{a^2 + x^2}$; $u^2x^2 = v^2(a^2 + x^2)$; $u^2x^2 = a^2v^2 + v^2x^2$;
$u^2x^2 - v^2x^2 = a^2v^2$; $(u^2 - v^2)x^2 = a^2v^2$; $x^2 = \dfrac{a^2v^2}{u^2 - v^2}$; $x = \pm\dfrac{av}{\sqrt{u^2 - v^2}}$

If the absolute minimum value of f occurs in the interval $I = (0, b)$ then the critical number must be in I, that is, the positive root must be less than b.

$\dfrac{av}{\sqrt{u^2 - v^2}} < b$; $av < \sqrt{u^2 - v^2}$

3.3 ROLLE'S THEOREM AND THE MEAN-VALUE THEOREM

3.3.1 Rolle's Theorem Let f be a function such that

(i) it is continuous on the closed interval $[a, b]$
(ii) it is differentiable on the open interval (a, b)
(iii) $f(a) = 0$ and $f(b) = 0$

Then there is a number c in the open interval (a, b) such that $f'(c) = 0$.

3.3.2 Mean-Value Theorem Let f be a function such that

(i) it is continuous on the closed interval $[a,b]$
(ii) it is differentiable on the open interval (a,b)

Then there is a number c in the open interval (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}, \text{ or equivalently, } f(b)-f(a) = (b-a)f'(c)$$

The conclusion of Rolle's theorem states that on the graph of f, between the points where $x=a$ and $x=b$, there is a point at which the line tangent to the curve is horizontal. The conclusion of the mean-value theorem states that on the graph of f, between the points where $x=a$ and $x=b$, there is a point at which the line tangent to the curve is parallel to the line joining these points. Note that Rolle's theorem is actually a special case of the mean-value theorem. The mean-value theorem is among the most important theorems of the calculus because it is used in the proof of many theorems. In particular:

3.3.3 Theorem If f is a function such that $f'(x)=0$ for all values of x in an interval I, then f is constant on I.

Also, taking the limit as b approaches a, and so c approaches a, we have

3.3.4 Theorem If f is continuous at a and $\lim\limits_{x\to a} f'(x)$ exists, then $f'(a)$ exists and $\lim\limits_{x\to a} f'(x) = f'(a)$.

Theorem 3.3.4 also applies to one-sided limits as stated in Section Summary 2.2.

Exercises 3.3

In Exercises 1–4, verify that conditions (i), (ii), and (iii) of the hypothesis of Rolle's theorem are satisfied by the function on the indicated interval. Then find a suitable value for c that satisfies the conclusion of Rolle's theorem. Support your choice of c by plotting the graph of f and the horizontal tangent line at $(c, f(c))$.

1. $f(x) = x^2 - 4x + 3$, $[1,3]$; $f'(x) = 2x - 4$
(i) Because f is a polynomial function, it is continuous everywhere. Thus f is continuous on $[1,3]$.
(ii) f is differentiable everywhere. Thus f is differentiable on $(1,3)$.
(iii) $f(1) = 1 - 4 + 3 = 0$, and $f(3) = 9 - 12 + 3 = 0$.
By Rolle's theorem there exists a number c in $(1,3)$ such that $f'(c) = 0$.
$f'(c) = 2c - 4$. Set $f'(c) = 0$: $2c - 4 = 0$; $c = 2$. Because $2 \in (1,3)$, 2 qualifies as c.

2. $f(x) = x^3 - 2x^2 - x + 2$, $[1,2]$; $f'(x) = 3x^2 - 4x - 1$
(i) Because f is a polynomial function, it is continuous everywhere. Thus f is continuous on $[1,2]$.
(ii) f is differentiable everywhere. Thus f is differentiable on $(1,2)$.
(iii) $f(1) = 1 - 2 - 1 + 2 = 0$, and $f(2) = 8 - 8 - 2 + 2 = 0$
By Rolle's theorem there exists a number c in $(1,3)$ such that $f'(c) = 0$.
Set $f'(c) = 0$: $3c^2 - 4c - 1 = 0$; $c = c_1 = \frac{1}{3}(4 - \sqrt{7}) \approx 0.45$ and $c = c_2 = \frac{1}{3}(4 + \sqrt{7}) \approx 2.21$.
Because $c_1 \in (1,2)$, c_1 qualifies as c.

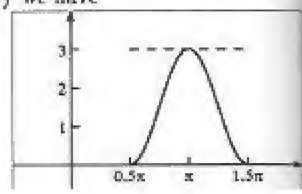
3. $f(x) = \sin 2x$, $[0, \frac{1}{2}\pi]$; $f'(x) = 2\cos 2x$
(i) Because the sine function is continuous everywhere, then f is continuous everywhere. Therefore f is continuous on $[0, \frac{1}{2}\pi]$.
(ii) f is differentiable everywhere. Thus f is differentiable on $(0, \frac{1}{2}\pi)$.
(iii) $f(0) = \sin 0 = 0$, and $f(\frac{1}{2}\pi) = \sin \pi = 0$
By Rolle's theorem there exists a number c in $(0, \frac{1}{2}\pi)$ such that $f'(c) = 0$.
$f'(c) = 2\cos 2c$. Set $f'(c) = 0$: $2\cos 2c = 0$; $2c = \frac{1}{2}\pi$; $c = \frac{1}{4}\pi$.
Because $\frac{1}{4}\pi \in (0, \frac{1}{2}\pi)$, $\frac{1}{4}\pi$ qualifies as c.

4. $f(x) = 3\cos^2 x$; $[\frac{1}{2}\pi, \frac{3}{2}\pi]$

▸ Because the cosine function is continuous for all x, then f is continuous on the closed interval $[\frac{1}{2}\pi, \frac{3}{2}\pi]$ and condition (i) of the hypothesis of Rolle's theorem is satisfied. Differentiating f we have

$$f'(x) = -6\cos x \sin x$$

Because both the sine and cosine function are defined everywhere, then $f'(x)$ is defined in the open interval $(\frac{1}{2}\pi, \frac{3}{2}\pi)$ and condition (ii) of the hypothesis is satisfied. Furthermore,



$$f(\tfrac{1}{2}\pi) = 3\cos^2 \tfrac{1}{2}\pi = 3(0^2) \qquad f(\tfrac{3}{2}\pi) = 3\cos^2 \tfrac{3}{2}\pi = 3(0^2)$$

and so condition (iii) is satisfied. To find a suitable value for c, we set $f'(c) = 0$ and get

$$0 = -6\cos c \sin c$$
$$\cos c = 0 \text{ or } \sin c = 0$$

Because $\sin \pi = 0$ and π is in the open interval $(\frac{1}{2}\pi, \frac{3}{2}\pi)$, we take $c = \pi$.

Exercises 5–10, (a) Plot the graph of the function on the indicated interval; (b) test the three conditions (i), (ii), and (iii) of the hypothesis of Rolle's theorem, and determine which conditions are satisfied and which, if any, are not satisfied; and (c) if the three conditions in part (b) are satisfied, determine a point at which there is a horizontal tangent line.

5. $f(x) = x^{4/3} - 3x^{1/3}$, $[0,3]$; $f'(x) = \frac{4}{3}x^{1/3} - x^{-2/3}$

(b) (i) f is continuous everywhere. Thus f is continuous on $[0,3]$.

(ii) f is differentiable at every number but 0, so f is differentiable on $(0,3)$.

(iii) $f(0) = 0$ and $f(3) = 0$.

(c) By Rolle's theorem there is a number c in $(0,3)$ such that $f'(c) = 0$; that is $\frac{4}{3}c^{1/3} - c^{-2/3} = 0$; $4c - 3 = 0$; $c = \frac{3}{4}$

Thus there is a horizontal tangent line at the point $(\frac{3}{4}, -\frac{9}{8}\sqrt[3]{6})$.

6. $f(x) = x^{3/4} - 2x^{1/4}$, $[0,4]$; $f'(x) = \frac{3}{4}x^{-1/4} - \frac{1}{2}x^{-3/4}$

(b) (i) f is continuous everywhere. Thus f is continuous on $[0,4]$.

(ii) f is differentiable at every number but 0, so f is differentiable on $(0,3)$.

(iii) $f(0) = 0$ and $f(4) = 2\sqrt{2} - 2\sqrt{2} = 0$.

(c) By Rolle's theorem there is a number c in $(0,4)$ such that $f'(c) = 0$; that is

$\frac{3}{4}c^{-1/4} - \frac{1}{2}c^{-3/4} = 0$; $3c^{1/2} - 2 = 0$; $c^{1/2} = \frac{2}{3}$; $c = \frac{4}{9}$

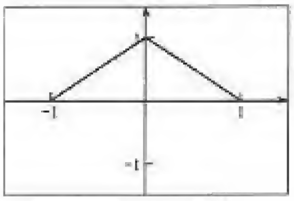
Thus there is a horizontal tangent line at the point $(\frac{4}{9}, -\frac{4}{9}\sqrt{6})$

7. $f(x) = \dfrac{x^2 - x - 12}{x - 3}$, $[-3,4]$

(b) Because $f(3)$ does not exist, f is not continuous on $[-3,4]$.
Thus condition (i) of Rolle's theorem is not satisfied.

8. $f(x) = 1 - |x|$; $[-1,1]$

(a) A plot of the graph of f on $[-1,1]$ is shown at the right.



(b) Because f is continuous in $[-1,1]$, condition (i) of the hypothesis of Rolle's theorem is satisfied. Because the absolute-value function is not differentiable at 0, f is not differentiable on $(-1,1)$, and thus condition (ii) is not satisfied. Because $f(1) = 0$ and $f(-1) = 0$, condition (iii) is satisfied.

(c) Because one of the conditions of the hypothesis of Rolle's theorem is not satisfied, we cannot conclude that there is a point at which there is a horizontal tangent. Indeed, as the figure illustrates, there is no such point.

9. $f(x) = \begin{cases} x^2 - 4 & \text{if } x < 1 \\ 5x - 8 & \text{if } 1 \le x \end{cases}$, $[-2, \frac{4}{5}]$; $f'(x) = \begin{cases} 2x & \text{if } x < 1 \\ 5 & \text{if } 1 < x \end{cases}$

(b) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^2 - 4) = -3$; $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (5x - 8) = -3$

Because $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$, then $\lim\limits_{x \to 1} f(x) = -3 = f(1)$.

Thus f is continuous at 1 and so f is continuous on $[-2, \frac{4}{5}]$.

$$f'_-(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 4 - (-3)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^-} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1^-} (x + 1) = 2$$

$$f'_+(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{5x - 8 - (-3)}{x - 1} = \lim_{x \to 1^+} \frac{5x - 5}{x - 1} = \lim_{x \to 1^+} \frac{5(x - 1)}{x - 1} = \lim_{x \to 1^+} 5 = 5$$

Alternative computation for $f'_-(1)$ and $f'_+(1)$:

$f'_-(1) = \lim\limits_{x \to 1^-} f'(x) = \lim\limits_{x \to 1^-} 2x = 2$; $f'_+(1) = \lim\limits_{x \to 1^+} f'(x) = \lim\limits_{x \to 1^+} 5 = 5$

Because $f'_-(1) \neq f'_+(1)$, then $f'(1)$ does not exist. Therefore f is not differentiable on$(-2, \frac{8}{5})$. Thus condition (ii) of Rolle's theorem is not satisfied. However, $f'(0) = 0$.

10. $f(x) = \begin{cases} 3x+6 & \text{if } x < 1 \\ x-4 & \text{if } x \geq 1 \end{cases}$; $[-2,4]$

(b) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (3x+6) = 9$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x-4) = -3$

Because f is not continuous at 1, then f is not continuous on $[-2,4]$.

Thus condition (i) of Rolle's theorem is not satisfied.

In Exercises 11–20, verify that the hypothesis of the mean-value theorem is satisfied for the function on the indicated interval. Then find a suitable value for c that satisfies the conclusion of the mean-value theorem. Check by plotting the graph of f on $[a,b]$, the tangent line at $(c, f(c))$, and the secant line through $(a, f(a))$ and $(b, f(b))$ and noting that the lines are parallel.

▷ On your graphics calculator, use TanLn(expression,value) and Line(xbeg,ybeg,xend,yend), respectively.

11. $f(x) = x^2 + 2x - 1$, $[0,1]$; $f'(x) = 2x + 2$

(i) Because f is a polynomial function, it is continuous everywhere. Thus f is continuous on $[0,1]$.

(ii) f is differentiable everywhere. Hence f is differentiable on $(0,1)$.

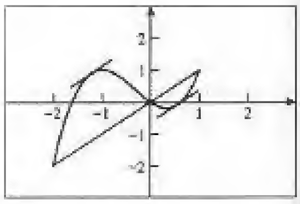
By the mean-value theorem, there exists a number c in $(0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0};\ 2c + 2 = \frac{2 - (-1)}{1};\ 2c + 2 = 3;\ c = \tfrac{1}{2}$$

Because $\frac{1}{2} \in (0,1)$, $\frac{1}{2}$ qualifies as c.

12. $f(x) = x^3 + x^2 - x$, $[-2,1]$

▷ $f'(x) = 3x^2 + 2x - 1$



(i) Because f is a polynomial function, it is continuous everywhere. Thus f is continuous on $[-2,1]$.

(ii) f is differentiable everywhere. Hence f is differentiable on $(-2,1)$.

By the mean-value theorem, there exists a number c in $(0,1)$ such that

$f'(c) = \dfrac{f(1) - f(-2)}{1 - (-2)};\ 3c^2 + 2c - 1 = \dfrac{1 - (-2)}{3} = 1;\ 3c^2 + 2c - 2 = 0;$

$c = c_+ = \frac{1}{3}(-1 + \sqrt{7}) \approx 0.55$ and $c = c_- = \frac{1}{3}(-1 - \sqrt{7}) \approx -1.22$

Because c_+ and $c_- \in (-2,1)$, both c_+ and c_- qualify as c.

13. $f(x) = x^{2/3}$, $[0,1]$; $f'(x) = \frac{2}{3}x^{-1/3}$

(i) f is continuous everywhere, and so continuous on $[0,1]$.

(ii) f is differentiable everywhere except at 0. Thus f is differentiable on $(0,1)$.

By the mean-value theorem, there exists a number c in $(0,1)$ such that

$f'(c) = \dfrac{f(1) - f(0)}{1 - 0};\ \dfrac{2}{3c^{1/3}} = \dfrac{1 - 0}{1};\ \frac{2}{3} = c^{1/3};\ c = \frac{8}{27}$. Because $\frac{8}{27} \in (0,1)$, $\frac{8}{27}$ qualifies as c.

14. $f(x) = \dfrac{x^2 + 4x}{x - 7}$, $[2,6]$. $f'(x) = \dfrac{(2x+4)(x-7) - (x^2+4x)}{(x-7)^2} = \dfrac{x^2 - 14x - 28}{(x-7)^2}$

(i) f is continuous at every number but 7, and so continuous on $[2,6]$.

(ii) f is differentiable everywhere except at 7. Thus f is differentiable on $(2,6)$.

By the mean-value theorem, there exists a number c in $(2,6)$ such that

$f'(c) = \dfrac{f(6) - f(2)}{6 - 2};\ \dfrac{c^2 - 14c - 28}{(c-7)^2} = \dfrac{-60 + 12/5}{4};\ 1 - \dfrac{77}{(c-7)^2} = -\dfrac{72}{5};\ -\dfrac{77}{(c-7)^2} = -\dfrac{77}{5};\ (c-7)^2 = 5$

$c = 7 + \sqrt{5}$ and $c = 7 - \sqrt{5}$. Because $7 - \sqrt{5} \in (2,6)$, $7 - \sqrt{5}$ qualifies as c.

15. $f(x) = \sqrt{1 + \cos x}$, $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$; $f'(x) = \dfrac{-\sin x}{2\sqrt{1 + \cos x}}$

(i) $1 + \cos x \geq 0$ on $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Thus f is continuous on $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

(ii) $f'(x)$ exists for all x in $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Thus f is differentiable on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

By the mean-value theorem, there exists a number c in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ such that

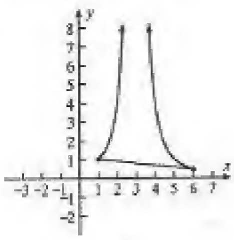
$$f'(c) = \frac{f(\frac{1}{2}\pi) - f(-\frac{1}{2}\pi)}{\frac{1}{2}\pi - (-\frac{1}{2}\pi)};\ \frac{-\sin c}{2\sqrt{1 + \cos c}} = \frac{1 - 1}{\pi};\ \frac{-\sin c}{2\sqrt{1 + \cos c}} = 0;\ \sin c = 0;\ c = 0$$

Because $0 \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, 0 qualifies as c.

In Exercises 21–24, for the given function, there is no number c in the open interval (a,b) that satisfies the conclusion of the mean-value theorem. Determine which part of the hypothesis of the mean-value theorem fails to hold. Sketch the graph of f and the line through the points $(a, f(a))$ and $(b, f(b))$.

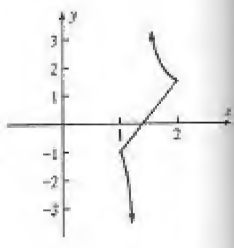
21. $f(x) = \dfrac{4}{(x-3)^2}$, $a = 1$, $b = 6$

▷ f is not defined at 3.
Thus f is discontinuous on $[1, 6]$.



22. $f(x) = \dfrac{2x-1}{3x-4}$; $a = 1$, $b = 2$.

▷ f is not defined at $\frac{4}{3}$.
Thus f is discontinuous on $[1, 2]$.



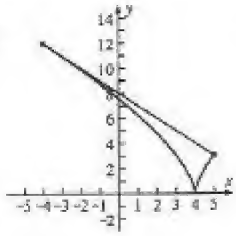
23. $f(x) = 3(x-4)^{2/3}$, $a = -4$, $b = 5$

$f'(x) = 2(x-4)^{-1/3}$

$$f'_-(4) = \lim_{x \to 4^-} \frac{3(x-4)^{2/3} - 0}{x-4} = \lim_{x \to 4^-} \frac{3}{(x-4)^{1/3}} = -\infty$$

$$f'_+(4) = \lim_{x \to 4^+} \frac{3(x-4)^{2/3}}{x-4} = \lim_{x \to 4^+} \frac{3}{(x-4)^{1/3}} = +\infty$$

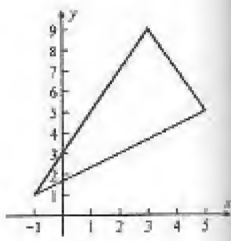
Therefore, f is not differentiable at 4;
so f is not differentiable on $(-4, 5)$.



24. $f(x) = \begin{cases} 2x+3 & \text{if } x < 3 \\ 15-2x & \text{if } 3 \le x \end{cases}$; $a = -1$, $b = 5$

▷ $f'(x) = \begin{cases} 2 & \text{if } x < 3 \\ -2 & \text{if } x > 3 \end{cases}$

By Theorem 3.3.4, $f'_-(3) = 2$ and $f'_+(3) = -2$. Thus $f'(3)$ does not exist and so f is not differentiable on the open interval $(-1, 5)$. Hence, condition (ii) of the hypothesis of the mean-value theorem is not satisfied. The graph of f on $[-1, 5]$ is shown at the right. Note that there is no point on the graph of f at which the tangent line to the graph is parallel to L, the line through the endpoints of the graph.



25. $f(x) = x^4 - 2x^3 + 2x^2 - x$; $f'(x) = 4x^3 - 6x^2 + 4x - 1$
Because f is a polynomial function, f is continuous and differentiable everywhere; $f(0) = 0$ and $f(1) = 0$. Thus Rolle's theorem holds on $[0, 1]$. Hence, there is some number $c \in (0, 1)$ such that $f'(c) = 0$. Thus, there is at least one real root of the equation $4x^3 - 6x^2 + 4x - 1 = 0$ in the open interval $(0, 1)$.

26. $f(x) = x^3 + 2x + k$; $f'(x) = 3x^2 + 2$
Because f is a polynomial function, f is continuous and differentiable everywhere. Suppose $f(a) = 0$ and $f(b) = 0$. Thus Rolle's theorem holds on $[a, b]$. Hence, there is some number $c \in (a, b)$ such that $f'(c) = 0$, that is, $3c^2 + 2 = 0$. Because this is impossible, f cannot have two real roots.

27. $f(x) = 4x^5 + 3x^3 + 3x - 2$; $f'(x) = 20x^4 + 9x^2 + 3$
$f(0) = -2$ and $f(1) = 8$. Because f is continuous on $[0, 1]$ and 0 is between -2 and 8, $f(x_1) = 0$ for some number x_1 between 0 and 1. Hence x_1 is a root of the equation. Now suppose that the equation has another root; call this root r. Then $f(r) = 0$. If $r < x_1$, consider the interval $[r, x_1]$ and if $r > x_1$, consider the interval $[x_1, r]$. In either case $f(r) = f(x_1) = 0$ and Rolle's theorem holds. Therefore there is a number c between r and x_1 such that $f'(c) = 0$. But $f'(x)$ is always greater than or equal to 3. Hence our assumption leads to a contradiction. Therefore, the equation cannot have another root.

28. Use the mean-value theorem to prove that if $x > 0$, then $\cos x > 1 - \frac{1}{2}x^2$.

- Let $f(x) = \cos x - \left(1 - \frac{1}{2}x\right)$. Then $f'(x) = x - \sin x$. Because f is continuous and differentiable everywhere, the mean-value theorem holds on $[0, x]$. Hence there is a number $c \in (0, x)$ such that

$$f(x) - f(0) = (x - 0)f'(c)$$

$$[\cos x - (1 - \tfrac{1}{2}x^2)] - 0 = x(c - \sin c)$$

Because $c > 0$, then $\sin c < c$ and so $x(c - \sin c) > 0$. Therefore

$$\cos x - (1 - \tfrac{1}{2}x^2) > 0$$

$$\cos x > 1 - \tfrac{1}{2}x^2$$

Because $\cos x - (1 - \frac{1}{2}x)$ is an even function, the above inequality is also true if $x < 0$.

29. Use the mean-value theorem to prove that if $x > 0$, then $\sin x > x - \frac{1}{6}x^3$.

- Let $f(x) = \sin x - (x - \frac{1}{6}x^3)$. Then $f'(x) = \cos x - (1 - \frac{1}{2}x^2)$. Because f is continuous and differentiable everywhere, the mean-value theorem holds on $[0, x]$. Hence there is a number $c \in (0, x)$ such that

$$f(x) - f(0) = (x - 0)f'(c)$$

$$[\sin x - (x - \tfrac{1}{6}x^3)] - 0 = x[\cos c - (1 - \tfrac{1}{2}c^2)]$$

Because $c > 0$, by Exercise 28 , $\cos c - (1 - \frac{1}{2}c^2) > 0$. Because $x > 0$,

$$\sin x - (x - \tfrac{1}{6}x^3) > 0$$

$$\sin x > x - \tfrac{1}{6}x^3$$

Because $\sin x - (x - \frac{1}{6}x^3)$ is an odd function, the above inequality is reversed if $x < 0$.

30. Use the mean-value theorem to prove Bernoulli's inequality: if $x > 0$ and $r > 1$, where r is rational, then $(1 + x)^r > 1 + rx$.

- Let $f(x) = (1 + x)^r - (1 + rx)$. Then $f'(x) = r(1 + x)^{r-1} - r = r[(1 + x)^{r-1} - 1]$. Because f is continuous and differentiable if $x > -1$, the mean-value theorem holds on $[0, x]$. Hence there is a number $c \in (0, x)$ such that

$$f(x) - f(0) = (x - 0)f'(c)$$

$$[(1 + x)^r - (1 + rx)] - 0 = xr[(1 + c)^{r-1} - 1]$$

Because $c > 0$, then $1 + c > 1$; because $r > 1$ then $r - 1 > 0$. Therefore $(1 + c)^{r-1} > 1$. Hence

$$(1 + x)^r - (1 + rx) > 0$$

$$(1 + x)^r > 1 + rx$$

If $-1 < x < 0$, then $-1 < c < 0$, $0 < 1 + c < 1$ and $(1 + c)^{r-1} < 1$ so that $xr[(1 + c)^{r-1} - 1] > 0$ and the above inequality is still true. The inequality remains true if $r < 0$. The inequality is reversed if $0 < r < 1$.

31. Use the mean-value theorem to prove that if $a < b$, the arithmetic mean $\frac{1}{2}(a + b)$ is in the open interval (a, b).

- Let $f(x) = x^2$. Then $f'(x) = 2x$. Because f is continuous and differentiable everywhere, the mean-value theorem holds on $[a, b]$. Hence there is a number $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c)$$

$$b^2 - a^2 = (b - a) \cdot 2c$$

$$\tfrac{1}{2}(b + a) = c$$

That is, $\frac{1}{2}(a + b)$ is in the open interval (a, b).

32. Use the mean-value theorem to prove that if $0 < a < b$, the geometric mean $\sqrt{ab}$ of a and b is in the open interval (a, b).

- Let $f(x) = 1/x$. Then $f'(x) = -1/x^2$. Because f is continuous and differentiable for $x \neq 0$, the mean-value theorem holds on $[a, b]$. Hence there is a number $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c)$$

$$\frac{1}{b} - \frac{1}{a} = (b - a) \cdot \frac{-1}{c^2}$$

$$\frac{a - b}{ab} = \frac{a - b}{c^2}$$

$$ab = c^2$$

$$\sqrt{ab} = c$$

That is, $\sqrt{ab}$ is in the open interval (a, b).

33. The mean-value theorem says that at some point of an interval the instantaneous velocity is equal to the average velocity. (See Exercise 39.) Because the average velocity is $35/\frac{1}{2} = 70$ mph. at some point the velocity was 70 mph.

34. $f(x) = \sin^2 x + \cos^2 x$; $f'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) \equiv 0$ in any interval I. By Theorem 3.3.3, f is constant in I. Because $f(0) = 1$, then $f(x) = 1$ in I.

35. f is continuous on $[a,b]$ and $f'(x) = 1$ for all x in (a,b). Let x be in the interval $(a,b]$. Then the hypothesis of the mean-value theorem is satisfied by f on $[a,x]$. Hence, there is a number c in (a,x) such that $f(x) - f(a) = (x-a)f'(c)$. But $f'(c) = 1$ because $a < c < x$. Therefore, if x is in $(a,b]$, $f(x) - f(a) = x - a$; $f(x) = x - a + f(a)$. The last equation is also valid if $x = a$. Hence for all x in $[a,b]$, $f(x) = x - a + f(a)$.

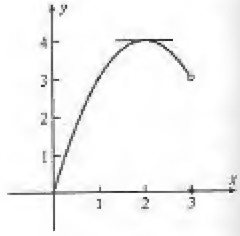
36. The converse of Rolle's theorem is not true. Make up an example of a function and an interval for which the conclusion of Rolle's theorem is true and for which
(a) condition (i) is not satisfied but conditions (ii) and (iii) are satisfied;
(b) condition (ii) is not satisfied but conditions (i) and (iii) are satisfied;
(c) condition (iii) is not satisfied but conditions (i) and (ii) are satisfied.
Sketch the graph showing the horizontal tangent line for each case.

(a) We want to define f so that f is discontinuous on the closed interval $[a,b]$; f is differentiable on the open interval (a,b); $f(a) = f(b) = 0$; and so that there is some number c in the open interval (a,b) for which $f'(c) = 0$. If f is differentiable in the open interval (a,b), then f is continuous in the open interval (a,b). Thus f must be either discontinuous from the right at a or discontinuous from the left at b. Let f be defined by

$$f(x) = \begin{cases} 4x - x^2 & \text{if } 0 \le x < 3 \\ 0 & \text{if } x = 3 \end{cases}$$

The graph of f is shown in at the right. We note that

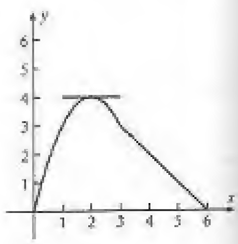
$$f'(x) = 4 - 2x \quad \text{if } 0 < x < 3$$



Thus, $f'(2) = 0$ and so the number $c = 2$ satisfies the conclusion of Rolle's theorem for the interval $[0,3]$. Furthermore, f is discontinuous from the left at $x = 3$; f is differentiable in the open interval $(0,3)$; $f(0) = 0$ and $f(3) = 0$.

(b) We want to define f so that f is continuous on the closed interval $[a,b]$; f is not differentiable on the open interval (a,b); $f(a) = f(b) = 0$; and there is a number c in (a,b) such that $f'(c) = 0$. f will not be differentiable at a point where its graph has a corner. Let

$$f(x) = \begin{cases} 4x - x^2 & \text{if } 0 \le x \le 3 \\ 6 - x & \text{if } 3 < x \le 6 \end{cases}$$



The graph of f is shown at the right. There is a horizontal tangent line at the point where $x = 2$; f is continuous on the closed interval $[0,6]$; $f(0) = 0$ and $f(6) = 0$. However, because

$$f'(x) = \begin{cases} 4 - 2x & \text{if } 0 < x < 3 \\ -1 & \text{if } 3 < x < 6 \end{cases}$$

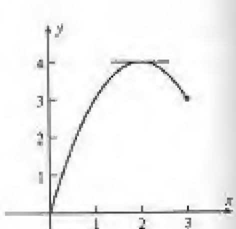
then by Theorem 3.3.4

$$f'_-(3) = 4 - 2(3) = -2 \text{ and } f'_+(3) = -1$$

Thus, $f'(3)$ does not exist and f is not differentiable at $x = 3$.

(c) We want to define f so that f is continuous on the closed interval $[a,b]$; f is differentiable in the open interval (a,b); either $f(a) \ne 0$ or $f(b) \ne 0$; and so there is a number c in the open interval (a,b) such that $f'(c) = 0$. Let

$$f(x) = 4x - x^2; \; 0 \le x \le 3$$



The graph of f is shown at the right. Note that $f'(x) = 4 - 2x$ and thus there is a horizontal tangent line at the point where $x = 2$. Moreover, f is continuous on the closed interval $[0,3]$; f is differentiable in the open interval $(0,3)$; $f(0) = 0$, but $f(3) \ne 0$.

Use Rolle's theorem to prove that if every polynomial function of the second degree has at most two real roots, then every polynomial of the third degree has at most three real roots.

Assume $f(x)$ is a third-degree polynomial with four real roots: x_1, x_2, x_3, x_4 and $x_1 < x_2 < x_3 < x_4$. Because the degree of f is 3, the degree of f' is 2. Because f is a polynomial, f is continuous and differentiable everywhere. Because $f(x_1) = f(x_2) = 0$, by Rolle's theorem there is a number c_1 in (x_1, x_2) such that $f'(c_1) = 0$. Because $f(x_2) = f(x_3) = 0$, there is a number c_2 in (x_2, x_3) such that $f'(c_2) = 0$. Because $f(x_3) = f(x_4) = 0$, there is a number c_3 in (x_3, x_4) such that $f'(c_3) = 0$. We have numbers c_1, c_2, and c_3, $c_1 < c_2 < c_3$ such that $f'(c_1) = f'(c_2) = f'(c_3) = 0$. Hence f' is a polynomial of degree 2 which has three real roots. This contradicts the hypothesis that a second-degree polynomial has at most two real roots. Hence our assumption is false. Therefore a third-degree polynomial has at most three roots.

Use mathematical induction to prove that a polynomial of the nth degree has at most n real real roots.

A linear polynomial has 1 real root and a quadratic polynomial has at most 2 real roots. Suppose the result is true for a polynomial of degree k. Assume $f(x)$ is a polynomial of degree $k+1$ with $k+2$ real roots: x_1, ..., x_{k+2} and $x_1 < \ldots < x_{k+2}$. Because the degree of f is $k+1$, the degree of f' is k. Because f is a polynomial, f is continuous and differentiable everywhere. Let j be an integer from 1 to $k+1$. Because $f(x_j) = f(x_{j+1}) = 0$, by Rolle's theorem there is a number c_j in (x_j, x_{j+1}) such that $f'(c_j) = 0$. We have numbers $c_1, \ldots, c_{k+1}$, $c_1 < \ldots < c_{k+1}$ such that $f'(c_1) = \ldots = f'(c_{k+1}) = 0$. Hence f' is a polynomial of degree k which has $k+1$ real roots. This contradicts the hypothesis that a polynomial of degree k has at most k real roots. Hence our assumption is false. Therefore a polynomial degree $k+1$ has at most $k+1$ roots. Because we have shown the result is true when $n = 1$ and if it is true when $n = k$, then it is true for $n = k+1$, it follows from the principle of mathematical induction the the result is true for every positive integer n.

Suppose $s = f(t)$ is an equation of the motion of a particle moving in a straight line. Show that the conclusion of the mean-value theorem assures that during any time interval where f satisfies the hypothesis of the mean-value theorem there will be some instant when the instantaneous velocity will equal the average velocity during that time interval.

Suppose f satisfies the hypothesis of the mean-value theorem on the time interval $[t_1, t_2]$. Therefore there is some c with $t_1 < c < t_2$ such that

$$f'(c) = \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{\Delta s}{\Delta t}$$

Now, $f'(c)$ is the instantaneous velocity at the instant when $t = c$ and $\Delta s/\Delta t$ is the average velocity of the particle during the time interval $[t_1, t_2]$. Therefore, there is some instant during the time interval when the instantaneous velocity equals the average velocity during that time interval.

3.4 INCREASING AND DECREASING FUNCTIONS AND THE FIRST-DERIVATIVE TEST

3.4.1 Definition A function f defined on an interval is said to be *increasing* on that interval if and only if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, where x_1 and x_2 are any numbers in the interval.

3.4.2 Definition A function f defined on an interval is said to be *decreasing* on that interval if and only if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, where x_1 and x_2 are any numbers in the interval.

If a function is either increasing on an interval or decreasing on an interval then it is said to be *monotonic* on the interval.

3.4.3 Theorem Let the function f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b):

(i) if $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$
(ii) if $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$

Note that the hypothesis of Theorem 3.4.3 may be satisfied even when $f'(a) = 0$ and $f'(b) = 0$ or when f is not differentiable at a or b. By letting points where $f'(x)$ is zero or undefined be endpoints of subintervals, we have:

(i) if $f'(x) > 0$ at all but finitely many points of an interval I, then f is increasing on I
(ii) if $f'(x) < 0$ at all but finitely many points of an interval I, then f is decreasing on I

3.4.4 First-Derivative Test for Relative Extrema Let the function f be continuous at all points of the open interval (a, b) containing the number c, and suppose that f' exists at all points of (a, b) except possibly at c:

(i) if $f'(x) > 0$ for all values of x in some open interval having c as its right endpoint, and if $f'(x) < 0$ for all values of x in some open interval having c as its left endpoint, then f has a relative maximum value at c

(ii) if $f'(x) < 0$ for all values of x in some open interval having c as its right endpoint, and if $f'(x) > 0$ for all values of x in some open interval having c as its left endpoint, then f has a relative minimum value at c

The following steps make use of the first-derivative test to sketch the graph of a function f.

1. Find each number at which $f'(x)$ is either zero or undefined, often by factoring $f'(x)$. Arrange the numbers in increasing order.
2. Use these numbers to partition the number line into open intervals.
3. For each interval (number) determine whether $f'(x)$ is positive or negative (zero or undefined). $f'(x)$ will be positive on an interval if the number of negative factors is even, and negative if the number of negative factors is odd.
4. For each interval state whether f is increasing or decreasing.
5. For each number, state whether f has a maximum, a minimum or neither, and evaluate $f(x)$. Determine if the graph of f has a horizontal or vertical tangent line, a corner, or a vertical asymptote.

In step 5 we may need to use Definition 3.1.1(ii) which states that $x = c$ is a vertical tangent to the graph of f at $P = (c, f(c))$ if both $f'_-(c)$ and $f'_+(c)$ are infinite. If, in some open interval containing c, the curve lies on one side of the normal line at P, then the graph is said to have a *cusp* at P. Figures 3.4.24 and 3.4.28 illustrate vertical tangents; in the latter there is a cusp.

Intermediate-Value Theorem for Derivatives. If $f'(x)$ exists at each point of a closed in the closed interval $[a, b]$ and k is a number strictly between $f'(a)$ and $f'(b)$ then there exists a number c in (a, b) such that $f(c) = k$. We do *not* assume that f' is continuous. See Exercise 56.

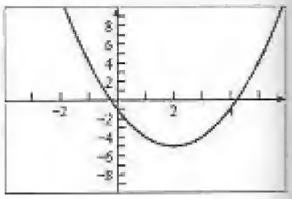
Exercises 3.4

In Exercises 1–18, (a) plot the graph, and determine (b) the relative extrema of f, (c) the values of x at which the relative extrema occur, (d) the intervals on which f is increasing, and (e) the intervals on which f is decreasing. Confirm by calculus.

1. $f(x) = x^2 - 4x - 1$; $f'(x) = 2x - 4$

Set $f'(x) = 0$ and obtain the critical number 2.

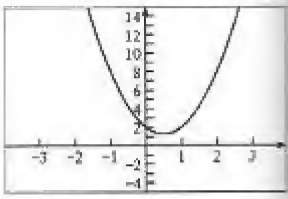
	$f(x)$	$f'(x)$	Conclusion
$x < 2$		$-$	f is decreasing on $(-\infty, 2]$
$x = 2$	-5	0	f has a relative minimum value
$2 < x$		$+$	f is increasing on $[2, +\infty)$



2. $f(x) = 3x^2 - 3x + 2$; $f'(x) = 6x - 3$

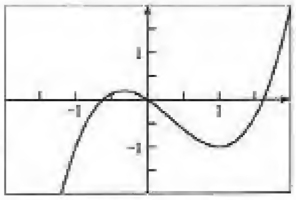
Set $f'(x) = 0$ and obtain the critical number $\frac{1}{2}$.

	$f(x)$	$f'(x)$	Conclusion
$x < \frac{1}{2}$		$-$	f is decreasing on $(-\infty, \frac{1}{2}]$
$x = \frac{1}{2}$	$\frac{5}{4}$	0	f has a relative minimum value
$\frac{1}{2} < x$		$+$	f is increasing on $[\frac{1}{2}, +\infty)$



$f(x) = x^3 - x^2 - x$; $f'(x) = 3x^2 - 2x - 1$

Set $f'(x) = 0$: $(3x+1)(x-1) = 0$; $x = -\frac{1}{3}$, $x = 1$

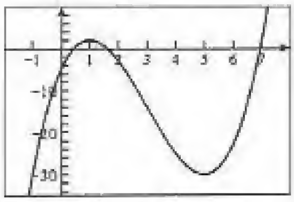


	$f(x)$	$f'(x)$	Conclusion
$x < -\frac{1}{3}$		$+$	f is increasing on $(-\infty, -\frac{1}{3}]$
$x = -\frac{1}{3}$	$\frac{5}{27}$	0	f has a relative maximum value
$-\frac{1}{3} < x < 1$		$-$	f is decreasing on $[-\frac{1}{3}, 1]$
$x = 1$	-1	0	f has a relative minimum value
$1 < x$		$+$	f is increasing on $[1, +\infty)$

$f(x) = x^3 - 9x^2 + 15x - 5$

A plot of f is shown at the right. We find the derivative of f and factor it.

$$f'(x) = 3x^2 - 18x + 15 = 3(x-1)(x-5)$$

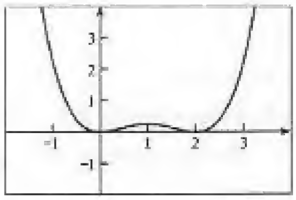


Because $f'(x) = 0$ when $x = 1$ or $x = 5$, the critical numbers of f are 1 and 5. We consider the three intervals $x < 1$, $1 < x < 5$, and $x > 5$ that are determined by the critical numbers; these are shown in the first row of the table. The factor $x - 1$ is 0 for $x = 1$, negative for smaller values of x, and positive for larger values of x; this is shown in the second row of the table. The third row shows the signs of the factor $x - 5$. In the fourth row, the sign of $f'(x)$ is the product of the signs in rows 2 and 3. The conclusion for the intervals is found using Theorem 3.4.3 and the conclusion for the critical numbers is found using Theorem 3.4.4. $f(x)$ is evaluated at the critical numbers and at one point of each infinite branch.

	$x < 1$	$x = 1$	$1 < x < 5$	$x = 5$	$x > 5$
$3(x-1)$	$-$	0	$+$	$+$	$+$
$x - 5$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
conclusion	f is increasing on $(-\infty, 1]$	f has a relative maximum	f is decreasing on $[1, 5]$	f has a relative minimum	f is increasing on $[5, -\infty)$
$f(x)$	$f(0) = 5$	2		-30	$f(7) = 2$

$f(x) = \frac{1}{4}x^4 - x^3 + x^2$; $f'(x) = x^3 - 3x^2 + 2x$

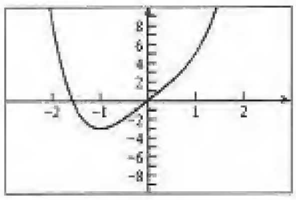
Set $f'(x) = 0$: $x(x-1)(x-2) = 0$; $x = 0$, $x = 1$, $x = 2$



	$f(x)$	$f'(x)$	Conclusion
$x < 0$		$-$	f is decreasing on $(-\infty, 0]$
$x = 0$	0	0	f has a relative minimum value
$0 < x < 1$		$+$	f is increasing on $[0, 1]$
$x = 1$	$\frac{1}{4}$	0	f has a relative maximum value
$1 < x < 2$		$-$	f is decreasing on $[1, 2]$
$x = 2$	0	0	f has a relative minimum value
$2 < x$		$+$	f is increasing on $[2, +\infty)$

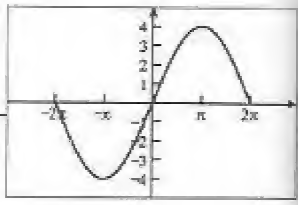
$f(x) = x^4 + 4x$; $f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x+1)(x^2 - x + 1)$

Set $f'(x) = 0$: $x = -1$



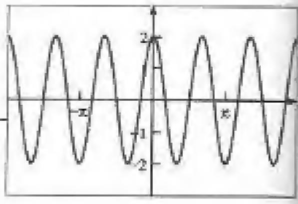
	$f(x)$	$f'(x)$	Conclusion
$x < -1$		$-$	f is decreasing on $(-\infty, -1]$
$x = -1$	-3	0	f has a relative minimum value
$-1 < x$		$+$	f is increasing on $[-1, +\infty)$

7. $f(x) = 4\sin\frac{1}{2}x,\ x \in [-2\pi, 2\pi];\ f'(x) = 2\cos\frac{1}{2}x$
Because the sine function has period 2π, f has period 4π.
Set $f'(x) = 0$: $2\cos\frac{1}{2}x = 0$; $\frac{1}{2}x = -\frac{1}{2}\pi$ or $\frac{1}{2}x = \frac{1}{2}\pi$; $x = -\pi$, $x = \pi$.



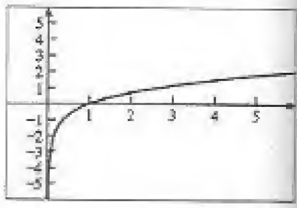
	$f(x)$	$f'(x)$	f is/has
$-2\pi \le x < -\pi$		$-$	decreasing on $[-2\pi, -\pi]$
$x = -\pi$	-4	0	a relative minimum value
$-\pi < x < \pi$		$+$	increasing on $[-\pi, \pi]$
$x = \pi$	4	0	a relative maximum value
$\pi < x \le 2\pi$		$-$	decreasing on $[\pi, 2\pi]$

8. $f(x) = 2\cos 3x,\ x \in [-2\pi, 2\pi];\ f'(x) = -6\sin 3x$
Set $f'(x) = 0$: $-6\sin 3x = 0$; $3x = 0$ or $3x = \pi$; $x = 0$, $x = \frac{1}{3}\pi$.
Because the cosine function has period 2π, f has period $\frac{2}{3}\pi$.
Thus we add $\frac{2}{3}k\pi$ where k is any integer.



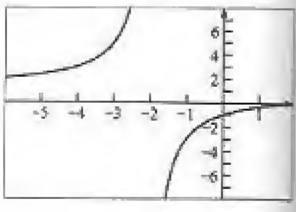
	$f(x)$	$f'(x)$	f is/has
$\frac{2}{3}k\pi < x < \frac{1}{3}\pi + \frac{2}{3}k\pi$		$-$	decreasing on $[\frac{2}{3}k\pi, \frac{1}{3}\pi + \frac{2}{3}k\pi]$
$x = \frac{1}{3}\pi + \frac{2}{3}k\pi$	-2	0	a relative minimum value
$\frac{1}{3}\pi + \frac{2}{3}k\pi < x < \frac{2}{3}\pi + \frac{2}{3}k\pi$		$+$	increasing on $[\frac{1}{3}\pi + \frac{2}{3}k\pi, \frac{2}{3}\pi + \frac{2}{3}k\pi]$
$x = \frac{2}{3}\pi + \frac{2}{3}k\pi$	2	0	a relative maximum value
$\frac{2}{3}\pi + \frac{2}{3}k\pi < x < \pi + \frac{2}{3}k\pi$		$-$	decreasing on $[\frac{2}{3}\pi + \frac{2}{3}k\pi, \pi + \frac{2}{3}k\pi]$

9. $f(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}} = x^{1/2} - x^{-1/2};\ f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} = \frac{1}{2}x^{-3/2}(x+1)$
Set $f'(x) = 0$: $x = -1$. The domain of f is $(0, +\infty)$ and $f'(x)$ exists for all x in domain of f. -1 is not in the domain so there are no critical numbers and no relative extrema.

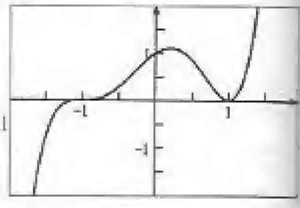


	$f(x)$	$f'(x)$	Conclusion
$0 < x$		$+$	f is increasing on $(0, +\infty)$

10. $f(x) = \dfrac{x-2}{x+2} = 1 - \dfrac{4}{x+2};\ f'(x) = \dfrac{4}{(x+2)^2}$
f and f' are not defined at 2.
Because $f'(x) > 0$ at every number in its domain, f is increasing on $(-\infty, -2)$ and $(2, +\infty)$.
There are no relative extrema.



11. $f(x) = (1-x)^2(1+x)^3,\ f'(x) = -2(1-x)(1+x)^3 + 3(1+x)^2(1-x)^2$
$= (1-x)(1+x)^2[-2(1+x) + 3(1-x)] = (1-x)(1+x)^2(-5x+1)$
Set $f'(x) = 0$: $x = 1$, $x = -1$, $x = \frac{1}{5}$



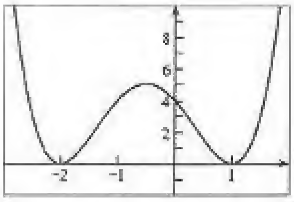
	$f(x)$	$f'(x)$	Conclusion
$x < -1$		$+$	f is increasing on $(-\infty, \frac{1}{5}]$
$x = -1$	0	0	no relative extremum at $x = -1$
$-1 < x < \frac{1}{5}$		$+$	
$x = \frac{1}{5}$	$\frac{3456}{3125}$	0	f has a relative maximum value
$\frac{1}{5} < x < 1$		$-$	f is decreasing on $[\frac{1}{5}, 1]$
$x = 1$	0	0	f has a relative minimum value
$1 < x$		$+$	f is increasing $[1, +\infty)$

$f(x) = (x+2)^2(x-1)^2$

A plot of the graph is shown at the right.

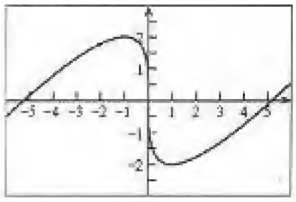
$$\begin{aligned} f'(x) &= 2(x+2)(x-1)^2 + 2(x+2)^2(x-1) \\ &= 2(x+2)(x-1)+[(x-1)+(x+2)] \\ &= 2(x+2)(x-1)(2x+1) \end{aligned}$$

The critical numbers are -2, $-\frac{1}{2}$ and 1.
The table is filled in one row at a time.



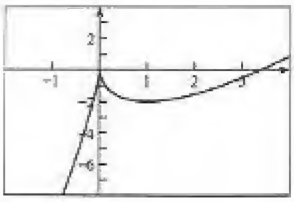
	$x<-2$	$x=-2$	$-2<x<-\frac{1}{2}$	$x=-\frac{1}{2}$	$-\frac{1}{2}<x<1$	$x=1$	$x>1$
$2(x+2)$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$2x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
conclusion	f is decreasing on $(-\infty,-2]$	f has a relative minimum	f is increasing on $[-2,-\frac{1}{2}]$	f has a relative maximum	f is decreasing on $[-\frac{1}{2},1]$	f has a relative minimum	f is increasing on $[1,+\infty)$
$f(x)$	$f(-3)=16$	0	$f(-1)=4$	$\frac{81}{16}$=5.0625	$f(0)=4$	0	$f(2)=16$

$f(x) = x - 3x^{1/3}$; $f'(x) = 1 - x^{-2/3}$. Set $f'(x)=0$: $x^{-2/3}=1$; $x^{2/3}=1$; $x^2=1$; $x=\pm 1$
$f'(x)$ does not exist when $x=0$. The critical numbers of f are -1, 0, and 1.



	$f(x)$	$f'(x)$	Conclusion
$x<-1$		$+$	f is increasing on $(-\infty,-1]$
$x=-1$	2	0	f has a relative maximum value
$-1<x<0$		$-$	
$x=0$	0	does not exist	f is decreasing on $[-1,1]$; no relative extremum at 0
$0<x<1$		$-$	
$x=1$	-2	0	f has a relative minimum value
$1<x$		$+$	f is increasing

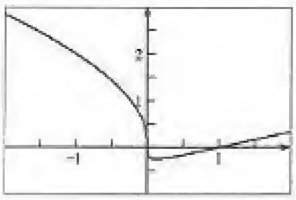
$f(x) = 4x - 6x^{2/3}$; $f'(x) = 4 - 4x^{-1/3} = 4(1-x^{-1/3})$. $f'(x)=0$ when $x=1$ and $f'(x)$ does not exist when $x=0$. The critical numbers of f are 0 and 1. There is a cusp at the origin.



	$x<0$	$x=0$	$0<x<1$	$x=1$	$x>1$
$f'(x)$	$+$	d.n.e.	$-$	0	$+$
f is/has a	increasing	relative maximum	decreasing	relative minimum	increasing
$f(x)$		0		-2	

$f(x) = x^{2/3} - x^{1/3}$; $f'(x) = \frac{2}{3}x^{-1/3} - \frac{1}{3}x^{-2/3} = \frac{1}{3}x^{-2/3}(2x^{1/3}-1)$

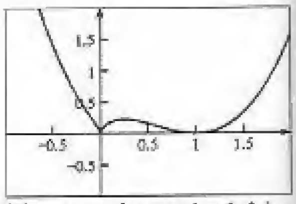
Set $f'(x)=0$: $2x^{1/3}=1$; $x^{1/3}=\frac{1}{2}$; $x=\frac{1}{8}$
$f'(0)$ does not exist and 0 is in the domain of f.
The critical numbers are 0 and $\frac{1}{8}$.



	$f(x)$	$f'(x)$	Conclusion
$x<0$		$-$	
$x=0$	0	does not exist	f is decreasing on $(-\infty,\frac{1}{8}]$; no relative extremum at 0
$0<x<\frac{1}{8}$		$-$	
$x=\frac{1}{8}$	$-\frac{1}{4}$	0	f has a relative minimum value
$\frac{1}{8}<x$		$+$	f is increasing on $[\frac{1}{8},+\infty)$

16. $f(x) = x^{2/3}(x-1)^2$

▹ A plot is shown at the right. f is continuous for all x.

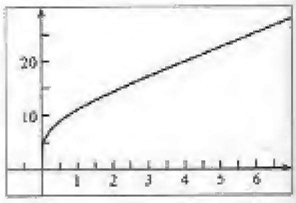


$$\begin{aligned} f'(x) &= x^{2/3}(2)(x-1) + (x-1)^2(\tfrac{2}{3}x^{-1/3}) \\ &= \tfrac{2}{3}x^{-1/3}(x-1)[3x + (x-1)] \\ &= \frac{2(x-1)(4x-1)}{3x^{1/3}} \end{aligned}$$

The critical numbers of f are 0, $\frac{1}{4}$, and 1. Because $f'_-(0) = -\infty$ and $f'_+(0) = +\infty$, the graph of f has a vertical tangent line at $x = 0$. Because the x axis is the normal line at $x = 0$ and the curve lies on one side of the normal line, the graph has a cusp at the origin. The table is filled in by rows.

	$x < 0$	$x = 0$	$0 < x < \frac{1}{4}$	$x = \frac{1}{4}$	$\frac{1}{4} < x < 1$	$x = 1$	$x > 1$
$2(x-1)$	$-$	$-$	$-$	$-$	$-$	0	$+$
$4x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$1/3x^{1/3}$	$-$	doesn't exist	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	doesn't exist	$+$	0	$-$	0	$+$
f is/has a	decreasing on $(-\infty, 0]$	relative minimum	increasing on $[0, \frac{1}{4}]$	relative maximum	decreasing on $[\frac{1}{4}, 1]$	relative minimum	increasing on $[1, +\infty)$
$f(x)$	$f(-.25) \approx .62$	0		$9/2^{16/3} \approx .22$		0	$f(1.5) \approx .32$

17. $f(x) = x^{5/4} + 10x^{1/4}$

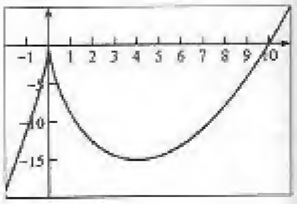


$f'(x) = \frac{5}{4}x^{1/4} + \frac{10}{4}x^{-3/4}$

The domain of f is $[0, \infty)$. $f'(x) > 0$ if $x > 0$.

f is increasing on $[0, \infty)$.

18. $f(x) = x^{5/3} - 10x^{2/3}$; $f'(x) = \frac{5}{3}x^{2/3} - \frac{20}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-4)$



$f'(4) = 0$ and $f'(0)$ does not exist. The graph has a cusp at the origin.

	$x < 0$	$x = 0$	$0 < x < 4$	$x = 4$	$x > 4$
$\frac{5}{3}x^{-1/3}$	$-$	doesn't exist	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	doesn't exist	$-$	0	$+$
f is/has a	increasing	relative maximum	decreasing	relative minimum	increasing
$f(x)$		0		$-6 \cdot 4^{2/3} \approx -15.1$	

In Exercises 19–32, compute (a) the relative extrema of f, (b) the values of x at which the relative extrema occur, (c) the intervals on which f is increasing, and (d) the intervals on which f is decreasing. Check by plotting.

19. $f(x) = 2x^3 - 9x^2 + 2$; $f'(x) = 6x^2 - 18x$

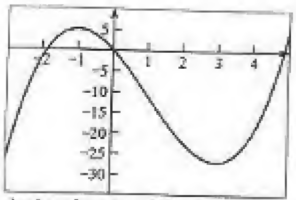
Set $f'(x) = 0$: $6x(x-3) = 0$; $x = 0$, $x = 3$

	$f(x)$	$f'(x)$	Conclusion
$x < 0$		$+$	f is increasing on $(-\infty, 0]$
$x = 0$	2	0	f has a relative maximum value
$0 < x < 3$		$-$	f is decreasing on $[0, 3]$
$x = 3$	-25	0	f has a relative minimum value
$3 < x$		$+$	f is increasing on $[3, +\infty)$

$f(x) = x^3 - 3x^2 - 9x$

A plot of f is shown at the right. We find the derivative of f and factor it.

$$f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$$



Because $f'(x) = 0$ when $x = -1$ or $x = 3$, the critical numbers of f are -1 and 3. We consider the three intervals $x < -1$, $-1 < x < 3$, and $x > 3$ that are determined by the critical numbers; these are shown in the first row of the table. The sign of each factor is show in the next two rows. In the fourth row, the sign of $f'(x)$ is the product of the signs in rows 2 and 3. The conclusion for the intervals is found using Theorem 3.4.3 and the conclusion for the critical numbers is found using Theorem 3.4.4. $f(x)$ is evaluated at the critical numbers and at one point of each infinite branch.

	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
$3(x+1)$	$-$	0	$+$	$+$	$+$
$x - 3$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
f is/has a	increasing on $(-\infty, 1]$	relative maximum	decreasing on $[1, 5]$	relative minimum	increasing on $[5, -\infty)$
$f(x)$	$f(0) = 5$	2		-30	$f(7) = 2$

$f(x) = \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x + 1$; $f'(x) = x^4 - 5x^2 + 4$

Set $f'(x) = 0$: $(x^2-1)(x^2-4) = 0$; $(x+1)(x-1)(x+2)(x-2) = 0$;

$x = -1$, $x = 1$, $x = -2$, $x = 2$

	$f(x)$	$f'(x)$	Conclusion
$x < -2$		$+$	f is increasing on $(-\infty, -2]$
$x = -2$	$-\frac{1}{15}$	0	f has a relative maximum value
$-2 < x < -1$		$-$	f is decreasing on $[-2, -1]$
$x = -1$	$-\frac{23}{15}$	0	f has a relative minimum value
$-1 < x < 1$		$+$	f is increasing on $[-1, 1]$
$x = 1$	$\frac{53}{15}$	0	f has a relative maximum value
$1 < x < 2$		$-$	f is decreasing on $[1, 2]$
$x = 2$	$\frac{31}{15}$	0	f has a relative minimum value
$2 < x$		$+$	f is increasing on $[2, +\infty)$

$f(x) = x^5 - 5x^3 - 20x - 2$

$$f'(x) = 5x^4 - 15x^2 - 20 = 5(x+2)(x-2)(x^2+1)$$

Because $x^2 + 1 > 0$, the critical numbers are -2 and 2. The table is filled in one row at a time.

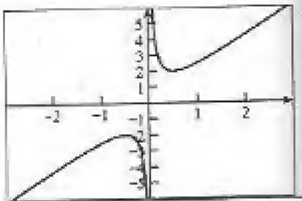
	$x < -2$	$x = -2$	$-2 < x < 2$	$x = 2$	$x > 2$
$5(x+2)$	$-$	0	$+$	$+$	$+$
$x - 2$	$-$	$-$	$-$	0	$+$
$x^2 + 1$	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	0	$+$
f is/has a	increasing on $(-\infty, -2]$	relative maximum	decreasing on $[-2, 2]$	relative minimum	increasing on $[2, +\infty)$
$f(x)$	$f(-2.7) \approx -7$	46		-50	$f(2.7) \approx -11$

23. $f(x) = x + \frac{1}{x^2} = x + x^{-2}$; $f'(x) = 1 - 2x^{-3}$. Set $f'(x) = 0$: $\frac{x^3 - 2}{x^3} = 0$; $x = \sqrt[3]{2}$

The domain of f is $\{x \mid x \neq 0\}$, and $f'(x)$ exists for all x in its domain.

The only critical number is $\sqrt[3]{2} \approx 1.26$.

	$f(x)$	$f'(x)$	Conclusion
$x < 0$		$+$	f is increasing on $(-\infty, 0)$
$x = 0$	not defined	not defined	vertical asymptote
$0 < x < \sqrt{2}$		$-$	f is decreasing on $(0, \sqrt{2}]$
$x = \sqrt{2}$	$\frac{3}{2}\sqrt{2} \approx 1.89$	0	f has a relative minimum value
$\sqrt{2} < x$		$+$	f is increasing on $[\sqrt{2}, +\infty)$



24. $f(x) = 2x + \frac{1}{2x}$

▸ A plot is shown at the right. Note that $f(0)$ is not defined.

$$f'(x) = 2 - \frac{1}{2x^2} = \frac{4x^2 - 1}{2x^2} = \frac{(2x+1)(2x-1)}{2x^2}$$

The critical numbers are $-\frac{1}{2}$ and $\frac{1}{2}$. The table is filled in one row at a time.

	$x < -\frac{1}{2}$	$x = -\frac{1}{2}$	$-\frac{1}{2} < x < 0$	$x = 0$	$0 < x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
$2x + 1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$2x - 1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$1/2x^2$	$+$	$+$	$+$	doesn't exist	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	doesn't exist	$-$	0	$+$
f is/has a	increasing on $(-\infty, -\frac{1}{2}]$	relative maximum	decreasing on $[-\frac{1}{2}, 0)$	vertical asymptote	decreasing on $(0, \frac{1}{2}]$	relative minimum	increasing on $[\frac{1}{2}, +\infty)$
$f(x)$	$f(-2)$=-4.25	-2	$f(-.5)$=-4.25		$f(.5)$=4.25	2	$f(2)$=4.25

25. $f(x) = 2x\sqrt{3 - x}$; $f'(x) = 2\sqrt{3 - x} + 2x\left(-\frac{1}{2\sqrt{3 - x}}\right) = \frac{2(3 - x) - x}{\sqrt{3 - x}} = \frac{6 - 3x}{\sqrt{3 - x}}$

Set $f'(x) = 0$: $6 - 3x = 0$; $x = 2$

	$f(x)$	$f'(x)$	Conclusion
$x < 2$		$+$	f is increasing on $(-\infty, 2]$
$x = 2$	4	0	f has a relative maximum value
$2 < x < 3$		$-$	f is decreasing on $[2, 3]$

26. $f(x) = x\sqrt{5 - x^2}$. Because $5 - x^2 \geq 0$ if $|x| \leq \sqrt{5}$, the domain of f is $[-\sqrt{5}, \sqrt{5}]$. $f(x) = x(5 - x^2)^{1/2}$

$$f'(x) = (5 - x^2)^{1/2}(1) + x(\tfrac{1}{2})(5 - x^2)^{-1/2}(-2x) = (5 - x^2)^{-1/2}[(5 - x^2) - x^2] = \frac{5 - 2x^2}{\sqrt{5 - x^2}}$$

Let $c = \sqrt{\frac{5}{2}}$. The critical numbers are $\pm c$. The table is filled in one row at a time. Values at the endpoints in rows 3 and 4 are one-sided limits.

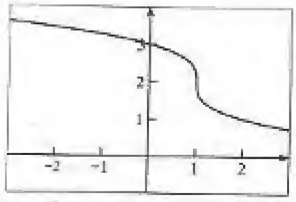
	x=$-\sqrt{5}$	$-\sqrt{5} < x < -c$	x=$-c$	$-c < x < c$	x=c	$c < x < \sqrt{5}$	x=$\sqrt{5}$
$5 - 2x^2$	$-$	$-$	0	$+$	0	$-$	$-$
$1/\sqrt{5 - x^2}$	$+\infty$	$+$	$+$	$+$	$+$	$+$	$+\infty$
$f'(x)$	$-\infty$	$-$	0	$+$	0	$-$	$-\infty$
f is/has a	vertical tangent	decreasing on $[-\sqrt{5}, -c]$	relative minimum	increasing on $[-c, c]$	relative maximum	decreasing on $[c, \sqrt{5}]$	vertical tangent
$f(x)$	0		-2.5		2.5		0

$f(x) = 2 - 3(x-4)^{2/3}$; $f'(x) = -2(x-4)^{-1/3}$. $f'(x) \neq 0$ for any value of x.
$f'(4)$ does not exist and 4 is in the domain of f. Therefore 4 is a critical number.

	$f(x)$	$f'(x)$	Conclusion
$x < 4$		$+$	f is increasing on $(-\infty, 4]$
$x = 4$	2	doesn't exist	f has a relative maximum value
$4 < x$		$-$	f is decreasing on $[4, +\infty)$

$f(x) = 2 - (x-1)^{1/3}$

f is continuous at all real numbers.



$$f'(x) = \frac{-1}{3(x-1)^{2/3}}$$

Because $f'(x) < 0$ if $x \neq 1$, we conclude that f is decreasing on $(-\infty, +\infty)$ and does not have a relative extremum. Because

$$\lim_{x \to 1} f'(x) = -\infty$$

the tangent line to the graph of f is vertical at the point $(1, 2)$. Other points on the graph are $(0, 3)$ and $(2, 1)$. A plot of the graph is shown at the right.

$f(x) = \frac{1}{2} \sec 4x$, $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$; $f'(x) = \frac{1}{2} \sec 4x \tan 4x(4) = 2 \sec 4x \tan 4x$
Set $f'(x) = 0$: $\tan 4x = 0$; $4x = 0$ or $4x = \pi$; $x = 0$, $x = \frac{1}{4}\pi$.
Because the secant function has period 2π, f has period $\frac{1}{2}\pi$. Thus we add $\frac{1}{2}k\pi$, where k is -1 or 0.

	$f(x)$	$f'(x)$	Conclusion
$x = \frac{1}{2}k\pi$	$\frac{1}{2}$	0	f has a relative minimum value
$\frac{1}{2}k\pi < x < \frac{1}{8}\pi + \frac{1}{2}k\pi$		$+$	f is increasing
$x = \frac{1}{8}\pi + \frac{1}{2}k\pi$	not defined	not defined	
$\frac{1}{8}\pi + \frac{1}{2}k\pi < x < \frac{1}{4}\pi + \frac{1}{2}k\pi$		$+$	f is increasing
$x = \frac{1}{4}\pi + \frac{1}{2}k\pi$	$-\frac{1}{2}$	0	f has a relative maximum value
$\frac{1}{4}\pi + \frac{1}{2}k\pi < x < \frac{3}{8}\pi + \frac{1}{2}k\pi$		$-$	f is decreasing
$x = \frac{3}{8}\pi + \frac{1}{2}k\pi$	not defined	not defined	
$\frac{3}{8}\pi + \frac{1}{2}k\pi < x < \frac{1}{2}\pi + \frac{1}{2}k\pi$		$-$	f is decreasing

$f(x) = 3 \csc 2x$, $[-\pi, \pi]$; $f'(x) = -6 \csc 2x \cot 2x = -6 \cos 2x / \sin^2 2x$
Set $f'(x) = 0$: $\cos 2x = 0$; $2x = \frac{1}{2}\pi$ or $2x = \frac{3}{2}\pi$; $x = \frac{1}{4}\pi$, $x = \frac{3}{4}\pi$.
Because the cosecant function has period 2π, f has period π. Thus we add $k\pi$, where k is any integer.

	$f(x)$	$f'(x)$	Conclusion
$x = k\pi$	not defined	not defined	f has a vertical asymptote
$k\pi < x < \frac{1}{4}\pi + k\pi$		$-$	f is decreasing
$x = \frac{1}{4}\pi + k\pi$	3	0	f has a relative minimum value
$\frac{1}{4}\pi + k\pi < x < \frac{1}{2}\pi + k\pi$		$+$	f is increasing
$x = \frac{1}{2}\pi + k\pi$	not defined	not defined	f has a vertical asymptote
$\frac{1}{2}\pi + k\pi < x < \frac{3}{4}\pi + k\pi$		$+$	f is increasing
$x = \frac{3}{4}\pi + k\pi$	-3	0	f has a relative maximum value
$\frac{3}{4}\pi + k\pi < x < \pi + k\pi$		$-$	f is decreasing

31. $f(x) = x^{1/3}(x+4)^{-2/3}$. $f(-4)$ is not defined.

$f'(x) = \frac{1}{3}x^{-2/3}(x+4)^{-2/3} - \frac{2}{3}x^{1/3}(x+4)^{-5/3} = \frac{1}{3}x^{-2/3}(x+4)^{-5/3}[(x+4)-2x] = \frac{1}{3}x^{-2/3}(x+4)^{-5/3}(4-x)$

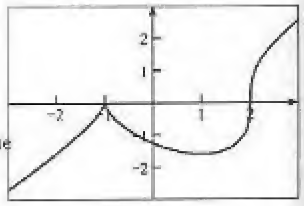
$f'(4) = 0$ and $f'(-4)$ and $f'(0)$ do not exist.

	$x<-4$	$x=-4$	$-4<x<0$	$x=0$	$0<x<4$	$x=4$	$x>4$
$\frac{1}{3}x^{-2/3}$	+	+	+	doesn't exist	+	+	+
$(x+4)^{-5/3}$	−	doesn't exist	+	+	+	+	+
$4-x$	+	+	+	+	+	0	−
$f'(x)$	−	doesn't exist	+	doesn't exist	+	0	−
f is/has a	decreasing	vertical asymptote	increasing	vertical tangent	increasing	relative maximum	decreasing

32. $f(x) = (x+1)^{2/3}(x-2)^{1/3}$. $f'(x) = \frac{2}{3}(x+1)^{-1/3}(x-2)^{1/3} + \frac{1}{3}(x-2)^{-2/3}(x+1)^{2/3}$

$= \frac{1}{3}(x+1)^{-1/3}(x-2)^{-2/3}[2(x-2)+(x+1)] = (x+1)^{-1/3}(x-2)^{-2/3}(x-1)$

Set $f'(x) = 0$: $x = 1$. $f'(-1)$ and $f'(2)$ do not exist and -1 and 2 are in the domain of f. The critical numbers of f are -1, 1, and 2.

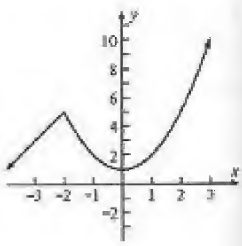


	$f(x)$	$f'(x)$	Conclusion
$x<-1$		+	f is increasing on $(-\infty,-1]$
$x=-1$	0	does not exist	f has a relative maximum value; graph has a cusp
$-1<x<1$		−	f is decreasing on $[-1,1]$
$x=1$	$-\sqrt[3]{4}$	0	f has a relative minimum value
$1<x<2$		+	f is increasing on $[1,+\infty)$; no relative extremum at 2
$x=2$	0	does not exist	
$2<x$		+	

In Exercises 33–38, compute (a) the relative extrema of f, (b) the values of x at which the relative extrema occur, (c) the intervals on which f is increasing, and (d) the intervals on which f is decreasing. (e) Use (a)-(d) to sketch

33. $f(x) = \begin{cases} 2x+9 & \text{if } x \le -2 \\ x^2+1 & \text{if } -2 < x \end{cases}$; $f'(x) = \begin{cases} 2 & \text{if } x < -2 \\ 2x & \text{if } -2 < x \end{cases}$

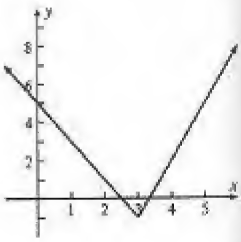
$f'_-(-2) = 2$ but $f'_+(-2) = -4$. Thus $f'(-2)$ does not exist.
Also, $f'(0) = 0$. Hence -2 and 0 are critical numbers of f.



	$f(x)$	$f'(x)$	Conclusion
$x<-2$		+	f is increasing on $(-\infty,-2]$
$x=-2$	5	doesn't exist	f has a relative maximum value
$-2<x<0$		−	f is decreasing on $[-2,0]$
$x=0$	1	0	f has a relative minimum value
$0<x$		+	f is increasing on $[0,+\infty)$

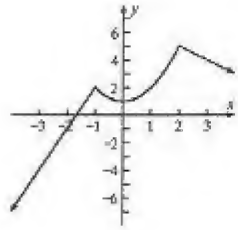
34. $f(x) = \begin{cases} 5-2x & \text{if } x < 3 \\ 3x-10 & \text{if } x \ge 3 \end{cases}$, $f'(x) = \begin{cases} -2 & \text{if } x < 3 \\ 3 & \text{if } x > 3 \end{cases}$

$f'_-(3) = -2$ but $f'_+(3) = 3$. Thus $f'(3)$ does not exist. 3 is the critical number.



	$x<3$	$x=3$	$x>3$
$f'(x)$	−	doesn't exist	+
f is/has a	decreasing	relative minimum	increasing

$f(x) = \begin{cases} 3x+5 & \text{if } x < -1 \\ x^2+1 & \text{if } -1 \le x < 2 \\ 7-x & \text{if } 2 \le x \end{cases}$; $f'(x) = \begin{cases} 3 & \text{if } x < -1 \\ 2x & \text{if } -1 < x < 2 \\ -1 & \text{if } 2 < x \end{cases}$



$f'(-1)$ and $f'(2)$ do not exist, and $f'(0) = 0$.
The critical numbers of f are -1, 0, 2.

	$f(x)$	$f'(x)$	Conclusion
$x < -1$		$+$	f is increasing on $(-\infty, -1]$
$x = -1$	2	doesn't exist	f has a relative maximum value
$-1 < x < 0$		$-$	f is decreasing on $[-1, 0]$
$x = 0$	1	0	f has a relative minimum value
$0 < x < 2$		$+$	f is increasing on $[0, 2]$
$x = 2$	5	doesn't exist	f has a relative maximum value
$2 < x$		$-$	f is decreasing on $[2, +\infty)$

$f(x) = \begin{cases} 12-(x+5)^2 & \text{if } x \le -3 \\ 5-x & \text{if } -3 < x \le -1 \\ \sqrt{100-(x-7)^2} & \text{if } -1 < x \le 17 \end{cases}$

The function f is defined on $(-\infty, 17]$. Because

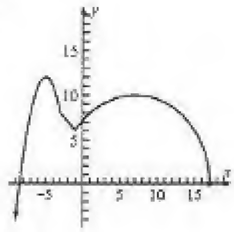
$$f_-(-3) = 12 - 2^2 = 8 \quad \text{and} \quad f_+(-3) = 5 + 3 = 8$$

then f is continuous at -3. Because

$$f_-(-1) = 5 + 1 = 6 \quad \text{and} \quad f_+(-1) = \sqrt{100 - 8^2} = 6$$

then f is continuous at -1. We conclude that f is continuous on $(-\infty, 17]$. Now

$$f'(x) = \begin{cases} -2(x+5) & \text{if } x < -3 \\ -1 & \text{if } -3 < x < -1 \\ \dfrac{-x+7}{\sqrt{100-(x-7)^2}} & \text{if } -1 < x < 17 \end{cases}$$



Thus, $f'(x) = 0$ if $x = -5$ or $x = 7$. Furthermore, by Theorem 4.3.4,

$$f'_-(-3) = -2(2) \quad \text{and} \quad f'_+(-3) = -1$$

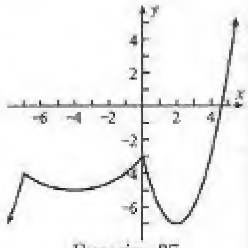
and so $f'(-3)$ does not exist and the graph has a corner at $x = -3$. Similarly, because

$$f'_-(-1) = -1 \quad \text{and} \quad f'_+(-1) = 6/\sqrt{100 - 8^2} = 4/3$$

then $f'(-1)$ does not exist and the graph has a corner at $x = -1$. The table is filled in one row at a time. Note that at $x = -3$ and $x = -1$ we compare values rather than multiply signs. The information in the last two rows is used to sketch the graph shown at the right.

	$x < -5$	$x = -5$	$-5 < x < -3$	$x = -3$	$-3 < x < -1$	$x = -1$	$-1 < x < 7$	$x = 7$	$7 \le x < 17$	$x = 17$
$-2(x+5)$	$+$	0	$-$	-4						
-1				-1	$-$	-1				
$\frac{-x+7}{\sqrt{100-(x-7)^2}}$						$\frac{4}{3}$	$+$	0	$-$	$-\infty$
$f'(x)$	$+$	0	$-$	doesn't exist	$-$	doesn't exist	$+$	0	$-$	$-\infty$
conclusion	f is increasing $(-\infty,-5]$	f has a relative maximum		f is decreasing on $[-5,-1]$ no relative extremum		f has a relative minimum	f is increasing $[-1,7]$	f has a relative maximum	f is decreasing $[7,17]$	f has a vertical tangent
$f(x)$	$f(-8)=3$	12		8		6		10		0

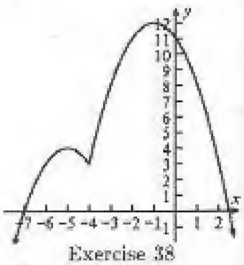
37. $f(x) = \begin{cases} (x+9)^2 - 8 & \text{if } x < -7 \\ -\sqrt{25-(x+4)^2} & \text{if } -7 \le x \le 0 \\ (x-2)^2 - 7 & \text{if } 0 < x \end{cases}$; $f'(x) = \begin{cases} 2(x+9) & \text{if } x < -7 \\ \dfrac{x+4}{\sqrt{25-(x+4)^2}} & \text{if } -7 < x < 0 \\ 2(x-2) & \text{if } 0 < x \end{cases}$



Exercise 37

$f'(-7)$ and $f'(0)$ do not exist. Also, $f'(-9) = 0$, $f'(-4) = 0$, $f'(2) = 0$.
The critical numbers of f are -9, -7, -4, 0, and 2.

	$f(x)$	$f'(x)$	Conclusion
$x < -9$		$-$	f is decreasing on $[-\infty, -9]$
$x = -9$	-8	0	f has a relative minimum value
$-9 < x < -7$		$+$	f is increasing on $[-9, -7]$
$x = -7$	-4	doesn't exist	f has a relative maximum value
$-7 < x < -4$		$-$	f is decreasing on $[-7, -4]$
$x = -4$	-5	0	f has a relative minimum value
$-4 < x < 0$		$+$	f is increasing on $[-4, 0]$
$x = 0$	-3	doesn't exist	f has a relative maximum value
$0 < x < 2$		$-$	f is decreasing on $[0, 2]$
$x = 2$	-7	0	f has a relative minimum value
$2 < x$		$+$	f is increasing on $[2, +\infty)$



Exercise 38

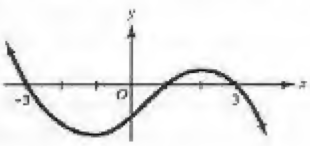
38. $f(x) = \begin{cases} 4-(x+5)^2 & \text{if } x < -4 \\ 12-(x+1)^2 & \text{if } x \ge -4 \end{cases}$; $f'(x) = \begin{cases} -2(x+5) & \text{if } x < -4 \\ -2(x+1) & \text{if } x > -4 \end{cases}$

$f'(-4)$ does not exist. Also $f'(-5) = 0$, $f'(-1) = 0$.
The critical numbers of f are -5, -4, -1.

	$x < -5$	$x = -5$	$-5 < x < -4$	$x = -4$	$-4 < x < -1$	$x = -1$	$x > -1$
$-2(x+5)$	$+$	0	$-$	-2			
$-2(x+1)$				6	$+$	0	$-$
$f'(x)$	$+$	0	$-$	doesn't exist	$+$	0	$-$
f is/has a	increasing	relative maximum	decreasing	relative minimum	increasing	relative minimum	decreasing

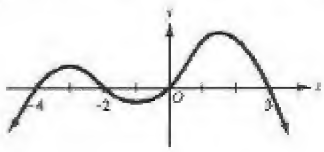
In Exercises 39–44, the figure shows the graph of the derivative of a function f continuous on R. Determine (a) the critical numbers of f, the intervals on which f is (b) increasing, (c) decreasing, and (d) any relative extrema.

39.



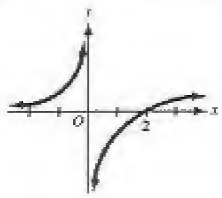
(a) $f'(x) = 0$ at -3, 1, 3
(b) $f'(x) \ge 0$ on $(-\infty, -3]$, $[1, 3]$
(c) $f'(x) \le 0$ on $[-3, 1]$, $[3, +\infty)$
(d) relative maxima at -3, 3
relative minimum at 1

40.



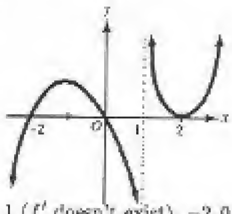
$f'(x) = 0$ at -4, -2, 0, 3
$f'(x) \ge 0$ on $[-4, -2]$, $[0, 3]$
$f' \le 0$ on $(-\infty, -4]$, $[-2, 0]$, $[3, +\infty)$
relative maxima at -2, 3
relative minima at -4, 0

41.

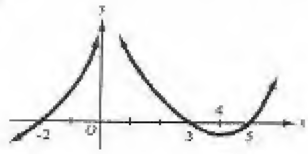


0 (f' doesn't exist), 2 ($f' = 0$)
$f'(x) \ge 0$ on $(-\infty, 0]$, $[2, +\infty)$
$f'(x) \le 0$ on $[0, 2]$
relative maximum at 0
relative minimum at 2

43.



44.



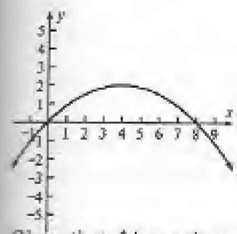
0 (f' doesn't exist), -1, 1 ($f' = 0$)
$f'(x) \geq 0$ on $[0,1]$
$f'(x) \leq 0$ on $[-\infty,0]$, $[1,+\infty)$
relative maximum at 1
relative minimum at 0
(-1 is an inflection point)

1 (f' doesn't exist), $-2, 0, 2$ ($f' = 0$)
$f'(x) \geq 0$ on $[-2,0]$, $[1,+\infty)$
$f' \leq 0$ on $(-\infty,-2]$, $[0,1]$
relative maximum at 0
relative minima at -2, 1
(2 is an inflection point)

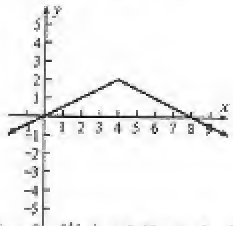
0 (f' doesn't exist), $2, 3, 5$ ($f' = 0$)
$f'(x) \geq 0$ on $[-2,3]$, $[5,+\infty)$
$f'(x) \leq 0$ on $(-\infty,-2]$, $[3,5]$
relative maximum at 0
relative minimum at 2
(0: inflection point, vertical tangent)

Given that f is continuous on R, $f(0) = 0$, $f(4) = 2$, $f(8) = 0$, $f'(x) > 0$ if $x < 4$, and $f'(x) < 0$ if $x \geq 4$, sketch a graph of f under the following additional hypothesis:

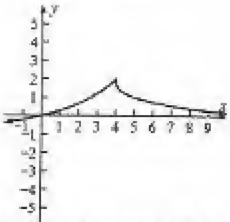
f' is continuous at 4



(b) $f'(x) = \frac{1}{2}$ if $x < 4$,
$f'(x) = -\frac{1}{2}$ if $x > 4$

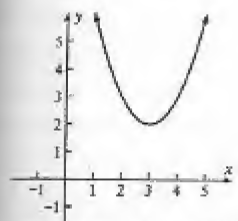


(c) $\lim_{x \to 4^-} f'(x) = 1$, $\lim_{x \to 4^+} f'(x) = -\infty$
$f'(a) \neq f'(b)$ if $a \neq b$

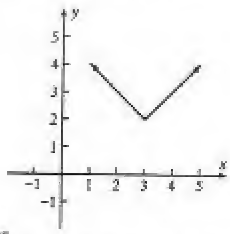


Given that f is continuous on R, $f(3) = 2$, $f'(x) < 0$ if $x < 3$, $f'(x) \geq 0$ if $x > 3$, sketch a graph of f under the following additional hypothesis:

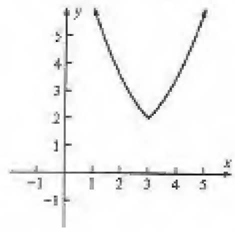
f' is continuous at 3



(b) $f'(x) = -1$ if $x < 3$,
$f'(x) = 1$ if $x > 3$



(c) $\lim_{x \to 3^-} f'(x) = -1$, $\lim_{x \to 3^+} f'(x) = 1$
$f'(a) \neq f'(b)$ if $a \neq b$



$f(x) = x^3 + ax^2 + b$; $f'(x) = 3x^2 + 2ax$
Because f has a relative extremum at $(2,3)$ and $f'(2)$ exists then
$f'(2) = 0$; $3(2)^2 + 2a(2) = 0$; $4a = -12$; $a = -3$
Because $(2,3)$ is on the graph of f
$f(2) = 3$; $2^3 + a(2)^2 + b = 3$; $8 + (-3)4 + b = 3$; $b = 7$

48. Find a, b, and c such that the function defined by $f(x) = ax^2 + bx + c$ will have a relative maximum value of 7 at 1 and the graph of $y = f(x)$ will go through the point $(2, -2)$.

▷ $f'(x) = 2ax + b$ is defined for all x. Therefore, a relative extremum of f must occur at a point where $f'(x) = 0$. Because we are given that a relative maximum value occurs at $x = 1$, then

$$f'(1) = 0$$
$$2a + b = 0 \tag{1}$$

Because 7 is the value of the function at 1, then

$$f(1) = 7$$
$$a + b + c = 7 \tag{2}$$

Because the graph of $y = f(x)$ contains the point $(2, -2)$, then

$$f(2) = -2$$
$$4a + 2b + c = -2 \tag{3}$$

Subtracting Eq. (2) from Eq. (3) we obtain

$$3a + b = 9 \tag{4}$$

and subtracting Eq. (1) from Eq. (4) we get $a = -9$. Substituting in Eq. (1) and Eq. (2) we get $b = 18$ and $c = 2$. Therefore $f(x) = -9x^2 + 18x - 2$ and $f'(x) = -18x + 18 = 18(1 - x)$. Because $f'(x) > 0$ if $x < 1$ and $f'(x) < 0$ if $x > 1$, then f has a relative maximum at $x = 1$.

49. $f(x) = ax^3 + bx^2 + cx + d$; $f'(x) = 3ax^2 + 2bx + c$

Because f has relative extrema at $(1, 2)$ and $(2, 3)$ and $f'(1)$ and $f'(2)$ exist, then

$f'(1) = 0$	$f'(2) = 0$	$f(1) = 2$	$f(2) = 3$
$3a \cdot 1^2 + 2b \cdot 1 + c = 0$	$3a \cdot 2^2 + 2b \cdot 2 + c = 0$	$a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = 2$	$a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d = 3$
$3a + 2b + c = 0$	$12a + 4b + c = 0$	$a + b + c + d = 0$	$8a + 4b + 2c + d = 3$

Solving these four equations simultaneously, we obtain $a = -2$, $b = 9$, $c = -12$, and $d = 7$.

50. $f(x) = x^p(1 - x)^q$, where p and q are positive integers.

$f'(x) = px^{p-1}(1 - x)^q - qx^p(1 - x)^{q-1} = x^{p-1}(1 - x)^{q-1}[p(1 - x) - qx] = x^{p-1}(1 - x)^{q-1}[p - (p + q)x]$

(a) If p is even, 0 is a critical number and $p - 1$ is odd. Thus $x^{p-1} < 0$ for $x < 0$, and $x^{p-1} > 0$ for $x > 0$. Also, for x sufficiently near 0, $(1 - x)^{q-1} > 0$ and $p - (p + q)x > 0$. Thus f' changes sign from "$-$" to "$+$" at $x = 0$. Hence by the first-derivative test, f has a relative minimum value at 0.

(b) If q is even, 1 is a critical number and $q - 1$ is odd. Thus $(1 - x)^{q-1} > 0$ for $x < 1$ and $(1 - x)^{q-1} < 0$ for $x > 1$. Further, for x sufficiently near 1, $x^{p-1} > 0$ and $p - (p + q)x < 0$. Thus f' changes sign from "$-$" to "$+$" at $x = 1$. Hence by the first-derivative test, f has a relative minimum value at 1.

(c) Because p and q are positive, $\dfrac{p}{p+q}$ is a critical number and $0 < \dfrac{p}{p+q} < 1$. For x in $(0, 1)$, x^{p-1} and $(1 - x)^{q-1}$ are positive. For $x < \dfrac{p}{p+q}$, $p - (p + q)x > 0$ and for $x > \dfrac{p}{p+q}$, $p - (p + q)x < 0$. So f' changes sign from "$+$" to "$-$" at $x = \dfrac{p}{p+q}$. Hence by the first-derivative test, f has a relative maximum value at $x = \dfrac{p}{p+q}$.

51. If $f'(x) < 0$ for all x in (a,b) then $D[-f(x)] > 0$ for all x in (a,b). By the first part of the proof, $-f$ is increasing on $[a,b]$ so that f is decreasing on $[a,b]$.

52. Prove Theorem 3.4.4(i).

▷ If $f'(x) < 0$ then $-f'(x) > 0$; if $f'(x) > 0$ then $-f'(x) < 0$. Therefore $-f$ satisfies the hypothesis of part (i). Hence $-f$ has a relative maximum value at c and so f has a relative minimum value at c.

53. $f(x) = x^k$, where k is an odd positive integer; $f'(x) = kx^{k-1}$. Since $k \geq 1$ and $k - 1$ is an even integer, $f'(x)$ is never negative. Because $f'(x)$ never changes sign, f has no relative extrema.

54. Prove that if f is increasing on $[a, b]$, g is increasing on $[f(a), f(b)]$, then $g \circ f$ is increasing on $[a, b]$.

▷ Let $a \leq x_1 < x_2 \leq b$. Because f is increasing on $[a, b]$ then $f(a) \leq f(x_1) \leq f(x_2) \leq f(b)$. Because g is increasing on $[f(a), f(b)]$ then $g(f(a)) \leq g(f(x_1)) \leq g(f(x_2)) \leq g(f(b))$. Hence $g \circ f$ is increasing on $[a, b]$.

[illegible] (a) For any x_1 and x_2 in I such that $x_1 < x_2$ then $f(x_1) < f(x_2)$ because f is increasing. But $f(x_1) < f(x_2)$ is equivalent to $-f(x_1) > -f(x_2)$ which states that $g(x_1) > g(x_2)$ for $x_1 < x_2$ in I. Thus g is decreasing on I.
(b) For any x_1 and x_2 in I such that $x_1 < x_2$, then $f(x_1) < f(x_2)$. Because $f(x)$ is positive on I we may divide by $f(x_1) \cdot f(x_2)$ and preserve the direction of the inequality. (This is also true if $f(x)$ is negative on I). Thus
$\dfrac{f(x_1)}{f(x_1) \cdot f(x_2)} < \dfrac{f(x_2)}{f(x_1) \cdot f(x_2)}$ which is equivalent to $\dfrac{1}{f(x_2)} < \dfrac{1}{f(x_1)}$
which states that $h(x_2) < h(x_1)$ for $x_1 < x_2$ in I. Thus h is decreasing on I.

[illegible] The intermediate-value theorem for derivatives. The function f is differentiable at each number in the closed interval $[a, b]$. Prove that if $f'(a) \cdot f'(b) < 0$, there is a number c in the open interval (a, b) such that $f'(c) = 0$.
Because f is differentiable at each number in $[a, b]$, then f is continuous on $[a, b]$ and by Theorem 3.1.7, f has an absolute maximum value and an absolute minimum value on $[a, b]$. If either of these absolute extrema occurs at some number c with $a < c < b$, then by Theorem 3.1.3, $f'(c) = 0$, which is the desired result. Otherwise, both of the absolute extrema occur at the endpoints of $[a, b]$. We show that this is impossible. We consider two cases.

Case 1: The absolute maximum value of f is at a and the absolute minimum value of f is at b. Because f is differentiable at a, then the derivative from the right of f at a exists. Furthermore, since the absolute maximum value of f is at a, then $f(a) \geq f(x)$ for all x in $[a, b]$, and thus

$$\frac{f(x) - f(a)}{x - a} \leq 0 \quad \text{if } a < x < b$$

Hence

$$f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0 \tag{1}$$

Because we are given that $f'(a) \cdot f'(b) < 0$, then $f'(a) \neq 0$, and thus (1) implies that

$$f'(a) < 0 \tag{2}$$

Because f is differentiable at b, the derivative from the left of f at b exists. Since the absolute minimum value of f is at b, then $f(b) \leq f(x)$ for all x in $[a, b]$, and thus

$$\frac{f(x) - f(b)}{x - b} \leq 0 \quad \text{if } a < x < b$$

Hence,

$$f'_-(b) = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b} \leq 0 \tag{3}$$

Because $f'(a) \cdot f'(b) < 0$, then $f'(b) \neq 0$, and thus (3) implies

$$f'(b) < 0 \tag{4}$$

But (2) and (4) contradict the hypothesis that $f'(a) \cdot f'(b) < 0$. Therefore, the Case 1 assumption is false. We consider the remaining possibility.

Case 2: The absolute maximum value of f is at b, and the absolute minimum value of f is at a. Then inequalities (1), (2), (3), and (4) of Case 1 are reversed, and we again contradict the hypothesis that $f'(a) \cdot f'(b) < 0$. Therefore both Case 1 and Case 2 are impossible, and we have proved that $f'(c) = 0$ for some c in (a, b).

[illegible] If $f'(x)$ exists at all numbers in an open interval containing c and $f'(c) = 0$, we may not conclude that f has a relative extremum at c. Consider $f(x) = x^3$, $f'(x) = 3x^2$, $f'(0) = 0$ but f is increasing on R.

3.5 CONCAVITY, POINTS OF INFLECTION, AND THE SECOND DERIVATIVE TEST

3.5.1 Definition The graph of a function f is said to be *concave upward* at the point $(c, f(c))$ if $f'(c)$ exists and if there is an open interval I containing c such that for all values of $x \neq c$ in I the point $(x, f(x))$ on the graph is above the tangent line to the graph at $(c, f(c))$.

3.5.2 Definition The graph of a function f is said to be *concave downward* at the point $(c, f(c))$ if $f'(c)$ exists and if there is an open interval I containing c such that for all values of $x \neq c$ in I the point $(x, f(x))$ on the graph is below the tangent line to the graph at $(c, f(c))$.

3.5.3 Theorem Let f be a function that is differentiable on some open interval containing c. Then

(i) if $f''(c) > 0$, the graph of f is concave upward at $(c, f(c))$;
(ii) if $f''(c) < 0$, the graph of f is concave downward at $(c, f(c))$.

The following theorem may be used if $f''(c)$ is zero or does not exist.

3.5.3′ Theorem Suppose $f'(c)$ exists and I is an open interval containing c.
(i) If $f''(x) > 0$ for all values of $x \neq c$ in I, the graph of f is concave upward at $(c, f(c))$.
(ii) If $f''(x) < 0$ for all values of $x \neq c$ in I, the graph of f is concave downward at $(c, f(c))$.

3.5.4 Definition The point $(c, f(c))$ is a *point of inflection* of the graph of the function f if the graph has a tangent line there, and if there exists an open interval I containing c such that if x is in I then either
(i) $f''(x) < 0$ if $x < c$ and $f''(x) > 0$ if $x > c$, or
(ii) $f''(x) > 0$ if $x < c$ and $f''(x) < 0$ if $x > c$.
That is, if $f'(x)$ or $1/f'(x)$ has a relative maximum or minimum at c. The inflectional tangent crosses the curve and is very close to it.

3.5.5 Theorem If the function f is differentiable on some open interval containing c, and if $(c, f(c))$ is a point of inflection of the graph of f, then if $f''(c)$ exists, $f''(c) = 0$.

However, if $f''(c) = 0$, we cannot conclude that the graph of f has a point of inflection at c. The following theorems are useful in sketching a graph.

Theorem If $f(x)$ is concave upward on the interval (a, b), then on (a, b) the graph of f lies below the chord joining $(a, f(a))$ and $(b, f(b))$.
If $f(x)$ is concave downward on the interval (a, b), then on (a, b) the graph of f lies above the chord joining $(a, f(a))$ and $(b, f(b))$.

Theorem The graph of a cubic polynomial is symmetric with respect to its point of inflection.

The first of these theorems suggests that we lightly draw the straight line connecting two points of a curve and then use concavity to determine whether the curve bows under or over the chord.

The following steps may be used to locate the points of inflection of the graph of f.
1. Find $f'(x)$ and $f''(x)$.
2. Find all numbers c for which $f''(c) = 0$ or $f''(c)$ is not defined.
3. If the graph of f has a tangent line, possibly vertical, at the point where $x = c$ and if $f''(x)$ changes sign as x increases through the value c, then the graph of f has a point of inflection at the point where $x = c$.

3.5.1 Second-Derivative Test for Relative Extrema Let c be a critical number of a function f at which $f'(c) = 0$, and let f' exist for all values of x in some open interval containing c. Then if $f''(c)$ exists and

(i) if $f''(c) < 0$, then f has a relative maximum value at c;
(ii) if $f''(c) > 0$, then f has a relative minimum value at c.

Suppose $f''(c) = 0$. If the first nonzero derivative at c is of odd order, then c is a point of inflection. If the first nonzero derivative at c is of even order, then f has a relative maximum or minimum at c according as that derivative is negative or positive.

Exercises 3.5

In Exercises 1–8, find any points of inflection of the graph of the function. Determine where the graph is concave upward and where it is concave downward. Check by plotting the graph of f and the inflectional tangents.

1. $f(x) = 2x^3 + 3x^2 - 12x + 1$; $f'(x) = 6x^2 + 6x - 12$;
$g''(x) = 12x + 6 = 12(x + \frac{1}{2})$. Set $g''(x) = 0$: $x = -\frac{1}{2}$

	$f(x)$	$f'(x)$	$g''(x)$	Conclusion
$x < -\frac{1}{2}$			$-$	graph is concave downward
$x = -\frac{1}{2}$	$-\frac{15}{2}$	$-\frac{27}{2}$	0	point of inflection
$-\frac{1}{2} < x$			$+$	graph is concave upward

2. $f(x) = x^3 - 6x^2 + 20$; $f'(x) = 3x^2 - 12x = 3x(x-4)$; $f''(x) = 6x - 12 = 6(x-2)$.

The critical number for f' is 2.

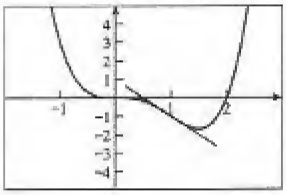
	$x < 2$	$x = 2$	$x > 2$
$f''(x)=6(x-2)$	$-$	0	$+$
graph is/ has a	concave downward	point of inflection	concave upward
$f(x)$; $f'(x)$		4; -12	

3. $g(x) = x^4 - 8x^3$; $g'(x) = 4x^3 - 24x^2$; $g''(x) = 12x^2 - 48x = 12x(x-4)$. Set $g''(x) = 0$: $x = 0$, $x = 4$

	$g(x)$	$g'(x)$	$g''(x)$	Conclusion
$x < 0$			$+$	graph is concave upward
$x = 0$	0	0	0	point of inflection
$0 < x < 4$			$-$	graph is concave downward
$x = 4$	-256	-128	0	point of inflection
$4 < x$			$+$	graph is concave upward

4. $f(x) = x^4 - 2x^3$; $f'(x) = 4x^3 - 6x^2$; $f''(x) = 12x^2 - 12x = 12x(x-1)$.
The critical numbers for f' are 0 and 1. A plot is shown at the right.

	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$12x$	$-$	0	$+$	$+$	$+$
$x - 1$	$-$	$-$	$-$	0	$+$
$f''(x)$	$+$	0	$-$	0	$+$
graph is/ has a	concave upward	point of inflection	concave downward	point of inflection	concave upward



5. $F(x) = 2(x^2+3)^{-1}$; $F'(x) = -4x(x^2+3)^{-2}$
$F''(x) = -4(x^2+3)^{-2} + 8x(x^2+3)^{-3}(2x) = 4(x^2+3)^{-3}[-(x^2+3) + 4x^2]$
$= 4(x^2+3)^{-3}(3x^2-3) = 12(x^2+3)^{-3}(x+1)(x-1)$
Set $F''(x) = 0$: $x = -1$, $x = 1$

	$F(x)$	$F'(x)$	$F''(x)$	Conclusion
$x < -1$			$+$	graph is concave upward
$x = -1$	$\frac{1}{2}$	$\frac{1}{4}$	0	point of inflection
$-1 < x < 1$			$-$	graph is concave downward
$x = 1$	$\frac{1}{2}$	$-\frac{1}{4}$	0	point of inflection
$1 < x$			$+$	graph is concave upward

6. $G(x) = \dfrac{x}{x^2+4} = x(x^2+4)^{-1}$; $G'(x) = x(-1)(x^2+4)^{-2}(2x) + (x^2+4)^{-1}(1) = (x^2+4)^{-2}(4-x^2)$

$= \dfrac{-(x+2)(x-2)}{(x^2+4)^2}$. $G''(x) = (x^2+4)^{-2}(-2x) + (-2)(x^2+4)^{-3}(4-x^2)$

$= 2x(x^2+4)^{-3}[(x^2+4)(-1) - 2(4-x^2)] = \dfrac{2x(x^2-12)}{(x^2+4)^3} = \dfrac{2x(x+2\sqrt{3})(x-2\sqrt{3})}{(x^2+4)^3}$

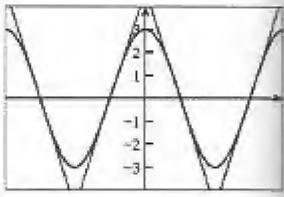
The critical numbers for G' are $\pm 2\sqrt{3} \approx \pm 3.5$.

	$x < -2\sqrt{3}$	$x = -2\sqrt{3}$	$-2\sqrt{3} < x < 0$	$x = 0$	$0 < x < 2\sqrt{3}$	$x = 2\sqrt{3}$	$x > 2\sqrt{3}$
$2x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x + 2\sqrt{3}$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x - 2\sqrt{3}$	$-$	$-$	$-$	$-$	$-$	0	$+$
$1/(x^2+4)^3$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$G''(x)$	$-$	0	$+$	0	$-$	0	$+$
graph is/ has a	concave downward	point of inflection	concave upward	point of inflection	concave downward	point of inflection	concave upward
$G(x)$; $G'(x)$		$-\frac{1}{8}\sqrt{3}$; $-\frac{1}{32}$		0; $\frac{1}{4}$		$\frac{1}{8}\sqrt{3}$; $-\frac{1}{32}$	

7. $f(x) = 2\sin 3x$, $x \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$; $f'(x) = 6\cos 3x$; $f''(x) = -18\sin 3x$
Set $f''(x) = 0$: $\sin 3x = 0$, $3x \in [-\frac{3}{2}\pi, \frac{3}{2}\pi]$
$3x = -\pi, 0, \pi$; $x = -\frac{1}{3}\pi, 0, \frac{1}{3}\pi$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\frac{1}{2}\pi \le x < -\frac{1}{3}\pi$			$-$	graph is concave downward
$x = -\frac{1}{3}\pi$	0	-6	0	point of inflection
$-\frac{1}{3}\pi < x < 0$			$+$	graph is concave upward
$x = 0$	0	6	0	point of inflection
$0 < x < \frac{1}{3}\pi$			$-$	graph is concave downward
$x = \frac{1}{3}\pi$	0	-6	0	point of inflection
$\frac{1}{3}\pi < x \le \frac{1}{2}\pi$			$+$	graph is concave upward

8. $f(x) = 3\cos 2x$, $x \in [-\pi, \pi]$; $f'(x) = -6\sin 2x$; $f''(x) = -12\cos 2x$
Because the graph is symmetric with respect to the y-axis, we consider $[0, \pi]$.
Because $f''(x) = 0$ if $2x = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, the critical numbers of f' are $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$.



	$x < \frac{1}{4}\pi$	$x = \frac{1}{4}\pi$	$\frac{1}{4}\pi < x < \frac{3}{4}\pi$	$x = \frac{3}{4}\pi$	$x > \frac{3}{4}\pi$
$f'' = -12\cos 2x$	$-$	0	$+$	0	$-$
graph is/ has a	concave downward	point of inflection	concave upward	point of inflection	concave downward
$f(x)$; $f'(x)$		0; -6		0; 6	

In Exercises 9–16, plot the graph and estimate the point of inflection and where the graph is concave upward and downward. Confirm by calculus.

9. $f(x) = x^3 + 9x$; $f'(x) = 3x^2 + 9$; $f''(x) = 6x$
Set $f''(x) = 0$: $6x = 0$; $x = 0$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$x < 0$		$+$	$-$	graph is concave downward
$x = 0$	0	9	0	point of inflection
$0 < x$		$+$	$+$	graph is concave upward

$g(x) = 2x^3 - 1$; $g'(x) = 6x^2$; $g''(x) = 12x$. Set $g''(x) = 0$: $12x = 0$; $x = 0$

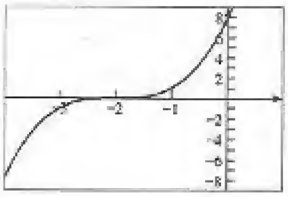
	$x < 0$	$x = 0$	$x > 0$
$g''(x) = 12x$	$-$	0	$+$
graph is/ has a	concave downward	point of inflection	concave upward

$G(x) = (x-1)^3$; $G'(x) = 3(x-1)^2$; $g''(x) = 6(x-1)$
Set $g''(x) = 0$: $x = 1$

	$G(x)$	$G'(x)$	$G''(x)$	Conclusion
$x < 1$			$-$	graph is concave downward
$x = 1$	0	0	0	point of inflection
$1 < x$			$+$	graph is concave upward

$F(x) = (x+2)^3$

A plot is shown at the right. $F'(x) = 3(x+2)^2$; $F''(x) = 6(x+2)$
The critical number for F' is -2.



	$x < -2$	$x = -2$	$x > -2$
$F''(x) = 6(x+2)$	$-$	0	$+$
graph is/ has a	concave downward	point of inflection	concave upward
$F(x)$; $F'(x)$		0; 0	

$f(x) = (x+2)^{1/3}$; $f'(x) = \frac{1}{3}(x+2)^{-2/3}$; $f''(x) = -\frac{2}{9}(x+2)^{-5/3}$
$f''(x)$ is never 0; $f''(-2)$ does not exist but f is continuous at -2
and $\lim_{x \to -2} f'(x) = +\infty$ so there is a vertical tangent line at $x = -2$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$x < -2$			$+$	graph is concave upward
$x = -2$	0	doesn't exist	doesn't exist	point of inflection
$-2 < x$			$-$	graph is concave downward

$g(x) = (x-1)^{1/3}$; $g'(x) = \frac{1}{3}(x-1)^{-2/3}$; $g''(x) = -\frac{2}{9}(x-1)^{-5/3}$
$g''(x)$ is never 0; $g''(1)$ does not exist but g is continuous at 1
and $\lim_{x \to 1} g'(x) = +\infty$ so there is a vertical tangent line at $x = 1$.

	$g(x)$	$g'(x)$	$g''(x)$	Conclusion
$x < 1$			$+$	graph is concave upward
$x = 1$	0	doesn't exist	doesn't exist	point of inflection
$1 < x$			$-$	graph is concave downward

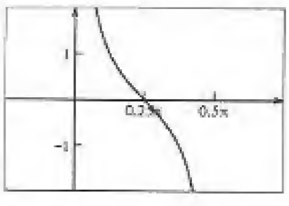
$f(x) = \tan \frac{1}{2}x$, $x \in (-\pi, \pi)$; $f'(x) = \frac{1}{2}\sec^2 \frac{1}{2}x$; $f''(x) = \frac{1}{2}\sec^2 \frac{1}{2}x \tan \frac{1}{2}x$

Set $f''(x) = 0$: $\tan \frac{1}{2}x = 0$, $\frac{1}{2}x \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$; $\frac{1}{2}x = 0$; $x = 0$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\pi < x < 0$			$-$	graph is concave downward
$x = 0$	0	$\frac{1}{2}$	0	point of inflection
$0 < x < \pi$			$+$	graph is concave upward

$g(x) = \cot 2x$; $x \in (0, \frac{1}{2}\pi)$

A plot is shown at the right.

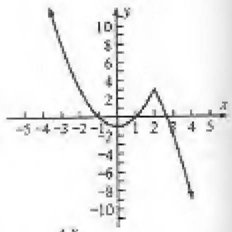


$$g'(x) = -2\csc^2 2x; \qquad g''(x) = 8\csc^2 2x \cot 2x$$

If $g''(x) = 0$, then $\cot 2x = 0$, so $2x = \frac{1}{2}\pi$ and $x = \frac{1}{4}\pi$.
If $0 < x < \frac{1}{4}\pi$ then $g''(x) > 0$, and the graph is concave upward.
If $\frac{1}{4}\pi < x < \frac{1}{2}\pi$ then $g''(x) < 0$, and the graph is concave downward.
Because $g'(\frac{1}{4}\pi) = -2$ and $g(\frac{1}{4}\pi) = 0$, the point $(\frac{1}{4}\pi, 0)$ is a point of inflection

In Exercises 17–22, find any point of inflection and determine where the graph is concave upward and concave downward. Sketch the graph.

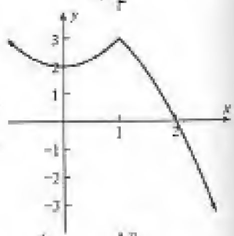
17. $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 2 \\ 7 - x^2 & \text{if } 2 \le x \end{cases}$; $f'(x) = \begin{cases} 2x & \text{if } x < 2 \\ -2x & \text{if } 2 < x \end{cases}$; $f''(x) = \begin{cases} 2 & \text{if } x < 2 \\ -2 & \text{if } 2 < x \end{cases}$



$f''(x)$ is never 0; $f'(2)$ does not exist so there is no tangent line there.

	$f(x)$	$f'(x)$	$f''(x)$	Graph is/has a
$x < 2$			+	concave upward
$x = 2$	3	doesn't exist	doesn't exist	not a point of inflection
$2 < x$			−	concave downward

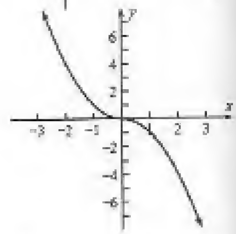
18. $f(x) = \begin{cases} 2 + x^2 & \text{if } x \le 1 \\ 4 - x^2 & \text{if } 1 < x \end{cases}$; $f'(x) = \begin{cases} 2x & \text{if } x < 1 \\ -2x & \text{if } 1 < x \end{cases}$; $f''(x) = \begin{cases} 2 & \text{if } x < 1 \\ -2 & \text{if } 1 < x \end{cases}$



$f''(x)$ is never 0; $f'(1)$ does not exist so there is no tangent line there.

	$f(x)$	$f'(x)$	$f''(x)$	Graph is/has a
$x < 1$			+	concave upward
$x = 1$	3	doesn't exist	doesn't exist	not a point of inflection
$1 < x$			−	concave downward

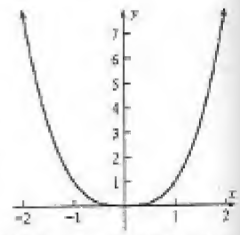
19. $g(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ -x^2 & \text{if } 0 < x \end{cases}$; $g'(x) = \begin{cases} 2x & \text{if } x \le 0 \\ -2x & \text{if } 0 < x \end{cases}$; $g''(x) = \begin{cases} 2 & \text{if } x < 0 \\ -2 & \text{if } 0 < x \end{cases}$



$g''(x)$ is never 0; $g''(0)$ does not exist but there is a tangent line at 0.

	$g(x)$	$g'(x)$	$g''(x)$	Graph is/has a
$x < 0$			+	concave upward
$x = 0$	0	0	doesn't exist	point of inflection
$0 < x$			−	concave downward

20. $g(x) = \begin{cases} -x^3 & \text{if } x < 0 \\ x^3 & \text{if } 0 \le x \end{cases}$



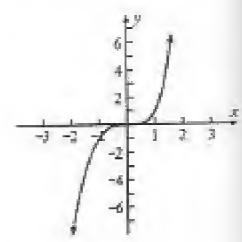
$$g'(x) = \begin{cases} -3x^2 & \text{if } x < 0 \\ 3x^2 & \text{if } 0 < x \end{cases} \quad g''(x) = \begin{cases} -6x & \text{if } x < 0 \\ 6x & \text{if } 0 < x \end{cases}$$

From Theorem 4.3.4 we have $g'_-(0) = -3(0^2) = 0$ and $g'_+(0) = 3(0^2) = 0$ and so $g'(0) = 0$. Similarly, $g''_-(0) = 0$ and $g''_+(0) = 0$ and so $g''(0) = 0$. Because $g''(x) > 0$ if $x \ne 0$, we conclude that the graph of g is concave upward at every point (including the point where $x = 0$). Thus there is no point of inflection. Because $g'(x) < 0$ if $x < 0$, then g is decreasing on the interval $(-\infty, 0]$. Because $g'(x) > 0$ if $x > 0$, the g is increasing on the interval $[0, +\infty)$. Thus $g(0) = 0$ is a relative minimum value of g, and because $g'(0) = 0$, the graph of g has a horizontal tangent line at the point $(0, 0)$. The graph is shown at the right.

21. $F(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^4 & \text{if } 0 \le x \end{cases}$; $F'(x) = \begin{cases} 3x^2 & \text{if } x < 0 \\ 4x^3 & \text{if } 0 \le x \end{cases}$; $F''(x) = \begin{cases} 6x & \text{if } x < 0 \\ 12x^2 & \text{if } 0 \le x \end{cases}$

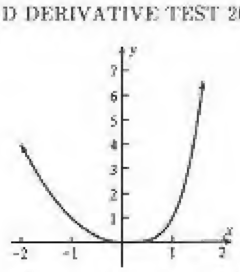
Set $F''(x) = 0$: $12x^2 = 0$; $x = 0$

	$F(x)$	$F'(x)$	$F''(x)$	Graph is/has a
$x < 0$			−	concave downward
$x = 0$	0	0	0	point of inflection
$0 < x$			+	concave upward



22. $G(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ x^4 & \text{if } 0 < x \end{cases}$; $G'(x) = \begin{cases} 2x & \text{if } x \le 0 \\ 4x^3 & \text{if } 0 < x \end{cases}$; $G''(x) = \begin{cases} 2 & \text{if } x \le 0 \\ 12x^2 & \text{if } 0 < x \end{cases}$

$G''(0)$ does not exist but $G''(x)$ does not change sign at 0,
G is concave upward on R.



In Exercises 23–30, sketch a portion of the graph of the continuous function f through the point where $x = c$ if the given conditions are satisfied.

23. (a) $f'(x) > 0$ if $x < c$; $f'(x) < 0$ if $x > c$; $f''(x) < 0$ if $x < c$; $f''(x) < 0$ if $x > c$

(b) $f'(x) > 0$ if $x < c$; $f'(x) < 0$ if $x > c$; $f''(x) > 0$ if $x < c$; $f''(x) > 0$ if $x > c$

24. (a) $f'(x) > 0$ if $x < c$; $f'(x) > 0$ if $x > c$; $f''(x) > 0$ if $x < c$; $f''(x) < 0$ if $x > c$

(b) $f'(x) < 0$ if $x < c$; $f'(x) > 0$ if $x > c$; $f''(x) > 0$ if $x < c$; $f''(x) < 0$ if $x > c$

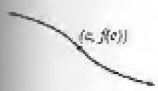
25. (a) $f''(c) = 0$; $f'(c) = 0$; $f''(x) > 0$ if $x < c$; $f''(x) < 0$ if $x > c$

(b) $f''(c) = 0$; $f'(c) = 0$; $f''(x) > 0$ if $x < c$; $f''x > 0$ if $x > c$

26. (a) $f'(c) = 0$; $f'(x) > 0$ if $x < c$; $f''(x) > 0$ if $x > c$

(b) $f'(c) = 0$; $f'(x) < 0$ if $x < c$; $f''(x) > 0$ if $x > c$

27. (a) $f''(c) = 0$; $f'(c) = -1$; $f''(x) < 0$ if $x < c$; $f''(x) > 0$ if $x > c$

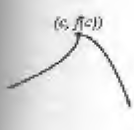


(b) $f'(c)$ doesn't exist; $f''(x) > 0$ if $x < c$; $f''(x) > 0$ if $x > c$

28. (a) $f''(c) = 0$; $f'(c) = \frac{1}{2}$; $f''(x) > 0$ if $x < c$; $f''(x) < 0$ if $x > c$

(b) $f'(c)$ doesn't exist; $f''(x) < 0$ if $x < c$; $f''(x) > 0$ if $x > c$

29. $\lim_{x \to c^-} f'(x) = +\infty$; $\lim_{x \to c^+} f'(x) = 0$;
$f''(x) > 0$ if $x < c$; $f''(x) < 0$ if $x > c$



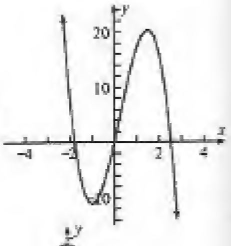
30. $\lim_{x \to c^-} f'(x) = +\infty$; $\lim_{x \to c^+} f'(x) = -\infty$;
$f''(x) > 0$ if $x < c$; $f''(x) > 0$ if $x > c$

In Exercises 31–38, find the relative extrema using the second derivative test and sketch. Check by plotting.

31. $f(x) = -4x^3 + 3x^2 + 18x$; $f'(x) = -12x^2 + 6x + 18 = -6(2x-3)(x+1)$;

$f''(x) = -24x + 6 = -24(x - \frac{1}{4})$. Set $f'(x) = 0$: $x = \frac{3}{2}$, $x = -1$.

	$f(x)$	$f'(x)$	$f''(x)$	f has a
$x = -1$	-11	0	+	relative minimum value
$x = \frac{3}{2}$	$\frac{81}{4}$	0	−	relative maximum value



32. $g(x) = 2x^3 - 9x^2 + 27$

$g'(x) = 6x^2 - 18x = 6x(x-3)$

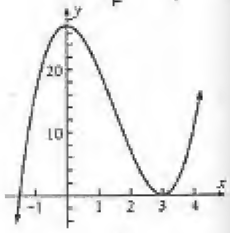
The critical numbers of g are 0 and 3.

$g''(x) = 12x - 18 = 12(x - \frac{3}{2})$

Because $g'(0) = 0$ and $g''(0) = -18 < 0$, by the second-derivative test $g(0) = 27$ is a relative maximum value.

Because $g'(3) = 0$ and $g''(3) = 18 > 0$, by the second-derivative test $g(3) = 0$ is a relative minimum value.

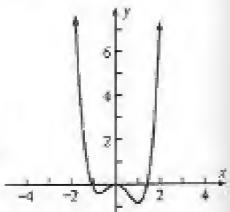
The graph of g is shown at the right.



33. $g(x) = x^4 - \frac{1}{3}x^3 - \frac{3}{2}x^2$; $g'(x) = 4x^3 - x^2 - 3x = x(4x+3)(x-1)$;

$g''(x) = 12x^2 - 2x - 3$. Set $g'(x) = 0$: $x = 0$, $x = -\frac{3}{4}$, $x = 1$

	$g(x)$	$g'(x)$	$g''(x)$	g has a
$x = -\frac{3}{4}$	$-\frac{99}{256}$	0	+	relative minimum value
$x = 0$	0	0	−	relative maximum value
$x = 1$	$-\frac{5}{6}$	0	+	relative minimum value

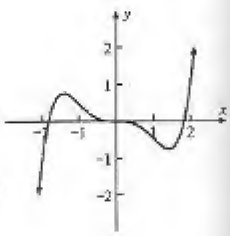


34. $f(x) = \frac{1}{5}x^5 - \frac{2}{3}x^3$; $f'(x) = x^4 - 2x^2 = x^2(x^2 - 2)$; $f''(x) = 4x^3 - 4x$

Set $f'(x) = 0$: $x = 0, \ \pm\sqrt{2}$

	$f(x)$	$f'(x)$	$f''(x)$	f has a
$x = -\sqrt{2}$	$\frac{8}{15}\sqrt{2}$	0	−	relative maximum value
$x = 0$	0	0	0	See below
$x = \sqrt{2}$	$-\frac{8}{15}\sqrt{2}$	0	+	relative minimum value

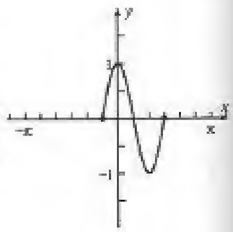
$f'''(x) = 12x^2 - 4$. $f'''(0) = -4 \neq 0$ so 0 is an inflection point.



35. $f(x) = \cos 3x$, $x \in [-\frac{1}{6}\pi, \frac{1}{2}\pi]$, $3x \in [-\frac{1}{2}\pi, \frac{3}{2}\pi]$; $f'(x) = -3\sin 3x$; $f''(x) = -9\cos 3x$

Set $f'(x) = 0$: $3x = 0$, π; $x = 0, \frac{1}{3}\pi$.

	$f(x)$	$f'(x)$	$f''(x)$	f has a
$x = 0$	1	0	−	relative maximum value
$x = \frac{1}{3}\pi$	-1	0	+	relative minimum value



$g(x) = 2\sin 4x$; $x \in [0, \frac{1}{2}\pi]$

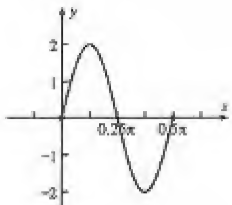
$$g'(x) = 8\cos 4x; \qquad g''(x) = -32\sin 4x$$

If $g'(x) = 0$, then $x = \frac{1}{8}\pi$ or $x = \frac{3}{8}\pi$. We apply the second-derivative test.

Because $g''(\frac{1}{8}\pi) = -32 < 0$, then $g(\frac{1}{8}\pi) = 2$ is a relative maximum value of g.

Because $g''(\frac{3}{8}\pi) = 32 > 0$, then $g(\frac{3}{8}\pi) = -2$ is a relative minimum value of g.

The graph appears at the right.

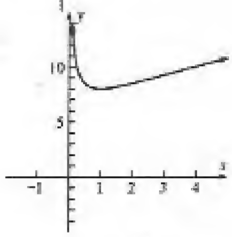


$h(x) = 4x^{1/2} + 4x^{-1/2}$; $h'(x) = 2x^{-1/2} - 2x^{-3/2}$

$h''(x) = -x^{-3/2} + 3x^{-5/2} = x^{-5/2}(3 - x)$

$h'(1) = 0$. $h'(0)$ does not exist but 0 is not in the domain of h.

Because $h''(1) = 2 > 0$, then $h(1) = 8$ is a relative minimum value of h.

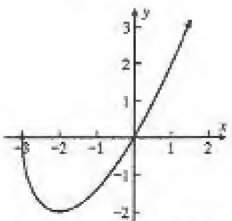


$f(x) = x\sqrt{x+3}$; $f'(x) = (x+3)^{1/2} + \frac{1}{2}x(x+3)^{-1/2} = (x+3)^{-1/2}(\frac{3}{2}x + 3)$

$f''(x) = -\frac{1}{2}(x+3)^{-3/2}(\frac{3}{2}x + 3) + (x+3)^{-1/2}(\frac{3}{2}) = (x+3)^{-3/2}(\frac{3}{4}x + 3)$

Set $f'(x) = 0$: $x = -2$.

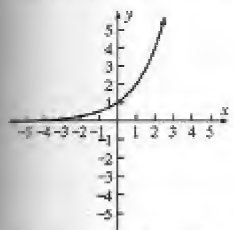
Because $f''(-2) = \frac{3}{2}$, then $f(-2) = -2$ is a relative minimum value of f.

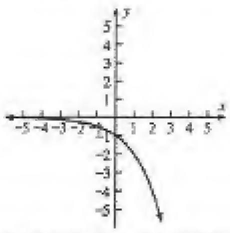


Sketch the graph of a function f for which $f(x)$, $f'(x)$, and $f''(x)$ exist and are
(a) positive for all x (b) negative for all x.

(a) Because $f(x) > 0$, the graph lies above the x axis.
Because $f'(x) > 0$, the function is always increasing.
Because $f''(x) > 0$, the graph is concave upward.
The graph has a horizontal asymptote.

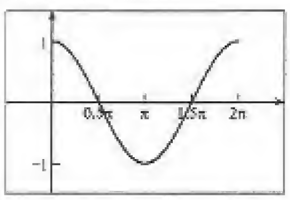
(b) Because $f(x) < 0$, the graph lies below the x axis.
Because $f'(x) < 0$, the function is always decreasing.
Because $f''(x) < 0$, the graph is concave downward.
The graph has a horizontal asymptote.





For the cosine function, find (a) the relative extrema by the second-derivative test; (b) the points of inflection; (c) the slopes of the inflectional tangents. (d) On an interval of length 2π and containing the point of inflection having the smallest positive abscissa, plot the cosine and the inflectional tangent.

$f(x) = \cos x$. (a) $f'(x) = -\sin x$; $f''(x) = -\cos x$. $f'(k\pi) = 0$ where k is any integer. If k is an even integer then $f''(k\pi) = -1$, so f has a relative maximum value. If k is an odd integer then $f''(k\pi) = 1$, so f has a relative minimum value.
(b) $f''((k + \frac{1}{2})\pi) = 0$ where k is any integer. Each such x gives a point of inflection because $f''(x)$ changes sign.
(c) The inflectional tangent has slope $f'((k + \frac{1}{2})\pi) = -\sin((k + \frac{1}{2}\pi))$ which is -1 if k is even and $+1$ if k is odd.



41. $f(x) = \tan x$

▷ (a) $f'(x) = \sec^2 x = \tan^2 x + 1$, $f''(x) = 2 \tan x \sec^2 x = 2 \tan x(\tan^2 x + 1)$
$f''(k\pi) = 0$ where k is any integer; each such x gives a point of inflection because $f''(x)$ changes sig
$f(k\pi) = 0$. (b) The slope of the inflectional tangent is $f'(k\pi) = 1$.

42. $f(x) = \cot x$; $f'(x) = -\csc^2 x$; $f''(x) = 2\csc^2 x \cot x$
(a) Set $f''(x) = 0$: $\cot x = 0$; $x = \frac{1}{2}\pi + k\pi$, where k is any integer. Each such x gives a point of inflecti
because $f''(x)$ changes sign. $f(\frac{1}{2}\pi + k\pi) = 0$
(b) The slope of the inflectional tangent is $f'(\frac{1}{2}\pi + k\pi) = -\csc^2(\frac{1}{2}\pi + k\pi) = -1$.

43. $f(x) = \csc x$; $f'(x) = -\csc x \cot x$; $f''(x) = \csc x(2\csc^2 x - 1)$
$f'(\frac{1}{2}\pi + k\pi) = 0$ where k is any integer.
If k is an even integer and $x = \frac{1}{2}\pi + k\pi$ then $f''(x) = 1$ so f has a relative minimum value.
If k is an odd integer and $x = \frac{1}{2}\pi + k\pi$ then $f''(x) = -1$ so f has a relative maximum value.

44. Find the relative extrema of the secant function by applying the second-derivative test.

▷ Let $f(x) = \sec x$. Then

$$f'(x) = \sec x \tan x$$

If $f'(x) = 0$, then $\tan x = 0$, so $x = k\pi$, where k is any integer.

$$\begin{aligned} f''(x) &= \sec^3 x + \tan^2 x \sec x \\ &= \sec x(2\tan^2 x + 1) \end{aligned}$$

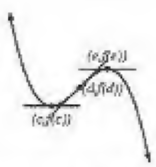
Substituting $x = k\pi$, we have

$$f''(k\pi) = \sec k\pi = \begin{cases} 1 & \text{if } k \text{ is an even integer} \\ -1 & \text{if } k \text{ is an odd integer} \end{cases}$$

By the second-derivative test, we conclude that if k is an odd integer then the secant function f has a relati
maximum value at $x = k\pi$ of -1 and if k is an even integer then the secant function f has a relati
minimum value at $x = k\pi$ of 1.

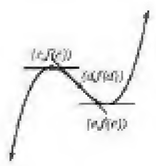
In Exercises 45–50, sketch the graph of the continuous function f through $(c, f(c))$, $(d, f(d))$, $(e, f(e))$, $c < d < e$, i
the given conditions are satisfied. At each point, draw the tangent line if there is one.

45. (a) $f'(c) = 0$; $f'(d) = 1$; $f''(d) = 0$; $f'(e) = 0$; $f''(x) > 0$ if $x < d$; $f''(x) < 0$ if $x > d$



(b) $f'(c) = 0$; $f'(d) = -1$; $f''(d) = 0$; $f'(e) = 0$; $f''(x) < 0$ if $x < d$; $f''(x) > 0$ if $x > d$

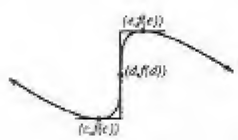
46. (a) $f'(c) = 0$; $f'(d) = -1$; $f''(d) = 0$; $f'(e) = 0$; $f''(x) < 0$ if $x < d$; $f''(x) > 0$ if $x > d$



(b) $f'(c) = 0$; $f'(d) = 1$; $f''(d) = 0$; $f'(e) = 0$; $f''(x) > 0$ if $x < d$; $f''(x) < 0$ if $d < x < e$; $f''(x) > 0$ if $x > e$

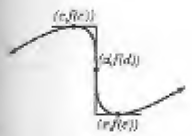
47. (a) $f'(c) = 0$; $f''(c) = 0$; $f'(d) = -1$; $f''(d) = 0$; $f'(e) = 0$; $f''(x) > 0$ if $x < c$; $f''(x) < 0$ if $c < x < d$; $f''(x) > 0$ if $x > d$

(b) $f'(c) = 0$; $\lim_{x \to d^-} f'(x) = +\infty$; $\lim_{x \to d^+} f'(x) = +\infty$; $f'(e) = 0$; $f''(x) > 0$ if $x < d$; $f''(x) < 0$ if $x > d$

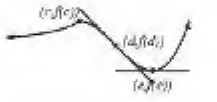


48. (a) $f'(c) = 0$; $f''(c) = 0$; $f'(d) = 1$; $f''(d) = 0$; $f'(e) = 0$; $f''(x) < 0$ if $x < c$; $f''(x) > 0$ if $c < x < d$; $f''(x) < 0$ if $x > d$

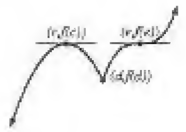
(b) $f'(c) = 0$; $\lim_{x \to d^-} f'(x) = -\infty$; $\lim_{x \to d^+} f'(x) = -\infty$; $f'(e) = 0$; $f''(x) < 0$ if $x < d$; $f''(x) > 0$ if $x > d$



49. (a) $f'(c)$ doesn't exist; $f'(d) = -1$; $f''(d) = 0$; $f'(e) = 0$; $f''(x) > 0$ if $x < c$; $f''(x) < 0$ if $c < x < d$; $f''(x) > 0$ if $x > d$



(b) $f'(c) = 0$; $f'(d)$ doesn't exist; $f'(e) = 0$; $f''(e) = 0$; $f''(x) < 0$ if $x < d$; $f''x < 0$ if $d < x < e$; $f''(x) > 0$ if $x > e$



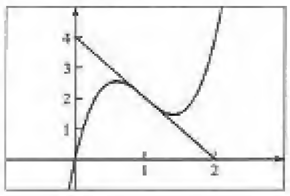
50. (a) $f'(c) = 0$; $f'(d) = -1$; $f''(d) = 0$; $f'(e)$ doesn't exist; $f''(x) < 0$ if $x < d$; $f''(x) > 0$ if $d < x < e$; $f''(x) < 0$ if $x > e$

(b) $f'(c) = 0$; $f''(c) = 0$; $f'(d)$ doesn't exist; $f'(e) = 0$; $f''(x) < 0$ if $x < c$; $f''(x) > 0$ if $c < x < d$; $f''(x) > 0$ if $x > d$

51. $f(x) = ax^3 + bx^2$; $f'(x) = 3ax^2 + 2bx$; $f''(x) = 6ax + 2b$
Since f is a polynomial, $f'(x)$ and $f''(x)$ exist everywhere. If f has a point of inflection at $(1,2)$ then $f(1) = 2$ and $f''(1) = 0$, that is $a + b = 2$ and $6a + 2b = 0$. Solving these two equations simultaneously, we get $a = -1$ and $b = 3$.

52. If $f(x) = ax^3 + bx^2 + cx$, determine a, b, and c, so that the graph of f will have a point of inflection at $(1,2)$ and so that the slope of the inflectional tangent there will be -2. Support your answer graphically.

- $f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$.
Since f is a polynomial, $f'(x)$ and $f''(x)$ exist everywhere.
Because $(1,2)$ is a point on the graph, then $f(1) = 2$ and so $2 = a + b + c$ (1)
Because $(1,2)$ is an inflection point, then $f''(1) = 0$ and so $0 = 6a + 2b$ (2)
Because the slope of the tangent at $(1,2)$ is -2, then $f'(1) = -2$ and so
$-2 = 3a + 2b + c$ (3)
Solving (1), (2) and (3) simultaneously, we find $a = 4$, $b = -12$, $c = 10$.
Because $f''(x)$ changes sign at $x = 1$, then the graph of f has a point of inflection when $x = 1$. A plot of $y = 4x^3 - 12x^2 + 10x$ is shown at the right.



53. $f(x) = ax^3 + bx^2 + cx + d$; $f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$
Since f is a polynomial, $f'(x)$ and $f''(x)$ exist everywhere. Because f has a relative extremum at $(0,3)$, $f(0) = 3$ and $f'(0) = 0$. Hence $d = 3$ and $c = 0$. Because the graph of f has a point of inflection at $(1,-1)$, $f(1) = -1$ and $f''(1) = 0$. Therefore $a + b + 3 = 0$ and $6a + 2b = 0$. Solving these equations simultaneously gives $a = 2$, $b = -6$, $c = 0$, $d = 3$.

54. If $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, determine the values of a, b, c, d, and e so the graph of f will have a point of inflection at $(1,-1)$, have the origin on it, and be symmetric with respect to the y axis.

- Because the graph contains the origin, then $f(0) = 0$. Thus, $e = 0$, and
$$f(x) = ax^4 + bx^3 + cx^2 + dx$$
The graph is symmetric with respect to the y axis and so $f(x) = f(-x)$. Thus,
$$ax^4 + bx^3 + cx^2 + dx = ax^4 - bx^3 + cx^2 - dx;\ 2bx^3 + 2dx = 0$$

This is an identity for all x if and only if $b = 0$ and $d = 0$. Thus,

$$f(x) = ax^4 + cx^2; \quad f'(x) = 4ax^3 + 2cx; \quad f''(x) = 12ax^2 + 2c$$

Because the graph contains the point $(1, -1)$, then $f(1) = -1$. Thus,

$$a + c = -1 \qquad (1)$$

Because f is a polynomial, $f''(x) = 0$ at each point of inflection. We are given that $(1, -1)$ is a point of inflection. Thus, $f''(1) = 0$, and

$$12a + 2c = 0 \qquad (2)$$

Subtracting twice Eq. (1) from Eq. (2) we get $10a = 2$, $a = \frac{1}{5}$ and substituting into Eq. (1) gives $c = -$
Therefore,

$$f(x) = \tfrac{1}{5}x^4 - \tfrac{6}{5}x^2 \quad \text{and} \quad f''(x) = \tfrac{12}{5}x^2 - \tfrac{12}{5} = \tfrac{12}{5}(x^2 - 1)$$

Because $f''(x)$ changes sign at $x = 1$, then the graph of f has a point of inflection when $x = 1$.

55. The numbers $\frac{1}{2}\sqrt{2}$ and $-\frac{1}{2}\sqrt{3}$ are critical numbers of f and $f''(x) = x[\frac{1}{2}x^2 + 1]$.

▸ Because $\frac{1}{2}x^2 + 1 \geq 1$, $[\frac{1}{2}x^2 + 1] \geq 1$ and $f''(x)$ has the same sign as x.

At $x = -\frac{1}{2}\sqrt{3}$, $f''(x) < 0$ and f has a relative maximum value.

At $x = \frac{1}{2}\sqrt{2}$, $f''(x) > 0$ and f has a relative minimum value.

56. Prove part(ii) of the second-derivative test for relative extrema.

▸ We must prove that if $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum value at c.
Because $-f'(c) = 0$ and $-f''(c) < 0$, it follows from part (i) that $-f$ has a relative maximum value at c.
Therefore f has a relative minimum value at c.

57. The graph of a function f has a point of inflection at $(c, f(c))$. Because we require a tangent line, then f must be continuous at c. The tangent may be vertical, so f' and f'' need not exist at c, and hence need not be continuous at c.

58. $f'''(x)$ exists for all values of x in I, and at a number c in I, $f''(c) = 0$ and $f'''(c)$ exists and is not 0. Because $f'''(c)$ exists, then $f'(c)$ exists, so the graph has a tangent line at c.

Suppose $f'''(c) < 0$. Then

$$f'''(c) = \lim_{x \to c} \frac{f''(x) - f''(c)}{x - c} < 0$$

Hence for some open interval I_1 containing c

$$\frac{f''(x) - f''(c)}{x - c} < 0; \ \frac{f''(x)}{x - c} < 0$$

If x is in I_1 and $x < c$ then $f''(x) > 0$.
If x is in I_1 and $x > c$ then $f''(x) < 0$.
Thus by Definition 3.5.4(ii),
$(c, f(c))$ is a point of inflection.

Suppose $f'''(c) > 0$. Then

$$f'''(c) = \lim_{x \to c} \frac{f''(x) - f''(c)}{x - c} > 0$$

Hence for some open interval I_2 containing c

$$\frac{f''(x) - f''(c)}{x - c} > 0; \ \frac{f''(x)}{x - c} > 0$$

If x is in I_2 and $x < c$ then $f''(x) < 0$.
If x is in I_2 and $x > c$ then $f''(x) > 0$.
Thus by Definition 3.5.4(i),
$(c, f(c))$ is a point of inflection.

59. If $f(t)$ represents the total units of work done after t hours, then $f'(t)$ is the rate of doing work in units/per hour, and this is maximum when $f''(t) = 0$, that is, at a point of inflection. (b) A worker can paint y frames x hours after starting work at 8 a.m. and
$y(x) = 3x + 8x^2 - x^3$, $0 \leq x \leq 4$; $y'(x) = 3 + 16 - 3x^2$; $y''(x) = 16 - 6x$. Set $y''(x) = 0$: $x = \frac{8}{3}$

	$y''(x)$	Conclusion
$0 < x < \frac{8}{3}$	+	y' is increasing and the worker is painting at an increasing rate
$x = \frac{8}{3}$	0	The worker is producing most efficiently
$\frac{8}{3} < x < 4$	−	y' is decreasing and the worker is painting at a decreasing rate

When $x = \frac{8}{3}$, the time is 2 hours and 40 minutes after 8 A.M.

▪ The worker is producing most efficiently at 10:40 A.M.

3.6 SKETCHING GRAPHS OF FUNCTIONS AND THEIR DERIVATIVES

Concavity from f' The graph of a function f is concave downward where its derivative f' is decreasing, and concave upward where where f' is increasing.

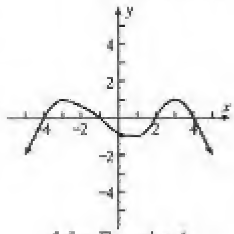
Exercises 3.6

In Exercises 1–18, determine from the figure, the graph of the derivative of a function f continuous on R, the following information and incorporate it into a table: the intervals on which f is increasing, decreasing; its relative extrema; intervals of concave upward and downward and abscissas of points of inflection. Sketch a graph of f if its only zeros are those stated. The functions of Exercises 1–6 are the same as in the indicated Exercise of §3.4.

1. Ex. 3.4.39. Zeros of f are -4, -1, 2, and 4.

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -3$	+	−	increasing	concave downward
$x = -3$	0	−	relative maximum	concave downward
$-3 < x < -1$	−	−	decreasing	concave downward
$x = -1$	−	0	decreasing	point of inflection
$-1 < x < 1$	−	+	decreasing	concave upward
$x = 1$	0	+	relative minimum	concave upward
$1 < x < 2$	+	+	increasing	concave upward
$x = 2$	+	0	increasing	point of inflection
$2 < x < 3$	+	−	increasing	concave downward
$x = 3$	0	−	relative maximum	concave downward
$x > 3$	−	−	decreasing	concave downward

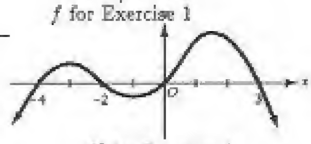
f' for Exercise 1



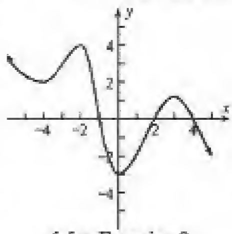
f for Exercise 1

2. Ex. 3.4.40. Zeros of f are -1, 2, and 4.

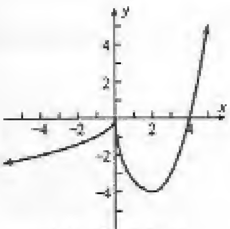
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -4$	−	+	decreasing	concave upward
$x = -4$	0	+	relative minimum	concave upward
$-4 < x < -3$	+	+	increasing	concave upward
$x = -3$	+	0	increasing	point of inflection
$-3 < x < -2$	+	−	increasing	concave downward
$x = -2$	0	−	relative maximum	concave downward
$-2 < x < -1$	−	−	decreasing	concave downward
$x = -1$	−	0	decreasing	point of inflection
$-1 < x < 0$	−	+	decreasing	concave upward
$x = 0$	0	+	relative minimum	concave upward
$0 < x < 1.5$	+	+	increasing	concave upward
$x = 1.5$	+	0	increasing	point of inflection
$1.5 < x < 3$	+	−	increasing	concave downward
$x = 3$	0	−	relative maximum	concave downward
$x > 3$	−	−	decreasing	concave downward

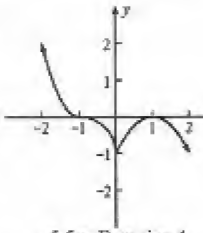


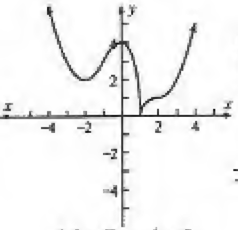
f' for Exercise 2

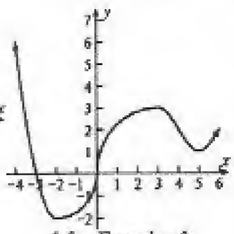


f for Exercise 2


f for Exercise 3

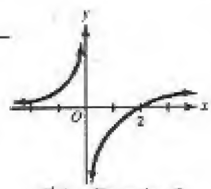

f for Exercise 4


f for Exercise 5


f for Exercise 6

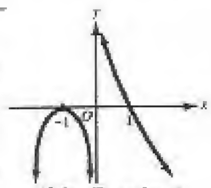
3. Ex. 3.4.41. Zeros of f are 0 and 4.

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$	$+$	$+$	increasing	concave upward
$x = 0$	d.n.e	d.n.e	relative maximum	vertical tangent
$0 < x < 2$	$-$	$+$	decreasing	concave upward
$x = 2$	0	$+$	relative minimum	concave upward
$x > 2$	$+$	$+$	increasing	concave upward


f' for Exercise 3

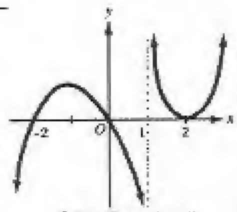
4. Ex. 3.4.42. Zeros of f are -1 and 1.

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$	$-$	$+$	decreasing	concave upward
$x = -1$	0	0	stationary	point of inflection
$-1 < x < 0$	$-$	$-$	decreasing	concave downward
$x = 0$	d.n.e.	d.n.e.	relative minimum	vertical tangent
$0 < x < 1$	$+$	$-$	increasing	concave downward
$x = 1$	0	$-$	relative maximum	concave downward
$x > 1$	$-$	$-$	decreasing	concave downward


f' for Exercise 4

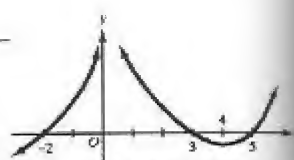
5. Ex. 3.4.43. Zero of f is 1.

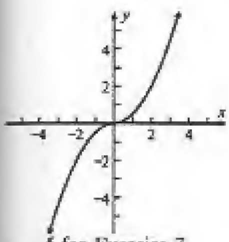
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	$-$	$+$	decreasing	concave upward
$x = -2$	0	$+$	relative minimum	concave upward
$-2 < x < -1$	$+$	$+$	increasing	concave upward
$x = -1$	$+$	0	increasing	point of inflection
$-1 < x < 0$	$+$	$-$	increasing	concave downward
$x = 0$	0	$-$	relative maximum	concave downward
$0 < x < 1$	$-$	$-$	decreasing	concave downward
$x = 1$	d.n.e.	d.n.e.	relative minimum	vertical tangent
$1 < x < 2$	$+$	$-$	increasing	concave downward
$x = 2$	0	0	stationary	point of inflection
$x > 2$	$+$	$+$	increasing	concave upward


f' for Exercise 5

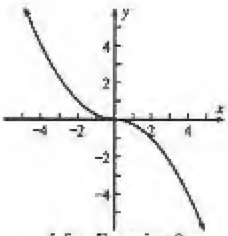
6. Ex. 3.4.44. Zeros of f are -3 and 0.

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	$-$	$+$	decreasing	concave upward
$x = -2$	0	$+$	relative minimum	concave upward
$-2 < x < 0$	$+$	$+$	increasing	concave upward
$x = 0$	d.n.e.	d.n.e.	vertical tangent	point of inflection
$0 < x < 3$	$+$	$-$	increasing	concave downward
$x = 3$	0	$-$	relative maximum	concave downward
$3 < x < 4$	$-$	$-$	decreasing	concave downward
$x = 4$	$-$	0	decreasing	point of inflection
$4 < x < 5$	$-$	$+$	decreasing	concave upward
$x = 5$	0	$+$	relative minimum	concave upward
$x > 5$	$+$	$+$	increasing	concave upward

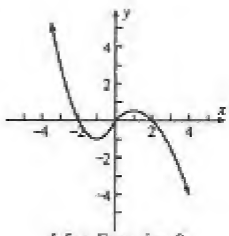

f' for Exercise 6



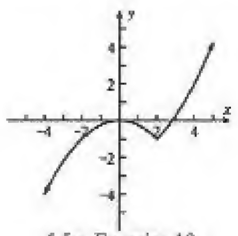
f for Exercise 7



f for Exercise 8



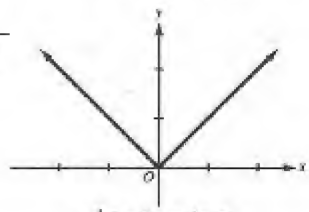
f for Exercise 9



f for Exercise 10

7. Zero of f is 0.

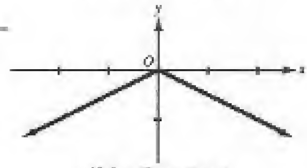
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$	$+$	$-$	increasing	concave downward
$x = 0$	0	d.n.e.	stationary	point of inflection
$x > 0$	$-$	$+$	increasing	concave upward



f' for Exercise 7

8. Zero of f is 0.

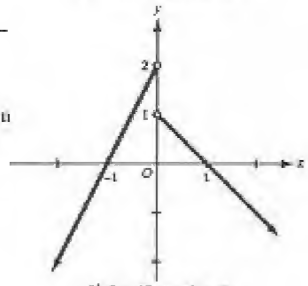
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$	$-$	$+$	decreasing	concave upward
$x = 0$	0	d.n.e.	stationary	point of inflection
$x > 0$	$-$	$-$	decreasing	concave downward



f' for Exercise 8

9. Zeros of f are -2, 0, and 2.

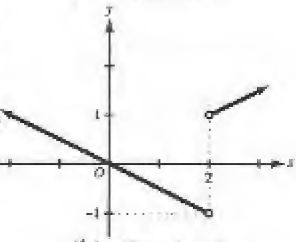
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$	$-$	$+$	decreasing	concave upward
$x = -1$	0	$+$	relative minimum	concave upward
$-1 < x < 0$	$+$	$+$	increasing	concave upward
$x = 0$	d.n.e.	d.n.e.	no tangent line	not a point of inflection
$0 < x < 1$	$+$	$-$	increasing	concave downward
$x = 1$	0	$-$	relative maximum	concave downward
$x > 1$	$-$	$-$	decreasing	concave downward



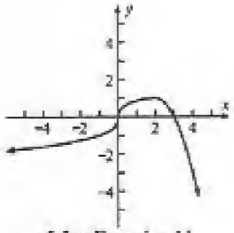
f' for Exercise 9

10. Zeros of f are 0 and 4.

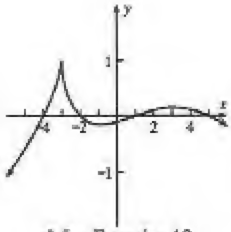
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$	$+$	$-$	increasing	concave downward
$x = 0$	0	$-$	relative maximum	concave downward
$0 < x < 2$	$-$	$-$	decreasing	concave downward
$x = 2$	d.n.e.	d.n.e.	no tangent line	not a point of inflection
$x > 2$	$+$	$+$	increasing	concave upward



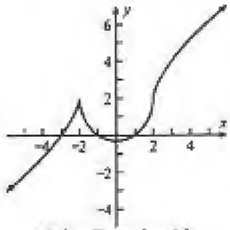
f' for Exercise 10



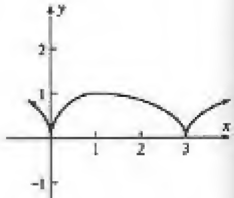
f for Exercise 11



f for Exercise 12



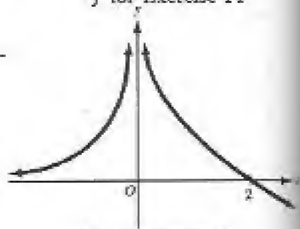
f for Exercise 13



f for Exercise 14

11. Zeros of f are 0 and 3.

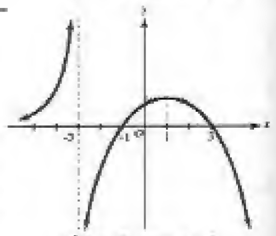
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$	+	+	increasing	concave upward
$x = 0$	d.n.e.	d.n.e.	vertical tangent	point of inflection
$0 < x < 2$	+	−	increasing	concave downward
$x = 2$	0	−	relative maximum	concave downward
$x > 2$	−	−	decreasing	concave downward



f' for Exercise 11

12. Zeros of f are −4, −2, 1 and 5.

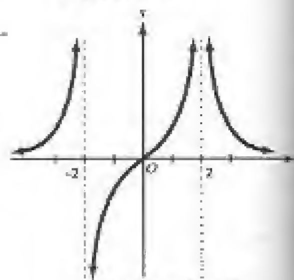
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -3$	+	+	increasing	concave upward
$x = -3$	d.n.e.	d.n.e	relative maximum	vertical tangent
$-3 < x < -1$	−	+	decreasing	concave upward
$x = -1$	0	+	relative minimum	concave upward
$-1 < x < 1$	+	+	increasing	concave upward
$x = 1$	+	0	increasing	point of inflection
$1 < x < 3$	+	−	increasing	concave downward
$x = 3$	0	−	relative maximum	concave downward
$x > 3$	−	+	decreasing	concave downward



f' for Exercise 12

13. Zeros of f are −3, −1, and 1.

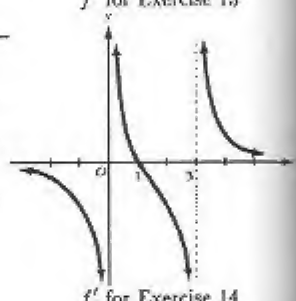
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	+	+	increasing	concave upward
$x = -2$	d.n.e.	d.n.e.	relative maximum	vertical tangent
$-2 < x < 0$	−	+	decreasing	concave upward
$x = 0$	0	+	relative minimum	concave upward
$0 < x < 2$	+	+	increasing	concave upward
$x = 2$	d.n.e.	d.n.e.	vertical tangent	point of inflection
$x > 2$	+	−	increasing	concave downward



f' for Exercise 13

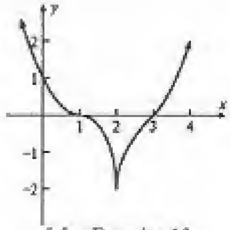
14. Zeros of f are 0 and 3.

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$	−	−	decreasing	concave downward
$x = 0$	d.n.e.	d.n.e.	relative minimum	vertical tangent
$0 < x < 1$	+	−	increasing	concave downward
$x = 1$	0	−	relative maximum	concave downward
$1 < x < 3$	−	−	decreasing	concave downward
$x = 3$	d.n.e.	d.n.e.	relative minimum	vertical tangent
$x > 3$	+	−	increasing	concave downward

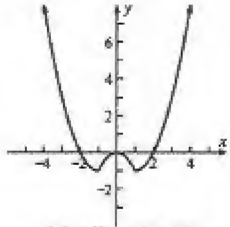


f' for Exercise 14

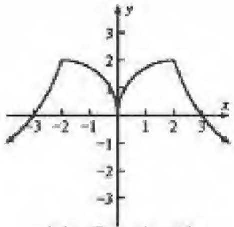
f for Exercise 15



f for Exercise 16



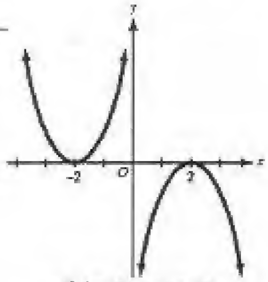
f for Exercise 17



f for Exercise 18

15. Zeros of f are -2 and 2.

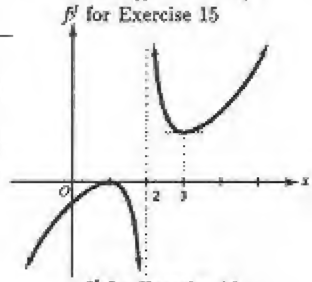
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	$+$	$-$	increasing	concave downward
$x = -2$	0	0	stationary	point of inflection
$-2 < x < 0$	$+$	$+$	increasing	concave upward
$x = 0$	d.n.e.	d.n.e.	relative maximum	vertical tangent
$0 < x < 2$	$-$	$+$	decreasing	concave upward
$x = 2$	0	0	stationary	point of inflection
$x > 2$	$-$	$-$	decreasing	concave downward



f' for Exercise 15

16. Zeros of f are 1 and 3.

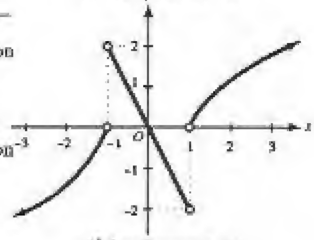
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 1$	$-$	$+$	decreasing	concave upward
$x = 1$	0	0	stationary	point of inflection
$1 < x < 2$	$-$	$-$	decreasing	concave downward
$x = 2$	d.n.e.	d.n.e.	relative minimum	vertical tangent
$2 < x < 3$	$+$	$-$	increasing	concave downward
$x = 3$	$+$	0	increasing	point of inflection
$x > 3$	$+$	$+$	increasing	concave upward



f' for Exercise 16

17. Zeros of f are -2, 0 and 2.

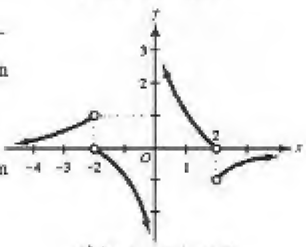
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$	$-$	$+$	decreasing	concave upward
$x = -1$	d.n.e.	d.n.e.	relative minimum	not a point of inflection
$-1 < x < 0$	$+$	$-$	increasing	concave downward
$x = 0$	0	$-$	relative maximum	concave downward
$0 < x < 1$	$-$	$-$	decreasing	concave downward
$x = 1$	d.n.e.	d.n.e.	relative minimum	not a point of inflection
$x > 1$	$+$	$+$	increasing	concave upward



f' for Exercise 17

18. Zeros of f are -3, 0 and 3.

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	$+$	$+$	increasing	concave upward
$x = -2$	d.n.e.	d.n.e.	relative maximum	not a point of inflection
$-2 < x < 0$	$-$	$-$	decreasing	concave downward
$x = 0$	d.n.e.	d.n.e.	relative minimum	vertical tangent
$0 < x < 2$	$+$	$-$	increasing	concave downward
$x = 2$	d.n.e.	d.n.e.	relative maximum	not a point of inflection
$x > 2$	$-$	$+$	decreasing	concave upward

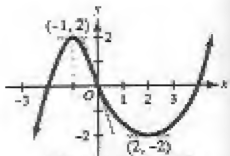


f' for Exercise 18

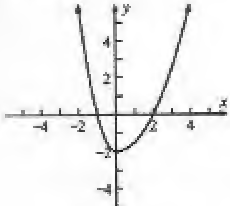
In Exercises 19–26, the graph of f and segments of the inflectional tangents appear in the figure. Make a table as in the previous Exercises and sketch possible graphs of f' and f''.

19.

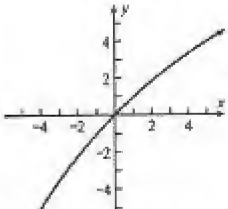
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$	+	−	increasing	concave downward
$x = -1$	0	−	relative maximum	concave downward
$-1 < x < 0$	−	−	decreasing	concave downward
$x = 0$	−2	0	decreasing	point of inflection
$0 < x < 2$	−	+	decreasing	concave upward
$x = 2$	0	+	relative minimum	concave upward
$x > 2$	+	+	increasing	concave upward



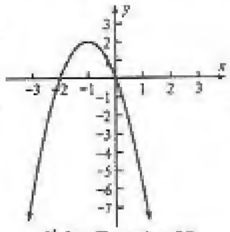
f for Exercise 19



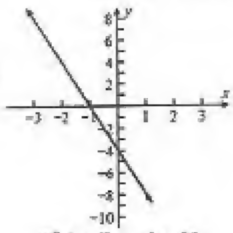
f' for Exercise 19



f'' for Exercise 19



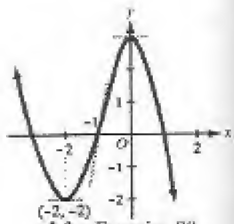
f' for Exercise 20



f'' for Exercise 20

20.

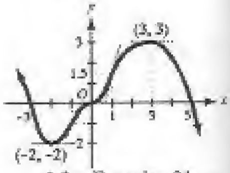
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	−	+	decreasing	concave upward
$x = -2$	0	+	relative minimum	concave upward
$-2 < x < -1$	+	+	increasing	concave upward
$x = -1$	+2	0	increasing	point of inflection
$-1 < x < 0$	+	−	increasing	concave downward
$x = 0$	0	−	relative maximum	concave downward
$x > 0$	−	−	decreasing	concave downward



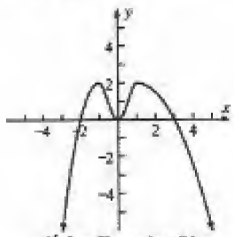
f for Exercise 20

21.

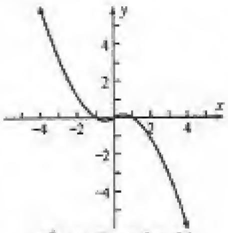
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	−	+	decreasing	concave upward
$x = -2$	0	+	relative minimum	concave upward
$-2 < x < -1$	+	+	increasing	concave upward
$x = -1$	+2	0	increasing	point of inflection
$-1 < x < 0$	+	−	increasing	concave downward
$x = 0$	0	0	stationary	point of inflection
$0 < x < 1$	+	+	increasing	concave upward
$x = 1$	+2	0	increasing	point of inflection
$1 < x < 3$	+	−	increasing	concave downward
$x = 3$	0	−	relative maximum	concave downward
$x > 3$	−	−	decreasing	concave downward



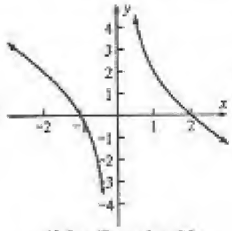
f for Exercise 21



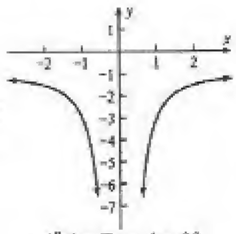
f' for Exercise 21



f'' for Exercise 21

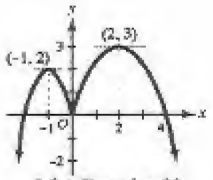


f' for Exercise 22



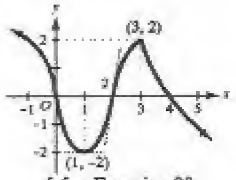
f'' for Exercise 22

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$	+	−	increasing	concave downward
$x = -1$	0	−	relative maximum	concave downward
$-1 < x < 0$	−	−	decreasing	concave downward
$x = 0$	d.n.e.	d.n.e.	relative minimum	vertical tangent
$0 < x < 2$	+	−	increasing	concave downward
$x = 2$	0	−	relative maximum	concave downward
$x > 2$	−	−	decreasing	concave downward

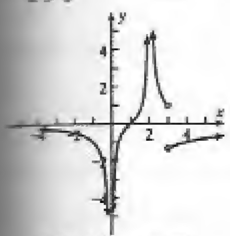


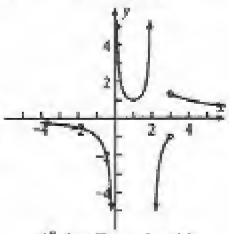
f for Exercise 22

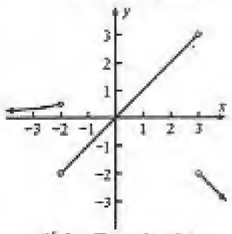
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$	−	−	decreasing	concave downward
$x = 0$	d.n.e.	d.n.e	vertical tangent	point of inflection
$0 < x < 1$	−	+	decreasing	concave upward
$x = 1$	0	+	relative minimum	concave upward
$1 < x < 2$	+	+	increasing	concave upward
$x = 2$	d.n.e.	d.n.e.	vertical tangent	point of inflection
$2 < x < 3$	+	−	increasing	concave downward
$x = 3$	d.n.e.	d.n.e.	relative maximum	not a point of inflection
$x > 3$	−	+	decreasing	concave upward

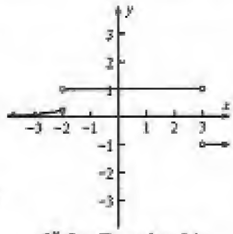


f for Exercise 23

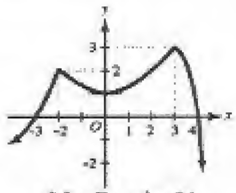

f' for Exercise 23


f'' for Exercise 23

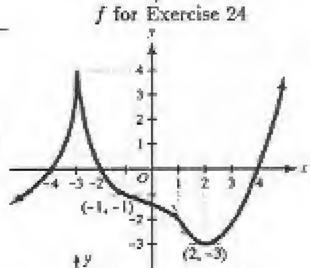

f' for Exercise 24


f'' for Exercise 24

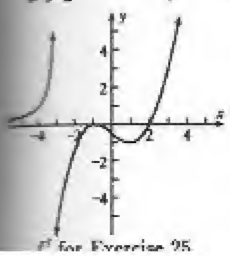
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	+	+	increasing	concave upward
$x = -2$	d.n.e.	d.n.e.	relative maximum	not a point of inflection
$-2 < x < 0$	−	+	decreasing	concave upward
$x = 0$	0	+	relative minimum	concave upward
$0 < x < 3$	+	+	increasing	concave upward
$x = 3$	d.n.e.	d.n.e.	relative maximum	not a point of inflection
$x > 3$	−	−	decreasing	concave downward

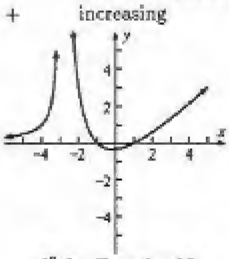

f for Exercise 24

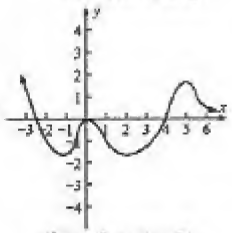
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -3$	+	+	increasing	concave upward
$x = -3$	d.n.e.	d.n.e.	relative maximum	vertical tangent
$-3 < x < -1$	−	+	decreasing	concave upward
$x = -1$	0	0	stationary	point of inflection
$-1 < x < 1$	−	−	decreasing	concave downward
$x = 1$	−1	0	decreasing	point of inflection
$1 < x < 2$	−	+	decreasing	concave upward
$x = 2$	0	+	relative minimum	concave upward
$x > 2$	+	+	increasing	concave upward

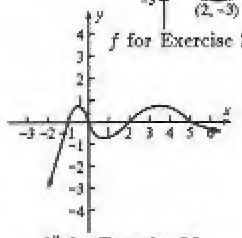


f for Exercise 25


f' for Exercise 25

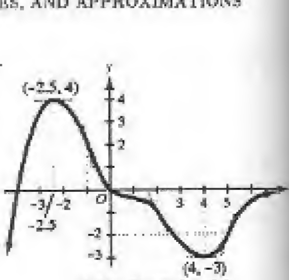

f'' for Exercise 25


f' for Exercise 26


f'' for Exercise 26

26.

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2.5$	+	−	increasing	concave downward
$x = -2.5$	0	−	relative maximum	concave downward
$-2.5 < x < -1$	−	−	decreasing	concave downward
$x = -1$	−2	0	decreasing	point of inflection
$-1 < x < 0$	−	+	decreasing	concave upward
$x = 0$	0	0	stationary	point of inflection
$0 < x < 2$	−	−	decreasing	concave downward
$x = 2$	−2	0	decreasing	point of inflection
$2 < x < 4$	−	+	decreasing	concave upward
$x = 4$	0	+	relative minimum	concave upward
$4 < x < 5$	+	+	increasing	concave upward
$x = 5$	+2	0	increasing	point of inflection
$x > 5$	+	−	increasing	concave downward



f for Exercise 26

27. $f(x) = 3x^2 + x|x| = \begin{cases} 3x^2 - x^2 = 2x^2 & \text{if } x < 0 \\ 3x^2 + x^2 = 4x^2 & \text{if } x \geq 0 \end{cases}$, $f'(x) = \begin{cases} 4x & \text{if } x < 0 \\ 8x & \text{if } x \geq 0 \end{cases}$, $f''(x) = \begin{cases} 4 & \text{if } x < 0 \\ 8 & \text{if } x > 0 \end{cases}$

Because $f''_-(0) = 4$ and $f''_+(0) = 8$, $f''(0)$ does not exist. Because $f''(x) > 0$ if $x \neq 0$ and $f'(0) = 0$ exists, graph of f is concave upward everywhere by Theorem 3.5.3′.

28. Given $f(x) = x^r - rx + k$, where r is a rational number, prove that (a) if $0 < r < 1$, f has a relative maxim value at 1; (b) if $r < 0$ or $r > 1$, f has a relative minimum value at 1.

▹ $f'(x) = rx^{r-1} - r$

We note that $f'(1) = r \cdot 1^{r-1} - r = 0$ for all r.

$f''(x) = r(r-1)x^{r-2}$; $\quad f''(1) = r(r-1) \cdot 1^{r-2} = r(r-1)$

(a) If $0 < r < 1$, then $r(r-1) < 0$. Because $f'(1) = 0$ and $f''(1) < 0$, by the second-derivative test f has relative maximum value at 1.

(b) If $r < 0$ or $r > 1$, then $r(r-1) > 0$. Because $f'(1) = 0$ and $f''(1) > 0$, by the second-derivative test f has relative minimum value at 1.

29. $f(x) = x^3 + 3rx + 5$; $f'(x) = 3x^2 + 3r$; $f''(x) = 6x$

(a) $f'(x) = 3x^2 + 3r > 0$ for all x if $r > 0$. Therefore f is increasing on $(-\infty, +\infty)$ and has no relative extrema.

(b) If $r < 0$, then $f'(\sqrt{-r}) = 0$ and $f'(-\sqrt{-r}) = 0$.

At $x = \sqrt{-r}$, $f''(x) = 6\sqrt{-r} > 0$ which implies that f has a relative minimum value.

At $x = -\sqrt{-r}$, $f''(x) = -6\sqrt{-r} < 0$ which implies that f has a relative maximum value.

30. $f(x) = x^2 + rx^{-1}$; $f'(x) = 2x - rx^{-2} = x^{-2}(2x^3 - r)$; $f''(x) = 2 + 2rx^{-3}$.

$f'\left(\sqrt[3]{\tfrac{1}{2}r}\right) = 0$ and $f''\left(\sqrt[3]{\tfrac{1}{2}r}\right) = 2 + 2r \cdot \frac{2}{r} = 6 > 0$. Thus f has a relative minimum value.

Because f has no other critical number, f has no other relative extrema.

31. Sketch the graph of the astroid $x^{2/3} + y^{2/3} = 1$. The portion in the first quadrant is the graph of a function. Obtain this portion and then complete the graph by symmetry properties.

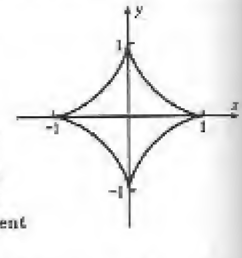
▹ The graph is symmetric with respect to the x and y axes and to the line $y = x$.

We solve for y when $x \geq 0$ and $y \geq 0$

$y^{2/3} = 1 - x^{2/3}$; $y = (1 - x^{2/3})^{3/2}$

Thus y is defined for $0 \leq x \leq 1$. Differentiating with respect to x we obtain

$$y' = \tfrac{3}{2}(1 - x^{2/3})^{1/2}(-\tfrac{2}{3}x^{-1/3}) = -x^{1/3}(1 - x^{2/3})^{1/2} \qquad (1)$$



Thus y' is negative and so the graph decreases from the point $(0, 1)$ to the point $(1, 0)$. Furthermore, y' approaches $-\infty$ as x approaches 0 from the right and so the graph has a vertical tangent at $(0, 1)$; by symmetry, it has a horizontal tangent at $(1, 0)$. Differentiating Eq. (1) with respect to x we get

$$y'' = \tfrac{1}{3}x^{-4/3}(1 - x^{2/3})^{1/2} - x^{-1/3}(\tfrac{1}{2})(1 - x^{2/3})^{-1/2}(-\tfrac{2}{3}x^{-1/3}) = \tfrac{1}{3}x^{-4/3}(1 - x^{2/3})^{1/2}[(1 - x^{2/3}) + x^{2/3}]$$
$$= \tfrac{1}{3}x^{-4/3}(1 - x^{2/3})^{-1/2}$$

Because y'' is positive if $0 < x < 1$, the graph is concave upward in the first quadrant.

3.7 LIMITS AT INFINITY

3.7.1 Definition Let f be a function that is defined at every number in some interval $(a, +\infty)$. The *limit of $f(x)$, as x increases without bound*, is L, written

$$\lim_{x \to +\infty} f(x) = L$$

if for any $\epsilon > 0$, however small, there exists a number $N > 0$ such that if $x > N$, then $|f(x) - L| < \epsilon$.

3.7.2 Definition Let f be a function that is defined at every number in some interval $(-\infty, a)$. The *limit of $f(x)$, as x decreases without bound*, is L, written

$$\lim_{x \to -\infty} f(x) = L$$

if for any $\epsilon > 0$, however small, there exists a number $N < 0$ such that if $x < N$, then $|f(x) - L| < \epsilon$.

Limit Theorem 13 If r is any positive integer, then

(i) $\displaystyle\lim_{x \to +\infty} \frac{1}{x^r} = 0$

(ii) $\displaystyle\lim_{x \to -\infty} \frac{1}{x^r} = 0$

Limit Theorems 2, 4, 5, 6, 7, 8, 9, and 10, given in Sec. 1.5, and Limit Theorems 11 and 12, given in Sec. 1.7, remain valid when "$x \to a$" is replaced by "$x \to +\infty$" or "$x \to -\infty$".

We will use the following theorem whose proof is similar to that of Limit Theorem 12.

Theorem B Let a be any real number or $+\infty$ or $-\infty$ and L any real number.

(i) If $\lim_{x \to a} g(x) > 0$ and $\lim_{x \to a} f(x) = +\infty$ then $\lim_{x \to a} f(x)g(x) = +\infty$

(ii) If $\lim_{x \to a} g(x) > 0$ and $\lim_{x \to a} f(x) = -\infty$ then $\lim_{x \to a} f(x)g(x) = -\infty$

(iii) If $\lim_{x \to a} g(x) < 0$ and $\lim_{x \to a} f(x) = +\infty$ then $\lim_{x \to a} f(x)g(x) = -\infty$

(iv) If $\lim_{x \to a} g(x) < 0$ and $\lim_{x \to a} f(x) = -\infty$ then $\lim_{x \to a} f(x)g(x) = +\infty$

(v) If $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} f(x) = +\infty$ then $\lim_{x \to a} [f(x) + g(x)] = +\infty$

(vi) If $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} f(x) = -\infty$ then $\lim_{x \to a} [f(x) + g(x)] = -\infty$

(vii) If $\lim_{x \to a} f(x) = +\infty$ then $\lim_{x \to a} \sqrt{f(x)} = +\infty$

Suppose that f is a function defined by

$$f(x) = \frac{g(x)}{h(x)}$$

where $g(x)$ and $h(x)$ are polynomials. To find the limit of f as $x \to +\infty$ or as $x \to -\infty$, first divide both $g(x)$ and $h(x)$ by the highest power of x that appears in the denominator. Then you may use Limit Theorem 13 together with the other limit theorems to find the limit of f.

It can be shown that if g and h are both polynomial functions, then

(i) If the degree of g is less than the degree of h, then

$$\lim_{x \to +\infty} \frac{g(x)}{h(x)} = \lim_{x \to -\infty} \frac{g(x)}{h(x)} = 0$$

(ii) If the degree of g and h is each equal to n, a is the coefficient of x^n in $g(x)$, and b is the coefficient of x^n in $h(x)$, then

$$\lim_{x \to +\infty} \frac{g(x)}{h(x)} = \lim_{x \to -\infty} \frac{g(x)}{h(x)} = \frac{a}{b}$$

(iii) If the degree of g is greater than the degree of h, then

$$\lim_{x \to +\infty} \frac{g(x)}{h(x)} = \lim_{x \to -\infty} \frac{g(x)}{h(x)} = \pm\infty$$

Exercise 20 illustrates part (i); Exercises 12 and 16 illustrate part (ii); and Exercise 24 illustrates part (iii) of the above discussion.

Asymptotes A line is an *asymptote* of a curve if the distance between the line and a point P on the cur approaches zero as the distance of P from the origin approaches infinity.

Let $f(x) = g(x)/h(x)$ be a rational function. The graph of f has a vertical asymptote $x =$ for each number a such that $h(a) = 0$ and $g(a) \neq 0$. If the degree of g is less than the deg of h then $y = 0$ (the x axis) is a horizontal asymptote of the graph of f, corresponding to ca (i) above. If the degree of g and h is equal to n, a is the coefficient of x^n in $g(x)$ and b is coefficient of x^n in $h(x)$, then $y = a/b$ is a horizontal asymptote, corresponding to case above. If the degree of g is one more than the degree of h and long division gives a quotient $mx + b$, then $y = mx + b$ is an oblique asymptote.

3.7.4 Definition The line $y = b$ is said to be *a horizontal asymptote* of the graph of the function f if at le one of the following statements is true:

(i) $\lim\limits_{x \to +\infty} f(x) = b$, and for some number N, if $x > N$ then $f(x) \neq b$

(ii) $\lim\limits_{x \to -\infty} f(x) = b$, and for some number N, if $x < N$ then $f(x) \neq b$

3.7.5 Definition The line $y = mx + b$ is said to be *an oblique asymptote* of the graph of the function f if least one of the following statements is true:

(i) $\lim\limits_{x \to +\infty} [f(x) - (mx + b)] = 0$, and for some number N, if $x > N$ then $f(x) \neq mx + b$

(ii) $\lim\limits_{x \to -\infty} [f(x) - (mx + b)] = 0$, and for some number N, if $x < N$ then $f(x) \neq mx + b$

Exercises 3.7

In Exercises 1–10, do the following: Use a calculator to tabulate the values of $f(x)$ for the specified values of (a) What does $f(x)$ appear to be approaching as x increases without bound? (b) What does $f(x)$ appear to approaching as x decreases without bound? (c) Check by plotting. (d) Find $\lim\limits_{x \to +\infty} f(x)$. (e) Find $\lim\limits_{x \to -\infty} f(x)$.

1. $f(x) = \frac{4}{x^2}$ ▷

x	1	2	4	6	8	10	100	1000
f	4	1	.25	.1111	.0625	.0400	.0004	.000004

x	−1	−2	−4	−6	−8	−10	−100	−1000
f	4	1	.25	.1111	.0625	.0400	.0004	.000004

(a) 0 (d) $\lim\limits_{x \to +\infty} \frac{4}{x^2} =$

(b) 0 (e) $\lim\limits_{x \to -\infty} \frac{4}{x^2} =$

2. $f(x) = \frac{3}{x^4}$ ▷

x	1	2	4	6	8	10	100	1000
f	3	.1875	.0117	.0023	.0007	.0003	3×10^{-8}	3×10^{-12}

x	−1	−2	−4	−6	−8	−10	−100	−1000
f	3	.1875	.0117	.0023	.0007	.0003	3×10^{-8}	3×10^{-12}

(a) 0 (d) $\lim\limits_{x \to +\infty} \frac{4}{x^2} =$

(b) 0 (e) $\lim\limits_{x \to -\infty} \frac{4}{x^2} =$

3. $f(x) = \frac{1}{x^3}$ ▷

x	1	2	4	6	8	10	100	1000
f	1	.1250	.0156	.0046	.0020	.0010	10^{-6}	10^{-9}

x	−1	−2	−4	−6	−8	−10	−100	−1000
f	−1	−.1250	−.0156	−.0046	−.0020	−.0010	-10^{-6}	-10^{-9}

(a) 0 (d) $\lim\limits_{x \to +\infty} \frac{1}{x^3} =$

(b) 0 (e) $\lim\limits_{x \to -\infty} \frac{1}{x^3} =$

4. $f(x) = -\frac{2}{x^3}$; x is 1, 2, 4, 6, 8, 10, 100, 1000 and x is −1, −2, −4, −6, −8, −10, −100, −1000.

▷ (a) See Table 4a. $f(x)$ appears to be approaching 0 as x increases without bound.

Table 4a

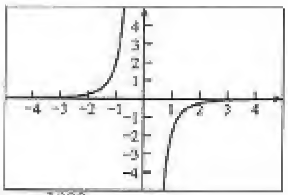
x	1	2	4	6	8	10	100	1000
$f(x) = -2/x^3$	−2	−0.25	−0.0313	−0.0093	−0.0039	−0.0020	-2×10^{-6}	-2×10^{-9}

(b) See Table 4b. $f(x)$ appears to be approaching 0 as x decreases without bound.

Table 4b

x	−1	−2	−4	−6	−8	−10	−100	−1000
$f(x) = -2/x^3$	2	0.25	0.0313	0.0093	0.0039	0.0020	2×10^{-6}	2×10^{-9}

(c) A plot is given at the right.



(d) Using Limit Theorem 13(i) with $r = 3$ we have

$$\lim_{x\to+\infty} -\frac{2}{x^3} = \lim_{x\to+\infty} (-2)\cdot \lim_{x\to+\infty} \frac{1}{x^3} = (-2)0 = 0$$

(e) Using Limit Theorem 13(ii) with $r = 3$ we have

$$\lim_{x\to-\infty} -\frac{2}{x^3} = \lim_{x\to-\infty} (-2)\cdot \lim_{x\to-\infty} \frac{1}{x^3} = (-2)0 = 0$$

5. $f(x) = \dfrac{-3x^2}{x^2+1}$.

x	0	1	2	4	6	8	10	100	1000
f	0	−1.5	−2.4	−2.823	−2.919	−2.953	−2.970	−2.9997	−2.999997

(a) −3

x	−1	−2	−4	−6	−8	−10	−100	−1000
f	−1.5	−2.4	−2.823	−2.919	−2.953	−2.970	−2.9997	−2.999997

(b) −3

(d) $\displaystyle\lim_{x\to+\infty} \frac{-3x^2}{x^2+1} = \lim_{x\to+\infty} \frac{-3}{1+1/x^2} = \frac{-3}{1} = -3$ (e) $\displaystyle\lim_{x\to-\infty} \frac{-3x^2}{x^2+1} = \lim_{x\to-\infty} \frac{-3}{1+1/x^2} = \frac{-3}{1} = -3$

6. $f(x) = \dfrac{x^3}{x^3+2}$.

x	2	4	6	8	10	100	1000
f	0.8	0.9697	0.9908	0.9961	0.9980	0.999998	1

(a) 1

x	−2	−4	−6	−8	−10	−100	−1000
f	1.3333	1.0323	1.0093	1.0039	1.0020	1.000002	1

(b) 1

(d) $\displaystyle\lim_{x\to+\infty} \frac{-3x^2}{x^2+1} = \lim_{x\to+\infty} \frac{-3}{1+1/x^2} = \frac{-3}{1} = -3$ (e) $\displaystyle\lim_{x\to-\infty} \frac{-3x^2}{x^2+1} = \lim_{x\to-\infty} \frac{-3}{1+1/x^2} = \frac{-3}{1} = -3$

7. $f(x) = \dfrac{4x+1}{2x-1}$.

x	2	6	10	100	1000	10,000	100,000
f	3	2.273	2.158	2.015	2.0015	2.00015	2.000015

(a) 2

x	−2	−6	−10	−100	−1000	−10,000	−100,000
f	1.4	1.769	1.857	1.985	1.9985	1.99985	1.999985

(b) 2

(d) $\displaystyle\lim_{x\to+\infty} \frac{4x+1}{2x-1} = \lim_{x\to+\infty} \frac{4+\frac{1}{x}}{2-\frac{1}{x}} = \frac{4}{2} = 2$ (e) $\displaystyle\lim_{x\to-\infty} \frac{4x+1}{2x-1} = \lim_{x\to-\infty} \frac{4+\frac{1}{x}}{2-\frac{1}{x}} = \frac{4}{2} = 2$

8. $f(x) = \dfrac{5x-3}{10x+1}$;

x is 2, 6, 10, 100, 1000, 10,000, 100,000 and x is −2, −6, −10, −100, −1000, −10,000, −100,000.

(a) See Table 8a. $f(x)$ appears to be approaching 0.5 as x increases without bound.

Table 8a

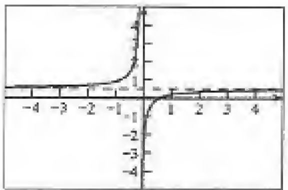
x	2	6	10	100	1000	10,000	100,000
$f(x) = \frac{5x-3}{10x+1}$	0.33333	0.44262	0.46535	0.49650	0.49965	0.49997	0.50000

(b) See Table 8b. $f(x)$ appears to be approaching 0.5 as x decreases without bound.

Table 8b

x	−2	−6	−10	−100	−1000	−10,000	−100,000
$f(x) = \frac{5x-3}{10x+1}$	0.68421	0.55932	0.53535	0.50350	0.50035	0.50004	0.50000

(c) A plot is given at the right.



(d) To use Limit Theorem 13, divide numerator and denominator by x:

$$\lim_{x\to+\infty} \frac{5x-3}{10x+1} = \lim_{x\to+\infty} \frac{5-\frac{3}{x}}{10+\frac{1}{x}} = \frac{\lim\limits_{x\to+\infty} 5 - \lim\limits_{x\to+\infty} 3 \cdot \lim\limits_{x\to+\infty} \frac{1}{x}}{\lim\limits_{x\to+\infty} 10 - \lim\limits_{x\to+\infty} \frac{1}{x}} = \frac{5-3(0)}{10-0} = \frac{1}{2}$$

(e) Because Limit Theorem 13 gives the same result for $x \to -\infty$ as for $x \to +\infty$, $\displaystyle\lim_{x\to-\infty} \frac{5x-3}{10x+1} = \frac{1}{2}$

9. $f(x) = \dfrac{x+1}{x^2}$.

x	2	6	10	100	1000	10,000	100,000
f	.75	.1944	.1100	.0101	.0010	.0001	.00001

(a) 0

x	−2	−6	−10	−100	−1000	−10,000	−100,000
f	−.25	−.1389	−.0900	−.0099	−.0010	−.0001	−.00001

(b) 0

(d) $\displaystyle\lim_{x\to+\infty}\frac{x+1}{x^2} = \lim_{x\to+\infty}\left(\frac{1}{x}+\frac{1}{x^2}\right) = 0+0 = 0$ (e) $\displaystyle\lim_{x\to-\infty}\frac{x+1}{x^2} = \lim_{x\to-\infty}\left(\frac{1}{x}+\frac{1}{x^2}\right) = 0+0 = 0$

10. $f(x) = \dfrac{x^2}{x+1}$.

x	2	6	10	100	1000	10,000	100,000
f	1.333	5.143	9.091	99.01	999.0	9999	99,999

(a) $+\infty$

x	−2	−6	−10	−100	−1000	−10,000	−100,000
f	−4	−7.2	−11.11	−101.0	−1001	−10,001	−100,001

(b) $-\infty$

(d) $\displaystyle\lim_{x\to+\infty}\frac{x+1}{x^2} = \lim_{x\to+\infty}\left(\frac{1}{x}+\frac{1}{x^2}\right) = 0+0 = 0$ (e) $\displaystyle\lim_{x\to-\infty}\frac{x+1}{x^2} = \lim_{x\to-\infty}\left(\frac{1}{x}+\frac{1}{x^2}\right) = 0+0 = 0$

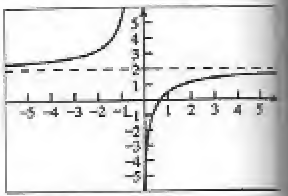
In Exercises 11–30, find the limit and support your answer graphically.

11. $\displaystyle\lim_{t\to+\infty}\frac{2t+1}{5t-2} = \lim_{t\to+\infty}\frac{2+\frac{1}{t}}{5-\frac{2}{t}} = \frac{2}{5}$ **13.** $\displaystyle\lim_{x\to-\infty}\frac{2x+7}{4-5x} = \lim_{x\to-\infty}\frac{2+\frac{7}{x}}{3-\frac{5}{x}} = -\frac{2}{5}$ **14.** $\displaystyle\lim_{x\to+\infty}\frac{1+5x}{2-3x} = \lim_{x\to+\infty}\frac{\frac{1}{x}+5}{\frac{2}{x}-3} = -\frac{5}{3}$

12. $\displaystyle\lim_{x\to-\infty}\frac{6x-4}{3x+1}$

▹ To apply Limit Theorem 13 we divide the numerator and denominator by x.

$$\lim_{x\to-\infty}\frac{6x-4}{3x+1} = \lim_{x\to-\infty}\frac{6-\frac{4}{x}}{3+\frac{1}{x}} = \frac{\displaystyle\lim_{x\to-\infty}6-\lim_{x\to-\infty}4\cdot\lim_{x\to-\infty}\frac{1}{x}}{\displaystyle\lim_{x\to-\infty}3+\lim_{x\to-\infty}\frac{1}{x}} = \frac{6-4(0)}{3+0} = 2$$

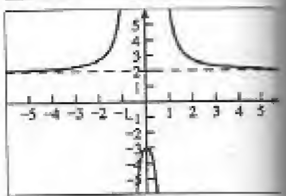


15. $\displaystyle\lim_{x\to+\infty}\frac{7x^2-2x+1}{3x^2+8x+5} = \lim_{x\to+\infty}\frac{7-\frac{2}{x}+\frac{1}{x^2}}{3+\frac{1}{x}+\frac{5}{x^2}} = \frac{7}{3}$

16. $\displaystyle\lim_{s\to-\infty}\frac{4s^2+3}{2s^2-1}$

▹ We divide the numerator and denominator by s^2. Thus,

$$\lim_{s\to-\infty}\frac{4s^2+3}{2s^2-1} = \lim_{s\to-\infty}\frac{4+\frac{3}{s^2}}{2-\frac{1}{s^2}} = \frac{\displaystyle\lim_{s\to-\infty}4+\lim_{s\to-\infty}3\cdot\lim_{s\to-\infty}\frac{1}{s^2}}{\displaystyle\lim_{s\to-\infty}2-\lim_{s\to-\infty}\frac{1}{s^2}} = \frac{4+3(0)}{2-0} = 2$$



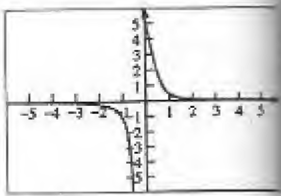
17. $\displaystyle\lim_{x\to+\infty}\frac{x+4}{3x^2-5} = \lim_{x\to+\infty}\frac{\frac{1}{x}+\frac{4}{x^2}}{3-\frac{5}{x^2}} = 0$ **18.** $\displaystyle\lim_{x\to+\infty}\frac{x^2+5}{x^3} = \lim_{x\to+\infty}\left(\frac{1}{x}+\frac{1}{x^3}\right) = 0$

19. $\displaystyle\lim_{y\to+\infty}\frac{2y^2-3y}{y+1} = \lim_{y\to+\infty}\frac{2-\frac{3}{y}}{\frac{1}{y}+\frac{1}{y^2}} = +\infty$

20. $\displaystyle\lim_{x\to+\infty}\frac{x^2-2x+5}{7x^3+x+1}$ ▹ Dividing the numerator and denominator by x^3, the highest power of x which appears, we have

$$\lim_{x\to+\infty}\frac{x^2-2x+5}{7x^3+x+1} = \lim_{x\to+\infty}\frac{\frac{1}{x}-\frac{2}{x^2}+\frac{5}{x^3}}{7+\frac{1}{x^2}+\frac{1}{x^3}}$$

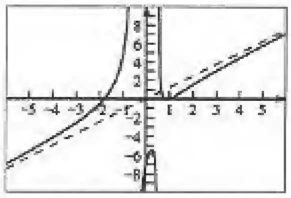
$$= \frac{\displaystyle\lim_{x\to+\infty}\frac{1}{x}-2\lim_{x\to+\infty}\frac{1}{x^2}+5\lim_{x\to+\infty}\frac{1}{x^3}}{\displaystyle 7+\lim_{x\to+\infty}\frac{1}{x^2}+\lim_{x\to+\infty}\frac{1}{x^3}} = \frac{0-2(0)+5(0)}{7+0+0} = 0$$



21. $\displaystyle\lim_{x\to-\infty}\frac{4x^3+2x^2-5}{8x^3+x+2} = \lim_{x\to-\infty}\frac{4+\frac{2}{x}-\frac{5}{x^3}}{8+\frac{1}{x^2}+\frac{2}{x^3}} = \frac{1}{2}$ **22.** $\displaystyle\lim_{x\to+\infty}\frac{3x^4-7x^2+2}{2x^4+1} = \lim_{x\to+\infty}\frac{3-\frac{7}{x^2}+\frac{2}{x^4}}{2+\frac{1}{x^4}} = \frac{3}{2}$

$\lim_{y\to+\infty} \frac{2y^3-4}{5y+3} = \lim_{y\to+\infty} \frac{2-\frac{4}{y^3}}{\frac{5}{y^2}+\frac{3}{y^3}} = +\infty$ **25.** $\lim_{x\to-\infty}\left(3x+\frac{1}{x^2}\right) = -\infty$ **26.** $\lim_{t\to+\infty}\left(\frac{2}{t^2}-4t\right) = -\infty$ (Th B(iv))

$\lim_{x\to-\infty} \frac{5x^3-12x+7}{4x^2-1}$

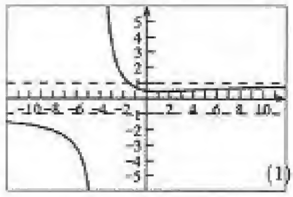


Because the numerator has the higher degree, we first remove a factor of x from the numerator. Then we divide the numerator and denominator by x^2.

$$\lim_{x\to-\infty} \frac{5x^3-12x+7}{4x^2-1} = \lim_{x\to-\infty} \frac{5x^2+12+\frac{7}{x}}{4x^2-1}\cdot x = \lim_{x\to-\infty} \frac{5-\frac{12}{x^2}+\frac{7}{x^3}}{4-\frac{1}{x^2}} \cdot \lim_{x\to-\infty} x$$

$$= \frac{5-12\cdot\lim_{x\to-\infty}\frac{1}{x^2}+7\cdot\lim_{x\to-\infty}\frac{1}{x^3}}{4-\lim_{x\to-\infty}\frac{1}{x^2}}\cdot\lim_{x\to-\infty} x = \frac{5-12(0)+7(0)}{4-0}\cdot\lim_{x\to-\infty} x = \frac{5}{4}\cdot\lim_{x\to-\infty} x = -\infty \quad \text{(Theorem B(ii))}$$

$\lim_{x\to+\infty} \frac{\sqrt{x^2+4}}{x+4} = \lim_{x\to+\infty} \frac{\sqrt{1+\frac{4}{x^2}}}{1+\frac{4}{x}} = 1$



$\lim_{x\to-\infty} \frac{\sqrt{x^2+4}}{x+4}$

We divide the numerator and denominator by x.
Because $x \to -\infty$, then $x < 0$ and $\sqrt{x^2} = -x$ or, equivalently, $x = -\sqrt{x^2}$.

$$\frac{\sqrt{x^2+4}}{x} = \frac{\sqrt{x^2+4}}{-\sqrt{x^2}} = -\sqrt{\frac{x^2+4}{x^2}} = -\sqrt{1+\frac{4}{x^2}} \qquad (1)$$

By Eq. (1) we have

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+4}}{x+4} = \lim_{x\to-\infty} \frac{\frac{\sqrt{x^2+4}}{x}}{\frac{x+4}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{4}{x^2}}}{1+\frac{4}{x}} = \frac{-\sqrt{1+\lim_{x\to-\infty}\frac{1}{x^2}}}{1+4\lim_{x\to-\infty}\frac{1}{x}} = \frac{-\sqrt{1+4(0)}}{1+4(0)} = -1$$

$\lim_{w\to-\infty} \frac{\sqrt{w^2-2w+3}}{w+5} = \lim_{w\to-\infty} \frac{\sqrt{1-\frac{2}{w}+\frac{3}{w^2}}}{2} = -1$ **30.** $\lim_{y\to-\infty} \frac{\sqrt{y^4+1}}{2y^2-3} = \lim_{y\to-\infty} \frac{\sqrt{1+\frac{1}{y^4}}}{2-\frac{3}{y^2}} = \frac{1}{2}$

Exercises 31–34, conjecture the behavior of $\lim_{x\to+\infty} f(x)$ by plotting and then compute the limit.

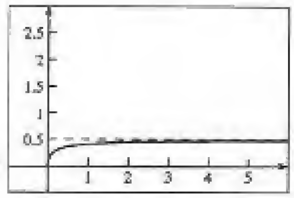
$\lim_{x\to+\infty} (\sqrt{x^2+1}-x) = \lim_{x\to+\infty}\left[(\sqrt{x^2+1}-x)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right] = \lim_{x\to+\infty} \frac{1}{\sqrt{x^2+1}+x} = 0$

$\lim_{x\to+\infty} (\sqrt{x^2+x}-x)$

Because both terms have infinite limits, we rationalize.

$$\lim_{x\to+\infty} (\sqrt{x^2+x}-x) = \lim_{x\to+\infty} \frac{\sqrt{x^2+x}-x}{1}\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to+\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to+\infty} \frac{x\cdot\frac{1}{x}}{(\sqrt{x^2+x}+x)\cdot\frac{1}{x}} \qquad (1)$$



Because $x \to +\infty$, then $x > 0$ and $x = \sqrt{x^2}$ so that

$$\frac{\sqrt{x^2+x}+x}{x} = \frac{\sqrt{x^2+x}}{x}+\frac{x}{x} = \frac{\sqrt{x^2+x}}{\sqrt{x^2}}+1 = \sqrt{\frac{x^2+x}{x^2}}+1 = \sqrt{1+\frac{1}{x}}+1 \qquad (2)$$

By Equations (1) and (2), we have

$$\lim_{x\to+\infty} (\sqrt{x^2+x}-x) = \lim_{x\to+\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{\sqrt{1+\lim_{x\to+\infty}\frac{1}{x}}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

$\lim_{x\to+\infty} \sqrt{3x^2+x}-2x = \lim_{x\to+\infty} x\left(\sqrt{3+\frac{1}{x}}-2\right) = \lim_{x\to+\infty} x\cdot(\sqrt{3}-2) = -\infty$

34. $\displaystyle\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}=\lim_{x\to+\infty}\frac{\sqrt{x(1+x^{-1}\sqrt{x+x^{1/2}})}}{\sqrt{x(1+x^{-1})}}=\lim_{x\to+\infty}\frac{\sqrt{x}\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}}{\sqrt{x}\sqrt{1+x^{-1}}}$

$\displaystyle=\lim_{x\to+\infty}\frac{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}}{\sqrt{1+x^{-1}}}=\frac{\sqrt{1+0}}{\sqrt{1+0}}=1$

In Exercises 35–46, find the the horizontal and vertical asymptotes and sketch the graph. Check by plotting.

35. $f(x)=\dfrac{2x+1}{x-3}$

▷ Because $\displaystyle\lim_{x\to3+}\frac{2x+1}{x-3}=+\infty$ or $\displaystyle\lim_{x\to3-}\frac{2x+1}{x-3}=-\infty$, $x=3$ is a vertical asymptote.

Because $\displaystyle\lim_{x\to+\infty}\frac{2x+1}{x-3}=\lim_{x\to+\infty}\frac{2+1/x}{1-3/x}=2$ or $\displaystyle\lim_{x\to-\infty}f(x)=2$, $y=2$ is a horizontal asymptote.

The graph is symmetric with respect to the point $(3,2)$.

36. $h(x)=1+\dfrac{1}{x^2}$

▷ Because $h(-x)=h(x)$, the graph is symmetric with respect to the y axis. We sketch the graph for $x>0$ and use symmetry to complete the sketch. Because

$$\lim_{x\to+\infty}h(x)=\lim_{x\to+\infty}\left(1+\frac{1}{x^2}\right)=1+0=1$$

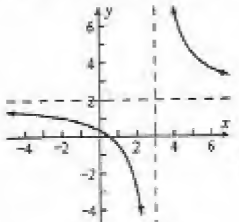
and $h(x)\neq 1$ if $x>0$, then the line $y=1$ is a horizontal asymptote. Because

$$h(x)=1+\frac{1}{x^2}=\frac{x^2+1}{x^2}$$

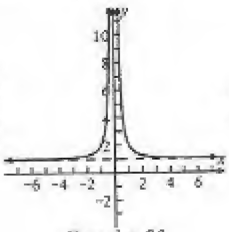
and $x^2=0$ if $x=0$, we test the line $x=0$ as a possible vertical asymptote. Because x^2 approaches 0 through positive numbers as x approaches 0, and x^2+1 approaches 1 as x approaches 0, then

$$\lim_{x\to0}h(x)=\lim_{x\to0}\frac{x^2+1}{x^2}=+\infty$$

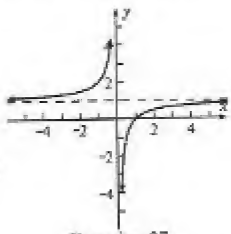
and we conclude that the line $x=0$ is a vertical asymptote. We use the above limits to sketch the graph for $x>0$ and use symmetry with respect to the y axis to complete the sketch, show below.



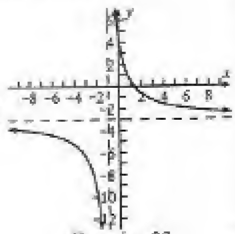
Exercise 35



Exercise 36



Exercise 37



Exercise 38

37. $g(x)=1-\dfrac{1}{x}$

▷ Because $\displaystyle\lim_{x\to0^+}\left(1-\frac{1}{x}\right)=-\infty$ or $\displaystyle\lim_{x\to0^-}\left(1-\frac{1}{x}\right)=+\infty$, $x=0$ is a vertical asymptote.

Because $\displaystyle\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)=1$ or $\displaystyle\lim_{x\to-\infty}\left(1-\frac{1}{x}\right)=1$, $y=1$ is a horizontal asymptote.

The graph is symmetric with respect to the point $(0,1)$.

38. $f(x)=\dfrac{4-3x}{x+1}$

▷ Because $\displaystyle\lim_{x\to-1^+}\frac{4-3x}{x+1}=+\infty$ or $\displaystyle\lim_{x\to-1^-}\frac{4-3x}{x+1}=-\infty$, $x=-1$ is a vertical asymptote.

Because $\displaystyle\lim_{x\to+\infty}\frac{4-3x}{x+1}=\lim_{x\to+\infty}\frac{\frac{4}{x}-3}{1+\frac{1}{x}}=-3$ or $\displaystyle\lim_{x\to-\infty}\frac{4-3x}{x+1}=-3$, $y=-3$ is a horizontal asymptote.

The graph is symmetric with respect to the point $(-1,-3)$.

39. $f(x) = \dfrac{1}{\sqrt{x^2 - 4}}$. Domain: $\{x: |x| > 2\}$. Because $\lim\limits_{x \to 2^+} f(x) = +\infty, x = 2$ is a vertical asymptote.
Because $\lim\limits_{x \to -2^-} f(x) = +\infty, x = -2$ is a vertical asymptote.
Because $\lim\limits_{x \to +\infty} f(x) = 0$ or $\lim\limits_{x \to -\infty} f(x) = 0, y = 0$ is a horizontal asymptote.
Because $f(x)$ is an even function, the graph is symmetric with respect to the y axis.

40. $g(x) = \dfrac{x^2}{4 - x^2}$
Because $g(-x) = g(x)$, the graph of g is symmetric with respect to the y axis. We sketch the graph for $x \geq 0$ and use symmetry to complete the sketch. Because

$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} \frac{x^2}{4 - x^2} = \lim_{x \to +\infty} \frac{1}{\frac{4}{x^2} - 1} = \frac{1}{0 - 1} = -1$$

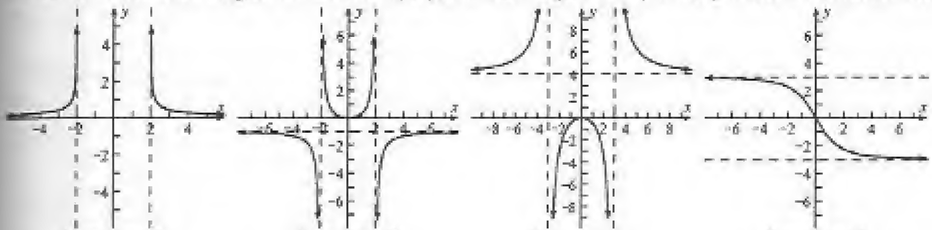
and $g(x) \neq -1$ if $x > 2$, we conclude that the line $y = -1$ is a horizontal asymptote. Because $4 - x^2 = 0$ if $x = \pm 2$, we test the line $x = 2$ as a possible vertical asymptote. (And by symmetry $x = -2$.) Since $4 - x^2$ approaches 0 through negative numbers as $x \to 2^+$, then

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} \frac{x^2}{4 - x^2} = -\infty$$

We conclude that the line $x = 2$ is a vertical asymptote, and the curve approaches the asymptote from the right in the downward direction. Since $4 - x^2$ approaches 0 through positive numbers as $x \to 2^-$, then

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} \frac{x^2}{4 - x^2} = +\infty$$

We conclude that the curve approaches the asymptote from the left in the upward direction. Because $g(0) = 0$, the curve contains the origin. We use the asymptotes, the origin, and symmetry to draw the sketch below.



Exercise 39 Exercise 40 Exercise 41 Exercise 42

41. $G(x) = \dfrac{4x^2}{x^2 - 9} = \dfrac{4x^2}{(x - 3)(x + 3)}$
Because $\lim\limits_{x \to 3^+} G(x) = +\infty$ or $\lim\limits_{x \to 3^-} G(x) = -\infty, x = 3$ is a vertical asymptote.
Because $\lim\limits_{x \to -3^+} G(x) = -\infty$ or $\lim\limits_{x \to -3^-} G(x) = +\infty, x = -3$ is a vertical asymptote.
Because $\lim\limits_{x \to +\infty} G(x) = \lim\limits_{x \to +\infty} \dfrac{4}{1 - 9/x^2} = 4$ or $\lim\limits_{x \to -\infty} G(x) = 4, y = 4$ is a horizontal asymptote.
Because $G(x)$ is an even function, the graph is symmetric with respect to the y axis.

42. $F(x) = \dfrac{-3x}{\sqrt{x^2 + 3}}$
Because $\lim\limits_{x \to +\infty} F(x) = \lim\limits_{x \to +\infty} \dfrac{-3}{\sqrt{1 + 3/x^2}} = -3$, $y = -3$ is a horizontal asymptote.
Because $\lim\limits_{x \to -\infty} F(x) = \lim\limits_{x \to -\infty} \dfrac{3}{\sqrt{1 + 3/x^2}} = 3$, $y = 3$ is a horizontal asymptote.
Because $F(x)$ is an odd function, the graph is symmetric with respect to the origin.

43. $h(x) = \dfrac{2x}{6x^2 + 11x - 10} = \dfrac{2x}{(2x+5)(3x-2)}$

Because $\lim\limits_{x \to -5/2^+} h(x) = +\infty$ or $\lim\limits_{x \to -5/2^-} h(x) = -\infty$, $x = -\frac{5}{2}$ is a vertical asymptote.

Because $\lim\limits_{x \to 2/3^+} h(x) = +\infty$ or $\lim\limits_{x \to 2/3^-} h(x) = -\infty$, $x = \frac{2}{3}$ is a vertical asymptote.

Because $\lim\limits_{x \to +\infty} h(x) = \lim\limits_{x \to +\infty} \dfrac{2/x}{6 + 11/x - 10/x^2} = 0$ or $\lim\limits_{x \to -\infty} h(x) = 0$, $y = 0$ is a horizontal asymptote.

44. $h(x) = \dfrac{x}{\sqrt{x^2 - 9}}$

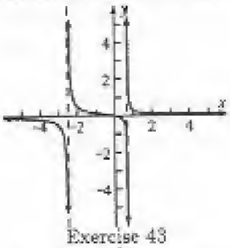
▷ Because $h(-x) = -h(x)$, the graph of h is symmetric with respect to the origin. Because $x^2 - 9 > 0$ if $|x| > 3$ then the domain of h is $(-\infty, -3) \cup (3, +\infty)$.

$$\lim_{x \to +\infty} \frac{x}{\sqrt{x^2 - 9}} = \lim_{x \to +\infty} \frac{1}{\sqrt{1 - \frac{9}{x^2}}} = \frac{1}{\sqrt{1 - 0}} = 1$$

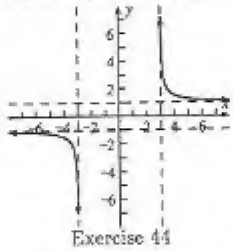
and $h(x) \neq 1$ if $x > 3$. Hence, the line $y = 1$ is a horizontal asymptote. By symmetry, the line $y = -1$ is also a horizontal asymptote. Because $x^2 - 9 = 0$ if $x = \pm 3$, we test the line $x = 3$ as a possible vertical asymptote. (And by symmetry $x = -3$.) Because $\sqrt{x^2 - 9}$ approaches 0 through positive numbers as $x \to 3^+$ then

$$\lim_{x \to 3^+} \frac{x}{\sqrt{x^2 - 9}} = +\infty$$

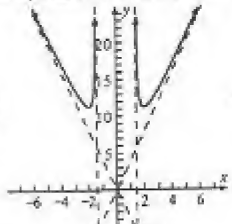
Therefore, the line $x = 3$ is a vertical asymptote and furthermore the curve approaches the asymptote from the right in the upward direction. Because $h(x)$ is not defined for $-3 \leq x \leq 3$, the curve does not approach the asymptote from the left. We use the domain, symmetry, and the asymptotes to sketch the graph below.



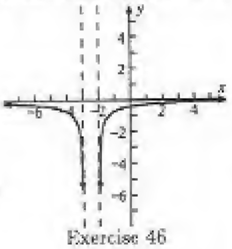
Exercise 43



Exercise 44



Exercise 45



Exercise 46

45. $f(x) = \dfrac{4x^2}{\sqrt{x^2 - 2}}$. Domain: $\{x \mid |x| > \sqrt{2}\}$

Because $\lim\limits_{x \to \sqrt{2}^+} f(x) = +\infty$, $x = \sqrt{2}$ is a vertical asymptote.

Because $\lim\limits_{x \to -\sqrt{2}^-} f(x) = +\infty$, $x = -\sqrt{2}$ is a vertical asymptote.

$$\lim_{x \to +\infty} f(x) - 4x = \lim_{x \to +\infty} \frac{4x^2}{\sqrt{x^2 - 2}} - 4x = \lim_{x \to +\infty} 4x\left(\frac{x}{\sqrt{x^2 - 2}} - 1\right) = \lim_{x \to +\infty} 4x \cdot \frac{x - \sqrt{x^2 - 2}}{\sqrt{x^2 - 2}} \cdot \frac{x + \sqrt{x^2 - 2}}{x + \sqrt{x^2 - 2}}$$

$\lim\limits_{x \to +\infty} \dfrac{8x}{\sqrt{x^2 - 2}(x + \sqrt{x^2 - 2})} = \lim\limits_{x \to +\infty} \dfrac{8/x}{\sqrt{1 - 2/x^2}(1 + \sqrt{1 - 2/x^2})} = \frac{0}{1}$ and so $y = 4x$ is an oblique asymptote.

Because $f(-x) = f(x)$, the graph is symmetric about the y axis. Hence $y = -x$ is also an oblique asymptote.

46. $f(x) = \dfrac{-1}{\sqrt{x^2 + 5x + 6}} = \dfrac{-1}{\sqrt{(x+2)(x+3)}}$. domain: $(-\infty, -3) \cup (-2, +\infty)$

Because $\lim\limits_{x \to -3^-} f(x) = -\infty$, $x = -3$ is a vertical asymptote.

Because $\lim\limits_{x \to -2^+} f(x) = -\infty$, $x = -2$ is a vertical asymptote.

Because $\lim\limits_{x \to +\infty} f(x) = 0$ or $\lim\limits_{x \to -\infty} f(x) = 0$, $y = 0$ is a horizontal asymptote.

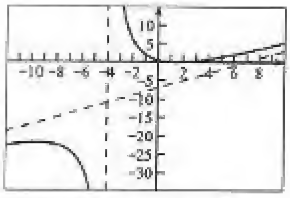
The graph is symmetric with respect to the line $x = -\frac{5}{2}$.

In Exercises 47–54, find the asymptotes of the graph of f. Check by plotting the graph and the asymptotes.

47. $f(x) = \dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1} = x + 1 + g(x)$

Because $\lim\limits_{x\to 1^+} f(x) = +\infty$ or because $\lim\limits_{x\to 1^-} f(x) = -\infty$, $x = 1$ is a vertical asymptote.
Because $\lim\limits_{x\to +\infty} g(x) = 0$ or because $\lim\limits_{x\to -\infty} g(x) = 0$, $y = x + 1$ is an oblique asymptote.

48. $f(x) = \dfrac{x^2 - 3x + 2}{x+4}$



Because $\lim\limits_{x\to -4^+} f(x) = +\infty$ and $\lim\limits_{x\to -4^-} f(x) = -\infty$, the line $x = -4$ is a vertical asymptote. Because the degree of the numerator is 1 more than the degree of the denominator, there is an oblique asymptote. Dividing, we get

$$f(x) = x - 7 + \frac{30}{x+4} \tag{1}$$

The line $y = x - 7$ is an oblique asymptote. The curve lies above the asymptote if $30/(x+4) > 0$, that is if $x > -4$, and below the asymptote if $x < -4$. A plot is shown above.

49. $f(x) = \dfrac{x^2-8}{x-3} = x + 3 + \dfrac{1}{x-3} = x + 3 + g(x)$

Because $\lim\limits_{x\to 3^+} f(x) = +\infty$ or because $\lim\limits_{x\to 3^-} f(x) = -\infty$, $x = 3$ is a vertical asymptote.
Because $\lim\limits_{x\to +\infty} g(x) = 0$ or because $\lim\limits_{x\to -\infty} g(x) = 0$, $y = x + 3$ is an oblique asymptote.

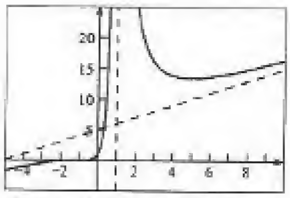
50. $f(x) = \dfrac{x^2-3}{x-2} = x + 2 + \dfrac{1}{x-2} = x + 2 + g(x)$

Because $\lim\limits_{x\to 2^+} f(x) = +\infty$ or because $\lim\limits_{x\to 2^-} f(x) = -\infty$, $x = 2$ is a vertical asymptote.
Because $\lim\limits_{x\to +\infty} g(x) = 0$ or because $\lim\limits_{x\to -\infty} g(x) = 0$, $y = x + 2$ is an oblique asymptote.

51. $f(x) = \dfrac{x^2-4x-5}{x+2} = x - 6 + \dfrac{7}{x+2} = x - 6 + g(x)$

Because $\lim\limits_{x\to -2^+} f(x) = +\infty$ or because $\lim\limits_{x\to -2^+} f(x) = -\infty$, $x = -2$ is a vertical asymptote.
Because $\lim\limits_{x\to +\infty} g(x) = 0$ or because $\lim\limits_{x\to -\infty} g(x) = 0$, $y = x - 6$ is an oblique asymptote.

52. $f(x) = \dfrac{(x+1)^3}{(x-1)^2}$



Because $\lim\limits_{x\to 1^+} f(x) = +\infty$ and $\lim\limits_{x\to 1^-} f(x) = +\infty$, the line $x = 1$ is a vertical asymptote and the curve approaches the asymptote in the upward direction both from the right and from the left. Because

$$f(x) = \frac{x^3 + 3x^2 + 3x + 1}{x^2 - 2x + 1}$$

and the degree of the numerator is one more than the degree of the denominator, we divide and obtain

$$f(x) = x + 5 + \frac{12x - 4}{(x-1)^2}$$

Therefore, the line $y = x + 5$ is an oblique asymptote. The curve lies above this asymptote if $(12x-4)/(x-1)^2 > 0$, that is, if $x > \frac{1}{3}$, and below it if $x < \frac{1}{3}$. A plot is shown above.

53. $f(x) = \dfrac{x^3 + 2x^2 + 4}{x^2} = x + 2 + \dfrac{4}{x^2} = x + 2 + g(x)$

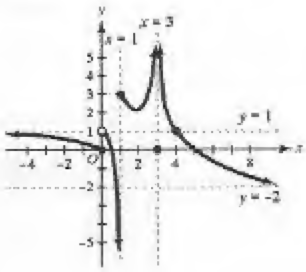
Because $\lim\limits_{x\to 0^+} f(x) = +\infty$ or because $\lim\limits_{x\to 0^-} f(x) = -\infty$, $x = 0$ is a vertical asymptote.
Because $\lim\limits_{x\to +\infty} g(x) = 0$ or because $\lim\limits_{x\to -\infty} g(x) = 0$, $y = x + 2$ is an oblique asymptote.

54. $f(x) = \dfrac{x^3 - 4}{x^2} = x - \dfrac{4}{x^2} = x + g(x)$

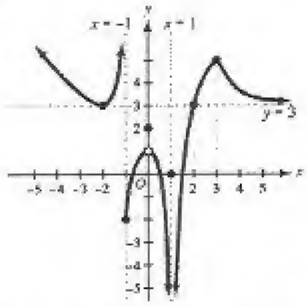
Because $\lim\limits_{x\to 0^+} f(x) = -\infty$ or because $\lim\limits_{x\to 0^-} f(x) = +\infty$, $x = 0$ is a vertical asymptote.
Because $\lim\limits_{x\to +\infty} g(x) = 0$ or because $\lim\limits_{x\to -\infty} g(x) = 0$, $y = x$ is an oblique asymptote.

In Exercises 55 and 56, evaluate each limit from the graph of the function f in the sketch.

55.



56.



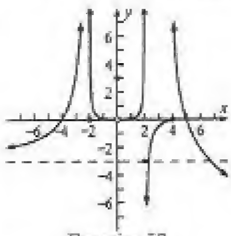
55. (a) $\lim_{x \to -\infty} f(x) = 1$ (b) $\lim_{x \to 0^-} f(x) = 0$
(c) $\lim_{x \to 0^+} f(x) = 1$ (d) $\lim_{x \to 1^-} f(x) = -\infty$
(e) $\lim_{x \to 1^+} f(x) = 3$ (f) $\lim_{x \to 3} f(x) = +\infty$
(g) $\lim_{x \to 4} f(x) = 1$ (h) $\lim_{x \to +\infty} f(x) = -2$

56. (a) $\lim_{x \to -\infty} f(x) = +\infty$ (b) $\lim_{x \to -1^-} f(x) = +\infty$
(c) $\lim_{x \to -1^+} -2$ (d) $\lim_{x \to 0} f(x) = 1$
(e) $\lim_{x \to 1} f(x) = -\infty$ (f) $\lim_{x \to 2} f(x) = 3$
(g) $\lim_{x \to 3} f(x) = 5$ (h) $\lim_{x \to +\infty} = 3$

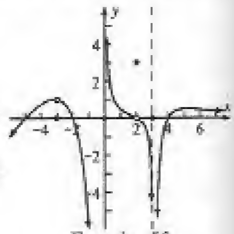
In Exercises 57 and 58, sketch the graph of a function f satisfying the given properties and whose domain is $\mathbb{R}$.

57. $f(-4) = 0$; $f(-2) = 0$; $f(0) = 3$; $f(2) = -3$; $f(4) = 0$; $f(5) = 0$; $\lim_{x \to -\infty} f(x) = -3$; $\lim_{x \to -4} f(x) = 0$; $\lim_{x \to -2} f(x) = +\infty$; $\lim_{x \to 0} f(x) = 0$; $\lim_{x \to 2^-} f(x) = +\infty$; $\lim_{x \to 2^+} = -\infty$; $\lim_{x \to 4^-} f(x) = 0$; $\lim_{x \to 4^+} f(x) = +\infty$; $\lim_{x \to 5} f(x) = 0$; $\lim_{x \to +\infty} f(x) = -\infty$

58. $f(-5) = 0$; $f(-3) = 0$; $f(-2) = 0$; $f(0) = 0$; $f(2) = 3$; $f(3) = 0$; $f(4) = 0$; $\lim_{x \to -\infty} = -\infty$; $\lim_{x \to -5} f(x) = 0$; $\lim_{x \to -3} f(x) = 1$; $\lim_{x \to -2} f(x) = 0$; $\lim_{x \to 0^-} f(x) = -\infty$; $\lim_{x \to 0^+} f(x) = +\infty$; $\lim_{x \to 2} f(x) = 0$; $\lim_{x \to 3} f(x) = -\infty$; $\lim_{x \to +\infty} f(x) = 0$



Exercise 57



Exercise 58

59. $f(x) = \frac{x}{x-1}$. Given an $\epsilon > 0$ we seek an $N > 0$ such that

If $x > N$ then $\left|\frac{x}{x-1} - 1\right| < \epsilon$

$\Leftrightarrow$ If $x > N$ then $\frac{1}{x-1} < \epsilon$

$\Leftrightarrow$ If $x > N$ then $x - 1 > \frac{1}{\epsilon}$

$\Leftrightarrow$ If $x > N$ then $x > \frac{1}{\epsilon} + 1$

The above holds if we take $N = \frac{1}{\epsilon} + 1$.

60. Let $f(x) = \frac{8x+3}{2x-1}$. Given $\epsilon > 0$ we seek an $N < 0$ such that

If $x < N$ then $\left|\frac{8x+3}{2x-1} - 4\right| < \epsilon$

$\Leftrightarrow$ If $x < N$ then $\left|\frac{7}{2x-1}\right| < \epsilon$

$\Leftrightarrow$ If $x < N$ then $2x - 1 < -\frac{7}{\epsilon}$

$\Leftrightarrow$ If $x < N$ then $x < \frac{1}{2}\left(-\frac{7}{\epsilon} + 1\right)$

The above holds if we take $N = -\frac{7}{2\epsilon} + \frac{1}{2} = \frac{\epsilon - 7}{2\epsilon}$.

61. (a) $\lim_{x \to +\infty} f(x) = -\infty$: given any number $N < 0$ there is a number $M > 0$ such that if $x > M$ then $f(x) < N$.
(b) $\lim_{x \to +\infty} f(x) = +\infty$: given any number $N > 0$ there is a number $M < 0$ such that if $x < M$ then $f(x) > N$.
(c) $\lim_{x \to -\infty} f(x) = -\infty$: given any number $N < 0$ there is a number $M < 0$ such that if $x < M$ then $f(x) < N$.

62. Given an $\epsilon > 0$ we seek an $N < 0$ such that if $x < N$ then $\left|\frac{1}{x^r} - 0\right| < \epsilon \Leftrightarrow$ if $x < N$ then $|x|^r > \frac{1}{\epsilon}$. Because $r >$
$\Leftrightarrow$ if $x < N$ then $|x| > \left(\frac{1}{\epsilon}\right)^{1/r}$. This holds if we take $N = -\left(\frac{1}{\epsilon}\right)^{1/r}$.

[illegible]. Prove that $\lim_{x\to+\infty}(x^2-4)=+\infty$ by showing that

for any $N>0$ there exists an $M>0$ such that if $x>M$ then $x^2-4>N$. (1)

▸ Statement (1) is equivalent to

if $x>M$ then $x^2>4+N$

Because $M>0$ then $x>0$ and we have

if $x>M$ then $x>\sqrt{4+N}$

This holds if we take $M=\sqrt{4+N}$.

[illegible]. Prove that $\lim_{x\to+\infty}(6-x-x^2)=-\infty$.

▸ Using 61(a), given a number $N<0$ we seek an $M>0$ such that

If $x>M$ then $6-x-x^2<N$

$\Leftarrow$ If $x>M$ then $6-x<N$

$\Leftrightarrow$ If $x>M$ then $x>6-N$

The above holds if we take $M=6-N$.

3.8 SUMMARY OF SKETCHING GRAPHS OF FUNCTIONS

To obtain a sketch of the graph of a function f, proceed as follows:

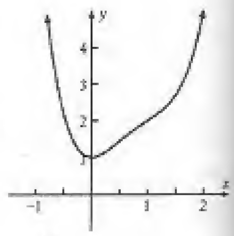
1. Determine the domain of f.
2. Find any x and y intercepts. You may need to approximate the roots of $f(x)=0$ on your calculator.
3. Test for symmetry. The graph is symmetric with respect to the y axis if f is an even function and with respect to the origin if f is an odd function. The graph of a cubic polynomial is symmetric with respect to its point of inflection.
4. Check for asymptotes. If $f(x)=g(x)/h(x)$:
   a. $x=a$ is a vertical asymptote if $h(a)=0$ and $g(a)\neq 0$.
   b. $y=b$ is a horizontal asymptote if $\lim_{x\to-\infty}f(x)=b$ or $\lim_{x\to+\infty}f(x)=b$.
   c. $y=mx+b$ is an oblique asymptote if g and h are polynomials, the degree of g is one more than the degree of h, and long division gives a quotient of $mx+b$.
5. Compute $f'(x)$ and $f''(x)$. Factoring is usually helpful.
6. Determine the critical numbers of f. These are the values of x in the domain of f for which either $f'(x)$ does not exist or $f'(x)=0$.
7. Use the numbers of step 6, as well as numbers at which f is not defined, and the first derivative to locate intervals where f is increasing $(f'(x)>0)$ and decreasing $(f'(x)<0)$ and relative maxima (where f changes from increasing to decreasing) and minima (where f changes from decreasing to increasing). A table may be helpful. Include 1-sided limits at $x=a$ which differ from $f(a)$. See Exercise 16.
8. Determine the critical numbers of f', that is, the values of x for which $f''(x)$ does not exist or $f''(x)=0$.
9. Use the numbers of step 6, as well as numbers at which f is not defined, and the second derivative to locate intervals where the graph is concave upward $(f''(x)>0)$ and downward $(f''(x)<0)$ and points of inflection (where concavity changes, provided the graph has a tangent line there). A table may be helpful.
10. Evaluate $f(x)$ at each number determined in steps 6 and 8 and at one number in each interval where the graph is unbounded. Join consecutive points where it is possible to do so without leaving the domain of f. Bow the curve below the chord in an interval where the graph is concave upward, and above the chord where it is concave downward.

Exercises 3.8

In Exercises 1–24, sketch the graph of f by first finding the following: the relative extrema of f; the points of inflection of the graph of f; the intervals on which f is increasing and those on which f is decreasing; where the graph is concave upward; where it is concave downward; the slope of any inflectional tangent; and the horizontal, vertical and oblique asymptotes, if there are any. Incorporate this information into a table.

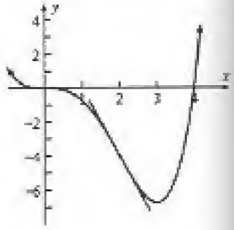
1. $f(x) = x^4 - 3x^3 + 3x^2 + 1$; $f'(x) = 4x^3 - 9x^2 + 6x$; $f''(x) = 12x^2 - 18x + 6 = 6(2x-1)(x-1)$
Set $f'(x) = 0$: $x = 0$. Set $f''(x) = 0$: $x = \frac{1}{2}, 1$.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$		$-$	$+$	decreasing	concave upward
$x = 0$	1	0	$+$	relative minimum	concave upward
$0 < x < \frac{1}{2}$		$+$	$+$	increasing	concave upward
$x = \frac{1}{2}$	$\frac{23}{16}$	$\frac{5}{4}$	0	increasing	point of inflection
$\frac{1}{2} < x < 1$		$+$	$-$	increasing	concave downward
$x = 1$	2	1	0	increasing	point of inflection
$1 < x$		$+$	$+$	increasing	concave upward



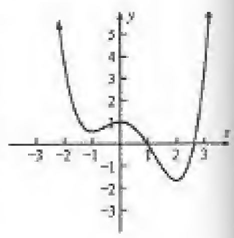
2. $f(x) = \frac{1}{4}x^4 - x^3$; $f'(x) = x^3 - 3x^2 = x^2(x-3)$; $f''(x) = 3x^2 - 6x = 3x(x-2)$
Set $f'(x) = 0$: $x = 0, 3$. Set $f''(x) = 0$: $x = 0, 2$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$		$+-=-$	$--=+$	decreasing	concave upward
$x = 0$	0	0	0	stationary	point of inflection
$0 < x < 2$		$+-=-$	$+-=-$	decreasing	concave downward
$x = 2$	-4	$-$	0	decreasing	point of inflection
$2 < x < 3$		$-$	$++=+$	decreasing	concave upward
$x = 3$	$-\frac{27}{4}$	0	$+$	relative minimum	concave upward
$x > 3$		$++=+$	$+$	increasing	concave upward



3. $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 1$; $f'(x) = x^3 - x^2 - 2x = x(x+1)(x-2)$; $f''(x) = 3x^2 - 2x - 2$
Set $f'(x) = 0$: $x = 0, -1, 2$. Set $f''(x) = 0$: $x = x_1 = \dfrac{1-\sqrt{7}}{3} \approx -0.55$, $x = x_2 = \dfrac{1+\sqrt{7}}{3} \approx 1.21$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$		$-$	$+$	decreasing	concave upward
$x = -1$	0.58	0	$+$	relative minimum	concave upward
$-1 < x < x_1$		$+$	$+$	increasing	concave upward
$x = x_1$	0.78	0.63	0	increasing	point of inflection
$x_1 < x < 0$		$+$	$-$	increasing	concave downward
$x = 0$	1	0	$-$	relative maximum	concave downward
$0 < x < x_2$		$-$	$-$	decreasing	concave downward
$x = x_2$	-0.11	-2.11	0	decreasing	point of inflection
$x_2 < x < 2$		$-$	$+$	decreasing	concave upward
$x = 2$	-1.67	0	$+$	relative minimum	concave upward
$2 < x$		$+$	$+$	increasing	concave upward



4. $f(x) = x^4 - 4x^3 + 16x$

$$f'(x) = 4x^3 - 12x^2 + 16 = 4(x^3 - 3x^2 + 4)$$

Because $f'(-1) = 0$, then by the factor theorem $x+1$ is a factor of $f'(x)$ and

$$f'(x) = (x+1)(x-2)^2$$

The critical numbers of f are -1 and 2. See Table a.

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

The critical numbers of f' are 0 and 2. See Table b. Because f is a polynomial function, there are no asymptotes. There is no symmetry. A sketch is at the right.

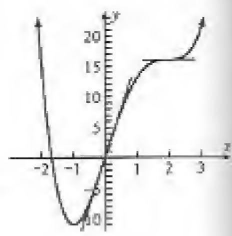


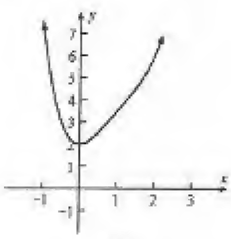
Table a

	$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
$4(x+1)$	$-$	0	$+$	$+$	$+$
$(x-2)^2$	$+$	$+$	$+$	0	$+$
$f'(x)$	$-$	0	$+$	0	$+$
f is/ has a	decreasing on $(-\infty, -1]$	relative minimum	increasing on $[-1, +\infty)$ no relative extremum at 2		
$f(x)$	$f(-2) = 16$	-11	$f(0) = 0$	16	$f(3) = 21$

Table b

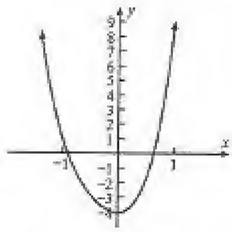
	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$12x$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	0	$+$
$f''(x)$	$+$	0	$-$	0	$+$
graph is/ has a	concave upward	point of inflection	concave downward	point of inflection	concave upward
$f(x)$; $f'(x)$		0; 4		16; 0	

5. $f(x) = \frac{1}{2}x^4 - 2x^3 + 3x^2 + 2$; $f'(x) = 2x^3 - 6x^2 + 6x = 2x(x^2 - 3x + 3)$;
$f''(x) = 6x^2 - 12x + 6 = 6(x-1)^2$. Set $f'(x) = 0$: $x = 0$. Set $f''(x) = 0$: $x = 1$.



	$f(x)$	$f'(x)$	$f''(x)$	Conclusion	
$x < 0$		$-$	$+$	decreasing	concave upward
$x = 0$	2	0	$+$	relative minimum	concave upward
$0 < x < 1$		$+$	$+$	increasing	concave upward
$x = 1$	$\frac{7}{2}$	2	0	increasing; not a point of inflection	
$1 < x$		$+$	$+$	increasing	concave upward

6. $f(x) = 3x^4 + 4x^3 + 6x^2 - 4$; $f'(x) = 2x^3 + 12x^2 + 12x = 12x(x^2 + x + 1)$



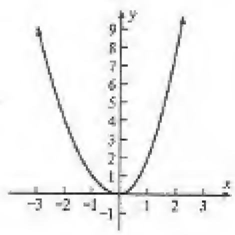
Because $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} > 0$, $x = 0$ is the only critical number of f. If $x < 0$ then $f'(x) < 0$ and if $x > 0$ then $f'(x) > 0$. Thus f is decreasing on $(-\infty, 0]$, increasing on $[0, +\infty)$ and f has a relative minimum value at 0 of-4.

$f''(x) = 36x^2 + 24x + 12 = 36(x^2 + \frac{2}{3}x + \frac{1}{3}) = 36[(x + \frac{1}{3})^2 + \frac{2}{9}] > 0$

Thus the graph of f is concave upward everywhere and has no point of inflection. There are no asymptotes or symmetry. $f(-1) = 1$ and $f(1) = 9$.

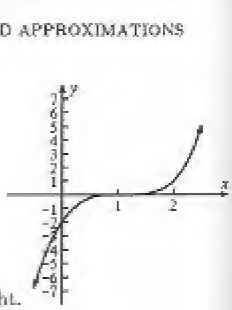
7. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \le x \end{cases}$; $f'(x) = \begin{cases} 2x & \text{if } x < 0 \\ 4x & \text{if } 0 \le x \end{cases}$; $f''(x) = \begin{cases} 2 & \text{if } x < 0 \\ 4 & \text{if } 0 \le x \end{cases}$

Set $f'(x) = 0$: $x = 0$. $f''(0)$ does not exist because $f''_-(0) = 2$ and $f''_+(0) = 4$.



	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$		$-$	$+$	decreasing	concave upward
$x = 0$	0	0	d.n.e.	relative minimum	concave upward
$0 < x$		$+$	$+$	increasing	concave upward

8. $f(x) = \begin{cases} 2(x-1)^3 & \text{if } x < 1 \\ (x-1)^4 & \text{if } x \geq 1 \end{cases}$



▷ $f'(x) = \begin{cases} 6(x-1)^2 & \text{if } x < 1 \\ 4(x-1)^3 & \text{if } x > 1 \end{cases}$

Because $f'_-(1) = f'_+(1) = 0$, then $f'(1) = 0$. $x = 1$ is the only critical number for f. See Table a.

$f''(x) = \begin{cases} 12(x-1) & \text{if } x < 1 \\ 12(x-1)^2 & \text{if } x > 1 \end{cases}$

Because $f''_-(1) = f''_+(1) = 0$, then $f''(1) = 0$. $x = 1$ is the only critical number for f'. See Table b. There are no asymptotes or symmetry. The graph is at the right.

Table a

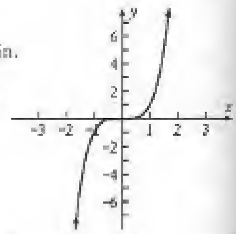
	$x < 1$	$x = 1$	$x > 1$
$6(x-1)^2$	+	0	
$4(x-1)^3$		0	+
$f'(x)$	+	0	+
f is/ has		increasing on $(-\infty, +\infty)$ no relative extremum at 1	
$f(x)$	$f(0) = -2$	0	$f(2) = 1$

Table b

	$x < 1$	$x = 1$	$x > 1$
$12(x-1)$	−	0	
$12(x-1)^2$		0	+
$f''(x)$	−	0	+
graph is/ has a	concave downward	point of inflection	concave upward
$f(x)$; $f'(x)$		0; 0	

9. $f(x) = \begin{cases} -x^4 & \text{if } x < 0 \\ x^4 & \text{if } 0 \leq x \end{cases}$; $f'(x) = \begin{cases} -4x^3 & \text{if } x < 0 \\ 4x^3 & \text{if } 0 \leq x \end{cases}$; $f''(x) = \begin{cases} -12x^2 & \text{if } x < 0 \\ 12x^2 & \text{if } 0 \leq x \end{cases}$

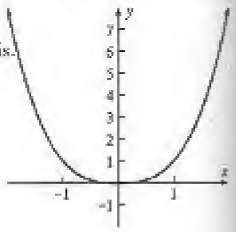
Set $f'(x) = 0$: $x = 0$. Set $f''(x) = 0$: $x = 0$. The graph is symmetric about the origin.



	$f(x)$	$f'(x)$	$f''(x)$	f is/has	graph is/has a
$x < 0$		+	−	increasing	concave downward
$x = 0$	0	0	0	stationary	point of inflection
$0 < x$		+	+	increasing	concave upward

10. $f(x) = \begin{cases} -x^3 & \text{if } x < 0 \\ x^3 & \text{if } 0 \leq x \end{cases}$; $f'(x) = \begin{cases} -3x^2 & \text{if } x < 0 \\ 3x^2 & \text{if } 0 \leq x \end{cases}$; $f''(x) = \begin{cases} -6x & \text{if } x < 0 \\ 6x & \text{if } 0 \leq x \end{cases}$

Set $f'(x) = 0$: $x = 0$. Set $f''(x) = 0$: $x = 0$. The graph is symmetric about the y axis.

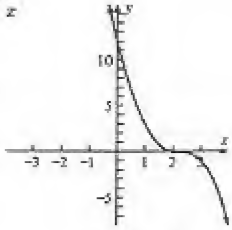


	$f(x)$	$f'(x)$	$f''(x)$	f is/has	graph is/has a
$x < 0$		−	+	decreasing	concave upward
$x = 0$	0	0	0	relative minimum	point of inflection
$0 < x$		+	+	increasing	concave upward

$f(x) = \begin{cases} 3(x-2)^2 & \text{if } x \le 2 \\ (2-x)^3 & \text{if } 2 < x \end{cases}$; $f'(x) = \begin{cases} 6(x-2) & \text{if } x \le 2 \\ -3(2-x)^2 & \text{if } 2 < x \end{cases}$; $f''(x) = \begin{cases} 6 & \text{if } x < 2 \\ 6(2-x) & \text{if } 2 < x \end{cases}$

Set $f'(x) = 0$: $x = 2$. $f''(x)$ is never 0. $f''(2)$ does not exist.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 2$		$-$	$+$	decreasing	concave upward
$x = 2$	0	0	d.n.e.	stationary	point of inflection
$2 < x$		$-$	$-$	decreasing	concave downward

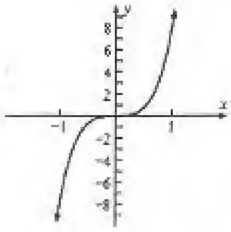


$f(x) = 3x^5 + 5x^3$

Because f is an odd function, the graph of f is symmetric with respect to the origin. There are no asymptotes.

$$f'(x) = 15x^4 + 15x^2 = 15x^2(x^2+1)$$
$$f''(x) = 60x^3 + 30x = 30x(2x^2+1)$$

If $f'(x) = 0$, then $x = 0$, and thus 0 is the only critical number of f. Because $f'(x) > 0$ if $x \neq 0$, then f is increasing on $(-\infty, +\infty)$ and so f has no relative extrema. Because $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$ then the graph of f is concave downward if $x < 0$, concave upward if $x > 0$, and $(0, 0)$ is a point of inflection. The tangent line at that point is the x axis because $f'(0) = 0$.



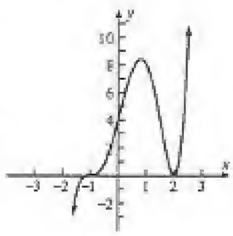
$f(x) = (x+1)^3(x-2)^2$

$f'(x) = 3(x+1)^2(x-2)^2 + 2(x-2)(x+1)^3 = (x+1)^2(x-2)[3(x-2)+2(x+1)] = (x+1)^2(x-2)(5x-4)$

$f''(x) = 2(x+1)(5x^2-14x+8) + (x+1)^2(10x-14) = 2(x+1)[(5x^2-14x+8)+(5x-7)(x+1)]$
$= 2(x+1)(10x^2-16x+1)$

Set $f'(x) = 0$: $x = -1, \frac{4}{5}, 2$. Set $f''(x) = 0$: $x = -1$, $x = x_1 = \dfrac{8-3\sqrt{6}}{10} \approx 0.065$, $x = x_2 = \dfrac{8+3\sqrt{6}}{10} \approx 1.53$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$		$+$	$-$	increasing	concave downward
$x = -1$	0	0	0	stationary	point of inflection
$-1 < x < x_1$		$+$	$+$	increasing	concave upward
$x = x_1$	4.52	8.07	0	increasing	point of inflection
$x_1 < x < \frac{4}{5}$		$+$	$-$	increasing	concave downward
$x = \frac{4}{5}$	8.40	0	$-$	relative maximum	concave downward
$\frac{4}{5} < x < x_2$		$-$	$-$	decreasing	concave downward
$x = x_2$	3.52	-11.0	0	decreasing	point of inflection
$x_2 < x < 2$		$-$	$+$	decreasing	concave upward
$x = 2$	0	0	$+$	relative minimum	concave upward
$2 < x$		$+$	$+$	increasing	concave upward

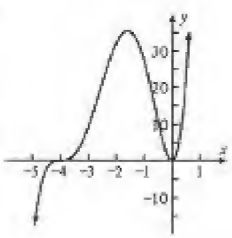


$f(x) = x^2(x+4)^3$; $f'(x) = 2x(x+4)^3 + 3x^2(x+4)^2 = x(x+4)^2[2(x+4)+3x] = x(x+4)^2(5x+8)$

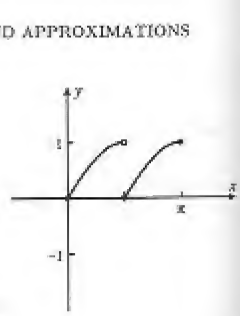
$f''(x) = 4(x+4)(5x^2+16x+8)$. Set $f'(x) = 0$: $x = -4, -\frac{8}{5}, 0$.

Set $f''(x) = 0$: $x = -4$, $x = x_1 = -\frac{2}{5}(4+\sqrt{6}) \approx -2.58$, $x = x_2 = -\frac{2}{5}(4-\sqrt{6}) \approx -0.62$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -4$		$+$	$-$	increasing	concave downward
$x = -4$	0	0	0	stationary	point of inflection
$-4 < x < x_1$		$+$	$+$	increasing	concave upward
$x = x_1$	19.1	$+$	0	increasing	point of inflection
$x_1 < x < -\frac{8}{5}$		$+$	$-$	increasing	concave downward
$x = -\frac{8}{5}$	35.4	0	$-$	relative maximum	concave downward
$-\frac{8}{5} < x < x_2$		$-$	$-$	decreasing	concave downward
$x = x_2$	14.9	$-$	0	decreasing	point of inflection
$x_2 < x < 0$		$-$	$+$	decreasing	concave upward
$x = 0$	0	0	$+$	relative minimum	concave upward
$0 < x$		$+$	$+$	increasing	concave upward



15. $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \frac{1}{2}\pi \\ \sin(x - \frac{1}{2}\pi) & \text{if } \frac{1}{2}\pi \le x \le \pi \end{cases}$



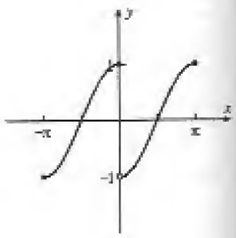
▹ $f'(x) = \begin{cases} \cos x & \text{if } 0 < x < \frac{1}{2}\pi \\ \cos(x - \frac{1}{2}\pi) & \text{if } \frac{1}{2}\pi < x < \pi \end{cases}$; $f''(x) = \begin{cases} -\sin x & \text{if } 0 < x < \frac{1}{2}\pi \\ -\sin(x - \frac{1}{2}\pi) & \text{if } \frac{1}{2}\pi < x < \pi \end{cases}$

Because $\lim\limits_{x \to \pi/2^-} f(x) = 1$ and $\lim\limits_{x \to \pi/2^+} f(x) = 0$, f is discontinuous at $\frac{1}{2}\pi$.

$f'(0)$ does not exist. $f''(x)$ is never 0.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x = 0$	0				
$0 < x < \frac{1}{2}\pi$		+	−	increasing	concave downward
$x = \frac{1}{2}\pi^-$	1			jump of −1	break
$x = \frac{1}{2}\pi$	0	d.n.e.	d.n.e.	relative minimum	no point of inflection
$\frac{1}{2}\pi < x < \pi$		+	−	increasing	concave downward
$x = \pi$	1				

16. $f(x) = \begin{cases} \cos x & \text{if } -\pi \le x \le 0 \\ \cos(\pi - x) & \text{if } 0 < x \le \pi \end{cases}$ ▹ $f(x) = \begin{cases} \cos x & \text{if } -\pi \le x \le 0 \\ -\cos x & \text{if } 0 < x \le \pi \end{cases}$;



$f'(x) = \begin{cases} -\sin x & \text{if } -\pi < x < 0 \\ \sin x & \text{if } 0 < x < \pi \end{cases}$; $f''(x) = \begin{cases} -\cos x & \text{if } -\pi < x < 0 \\ \cos x & \text{if } 0 < x < \pi \end{cases}$

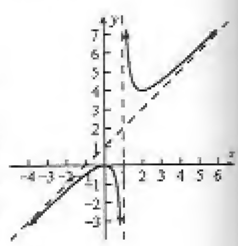
Because $\lim\limits_{x \to 0^-} f(x) = 1$ and $\lim\limits_{x \to 0^+} f(x) = -1$, f is discontinuous at 0.

$f'(0)$ does not exist. $f''(-\frac{1}{2}\pi) = 0$ and $f''(\frac{1}{2}\pi) = 0$.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x = -\pi$	−1				
$-\pi < x < -\frac{1}{2}\pi$		+	+	increasing	concave upward
$x = -\frac{1}{2}\pi$	0	+	0	increasing	point of inflection
$-\frac{1}{2}\pi < x < 0$		+	−	increasing	concave downward
$x = 0$	1	d.n.e.	d.n.e.	relative maximum	no point of inflection
$x = 0^+$	−1			jump −1	break
$0 < x < \frac{1}{2}\pi$		+	+	increasing	concave upward
$x = \frac{1}{2}\pi$	0	+	0	increasing	point of inflection
$\frac{1}{2} < x < \pi$		+	−	increasing	concave downward

17. $f(x) = \dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1} = x + 1 + g(x)$; $f'(x) = 1 - \dfrac{1}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2}$; $f''(x) = \dfrac{2}{(x-1)^3}$

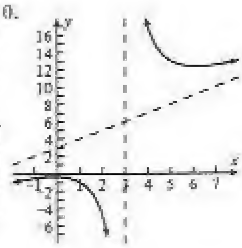
Set $f'(x) = 0$: $x = 0, 2$. There is no value of x for which $f''(x) = 0$. Because $\lim\limits_{x \to 1^+} f(x) = +\infty$, $x = 1$ is a vertical asymptote. Because $\lim\limits_{x \to -\infty} g(x) = 0$, $y = x + 1$ is an oblique asymptote. The graph is symmetric with respect to $(1, 2)$.



	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$		+	−	increasing	concave downward
$x = 0$	0	0	−	relative maximum	concave downward
$0 < x < 1$		−	−	decreasing	concave downward
$x = 1$	d.n.e.	d.n.e.	d.n.e.		vertical asymptote
$1 < x < 2$		−	+	decreasing	concave upward
$x = 2$	4	0	+	a relative minimum	concave upward
$2 < x$		+	+	increasing	concave upward

18. $f(x) = \dfrac{x^2+1}{x-3} = x + 3 + \dfrac{10}{x-3} = x + 3 + g(x)$; $f'(x) = 1 - \dfrac{10}{(x-3)^2}$; $f''(x) = \dfrac{20}{(x-3)^3}$

Set $f'(x) = 0$: $x = x_1 = 3 - \sqrt{10} \approx -.16$, $x = x_2 = 3 + \sqrt{10} \approx 6.16$. $f''(x)$ is never 0. Because $\lim\limits_{x \to 3^+} f(x) = +\infty$, $x = 3$ is a vertical asymptote. Because $\lim\limits_{x \to -\infty} g(x) = 0$, $y = x + 3$ is an oblique asymptote.. The graph is symmetric with respect to $(3, 6)$.



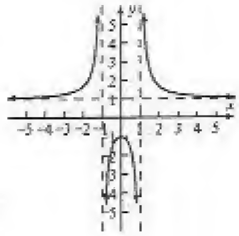
	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < x_1$		+	−	increasing	concave downward
$x = x_1$	−0.32	0	−	relative maximum	concave downward
$x_1 < x < 3$		−	−	decreasing	concave downward
$x = 3$	d.n.e.	d.n.e.	d.n.e.		vertical asymptote
$3 < x < x_2$		−	+	decreasing	concave upward
$x = x_2$	12.32	0	+	relative minimum	concave upward
$x_2 < x$		+	+	increasing	concave upward

19. $f(x) = \dfrac{x^2+1}{x^2-1}$; $f'(x) = \dfrac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2} = \dfrac{-4x}{(x^2-1)^2}$

$$f''(x) = \frac{-4(x^2-1)^2 - 2(x^2-1)(2x)(-4x)}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}$$

Set $f'(x) = 0$: $x = 0$. There are no values of x for which $f''(x) = 0$. Because $\lim\limits_{x \to -\infty} (x) = 1$, $y = 1$ is a horizontal asymptote. Because $\lim\limits_{x \to -1^+} f(x) = -\infty$ and $\lim\limits_{x \to 1^-} f(x) = -\infty$, $x = -1$ and $x = 1$ are vertical asymptotes.

Because f is even, the graph is symmetric with respect to the y axis.

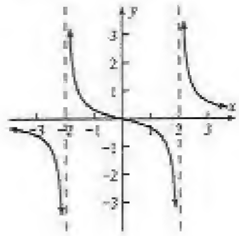


	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$		+	+	increasing	concave upward
$x = -1$	d.n.e.	d.n.e.	d.n.e.		vertical asymptote
$-1 < x < 0$		+	−	increasing	concave downward
$x = 0$	−1	0	−	relative maximum	concave downward
$0 < x < 1$		−	−	decreasing	concave downward
$x = 1$	d.n.e.	d.n.e.	d.n.e.		vertical asymptote
$1 < x$		−	+	decreasing	concave upward

20. $f(x) = \dfrac{x}{x^2-4}$

Because $x^2 - 4 = 0$ if $x = \pm 2$, the lines $x = 2$ and $x = -2$ are vertical asymptotes. Because

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{1/x}{1 - 4/x^2} = 0$$



the x axis is a horizontal asymptote. By the quotient rule,

$$f'(x) = \frac{(x^2-4)(1) - x(2x)}{(x^2-4)^2} = -\frac{x^2+4}{(x^2-4)^2}$$

Because $f'(x) < 0$ if $x \neq \pm 2$, then f is decreasing on the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, +\infty)$. Now

$$\begin{aligned} f'(x) &= -(x^2+4)(x^2-4)^{-2} \\ f''(x) &= -[(x^2+4)(-2)(x^2-4)^{-3}(2x) + (x^2-4)^{-2}(2x)] \\ &= -2x(x^2-4)^{-3}[-2(x^2+4) + (x^2-4)] \\ &= \frac{2x(x^2+12)}{(x+2)^3(x-2)^3} \end{aligned}$$

The only critical number of f' is 0. See the table. The origin is a point of inflection and the slope of the inflectional tangent is $f'(0) = -\frac{1}{4}$. Because f is an odd function, the graph, shown above, is symmetric with respect to the origin.

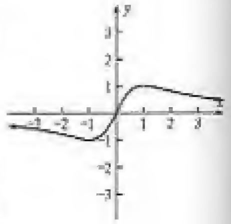
	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$2x$	−	−	−	0	+	+	+
$x^2 + 12$	+	+	+	+	+	+	+
$1/(x+2)^3$	−	doesn't exist	+	+	+	+	+
$1/(x-2)^3$	−	−	−	−	−	doesn't exist	+
$f''(x)$	−	doesn't exist	+	0	−	doesn't exist	+
graph is/ has a	concave downward	vertical asymptote	concave upward	point of inflection	concave downward	vertical asymptote	concave upward

21. $f(x) = \dfrac{2x}{x^2+1}$; $f'(x) = \dfrac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2} = \dfrac{-2x^2+2}{(x^2+1)^2}$

$$f''(x) = \frac{-4x(x^2+1)^2 - 2(x^2+1)(2x)(-2x^2+2)}{(x^2+1)^4} = \frac{-4x(-x^2+3)}{(x^2+1)^3}$$

Set $f'(x) = 0$: $x = \pm 1$. Set $f''(x) = 0$: $x = 0, \pm\sqrt{3}$.
Because $\lim_{x\to-\infty} f(x) = 0$ or because $\lim_{x\to+\infty} f(x) = 0$, $y = 0$ is a horizontal asymptote.
Because $f(x)$ is an odd function, the graph is symmetric with respect to the origin.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -\sqrt{3}$		−	−	decreasing	concave downward
$x = -\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{4}$	0	decreasing	point of inflection
$-\sqrt{3}<x<-1$		−	+	decreasing	concave upward
$x = -1$	-1	0	+	relative minimum	concave upward
$-1 < x < 0$		+	+	increasing	concave upward
$x = 0$	0	2	0	increasing	point of inflection
$0 < x < 1$		+	−	increasing	concave downward
$x = 1$	1	0	−	relative maximum	concave downward
$1 < x < \sqrt{3}$		−	−	decreasing	concave downward
$x = \sqrt{3}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{4}$	0	decreasing	point of inflection
$\sqrt{3} < x$		−	+	decreasing	concave downward

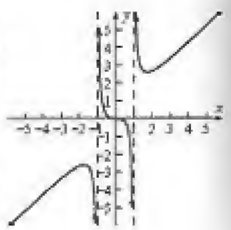


22. $f(x) = \dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1} = x + g(x)$; $f'(x) = \dfrac{3x^2(x^2-1) + x^3(2x)}{(x^2-1)^2} = \dfrac{x^4 - 3x^2}{(x^2-1)^2} = \dfrac{x^2(x^2-3)}{(x^2-1)^2}$

$$f''(x) = \frac{(4x^3 - 6x)(x^2-1)^2 - 2(x^4-3x^2)(x^2-1)2x}{(x^2-1)^4} = \frac{2x(x^2+3)}{(x^2-1)^3}$$

Set $f'(x) = 0$: $x = 0, \pm\sqrt{3}$. Set $f''(x) = 0$: $x = 0$. Because $\lim_{x\to-1^-} f(x) = +\infty$ and $\lim_{x\to1^+} f(x) = +\infty$, $x = -1$ and $x = 1$ are vertical asymptotes. Because $\lim_{x\to-\infty} g(x) = 0$ or because $\lim_{x\to+\infty} g(x) = 0$, $y = x$ is an oblique asymptote. Because f is an odd function, the graph is symmetric with respect to the origin.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -\sqrt{3}$		+	−	increasing	concave downward
$x = -\sqrt{3}$	$-\frac{3}{2}\sqrt{3}$	0	−	relative maximum	concave downward
$-\sqrt{3}<x<-1$		−	−	decreasing	concave downward
$x = -1$		d.n.e.	d.n.e.		vertical asymptote
$-1 < x < 0$		−	+	decreasing	concave upward
$x = 0$	0	0	0	stationary	point of inflection
$0 < x < 1$		−	−	decreasing	concave downward
$x = 1$		d.n.e	d.n.e		vertical asymptote
$1 < x < \sqrt{3}$		−	+	decreasing	concave upward
$x = \sqrt{3}$	$\frac{3}{2}\sqrt{3}$	0	+	relative minimum	concave upward
$\sqrt{3} < x$		+	+	increasing	concave upward



23. $f(x) = (x+1)^{2/3}(x-2)^{1/3}$

$f'(x) = \frac{2}{3}(x+1)^{-1/3}(x-2)^{1/3} + \frac{1}{3}(x+1)^{2/3}(x-2)^{-2/3}$

$\quad = \frac{1}{3}(x+1)^{-1/3}(x-2)^{-2/3}[2(x-2)+(x+1)] = (x+1)^{-1/3}(x-2)^{-2/3}(x-1)$

$f''(x) = -\frac{1}{3}(x+1)^{-4/3}(x-2)^{-2/3}(x-1) - \frac{2}{3}(x+1)^{-1/3}(x-2)^{-5/3}(x-1) + (x+1)^{-1/3}(x-2)^{-2/3}$

$\quad = -\frac{1}{3}(x+1)^{-4/3}(x-2)^{-5/3}[(x-2)(x-1) + 2(x+1)(x-1) - 3(x+1)(x-2)] = -2(x+1)^{-4/3}(x-2)^{-5/3}$

Set $f'(x) = 0$: $x = 1$. $f''(x)$ is never 0. $f'(-1)$, $f''(-1)$, $f'(2)$, $f''(2)$ do not exist.

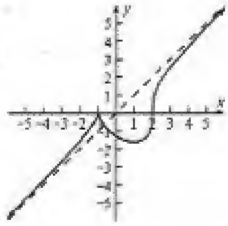
To show that $y = x$ is an oblique asymptote, we use the substitution $u = 1/x$ and the definition of derivative:

$$\lim_{x\to+\infty}[f(x)-x] = \lim_{x\to+\infty} x[(1+1/x)^{2/3}(1-2/x)^{1/3} - 1] = \lim_{x\to+\infty}\frac{(1+1/x)^{2/3}(1-2/x)^{1/3}-1}{1/x}$$

$$= \lim_{u\to0}\frac{(1+u)^{2/3}(1-2u)^{1/3}-1}{u} = \frac{d}{du}\Big[(1+u)^{2/3}(1-2u)^{1/3}\Big]_{u=0}$$

$$= \Big[\tfrac{2}{3}(1+u)^{-1/3}(1-2u)^{1/3} + (1+u)^{2/3}\cdot\tfrac{1}{3}(-2)(1-2u)^{-2/3}\Big]_{u=0} = \tfrac{2}{3} - \tfrac{2}{3} = 0$$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$		+	+	increasing	concave upward
$x = -1$	0	d.n.e.	d.n.e.	relative maximum	vertical tangent
$-1 < x < 1$		−	+	decreasing	concave upward
$x = 1$	$-4^{1/3}$	0	+	relative minimum	concave upward
$1 < x < 2$		+	+	increasing	concave upward
$x = 2$	0	d.n.e.	d.n.e.	vertical tangent	point of inflection
$2 < x$		+	−	increasing	concave downward



24. $f(x) = \dfrac{x^2-4}{x^2-9}$

► Because $x^2 - 9 = 0$ if $x = \pm 3$, the lines $x = 3$ and $x = -3$ are vertical asymptotes. The line $y = 1$ is a horizontal asymptote because

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{1-4/x^2}{1-9/x^2} = 1$$

Because f is an even function, the graph is symmetric with respect to the y axis.

$$f'(x) = \frac{(x^2-9)(2x)-(x^2-4)(2x)}{(x^2-9)^2} = \frac{-10x}{(x+3)^2(x-3)^2}$$

The only critical number is 0. See Table a. Now

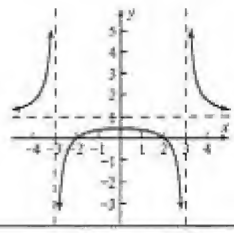


Table a

	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$0 < x < 3$	$x = 3$	$x > 3$
$-10x$	+	+	+	0	−	−	−
$1/(x+3)^2$	+	doesn't exist	+	+	+	+	+
$1/(x-3)^2$	+	+	+	+	+	doesn't exist	+
$f'(x)$	+	doesn't exist	+	0	−	doesn't exist	−
f is/ has a	increasing on $(-\infty,-3)$	does not exist	increasing on $(-3,0]$	relative minimum	decreasing on $[0,3)$	does not exist	decreasing on $(3,+\infty)$
$f(x)$	$f(-4) = \frac{12}{7}$		$f(-2) = 0$	$\frac{4}{9}$	$f(2) = 0$		$f(4) = \frac{12}{7}$

$$f'(x) = -10x(x^2-9)^{-2}$$

$$f''(x) = -10[x(-2)(x^2-9)^{-3}(2x) + (x^2-9)^{-2}] = -10(x^2-9)^{-3}[-4x^2 + (x^2-9)]$$

$$= 30(x^2-9)^{-3}(x^2+3) = \frac{30(x^2+3)}{(x+3)^3(x-3)^3}$$

Thus f' has no critical numbers so there are no points of inflection. See Table b. The graph is at the right.

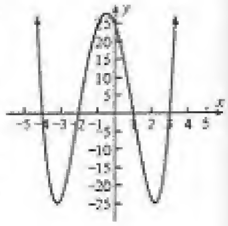
Table b

	$x < -3$	$x = -3$	$-3 < x < 3$	$x = 3$	$x > 3$
$30(x^2+3)$	+	+	+	+	+
$1/(x+3)^3$	−	doesn't exist	+	+	+
$1/(x-3)^3$	−	−	−	doesn't exist	+
$f''(x)$	+	doesn't exist	−	doesn't exist	+
graph is/has a	concave upward	vertical asymptote	concave downward	vertical asymptote	concave upward

In Exercises 25–32, (a) plot the graphs of f, f', and f'' and estimate the following: the relative extrema of f; the points of inflection of the graph of f; the intervals on which f is increasing and those on which f is decreasing; where the graph is concave upward; where it is concave downward. (b) Confirm analytically and incorporate this information into a table. Then sketch the graph of f.

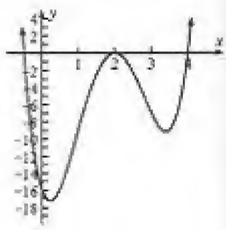
25. $f(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$; $f'(x) = 4x^3 + 6x^2 - 26x - 14 = 2(2x+1)(x^2+x-7)$;
$f''(x) = 12x^2 + 12x - 26$. Set $f'(x) = 0$: $x = -\frac{1}{2}$, $x = x_1 = -\frac{1}{2}\sqrt{29} - \frac{1}{2} \approx -3.19$, $x = x_2 = \frac{1}{2}\sqrt{29} - \frac{1}{2} \approx 2.19$
Set $f''(x) = 0$: $x = x_3 = -\frac{1}{6}(\sqrt{87}+3) \approx -2.05$, $x = x_4 = \frac{1}{6}(\sqrt{87}-3) \approx 1.05$. The graph appears to be symmetric with respect to the line $x = -\frac{1}{2}$. Substituting $x = u - \frac{1}{2}$ we get $f(x) = u^4 - \frac{29}{2}u^2 + \frac{441}{16}$ which is even.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < x_1$		−	+	decreasing	concave upward
$x = x_1$	−25	0	+	relative minimum	concave upward
$x_1 < x < x_3$		+	+	increasing	concave upward
$x = x_3$	$-\frac{59}{36}$	+	0	increasing	point of inflection
$x_3 < x < -\frac{1}{2}$		+	−	increasing	concave downward
$x = -\frac{1}{2}$	$\frac{441}{16}$	0	−	relative maximum	concave downward
$-\frac{1}{2} < x < x_4$		−	−	decreasing	concave downward
$x = x_4$	$-\frac{59}{36}$	−	0	decreasing	point of inflection
$x_4 < x < x_2$		−	+	decreasing	concave upward
$x = x_2$	−25	0	+	relative minimum	concave upward
$x_2 < x$		+	+	increasing	concave upward



26. $f(x) = 2x^4 - 15x^3 + 32x^2 - 12x - 16$; $f'(x) = 8x^3 - 45x^2 + 64x - 12 = (x-2)(8x^2 - 29x + 6)$
$f''(x) = 24x^2 - 90x + 64$. Set $f'(x) = 0$: $x = 2$, $x = x_1 = \frac{1}{16}(29 - \sqrt{649}) \approx 0.22$, $x = x_2 = \frac{1}{16}(29 + \sqrt{649}) \approx 3.40$
Set $f''(x) = 0$: $x = x_3 = \frac{1}{24}(45 - \sqrt{489}) \approx 0.95$, $x = x_4 = \frac{1}{24}(45 + \sqrt{489}) \approx 2.80$

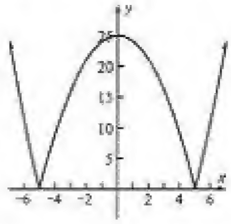
	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < x_1$		−	+	decreasing	concave upward
$x = x_1$	−9.17	0	+	relative minimum	concave upward
$x_1 < x < x_3$		+	+	increasing	concave upward
$x = x_3$	−5.033	+	0	increasing	point of inflection
$x_3 < x < 2$		+	−	increasing	concave downward
$x = 2$	0	0	−	relative maximum	concave downward
$2 < x < x_4$		−	−	decreasing	concave downward
$x = x_4$	−9.70	−	0	decreasing	point of inflection
$x_4 < x < x_2$		−	+	decreasing	concave upward
$x = x_2$	−17.25	0	+	relative minimum	concave upward
$x_2 < x$		+	+	increasing	concave upward



27. $f(x) = |25 - x^2| = \begin{cases} x^2 - 25 & \text{if } x < -5 \\ 25 - x^2 & \text{if } -5 \le x \le 5 \\ x^2 - 25 & \text{if } x > 5 \end{cases}$; $f'(x) = \begin{cases} 2x & \text{if } x < -5 \\ -2x & \text{if } -5 < x < 5 \\ 2x & \text{if } x > 5 \end{cases}$; $f''(x) = \begin{cases} 2 & \text{if } x < -5 \\ -2 & \text{if } -5 < x < 5 \\ 2 & \text{if } x > 5 \end{cases}$

$f'(0) = 0$. $f'(-5)$, $f''(-5)$, $f'(5)$, and $f''(5)$ do not exist.
Because f is even, the graph is symmetric with respect to the y axis.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -5$		$-$	$+$	decreasing	concave upward
$x = -5$	0	d.n.e.	d.n.e.	relative minimum	no inflection
$-5 < x < 0$		$+$	$-$	increasing	concave downward
$x = 0$	25	0	$-$	relative maximum	concave downward
$0 < x < 5$		$-$	$-$	decreasing	concave downward
$x = 5$	0	d.n.e.	d.n.e.	relative minimum	no inflection
$5 < x$		$+$	$+$	increasing	concave upward



28. $f(x) = 3x^{1/3} - x$

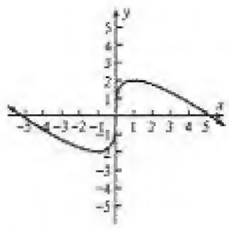
The domain of f is $(-\infty, +\infty)$. Because f is an odd function the graph is symmetric with respect to the origin. There are no asymptotes.

$$f'(x) = x^{-2/3} - 1 = \frac{1 - x^{2/3}}{x^{2/3}} = \frac{(1 + x^{1/3})(1 - x^{1/3})}{x^{2/3}}$$

The critical numbers for f are -1, 0, and 1. See the table.

$$f''(x) = -\tfrac{2}{3}x^{-5/3} = \frac{-2}{3x^{5/3}}$$

The critical number for f' is 0. Because $f''(x) > 0$ if $x < 0$, $f''(x) < 0$ if $x < 0$ and the graph of f has a vertical tangent at $x = 0$, then the graph of f is concave upward if $x < 0$, concave downward if $x > 0$, and the origin is a point of inflection. The graph is at the right.

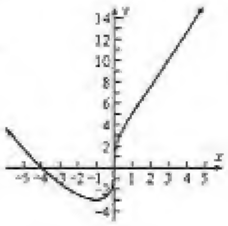


	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$1 + x^{1/3}$	$-$	0	$+$	$+$	$+$	$+$	$+$
$1 - x^{1/3}$	$+$	$+$	$+$	$+$	$+$	0	$-$
$1/x^{2/3}$	$+$	$+$	$+$	$+\infty$	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	$+\infty$	$+$	0	$-$
f is/ has a	decreasing on $(-\infty, -1]$	relative minimum	increasing on $[-1, 1]$ vertical tangent at $x = 0$			relative maximum	decreasing on $[1, +\infty)$
$f(x)$	$f(-3\sqrt{3}) = 0$	-2		0		2	$f(3\sqrt{3}) = 0$

29. $f(x) = 4x^{1/3} + x^{4/3}$; $f'(x) = \frac{4}{3}x^{-2/3} + \frac{4}{3}x^{1/3} = \frac{4}{3}x^{-2/3}(1 + x)$; $f''(x) = -\frac{8}{9}x^{-5/3} + \frac{4}{9}x^{-2/3} = \frac{4}{9}x^{-5/3}(x - 2)$

$f'(-1) = 0$. $f'(0)$ does not exist. $f''(2) = 2$. $f''(0)$ does not exist.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$		$-$	$+$	increasing	concave upward
$x = -1$	-3	0	$+$	relative minimum	concave upward
$-1 < x < 0$		$+$	$+$	increasing	concave upward
$x = 0$	0	d.n.e.	d.n.e.	vertical tangent	point of inflection
$0 < x < 2$		$+$	$-$	increasing	concave downward
$x = 2$	$6\sqrt[3]{2}$	$+$	0	increasing	point of inflection
$2 < x$		$+$	$+$	increasing	concave upward



30. $f(x) = x^2\sqrt{4-x}$. The domain of f is $(-\infty, 4]$.

$f'(x) = 2x(4-x)^{1/2} - \frac{1}{2}x^2(4-x)^{-1/2} = \frac{1}{2}(4-x)^{-1/2}[4x(4-x) - x^2] = \frac{1}{2}(4-x)^{-1/2}(16x - 5x^2)$

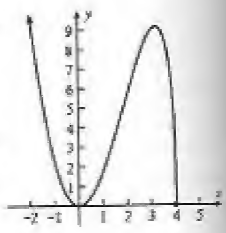
$f''(x) = \frac{1}{4}(4-x)^{-3/2}(16x - 5x^2) + \frac{1}{2}(4-x)^{-1/2}(16 - 10x)$

$= \frac{1}{4}(4-x)^{-3/2}[16x - 5x^2 + 2(4-x)(16-10x)] = \frac{1}{4}(4-x)^{-3/2}(15x^2 - 96x + 128)$

Set $f'(x) = 0$: $x = 0, 3.2$

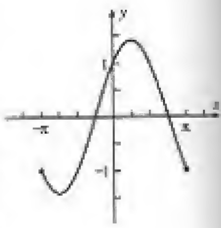
Set $f''(x) = 0$: $x = x_1 = \dfrac{48 - 8\sqrt{6}}{15} \approx 1.89$; $x_2 = \dfrac{48 + 8\sqrt{6}}{15} \approx 4.5$ is not in the domain of f.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$		$-$	$+$	decreasing	concave upward
$x = 0$	0	0	$+$	relative minimum	concave upward
$0 < x < x_1$		$+$	$+$	increasing	concave upward
$x = x_1$	5.20	$+$	0	increasing	point of inflection
$x_1 < x < 3.2$		$+$	$-$	increasing	concave downward
$x = 3.2$	9.16	0	$-$	relative maximum	concave downward
$3.2 < x < 4$		$-$	$-$	decreasing	concave downward
$x = 4$	0				



31. $f(x) = \sin x + \cos x$, $x = [-\pi, \pi]$; $f'(x) = \cos x - \sin x$; $f''(x) = -\sin x - \cos x$
Set $f'(x) = 0$: $x = -\frac{3}{4}\pi, \frac{1}{4}\pi$. Set $f''(x) = 0$: $x = -\frac{1}{4}\pi, \frac{3}{4}\pi$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x = -2\pi$	1				
$-\pi \le x < -\frac{3}{4}\pi$		$-$	$+$	decreasing	concave upward
$x = -\frac{3}{4}\pi$	$-\sqrt{2}$	0	$+$	relative minimum	concave upward
$-\frac{3}{4}\pi < x < -\frac{1}{4}\pi$		$+$	$+$	increasing	concave upward
$x = -\frac{1}{4}\pi$	0	$\sqrt{2}$	0	increasing	point of inflection
$-\frac{1}{4}\pi < x < \frac{1}{4}\pi$		$+$	$-$	increasing	concave downward
$x = \frac{1}{4}\pi$	$\sqrt{2}$	0	$-$	relative maximum	concave downward
$\frac{1}{4}\pi < x < \frac{3}{4}\pi$		$-$	$-$	decreasing	concave downward
$x = \frac{3}{4}\pi$	0	$-\sqrt{2}$	0	decreasing	point of inflection
$\frac{3}{4}\pi < x \le \pi$		$-$	$+$	decreasing	concave upward



32. $f(x) = 3\sin 2x - 5\cos 2x$, $x \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$

f is periodic with period π and amplitude $\sqrt{3^2 + 5^2} = \sqrt{34}$.

$f'(x) = 6\cos 2x + 10\sin 2x$

Set $f'(x) = 0$:

$6\cos 2x + 10\sin 2x = 0$

$\tan 2x = -\frac{3}{5}$

$2x = -\tan^{-1}\frac{3}{5}$, $2x = \pi - \tan^{-1}\frac{3}{5}$

$x = x_1 = -\frac{1}{2}\tan^{-1}\frac{3}{5} \approx -0.27$

$x = x_2 = \frac{1}{2}\pi - \frac{1}{2}\tan^{-1}\frac{3}{5} \approx 1.30$

$f''(x) = -12\sin 2x + 20\cos 2x$

Set $f''(x) = 0$:

$-12\sin 2x + 20\cos 2x = 0$

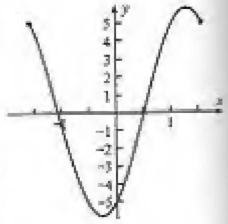
$\tan 2x = \frac{5}{3}$

$2x = \tan^{-1}\frac{5}{3}$, $2x = \tan^{-1}\frac{5}{3} - \pi$

$x = x_3 = \frac{1}{2}\tan^{-1}\frac{5}{3} - \frac{1}{2}\pi \approx -1.05$

$x = x_4 = \frac{1}{2}\tan^{-1}\frac{5}{3} \approx 0.52$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$-\frac{1}{2}\pi \le x < x_3$		$-$	$-$	decreasing	concave downward
$x = x_3$	0	$-$	0	decreasing	point of inflection
$x_3 < x < x_1$		$-$	$+$	decreasing	concave upward
$x = x_1$	$-\sqrt{34}$	0	$+$	relative minimum	concave upward
$x_1 < x < x_4$		$+$	$+$	increasing	concave upward
$x = x_4$	0	$+$	0	increasing	point of inflection
$x_4 < x < x_2$		$+$	$-$	increasing	concave downward
$x = x_2$	$\sqrt{34}$	0	$-$	relative maximum	concave downward
$x_2 < x < \frac{1}{2}\pi$		$-$	$-$	decreasing	concave downward



3.9 ADDITIONAL APPLICATIONS OF ABSOLUTE EXTREMA

3.9.1 Theorem Let the function f be continuous on the interval I containing the number c. If $f(c)$ is a relative extremum of f on I and c is the only number in I for which f has a relative extremum, then $f(c)$ is an absolute extremum of f on I. Furthermore,

(i) if $f(c)$ is a relative maximum value of f on I, then $f(c)$ is an absolute maximum value of f on I;
(ii) if $f(c)$ is a relative minimum value of f on I, then $f(c)$ is an absolute minimum value of f on I.

If f is differentiable on I we may apply the following result: If for all $x \in$ I,

(i) $f'(x) > 0$ if $x < c$ and $f'(x) < 0$ if $x > c$ then $f(c)$ is an absolute maximum value of f on I;
(ii) $f'(x) < 0$ if $x < c$ and $f'(x) > 0$ if $x > c$ then $f(c)$ is an absolute minimum value of f on I.

Exercises 3.9

In Exercises 1–4, use calculus to confirm the result of part (c) of the indicated exercise of Exercises 1.3.

1. (Ex. 21) r in. is the radius, $60/\pi r^2$ in. is the height, $120/r$ in^2 is the lateral area, $2\pi r^2$ in^2 is the ends, k/in^2 and \$$2k$/in^2 is the cost of the sides and ends. The total cost is $C(r) = k(120/r + 4\pi r^2)$, $r > 0$. $C'(r) = k(-120r^{-2} + 8\pi r) = 8\pi k r^{-2}(r^3 - 15/\pi)$. Because $C'(r) < 0$ if $0 < r < \sqrt[3]{15/\pi} = r_1$ and $C'(r) > 0$ if $r > r_1$ then C has an absolute minimum value when $r = r_1$.
   - The cost is least when the base radius is $\sqrt[3]{15/\pi}$ in. ≈ 1.68 in. and the height is $60/\sqrt[3]{225\pi}$ in. ≈ 6.73 in.
2. (Ex. 22) r in. is the radius, $60/\pi r^2$ in. is the height, $120/r$ in^2 is the lateral area, πr^2 in^2 is the end, k/in^2 and \$$2k$/in^2 is the cost of the sides and end. The total cost is $C(r) = k(120/r + 2\pi r^2)$, $r > 0$. $C'(r) = k(-120r^{-2} + 4\pi r) = 4k\pi r^{-2}(r^3 - 30/\pi)$. Because $C'(r) < 0$ if $0 < r < \sqrt[3]{30/\pi} = r_1$ and $C'(r) > 0$ if $r > r_1$ then C has an absolute minimum value when $r = r_1$.
   - The cost is least when the base radius is $\sqrt[3]{30/\pi}$ in. ≈ 2.12 in. and the height is $60/\sqrt[3]{900\pi}$ in. ≈ 4.24 in.
3. (Ex. 23) A page with margins of 1.5 in. at the top and bottom and 1 in. at the sides is to contain 24 in^2 of print. The length of the printed region is $\frac{24}{x}$ in. The area is A in^2 where $A(x) = (x+2)(\frac{24}{x} + 3) = 30 + 3x + \frac{48}{x}$, $x > 0$. $A'(x) = 3 - 48x^{-2} = 3x^{-2}(x^2 - 16)$. Because $A'(x) < 0$ if $0 < x < 4$ and $A'(x) > 0$ if $x > 4$, then A has an absolute minimum value when $x = 4$. $x + 2 = 4 + 2 = 6$, $\frac{24}{x} + 3 = \frac{24}{4} + 3 = 9$.
   - The smallest page is 6 in. wide and 9 in. long.
4. (Ex 24.) A lot with walkways 22 ft wide at the front and back and 15 ft at the sides is to contain a 13,200 ft^2 building. The length of the building is $\dfrac{13{,}200}{x}$ ft. The area is A ft^2 where $A(x) = (x+30)\left(\dfrac{13200}{x} + 44\right) = 14520 + 44x + 396000/x$, $x > 0$. $A'(x) = 44 - 396000x^{-2} = 44x^{-2}(x^2 - 9000)$. Because $A'(x) < 0$ if $0 < x < \sqrt{9000}$ and $A'(x) > 0$ if $x > \sqrt{9000}$ then A has an absolute minimum value when $x = \sqrt{9000} \approx 94.87$.
   - The smallest lot has area 22,868.4 ft^2. The field is 124.87 ft by 183.14 ft.
5. x meters is the length of the field; the width is $\frac{2700}{x}$ m. $C(x)$ dollars is the cost.
   $C(x) = 2x(36) + 2\left(\dfrac{2700}{x}\right)(36) + x(24) = 96x + \dfrac{194400}{x}$, $x > 0$. $C'(x) = 96 - 194400x^{-2} = 96x^{-2}(x^2 - 2025)$
   Because $C'(x) < 0$ if $x < 45$ and $C'(x) > 0$ if $x > 45$, then C has an absolute minimum value when $x = 45$.
   - The dimensions of the field are 45 m by $\frac{2700}{45} = 60$ m.
6. x meters is the width of the base and $\frac{125}{x^2}$ m is the height. $C(x)$ dollars is the cost.
   $C(x) = 24x^2 + 4(12)x \cdot \dfrac{125}{x^2} = 24x^2 + \dfrac{6000}{x}$, $x > 0$. $C'(x) = 48x - 6000x^{-2} = 48x^{-2}(x^3 - 125)$
   Because $C'(x) < 0$ if $x < 5$ and $C'(x) > 0$ if $x > 5$, then C has an absolute minimum value when $x = 5$.
   - The dimensions of the box are 5 m square by $\frac{125}{25} = 5$ m high.

In Exercises 7 and 8, a box has volume 288 in^3, where the base is a rectangle having length three times its width
(a) Use your graphics calculator to estimate the dimensions of the box constructed from the least amount of
material. (b) Confirm your estimate using calculus.

7. The box is closed. Let x in. be the width of the base of the box. Then the length of the base is $3x$ in., and the height of the box is $\frac{288}{3x^2} = \frac{96}{x^2}$ in. If $M(x)$ in^2 is the amount of material needed, $2 \times$ bottom $+ 2 \times$ front $+ 2 \times$ left,
$M(x) = 2(3x^2) + 2\left(\frac{96}{x^2} \cdot x\right) + 2\left(3x \cdot \frac{96}{x^2}\right) = 6x^2 + \frac{768}{x}$, $x > 0$. $M'(x) = 12x - 768x^{-2} = 12x^{-2}(x^3 - 64)$
Because $M'(x) < 0$ if $x < 4$ and $M'(x) > 0$ if $x > 4$, then M has an absolute minimum value when $M = 4$.
- The dimensions of the box are 4 in. by $3 \cdot 4 = 12$ in. by $\frac{96}{16} = 6$ in.

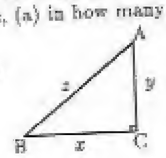
8. The box is open. Let x in. be the width of the base of the box. Then the length of the base is $3x$ in., and the height of the box is $\frac{288}{3x^2} = \frac{96}{x^2}$ in. If $M(x)$ in^2 is the amount of material needed, bottom $+ 2 \times$ front $+ 2 \times$ left,
$M(x) = (3x^2) + 2\left(\frac{96}{x^2} \cdot x\right) + 2\left(3x \cdot \frac{96}{x^2}\right) = 3x^2 + \frac{768}{x}$, $x > 0$. $M'(x) = 6x - 768x^{-2} = 6x^{-2}(x^3 - 128)$. Because
$M'(x) < 0$ if $x < 4\sqrt[3]{2}$ and $M'(x) > 0$ if $x > 4\sqrt[3]{2}$, then M has an absolute minimum value when $M = 4\sqrt[3]{2}$.
- The dimensions of the box are $4\sqrt[3]{2}$ in ≈ 5.77 in. by $3 \cdot 4\sqrt[3]{2} \approx 17.31$ in. by $96/(4\sqrt[3]{2})^2$in ≈ 2.88 in.

9. If x km/hr is the speed of the truck, the time to drive 1 km is $\frac{1}{x}$ hr. If $C(x)$ dollars is the total operating cost per kilometer, then $C(x) = 8 + \frac{1}{300}x + 27 \cdot \frac{1}{x}$, $x > 0$. $C'(x) = \frac{1}{300} - \frac{27}{x^2} = \frac{1}{300x^2}(x^2 - 8100)$. Because $C'(x) < 0$ if
$x < 90$ and $C'(x) > 0$ if $x > 90$, then C has an absolute minimum value when $x = 90$.
- The operating cost per km has an absolute minimum when the speed is 90 km/hr.

10. v knots is the speed of the ship and the time to sail 1 mi is $\frac{1}{v}$ hr. If $C(v)$ dollars is the total operating cost per mile, then $C(v) = .02v^3 + 400v^{-1}$. $C'(v) = .06v^2 - 400v^{-2} = .06v^{-2}(v^4 - \frac{20000}{3})$. Because $C'(v) < 0$ if
$v < \sqrt[4]{20000/3} \approx 9.04 = v_1$ and $C'(v) > 0$ if $v > v_1$, then C has an absolute minimum value when $v = v_1$.
- The operating cost per mile has an absolute minimum when the speed is about 9.04 knots.

11. In t seconds after the truck has left the intersection the truck has traveled $40t$ ft, the car has traveled $30t$ ft, and the distance between them is S ft. We wish to find the value of t that will make S an absolute minimum. This value of t will be the same as the value of t that makes S^2 an absolute minimum. Let $z = S^2$. Then
$z(t) = (40t)^2 + (120 - 30t)^2$, $t \geq 0$. $z'(t) = 2(40t)(40) + 2(120 - 30t)(-30) = 5000t - 7200 = 5000(t - 1.44)$
Because $z'(t) < 0$ if $0 \leq t < 1.44$ and $z'(t) > 0$ if $t > 1.44$, z has an absolute minimum value when $t = 1.44$.
- The truck and the car are closest 1.44 sec after the truck leaves the intersection.

12. Two airplanes A and B are flying horizontally at the same altitude. The position of plane B is southwest of plane A and 20 kilometers to the west and 20 kilometers to the south of A. If plane A is flying due west at 16 kilometers per minute and plane B is flying due north at $\frac{64}{3}$ kilometers per minute, (a) in how many seconds will they be closest, and (b) what will be their closest distance?



▸ Refer to the figure. Point C is due west of plane A and due north of plane B. Thus, at the start each plane is flying toward point C. t minutes after they start, let

x km be the directed distance from point C east to plane A
y km be the directed distance from point C south to plane B
z km be the distance between plane A and plane B

Because plane A is 20 km east of C when $t = 0$ and flying west at 16 km/min,

$$x = 20 - 16t \tag{1}$$

Because plane B is 20 km south of C when $t = 0$ and flying north at $\frac{64}{3}$ km/min,

$$y = 20 - \frac{64}{3}t \tag{2}$$

Because triangle ABC is a right triangle

$$z^2 = x^2 + y^2 \tag{3}$$

Now $s = z^2$ is minimized when z is minimized. Substituting for z^2, x, and y in Eq. (3), we get

$$s(t) = (20 - 16t)^2 + (20 - \tfrac{64}{3}t)^2$$

Differentiating with respect to t, we obtain

$$s'(t) = 2(20 - 16t)(-16) + 2(20 - \tfrac{64}{3}t)(-\tfrac{64}{3}) = \tfrac{640}{9}(20t - 21)$$

Because $s'(t) < 0$ if $t < \frac{21}{20}$ and $s'(t) > 0$ if $t > \frac{21}{20}$, then s and z have an absolute minimum value when $t = \frac{21}{20}$.

▪ (a) The airplanes are closest after $\frac{21}{20}$ min. Substituting $t = \frac{21}{20}$ into Equations (1) and (2), we obtain $x = 3.2$ and $y = -2.4$. Thus, plane A is 3.2 km east of point C and plane B is 2.4 km north of C when they are closest. Substituting for x and y into Eq. (3), we obtain $z^2 = 16$, $z = 4$, and so (b) the closest distance is 4 km.

13. $y = x^3 - 3x^2 + 5x$; $m(x) = y'(x) = 3x^2 - 6x + 5$, $x \in (-\infty, +\infty)$. We wish to find the value of x that makes $m(x)$ an absolute minimum. $m'(x) = 6x - 6 = 6(x - 1)$. Because $m'(x) < 0$ when $x < 1$ and $m'(x) > 0$ if $x > 1$, m has an absolute minimum value when $x = 1$ and $m(1) = 2$. Therefore, the least slope for a tangent line to the given curve is 2 when the point of tangency is $(1, 3)$.

▪ An equation of the tangent line is $y - 3 = 2(x - 1)$; $2x - y + 1 = 0$.

14. When r ohms and R ohms are the internal and external resistance, P watts is the power where $P(R) = E^2R(r + R)^{-2}$. $P'(R) = E^2[(r + r)^{-2} - 2R(r + R)^{-3}] = E^2(r + R)^{-3}[(r + R) - 2R] = E^2(r + R)^{-3}(r - R)$
Because $P'(R) > 0$ if $0 \le R < r$ and $P'(t) < 0$ if $R > r$, z has an absolute maximum value when $R = r$.

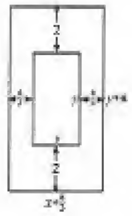
15. x months after the start of an epidemic, P percent of the population is infected, where
$P(x) = \dfrac{30x^2}{(1 + x^2)^2}$, $x \in [0, \infty)$; $P'(x) = \dfrac{60x(1 + x^2)^2 - 30x^2 \cdot 2(1 + x^2)(2x)}{(1 + x^2)^4} = \dfrac{60x(1 - x^2)}{(1 + x^2)^3}$
Because $P'(x) > 0$ if $0 \le x \le 1$ and $P'(x) < 0$ if $x > 1$, then P has an absolute maximum value when $x = 1$.

▪ In 1 month the most people will be infected, and 7.5% of the population will be infected.

16. A cardboard poster containing 32 in^2 of printed region is to have a margin of 2 in. at the top and bottom and $\frac{4}{3}$ in. at the sides. Find the dimensions of the smallest piece of cardboard that can be used to make the poster.

▪ Refer to the figure. We let the printed area have width x in. and height y in. Because the margin is $\frac{4}{3}$ in. at the sides, the cardboard has width $(x + \frac{8}{3})$ in. Because the margin is 2 in. at the top and bottom, the height of the cardboard is $(y + 4)$ in. Because the area of the printed region is 32 in.^{2}, then

$$xy = 32; \qquad y = \frac{32}{x} \qquad (1)$$



If A in.^{2} is the area of the cardboard, then

$$A(x) = \left(x + \frac{8}{3}\right)(y + 4) = \left(x + \frac{8}{3}\right)\left(\frac{32}{x} + 4\right) = 32 + \frac{256}{3x} + 4x + \frac{32}{3}$$

A(x) is defined for $x > 0$. Differentiating, we obtain

$$A'(x) = -\frac{256}{3x^2} + 4$$

If $A'(x) = 0$ then

$$x^2 = \frac{63}{3}; \qquad x = \frac{8}{3}\sqrt{3}$$

Because $A'(x) < 0$ if $0 < x < \frac{8}{3}\sqrt{3}$ and $A'(x) > 0$ if $x > \frac{8}{3}\sqrt{3}$, then $A(\frac{8}{3}\sqrt{3}) \approx 79.6$ is an absolute minimum value. Substituting $x = \frac{8}{3}\sqrt{3}$ into Eq. (1) we find $y = 4\sqrt{3}$.

▪ The dimensions of the smallest piece of cardboard are $\frac{8}{3}(1 + \sqrt{3})$ in. $\approx$ 7.3 in. by $4(1 + \sqrt{3})$ in. $\approx$ 10.9 in.
Note that the cardboard and the printed region are similar rectangles.

17. R(x) dollars is the total revenue when x units are sold: $R(x) = 200x$
C(x) dollars is the total cost of producing x units per day: $C(x) = 2x^2 + 40x + 1400$
P(x) dollars is the profit for x units sold: $P(x) = R(x) - C(x) = -2x^2 + 160x - 1400$, $x \ge 0$
$P'(x) = -4x + 160 = 4(40 - x)$. $P'(x) > 0$ if $x < 40$ and $P'(x) < 0$ if $x > 40$. Hence P(x) has an absolute maximum value if $x = 40$. ▪ The firm should produce 40 units daily.

18. R(x) dollars is the total revenue when x units are sold: $R(x) = 400x$
C(x) dollars is the total cost of producing x units per day: $C(x) = 2x^2 + 80x + 6000$
P(x) dollars is the profit for x units sold: $P(x) = R(x) - C(x) = -2x^2 + 320x - 6000$, $x \ge 0$
$P'(x) = -4x + 320 = 4(80 - x)$. $P'(x) > 0$ if $x < 80$ and $P'(x) < 0$ if $x > 80$. Hence P(x) has an absolute maximum value if $x = 80$. ▪ The company should produce 80 desks daily.

19. R(x) dollars is the revenue when x units are sold: $R(x) = px = (140 - x)x = 140x - x^2$
C(x) dollars is the total cost of producing x units per day: $C(x) = x^2 + 20x + 300$
P(x) dollars is the profit for x units sold: $P(x) = R(x) - C(x) = -2x^2 + 120x - 300$, $x \ge 0$
$P'(x) = -4x + 120 = 4(30 - x)$. $P'(x) > 0$ if $x < 30$ and $P'(x) < 0$ if $x > 30$. Hence P(x) has an absolute maximum value if $x = 30$ and $P(30) = -2(30)^2 + 120(30) - 300 = 1500$. ▪ The maximum profit is \$1500.

20. Find the shortest distance from the point P(2,0) to a point on the curve $y^2 - x^2 = 1$, and find the point on the curve closest to P.

▹ A distance is least when its square is least and the square of the distance from point P(2,0) to the point Q(x, y) on the curve $y^2 - x^2 = 1$ is given by

$$s = (x-2)^2 + y^2 \qquad (1)$$

Because Q is on the curve, then $y^2 = 1 + x^2$. Substituting into Eq. (1), we have

$$s(x) = (x-2)^2 + (1+x^2) = 2x^2 - 4x + 5$$
$$s'(x) = 4x - 4 = 4(x-1)$$

Because $s'(x) < 0$ if $x < 1$ and $s'(x) > 0$ if $x > 1$ then $s(1) = 3$ is an absolute minimum value. Also, when $x = 1$ then $y^2 = 2, y = \pm\sqrt{2}$.

▪ The shortest distance is $\sqrt{s} = \sqrt{3}$ and the points on the curve that are closest are $(1, \pm\sqrt{2})$.

21. Let z be the square of the number of units in the distance from the origin to the point (x,y) on the line $3x + y = 6$; $y = 6 - 3x$. Then
$z = x^2 + y^2 = x^2 + (6-3x)^2 = 10x^2 - 36x + 36$, $x \in (-\infty, +\infty)$; $z'(x) = 20x - 36 = 20(x - \frac{9}{5})$
Because $z'(x) < 0$ if $x < \frac{9}{5}$ and $z'(x) > 0$ if $x > \frac{9}{5}$ then $z(\frac{9}{5}) = \frac{18}{5}$ is an absolute minimum.
A value of x that makes z an absolute minimum makes the distance an absolute minimum.
When $x = \frac{9}{5}$, $y = 6 - 3x = \frac{3}{5}$, so the closest point to the origin is $P(\frac{9}{5}, \frac{3}{5})$ at a distance of $\frac{3}{5}\sqrt{10}$ units. The given line has slope -3. The line joining P to the origin has slope $\frac{3}{5}/\frac{9}{5} = \frac{1}{3}$. Hence the two lines are perpendicular.

22. Find the shortest distance from the point $A(2, \frac{1}{2})$ to a point on the parabola $y = x^2$, and find the point B on the parabola closest to A. Then show that A lies on the normal line of the parabola at B.

▹ A distance is least when its square is least and the square of the distance from point A to point (x, y) on the parabola is given by $s = (x-2)^2 + (y - \frac{1}{2})^2 = (x-2)^2 + (x^2 - \frac{1}{2})^2 = x^4 - 4x + \frac{17}{4}$. $s'(x) = 4x^3 - 4 = 4(x^3 - 1)$
Because $s'(x) < 0$ if $x < 1$ and $s'(x) > 0$ if $x > 1$, then $s(x)$ has an absolute minimum value when $x = 1$. Then $s = \frac{5}{4}$ and the shortest distance is $\frac{1}{2}\sqrt{5}$. On the parabola $y' = 2x$ and so, when $x = 1$ the slope is 2. The slope of segment $(1,1)$ $(2, \frac{1}{2})$ is $(\frac{1}{2} - 1)/(2 - 1) = -\frac{1}{2}$. Thus A lies on the normal line of the parabola at B.

In Exercises 23 and 24, a Norman window consists of a rectangle surmounted by a semicircle. If the perimeter of a Norman window is to be 32 ft, determine what should be the radius of the semicircle and the height of the rectangle so that the window will admit the most light.

23. Let r ft be the radius of the semicircle so that $2r$ ft is the width of the rectangle. Then the height of the rectangle is $\frac{1}{2}(32 - \pi r - 2r)$. The window will admit the most light when its area is greatest. If $A(r)$ ft^2 is the total area of the window then
$A(r) = \frac{1}{2}\pi r^2 + 2r \cdot \frac{1}{2}(32 - \pi r - 2r) = 32r - (\frac{1}{2}\pi + 2)r^2$, $r \in \left[0, \dfrac{32}{\pi + 2}\right] = I$; $A'(r) = 32 - (\pi + 4)r$; $A''(r) = -(\pi + 4)$
Set $A'(r) = 0$: $32 = (\pi + 4)r$; $r = \dfrac{32}{\pi + 4}$ is the only critical number. The radius of the semicircle is $\dfrac{32}{\pi + 4}$ ft and the height of the rectangle is $\dfrac{1}{2}\left[32 - \dfrac{32(\pi + 2)}{\pi + 4}\right] = \dfrac{32}{\pi + 4}$ ft too. Since $A''\left(\dfrac{32}{\pi + 4}\right) < 0$, these dimensions give a relative maximum area, and by Theorem 3.9.1 they give the absolute maximum area. Because I is a closed interval, the extreme-value theorem also applies.

24. Assume that the semicircle transmits only half as much light per square foot of area as the rectangle.

▹ Let r ft be the radius of the semicircle and let h ft be the height of the rectangle.
See the figure. The width of the rectangle is $2r$ ft. Then

perimeter of window = perimeter of semicircle + perimeter of rectangle part

$$32 = \pi r + (2h + 2r)$$
$$h = 16 - \tfrac{1}{2}\pi r - r \qquad (1)$$

If L units is the amount of light admitted, we have

$$L = \text{area of rectangle} + \tfrac{1}{2}(\text{area of semicircle})$$
$$= (2r)h + \tfrac{1}{2}(\tfrac{1}{2}\pi r^2) = 2rh + \tfrac{1}{4}\pi r^2 \qquad (2)$$

Substituting from Eq. (1) into Eq. (2), we get

$$L(r) = 2r(16 - r - \tfrac{1}{2}\pi r) + \tfrac{1}{4}\pi r^2 = 32r - 2r^2 - \tfrac{3}{4}\pi r^2$$

Differentiating, we get

$$L'(r) = 32 - 4r - \tfrac{3}{2}\pi r = 32 - (4 + \tfrac{3}{2}\pi)r$$

Because $L'(r) > 0$ if $r < \dfrac{32}{4 + \frac{3}{2}\pi} = \dfrac{64}{8 + 3\pi}$ and $L'(r) < 0$ if $r > \dfrac{64}{8 + 3\pi}$
then L has an absolute maximum value when

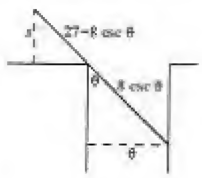
$$r = \frac{64}{8 + 3\pi} \tag{3}$$

Substituting from Eq. (3) into Eq. (1) we obtain

$$h = 16 - (1 + \tfrac{1}{2}\pi)\frac{64}{8 + 3\pi} = \frac{64 + 16\pi}{8 + 3\pi}$$

- A window of perimeter 32 ft will admit the most light if the radius of the semicircle is $\dfrac{64}{8 + 3\pi}$ ft and the altitude of the rectangle is $\dfrac{64 + 16\pi}{8 + 3\pi}$ ft.

25. Let one end of the 27-ft girder touch the corner and the opposite wall of the 8-ft passageway. See the figure. If θ is the measure of the acute angle between the girder and the passageway, then $8 \csc \theta$ ft of the girder is in the passageway so $27 - 8 \csc \theta$ is in the corridor at right angles to the passageway. Let s ft be the distance of the other end of the girder from the side of the corridor. We want to find the absolute maximum value of s. Then



$s(\theta) = (27 - 8 \csc \theta)\cos \theta = 27 \cos \theta - 8 \cot \theta$, $\sin \theta \in [\frac{8}{27}, 1]$

$s'(\theta) = -27 \sin \theta + 8 \csc^2\theta = \dfrac{8 - 27 \sin^3\theta}{\sin^2\theta}$

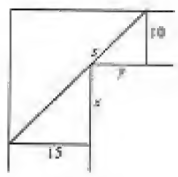
Set $s'(\theta) = 0$: $27 \sin^3\theta = 8$; $\sin^3\theta = \frac{8}{27}$; $\sin \theta = \frac{2}{3}$ so that $\cos \theta = \frac{1}{3}\sqrt{5}$, $\cot \theta = \frac{1}{2}\sqrt{5}$.

$s(\theta)\Big|_{\sin \theta = 8/27} = 0$, $s(\theta)\Big|_{\sin \theta = 2/3} = 27(\frac{1}{3}\sqrt{5}) - 8(\frac{1}{2}\sqrt{5}) = 5\sqrt{5}$, $s(\theta)\Big|_{\sin \theta = 1} = 0$

$s(\theta)$ is continuous on a closed interval so the absolute maximum value of s is $5\sqrt{5}$.

- The passageway must be at least $5\sqrt{5} \approx 11.2$ ft wide.

26. Let the girder of length s ft touch the corner and two opposite walls, x ft from the corner of the 15 ft corridor and y ft from the corner of the 10 ft corridor. See the figure. We seek the absolute minimum value of s. From similar right triangles we have $y/10 = 15/x$; $y = 150x^{-1}$. Adding the hypotenuses, we get



$s(x) = \sqrt{x^2 + 225} + \sqrt{22500x^{-2} + 100}$. $s'(x) = \dfrac{x}{\sqrt{x^2 + 225}} - \dfrac{22500x^{-3}}{\sqrt{22500x^{-2} + 100}}$

$= \dfrac{x}{\sqrt{x^2 + 225}} - \dfrac{22500x^{-3}}{\sqrt{100(225 + x^2)}} = \dfrac{x^{-2}}{\sqrt{x^2 + 225}}(x^3 - 2250)$

Because $s'(x) < 0$ if $x < \sqrt[3]{2250} = 5\sqrt[3]{18}$ and $s'(x) > 0$ if $x > 5\sqrt[3]{18}$, then s has an absolute minimum value when $x = 5\sqrt[3]{18}$.

- The beam may have length $\sqrt{50\sqrt[3]{18} + 100} + \sqrt{75\sqrt[3]{12} + 225}$ ft ≈ 35.1 ft.

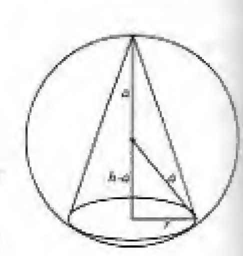
27. Let V cubic units be the specific volume of the cone. Let r units be the radius of the base, let h units be the height, and A square units be the surface of the cone. We want to find the value of h/r when A has an absolute minimum value.

$V = \frac{1}{3}\pi r^2 h$; $h = 3V\pi^{-1}r^{-2}$. $A = \pi r\sqrt{r^2 + h^2} = \sqrt{\pi^2 r^4 + \pi^2 r^2 h^2} = (\pi^2 r^4 + 9V^2 r^{-2})^{1/2}$, $r \in (0, \infty)$.

$D_r A = \frac{1}{2}(\pi^2 r^4 + 9V^2 r^{-2})^{-1/2}(4\pi^2 r^3 - 18V^2 r^{-3}) = 2\pi^2 r^{-3}(\pi^2 r^4 + 9V^2 r^{-2})^{-1/2}(r^6 - \frac{9}{2}\pi^{-2}V^2)$

Because $D_r A < 0$ if $r < (\frac{9}{2}\pi^{-2}V^2)^{1/6} = r_0$ and $D_r A > 0$ if $r > r_0$, A has an absolute minimum value when $r = r_0$. Then $\frac{h}{r} = \dfrac{3V}{\pi r^3} = \dfrac{3V}{\pi\sqrt{\frac{9}{2}\pi^{-2}V^2}} = \sqrt{2}$.

28. A right-circular cone is to be inscribed in a sphere of given radius. Find the ratio of the altitude to the base radius of the cone of largest possible volume.



▷ Refer to the figure. Let

r units be the radius of the cone
h units be the altitude of the cone
V cubic units the volume of the cone

We want to find h/r when V has an absolute maximum value. We have

$$V = \tfrac{1}{3}\pi r^2 h \qquad (1)$$

If the radius of the sphere is a units, then by the Pythagorean theorem

$$(h-a)^2 + r^2 = a^2$$
$$r^2 = 2ah - h^2 \qquad (2)$$

Substituting for r^2 into Eq. (1), we obtain V as a function of h:

$$V(h) = \tfrac{1}{3}\pi(-h^3 + 2ah)$$

We note that V is continuous on $(0, 2a)$. Differentiating, we get

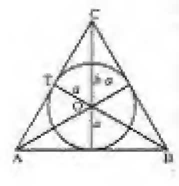
$$V'(h) = \tfrac{1}{3}\pi(-3h^2 + 4ah) = -3ah(h - \tfrac{4}{3}a)$$

Because $V'(h) > 0$ if $0 < h < \frac{4}{3}a$ and $V'(h) < 0$ if $\frac{4}{3}a < h < 2a$ then V has an absolute maximum value wh $h = \frac{4}{3}a$. Substituting into Eq. (2) we find

$$r^2 = \tfrac{8}{9}a^2; \qquad r = \tfrac{2}{3}\sqrt{2}a; \qquad \frac{h}{r} = \frac{\frac{4}{3}a}{\frac{2}{3}\sqrt{2}a} = \sqrt{2}$$

■ For the cone of largest volume, the ratio of altitude to base radius is $\sqrt{2}$.

29. Let a units be the radius of the given sphere, and a is a constant. Let r units be the radius of the base of the cone and h units be the height of the cone. See the figure for a cross section of the cone and inscribed sphere. Because right triangle ABC is similar to right triangle OTC, then



$$\frac{h}{r} = \frac{\sqrt{(h-a)^2 - a^2}}{a}; \; r = \frac{ah}{\sqrt{h^2 - 2ah}}; \; r^2 = \frac{a^2h^2}{h^2 - 2ah}$$

If V(h) cubic units is the volume of the cone, then

$$V(h) = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi \cdot \frac{a^2h^2}{h^2 - 2ah} \cdot h = \frac{\pi a^2 h^2}{3h - 6a}, \; h \in (2a, +\infty)$$

$$V'(h) = \frac{2\pi a^2 h(3h - 6a) - 3(\pi a^2 h^2)}{(3h - 6a)^2} = \frac{3\pi a^2 h(2h - 4a - h)}{(3h - 6a)^2} = \frac{3\pi a^2 h(h - 4a)}{(3h - 6a)^2}$$

Because $V'(h) < 0$ if $2a < h < 4a$ and $V'(h) > 0$ if $h > 4a$, h has an absolute minimum value when $h = 4a$.

Then $\dfrac{h}{r} = \dfrac{\sqrt{(h-a)^2 - a^2}}{a} = \dfrac{\sqrt{(3a)^2 - a^2}}{a} = 2\sqrt{2}$.

30. Let f be the square of the distance from the point (x, y) of line $Ax + By + C = 0$. Differentiating implicit with respect to x: $A + By' = 0$, $y' = -A/B$. Then $f(x) = (x - x_1)^2 + (y - y_1)^2$.
$f'(x) = 2(x - x_1) + 2(y - y_1)(-A/B) = 0$ when $y - y_1 = (B/A)(x - x_1)$. Writing the equation of the line as $A(x - x_1) + B(y - y_1) = -(Ax_1 + By_1 + C) = -d$, and substituting: $A(x - x_1) + (B^2/A)(x - x_1) = -d$,

$x - x_1 = \dfrac{Ad}{A^2 + B^2}$ and so $y - y_1 = \dfrac{Bd}{A^2 + B^2}$. Thus $f = \dfrac{(A^2 + B^2)d^2}{(A^2 + B^2)^2} = \dfrac{d^2}{A^2 + B^2}$. The minimum distance is

$\dfrac{|d|}{\sqrt{A^2 + B^2}} = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$. $f''(x) = 2 + 2(-A/B)(-A/B) = 2 + 2A^2/B^2 > 0$ so the critical point is a local minimum and hence an absolute minimum.

31. The cross section of a trough has the shape of an inverted isosceles triangle. If the lengths of the equal sid are 15 inches, find the size of the vertex angle that will give the maximum capacity for the trough.

▷ If A square units is the area of the cross section, then the capacity is a maximum when A is a maximum an $A = \frac{1}{2}ab \sin\theta$ where a and b are the measures of the lengths of two sides of the triangle and $\theta \in (0, \pi)$ is th measure of the angle between these sides. Because $\sin\theta$ has an absolute maximum value when $\theta = \frac{1}{2}\pi$, then has an absolute maximum value when $\theta = \frac{1}{2}\pi$. Thus the vertex angle should have radian measure $\frac{1}{2}\pi$.

3.10 APPROXIMATIONS BY NEWTON'S METHOD, THE TANGENT LINE, AND DIFFERENTIALS

Newton's Method When using Newton's method to solve an equation of the form $f(x) = 0$, do the following:

1. Make a *good* guess for the first approximation x_1. A rough sketch of the graph of f will help to obtain a reasonable choice.
2. With the value of x_1 in formula (1), get a second approximation x_2. Then use x_2 in (1) to get a third approximation x_3, and so on until $x_{n+1} = x_n$ to the required degree of accuracy. If $f''(x)$ exists in an open interval I containing the root and $f'(x)$ is not zero on I (cf. Exercise 60) then the number of correct digits is approximately doubled at each iteration. Thus, if x_n and x_{n+1} agree to four digits, then x_{n+1} is correct to eight digits.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

Be sure to switch into radian mode if a problem involves trigonometric functions. The value of x should be stored in memory (as a variable) at the beginning of the problem, and again when the new value of x is in the display at the end of each iteration. Each time you need x, just press the recall button. Thus the keystrokes for each iteration are the same. Remember to use the multiplication key for implied multiplications. Always use parentheses around the numerator and denominator, and end each iteration with "= STO" as shown in the Solutions. Note that the numerator shows the function values and should progress toward 0.

Differential If the function f is defined by $y = f(x)$ then the *differential of y*, denoted by dy, is given by

$$dy = f'(x)\Delta x$$

and the *differential of x*, denoted by dx, is given by

$$dx = \Delta x$$

where x is in the domain of f' and Δx is an arbitrary increment of x.

When dx is small then dy is a good approximation to Δy, where $\Delta y = f(x + \Delta x) - f(x)$. That is

$$dy \approx f(x + \Delta x) - f(x) \text{ or } f'(x)dx \approx f(x + \Delta x) - f(x) \text{ or } (\textit{the linear approximation})$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

If $y = f(x)$, then when $f'(x)$ exists,

$$dy = f'(x)dx$$

whether or not x is an independent variable.

Hence, the ratio dy/dx is the derivative of y with respect to x. That is, if $y = f(x)$, then

$$f'(x) = D_x y = \frac{dy}{dx} \text{ if } dx \neq 0$$

Warning: the symbol d^2y/dx^2 for the second derivative of y is *not* the quotient of two differentials.

Each of the formulas from Section 2.4 for the derivatives of algebraic functions is now stated in the Leibniz notation along with the corresponding formula for the differential. In these formulas u and v are differentiable functions of x and c is a constant.

I $\frac{d(c)}{dx} = 0$	I′ $d(c) = 0$
II $\frac{d(x^n)}{dx} = nx^{n-1}$	II′ $d(x^n) = nx^{n-1}\,dx$
III $\frac{d(cu)}{dx} = c\frac{du}{dx}$	III′ $d(cu) = c\,du$
IV $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$	IV′ $d(u+v) = du + dv$
V $\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$	V′ $d(uv) = v\,du + u\,dv$

$$\text{VI} \quad \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad\qquad \text{VI}' \quad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

$$\text{VII} \quad \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx} \qquad\qquad \text{VII}' \quad d(u^n) = nu^{n-1}\,du$$

The differential formulas are particularly useful when doing implicit differentiation because w do not need to distinguish the independent variable from the dependent variable.

Exercises 3.10

In Exercises 1–10, use Newton's method to find the real root. In 1–4 to four places; in 5–10 to 3 places.

1. $x^3 - 4x^2 - 2 = 0$. Let $f(x) = x^3 - 4x^2 - 2$; $f'(x) = 3x^2 - 8x$
Since $f(4) = -2$ and $f(5) = 23$, $f(x) = 0$ has a root between 4 and 5; take $x_1 = 4$.
$x_3 = x_4 = x = 4.1179$ to four decimal places.

n	x_n −	$(x_n{}^3 - 4x_n - 2)$ ÷	$(3x_n{}^2 - 8x_n)$	= STO
1	4	−2	16	4.125
2	4.125	0.126961	18.0469	4.11797
3	4.11797	0.00041	17.9292	4.11794

2. $6x^3 + 9x + 1 = 0$. Let $f(x) = 6x^3 + 9x + 1$; $f'(x) = 18x^2 + 9$
Since $f(-1) = -14$ and $f(0) = 1$, $f(x) = 0$ has a root between −1 and 0; take $x_1 = 0$
$x_2 = x_3 = -.1102$ to four decimal places.

n	x_n −	$(6x_n{}^3 + 9x_n + 1)$ ÷	$(18x_n{}^2 + 9)$	= STO
1	0	1	9	−.11111
2	−.11111	−.00823	9.22222	−.11022
3	−.11022	-1.59×10^{-6}	9.21867	−.11022

3. $x^5 - x + 1 = 0$. Let $f(x) = x^5 - x + 1$; $f'(x) = 5x^4 - 1$
Since $f(-2) = -29$ and $f(-1) = 1$, $f(x) = 0$ has a root between −2 and −1; take $x_1 = -1$.
$x_5 = x_4 = x = -1.1673$ to four decimal places.

n	x_n −	$(x_n{}^5 - x_n + 1)$ ÷	$(5x_n{}^4 - 1)$	= STO
1	−1	−0.80116	11.207	−1.17846
2	−1.17846	−0.09440	8.6434	−1.16754
3	−1.16754	−0.00193	8.2900	−1.16730
4	−1.16730	0	8.2834	−1.16730

4. $x^5 + x - 1 = 0$

▷ Let

$$f(x) = x^5 + x - 1; \; f'(x) = 5x^4 + 1$$

Because $f(0) = -1$ and $f(1) = 1$, there is a zero of f between 0 and 1. We take $x_1 = 0.5$ and apply Newto method as shown in the table, where the numbers are rounded off to five decimal places. Note that x_4 and agree to 2 digits and, as predicted, x_5 and x_6 agree to (more than) 4 digits. To four decimal places, the roo 0.7549.

n	$x_n -$	$(x_n{}^5 + x_n - 1)$ ÷	$(5x_n{}^4 + 1)$	$=$ STO
1	0.5	−0.46875	1.3125	0.85714
2	0.85714	0.31980	3.69884	0.77068
3	0.77068	0.04256	2.76387	0.75528
4	0.75528	0.00106	2.62358	0.75488
5	0.75488	0.00000	2.62358	0.75488

5. $x^3-4x-8=0$; the positive root. Let $f(x)=x^3-4x-8$; $f'(x)=3x^2-4$
Since $f(2)=-8$ and $f(3)=7$, $f(x)=0$ has a positive root between 2 and 3; take $x_1=3$.
$x_4=x_5=x=2.6494$ to four decimal places.

n	x_n	− $(x_n^3-4x_n-8)$	÷ $(3x_n^2-4)$	= STO
1	3	7	23	2.69565
2	2.69565	0.80546	17.7996	2.65040
3	2.65040	0.01646	17.0739	2.64944
4	2.64944	0	17.0585	2.64944

6. $x^3-2x+7=0$; the negative root. Let $f(x)=x^3-2x+7$; $f'(x)=3x^2-2$
Since $f(-3)=-14$ and $f(-2)=3$, $f(x)$ has a negative root between -3 and -3; take $x_1=-2$.
$x_4=x_5=-2.2583$ to four decimal places

n	x_n	− $(x_n^3-2x_n+7)$	÷ $(3x_n^2-2)$	= STO
1	−2	3	10	−2.3
2	−2.3	−.57	13.87	−2.25912
3	−2.25912	−.011	13.31087	−2.25825
4	−2.25826	$-.50\times10^{-5}$	13.29920	−2.25826
5	−2.25826	$-.97\times10^{-12}$	13.29920	−2.25826

7. $x^4-10x+5=0$. Let $f(x)=x^4-10x+5$; $f'(x)=4x^3-10$
Since $f(0)=5$ and $f(1)=-4$, $f(x)=0$ has a positive root between 0 and 1; take $x_1=0$.
$x_3=x_4=x=0.5066$ to four decimal places.

n	x_n	− $(x_n^4-10x_n+5)$	÷ $(4x_n^3-10)$	= STO
1	0	5	−10	0.5
2	0.5	0.0625	−9.5	0.506579
3	0.506579	0.00006	−9.48	0.506586

8. $x^4-10x+5=0$; the largest positive root
Let
$$f(x)=x^4-10x+5; \qquad f'(x)=4x^3-1$$
Because $f(1)=-4, f(2)=1$, and $f'(x)>0$ if $x>2$ then f has a zero between 1 and 2 and is positive if $x>2$. We take $x_1=1.9$. The calculations are shown in the table where the numbers are rounded off to five digits. The root is 1.952, rounded to the nearest thousandth.

n	x_n	− $(x_n^4-10x_n+5)$	÷ $(4x_n^3-10)$	= STO
1	1.9	−0.968	17.436	1.9555
2	1.9555	0.0678	19.991	1.9521
3	1.9521	0.0004	19.755	1.9521

9. $2x^4-2x^3+x^2+3x-4=0$; the negative root. Let $f(x)=2x^4-2x^3+x^2+3x-4$; $f'(x)=8x^3-6x^2+2x+3$
Since $f(-2)=42$ and $f(-1)=-2$, $f(x)=0$ has a negative root between -2 and -1.
Take $x_1=-1$. Then $x_4=x_5=x=-1.1282$ to four decimal places.

n	x_n	− $(2x_n^4-2x_n^3+x_n^2+3x_n-4)$	÷ $(8x_n^3-6x_n^2+24x_n+3)$	= STO
1	−1	−2	−13	−1.15385
2	−1.15835	0.407238	−19.5853	−1.12897
3	−1.12897	0.014619	−18.4169	−1.12817
4	−1.12817	0.000013	−18.3803	−1.12817

10. $x^4+x^3-3x^2-x-4=0$; the positive root. Let $f(x)=x^4+x^3-3x^2-x-4$; $f'(x)=4x^3+3x^2-6x-1$
Since $f(1)=-6$ and $f(2)=6$, $f(x)=0$ has a positive root between 1 and 2. Take $x_1=1.5$.
$x_5=x_6=x=1.7603$ to four decimal places.

n	x_n −	$(x_n^4+x_n^3-3x_n^2-x_n-4)$ ÷	$(4x_n^3+3x_n^2-6x_n-1)$	= STO
1	1.5	−3.8	10.25	1.87195
2	1.87195	2.45	24.51967	1.77185
3	1.77184	.228	20.03761	1.76044
4	1.76044	.002274	19.55835	1.76030
5	1.76030	4.09×10^{-7}	19.55250	1.76030

In Exercises 11–14, use Newton's method to find the value of the radical to five decimal places.

11. $x=\sqrt{3}$ by solving the equation $x^2-3=0$. Let $f(x)=x^2-3$; $f'(x)=2x$
Since $f(1)=-2$ and $f(2)=1$, $f(x)=0$ has a positive root between 1 and 2; take $x_1=2$.
$x_4=x_5=x=1.73205$ to five decimal places.

n	x_n −	(x_n^2-3) ÷	$(2x_n)$	= STO
1	2	1	4	1.75
2	1.75	0.0625	3.5	1.73214
3	1.73214	0.000319	3.464	1.73205
4	1.73205	0	3.464	1.73205

12. $\sqrt{10}$ by solving the equation $x^2-10=0$.

▷ $f(x)=x^2-10 \qquad f'(x)=2x$

We take 3.1 as a starting value and calculate as shown in the table. We conclude that $\sqrt{10}=3.16228$, correct to five decimal places.

n	x_n −	(x_n^2-10) ÷	$(2x_n)$	= STO
1	3.1	−0.39	6.2	3.162903
2	3.162903	0.003955	6.325806	3.162278
3	3.162278	0.000000	6.324556	3.162278

13. $x=\sqrt[3]{6}$ by solving the equation $x^3-6=0$. Let $f(x)=x^3-6$; $f'(x)=3x^2$
Since $f(1)=-5$ and $f(2)=2$, $f(x)=0$ has a root between 1 and 2; take $x_1=2$.
$x_4=x_5=x=1.81712$ to five decimal places.

n	x_n −	(x_n^3-6) ÷	$(3x_n^2)$	= STO
1	2	2	12	1.83333
2	1.83333	0.162039	10.083	1.81726
3	1.87126	0.001417	9.907	1.81712
4	1.81712	0	9.906	1.817

14. $x=\sqrt[3]{7}$ by solving the equation $x^3-7=0$. Let $f(x)=x^3-7$; $f'(x)=3x^2$.
Since $f(1)=-6$ and $f(2)=1$, $f(x)=0$ has a root between 1 and 2; take $x_1=2$.
$x_4=x_5=x=1.91293$ to five decimal places.

n	x_n −	(x_n^3-7) ÷	$(3x_n^2)$	= STO
1	2	1	12	1.91667
2	1.91667	0.0411	11.0208	1.91294
3	1.91294	7.99×10^{-5}	10.9780	1.91293
4	1.91293	3.04×10^{-10}	10.9779	1.91293

Exercises 15–18, use Newton's method to find to 4 places the x coordinate of the first quadrant intersection.

$y = x$; $y = \cos x$. At the point of intersection $x - \cos x = 0$. Let $f(x) = x - \cos x$; $f'(x) = 1 + \sin x$. Since $f(0) = 0$ and $f(\frac{1}{2}\pi) = \frac{1}{2}\pi$, the abscissa of the point of intersection is between 0 and $\frac{1}{2}\pi$; take $x_1 = 0$.
$x_4 = x_5 = x = 0.7391$ to four decimal places.

n	x_n −	$(x - \cos x)$ ÷	$(1 + \sin x)$	= STO
1	0	−1	1	1
2	1	0.459698	1.84147	0.75036
3	0.75036	0.018923	1.68191	0.73911
4	0.73911	0.000047	1.67363	0.73908

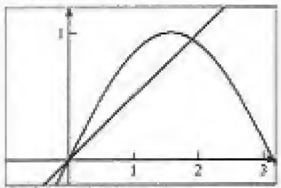
$y = \frac{1}{2}x$; $y = \sin x$

The graphs of the two equations are shown in the plot. Let

$$f(x) = \tfrac{1}{2}x - \sin x \qquad f'(x) = \tfrac{1}{2} - \cos x$$

The x coordinate of the first quadrant point of intersection of the graphs is the zero of f that is between 0 and π. Because the root appears to be greater than $\pi \approx 1.57$, we take $x_1 = 1.9$, put our calculator into radian mode, and calculate as shown in the table.
We conclude that the x coordinate is 1.8955, correct to four decimal places.

n	x_n −	$(\frac{1}{2}x - \sin x)$ ÷	$(\frac{1}{2} - \cos x)$	= STO
1	1.9	0.0037	0.82329	1.89551
2	1.89551	0.00001	0.81904	1.89549
3	1.89549	0.00000	0.81902	1.89549



$y = x^2$, $y = \sin x$. At the point of intersection, $x^2 - \sin x = 0$. Let $f(x) = x^2 - \sin x$; $f'(x) = 2x - \cos x$. Since $f(\frac{1}{4}\pi) = -0.0902$ and $f(\frac{1}{2}\pi) = 1.4674$, $f(x) = 0$ has a root between $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$; take $x_1 = \frac{1}{4}\pi \approx .7854$.
$x_4 = x_5 = x = 0.8767$ to four decimal places.

n	x_n −	$(x_n{}^2 - \sin x_n)$ ÷	$(2x_n - \cos x_n)$	= STO
1	0.7854	−0.090255	0.8637	0.889899
2	0.889899	0.014912	1.1503	0.876935
3	0.876935	0.000733	1.1144	0.876726
4	0.876726	0	1.1138	0.876726

$y = x^2$; $y = \cos x$. At the point of intersection, $x^2 - \cos x = 0$. Let $f(x) = x^2 - \cos x$; $f'(x) = 2x + \sin x$. Based on a sketch of the functions, we take $x_1 = 1$.
$x_4 = x_5 = x = 0.8241$ to four decimal places.

n	x_n −	$(x_n{}^2 - \cos x_n)$ ÷	$(2x_n + \sin x_n)$	= STO
1	1	.46	2.84147	.838218
2	.838218	.0338	2.41989	.824242
3	.824242	.000261	2.38252	.824132
4	.824132	1.61×10^{-8}	2.38222	.824132

Exercises 19–24, for the given f (a) find the linear approximation L at $x = 1$; (b) check by plotting; (c) compare f and L at 0.9, 0.99, 1, 1.01, and 1.1.

$f(x) = x^2$, $f(1) = 1$
$f'(x) = 2x$, $f'(1) = 2$

x	0.9	0.99	1	1.01	1.1
$f(x) = x^2$	0.81	0.9801	1	1.0201	1.21
$L(x) = 1 + 2(x - 1)$	$1 - .2 = .8$	$1 - .02 = .98$	1	$1 + .02 = 1.02$	$1 + .2 = 1.2$

$f(x) = x^3$, $f(1) = 1$
$f'(x) = 3x^2$, $f'(1) = 3$

x	0.9	0.99	1	1.01	1.1
$f(x) = x^3$	0.729	0.9703	1	1.0303	1.331
$L(x) = 1 + 3(x - 1)$	$1 - .3 = .7$	$1 - .03 = .97$	1	$1 + .03 = 1.03$	$1 + .3 = 1.3$

$f(x) = 2\sqrt{x}$, $f(1) = 2$
$f'(x) = 1/\sqrt{x}$, $f'(1) = 1$

x	0.9	0.99	1	1.01	1.1
$f(x) = 2\sqrt{x}$	1.897	1.98997	2	2.00998	2.0976
$L(x) = 2 + (x - 1)$	$2 - .1 = 1.9$	$2 - .01 = 1.99$	2	$2 + .01 = 2.01$	$2 + .1 = 2.1$

22. $f(x) = 2/x^2$, $f(1) = 2$
$f'(x) = -4/x^3$, $f'(1) = -4$

x	0.9	0.99	1	1.01	1.1
$f(x) = 2/x^2$	2.469	2.0406	2	1.9606	1.653
$L(x) = 2 - 4(x-1)$	$2 + .4 = 2.4$	$2 + .04 = 2.04$	2	$2 - .04 = 1.96$	$2 - .4 =$ 1[illegible]

23. $f(x) = \cos x$. $f(1) = 0.54030$; $f'(x) = -\sin x$, $f'(1) = -0.84147$. $L(x) = 0.54030 - 0.84147(x-1)$

x	0.9	0.99	1	1.01	1.1
$f(x)$	0.6216	0.54869	0.54030	0.53186	0.4536
$L(x)$	$.5403 + .0841 = 0.6244$	$.54030 + .00841 = 0.54871$	0.54030	$.54030 - .00841 = 0.53189$	$.5403 - .0841 = 0.4562$

24. $f(x) = \sin x$. $f(1) = 0.84147$; $f'(x) = \cos x$, $f'(1) = 0.54030$. $L(x) = 0.84147 + 0.54030(x-1)$

x	0.9	0.99	1	1.01	1.1
$f(x)$	0.7833	0.83603	0.84147	0.84683	0.8912
$L(x)$	$.8415 - .0540 = 0.7875$	$.84147 - .00540 = 0.83607$	0.84147	$.84147 + .00540 = 0.84687$	$.8415 + .0540 = 0.8955$

In Exercises 25–28, (a) find dy and Δy for the values of x and Δx. (b) Sketch the graph indicating dy and Δy.

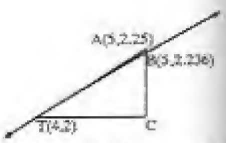
25. $y = x^2$, $x = 2$, $\Delta x = 0.5$ ▷ $y'(x) = 2x$. $dy = y'(2)\Delta x = 2(2)(0.5) = 2$. $\Delta y = 2.5^2 - 2^2 = 2.25$

26. $y = x^3$, $x = 2$, $\Delta x = 0.5$ ▷ $y'(x) = 3x^2$. $dy = y'(2)\Delta x = 3(2)^2(0.5) = 6$. $\Delta y = 2.5^3 - 2^3 = 7.625$]

27. $y = \sqrt[3]{x}$, $x = 8$, $\Delta x = 1$

▷ $y'(x) = \frac{1}{3}x^{-2/3}$. $dy = y'(8)\Delta x = \frac{1}{3}\cdot\frac{1}{4}(1) = \frac{1}{12} \approx .083$ $\Delta y = \sqrt[3]{9} - 2 = .080$

28. $y = \sqrt{x}$, $x = 4$, $\Delta x = 1$



▷ $y'(x) = \dfrac{1}{2\sqrt{x}}$. $dy = y'(4)\Delta x = \dfrac{1}{2\sqrt{4}}\cdot 1 = \frac{1}{4} = 0.25$

$\Delta y = y(5) - y(4) = \sqrt{5} - \sqrt{4} = 0.236$

In the figure, TA is the tangent at $(4, 2)$. TC is parallel to the x axis; its length is $\Delta x = 1$. CA is parallel [illegible] the y axis; its length is $dy = 0.25$. The length of CB is $\Delta y = 0.236$.

In Exercises 29–34, find (a) Δy; (b) dy; (c) $\Delta y - dy$

29. $y = x^2 - 3x$; $x = 2$; $\Delta x = 0.03$. Let $f(x) = x^2 - 3x$.
(a) $\Delta y = f(x + \Delta x) - f(x) = f(2.03) - f(2) = [(2.03)^2 - 3(2.03)] - (4 - 6) = -1.961 + 2 = 0.0309$
(b) $dy = f'(x)dx = (2x - 3)\Delta x = (4 - 3)(0.03) = 0.03$ (c) $\Delta y - dy = 0.0309 - 0.03 = 0.0009$

30. $y = x^2 - 3x$; $x = -1$; $\Delta x = 0.02$. Let $f(x) = x^2 - 3x$. If $x = -1$ and $\Delta x = 0.02$ then
(a) $\Delta y = f(x + \Delta x) - f(x) = f(-0.98) - f(-1) = [(-.98)^2 - 3(-.98)] - [(-1)^2 - 3(-1)] = 3.9004 - 4 = -.0$[illegible]
(b) $dy = f'(x)\Delta x = (2x - 3)\Delta x = [2(-1) - 3](0.02) = -0.1$. (c) $\Delta y - dy = (0.0996) - (-0.1) = 0.0004$

31. $y = \frac{1}{x}$; $x = -2$; $\Delta x = -0.1$. Let $f(x) = \frac{1}{x}$.
(a) $\Delta y = f(x + \Delta x) - f(x) = f(-2.1) - f(-2) = -\frac{1}{2.1} + \frac{1}{2} = \frac{1}{42} \approx 0.0238$
(b) $dy = f'(x)dx = -\frac{1}{x^2}\Delta x = -\dfrac{(-0.1)}{4} = \frac{1}{40} = 0.025$ (c) $\Delta y - dy = \frac{1}{42} - \frac{1}{40} = -\frac{2}{1680} = -\frac{1}{840} \approx -0.0012$

32. $y = \frac{1}{x}$; $x = 3$; $\Delta x = -0.2$. Let $f(x) = \frac{1}{x}$.
(a) $\Delta y = f(x + \Delta x) - f(x) = f(2.8) - f(3) = \frac{1}{2.8} + \frac{1}{3} = \frac{1}{42} \approx 0.0238$
(b) $dy = f'(x)dx = -\frac{1}{x^2}\Delta x = -\dfrac{(-0.2)}{9} = \frac{1}{45} = 0.022$ (c) $\Delta y - dy = \frac{1}{42} - \frac{1}{45} = \frac{1}{630} \approx 0.0016$

33. $y = x^3 + 1$; $x = 1$; $\Delta x = -0.5$. Let $f(x) = x^3 + 1$.
(a) $\Delta y = f(x + \Delta x) - f(x) = f(0.5) - f(1) = [(0.5)^3 + 1] - 2 = 1.125 - 2 = -0.875$
(b) $dy = f'(x)dx = 3x^2\Delta x = 3(1)^2(-0.5) = -1.5$ (c) $\Delta y - dy = -0.875 - (-1.5) = 0.625$

34. $y = x^3 + 1$; $x = -1$; $\Delta x = 0.1$. Let $f(x) = x^3 + 1$. If $x = -1$ and $\Delta x = 0.1$ then
(a) $\Delta y = f(x + \Delta x) - f(x) = f(-0.4) - f(-1) = [(-0.9)^3 + 1] - [(-1)^3 + 1] = 0.271 - 0 = 0.271$
(b) $dy = f'(x)\Delta x = 3x^2\Delta x = 3(-1)^2(0.1) = 0.3$ (c) $\Delta y - dy = 0.271 - 0.3 = -0.029$

In Exercises 35–42, find dy.

35. $y = (3x^2 - 2x + 1)^3$; $dy = 3(3x^2 - 2x + 1)^2(6x - 2)dx$

36. $y = \dfrac{3x}{x^2 + 2}$

$$dy = \frac{(x^2+2)\, d(3x) - 3x\, d(x^2+2)}{(x^2+2)^2} = \frac{(x^2+2)(3\, dx) - 3x(2x\, dx)}{(x^2+2)^2} = \frac{3(2-x^2)\, dx}{(x^2+2)^2}$$

37. $y = x^2\sqrt{2x+3}$

$$dy = \left\{2x(2x+3)^{1/2} + x^2[\tfrac{1}{2}(2x+3)^{-1/2}(2)]\right\}dx = x(2x+3)^{-1/2}[2(2x+3) + x]dx = \frac{x(5x+6)dx}{(2x+3)^{1/2}}$$

38. $y = \sqrt{4 - x^2}$, $dy = \dfrac{-x\, dx}{\sqrt{4-x^2}}$

39. $y = \dfrac{2 + \cos x}{2 - \sin x}$; $dy = \dfrac{-\sin x(2 - \sin x) - (-\cos x)(2 + \cos x)}{(2 - \sin x)^2}\, dx$

$$= \frac{-2\sin x + \sin^2 x + 2\cos x + \cos^2 x}{(2-\sin x)^2}dx = \frac{1 - 2\sin x + 2\cos x}{(2 - \sin x)^2}dx$$

40. $y = x^2 \sin \frac{1}{x} - x\cos\frac{1}{x}$

$$dy = \left[\sin\tfrac{1}{x}\, D_x(x^2) + x^2\, D_x\!\left(\sin\tfrac{1}{x}\right) - \cos\tfrac{1}{x} - x\, D_x\!\left(\cos\tfrac{1}{x}\right)\right]dx$$

$$= \left[2x\sin\tfrac{1}{x} + x^2\cos\tfrac{1}{x}\left(-\tfrac{1}{x^2}\right) - \cos\tfrac{1}{x} + x\sin\tfrac{1}{x}\left(-\tfrac{1}{x^2}\right)\right]dx = \left[2\sin\tfrac{1}{x} - 2\cos\tfrac{1}{x} - \tfrac{1}{x}\sin\tfrac{1}{x}\right]dx$$

41. $y = \tan^2 x \sec^2 x$; $dy = [(2\tan x \sec^2 x)\sec^2 x + \tan^2 x(2\sec x \sec x \tan x)]dx = 2\tan x \sec^2 x(\sec^2 x + \tan^2 x)dx$

42. $y = \cot 2x \csc 2x$

$dy = -2\csc^2 2x \cdot \csc 2x + \cot 2x(-2\cot 2x \csc 2x) = -2\csc^3 2x - 2\cot^2 2x \csc 2x = -4\csc^3 2x + 2\csc 2x$

43. (a) Let x cm be the length of an edge of the cube and V cm^3 the volume of the cube.
Then $V = x^3$; $dV = 3x^2 dx$. If $x = 15$ and $dx = \Delta x = 0.01$, then $\Delta V \approx dV = 3(15)^2(0.01) = 6.75$.
- The approximate error in computing the volume is 6.75 cm^3.

(b) Let A cm^2 be the area of a face of the cube.
Then $A = x^2$; $dA = 2x\, dx$. If $x = 15$ and $dx = \Delta x = 0.01$, then $\Delta A \approx dA = 2(15)(0.01) = 0.3$.
- The approximate error in computing the area of a face is 0.3 cm^2.

44. A metal box in the form of a cube is to have an interior volume of 1000 cm^3. The six sides are to be made of metal $\frac{1}{2}$ cm thick. If the cost of the metal to be used is \$0.20 per cubic centimeter, use differentials to find the approximate cost of the metal to be used in the manufacture of the box.

Let x cm be the length of each inside edge of the cube. Let V cm^3 be the interior volume of the cube. Because

$$V = x^3$$

and we are given that the interior volume is 1000 cm^3, then

$$1000 = x^3; \qquad x = 10$$

Because the thickness of the metal is 0.5 cm, then each outside edge of the cube is 1 cm longer than each inside edge. Therefore, the volume of the metal is ΔV cm^3, with $x = 10$ and $\Delta x = 1$. We use dV to approximate ΔV. Thus

$$dV = 3x^2(\Delta x) = 3(10^2)1 = 300$$

The volume of the metal is approximately 300 cm^3. Because the metal costs \$0.20 per cubic centimeter, and $0.20(300) = 60$, the cost of the metal is approximately \$60. ($\Delta V = 11^3 - 10^3 = 331$ and the actual cost is \$0.20(331)=\$66.20.)

45. Let r meters be the radius and V m^3 be the volume of the cylindrical tank. The altitude of the tank is 10 m.
Then $V = 10\pi r^2$; $dV = 20\pi r\, dr$. If $r = 6$ and $dr = \Delta r = \frac{1}{50}$, then $\Delta V \approx dV = 20\pi(6)\frac{1}{50} = \frac{12}{5}\pi$.
- The approximate amount of coating material is $\frac{12}{5}\pi$ m^3.

46. Let r cm be the radius and V m^3 be the volume of the cylindrical stem. The height of the stem is 2 cm.
Then $V = 2\pi r^2$; $dV = 4\pi r\, dr$. If $r = .4$ and $dr = \Delta r = .1$, then $\Delta V \approx dV = 4\pi(.4)(.1) = .16\pi$
- The approximate increase in volume is $.16\pi$ cm^3.

47. A cm^2 is the area of the circular burn when the radius is r cm.
Then $A = \pi r^2$; $dA = 2\pi r\, dr$. If $r = 1$ and $dr = -0.2$, then $\Delta A \approx dA = 2\pi(1)(-.02) = -0.4\pi$.

- The approximate decrease in the area of the burn is $0.4\pi\ \text{cm}^2$.

48. A certain bacterial cell is spherical in shape such that if r micrometers is its radius and V cubic micrometers is its volume, then $V = \frac{4}{3}\pi r^3$. Use the differential to find the approximate increase in the volume of the cell when the radius increases from 2.2 μm to 2.3 μm.

▷ We find the value of dV when $r = 22$ and $\Delta r = 0.1$. Thus,

$$V = \tfrac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\, \Delta r = 4\pi(2.2)^2(0.1) = 6.08$$

- The increase in volume is approximately 6 μm^3.

49. V cm^3 is the volume of the spherical tumor when the radius is r cm.
Then $V = \frac{4}{3}\pi r^3$; $dV = 4\pi r^2 dr$. If $r = 1.5$ and $dr = 0.1$ then $\Delta V \approx dV = 4\pi(1.5)^2(0.1) = 0.9\pi$.

- The approximate increase in the volume of the tumor is $0.9\pi\ \text{cm}^3$.

50. t seconds is the period of a pendulum when the length is ℓ feet. Then $4\pi^2\ell = gt^2$. Dividing the differential by the given, we get $\frac{4\pi^2 d\ell}{4\pi^2\ell} = \frac{2gt\, dt}{gt^2}$, $\frac{d\ell}{\ell} = 2\frac{dt}{t} = 2 \cdot \frac{5}{24 \cdot 60} = \frac{1}{144}$, $d\ell = \frac{\ell}{144} = \frac{1}{144}$

- The pendulum should be lengthened $\frac{1}{144}$ft $= \frac{1}{12}$in.

51. Let R be the measure of the electrical resistance of a wire, x be the measure of its diameter, and L be the measure of its length. L is constant. For some constant k,
$R = \frac{kL}{x^2}$; $dR = \frac{-2kL}{x^3}\, dx$; $\frac{dR}{R} = \frac{-2kL}{x^3}\, dx \cdot \frac{x^2}{kL} = -2\frac{dx}{x}$. If $\left|\frac{dx}{x}\right| = \left|\frac{\Delta x}{x}\right| = 2\% = 0.02$, then $\left|\frac{\Delta R}{R}\right| \approx \left|\frac{dR}{R}\right| -2(0.02)$
$= 0.04 = 4\%$. Hence, the approximate error in the resistance is 4%.

52. A contractor agrees to paint on both sides of 1000 circular signs each of radius 3 m. Upon receiving the signs it is discovered that the radius of each sign is 1 cm too large. Use differentials to find the approximate percent increase of paint that will be needed.

▷ Let r meters be the radius of each sign; A square meters the total area that must be painted.
Because there are 1000 signs, each to be painted on both sides, by the formula for the area of a circle we have

$$A = 2000\pi r^2$$

Because ΔA is the increase in the paint required to do the job, then

$$\frac{100\Delta A}{A}\%$$

is the *percent* increase in paint needed. By Eq. (1) we have

$$dA = 2(2000)\pi r\, \Delta r$$

$$\frac{dA}{A} = \frac{2(2000)\pi r\, \Delta r}{2000\pi r^2} = \frac{2\,\Delta r}{r}$$

When $r = 3$ and $\Delta r = 0.01$, since 1 cm = 0.01 m, we get

$$\frac{dA}{A} = \frac{2(0.01)}{3} = 0.0067 = 0.67\%$$

Because $\Delta A \approx dA$, we conclude 0.67% is the approximate percent increase of paint needed. The result is the same for any number of signs.

53. If V ft^3 is the volume of a gas, and P lb/ft^2 is the pressure of the gas, Boyle's law states that for some constant C, $P = \frac{C}{V}$; $dP = -\frac{C}{V^2}\, dV$; $V^2 = -\frac{C}{dP}\, dV$. To find the smallest possible V we take $dP = {}^0 0.00$
Then with $dV = \pm 0.1$, $V^2 = -\frac{C}{\mp 0.001C}(\pm 0.1) = 100$; $V = 10$.

- The smallest container has a volume of 10 ft^3.

54. The adiabatic law is $PV^{1.4} = C$; $P = CV^{-1.4}$. Divide the differential by the given to get
$\frac{dP}{P} = \frac{-1.4CV^{-2.4}}{CV^{-1.4}} = -1.4\frac{dV}{V}$

55. Boyle's law is $PV = C$; $P = CV^{-1}$. Divide the differential by the given to get $\frac{dP}{P} = \frac{-CV^{-2}}{CV^{-1}} = -\frac{dV}{V}$

56. A tightly-wound flexible tape of length L feet, fastened at the top of an incline that makes an angle θ with the horizontal, is allowed to unwind down the incline. If T seconds is the time for the tape to completely unwind, then $T = \sqrt{\frac{3L}{g}\csc\theta}$. Show that $\frac{dT}{T} = -\frac{d\theta}{2\tan\theta}$

» We have

$$T^2 = \frac{3L}{g}\csc\theta \tag{1}$$

Taking the differential on each side, we get

$$2T\,dT = -\frac{3L}{g}\csc\theta\cot\theta\,d\theta \tag{2}$$

Dividing equation (2) by equation (1) gives

$$\frac{2T\,dT}{T^2} = \frac{-(3L/g)\csc\theta\cot\theta\,d\theta}{(3L/g)\csc\theta}$$

Simplifying each side, we get the required result.

In Exercises 57 and 58, equations of the form $\tan x + ax = 0$ arise in heat conduction problems. The positive roots of the equation in increasing order are $\alpha_1, \alpha_2, \alpha_3, \ldots$.

57. If $a = 1$, find α_1 and α_2 to four decimal places.

» $\tan x + x = 0$. Let $f(x) = \tan x + x$; $f'(x) = \sec^2 x + 1 = \tan^2 x + 2$ (the secant function is less available). It is clear from a sketch of the graphs of $y = \tan x$ and $y = -x$ that $\alpha_1 \in (\frac{1}{2}\pi, \frac{3}{2}\pi)$, $\alpha_2 \in (\frac{3}{2}\pi, \frac{5}{2}\pi)$, ..., $\alpha_n \in (n\pi - \frac{1}{2}\pi, n\pi + \frac{1}{2}\pi)$ and that the roots are getting closer to the left endpoint. To find α_n, take

$x_1 = n\pi - \frac{1}{2}\pi + \frac{1}{n\pi}$; Newton's method will fail if x_1 is taken much further from $n\pi - \frac{1}{2}\pi$.

To find α_1, take $x_1 = \pi - \frac{1}{2}\pi + \frac{1}{\pi} \approx 1.8891$ (x_1 must be $\leq$ 3.0). Then $x_4 = x_5 = \alpha_1 = 2.0288$.

To find α_2, take $x_1 = 2\pi - \frac{1}{2}\pi + \frac{1}{2\pi} \approx 4.8715$ (x_1 must be $\leq$ 5.1). Then $x_4 = x_5 = \alpha_2 = 4.9132$.

n	x_n $-$	$(\tan x_n + x_n)$ $\div$	$(\tan^2 x_n + 2)$	$=$ STO
1	1.88910	−1.14573	11.2102	1.99130
2	1.99130	−0.24492	7.0007	2.02629
3	2.02629	−0.01515	6.1675	2.02875
4	2.02875	−0.00006	6.1161	2.02876

n	x_n $-$	$(\tan x_n + x_n)$ $\div$	$(\tan^2 x_n + 2)$	$=$ STO
1	4.87150	−1.36030	40.8354	4.90481
2	4.90481	−0.22777	28.3434	4.91285
3	4.91285	−0.00871	26.2218	4.91318
4	4.91318	0	26.1394	4.91318

58. If $a = -2$, find α_1 and α_2 to four decimal places.

» We wish to find the first two positive roots of $\tan x - 2x = 0$. If we sketch the graphs of $y = \tan x$ and $y = 2x$ on the same axes, we see that there are points of intersection with positive x coordinates slightly less than $\frac{1}{2}\pi \approx 1.57$ and $\frac{3}{2}\pi \approx 4.71$. However, because the tangent function is unbounded near these roots, values of x_1 quite close to the root do not lead to the root, and those which do lead to the root require many iterations. Because $\tan x = \sin x/\cos x$, we multiply the given equation by $\cos x$ to get the equivalent equation

$$\sin x - 2x\cos x = 0$$

with continuous function and derivative. Let

$$f(x) = \sin x - 2x\cos x$$
$$f'(x) = 2x\sin x - \cos x$$

We put our calculator into radian mode. To find α_1, we let $x_1 = 1.4$ and conclude from Table a that $\alpha_1 = 1.1656$ to four decimal places. To find α_2, we let $x_1 = 4.7$ and see from Table b that $\alpha_2 = 4.6042$ to four decimal places.

Table a

n	$x_n - (\sin x_n - 2x_n\cos x_n)$	$\div$	$(2x_n\sin x_n - \cos x_n)$	$=$ STO
1	1.4	0.509542	2.58929	1.203212
2	1.203212	0.068420	1.88631	1.166940
3	1.166940	0.002414	1.75316	1.165563
4	1.165561	0.000003	1.74810	1.165561

Table b

n	x_n −	$(\sin x_n - 2x_n \cos x_n)$ ÷	$(2x_n \sin x_n - \cos x_n)$	= STO
1	4.7	−0.883470	−9.38689	4.605883
2	4.605883	−0.015076	−9.05326	4.604217
3	4.604217	−0.000006	−9.04665	4.604217

In Exercises 59 and 60, approximate π to five places by using Newton's method to solve the equation.

59. $\tan x = 0$. Let $f(x) = \tan x$; $f'(x) = \sec^2 x$. Then $\dfrac{f(x)}{f'(x)} = \dfrac{\tan x}{\sec^2 x} = \sin x \cos x$.

Take $x_1 = 3$. Then $x_2 = x_3 = \pi = 3.14159$ to five decimal places.

n	x_n −	$(\sin x_n \cos x_n)$	= STO
1	3	−0.139708	3.13971
2	3.13971	−0.001885	3.14159
3	3.14159	0	3.14159

60. $\cos x + 1 = 0$. Let $f(x) = \cos x + 1$; $f'(x) = -\sin x$.
We take 3.1 as a starting value and calculate as shown in table a. Because $f'(x)$ also has a zero at π, the convergence is very slow and it will take very many iterations. We start again with $f(x) = \sin x$, $f'(x) =$ … and calculate as show in table b. We conclude that $\pi = 3.14159265$, correct to eight decimal places.

Table a

n	x_n ÷	$(\cos x_n + 1)$ ÷	$\sin x_n$	= STO
1	3	1.00×10^{-2}	.141120	3.070915
2	3.070915	2.50×10^{-3}	.070619	3.106268
3	3.106268	6.24×10^{-4}	.035317	3.123932
4	3.123932	1.56×10^{-4}	.017659	3.132763

Table b

n	x_n −	$\sin x_n$ ÷	$\cos x_n$	= STO
1	3	1.41×10^{-1}	−.989993	3.14254654
2	3.14254654	-9.54×10^{-4}	−.99999955	3.14159265
3	3.14159265	2.89×10^{-10}	−1.0000000	3.14159265

Miscellaneous Exercises for Chapter 3

In Exercises 1–10, for the given interval I, (a) sketch the graph and (b) find any absolute extrema.

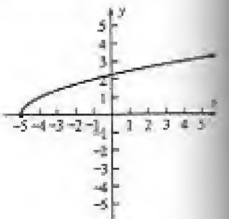
1. $f(x) = \sqrt{5+x}$; $x \in [-5, +\infty)$; $f'(x) = \dfrac{1}{2\sqrt{5+x}}$

There are no critical numbers of f. $f'(x) > 0$ for all x in $(-5, +\infty)$.

Therefore f is increasing on $[-5, +\infty)$. Because $f(-5) = 0$,

the absolute minimum value of f is 0 occurring when $x = -5$.

There is no absolute maximum value.



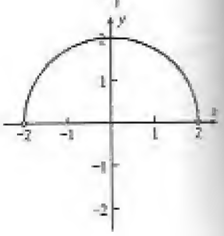
2. $f(x) = \sqrt{4 - x^2}$; $x \in (-2, 2)$; $f'(x) = -\dfrac{x}{\sqrt{4-x^2}}$

Set $f'(x) = 0$: $x = 0$. The critical number of f is 0.

Because $f'(x) > 0$ if $x < 0$ and $f'(x) < 0$ if $x > 0$, $f(0) = 4$

is an absolute maximum value.

I is not a closed interval and f does not have an absolute minimum value on I.

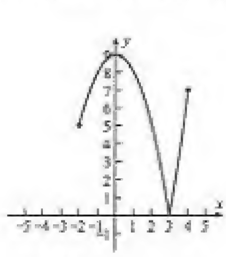


3. $f(x) = |9 - x^2|$, $x \in [-2, 4] = I$; $f'(x) = \operatorname{sgn}(9 - x^2)(-2x)$, $x \neq \pm 3$

$f'(0) = 0$, and $f'(-3)$ and $f'(3)$ do not exist. The critical numbers of f are 0 and 3. f is continuous on I so absolute extrema occur at an end point or a critical number.

$f(-2) = 5$, $f(0) = 9$, $f(3) = 0$, $f(4) = 7$

The absolute minimum value is $f(3) = 0$ and the absolute maximum value is $f(0) = 9$.

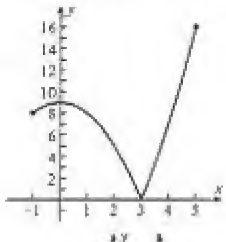


4. $f(x) = |9 - x^2|$, $x \in [-1, 5] = I$; $f'(x) = \operatorname{sgn}(9 - x^2)(-2x)$, $x \neq \pm 3$

$f'(0) = 0$, and $f'(-3)$ and $f'(3)$ do not exist. The critical numbers of f are 0 and 3. f is continuous on I so absolute extrema occur at an end point or a critical number.

$f(-1) = 8$, $f(0) = 9$, $f(3) = 0$, $f(5) = 16$

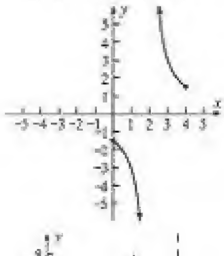
The absolute minimum value is $f(3) = 0$ and the absolute maximum value is $f(5) = 16$.



5. $f(x) = \dfrac{3}{x - 2}$; $[0, 4]$

Because $\lim\limits_{x \to 0^+} f(x) = +\infty$,
f does not have an absolute maximum value on I.

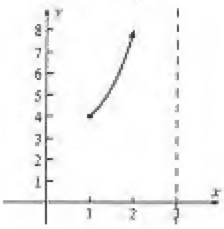
Because $\lim\limits_{x \to 0^-} f(x) = -\infty$,
f does not have an absolute minimum value on I.



6. $f(x) = \dfrac{8}{3 - x}$; $[1, 3]$; $f'(x) = \dfrac{8}{(3 - x)^2}$

Because $\lim\limits_{x \to 3^-} f(x) = +\infty$,
f does not have an absolute maximum value on I.

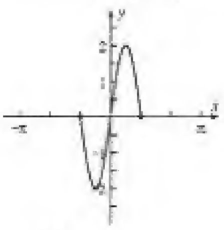
Because $f'(x) > 0$, $f(x)$ is increasing on I. Thus the absolute minimum value occurs at the left endpoint: $f(1) = 4$.



7. $f(x) = 2 \sin 3x$, $x \in [-\frac{1}{3}\pi, \frac{1}{3}\pi] = I$, $3x \in [-\pi, \pi]$; $f'(x) = 6 \cos 3x$

Set $f'(x) = 0$: $\cos 3x = 0$; $3x = -\frac{1}{2}\pi, \frac{1}{2}\pi$; $x = -\frac{1}{6}\pi, \frac{1}{6}\pi$. The critical numbers are $-\frac{1}{6}\pi$ and $\frac{1}{6}\pi$. f is continuous on I so absolute extrema occur at an end point or a critical number.
$f(-\frac{1}{3}\pi) = 0$, $f(-\frac{1}{6}\pi) = -2$, $f(\frac{1}{6}\pi) = 2$, $f(\frac{1}{3}\pi) = 0$

The absolute minimum value is $f(-\frac{1}{6}\pi) = -2$ and the absolute maximum value is $f(\frac{1}{6}\pi) = 2$.

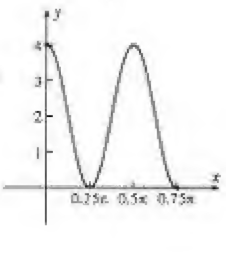


8. $f(x) = 4 \cos^2 2x$; $x \in [0, \frac{3}{4}\pi] = I$. $f(x) = 2(1 + \cos 4x)$, $4x \in [0, 3\pi]$

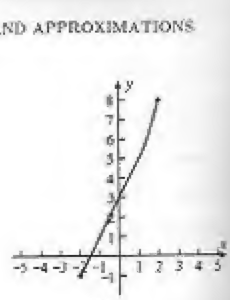
$f'(x) = -8 \sin 4x$. Set $f'(x) = 0$: $4x = 0, \pi, 2\pi, 3\pi$; $x = 0, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi$.

f is continuous on I so absolute extrema occur at end points or a critical number.
$f(0) = 4$, $f(\frac{1}{4}\pi) = 0$, $f(\frac{1}{2}\pi) = 4$, $f(\frac{3}{4}\pi) = 0$

The absolute maximum value is $f(0) = f(\frac{1}{2}\pi) = 4$ and the absolute minimum value is $f(\frac{1}{4}\pi) = f(\frac{3}{4}\pi) = 0$.



9. $f(x) = \begin{cases} 2x+3 & \text{if } -2 \le x < 1 \\ x^2+4 & \text{if } 1 \le x \le 2 \end{cases}$; $[-2,2]$. $f'(x) = \begin{cases} 2 & \text{if } -2 < x < 1 \\ 2x & \text{if } 1 < x < 2 \end{cases}$

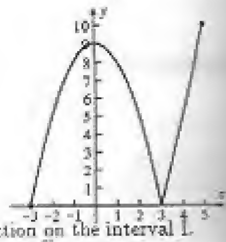


$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-}(2x+3) = 5$; $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+}(x^2+4) = 5$

and so f is continuous on I. $f'(x)$ is never 0 and $f'(1)$ may not exist.
$f(-2) = 2(-2)+3 = -1$, $f(1) = 5$, $f(2) = 2^2+4 = 8$

The absolute minimum value is $f(-2) = -1$ and
the absolute maximum value is $f(2) = 8$.

10. $f(x) = \begin{cases} 9-x^2 & \text{if } -3 \le x < 3 \\ 5x-15 & \text{if } 3 \le x \le 5 \end{cases}$; $[-3,5]$. $f'(x) = \begin{cases} -2x & \text{if } -3 < x < 3 \\ 5 & \text{if } 3 < x < 5 \end{cases}$



$\lim\limits_{x\to3^-} f(x) = \lim\limits_{x\to3^-}(9-x^2) = 0$; $\lim\limits_{x\to3^+} f(x) = \lim\limits_{x\to3^+}(5x-15) = 0$

and so f is continuous on I. $f'(0) = 0$ and $f'(3)$ may not exist.

$f(-3) = 9-(-3)^2 = 0$, $f(0) = 9-0 = 9$, $f(3) = 0$, $f(5) = 5(5)-15 = 10$

The absolute minimum value is $f(-3) = f(3) = 0$ and
the absolute maximum value is $f(5) = 10$.

In Exercises 11–14, find graphically and analytically the absolute extrema of the function on the interval I.

11. $f(x) = x^4 - 12x^2 + 36$; $f'(x) = 4x^3 - 24x = 4x(x^2-6)$. Set $f'(x) = 0$: $x = 0$, $x = \pm\sqrt{6}$.
f is continuous on any I so absolute extrema occur at an end point or a critical number.
(a) $\mathrm{I} = [-2,3]$. The critical number is $\sqrt{6}$. $f(-2) = 4$, $f(\sqrt{6}) = 0$, $f(3) = 9$
The absolute minimum value is $f(\sqrt{6}) = 0$ and the absolute maximum value is $f(3) = 9$.
(b) $\mathrm{I} = [-4,2]$. The critical number is $-\sqrt{6}$. $f(-4) = 100$, $f(-\sqrt{6}) = 0$, $f(2) = 4$
The absolute minimum value is $f(\sqrt{6}) = 0$ and the absolute maximum value is $f(-4) = 100$.

12. $f(x) = x^5 - 9x^2 + 5$; $f'(x) = 5x^4 - 18x = 5x(x^3-3.6)$. Set $f'(x) = 0$: $x = 0$, $\sqrt[3]{3.6} \approx 1.53$.
f is continuous on any I so absolute extrema occur at an end point or a critical number.
(a) $\mathrm{I} = [-1,2]$. The critical number is $\sqrt[3]{3.6}$. $f(-1) = -5$, $f(\sqrt[3]{3.6}) = 5 - 5.4(3.6)^{2/3} \approx -7.68$, $f(2) = 1$
The absolute minimum value is $f(\sqrt[3]{3.6}) \approx -7.68$ and the absolute maximum value is $f(2) = 1$.
(b) $\mathrm{I} = [-2,1]$. There is no critical number. $f(-2) = -63$, $f(1) = -3$
The absolute minimum value is $f(-2) = -63$ and the absolute maximum value is $f(1) = -3$.

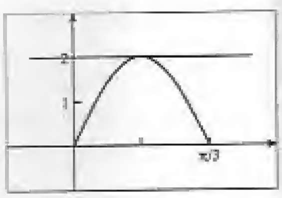
13. $f(x) = \sin x + \cos x$; $\mathrm{I} = [-1,1]$. $f'(x) = \cos x - \sin x$. Set $f'(x) = 0$: $\tan x = 1$, $x = \frac{1}{4}\pi \approx .79$
f is continuous on I so absolute extrema occur at an end point or a critical number.
$f(-1) = \sin(-1) + \cos(-1) \approx -.301$, $f(\frac{1}{4}\pi) = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = \sqrt{2} \approx 1.41$, $f(1) = \sin 1 + \cos 1 \approx 1.38$
The absolute minimum value is $f(-1) \approx -.301$ and the absolute maximum value is $f(\frac{1}{4}\pi) = \sqrt{2}$.

14. $f(x) = 2\cos x + x$; $\mathrm{I} = [-1,3]$. $f'(x) = 1 - 2\sin x$. Set $f'(x) = 0$: $\sin x = \frac{1}{2}$, $x = \frac{1}{6}\pi$, $\frac{5}{6}\pi \approx 2.62$
f is continuous on I so absolute extrema occur at an end point or a critical number.
$f(-1) = 2\cos(-1) - 1 \approx .081$, $f(\frac{1}{6}\pi) = \sqrt{3} + \frac{1}{6}\pi \approx 2.256$, $f(\frac{5}{6}\pi) = -\sqrt{3} + \frac{5}{6}\pi \approx .886$, $f(3) = 2\cos 3 + 3 \approx 1.02$
The absolute minimum value is $f(-1) \approx .081$ and the absolute maximum value is $f(\frac{1}{6}\pi) = \sqrt{3} + \frac{1}{6}\pi$.

In Exercises 15 and 16, verify the 3 hypotheses of Rolle's theorem; find a c for the conclusion. Check by plotting.

15. $f(x) = x^3 - x^2 - 4x + 4$, $x \in [-2,1]$; $f'(x) = 3x^2 - 2x - 4$
$f(-2) = (-2)^3 - (-2)^2 - 4(-2) + 4 = 0$; $f(1) = (1)^3 - (1)^2 - 4(1) + 4 = 0$
Because f is a polynomial it is continuous and differentiable everywhere so the hypothesis of Rolle's theorem is satisfied. Thus, there exists a c in $(-2,1)$ for which $f'(c) = 0$; $3c^2 - 2c - 4 = 0$; $c = \frac{1}{3}(1 \pm \sqrt{13})$
Because $\frac{1}{3}(1 - \sqrt{13}) \approx -0.86$ is in $(-1,2)$, $\frac{1}{3}(1 - \sqrt{13})$ qualifies as c.

16. $f(x) = 2\sin 3x$; $[0, \frac{1}{3}\pi]$. $f'(x) = 6\cos 3x$
$f(0) = 2\sin 0 = 0$; $f(\frac{1}{3}\pi) = 2\sin\pi = 0$
Because f is continuous and differentiable everywhere,
the hypothesis of Rolle's theorem is satisfied.
Thus, there exists a c in $(0, \frac{1}{3}\pi)$ for which $f'(c) = 0$:
$6\cos 3c = 0$; $3c = \frac{1}{2}\pi$; $c = \frac{1}{6}\pi$.
Because $\frac{1}{6}\pi$ is in $(0, \frac{1}{3}\pi)$, $\frac{1}{6}\pi$ qualifies as c.



In Exercises 17–20, verify the hypotheses of the mean-value theorem; find a c for the conclusion. Check by plotting.

17. $f(x) = \sqrt{3-x}$, $x \in [-6,-1]$; $f'(x) = \dfrac{-1}{2\sqrt{3-x}}$

f is continuous on $(-\infty, 3]$ and differentiable on $(-\infty, 3)$. Thus f is continuous on $[-6,-1]$ and differentiable on $(-6,-1)$, so the hypothesis of the mean-value theorem is satisfied. Hence there exists a c in $(-6,-1)$ for which $f'(c) = \dfrac{f(-1)-f(-6)}{-1-(-6)} = \dfrac{2-3}{5} = -\frac{1}{5}$; $\dfrac{-1}{2\sqrt{3-c}} = -\frac{1}{5}$; $2\sqrt{3-c} = 5$; $12-4c = 25$; $c = -\frac{13}{4}$

Because $-\frac{13}{4}$ is in $(-6,-1)$, $-\frac{13}{4}$ qualifies as c.

18. $f(x) = x^3$, $x \in [-2,2]$; $f'(x) = 3x^2$.

Because f is a polynomial function, it is continuous and differentiable for all x. In particular, f is continuous on $[-2,2]$ and differentiable on $(-2,2)$. Thus, the hypothesis of the mean-value theorem is satisfied. Hence there exists a c in $(-2,2)$ for which $f'(c) = \dfrac{f(2)-f(-2)}{2-(-2)} = \dfrac{8-(-8)}{4} = 4$; $3c^2 = 4$, $c^2 = \frac{4}{3}$, $c = \pm\frac{2}{3}\sqrt{3}$

Both values of c are suitable.

19. $f(x) = 4\cos x$, $x \in [\frac{1}{3}\pi, \frac{2}{3}\pi]$; $f'(x) = -4\sin x$.

f is continuous and differentiable for all x. Thus, f is continuous on $[\frac{1}{3}\pi, \frac{2}{3}\pi]$ and differentiable on $(\frac{1}{3}\pi, \frac{2}{3}\pi)$. Thus, the hypothesis of the mean-value theorem is satisfied. Hence there exists a c in $(\frac{1}{3}\pi, \frac{2}{3}\pi)$ for which

$f'(c) = \dfrac{f(\frac{2}{3}\pi) - f(\frac{1}{3}\pi)}{\frac{2}{3}\pi - \frac{1}{3}\pi} = \dfrac{-2-2}{\frac{1}{3}\pi} = -\dfrac{12}{\pi}$; $-4\sin c = -\dfrac{12}{\pi}$; $\sin c = \dfrac{3}{\pi} \approx .955$, $c \approx 1.269$, $\pi - 1.269 \approx 1.872$

Both values of c are suitable.

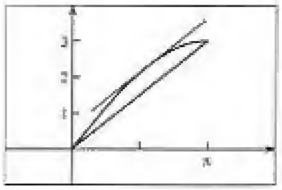
20. $f(x) = 3\sin\frac{1}{2}x$, $x \in [0,\pi]$

▸ f is continuous and differentiable for all x. Thus, f is continuous on $[0,\pi]$ and differentiable on $(0,\pi)$. Thus, the hypothesis of the mean-value theorem is satisfied. Hence there exists a c in $(0,\pi)$ for which

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{3}{2}\cos\tfrac{1}{2}c = \frac{3\sin\frac{1}{2}\pi - 3\sin 0}{\pi - 0} = \frac{3}{\pi}$$

$$\cos\tfrac{1}{2}c = \frac{2}{\pi}; \qquad \tfrac{1}{2}c \approx 0.881; \qquad c = 1.761$$



Because 1.761 is in $(0,\pi)$, 1.761 qualifies as c.
The plot at the right shows that the tangent line to the curve at this value of c is parallel to the secant line that contains the endpoints.

21. (a) If f is a polynomial function, it is continuous and differentiable everywhere, in particular on $[a,b]$ and (a,b). Further, f' is a polynomial function for which the above is also true. Because $f(a) = f(b) = 0$, Rolle's theorem holds, and so there is a number c in (a,b) such that $f'(c) = 0$. Then $f'(a) = f'(c) = 0$ and $f'(c) = f'(b) = 0$. Hence, by Rolle's theorem, there is a number d in (a,c) and a number e in (c,b) such that $f''(d) = 0$ and $f''(e) = 0$.

(b) If $f(x) = (x^2-4)^2$, $f'(x) = x(x^2-4)$, and $f''(x) = 12x^2 - 16$. Hence $f(-2) = f(2) = f'(-2) = f'(2) = 0$. Also, $f'(0) = 0$ and $f''(-\frac{2}{3}\sqrt{3}) = f''(\frac{2}{3}\sqrt{3}) = 0$.
So f satisfies part (a) where $a = -2$, $b = 2$, $c = 0$, $d = -\frac{2}{3}\sqrt{3}$, and $e = \frac{2}{3}\sqrt{3}$.

22. Rolle's theorem does not apply to $f(x) = |2x-4| - 6$ on $[-1,5]$ because $f'(2)$ does not exist and $2 \in (-1,5)$.

In Exercises 23 and 24, find why the mean-value theorem fails. Sketch the graph and the secant line.

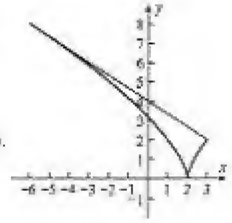
23. $f(x) = \begin{cases} 4-x^2 & \text{if } x < 1 \\ 6-3x & \text{if } x \geq 1 \end{cases}$; $a = 0$, $b = 3$

▸ $f'_-(1) = \lim\limits_{x\to1^-}(-2x) = -2$ and $f'_+(1) = \lim\limits_{x\to1^+}(-3) = -3$ and so $f'(1)$ does not exist. Thus f is not differentiable on $(0,3)$.

24. $f(x) = 2(x-2)^{2/3}$; $a = -6$, $b = 3$

▸ $f'(x) = \frac{4}{3}(x-2)^{-1/3}$. $f'(2)$ does not exist and so f is not differentiable on $(-6,3)$.

A graph of f and the line through $(-6,8)$ and $(3, 2\cdot5^{2/3})$ appears at the right.



In Exercises 25–32, (a) plot the graph; determine (b) the relative extrema of f, (c) the values of x at which they occur, the intervals on which f is (d) increasing and (e) decreasing. In Exercises 37–44, locate the points of inflection and where the graph is concave upward and downward. Confirm analytically.

25 and 37. $f(x) = x^3 + 3x^2 - 4$

▹ $f'(x) = 3x^2 + 6x = 3x(x+2)$; $f''(x) = 6x + 6 = 6(x+1)$

Set $f'(x) = 0$: $x = 0$, $x = -2$. Set $f''(x) = 0$: $x = -1$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$		$+$	$-$	increasing	concave downward
$x = -2$	0	0	$-$	relative maximum	concave downward
$-2 < x < -1$		$-$	$-$	decreasing	concave downward
$x = -1$	-2	-3	0	decreasing	point of inflection
$-1 < x < 0$		$-$	$+$	decreasing	concave upward
$x = 0$	-4	0	$+$	relative minimum	concave upward
$0 < x$		$+$	$+$	increasing	concave upward

26 and 38. $f(x) = x^3 + 2x^2 + x - 5$

▹ $f'(x) = 3x^2 + 4x + 1 = (3x+1)(x+1)$; $f''(x) = 6x + 4$

Set $f'(x) = 0$: $x = -1$, $x = -\frac{1}{3}$. Set $f''(x) = 0$: $x = -\frac{2}{3}$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$		$+$	$-$	increasing	concave downward
$x = -1$	-5	0	$-$	relative maximum	concave downward
$-1 < x < -\frac{2}{3}$		$-$	$-$	decreasing	concave downward
$x = -\frac{2}{3}$	$-5\frac{2}{27}$	$-$	0	decreasing	point of inflection
$-\frac{2}{3} < x < -\frac{1}{3}$		$-$	$+$	decreasing	concave upward
$x = -\frac{1}{3}$	$-5\frac{4}{27}$	0	$+$	relative minimum	concave upward
$-\frac{1}{3} < x$		$+$	$+$	increasing	concave upward

27 and 39. $f(x) = (x-3)^{5/3} + 1$; $f'(x) = \frac{5}{3}(x-3)^{2/3}$; $f''(x) = \frac{10}{9}(x-3)^{-1/3}$

Set $f'(x) = 0$: $x = 3$. $f''(x)$ is never 0. $f''(3)$ does not exist.

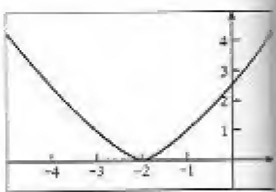
	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 3$		$+$	$-$	increasing	concave downward
$x = 3$	1	0	d.n.e.	stationary	point of inflection
$3 < x$		$+$	$+$	increasing	concave upward

28 and 40. $f(x) = (x+2)^{4/3}$

▹ There are no asymptotes. The graph is symmetric about the line $x = -2$.

$$f'(x) = \tfrac{4}{3}(x+2)^{1/3}; \quad f''(x) = \tfrac{4}{9}(x+2)^{-2/3}$$

Because $f'(x) < 0$ if $x < -2$ and $f'(x) > 0$ if $x > -2$ then f is decreasing on $(-\infty, -2]$, increasing on $[-2, +\infty)$, and has a relative minimum of $f(-2) = 0$ at $x = 0$. Because $f''(x) > 0$ if $x \neq -2$, the graph is concave upward for all x and there are no points of inflection. The plot appears at the right.



29 and 41. $f(x) = x - \tan x$; $x \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$

▹ Because $\lim\limits_{x \to -\pi/2^+} f(x) = +\infty$ and $\lim\limits_{x \to \pi/2^-} f(x) = -\infty$, the line $x = -\frac{1}{2}\pi$ and $x = \frac{1}{2}\pi$ are vertical asymptotes. Because f is an odd function, the graph is symmetric with respect to the origin.

$$f'(x) = 1 - \sec^2 x = -\tan^2 x; \quad f''(x) = -2\sec^2 x \tan x$$

Because $f'(x) < 0$ if $x \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and $x \neq 0$, then f is decreasing on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and there are no relative extrema. Because $f''(x) > 0$ if $-\frac{1}{2}\pi < x < 0$ and $f''(x) < 0$ if $0 < x < \frac{1}{2}\pi$, then the graph is concave upward if $-\frac{1}{2}\pi < x < 0$, concave downward if $0 < x < \frac{1}{2}\pi$, and $(0, 0)$ is a point of inflection.

30 and 42. $f(x) = \sin 2x - \cos 2x,\ x \in [-\frac{3}{8}\pi, \frac{5}{8}\pi],\ 2x \in [-\frac{3}{4}\pi, \frac{5}{4}\pi]$

$f'(x) = 2 \cos 2x + 2 \sin 2x;\ f''(x) = -4 \sin 2x + 4 \cos 2x$

Set $f'(x) = 0$: $\sin 2x = -\cos 2x$; $\tan 2x = -1$; $2x = -\frac{1}{4}\pi, \frac{3}{4}\pi$; $x = -\frac{1}{8}\pi, \frac{3}{8}\pi$

Set $f''(x) = 0$: $\sin 2x = \cos 2x$; $\tan 2x = 1$; $2x = -\frac{3}{4}\pi, \frac{1}{4}\pi$; $x = -\frac{3}{8}\pi, \frac{1}{8}\pi$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x = -\frac{3}{8}\pi$	0				
$-\frac{3}{8}\pi < x < -\frac{1}{8}\pi$		−	+	decreasing	concave upward
$x = -\frac{1}{8}\pi$	$-\sqrt{2}$	0	+	relative minimum	concave upward
$-\frac{1}{8}\pi < x < \frac{1}{8}\pi$		+	+	increasing	concave upward
$x = \frac{1}{8}\pi$	0	$\sqrt{2}$	0	increasing	point of inflection
$\frac{1}{8}\pi < x < \frac{3}{8}\pi$		+	−	increasing	concave downward
$x = \frac{3}{8}\pi$	$\sqrt{2}$	0	−	relative maximum	concave downward
$\frac{3}{8}\pi < x < \frac{5}{8}\pi$		−	−	decreasing	concave downward
$x = \frac{5}{8}\pi$	0				

31 and 43. $f(x) = (x+1)^{2/3}(x-3)^2$

$f'(x) = \frac{2}{3}(x+1)^{-1/3}(x-3)^2 + (x+1)^{2/3}(x-3)$

$= \frac{2}{3}(x+1)^{-1/3}(x-3)[x-3+3(x+1)] = \frac{8}{3}(x+1)^{-1/3}(x-3)x$

$f''(x) = -\frac{8}{9}(x+1)^{-4/3}(x^2-3x) + \frac{8}{3}(x+1)^{-1/3}(2x-3)$

$= \frac{8}{9}(x+1)^{-4/3}[-x^2+3x+3(x+1)(2x-3)] = \frac{8}{9}(x+1)^{-4/3}(5x^2-9)$

Set $f'(x) = 0$: $x = 0$, $x = 3$. Set $f''(x) = 0$: $x = x_1 = -\frac{3}{5}\sqrt{5} \approx -1.346$, $x = x_2 = \frac{3}{5}\sqrt{5} \approx 1.346$.

$f'(-1)$ and $f''(-1)$ do not exist and -1 is in the domain of f.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < x_1$		−	+	decreasing	graph is concave upward
$x = x_1$	9.21	−22.2	0	decreasing	point of inflection
$x_1 < x < -1$		−	−	decreasing	concave downward
$x = -1$	0	d.n.e.	d.n.e.	relative minimum	vertical tangent
$-1 < x < 0$		+	−	increasing	concave downward
$x = 0$	9	0	−	relative minimum	concave downward
$0 < x < x_2$		−	−	decreasing	concave downward
$x = x_2$	4.85	−4.4	0	decreasing	point of inflection
$x_2 < x < 3$		−	+	decreasing	graph is concave upward
$x = 3$	0	0	+	relative maximum	graph is concave upward
$3 < x$		+	+	increasing	graph is concave upward

32 and 44. $f(x) = x\sqrt{25 - x^2}$

Because $25 - x^2 \geq 0$ if $|x| \leq 5$, the domain of f is $[-5, 5]$. $f(x) = x(25 - x^2)^{1/2}$

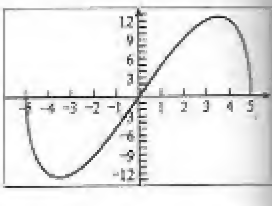
$f'(x) = (25-x^2)^{1/2}(1) + x(\frac{1}{2})(25-x^2)^{-1/2}(-2x) = (25-x^2)^{-1/2}[(25-x^2)-x^2] = (25-x^2)^{-1/2}(25-2x^2)$

$f''(x) = -\frac{1}{2}(25-x^2)^{-3/2}(-2x)(25-2x^2) + (25-x^2)^{-1/2}(-4x) = x(25-x^2)^{-3/2}[(25-2x^2)-4(25-x^2)]$

$= x(25-x^2)^{-3/2}(2x^2-75)$

Set $f'(x) = 0$: $x = \pm\frac{5}{2}\sqrt{2}$. Set $f''(x) = 0$: $x = 0$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x=-5$	0	d.n.e.	d.n.e.	end maximum	vertical tangent
$-5x<-\frac{5}{2}\sqrt{2}$		$-$	$+$	decreasing	concave upward
$x=-\frac{5}{2}\sqrt{2}$	$-\frac{25}{2}$	0	$+$	relative minimum	concave upward
$-\frac{5}{2}\sqrt{2}<x<0$		$+$	$+$	increasing	concave upward
$x=0$	0	$+$	0	increasing	point of inflection
$0<x<\frac{5}{2}\sqrt{2}$		$+$	$-$	increasing	concave downward
$x=\frac{5}{2}\sqrt{2}$	$\frac{25}{2}$	0	$-$	relative maximum	concave downward
$\frac{5}{2}\sqrt{2}<x<5$		$-$	$-$	decreasing	concave downward
$x=5$	0	d.n.e.	d.n.e.	end minimum	vertical tangent



In Exercises 33–36, find (a) the relative extrema; (b) the values of x at which they occur; the intervals on which f is (c) increasing and (d) decreasing. (e) Sketch.

33. $f(x)=(x-4)^2(x+2)^3$

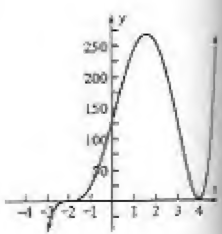
$f'(x)=2(x-4)(x+2)^3+3(x+2)^2(x-4)^2=(x-4)(x+2)^2[2(x+2)+3(x-4)]=(x-4)(x+2)^2(5x-8)$

$f''(x)=(x+2)^2(5x-8)+2(x-4)(x+2)(5x-8)+5(x-4)(x+2)^2$
$=(x+2)(5x^2+2x-16+10x^2-56+64+5x^2-10x-40)=(x+2)(20x^2-64x+8)$

Set $f'(x)=0$: $x=4$, $x=-2$, $x=\frac{8}{5}=1.6$

Set $f''(x)=0$: $x=-2$, $x=\frac{1}{5}(8-3\sqrt{6})=x_1\approx 0.130$, $x=\frac{1}{5}(8+3\sqrt{6})=x_2\approx 3.070$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x<-2$		$+$	$-$	increasing	concave downward
$x=-2$	0	0	0	stationary	point of inflection
$-2<x<x_1$		$+$	$+$	increasing	concave upward
$x=x_1$	144.7	129.1	0	increasing	point of inflection
$x_1<x<1.6$		$+$	$-$	increasing	concave downward
$x=1.6$	268.7	0	$-$	relative minimum	concave downward
$1.6<x<x_2$		$-$	$-$	decreasing	concave downward
$x=x_2$	112.8	-175.7	0	decreasing	point of inflection
$x_2<x<4$		$-$	$+$	decreasing	concave upward
$x=4$	0	0	$+$	relative maximum	concave upward
$4<x$		$+$	$+$	increasing	concave upward



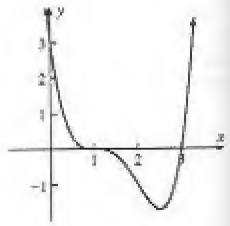
34. $f(x)=(x-1)^3(x-3)$

$f'(x)=(x-1)^3(1)+(x-3)(3)(x-1)^2=2(x-1)^2(2x-5)$

$f''(x)=4(x-1)(2x-5)+2(x-1)^2(2)=12(x-1)(x-2)$

The critical numbers of f are 1 and $\frac{5}{2}$. The critical numbers of f' are 1 and 2.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x<1$		$-$	$+$	decreasing	concave upward
$x=1$	0	0	0	stationary	point of inflection
$1<x<2$		$-$	$-$	decreasing	concave downward
$x=2$	-1	$-$	0	decreasing	point of inflection
$2<x<\frac{5}{2}$		$-$	$+$	decreasing	concave upward
$x=\frac{5}{2}$	$-\frac{27}{16}$	0	$+$	relative minimum	concave upward
$\frac{5}{2}<x$		$+$	$+$	increasing	concave upward

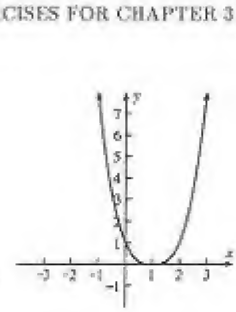


35. $f(x) = \begin{cases} (1-x)^3 & \text{if } x \le 1 \\ (x-1)^3 & \text{if } 1 < x \end{cases}$

▸ $f'(x) = \begin{cases} -3(1-x)^2 & \text{if } x \le 1 \\ 3(x-1)^2 & \text{if } 1 < x \end{cases}$; $f''(x) = \begin{cases} 6(1-x) & \text{if } x \le 1 \\ 6(x-1) & \text{if } 1 < x \end{cases}$

Set $f'(x) = 0$: $x = 1$. Set $f''(x) = 0$: $x = 1$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 1$		$-$	$+$	decreasing	concave upward
$x = 1$	0	0	0	relative minimum	no point of inflection
$1 < x$		$+$	$+$	increasing	concave upward



36. $f(x) = \begin{cases} x^3 - 3x & \text{if } x < 2 \\ 6 - x^2 & \text{if } x \ge 2 \end{cases}$

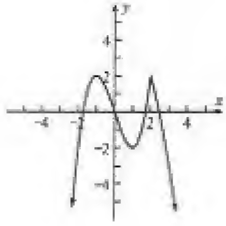
▸ $f'(x) = \begin{cases} 3x^2 - 3 & \text{if } x < 2 \\ -2x & \text{if } x > 2 \end{cases}$; $f''(x) = \begin{cases} 6x & \text{if } x < 2 \\ -2 & \text{if } x > 2 \end{cases}$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 3x) = 2$; $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (6 - x^2) = 2$.

f is continuous at 2. $f'(2)$ and $f''(2)$ do not exist.

Set $f'(x) = 0$: $x = \pm 1$. Set $f''(x) = 0$: $x = 0$.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$		$+$	$-$	increasing	concave downward
$x = -1$	2	0	$-$	relative maximum	concave downward
$-1 < x < 0$		$-$	$-$	decreasing	concave downward
$x = 0$	0	$-$	0	decreasing	point of inflection
$0 < x < 1$		$-$	$+$	decreasing	concave upward
$x = 1$	-2	0	$-$	relative minimum	concave upward
$1 < x < 2$		$+$	$+$	increasing	concave upward
$x = 2$	2	d.n.e.	d.n.e.	relative maximum	no inflection point
$2 < x$		$-$	$-$	decreasing	concave downward



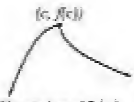
Exercises 37–44 are found with Exercises 25–32.

In Exercises 45 and 46, sketch part of the graph of the continuous function f near $x = c$ satisfying the conditions.

45. (a) $f'(x) > 0$ if $x < c$; $f'(x) < 0$ if $x > c$; $f''(x) < 0$ if $x < c$; $f''(x) < 0$ if $x > c$.
(b) $f'(x) < 0$ if $x < c$; $f'(x) > 0$ if $x > c$; $f''(x) < 0$ if $x < c$; $f''(x) < 0$ if $x > c$.
(c) $f'(x) > 0$ if $x < c$; $f'(x) < 0$ if $x > c$; $f''(x) < 0$ if $x < c$; $f''(x) > 0$ if $x > c$.
(d) $f'(c) = 0$; $f''(c) = 0$; $f'(x) < 0$ if $x < c$; $f'(x) < 0$ if $x > c$; $f''(x) > 0$ if $x < c$; $f''(x) < 0$ if $x > c$

Exercise 45(a)

Exercise 45(b)



Exercise 45(c)

Exercise 45(d)

46. (a) $f'(c) = -2$; $f''(c) = 0$; $f''(x) < 0$ if $x < c$; $f''(x) > 0$ if $x > c$.
(b) $f'(c)$ does not exist; $f''(x) > 0$ if $x < 0$; $f''(x) > 0$ if $x > c$.
(c) $f'(x) < 0$ if $x < c$; $f'(x) > 0$ if $x > c$; $f''(x) > 0$ if $x < c$; $f''(x) < 0$ if $x > c$.
(d) $\lim_{x \to c^-} f'(x) = 1$; $\lim_{x \to c^+} f'(x) = +\infty$; $f''(x) > 0$ if $x < c$; $f''(x) < 0$ if $x > c$.

Exercise 46(a)

Exercise 46(b)

Exercise 46(c)

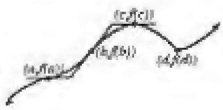
Exercise 46(d)

In Exercises 47 and 48, sketch part of the graph of the continuous function f on an interval containing $a < b < c < d$ satisfying the conditions. Also draw a segment of the tangent line at each point if there is one.

47. (a) $f'(a) = 0$; $f'(b) = -1$; $f'(c)$ does not exist; $f'(d) = 0$; $f''(x) < 0$ if $x < b$; $f''(x) > 0$ if $b < x < c$; $f''(x) < 0$ if $x > c$.

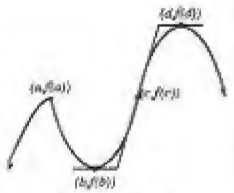
(b) $f'(a) = 0$; $f''(a) = 0$; $f'(b) = 1$; $f''(b) = 0$; $f'(c) = 0$; $f'(d)$ does not exist; $f''(x) < 0$ if $x < a$; $f''(x) > 0$ if $a < x < b$; $f''(x) < 0$ if $b < x < d$; $f''(x) > 0$ if $x > d$.

Exercise 47(a)



Exercise 47(b)

Exercise 48(a)

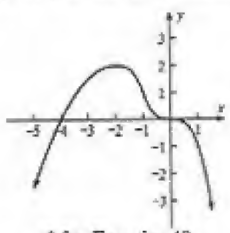


Exercise 48(b)

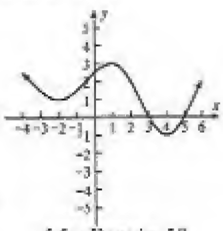
48. (a) $f'(a) = 0$; $f'(b) = -1$; $f''(b) = 0$; $f'(c) = 0$; $f''(c) = 0$; $f'(d) = -1$; $f''(d) = 0$; $f''(x) < 0$ if $x < b$; $f''(x) > 0$ if $b < x < c$; $f''(x) < 0$ if $c < x < d$; $f''(x) > 0$ if $x > d$.

(b) $f'(a)$ does not exist; $f'(b) = 0$; $f'(c) = 2$; $f''(c) = 0$; $f'(d) = 0$; $f''(x) < 0$ if $x < a$; $f''(x) > 0$ if $a < x < c$; $f''(x) < 0$ if $x > c$.

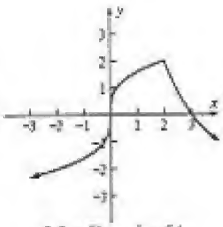
In Exercises 49–52, determine from the figure, the graph of the derivative of a function f continuous on R, the following information and incorporate it into a table as in §3.6: the intervals on which f is increasing, decreasing, its relative extrema; intervals of concave upward and downward and abscissas of points of inflection. Sketch a graph of f if the only zeros are those stated.



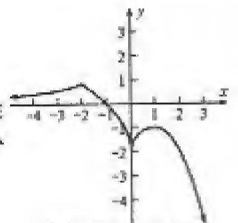
f for Exercise 49



f for Exercise 50



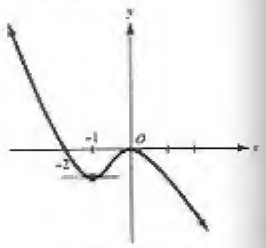
f for Exercise 51



f for Exercise 52

49. Zeros of f are -4 and 0.

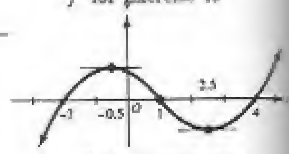
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	+	−	increasing	concave downward
$x = -2$	0	−	relative maximum	concave downward
$-2 < x < -1$	−	−	decreasing	concave downward
$x = -1$	−	0	decreasing	point of inflection
$-1 < x < 0$	−	+	decreasing	concave upward
$x = 0$	0	0	stationary	point of inflection
$x > 0$	−	−	decreasing	concave downward



f' for Exercise 49

50. Zeros of f are 3 and 5.

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	−	+	decreasing	concave upward
$x = -2$	0	+	relative minimum	concave upward
$-2 < x < -.5$	+	+	increasing	concave upward
$x = -.5$	+	0	increasing	point of inflection
$-.5 < x < 1$	+	−	increasing	concave downward
$x = 1$	0	−	relative maximum	concave downward
$1 < x < 2.5$	−	−	decreasing	concave downward
$x = 2.5$	−	0	decreasing	point of inflection

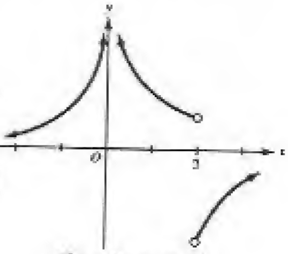


f' for Exercise 50

$2.5 < x < 4$	$-$	$+$	decreasing	concave upward
$x = 4$	0	$+$	relative minimum	concave upward
$x > 4$	$+$	$+$	increasing	concave upward

51. Zeros of f are 0 and 3.

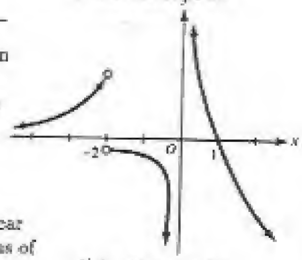
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 0$	$+$	$+$	increasing	concave upward
$x = 0$	d.n.e.	d.n.e	vertical tangent line	point of inflection
$0 < x < 2$	$+$	$-$	increasing	concave downward
$x = 2$	d.n.e.	d.n.e.	relative maximum	no point of inflection
$x > 2$	$-$	$+$	decreasing	concave upward



f' for Exercise 51

52. Zeros of f is -1.

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2$	$+$	$+$	increasing	concave upward
$x = -2$	d.n.e.	d.n.e.	relative maximum	no point of inflection
$-2 < x < 0$	$-$	$-$	decreasing	concave downward
$x = 0$	d.n.e.	d.n.e.	relative minimum	vertical tangent line
$0 < x < 1$	$+$	$-$	increasing	concave downward
$x = 1$	0	$-$	relative maximum	concave downward
$x > 1$	$-$	$-$	decreasing	concave downward

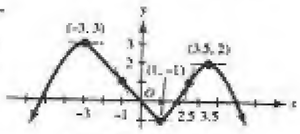


f' for Exercise 52

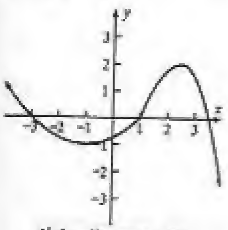
In Exercises 53–56, the graph of f and segments of the inflectional tangents appear in the figure. Make a table as in the previous Exercises and sketch possible graphs of f' and f''.

53.

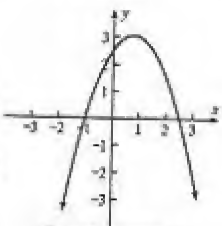
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -3$	$+$	$-$	increasing	concave downward
$x = -3$	0	$-$	relative maximum	concave downward
$-3 < x < -1$	$-$	$-$	decreasing	concave downward
$x = -1$	-1	0	decreasing	point of inflection
$-1 < x < 1$	$-$	$+$	decreasing	concave upward
$x = 1$	0	$+$	relative minimum	concave upward
$1 < x < 2.5$	$+$	$+$	increasing	concave upward
$x = 2.5$	$+2$	0	increasing	point of inflection
$2.5 < x < 3.5$	$+$	$-$	increasing	concave downward
$x = 3.5$	0	$-$	relative maximum	concave downward
$x > 3.5$	$-$	$-$	decreasing	concave downward



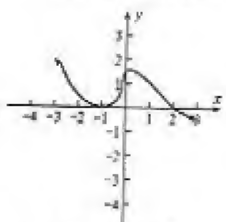
f for Exercise 53



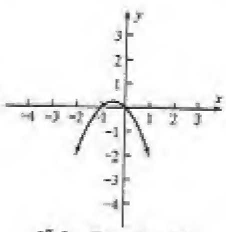
f' for Exercise 53



f'' for Exercise 53



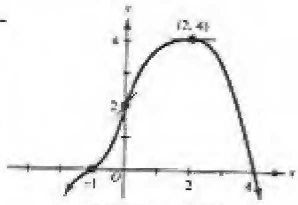
f' for Exercise 54



f'' for Exercise 54

54.

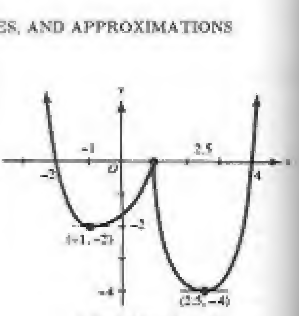
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$	$+$	$-$	increasing	concave downward
$x = -1$	0	0	stationary	point of inflection
$-1 < x < 0$	$+$	$+$	increasing	concave upward
$x = 0$	2	0	increasing	point of inflection
$0 < x < 2$	$+$	$-$	increasing	concave downward
$x = 2$	0	$-$	relative maximum	concave downward
$x > 2$	$-$	$-$	decreasing	concave downward



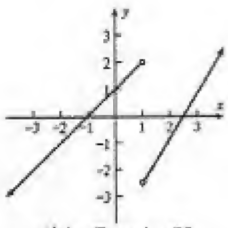
f for Exercise 54

55.

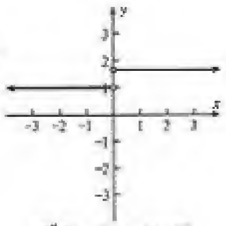
	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$	$-$	$+$	decreasing	concave upward
$x = -1$	0	$+$	relative minimum	concave upward
$-1 < x < 1$	$+$	$+$	increasing	concave upward
$x = 1$	d.n.e.	d.n.e.	relative maximum	not an inflection point
$1 < x < 2.5$	$-$	$+$	decreasing	concave upward
$x = 2.5$	0	$+$	relative minimum	concave upward
$x > 2.5$	$+$	$+$	increasing	concave upward



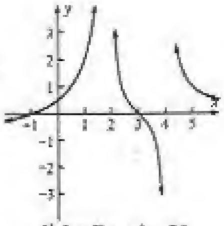
f for Exercise 55



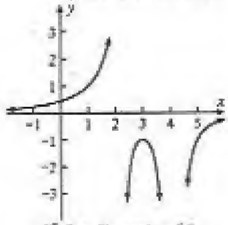
f' for Exercise 55



f'' for Exercise 55



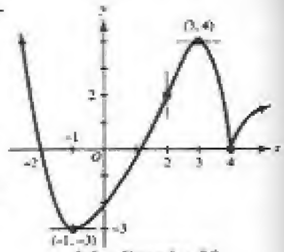
f' for Exercise 56



f'' for Exercise 56

56.

	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -1$	$-$	$+$	decreasing	concave upward
$x = -1$	0	$+$	relative minimum	concave upward
$-1 < x < 2$	$+$	$+$	increasing	concave upward
$x = 2$	d.n.e.	d.n.e.	vertical tangent line	point of inflection
$2 < x < 3$	$+$	$-$	increasing	concave downward
$x = 3$	0	$-$	relative maximum	concave downward
$3 < x < 4$	$-$	$-$	decreasing	concave downward
$x = 4$	d.n.e.	d.n.e	relative minimum	not an inflection point
$x > 4$	$+$	$-$	increasing	concave downward



f for Exercise 56

In Exercises 57–60, find the limit and support your answer graphically.

57. $\lim\limits_{x\to\infty} \dfrac{3x^2+2x-5}{x^2+4} = \lim\limits_{x\to\infty} \dfrac{3+2/x-5/x^2}{1+4/x^2} = \dfrac{3}{1} = 3$

58. $\lim\limits_{x\to-\infty} \dfrac{4x-3}{5x^2-x+1} = \lim\limits_{x\to-\infty} \dfrac{4/x-3/x^2}{5-1/x+1/x^2} = \dfrac{0}{5} = 0$

59. $\lim\limits_{x\to-\infty} \dfrac{x^2+5}{2x-4} = \lim\limits_{x\to-\infty} \dfrac{x+5/x}{2-4/x} = -\infty$

60. $\lim\limits_{x\to+\infty} \left(\dfrac{8x^3+7x-2}{7x^3+3x^2+5x}\right)^2$

$= \left(\lim\limits_{x\to+\infty} \dfrac{8x^3+7x-2}{7x^3+3x^2+5x}\right)^2$ limit of a power

$= \left(\lim\limits_{x\to+\infty} \dfrac{8+7/x^2-2/x^3}{7+3/x+5/x^2}\right)^2$ dividing by the highest power in the numerator

$= \left(\dfrac{8}{7}\right)^2 = \dfrac{64}{49}$ $\lim\limits_{x\to+\infty} \dfrac{a}{x^n} = 0$

In Exercises 61 and 62, (a) plot and conjecture the behavior as x increases without bound; (b) compute $\lim\limits_{x\to+\infty} f(x)$.

61. $\lim\limits_{x\to+\infty} (\sqrt{x+1}-\sqrt{x}) = \lim\limits_{x\to+\infty} \dfrac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}} = \lim\limits_{x\to+\infty} \dfrac{1}{\sqrt{x+1}+\sqrt{x}} = 0$

62. $\lim_{x\to+\infty}(\sqrt{x^2+x}-\sqrt{x^2+4}) = \lim_{x\to+\infty}\dfrac{(\sqrt{x^2+x}-\sqrt{x^2+4})(\sqrt{x^2+x}+\sqrt{x^2+4})}{\sqrt{x^2+x}+\sqrt{x^2+4}} = \lim_{x\to+\infty}\dfrac{(x^2+x)-(x^2+4)}{\sqrt{x^2+x}+\sqrt{x^2+4}}$

$= \lim_{x\to+\infty}\dfrac{x-4}{\sqrt{x^2+x}+\sqrt{x^2+4}} = \lim_{x\to+\infty}\dfrac{1-4/x}{\sqrt{1+1/x}+\sqrt{1+4/x^2}} = \frac{1}{2}$

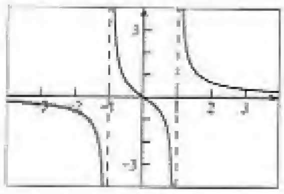
In Exercises 63–66, find the asymptotes of the graph of the function. Check by plotting.

63. $f(x) = \dfrac{5x^2}{x^2-4}$

▸ Because $\lim_{x\to 2^-} f(x) = -\infty$ or $\lim_{x\to 2^+} f(x) = +\infty, x = 2$ is a vertical asymptote.
Because $\lim_{x\to -2^-} f(x) = +\infty$ or $\lim_{x\to -2^+} f(x) = -\infty, x = -2$ is a vertical asymptote.
Because $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\dfrac{5}{1-4/x^2} = 5$ or $\lim_{x\to+\infty} f(x) = 5, y = 5$ is a horizontal asymptote.

64. $f(x) = \dfrac{x}{x^2-1}$

▸ Because $\lim_{x\to 1^-} f(x) = -\infty$ or $\lim_{x\to 1^+} f(x) = +\infty$, $x = 1$ is a vertical asymptote. Because $\lim_{x\to -1^-} f(x) = +\infty$ or $\lim_{x\to -1^+} f(x) = -\infty$, $x = -1$ is a vertical asymptote. Because $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\dfrac{1/x}{1-1/x^2} = \frac{0}{1} = 0$ or because $\lim_{x\to-\infty} f(x) = 0$, $y = 0$ is a horizontal asymptote.



65. $f(x) = \dfrac{x^2}{x-3}$

▸ Because $\lim_{x\to 3^-} f(x) = -\infty$ or because $\lim_{x\to 3^+} f(x) = +\infty$, $x = 3$ is a vertical asymptote.
Because $f(x) = x + 3 + \dfrac{9}{x-3} = x + 3 + g(x)$ and $\lim_{x\to+\infty} g(x) = 0$, $y = x + 3$ is an oblique asymptote.

66. $f(x) = \dfrac{x^2+9}{x}$

▸ Because $\lim_{x\to 0^+} f(x) = +\infty$ or because $\lim_{x\to 0^-} f(x) = -\infty$, $x = 0$ is a vertical asymptote.
Because $f(x) = x + 9/x = x + g(x)$ and $\lim_{x\to+\infty} g(x) = 0$, $y = x$ is an oblique asymptote.

In Exercises 67–70, (a) plot the graphs of f, f', and f'' in separate windows and estimate the relative extrema, the intervals on which f is increasing and those on which f is decreasing, where the graph is concave upward and where it is concave downward, and any points of inflection. (b) confirm analytically, make a table, and sketch.

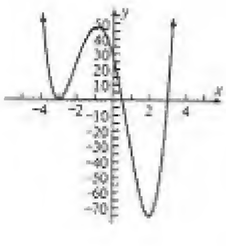
67. $f(x) = 2x^4 + 5x^3 - 21x^2 - 45x + 27$

▸ $f'(x) = 8x^3 + 15x^2 - 42x - 45 = (x+3)(8x^2 - 9x - 15)$; $f''(x) = 24x^2 + 30x - 42$.

Set $f'(x) = 0$: $x = -3$, $x = x_1 = \frac{1}{16}(9 - \sqrt{561}) \approx -0.918$, $x = x_2 = \frac{1}{16}(9 + \sqrt{561}) \approx 2.043$

Set $f''(x) = 0$: $x = x_3 = -\frac{1}{8}(\sqrt{137} + 5) \approx -2.088$; $x = x_4 = \frac{1}{8}(\sqrt{137} - 5) \approx 0.838$

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -3$		$-$	$+$	decreasing	concave upward
$x = -3$	0	0	$+$	relative minimum	concave upward
$-3 < x < x_3$		$+$	$+$	increasing	concave upward
$x = x_3$	21.9	$+$	0	increasing	point of inflection
$x_3 < x < x_1$		$+$	$-$	increasing	concave downward
$x = x_1$	48.1	0	$-$	relative maximum	concave downward
$x_1 < x < x_4$		$-$	$-$	decreasing	concave downward
$x = x_4$	-21.5	$-$	0	decreasing	point of inflection
$x_4 < x < x_2$		$-$	$+$	decreasing	concave upward
$x = x_2$	-75.1	0	$+$	relative minimum	concave upward
$x > x_2$		$+$	$+$	increasing	concave upward



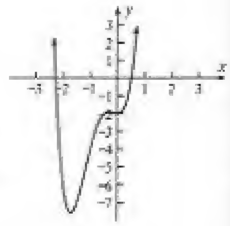
68. $f(x) = 3x^4 + 8x^3 + 3x^2 - 2$

▷ $f'(x) = 12x^3 + 24x^2 + 6x = 6x(2x^2 + 4x + 1)$; $f''(x) = 36x^2 + 48x + 6$

Set $f'(x) = 0$: $x = 0$, $x = x_1 = -\frac{1}{2}\sqrt{2} - 1 \approx -1.71$, $x = x_2 = \frac{1}{2}\sqrt{2} - 1 \approx -0.29$

Set $f''(x) = 0$: $x = x_3 = -\frac{1}{6}(\sqrt{10} + 4) \approx -1.19$, $x = x_4 = \frac{1}{6}(\sqrt{10} - 4) \approx -0.14$

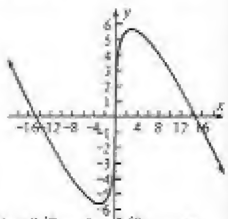
	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < x_1$		$-$	$+$	decreasing	concave upward
$x = x_1$	-7.58	0	$+$	relative minimum	concave upward
$x_1 < x < x_3$		$+$	$+$	increasing	concave upward
$x = x_3$	-5.24	$+$	0	increasing	point of inflection
$x_3 < x < x_2$		$+$	$-$	increasing	concave downward
$x = x_2$	-1.92	0	$-$	relative maximum	concave downward
$x_1 < x < x_4$		$-$	$-$	decreasing	concave downward
$x = x_4$	-1.96	$-$	0	decreasing	point of inflection
$x_4 < x < 0$		$-$	$+$	decreasing	concave upward
$x = 0$	-2	0	$+$	relative minimum	concave upward
$x > 0$		$+$	$+$	increasing	concave upward



69. $f(x) = 6\sqrt[3]{x} - x$; $f'(x) = 2x^{-2/3} - 1$; $f''(x) = -\frac{4}{3}x^{-5/3}$. $f(x)$ is odd; graph is symmetrical about the origin.

Set $f'(x) = 0$: $x^{2/3} = 2$; $x = \pm 2^{3/2} = \pm 2\sqrt{2} \approx \pm 2.83$. $f''(x)$ is never zero. $f'(0)$ and $f''(0)$ do not exist.

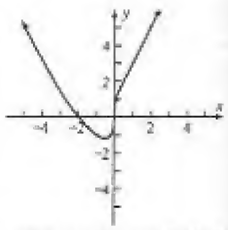
	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -2\sqrt{2}$		$-$	$+$	decreasing	concave upward
$x = -2\sqrt{2}$	$-4\sqrt{2}$	0	$+$	relative minimum	concave upward
$-2\sqrt{2}<x<0$		$+$	$+$	increasing	concave upward
$x = 0$	0	d.n.e.	d.n.e.	vertical tangent	point of inflection
$0 < x < 2\sqrt{2}$		$+$	$-$	increasing	concave downward
$x = 2\sqrt{2}$	$4\sqrt{2}$	0	$-$	relative maximum	concave downward
$x > 2\sqrt{2}$		$-$	$-$	decreasing	concave downward



70. $f(x) = 2x^{1/3} + x^{4/3}$; $f'(x) = \frac{2}{3}x^{-2/3} + \frac{4}{3}x^{1/3} = \frac{4}{3}x^{-2/3}(\frac{1}{2} + x)$; $f''(x) = -\frac{4}{9}x^{-5/3} + \frac{4}{9}x^{-2/3} = \frac{4}{9}x^{-5/3}(x - 1)$

Set $f'(x) = 0$: $x = -\frac{1}{2}$. Set $f''(x) = 0$: $x = 1$. $f'(0)$ and $f''(0)$ do not exist.

	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -\frac{1}{2}$		$-$	$+$	decreasing	concave upward
$x = -\frac{1}{2}$	$-\frac{3}{4}\sqrt[3]{4}$	0	$+$	relative minimum	concave upward
$-\frac{1}{2} < x < 0$		$+$	$+$	increasing	concave upward
$x = 0$	0	d.n.e.	d.n.e.	vertical tangent	point of inflection
$0 < x < 1$		$+$	$-$	increasing	concave downward
$x = 1$	3	$+$	0	increasing	point of inflection
$x > 1$		$+$	$+$	increasing	concave upward



71. $f(x) = \mathrm{A} \sin kx + \mathrm{B} \cos kx$. Let θ be the angle of inclination of the line joining the origin to the point (A, B).

Then $f(x) = \sqrt{\mathrm{A}^2 + \mathrm{B}^2}\left(\sin kx \cdot \frac{\mathrm{A}}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2}} + \cos kx \cdot \frac{\mathrm{B}}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2}}\right) = \sqrt{\mathrm{A}^2 + \mathrm{B}^2}(\sin kx \cos\theta + \cos kx \sin\theta)$
$= \sqrt{\mathrm{A}^2 + \mathrm{B}^2}\sin(kx + \theta)$. Because the absolute maximum value of the sine function is 1, the absolute maximum value of $f(x) = \sqrt{\mathrm{A}^2 + \mathrm{B}^2}$.

72. If $f(x) = ax^3 + bx^2$, determine a and b so that the graph of f will have a point of inflection at (2, 16). Plot.

▷ $f'(x) = 3ax^2 + 2bx$; $f''(x) = 6ax + 2b$

Because the graph of f has a point of inflection at the point where $x = 2$, then

$$f''(2) = 0; \qquad 12a + 2b = 0 \qquad (1)$$

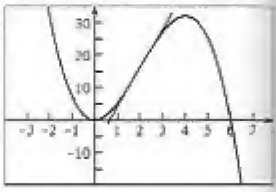
Because the graph contains the point (2, 16), then

$$f(2) = 16; \qquad 8a + 4b = 16 \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously, we obtain $a = -1$ and $b = 6$. Thus

$$f''(x) = -6x + 12 = 6(2 - x)$$

Because $f''(x)$ changes sign at $x = 2$, then (2, 16) is a point of inflection.



73. $f(x) = ax^3 + bx^2 + cx$; $f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$
The graph of f is to have a point of inflection at $(1, -1)$. Thus $f(1) = -1$ and $f''(1) = 0$;
$a + b + c = -1$ (1)
$6a + 2b = 0$ (2)
Because the slope of the inflectional tangent at $(1, -1)$ is -3, then $f'(1) = -3$;
$3a + 2b + c = -3$ (3)
Solving equations (1), (2), and (3) simultaneously, we obtain $a = 2$, $b = -6$, and $c = 3$.

74. If $f(x) = \dfrac{x+1}{x^2+1}$, prove that the graph of f has 3 points of inflection that are collinear. Check by plotting.

▸ $f'(x) = \dfrac{(x^2+1) - (x+1)(2x)}{(x^2+1)^2} = \dfrac{-x^2 - 2x + 1}{(x^2+1)^2}$. $f''(x) = \dfrac{(-2x-2)(x^2+1)^2 - (-x^2-2x+1)\cdot 2(x^2+1)(2x)}{(x^2+1)^4}$

$= \dfrac{2[(-x-1)(x^2+1) - (-x^2-2x+1)2x]}{(x^2+1)^3} = \dfrac{2(x^3+3x^2-3x-1)}{(x^2+1)^3} = \dfrac{2(x^2+4x+1)(x-1)}{(x^2+1)^3}$

Set $f''(x) = 0$: $x^2 + 4x + 1 = 0$, $x = x_1 = -\sqrt{3} - 2$, $x = x_2 = \sqrt{3} - 2$; $x = x_3 = 1$

$f(x_1) = \dfrac{x_1+1}{x_1{}^2+1} = \dfrac{x_1+1}{-4x_1} = \dfrac{-\sqrt{3}-1}{4(2+\sqrt{3})} = \dfrac{1-\sqrt{3}}{4}$; $f(x_2) = \dfrac{x_2+1}{x_2{}^2+1} = \dfrac{x_2+1}{-4x_2} = \dfrac{\sqrt{3}-1}{4(2-\sqrt{3})} = \dfrac{1+\sqrt{3}}{4}$. $f(1) = 1$

$m_{12} = \dfrac{\frac{1}{4}(1+\sqrt{3}) - \frac{1}{4}(1-\sqrt{3})}{(\sqrt{3}-2) - (-\sqrt{3}-2)} = \dfrac{\frac{1}{2}\sqrt{3}}{2\sqrt{3}} = \dfrac{1}{4}$. $m_{23} = \dfrac{1 - \frac{1}{4}(1+\sqrt{3})}{1 - (\sqrt{3}-2)} = \dfrac{\frac{1}{4}(3-\sqrt{3})}{3-\sqrt{3}} = \dfrac{1}{4}$

Because the slopes of the segment are equal, the three points of inflection are collinear.

75. $f(x) = x|x| = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$; $f'(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x & \text{if } x > 0 \end{cases}$; $f''(x) = \begin{cases} -2 & \text{if } x < 0 \\ 2 & \text{if } x > 0 \end{cases}$
$f'(0) = 0$ because $f'_-(0) = 0$ and $f'_+(0) = 0$. If $x < 0$, $f''(x) < 0$; and if $x > 0$, $f''(x) > 0$.
Hence, by Definition 3.5.4, the graph of f has a point of inflection at the origin.

76. Let $f(x) = x^n$, where n is a positive integer. (a) Prove that the graph of f has a point of inflection at the origin if and only if n is odd and $n > 1$. (b) Show that if n is even, f has a relative minimum value at 0.

▸ $f'(x) = nx^{n-1}$; $f''(x) = n(n-1)x^{n-2}$

Suppose that n is an odd integer greater than 1. Then $n - 1 > 0$, and thus $f'(0)$ exists. Hence the graph of f has a tangent line at $x = 0$. Furthermore, because $n \geq 3$, then $n(n-1) > 0$. Because $n - 2$ is also an odd integer, then $x^{n-2} < 0$ if $x < 0$. Therefore $f''(x) < 0$ if $x < 0$ and $f''(x) > 0$ if $x > 0$ and so the graph of f has a point of inflection at the origin.

Now, suppose that n is an even integer. Because n is positive, then f is continuous at $x = 0$. Furthermore, $n \geq 2$ and so $n(n-1) > 0$. Because $n - 1$ is an odd integer, then $x^{n-1} < 0$ if $x < 0$. Therefore $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$ and so f has a relative minimum value at $x = 0$.

If $n = 1$, then $f(x) = x$ and the graph of f is a straight line. Therefore, the graph of f has a point of inflection at the origin if and only if n is odd and $n > 1$, and f has a relative minimum value at 0 if n is even.

In Exercises 77 and 78 confirm by calculus the value obtained in part (d) of Miscellaneous Exercises for Chapter 1.

77. (a) Exercise 103. If x in. squares are cut from the corners of a 14 in × 18 in sheet and the sides turned up the volume is $V(x)$ in^3 where $V = (14 - 2x)(18 - 2x)x = 4x^3 - 64x^2 + 252x$, $0 \leq x \leq 7$. $V'(x) = 12x^2 - 128x + 252$ $= 4(3x^2 - 32x + 63)$. Set $V'(x) = 0$: $x = \frac{1}{3}(16 \pm \sqrt{67})$. Only $\frac{1}{3}(16 - \sqrt{67}) \approx 2.6049 \in [0, 7]$
$V(0) = 0$, $V(2.6049) \approx 292.9$, $V(7) = 0$. The volume has an absolute maximum value when $x \approx 2.6$ in.
(b) Exercise 104. An open box having a square base x in. is to have a volume of 4000 in^3. The total surface area of the box is S in^2, where $S(x) = x^2 + \dfrac{16{,}000}{x}$, $x > 0$. $S'(x) = 2x - \dfrac{16{,}000}{x^2} = \dfrac{2(x^3 - 8000)}{x^2}$.
Because $S'(x) < 0$ if $0 < x < 20$ and $S'(x) > 0$ if $x > 20$, then S has an absolute minimum value when $x = 20$.

78. (a) Exercise 105. A sign with margins of 4 m at the top and bottom and 2 m at the sides is to contain 50 m^2 of print. The total area of the sign is $A(x)$ m^2 when the width of the printed region is x m, where $A(x) = 92 + 8x + \dfrac{200}{x}$, $x > 0$. $A'(x) = 8 - \dfrac{200}{x^2} = \dfrac{8(x^2-25)}{x^2}$. Because $A'(x) < 0$ if $0 < x < 5$ and $A'(x) > 0$ if $x > 5$, then A has an absolute minimum value when $x = 5$. The smallest sign is 9 m wide and 18 m long.
(b) Exercise 106. The growth rate f fish/week is jointly proportional to the number x of fish and the number $10{,}000 - x$ of capacity. $f(x) = kx(10{,}000 - x) = k(10{,}000x - x^2)$, $0 \le x \le 10{,}000$. $f'(x) = 10{,}000 - 2x$. Because $f'(x) > 0$ if $x < 5000$ and $f'(x) < 0$ if $x > 5000$ then f has an absolute maximum value when $x = 5000$.

79. Profit on an item is \$200 if not more than 800 are produced each week and decreases \$0.20 per item for each item over 800. When x items are sold the profit is $f(x)$ dollars, where $f(x) = \begin{cases} 200x & \text{if } 0 \le x \le 800 \\ 360x - .2x^2 & \text{if } 800 < x \le 1800 \end{cases}$
$f'(x) = \begin{cases} 200 & \text{if } 0 < x < 800 \\ 360 - .4x & \text{if } 800 < x < 1800 \end{cases}$. Because $f'(x) > 0$ if $0 \le x < 900$ and $f'(x) < 0$ if $x > 900$ then f has an absolute maximum value when $x = 900$.

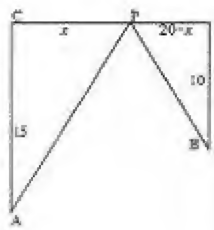
80. Find the dimensions of an open box, having a square base and a volume of k in^3 that can be constructed with the least amount of material.

▹ Let x in. be the length of a side of the base so its area is x^2 in^2. Then the height of the box is $\dfrac{k}{x^2}$ in. If $A(x)$ in^2 is the amount of material needed to construct the box,

$A(x) = x^2 + 4x \cdot \dfrac{k}{x^2} = x^2 + \dfrac{4k}{x}$, $0 < x$; $A'(x) = 2x - \dfrac{4k}{x^2} = \dfrac{2(x^3 - 2k)}{x^2}$

Because $A'(x) < 0$ if $0 < x < \sqrt[3]{2k}$ and $A'(x) > 0$ if $x > \sqrt[3]{2k}$ then A has an absolute minimum value when $x = \sqrt[3]{2k}$. Therefore, the dimensions of the required box are $\sqrt[3]{2k}$ in. by $\sqrt[3]{2k}$ in. by $\frac{1}{2}\sqrt[3]{2k}$ in.

81. Let C be the point on the bank nearest A. See the figure. If $f(x)$ km is the amount of piping when the pumping station is x km from C, then



$f(x) = |\overline{AP}| + |\overline{PB}| = \sqrt{x^2 + 15^2} + \sqrt{(20-x)^2 + 10^2}$, $x \in [0, 20] = I$

$f'(x) = \dfrac{x}{\sqrt{x^2 + 15^2}} - \dfrac{20 - x}{\sqrt{(20-x)^2 + 10^2}}$

Set $f'(x) = 0$: $\dfrac{\sqrt{x^2 + 15^2}}{x} = \dfrac{\sqrt{(20-x)^2}}{20 - x}$; $\dfrac{x^2 + 15^2}{x^2} = \dfrac{(20-x)^2 + 10^2}{(20-x)^2}$;

$1 + \dfrac{15^2}{x^2} = 1 + \dfrac{10^2}{(20-x)^2}$; $\dfrac{15}{x} = \dfrac{10}{20 - x}$

(x and $20 - x$ are positive); $300 - 15x = 10x$; $300 = 25x$; $x = 12$ is the critical number.
f is continuous on I so absolute extrema occur at an endpoint or a critical number.
$f(0) = 15 + 10\sqrt{5} \approx 37.36$, $f(12) = \sqrt{369} + \sqrt{164} = 3\sqrt{41} + 2\sqrt{41} = 5\sqrt{41} \approx 32.02$, $f(20) = 35$
The least amount of piping is required when the pumping station is 12 km from C.

82. If $C(x)$ is the total cost of x chairs, then $C(x) = [360 - (x - 300)]x = (660 - x)x = 660x - x^2$, $x \ge 300$.
$C'(x) = 660 - 2x$. Because $C'(x) > 0$ if $x < 330$ and $C'(x) > 0$ if $x > 330$ then $C(330) = 108{,}900$ is the absolute maximum value of T. The largest possible transaction is \$108,900.

83. x units are demanded when p dollars is the price per unit and $x^2 + p = 320$; $20x$ dollars is the cost of producing x units. The profit is $P(x)$ dollars, where $P(x) = xp - 20x = x(320 - x^2) - 20x = 300x - x^3$, $x \ge 0$.
$P'(x) = 300 - 3x^2 = 3(100 - x^2)$. Because $P'(x) > 0$ if $x < 10$ and $P'(x) > 0$ if $x > 10$ then $P(10) = 2000$ is the absolute maximum value of P.

84. A closed tin can having a volume of 27 in.^{3} is to be in the form of a right-circular cylinder. If the circular top and bottom are cut from square pieces of tin, find the radius and height of the can if the least amount of tin is to be used in its manufacture. Include the tin that is wasted when obtaining the top and bottom.

▹ Let r in. and h in. be the radius and height of the can. Because the volume of the can is 27 in.^{3} then

$$\pi r^2 h = 27; \qquad h = \frac{27}{\pi r^2} \tag{1}$$

The squares needed to make the top and bottom of the can are $2r$ in. long on each side. Therefore, the total area of both squares is $8r^2$ in^2. The lateral area of the can is $2\pi r h$ in^2. If S in^2 is the total area, then

$$S = 8r^2 + 2\pi rh \tag{2}$$

Substituting from Eq. (1) into Eq. (2), we obtain

$$S(r) = 8r^2 + 2\pi r\left(\frac{27}{\pi r^2}\right) = 8r^2 + 54r^{-1}$$

$$S'(r) = 16r - 54r^{-2} = \frac{16(r^3 - \frac{27}{8})}{r^2}$$

Because $S'(r) < 0$ if $0 < r < \sqrt[3]{\frac{27}{8}} = \frac{3}{2}$ and $S'(r) > 0$ if $r > \frac{3}{2}$, then S has an absolute minimum value when $r = \frac{3}{2}$. By substituting $r = \frac{3}{2}$ into Eq. (1) we obtain $h = \dfrac{27}{\pi(3/2)^2} = \dfrac{12}{\pi}$. Therefore, the radius is $\frac{3}{2}$ in. and the altitude is $\frac{12}{\pi}$ in. if the least amount of material is used to construct the can.

85. When p dollars is the price per unit, $100x$ units are demanded and $x^2 + p^2 = 36$; $p = \sqrt{36 - x^2}$.
$R(x) = (100x)p = 100x\sqrt{36 - x^2}$, $x \in [0, 6] = I$. $R'(x) = 100\sqrt{36 - x^2} + 100x \cdot \dfrac{-x}{\sqrt{36 - x^2}} = \dfrac{3600 - 200x^2}{\sqrt{36 - x^2}}$
Set $R'(x) = 0$: $3600 - 200x^2 = 0$; $x^2 = 18$; $x = 3\sqrt{2}$
R is continuous on I so absolute extrema occur at an endpoint or a critical number.
$$R(0) = 0,\ R(3\sqrt{2}) = 1800,\ R(6) = 0$$
Thus R has an absolute maximum value when $x = 3\sqrt{2}$. The maximum total revenue is \$1800.

86. The growth rate f is jointly proportional to the number x of infectives and the number $11{,}000 - x$ of susceptibles; $f(x) = kx(11{,}000 - x) = k(11{,}000x - x^2)$, $0 \le x \le 11{,}000$. $f'(x) = 11{,}000 - 2x$. Because $f'(x) > 0$ if $x < 5500$ and $f'(x) < 0$ if $x > 5500$ then f has an absolute maximum value when $x = 5500$.

87. The growth rate f is jointly proportional to the number x of inhabitants and the amount $3{,}000 - x$ of room; $f(x) = kx(3{,}000 - x) = k(3{,}000x - x^2)$, $0 \le x \le 3{,}000$. $f'(x) = 3{,}000 - 2x$. Because $f'(x) > 0$ if $x < 1500$ and $f'(x) < 0$ if $x > 1500$ then f has an absolute maximum value when $x = 1500$.

88. Find the shortest distance from the point $P(0,4)$ to a point on the curve $x^2 - y^2 = 16$, and find the point on the curve that is closest to P.

▸ The distance is least when its square is least, and the square of the distance from the point $P(0,4)$ to the point $Q(x,y)$ on the curve $x^2 - y^2 = 16$ is given by
$$s = x^2 + (y-4)^2 \tag{1}$$
Because Q is on the curve, then $x^2 = 16 + y^2$. Substituting into Eq. (1), we get
$$s(y) = 16 + y^2 + (y-4)^2 = 2y^2 - 8y + 32$$
$$s'(y) = 4y - 8 = 4(y-2)$$
Because $s'(y) < 0$ if $y < 2$ and $s'(y) > 0$ if $y > 2$, then $s(2) = 24$ is an absolute minimum value. Also, when $y = 2$, $x^2 = 16 + 2^2 = 20$, $x = \pm 2\sqrt{5}$. Hence, the shortest distance is $\sqrt{24} = 2\sqrt{6}$ and the points on the curve that are closest to P are $(\pm 2\sqrt{5}, 2)$.

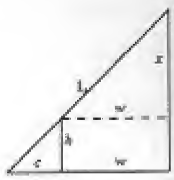
89. $P(x)$ dollars is the profit when x radios are sold where
$P(x) = 75x - (x^2 + 25x + 100) = -x^2 + 50x - 100$, $x \ge 0$; $P'(x) = -2x + 50 = 2(25 - x)$
$P'(x) > 0$ if $x < 25$; $P'(x) < 0$ if $x > 25$. Hence $P(x)$ has an absolute maximum value if $x = 25$.
25 radios should be produced daily. $P(25) = -(25)^2 + 50(25) - 100 = 525$. The profit is \$525.

90. After t sec the particles are at $(x,0)$ and $(0,y)$, where $x = t^2 - 2t$ and $y = t^2 - 2$, and let $F(t)$ cm be the distance between them.
$F(t) = \sqrt{(t^2 - 2t)^2 + (t^2 - 2)^2}$, $t \ge 0$. $F'(t) = \frac{1}{2}(2t^4 - 4t^3 + 4)^{-1/2}(8t^3 - 12t^2) = 4t^2(2t^4 - 4t^3 + 4)^{-1/2}(t - \frac{3}{2})$
Because $F'(t) \le 0$ if $0 < t < \frac{3}{2}$ and $F'(t) > 0$ if $t > \frac{3}{2}$, F has an absolute minimum value when $t = \frac{3}{2}$. Because $D_t x = 2t - 2$ and $D_t y = 2t$, when $t = \frac{3}{2}$ the velocity of the horizontal particle is $2(\frac{3}{2}) - 2 = 1$ and the velocity of the vertical particle is $2(\frac{3}{2}) = 3$. Hence the distance between the particles is least after $\frac{3}{2}$ sec, and the velocity of the horizontal particle is 1 cm/sec and the velocity of the vertical particle is 3 cm/sec.

91. See Exercise 92 with $h = \frac{27}{8}$ and $w = 8$: $L = ((\frac{27}{8})^{2/3} + 8^{2/3})^{3/2} = (\frac{9}{4} + 4)^{3/2} = (\frac{25}{4})^{3/2} = \frac{125}{8}$ m.

92. A ladder is to reach over a fence h meters high to a wall w meters behind the fence. Find the length of the shortest ladder that may be used.

▹ Let L meters be the length of the ladder. See the figure at the right.



From similar triangles we get $\frac{c}{h} = \frac{w}{x}$; $c = \frac{hw}{x}$. Then

$$L(x) = \sqrt{(c+w)^2 + (x+h)^2} = \sqrt{\left(\frac{hw}{x} + w\right)^2 + (x+h)^2},\ x > 0$$

$$L'(x) = \frac{\left(\frac{hw}{x} + w\right)\left(-\frac{hw}{x^2}\right) + (x+h)}{L(x)} = \frac{(x+h)\left(1 - \frac{hw^2}{x^3}\right)}{L(x)}$$

Set $L'(x) = 0$: $(x+h)\left(1 - \frac{hw^2}{x^3}\right) = 0$; $x = -h$, $x = h^{1/3}w^{2/3}$. The only critical number is $h^{1/3}w^{2/3}$.

Because $L'(x) < 0$ if $0 < x < h^{1/3}w^{2/3}$ and $L'(x) > 0$ if $x > h^{1/3}w^{2/3}$, L has an absolute minimum value when $x = h^{1/3}w^{2/3}$. When $x = h^{1/3}w^{2/3}$, the minimum length is

$$L(h^{1/3}w^{2/3}) = \sqrt{\left(\frac{hw}{h^{1/3}w^{2/3}} + w\right)^2 + (h^{1/3}w^{2/3} + h)^2} = \sqrt{w^{2/3}(h^{2/3} + w^{2/3})^2 + h^{2/3}(w^{2/3} + h^{2/3})^2}$$
$$= (h^{2/3} + w^{2/3})^{3/2}$$

93. Let the radius of the cylinder be r in. and the height h in; V in^3 is the volume. From similar right triangles, $\frac{8-h}{r} = \frac{8}{4} = 2$; $h = 8 - 2r$, $V = \pi r^2 h = \pi r^2(8 - 2r) = \pi(8r^2 - 2r^3)$, $0 \le r \le 4$. $V'(r) = \pi(16r - 6r^2) = 6\pi r(\frac{8}{3} - r)$

Because $V'(r) > 0$ if $0 < r < \frac{8}{3}$ and $V'(r) > 0$ if $r > \frac{8}{3}$ then $V(\frac{8}{3}) = \pi \cdot \frac{64}{9}(8 - 2 \cdot \frac{8}{3}) = \frac{512}{27}\pi$ is the absolute maximum value of V.

94. A tent is to be in the shape of a cone. Find the ratio of the measure of the radius to the measure of the altitude for a tent of given volume to require the least material.

▹ We want to minimize the lateral surface area of the cone. Let r units be the radius of the cone, h units be the altitude of the cone, A square units be the lateral surface area of the cone. We have the formula

$$A = \pi r\sqrt{r^2 + h^2}$$

Let $z = A^2$. Then

$$z = \pi^2 r^2(r^2 + h^2) \tag{1}$$

Because A is a minimum when z is a minimum, we want to find r/h when z has a minimum. If V cubic units is the volume of the cone, then

$$V = \tfrac{1}{3}\pi r^2 h; \quad r^2 = \frac{3V}{\pi h} \tag{2}$$

Substituting from Eq. (2) into Eq. (1), we express z as a function of h.

$$z(h) = \pi^2\left(\frac{3V}{\pi h}\right)\left(\frac{3V}{\pi h} + h^2\right) = 3V\pi\left(\frac{3V}{\pi}h^{-2} + h\right) \qquad z'(h) = 3V\pi\left(\frac{-6V}{\pi}h^{-3} + 1\right) = \frac{3V\pi\left(h^3 - \frac{6V}{\pi}\right)}{h^3}$$

Because $z'(h) < 0$ if $0 < h < \sqrt[3]{6V/\pi}$ and $z'(h) > 0$ if $h > \sqrt[3]{6V/\pi}$, then z and A have an absolute minimum value when $h = \sqrt[3]{6V/\pi}$. From Eq. (2) we have

$$r = \sqrt{\frac{3V}{\pi h}}; \quad \frac{r}{h} = \sqrt{\frac{3V}{\pi h^3}}$$

At the minimum, $h^3 = 6V/\pi$ and so

$$\frac{r}{h} = \sqrt{\frac{3V}{\pi(6V/\pi)}} = \tfrac{1}{2}\sqrt{2}$$

▪ The tent requires the least material if the ratio of the radius to the altitude is $\frac{1}{2}\sqrt{2}$.

95. Let R cm be the radius and H cm be the altitude of the right-circular cone. From similar triangles $\frac{R}{H} = \frac{r}{H-h}$ so that $R = \frac{Hr}{H-h}$. If V(H) cm^3 is the volume of the cone, then

$$V(H) = \tfrac{1}{3}\pi R^2 H = \tfrac{1}{3}\pi r^2 \frac{H^3}{(H-h)^2},\ H > h;\ V'(H) = \tfrac{1}{3}\pi r^2 \frac{3H^2(H-h)^2 - 2H^3(H-h)}{(H-h)^4} = \tfrac{1}{3}\pi r^2 \frac{H^2(H-3h)}{(H-h)^3}$$

Because $V'(H) < 0$ if $h < H < 3h$ and $V'(h) > 0$ if $H > 3h$, V has an absolute minimum value when $H = 3h$. Then $R = \frac{Hr}{H-h} = \frac{3hr}{2h} = \frac{3}{2}r$.

96. One of the acute angles of a triangle has measure $\frac{1}{6}\pi$ and the opposite side has length 10 in. Prove that the area is greatest when the triangle is isosceles.

▸ Let a and b be the measures of the other two sides. By the law of cosines,

$$a^2 + b^2 = 10^2 - 2ab \cos \tfrac{1}{6}\pi = 100 - \sqrt{3}ab \tag{1}$$

Differentiating implicitly with respect to a on both sides of (1), we get

$$2a + 2bb' = -\sqrt{3}(b + ab'); \qquad (2b + \sqrt{3}a)b' = -(\sqrt{3}b + 2a); \qquad b' = \frac{-\sqrt{3}b - 2a}{\sqrt{3}a + 2b} \tag{2}$$

We wish to maximize the measure of area where

$$A(a) = \tfrac{1}{2}ab \sin \tfrac{1}{6}\pi = \tfrac{1}{4}ab \tag{3}$$

Differentiating (3) with respect to a and substituting from (2), we get

$$A'(a) = \tfrac{1}{4}(b + ab') = \tfrac{1}{4}\left(b - a\frac{\sqrt{3}b + 2a}{\sqrt{3}a + 2b}\right) = \frac{2(b^2 - a^2)}{4(\sqrt{3}a + 2b)}$$

Because $A'(a) > 0$ if $a < b$ and $A'(a) > 0$ if $a > b$ then A has an absolute maximum value when $a = b$; that is, the area is greatest when the triangle is isosceles.

97. $F(\theta) = \dfrac{1000k}{k \sin \theta + \cos \theta}$, $\theta \in [0, \frac{1}{2}\pi] = I$; $F'(\theta) = \dfrac{-1000k(k \cos \theta - \sin \theta)}{(k \sin \theta + \cos \theta)^2}$

Set $F'(\theta) = 0$: $k \cos \theta = -\sin \theta$; $\tan \theta = k$ so that $\sin \theta = \dfrac{k}{\sqrt{k^2 + 1}}$, $\cos \theta = \dfrac{1}{\sqrt{k^2 + 1}}$

F is continuous on I so absolute extrema occur at an endpoint or a critical number.

$F(0) = 1000k$, $F(\theta)\Big|_{\tan \theta = k} = \dfrac{1000k}{\sqrt{k^2 + 1}}$, $F(\frac{1}{2}\pi) = 1000$

Because $0 < k < 1$, F is least when $\tan \theta = k$.

98. Let x in. be the length of two opposite sides of the rectangle.
(a) The area is 81 in^2. The the length of the other two sides is $81/x$. If P in. is the perimeter of the rectangle, then $P(x) = x + 81/x$, $x > 0$. $P'(x) = 1 - 81x^{-2} = x^{-2}(x^2 - 81)$
Because $P'(x) < 0$ if $0 < x < 9$ and $P'(a) > 0$ if $x < 9$ then A has an absolute minimum value when $x = 9$.
(b) The perimeter is 36. Then the length of the other two sides is $(18 - x)$in. If $A(x)$ in^2 is the area of the rectangle, then $A(x) = (18 - x)x = 18x - x^2$, $0 \le x \le 18$; $A'(x) = 18 - 2x$
Because $A'(x) > 0$ if $x < 9$ and $A'(x) > 0$ if $x > 9$ then A has an absolute maximum value when $x = 9$.

99. Let x cm be the length of one piece of wire. Then $(20 - x)$ cm is the length of the other piece of wire. If $A(x)$ cm^2 is the combined area of the two squares, then
$A(x) = (\frac{1}{4}x)^2 + [\frac{1}{4}(20 - x)]^2 = \frac{1}{8}x^2 - \frac{5}{2}x + 25$, $0 \le x \le 20$; $A'(x) = \frac{1}{4}x - \frac{5}{2} = \frac{1}{4}(x - 10)$
Because $A'(x) < 0$ if $x < 10$ and $A'(x) > 0$ if $x > 10$, A has an absolute minimum value when $x = 10$. Therefore, the total area of the two squares will be least if each piece of wire has length 10 cm; that is, the wire should be cut in half.

100. A piece of wire 80 cm long is bent to form a rectangle. Find the dimensions of the rectangle so that its area is as large as possible.

▸ Let x cm be the length of the rectangle and y cm be the width of the rectangle. If A cm^2 is the area of the rectangle, then $A = xy$, and we want to find x and y so that the value of A is an absolute maximum. Because there is 80 cm of wire, then

$$2x + 2y = 80; \qquad y = 40 - x \tag{1}$$

Substituting from Eq. (1) into $A = xy$, we get A as a function of x. Thus

$$A(x) = x(4 - x) = 40x - x^2; \qquad A'(x) = 40 - 2x = 2(20 - x)$$

Because $A'(x) > 0$ if $x < 20$ and $A'(x) < 0$ if $x > 0$, then $A(20) = 400$ is an absolute maximum value. If $x = 20$, then $y = 20$. Hence, if the area of the rectangle is as large as possible, it is a square 20 cm by 20 cm.

101. Solving the demand equation for p, and replacing p by P(x), we obtain
$P(x) = 10^3x^{-1} - 2 + 18 \cdot 10^{-3}x - 6 \cdot 10^{-6}x$, $x \geq 100$
$R(x) = xP(x) = 10^3 - 2x + 18 \cdot 10^{-3}x^2 - 6 \cdot 10^{-6}x^2$ and $C(x) = xQ(x) = 20 \cdot 10^{-3}x^2 - 24x + 11 \cdot 10^3$
If S(x) dollars is the weekly profit, then
$S(x) = R(x) - C(x) = -10^4 + 22x - 2 \cdot 10^{-3}x^2 - 6 \cdot 10^{-6}x^3$
$S'(x) = 22 - 4 \cdot 10^{-3}x - 18 \cdot 10^{-6}x^2 = (1 - 10^{-3}x)(22 + 18 \cdot 10^{-3}x) = 10^{-3}(1000 - x)(22 + 18 \cdot 10^{-3}x)$
Because $S'(x) > 0$ if $100 \leq x < 1000$ and $S'(x) < 0$ if $x > 1000$, S has an absolute maximum value when $x = 1000$. $P(1000) = 11$. 1000 units are produced at \$11 each.

102. $4x^4 - 3x^3 + 2x - 5 = 0$. Let $f(x) = 4x^4 - 3x^3 + 2x - 5$; $f'(x) = 16x^3 - 9x^2 + 2$.
Because $f(1) = -2$ and $f(2) = 39$, $f(x) = 0$ has a positive root between 1 and 2; take $x_1 = 1$.
$x_4 = x_5 = x = 1.168$ to three decimal places.

n	x_n	$-\ (4x_n^4 - 3x_n^3 + 2x_n - 5)$	$\div\ (16x_n^3 - 9x_n^2 + 2)$	= STO
1	1	−2	9	1.22222
2	1.22222	0.893154	17.768	1.17196
3	1.17196	0.060722	15.393	1.16801
4	1.16801	0.000346	15.217	1.16799

103. $3x^4 - 4x^3 + 36x^2 + 2x - 8 = 0$. $f(x) = 3x^4 - 4x^3 + 36x^2 + 2x - 8$; $f'(x) = 12x^3 - 12x^2 + 72x + 2$
Because $f(x)$ is continuous, $f(-1) = 33$ and $f(0) = -8$, then f has a zero between −1 and 0. We choose $x_1 = -0.5$. From the we conclude that the root is −0.482 to three decimal places.

n	$x_n - (3x_n^4 - 4x_n^3 + 36x_n^2 + 2x_n - 8) \div (12x_n^3 - 12x_n^2 + 72x_n + 2) = \text{STO}$			
1	−0.5	0.6875	−38.5	−0.48214
2	−0.48214	0.01477	−36.849	−0.48174
3	−0.48174	0.00000	−36.812	−0.48174

104. $y = \sin x$, $y = 2x - 3$. At the point of intersection $\sin x - 2x + 3 = 0$. Let $f(x) = \sin x - 2x + 3$;
$f'(x) = \cos x - 2$. Because $f(\frac{1}{2}\pi) = 0.8584$ and $f(\pi) = -3.2832$, $f(x) = 0$ has a root between $\frac{1}{2}\pi$ and π; take $x_1 = \frac{1}{2}\pi \approx 1.5708$. See Note to Solution of Exercise 4.10.15.
$x_4 = x_5 = x = 1.9622$ to four decimal places.

n	x_n	$-\ (\sin x_n - 2x_n + 3)$	$\div\ (\cos x_n - 2)$	= STO
1	1.5708	0.8584	−2.0000	2.0000
2	2.00000	−0.09070	−2.4161	1.96246
3	1.96246	−0.00064	−2.3817	1.96219
4	1.96219	0	−2.3814	1.96219

105. $\tan x = x$ in $(\frac{1}{2}\pi, \frac{3}{2}\pi)$. We avoid the discontinuity of $\tan x$ and solve the equivalent equation $\sin x = x \cos x$.
Let $f(x) = x \cos x - \sin x$; $f'(x) = \cos x - x \sin x - \cos x = -x \sin x$. We choose $x_1 = 4.5$. $x_2 = x_3 = 4.4934$ to four decimal places.

n	x_n	$-\ (x_n \sin x_n - \sin x_n)$	$\div\ (-\sin x_n)$	= STO
1	4.5	.0289	4.39889	4.49342
2	4.49342	4.19×10^{-5}	4.38612	4.49341

In Exercises 106 and 107, for the given f (a) find the linear approximation L at $x = 8$; (b) check by plotting; (c) compare f and L at 7.9, 7.99, 8, 8.01, and 8.1.

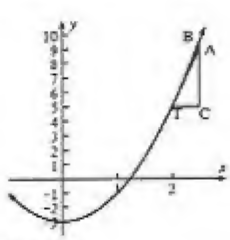
106. $f(x) = \sqrt[3]{x}$, $f(8) = 2$
$f'(x) = \frac{1}{3}x^{-2/3}$, $f'(8) = \frac{1}{12}$

x	7.9	7.99	8	8.01	8.1
$f(x) = \sqrt[3]{x}$	1.99163	1.9991663	2	2.0008330	2.00829
$L(x) = 2 + \frac{1}{12}(x - 2)$	$= 2 - \frac{1}{120}$	$2 - \frac{1}{1200}$	2	$2 + \frac{1}{1200}$	$2 + \frac{1}{120}$
	= 1.99167	= 1.9991667		= 2.0008333	= 2.00833

107. $f(x) = \sin \frac{1}{16}\pi x$, $f(8) = 1$
$f'(x) = \frac{1}{16}\pi \cos \frac{1}{16}\pi x$, $f'(8) = 0$

x	7.9	7.99	8	8.01	8.1
$f = \sin \frac{1}{16}\pi x$	0.9998	0.999998	1	0.999998	0.9998
L = 1	1	1	1	1	1

108. If $y = 2x^2 - 3$, (a) find dy and Δy for $x = 2$ and $\Delta x = 0.5$. (b) Sketch the graph, and indicate the line segments whose lengths are dy and Δy.



■ (a) Let $f(x) = 2x^2 - 3$. Because $x = 2$ and $\Delta x = 0.5$, we have

$dy = f'(x)\Delta x = 4x\ \Delta x = 4(2)(0.5) = 4$

$\Delta y = f(x + \Delta x) - f(x) = f(2.5) - f(2) = [2(2.5)^2 - 3] - [2(2)^2 - 3]$

$= 9.5 - 5 = 4.5$

(b) A sketch of the graph is shown at the right. The line segment TA is tangent to the graph at the point $(2,5)$ and the x coordinate of the point A is 2.5. Line segment TC is parallel to the x axis, and its length is $\Delta x = 0.5$. Line segment CA is parallel to the y axis and its length is $dy = 4$. Point B is the intersection of line CA and the graph of f, and the length of segment CB is $\Delta y = 4.5$.

109. $y = 80x - 16x^2$, $dy = (80 - 32x)\Delta x$.
$\Delta y = [80(x + \Delta x) - 16(x + \Delta x)^2] - (80x - 16x^2) = 80\Delta x - 32x\Delta x - 16(\Delta x^2)$. $\Delta y dy = -16(\Delta x)^2$
(a) If $\Delta x = 0.1$, $\Delta y - dy = -16(0.1)^2 = -0.16$ (b) If $\Delta x = -0.2$, $\Delta y - dy = -16(-0.2)^2 = -0.64$

110. $x^3 + y^3 - 3xy^2 + 1 = 0$. $3x^2dx + 3y^2dy - 3y^2dx - 6xydy = 0$, $(3y^2 - 6xy)dy = (3y^2 - 3x^2)dx$;
$dy = \dfrac{y^2 - x^2}{y^2 - 2xy}dx$. At $(1,1)$, $dy = 0$

111. The volume of material is the volume of a spherical shell which is an increment of the volume of a sphere. Let r in. be the radius of the sphere, let V in^3 be the volume of the sphere, and let ΔV in^3 be the volume of the spherical shell.
$V = \frac{4}{3}\pi r^3$; $dV = 4\pi r^2dr$. Let $r = 2$ and $dr = \frac{1}{8}$. Then $\Delta V \approx dV = 4\pi(2)^2 \cdot \frac{1}{8} = 2\pi$.
■ The volume of the material is approximately 2π in^3.

112. The time for one complete swing of a simple pendulum of length x ft is t sec. Then
$4\pi^2x = 32.2t^2$; $4\pi^2dx = 64.4t\ dt$; $dt = \dfrac{\pi^2}{16.1t}\,dx$
Let $dx = \Delta x = 0.01$. Then $\Delta t \approx \dfrac{\pi^2}{16.1t}(0.01) = \dfrac{\pi^2}{1610t}$.
■ The error in the time for one complete swing of the pendulum is approximately $\dfrac{\pi^2}{1610t}$ sec.

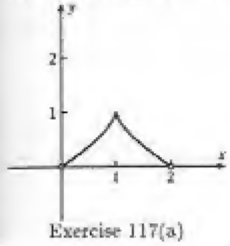
113. $V = \frac{1}{3}\pi r^2h = \frac{1}{3}\pi(\frac{4}{3}h)^2h = \frac{16}{27}\pi h^3$, $dV = \frac{16}{9}\pi h^2dh$. $\dfrac{dV}{V} = \dfrac{(16/9)\pi h^2dh}{(16/27)\pi h^3} = 3\dfrac{dh}{h} \le 3\%$ if $\dfrac{dh}{h} \le 1\%$.
■ The error in the altitude may not exceed 1%.

114. If $f'(x) = g'(x)$ in I then $(f - g)'(x) = 0$ in I, and so, by Theorem 3.3.3 $f - g$ is a constant in I. Hence $f(x) - g(x) = f(a) - g(a)$.

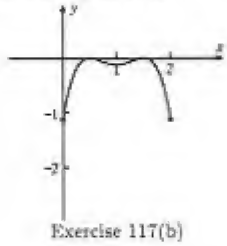
115. The functions f and g are differentiable at every number in the closed interval $[a,b]$.
Let $h(x) = f(x) - g(x)$. Then h is continuous on $[a,b]$ and differentiable on (a,b).
$h(a) = f(a) - g(a) = 0$ and $h(b) = f(b) - g(b) = 0$
Therefore, Rolle's theorem holds for the function h and the interval (a,b) so there is a number c in (a,b) such that $f'(c) - g'(c) = 0$, that is, $f'(c) = g'(c)$.

116. Let $a < b$ be consecutive zeros of $f'(x)$. Suppose c and d are zeros of $f(x)$ with $a < c < d < b$. By Rolle's theorem there is a root of $f'(x)$ between c and d, contradicting the hypothesis that a and b are consecutive. Hence there is at most one root of $f(x)$.

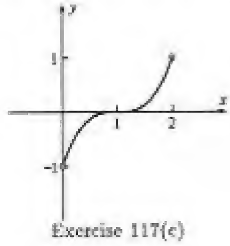
117. Sketch the graph of a function f on the interval I.
(a) I is $(0,2)$ and f is continuous on I. At 1, f has a relative maximum value but $f'(1)$ does not exist.
(b) I is $[0,2]$. f has a relative minimum value at 1, but the absolute minimum value of f is at 0.
(c) I is $(0,2)$, and f' has a relative minimum value at 1.



Exercise 117(a)



Exercise 117(b)



Exercise 117(c)

118. $f(x) = (x^2 + a^2)^p$; $f'(x) = 2px(x^2 + a^2)^{p-1}$

$$f''(x) = 2p(x^2 + a^2)^{p-1} + 4p(p-1)x^2(x^2 + a^2)^{p-2} = 2p(x^2 + a^2)^{p-2}[x^2 + a^2 + 2x^2(p-1)]$$
$$= 2p(x^2 + a^2)^{p-2}[a^2 - (1-2p)x^2]$$

If $p \geq \frac{1}{2}$, then $(1-2p) \leq 0$ and $f''(x) > 0$. Thus by Definition 4.5.4 the graph of f has no points of inflection. Suppose $p < \frac{1}{2}$. Set $f''(x) = 0$: $a^2 = (1-2p)x^2$; $x^2 = \dfrac{a^2}{1-2p}$; $x = x_1 = -\sqrt{\dfrac{a^2}{1-2p}}$ and $x = x_2 = \sqrt{\dfrac{a^2}{1-2p}}$

	$f''(x)$	Conclusion
$x < x_1$	$+$	the graph is concave up
$x = x_1$	0	the graph has a point of inflection
$x_1 < x < x_2$	$-$	the graph is concave down
$x = x_2$	0	the graph has a point of inflection
$x_2 < x$	$+$	the graph is concave up

Thus the graph of f has two points of inflection.

119. (a) $$f(x) = 3|x| + 4|x-1| = \begin{cases} -7x+4 & \text{if } x < 0 \\ 4 & \text{if } x = 0 \\ -x+4 & \text{if } 0 < x < 1 \\ 3 & \text{if } x = 1 \\ 7x-4 & \text{if } x > 1 \end{cases}; \quad f'(x) = \begin{cases} -7 & \text{if } x < 0 \\ -1 & \text{if } 0 < x < 1 \\ 7 & \text{if } x > 1 \end{cases}$$

Because f is continuous on $[0, 1]$ and there are no critical numbers in $[0, 1]$, the absolute minimum value of f on $[0, 1]$ is the smaller of $f(0) = 4$ and $f(1) = 3$. Hence $f(x) \geq 3$ for all x in $[0, 1]$. If $x < 0$, then $f'(x) < 0$ so f is decreasing on $(-\infty, 0]$. Hence $f(x) > f(0) = 4 > 3$ for all x in $(-\infty, 0)$. If $x > 1$, then $f'(x) > 0$ so f is increasing on $[1, +\infty)$. Hence $f(x) > f(1) = 3$ for all x in $(1, +\infty)$. Thus $f(x) \geq 3$ for all x, so 3 is the absolute minimum value.

(b) $$f(x) = 4|x| + 3|x-1| = \begin{cases} -7x+3 & \text{if } x < 0 \\ 3 & \text{if } x = 0 \\ x+3 & \text{if } 0 < x < 1 \\ 4 & \text{if } x = 1 \\ 7x-3 & \text{if } x > 1 \end{cases}; \quad f'(x) = \begin{cases} -7 & \text{if } x < 0 \\ -1 & \text{if } 0 < x < 1 \\ 7 & \text{if } x > 1 \end{cases}$$

Because g is continuous on $[0, 1]$ and there are no critical numbers in $[0, 1]$, the absolute minimum value of g on $[0, 1]$ is the smaller of $g(0) = 3$ and $f(1) = 4$. Hence $g(x) \geq 3$ for all x in $[0, 1]$. If $x < 0$, then $g'(x) < 0$ so g is decreasing on $(-\infty, 0]$. Hence $g(x) > g(0) = 3$ for all x in $(-\infty, 0)$. If $x > 1$, then $g'(x) > 0$ so g is increasing on $[1, +\infty)$. Hence $g(x) > g(1) = 4 > 3$ for all x in $(1, +\infty)$. Thus $g(x) \geq 3$ for all x, so 3 is the absolute minimum value.

(c) $$f(x) = a|x| + b|x-1| = \begin{cases} -ax-bx+b & \text{if } x < 0 \\ b & \text{if } x = 0 \\ x+b & \text{if } 0 < x < 1 \\ a & \text{if } x = 1 \\ ax+bx-b & \text{if } x > 1 \end{cases}; \quad f'(x) = \begin{cases} -a-b & \text{if } x < 0 \\ a-b & \text{if } 0 < x < 1 \\ a+b & \text{if } x > 1 \end{cases}$$

Because h is continuous on $[0, 1]$ and there are no critical numbers in $[0, 1]$, the absolute minimum value of h on $[0, 1]$ is the smaller of $h(0) = b$ and $h(1) = a$. Call this number m. Hence $h(x) \geq m$ for all x in $[0, 1]$. If $x < 0$, then $h'(x) < 0$ so h is decreasing on $(-\infty, 0]$. Hence $h(x) > h(0) = b \geq m$ for all x in $(-\infty, 0)$. If $x > 1$, then $h'(x) > 0$ so h is increasing on $[1, +\infty)$. Hence $h(x) > h(1) = a \geq m$ for all x in $(1, +\infty)$. Thus $h(x) \geq m$ for all x so m is the absolute minimum value.

120. If $f(x) = |x|^a \cdot |x-1|^b$, where a and b are positive rational numbers, prove that f has a relative maximum value of $a^a b^b/(a+b)^{a+b}$.

▶ $$f(x) = \begin{cases} (-x)^a(1-x)^b & \text{if } x < 0 \\ x^a(1-x)^b & \text{if } 0 \leq x \leq 1 \\ x^a(x-1)^b & \text{if } x > 1 \end{cases}$$

If $x < 0$ then $(-x)^a$ and $(1-x)^b$ are both positive and decreasing, and so $f(x)$ is decreasing and cannot have any extrema. Similarly, if $x > 0$ then f is increasing and cannot have any extrema. If $0 \leq x \leq 1$, then

$$f'(x) = ax^{a-1}(1-x)^b + x^a(b)(1-x)^{b-1}(-1) = x^{a-1}(1-x)^{b-1}[a(1-x) - bx]$$
$$= x^{a-1}(1-x)^{b-1}[a - (a+b)x] = (a+b)x^{a-1}(1-x)^{b-1}\left[\frac{a}{a+b} - x\right]$$

Now $a/(a+b) < 1$. Because $f'(x) > 0$ if $0 < x < a/(a+b)$ and $f'(x) < 0$ if $a/(a+b) < x < 1$, then f has a

relative maximum value at $x = a/(a+b)$ of

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(1 - \frac{a}{a+b}\right)^b = \left(\frac{a}{a+b}\right)^a \left(\frac{a}{a+b}\right)^b = \frac{a^a b^b}{(a+b)^{a+b}}$$

121. Show that if p and q are rational numbers whose sum is 1 then the graph of $f(x) = (x+a)^p (x+b)^q$ has the oblique asymptote $y = x + (pa + qb)$.

▶ We use the substitution $u = 1/x$ and the definition of derivative:

$$\lim_{x\to+\infty}[f(x) - x] = \lim_{x\to+\infty}[(x+a)^p (x+b)^q - x] = \lim_{x\to+\infty} x\left[\frac{(x+a)^p}{x^p} \cdot \frac{(x+b)^q}{x^p} - 1\right]$$

$$= \lim_{x\to+\infty} \frac{(1+a/x)^p (1+b/x)^q - 1}{1/x} = \lim_{u\to 0} \frac{(1+au)^p (1+bu)^q - 1}{u} = \frac{d}{du}\left[(1+au)^p (1+bu)^q\right]_{q=0}$$

$$= \left[pa(1+au)^{p-1}(1+bu)^q + qb(1+au)^p (1+bu)^{q-1}\right]_{q=0} = pa + qb$$

F O U R

THE DEFINITE INTEGRAL AND INTEGRATION

4.1 ANTIDIFFERENTIATION

4.1.1 Definition A function F is called an *antiderivative* of a function f on an interval I if $F'(x) = f(x)$ for every value of x in I. If f is discontinuous, we require the one-sided derivatives of F to equal the one-sided limits of f. See Exercise 36.

4.1.3 Theorem If F is a particular antiderivative of f on an interval I, then every antiderivative of f on I is given by $F(x) + C$, where C is an arbitrary constant, and all antiderivatives of f can be obtained from $F(x) + C$ by assigning particular values to C.

The symbol $\int$ denotes the operation of antidifferentiation. That is

$$\int F'(x)dx = \int d(F(x)) = F(x) + C \tag{1}$$

Eq. (1) states that when we antidifferentiate the derivative of a function, we obtain that function plus an arbitrary constant. We have the following formulas for antidifferentiation.

4.1.4 Theorem $\int dx = x + C$

4.1.5 Theorem Constant Multiple $\int af(x)\,dx = a\int f(x)\,dx$, where a is a constant.

4.1.6 Theorem Sum Rule If f_1 and f_2 are defined on the same interval, then

$$\int [f_1(x) + f_2(x)]\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx$$

4.1.7 Theorem If $f_1, f_2, \ldots, f_n$ are defined on the same interval and $c_1, c_2, \ldots, c_n$ are constants,

$$\int [c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x)]\,dx = c_1\int f_1(x)\,dx + c_2\int f_2(x)\,dx + \cdots + c_n\int f_n(x)dx$$

When applying Theorems 4.1.6 and 4.1.7, we use only one arbitrary constant.

To find the antiderivative of a product or quotient we may replace the given function by an equivalent sum and then use Theorem 4.1.7, as illustrated in Exercise 28. Following are some antidifferentiation formulas.

4.1.8 Theorem Power Rule If $n \neq -1$ is a rational number, $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$

Note that dividing by a fraction is the same as multiplying by its reciprocal.

4.1.9 Theorem $\int \sin x\,dx = -\cos x + C$

4.1.10 Theorem $\int \cos x\,dx = \sin x + C$

4.1.11 Theorem $\int \sec^2 x\,dx = \tan x + C$

4.1.12 Theorem $\int \csc^2 x\,dx = -\cot x + C$

4.1.13 Theorem $\int \sec x \tan x\,dx = \sec x + C$

4.1.14 Theorem $\int \csc x \cot dx = -\csc x + C$

Graphic Support Assign a value to the constant C and plot its numerical derivative in the same window as the original function to see if they appear identical.

Exercises 4.1

In Exercises 1–30, perform the antidifferentiation. In Exercises 1–8 and 25–28, check by finding the derivative of your answer. In Exercises 9–12, 29 and 30, support your answer graphically.

1. $\int 3x^4\,dx = 3\cdot\frac{1}{5}x^5 + C = \frac{3}{5}x^5 + C$

2. $\int 2x^7dx = 2\cdot\frac{1}{8}x^8 + C = \frac{1}{4}x^8 + C$

$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = -\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C$

$\int \frac{3}{t^5}\,dt$

We apply Theorem 4.1.5 and Theorem 4.1.8 with $n = -5$.

$$\int \frac{3}{t^5}\,dt = 3\int t^{-5}\,dt = 3\frac{t^{-5+1}}{-5+1} + C = \frac{-3}{4t^4} + C$$

Let $\quad F(t) = \frac{-3}{4t^4} + C = -\frac{3}{4}t^{-4} + C$

Then $\quad F'(t) = -\frac{3}{4}(-4)t^{-5} = \frac{3}{t^5}$

Because $F'(t)$ is the given function, we have shown that $F(t)$ is an antiderivative of the given function.

$\int 5u^{3/2}\,du = 5 \cdot \frac{2}{5}u^{5/2} + C = 2u^{5/2} + C$

6. $\int 10\sqrt[3]{x^2}\,dx = 10\int x^{2/3}dx = 10 \cdot \frac{3}{5}x^{5/3} + C = 6x^{5/3} + C$

$\int \frac{2}{\sqrt[3]{x}}\,dx = \int 2x^{-1/3}\,dx = 2 \cdot \frac{3}{2}x^{2/3} + C = 3x^{2/3} + C$

$\int \frac{3}{\sqrt{y}}\,dy$

We apply Theorem 4.1.5 and Theorem 4.1.8 with $n = -\frac{1}{2}$.

$$\int \frac{3}{\sqrt{y}}\,dy = 3\int y^{-1/2}\,dy = 3\frac{y^{-1/2+1}}{-\frac{1}{2}+1} + C = 6\sqrt{y} + C$$

Let $\quad D_x(6\sqrt{y} + C) = D_x(6y^{1/2} + C) = 6(\frac{1}{2})y^{-1/2} = \frac{3}{\sqrt{y}}$

$\int 6t^2\sqrt{t}\,dt = \int 6t^{7/3}\,dt = 6 \cdot \frac{3}{10}x^{10/3} + C = \frac{9}{5}t^{10/3} + C$

$\int (3u^5 - 2u^3)du = 3\int u^5\,du - 2\int u^3\,du = 3 \cdot \frac{u^6}{6} - 2 \cdot \frac{u^4}{4} + C = \frac{1}{2}u^6 - \frac{1}{2}u^4 + C$

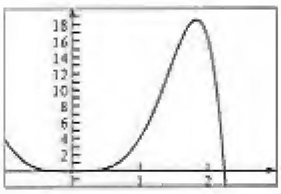
$\int y^3(2y^2 - 3)dy = \int (2y^5 - 3y^3)dy = \frac{1}{3}y^6 - \frac{3}{4}y^4 + C$

$\int x^4(5 - x^2)dx$

We multiply and then apply Theorems 4.1.7 and 4.1.8.

$$\int x^4(5 - x^2)dx = \int (5x^4 - x^6)dx = 5 \cdot \frac{1}{5}x^5 - \frac{1}{7}x^7 + C = x^5 - \frac{1}{7}x^7 + C$$

The figure shows a plot of $y = x^4(5 - x^2)$ and $y = \text{NDER}(x^5 - \frac{1}{7}x^7)$.



$\int (8x^4 + 4x^3 - 6x^2 - 4x + 5)dx = \frac{8}{5}x^5 + x^4 - 2x^3 - 2x^2 + 5x + C$

$\int (2 + 3x^2 - 8x^3)dx = 2x + 3 \cdot \frac{1}{3}x^3 - 8 \cdot \frac{1}{4}x^4 + C = 2x + x^3 - 2x^4 + C$

$\int \sqrt{x}(x + 1)dx = \int (x^{3/2} + x^{1/2})dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + C$

$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx$

We replace radicals with fractional exponents and then apply Theorem 4.1.6.

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx = \int (x^{1/2} - x^{-1/2})dx = \frac{2}{3}x^{3/2} - 2x^{1/2} + C$$

$\int \left(\frac{2}{x^3} + \frac{3}{x^2} + 5\right)dx = \int (2x^{-3} + 3x^{-2} + 5)dx = -x^{-2} - 3x^{-1} + 5x + C = -\frac{1}{x^2} - \frac{3}{x} + 5x + C$

$\int \left(3 - \frac{1}{x^4} + \frac{1}{x^2}\right)dx = \int (3 - x^{-4} + x^{-2})\,dx = 3x - \frac{x^{-3}}{-3} + \frac{x^{-1}}{-1} + C = 3x + \frac{1}{3}x^{-3} - x^{-1} + C$

$\int \frac{x^2 + 4x - 4}{\sqrt{x}}\,dx = \int (x^{3/2} + 4x^{1/2} - 4x^{-1/2})dx = \frac{2}{5}x^{5/2} + 4 \cdot \frac{2}{3}x^{3/2} - 4 \cdot 2x^{1/2} + C = \frac{2}{5}x^{5/2} + \frac{8}{3}x^{3/2} - 8x^{1/2} + C$

20. $\displaystyle\int \frac{y^4+2y^2-1}{\sqrt{y}}dy$

▷ We replace radicals with fractional exponents, then divide and apply Theorem 4.1.7.

$$\int \frac{y^4+2y^2-1}{\sqrt{y}}dy = \int \frac{y^4+2y^2-1}{y^{1/2}}dy = \int (y^{7/2}+2y^{3/2}-y^{-1/2})dy = \tfrac{2}{9}\cdot y^{9/2}+\tfrac{2}{5}\cdot 2y^{5/2}-2y^{1/2}+C$$
$$= \tfrac{2}{9}y^{9/2}+\tfrac{4}{5}y^{5/2}-2y^{1/2}+C \text{ since } \tfrac{7}{2}+1=\tfrac{9}{2} \text{ and } \tfrac{3}{2}+1=\tfrac{5}{2}.$$

21. $\displaystyle\int \left(\sqrt[3]{x}+\frac{1}{\sqrt[3]{x}}\right)dx = \int (x^{1/3}+x^{-1/3})dx = \tfrac{3}{4}x^{4/3}+\tfrac{3}{2}x^{2/3}+C$

22. $\displaystyle\int \frac{27t^3-1}{\sqrt[3]{t}}\,dt = \int (27t^3-1)t^{-1/3}\,dt = \int (27t^{8/3}-t^{-1/3})\,dt = \tfrac{11}{3}\cdot 27t^{11/3}-\tfrac{3}{2}\cdot t^{2/3}+C = \tfrac{81}{11}t^{11/3}-\tfrac{3}{2}t^{2/3}+C$

23. $\displaystyle\int (3\sin t-2\cos t)dt = -3\cos t-2\sin t+C$

24. $\displaystyle\int (5\cos x-4\sin x)dx$

▷ We apply Theorem 4.1.7 and then Theorems 4.1.9 and 4.1.10.

$$\int (5\cos x-4\sin x)dx = 5\int \cos x\,dx - 4\int \sin x\,dx = 5\sin x+4\cos x+C$$

25. $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{\sin x}{\cos x}\frac{1}{\cos x}\,dx = \int \tan x\sec x\,dx = \sec x+C$

26. $\displaystyle\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}\,dx = \int \cot x\csc x\,dx = -\csc x+C$

27. $\displaystyle\int (4\csc x\cot x+2\sec^2 x)dx = -4\csc x+2\tan x+C$

28. $\displaystyle\int (3\csc^2 t-5\sec t\tan t)dt$

▷ We apply Theorem 4.1.7 and then Theorems 4.1.12 and 4.1.13.

$$\int (3\csc^2 t-5\sec t\tan t)dt = 3\int \csc^2 t\,dt - 5\int \sec t\tan t\,dt = -3\cos t-5\sec t\,dt+C$$

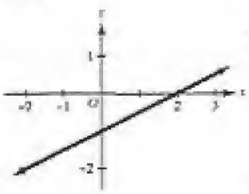
29. $\displaystyle\int (2\cot^2\theta-3\tan^2\theta)d\theta = [2(\csc^2\theta-1)-3(\sec^2\theta-1)]d\theta = \int (2\csc^2\theta-3\sec^2\theta+1)d\theta$

$= -2\cot\theta-3\tan\theta+\theta+C$

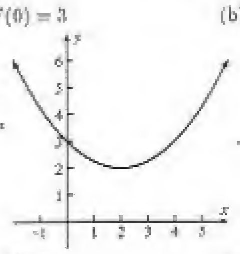
30. $\displaystyle\int \frac{3\tan\theta-4\cos^2\theta}{\cos\theta}\,d\theta = 3\int \frac{\tan\theta}{\cos\theta}\,d\theta - 4\int \frac{\cos^2\theta}{\cos\theta}\,d\theta = 3\int \tan\theta\sec\theta\,d\theta - 4\int \cos\theta\,d\theta = 3\sec\theta-4\sin\theta+C$

In Exercises 31–36, the graph of a function f is given. Sketch an antiderivative F, continuous everywhere, having the given values.

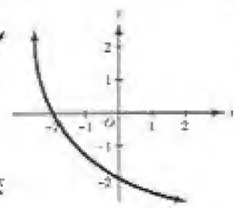
31. (a) f



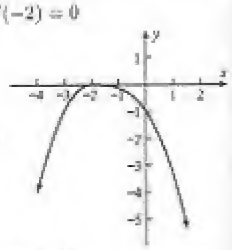
F(0) = 3



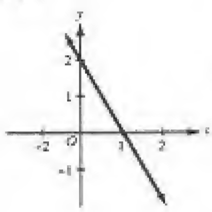
(b)



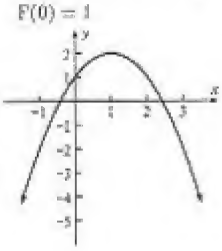
F(−2) = 0



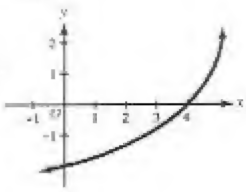
32. (a) f



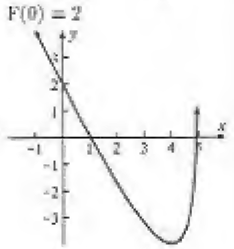
F(0) = 1



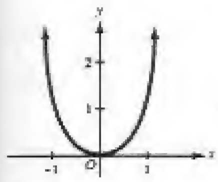
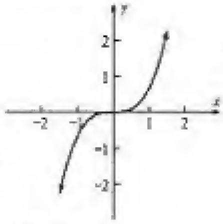
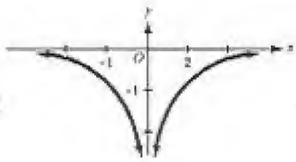
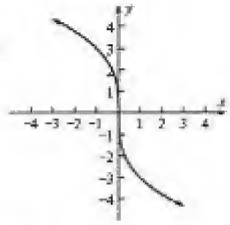
(b) f



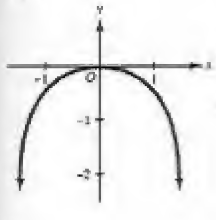
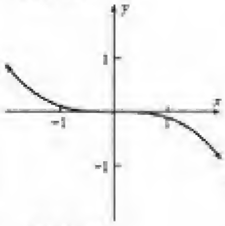
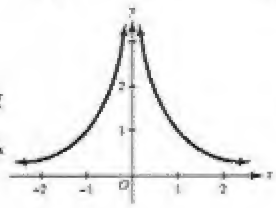
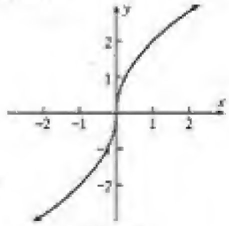
F(0) = 2



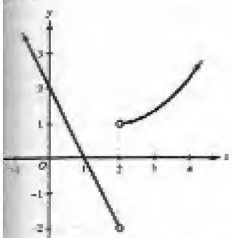
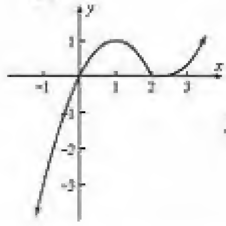
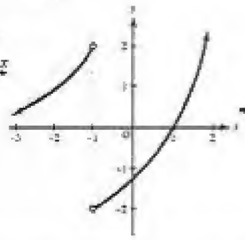
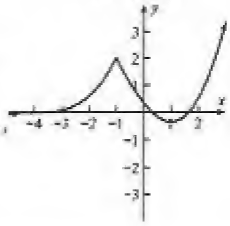
33. (a) f F(0) = 0 (b) f F(0) = 0

34. (a) f F(0) = 0 (b) f F(0) = 0

35. f F(2) = 0 **36.** f F(−4) = 0

37. $\frac{dy}{dx} = 2x - 3$; $y = \int (2x-3)dx$; $y = x^2 - 3x + C$. Substituting $(x,y) = (3,2)$ gives $2 = (3)^2 - 3(3) + C$; $C = 2$. Thus an equation of the curve is $y = x^2 - 3x + 2$.

38. $\frac{dy}{dx} = 3\sqrt{x}$; $y = \int 3x^{1/2}dx = 3 \cdot \frac{2}{3}x^{3/2} + C = 2x^{3/2} + C$. Substituting $(x,y) = (9,4)$ gives $4 = 2 \cdot 9^{3/2} + C = 54 + C$, $C = -50$. Thus an equation of the curve is $y = 2x^{3/2} - 50$.

39. $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = 2 - 4x$; $y' = \int (2-4x)dx$; $\frac{dy}{dx} = 2x - 2x^2 + C_1$; $y = \int (2x - 2x^2 + C_1)dx$; $y = x^2 - \frac{2}{3}x^3 + C_1x + C_2$ Substituting $(x,y) = (0,2)$ gives $2 = 0 + C_2$; $C_2 = 2$, so $y = x^2 - \frac{2}{3}x^3 + C_1x + 2$. Substituting $(x,y) = (-1,3)$ gives $3 = (-1)^2 - \frac{2}{3}(-1)^3 + C_1(-1) + 2$; $C_1 = \frac{2}{3}$. Thus, an equation of the curve is $y = x^2 - \frac{2}{3}x^3 + \frac{2}{3}x + 2$.

40. An equation of the tangent line to a curve at the point $(1,3)$ is $y = x + 2$. If at any point (x,y) on the curve, $d^2y/dx^2 = 6x$, find an equation of the curve.

▸ If $dy/dx = y'$, then $d^2y/dx^2 = dy'/dx$. Therefore, we are given that

$$\frac{dy'}{dx} = 6x$$

Antidifferentiating, we have

$$y' = \int 6x\, dx = 3x^2 + C \tag{1}$$

Because $y = x + 2$ is an equation of the tangent line to the curve at the point $(1,3)$, then the slope of the tangent line at $(1,3)$ is 1. Therefore, we may let $y' = 1$ and $x = 1$ in Eq. (1). Thus,

$$1 = 3 + C$$

$C = -2$

Substituting -2 for c in Eq. (1), we get

$$y' = 3x^2 - 2$$

$$y = \int (3x^2 - 2)\,dx = x^3 - 2x + K \qquad (2)$$

Because the curve contains the point (1,3), we may let $x = 1$ and $y = 3$ in Eq. (2). Thus,

$$3 = 1 - 2 + K$$

$$K = 4$$

Substituting $K = 4$ into Eq. (2) gives an equation of the curve

■ $$y = x^3 - 2x + 4$$

41. $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = 1 - x^2$; $y' = \int (1 - x^2)dx$; $\frac{dy}{dx} = x - \frac{1}{3}x^3 + C_1$. At (1,1), the slope of the tangent line is -1. Setting $\frac{dy}{dx} = -1$ when $x = 1$ gives $-1 = \frac{2}{3} + C_1$; $C_1 = -\frac{5}{3}$. Thus $\frac{dy}{dx} = x - \frac{1}{3}x^3 - \frac{5}{3}$; $y = \int (x - \frac{1}{3}x^3 - \frac{5}{3})dx$; $y = \frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{5}{3}x + C_2$. Substituting $(x,y) = (1,1)$ gives $1 = -\frac{5}{4} + C_2$; $C_2 = \frac{9}{4}$.

■ An equation of the curve is $y = \frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{5}{3}x + \frac{9}{4}$.

42. $y''' = 2$; $y'' = \int 2\,dx = 2x + C_1$. Because (1,3) is a point of inflection, $y''(1) = 0 = 2 + C_1$, $C_1 = -2$. $y'' = 2x - 2$. $y' = \int (2x - 2)dx = x^2 - 2x + C_2$. $y'(1) = -2 = -1 + C_2$, $C_2 = -1$, $y' = x^2 - 2x - 1$. $y = \int (x^2 - 2x - 1)dx = \frac{1}{3}x^3 - x^2 - x + C_3$. $y(1) = 3 = -\frac{5}{3} + C_3$, $C_3 = \frac{14}{3}$; $y = \frac{1}{3}x^3 - x^2 - x + \frac{14}{3}$

43. $C'(x) = 3x^2 + 8x + 4$, and $C(0) = 6$. $C(x) = \int (3x^2 + 8x + 4)dx = x^3 + 4x^2 + 4x + K$. Because $C(0) = 6$, then $K = 6$. Therefore $C(x) = x^3 + 4x^2 + 4x + 6$.

44. A company has determined that the marginal cost function for the production of a particular commodity is given by $C'(x) = 125 + 10x + \frac{1}{9}x^2$, where $C(x)$ dollars is the total cost of producing x units of the commodity. If the overhead cost is \$250, what is the cost of producing 15 units?

▷ The cost function is the antiderivative of the marginal cost, that is

$$C(x) = \int (125 + 10x + \tfrac{1}{9}x^2)\,dx = 125x + 5x^2 + \tfrac{1}{27}x^3 + K \qquad (1)$$

Because the overhead cost is $c(0)$, we let $x = 0$ in Eq. (1). Thus,

$$250 = K$$

Substituting $x = 15$ and $K = 250$ into Eq. (1), we have

$$C(15) = 125(15) + 5(15)^2 + \tfrac{1}{27}(15^3) + 250 = 3375$$

■ The cost of producing 15 units is \$3375.

45. $C'(x) = 6x$. $C(x) = \int 6x\,dx = 3x^2 + K$. Because $C(2) = 20$, then $20 = 3(2)^2 + K$; $K = 8$.

(a) $C(x) = 3x^2 + 8$ (b) $C(0) = 8$. Thus the overhead cost is \$800.

46. $R'(x) = 12 - 3x$. $R(x) = \int (12 - 3x)dx = 12x - \frac{3}{2}x^2 + C$. $R(0)$ is always 0, so $C = 0$. $R(x) = 12x - \frac{3}{2}x^2$ and $p = R/x = 12 - \frac{3}{2}x$

47. $R'(x) = 15 - 4x$. $R(x) = \int (15 - 4x)dx = 15x - 2x^2 + K$. Because $R(0) = 0$, $0 = K$.

(a) $R(x) = 15x - x^2$ (b) Because $R(x) = px$, then $px = 15x - 2x^2$ so that $p = 15 - 2x$.

48. The efficiency of a factory worker is expressed as a percent. For instance, if the worker's efficiency at a particular time is given as 70 percent, then the worker is performing at 70 percent of her full potential. Suppose that E percent is a factory worker's efficiency t hours after beginning work, and the rate at which E is changing is $(35 - 8t)$ percent per hour. If the worker's efficiency is 81 percent after working 3 hr, find her efficiency after working (a) 4 hr and (b) 8 hr.

▷ We have

$$E'(t) = 35 - 8t$$

$$E(t) = \int (35 - 8t)\,dt = 35t - 4t^2 + C \qquad (1)$$

Because $E = 81$ when $t = 3$, then

$$81 = 35(3) - 4(3^2) + C$$
$$C = 12$$

Substituting for C in Eq. (1), we have
$$E(t) = 35t - 4t^2 + 12$$
Thus,
$$E(4) = 35(4) - 4(4^2) + 12 = 88$$
$$E(8) = 35(8) - 4(8^2) + 12 = 36$$

- The efficiency is (a) 88 percent after 4 hr and (b) 36 percent after 8 hr.

49. The volume of water in a tank is V m^3 when the depth of the water is h meters.
$\frac{dV}{dh} = \pi(4h^2 + 12h + 9)$; $V = \int \pi(4h^2 + 12h + 9)dh$; $V = \pi(\frac{4}{3}h^3 + 6h^2 + 9h) + C$. Because $V = 0$ when4 $h = 0$, then $C = 0$, so $V = \pi(\frac{4}{3}h^3 + 6h^2 + 9h)$. When $h = 3$, $V = 117\pi$, so the volume is 117π m^3.

50. $\frac{dV}{dt} = 5t^{3/2} + 10t + 50$; $V = \int (5t^{3/2} + 10t + 50)dt = 2t^{5/2} + 5t^2 + 50t + C$. $V(0) = 1000 = C$; $V = 2t^{5/2} + 5t^2 + 50t + 1000$. $V(4) = 2 \cdot 4^{5/2} + 5 \cdot 4^2 + 50 \cdot 4 + 1000 = 1344$. The painting is worth \$1344.

51. $f(x) = |x|$. $F(x) = \begin{cases} -\frac{1}{2}x^2 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } x \geq 0 \end{cases}$. $F'(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} = |x|$. Because $F'(0) = \lim\limits_{x \to 0^-}(-x) = 0$ and $F'(0) = \lim\limits_{x \to 0^+}(x) = 0$ then $F'(0) = 0$. Hence F is an antiderivative of f on $(-\infty, +\infty)$.

52. Let $U(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$. Show that U does not have an antiderivative on $(-\infty, +\infty)$.

- $U(-1) = 0$ and $U(1) = 1$ and $\frac{1}{2}$ is between 0 and 1, but U does take the value $\frac{1}{2}$ on $(-1, 1)$. The Intermediate-Value theorem for derivatives (Exercise 3.4.56) implies that because U does not satisfy the intermediate-value property, then it is not the derivative of any function, that is, U is not an antiderivative.

53. $f(x) = 1$ for all x in $(-1, 1)$. Therefore $f'(x) = 0$ for all x in $(-1, 1)$. $g(x) = \begin{cases} -1 & \text{if } -1 < x \leq 0 \\ 1 & \text{if } 0 < x < 1 \end{cases}$.
$g'(x) = 0$ if x is in $(-1, 1)$ and $x \neq 0$. $g'(0)$ does not exist because g is discontinuous at 0. Hence g is not differentiable on $(-1, 1)$, and so Theorem 4.1.2 does not apply.

54. $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$. $F(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x > 0 \end{cases}$. $F'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$.
Because $F'(x) \neq f(x)$ for all x, F is not an antiderivative of f.

4.2 SOME TECHNIQUES OF ANTIDIFFERENTIATION

4.2.1 Theorem Chain Rule for Antidifferentiation Let g be a differentiable function, and let the range of g be an interval I. Suppose that f is a function defined on I and that F is an antiderivative of f on I. Then

$$\int f(g(x))[g'(x)\,dx] = F(g(x)) + C$$

Method of Substitution To apply Theorem 4.2.1 we may let $u = g(x)$. It follows that $du = g'(x)\,dx$ so

$$\int f(g(x))[g'(x)\,dx] = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

where F is any antiderivative of f. In easy cases, the work may be shortened to

$$\int f(g(x))\{g'(x)\,dx\} = \int f(x)dg(x) = F(g(x)) + C$$

Linear Argument In particular, if $g(x) = ax + b$, then $du = a\,dx$ and so

$$\int f(ax + b)dx = \int f(u) \cdot \tfrac{1}{a}du = \tfrac{1}{a}F(u) + C = \tfrac{1}{a}F(ax + b) + C$$

An important special case is when $f(x) = x^n$:

4.2.2 Theorem If g is a differentiable function and $n \neq -1$ is a rational number,

$$\int [g(x)]^n \{g'(x)\,dx\} = \frac{[g(x)]^{n+1}}{n+1} + C$$

Note the use of braces $\{\}$ to signal an implicit substitution.

Exercises 4.2

In Exercises 1–44, find the antiderivative I. Check your answer or support it graphically.

1. $\int \sqrt{1-4y}\, dy = -\frac{1}{4}\int (1-4y)^{1/2}\{-4\, dy\} = -\frac{1}{4}[\frac{2}{3}(1-4y)^{3/2}] + C = -\frac{1}{6}(1-4y)^{3/2} + C$

2. $\int \sqrt[3]{3x-4}\, dx = \frac{1}{3}\int (3x-4)^{1/3}\{3\, dx\} = \frac{1}{3}\cdot\frac{3}{4}(3x-4)^{4/3} + C = \frac{1}{4}(3x-4)^{4/3} + C$

3. $\int x\sqrt[3]{x^2-9}\, dx = \frac{1}{2}\int (x^2-9)^{1/3}\{2x\, dx\} = \frac{1}{2}[\frac{3}{4}(x^2-9)^{4/3}] + C = \frac{3}{8}(x^2-9)^{4/3} + C$

4. $\int x(2x^2+1)^6\, dx$

▸ We apply Theorem 4.2.2 with $n = 6$. Because $d(2x^2+1) = 4x\, dx$, then

$$\int x(2x^2+1)^6\, dx = \frac{1}{4}\int (2x^2+1)^6\{4x\, dx\} = \frac{1}{4}\,\frac{(2x^2+1)^7}{7} + C = \frac{1}{28}(2x^2+1)^7 + C$$

5. $\int x^2(x^3-1)^{10} dx = \frac{1}{3}\int (x^3-1)^{10}\{x^2\, dx\} = \frac{1}{3}[\frac{1}{11}(x^3-1)^{11}] + C = \frac{1}{33}(x^3-1)^{11} + C$

6. $\int 3x\sqrt{4-x^2}\, dx$. Let $u = 4-x^2$. Then $du = -2x\, dx$, $x\, dx = -\frac{1}{2}\, du$.

$\int 3x\sqrt{4-x^2}\, dx = 3(-\frac{1}{2})\int u^{1/2}\, du = -\frac{3}{2}\cdot\frac{2}{3}u^{3/2} + C = -(4-x^2)^{3/2} + C$

7. $\int \frac{y^3 dy}{(1-2y^4)^5} = -\frac{1}{8}\int (1-2y^4)^{-5}\{8y^3\, dy\} - \frac{1}{8}[-\frac{1}{4}(1-2y^4)^{-4}] + C = \frac{1}{32}(1-2y^4)^{-4} + C$

8. $\int \frac{s\, ds}{\sqrt{3s^2+1}}$

▸ Let $u = 3s^2+1$. Then $du = 6s\, ds$, so $s\, ds = \frac{1}{6}\, du$. Thus,

$$\int \frac{s\, ds}{\sqrt{3s^2+1}} = \int \frac{\frac{1}{6}\, du}{\sqrt{u}} = \frac{1}{6}\int u^{-1/2}\, du = \frac{1}{3}u^{1/2} + C = \frac{1}{3}\sqrt{3s^2+1} + C$$

9. $\int (x^2-4x+4)^{4/3} dx = \int [(x-2)^2]^{4/3} dx = \int (x-2)^{8/3} dx = \frac{3}{11}(x-2)^{11/3} + C$

10. $\int x^4\sqrt{3x^5-5}\, dx = \frac{1}{15}\int (3x^5-5)^{1/2}\{15x^4 dx\} = \frac{1}{15}\cdot\frac{2}{3}(3x^5-5)^{3/2} + C = \frac{2}{45}(3x^5-5)^{3/2} + C$

11. $\int x\sqrt{x+2}\, dx$. Let $u = x+2$. Then $du = dx$, and $x = u-2$. Thus

$\int x\sqrt{x+2}\, dx = \int (u-2)\sqrt{u}\, du = \int (u^{3/2} - 2u^{1/2}) du = \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$

12. $\int \frac{t\, dt}{\sqrt{t+3}}$

▸ Let $u = t+3$. Then $du = dt$, and $t = u-3$. Thus

$$\int \frac{t\, dt}{\sqrt{t+3}} = \int \frac{(u-3)\, du}{\sqrt{u}} = \int \frac{u\, du}{\sqrt{u}} - 3\int \frac{du}{\sqrt{u}} = \int u^{1/2}\, du - 3\int u^{-1/2}\, du = \frac{2}{3}u^{3/2} - 6u^{1/2} + C$$
$$= \frac{2}{3}(t+3)^{3/2} - 6(t+3)^{1/2} + C$$

13. $\int \frac{2r\, dr}{(1-r)^7}$. Let $u = 1-r$. Then $du = -dr$, and $r = 1-u$. Thus

$\int \frac{2r\, dr}{(1-r)^7} = \int 2(1-u)u^{-7}(-du) = \int (-2u^{-7} + 2u^{-6}) du = \frac{1}{3}u^{-6} - \frac{2}{5}u^{-5} + C = \frac{1}{3}(1-r)^{-6} - \frac{2}{5}(1-r)^{-5} + C$

14. $\int x^3(2-x^2)^{12} dx = \int x^2(2-x^2)^{12}(x\, dx)$. Let $u = 2-x^2$, $x^2 = 2-u$, $2x\, dx = -du$.

$I = \int (2-u)u^{12}(-\frac{1}{2}du) = \int (-u^{12} + \frac{1}{2}u^{13}) du = -\frac{1}{13}u^{13} + \frac{1}{28}u^{14} + C = \frac{1}{28}(2-x^2)^{14} - \frac{1}{13}(2-x^2)^{13} + C$

$\int \sqrt{3-2x}\, x^2\, dx$. Let $u = 3-2x$. Then $x = \frac{1}{2}(3-u)$ and $dx = -\frac{1}{2}du$. Thus

$\int \sqrt{3-2x}\, x^2\, dx = \int u^{1/2} \cdot \frac{1}{4}(3-u)^2(-\frac{1}{2}du) = -\frac{1}{8}\int u^{1/2}(9-6u+u^2)du = -\frac{1}{8}\int (9u^{1/2} - 6u^{3/2} + u^{5/2})du$

$= -\frac{1}{8}(6u^{3/2} - \frac{12}{5}u^{5/2} + \frac{2}{7}u^{7/2}) + C = -\frac{3}{4}(3-2x)^{3/2} + \frac{3}{10}(3-2x)^{5/2} - \frac{1}{28}(3-2x)^{7/2} + C$

$\int (x^3+3)^{1/4}\, x^5\, dx$

Because $(x^3+3)^{1/4}\, x^5\, dx = (x^3+3)^{1/4}(x^3)(x^2\, dx)$, we let $x^3+3 = u$ $x^3 = u-3$ $3x^2\, dx = du$ Then

$$\int (x^3+3)^{1/4}\, x^5\, dx = \int (x^3+3)^{1/4}(x^3)(x^2\, dx) = \int u^{1/4}(u-3)(\tfrac{1}{3}\, du) = \tfrac{1}{3}\int u^{5/4}\, du - \int u^{1/4}\, du$$

$$= \tfrac{1}{3}\left(\tfrac{4}{9}u^{9/4}\right) - \tfrac{4}{5}u^{5/4} + C = \tfrac{4}{27}(x^3+3)^{9/4} - \tfrac{4}{5}(x^3+3)^{5/4} + C$$

$\int \cos 4\theta\, d\theta = \frac{1}{4}\int \cos 4\theta(4\, d\theta) = \frac{1}{4}\sin 4\theta + C$

$\int \sin \frac{1}{3}x\, dx = 3\int \sin \frac{1}{3}x(\frac{1}{3}dx) = 3(-\cos \frac{1}{3}x) + C = -3\cos \frac{1}{3}x + C$

$\int 6x^2 \sin x^3\, dx = 2\int \sin x^3(3x^2\, dx) = -2\cos x^3 + C$

$\int \frac{1}{2}t \cos 4t^2\, dt$

Let $u = 4t^2$. Then $du = 8t\, dt$, so $t\, dt = \frac{1}{8}\, du$, and

$$\int \tfrac{1}{2}\cos 4t^2\, (t\, dt) = \int \tfrac{1}{2}\cos u\, (\tfrac{1}{8}\, du) = \tfrac{1}{16}\int \cos u\, du = \tfrac{1}{16}\sin u + C = \tfrac{1}{16}\sin 4t^2 + C$$

$\int \sec^2 5x\, dx = \frac{1}{5}\int \sec^2 5x(5\, dx) = \frac{1}{5}\tan 5x + C$

$\int \csc^2 2\theta\, d\theta = \frac{1}{2}\int \csc^2 2\theta(2\, d\theta) = -\frac{1}{2}\cot 2\theta + C$

$\int y \csc 3y^2 \cot 3y^2\, dy = \frac{1}{6}\int \csc 3y^2 \cot 3y^2\{6y\, dy\} = -\frac{1}{6}\csc 3y^2 + C$

$\int r^2 \sec^2 r^3\, dr$

Let $u = r^3$. Then $du = 3r^2\, dr$, so $r^2\, dr = \frac{1}{3}\, du$, and

$$\int \sec^2 r^3\, (r^2\, dr) = \int \tfrac{1}{3}\sec^2 u\, du = \tfrac{1}{3}\tan u + C = \tfrac{1}{3}\tan r^3 + C$$

$\int \cos x(2+\sin x)^5 dx$. Let $u = 2+\sin x$, $du = \cos x\, dx$.

$\int \cos x(2+\sin x)^5 dx = \int u^5 du = \frac{1}{6}u^6 + C = \frac{1}{6}(2+\sin x)^6 + C$

$\int \frac{4\sin x\, dx}{(1+\cos x)^2} = -4\int (1+\cos x)^{-2}\{-\sin x\, dx\} = -4[-(1+\cos x)^{-1}] + C = \frac{4}{1+\cos x} + C$

$\int \sqrt{1+\frac{1}{3x}}\frac{dx}{x^2}$. Let $u = 1+\frac{1}{3x}$; $du = -\frac{1}{3x^2}\, dx$; $\frac{dx}{x^2} = -3\, du$.

$\int \sqrt{1+\frac{1}{3x}}\frac{dx}{x^2} = -3\int \sqrt{u}\, du = -2u^{3/2} + C = -2\left(1+\frac{1}{3x}\right)^{3/2} + C$

$\int \sqrt{\frac{1}{t}-1}\,\frac{dt}{t^2}$

Let $u = \frac{1}{t}-1$. Then $du = -\frac{1}{t^2}\, dt$, and $\int \sqrt{\frac{1}{t}-1}\,\frac{dt}{t^2} = \int \sqrt{u}(-du) = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}\left(\frac{1}{t}-1\right)^{3/2} + C$

$\int 2\sin x\, \sqrt[3]{1+\cos x}\, dx$. Let $u = 1+\cos x$; $du = -\sin x\, dx$.

$\int 2\sin x(1+\cos x)^{1/3}dx = -2\int u^{1/3}du = -\frac{3}{2}u^{4/3} + C = -\frac{3}{2}(1+\cos x)^{4/3} + C$

$\int \sin 2x\sqrt{2-\cos 2x}\, dx = \frac{1}{2}\int (2-\cos 2x)^{1/2}(2\sin 2x\, dx) = \frac{1}{2}\cdot\frac{2}{3}(2-\cos 2x)^{3/2} + C = \frac{1}{3}(2-\cos 2x)^{3/2} + C$

31. $\int \cos^2 t \sin t\, dt$. Let $u = \cos t$; $du = -\sin t\, dt$. $\int \cos^2 t \sin t\, dt = -\int u^2 du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cos^3 t + C$

32. $\int \sin^3 \theta \cos \theta\, d\theta$

▸ Let $u = \sin \theta$. Then $du = \cos \theta\, d\theta$, and $\int \sin^3 \theta \cos \theta\, d\theta = \int u^3\, du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 \theta + C$

33. $\int (\tan 2x + \cot 2x)^2 dx = \int (\tan^2 2x + 2 + \cot^2 2x) dx = \int [(\sec^2 2x - 1) + 2 + (\csc^2 2x - 1)] dx$

$$= \int (\sec^2 2x + \csc^2 2x) dx = \tfrac{1}{2}\tan 2x - \tfrac{1}{2}\cot 2x + C$$

34. $\int \frac{\sec^2 3\sqrt{t}}{\sqrt{t}}\, dt$. Let $u = 3\sqrt{t}$. Then $du = \frac{3}{2}t^{-1/2}\, dt$ and $\frac{dt}{\sqrt{t}} = \frac{2}{3}\, du$. Thus,

$\int \frac{\sec^2 3\sqrt{t}}{\sqrt{t}}\, dt = \int \sec^2 u \left(\frac{2}{3}\, du\right) = \frac{2}{3}\tan u + C = \frac{2}{3}\tan 3\sqrt{t} + C$

35. $\int \frac{(x^2 + 2x)dx}{\sqrt{x^3 + 3x^2 + 1}}$. Let $u = x^3 + 3x^2 + 1$; $du = (3x^2 + 6x)dx = 3(x^2 + 2x)dx$

$\int \frac{(x^2 + 2x)dx}{\sqrt{x^3 + 3x^2 + 1}} = \frac{1}{3}\int u^{-1/2} du = \frac{2}{3}u^{1/2} + C = \frac{2}{3}\sqrt{x^3 + 3x^2 + 1} + C$

36. $\int x(x^2 + 1)\sqrt{4 - 2x^2 - x^4}\, dx$. Let $u = 4 - 2x^2 - x^4$. Then $du = (-4x - 4x^3)dx = -4x(x^2 + 1)dx$. Thus

$\int \sqrt{4 - 2x^2 - x^4}[x(x^2 + 1)dx] = -\frac{1}{4}\int u^{1/2} du = -\frac{1}{4}\cdot\frac{2}{3}u^{3/2} + C = -\frac{1}{6}(4 - 2x^2 - x^4)^{3/2} + C$

37. $\int \frac{(y + 3)dy}{(3 - y)^{2/3}}$. Let $u = 3 - y$. Then $y = 3 - u$, and $dy = -du$.

$\int \frac{(y + 3)dy}{(3 - y)^{2/3}} = \int \frac{(6 - u)(-du)}{u^{2/3}} = \int (u^{1/3} - 6u^{-2/3}) du = \frac{3}{4}u^{4/3} - 18u^{1/3} + C = \frac{3}{4}(3 - y)^{4/3} - 18(3 - y)^{1/3} + C$

38. $\int \sqrt{3 + s}\, (s + 1)^2\, ds = \int u[(u^2 - 3) + 1]^2(2u\, du) = 2\int u(u^2 - 2)^2 u\, du = 2\int (u^6 - 4u^4 + 4u^2)\, du$

$$= 2(\tfrac{1}{7}u^7 - \tfrac{4}{5}u^5 + \tfrac{4}{3}u^3) + C = \tfrac{2}{7}(3 + s)^{7/2} - \tfrac{8}{5}(3 + s)^{5/2} + \tfrac{8}{3}(3 + s)^{3/2} + C$$

39. $\int \frac{(r^{1/3} + 2)^4}{\sqrt[3]{r^2}}$. Let $u = r^{1/3} + 2$; $du = \frac{1}{3}r^{-2/3} dr$. $\int \frac{(r^{1/3} + 2)^4 dr}{\sqrt[3]{r^2}} = 3\int u^4 du = \frac{3}{5}u^5 + C = \frac{3}{5}(r^{1/3} + 2)^5 + C$

40. $\int \left(t + \frac{1}{t}\right)^{3/2} \frac{t^2 - 1}{t^2}\, dt$

▸ Let $\quad u = t + \frac{1}{t} \quad du = \left(1 - \frac{1}{t^2}\right) dt = \frac{t^2 - 1}{t^2}\, dt$

Then

$$\int \left(t + \tfrac{1}{t}\right)^{3/2} \frac{t^2 - 1}{t^2}\, dt = \int u^{3/2}\, du = \tfrac{2}{5}u^{5/2} + C = \tfrac{2}{5}\left(t + \tfrac{1}{t}\right)^{5/2} + C$$

41. $\int \frac{x^3 dx}{(x^2 + 4)^{3/2}} = \int \frac{x^2(x\, dx)}{(x^2 + 4)^{3/2}}$. Let $u = x^2 + 4$. Then $x^2 = u - 4$, and $du = 2x\, dx$. $\int \frac{x^3 dx}{(x^2 + 4)^{3/2}}$

$= \frac{1}{2}\int (u - 4)u^{-3/2} du = \frac{1}{2}\int (u^{-1/2} - 4u^{-3/2}) du = \frac{1}{2}(2u^{1/2} + 8u^{-1/2}) + C = (x^2 + 4)^{1/2} + 4(x^2 + 4)^{-1/2} + C$

42. $\int \frac{x^3 dx}{\sqrt{1 - 2x^2}} = \int (1 - 2x^2)^{-1/2} x^2 (x\, dx)$. Let $u = 1 - 2x^2$. Then $x^2 = \frac{1}{2}(1 - u)$, and $du = -4x\, dx$. I =

$\int u^{-1/2}\cdot\frac{1}{2}(1 - u)(-\frac{1}{4}du) = \int (\frac{1}{8}u^{1/2} - \frac{1}{8}u^{-1/2}) du = \frac{1}{8}\cdot\frac{2}{3}u^{3/2} - \frac{1}{8}\cdot 2u^{1/2} + C = \frac{1}{12}(1 - 2x^2)^{3/2} - \frac{1}{4}(1 - 2x^2)^{1/2} + C$

43. $\int \sin x \sin(\cos x) dx$. Let $u = \cos x$; $du = -\sin x\, dx$.

$\int \sin x \sin(\cos x) dx = -\int \sin u\, du = \cos u + C = \cos(\cos x) + C$

44. $\int \sec x \tan x \cos(\sec x) dx$

▸ Let $u = \sec x$. Then $du = \sec x \tan x\, dx$, so

$$\int \sec x \tan x \cos(\sec x)\, dx = \int \cos u\, du = \sin u + C = \sin(\sec x) + C$$

45. $C'(x) = 3(5x+4)^{-1/2}$ and $C(0) = 10$. Hence $C(x) = 3\int (5x+4)^{-1/2} dx = \frac{6}{5}(5x+4)^{1/2} + K$.
Because $C(0) = 10$, $10 = \frac{6}{5}(2) + K$; $K = \frac{38}{5}$. Thus $C(x) = \frac{6}{5}\sqrt{5x+4} + \frac{38}{5}$.

46. $C'(x) = 3\sqrt{2x+4}$ and $C(0) = 0$. Hence $C(x) = 3\int (2x+4)^{1/2} dx = 3 \cdot \frac{1}{2} \cdot \frac{2}{3}(2x+4)^{3/2} + k = (2x+4)^{3/2} + k$
Because $c(0) = 0$, then $0 = 4^{3/2} + K$ so $K = -8$ and the total cost function is $C(x) = (2x+4)^{3/2} - 8$.

47. $R'(x) = 4 + 10(x+5)^{-2}$. Therefore $R(x) = \int [4 + 10(x+5)^{-2}] dx = 4x - \frac{10}{x+5} + C$.
Because $R(0) = 0$, $0 = 0 - \frac{10}{5} + C$; $C = 2$. Hence $R(x) = 4x - \frac{10}{x+5} + 2$.
If x units are demanded when p dollars is the price per unit, then $px = R(x)$. Thus
$px = 4x - \frac{10}{x+5} + 2$; $px(x+5) = 4x^2 + 20x - 10 + 2x + 10$; $px(x+5) = 4x^2 + 22x$; $p(x+5) = 4x + 22$

48. The marginal revenue function for a particular article is given by $R'(x) = ab(x+b)^{-2} - c$. Find (a) the total revenue function R and (b) an equation involving p and x (the demand equation) where x units are demanded when p dollars is the price per unit.

▸ (a) $R(x) = \int [ab(x+b)^{-2} - c] dx = -ab(x+b)^{-1} - cx + k$. Because $R(0) = 0$, then $0 = k - ab(b)^{-1}$ and so $k = a$. Hence the revenue function is $R(x) = a - \frac{ab}{x+b} - cx = \frac{ax}{a+b} - cx$.
(b) Because $R(x) = px$, then $\pi x = \frac{ax}{a+b}$ and so $p = \frac{a}{a+b} - c$.

49. q coulombs is the charge of electricity at t sec and $i = \frac{dq}{dt} = 5 \sin 60t$. $q = \int 5 \sin 60t\, dt = -\frac{1}{12}\cos 60t + C$.
Because $q = 0$ when $t = \frac{1}{2}\pi$, then $0 = -\frac{1}{12}\cos 30\pi = -\frac{1}{12} + C$; $C = \frac{1}{12}$. Thus
$q = -\frac{1}{12}\cos 60t + \frac{1}{12} \leq \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$. Hence the greatest charge is $\frac{1}{6}$ coulombs.

50. q coulombs is the charge of electricity at t sec and $i = \frac{dq}{dt} = 4\cos 120t$. $q = \int 4\cos 120t\, dt = \frac{1}{30}\sin 120t + C$.
Because $q = 0$ when $t = 0$, then $0 = C$. Thus $q = \frac{1}{30}\sin 120t \leq \frac{1}{30}$. Hence the greatest charge is $\frac{1}{30}$ coulombs.

51. V dollars is the value of the machinery after t years, and $\frac{dV}{dt} = -500(t+1)^{-2}$.
$V = \int -500(t+1)^{-2} dt = 500(t+1)^{-1} + C$. Because $V = 700$ when $t = 0$, then $700 = 500 + C$; $C = 200$.
Thus $V(t) = 500(t+1)^{-1} + 200$, and $V(3) = 500(4)^{-1} + 200 = 325$.
Hence after 3 years the value of the machinery is \$325.

52. The volume of water in a tank is V cubic meters when the depth of the water is h meters. If the rate of change of V with respect to h is given by $\frac{dV}{dh} = \pi(2h+3)^2$, find the volume of water in the tank when the depth is 3 m.

▸ $$\frac{dV}{dh} = \pi(2h+3)^2; \quad V(h) = \int \pi(2h+3)^2\, dh$$

Because $d(2h+3) = 2\, dh$, then

$$V(h) = \frac{1}{2}\pi \int (2h+3)^2 (2\, dh) = \frac{1}{6}\pi(2h+3)^3 + C$$

Because $V = 0$ when $h = 0$, then

$$0 = \frac{1}{6}\pi(3)^3 + C$$

and so $C = -\frac{9}{2}\pi$, and thus

$$V(h) = \frac{1}{6}\pi(2h+3)^3 - \frac{9}{2}\pi; \quad V(3) = \frac{1}{6}\pi(9^3) - \frac{9}{2}\pi = 117\pi$$

• The volume of water in the tank when the depth is 3 m is 117π m^3.

53. If V μm^3 is the volume of the cell t days after December 1, then $\frac{dV}{dt} = (12-t)^{-2}$.
$V = \int (12-t)^{-2}dt = (12-t)^{-1} + C$. Because $V = 3$ when $t = 2$, then $3 = 10^{-1} + C$; $C = 2.9$.
Thus $V = (12-t)^{-1} + 2.9$. On December 8, $t = 7$, $V = 5^{-1} + 2.9 = 3.1$, and the volume is $3.1\ \mu m^3$.

54. If V cm^3 is the volume of the balloon at t seconds, then $\frac{dV}{dt} = \sqrt{t+1} + \frac{2}{3}t$. $V = \int [(t+1)^{1/2} + \frac{2}{3}t]dt$
$= \frac{2}{3}(t+1)^{3/2} + \frac{1}{3}t^2 + C$. Because $V = 33$ when $t = 3$, then $33 = \frac{2}{3}\cdot 4^{3/2} + \frac{1}{3}3^2 + C$; $C = \frac{74}{3}$;
(a) $V = \frac{2}{3}(t+1)^{3/2} + \frac{1}{3}t^2 + \frac{74}{3}$. (b) $V(8) = \frac{2}{3}\cdot 9^{3/2} + \frac{1}{3}\cdot 8^2 + \frac{74}{3} = 64$. The volume is $64\ cm^3$.

55. $\int (2x+1)^3 dx$. (a) $\int (2x+1)^3 dx = \int (8x^3 + 12x^2 + 6x + 1)dx = 2x^4 + 4x^3 + 3x^2 + x + C_1$
(b) Let $u = 2x+1$; $du = 2\ dx$. $\int (2x+1)^3 dx = \frac{1}{2}\int u^3 du = \frac{1}{8}u^4 + C_2 = \frac{1}{8}(2x+1)^4 + C_2$
$= \frac{1}{8}(16x^4 + 32x^3 + 24x^2 + 8x + 1) + C_2 = 2x^4 + 4x^3 + 3x^2 + x + \frac{1}{8} + C_2$.
(c) The answers to parts (a) and (b) are the same with $C_1 = \frac{1}{8} + C_2$.

56. Evaluate $\int x(x^2+2)^2 dx$ by two methods. (a) Expand $(x^2+2)^2$ and multiply the result by x; (b) let $u = x^2 + 2$. (c) Explain the difference in appearance of the answers obtained in (a) and (b).

▸ (a) $\int x(x^2+2)^2 dx = \int x(x^4 + 4x^2 + 4)dx = \int (x^5 + 4x^3 + 4x)dx = \frac{1}{6}x^6 + x^4 + 2x^2 + C_1$
(b) If $u = x^2 + 2$ then $du = 2x\ dx$. $\int (x^2+2)^2(x\ dx) = \int u^2(\frac{1}{2}\ du) = \frac{1}{6}u^3 + C_2 = \frac{1}{6}(x^2+2)^3 + C_2$
$= \frac{1}{6}(x^6 + 6x^4 + 12x^2 + 8) + C_2 = \frac{1}{6}x^6 + x^4 + 2x^2 + \frac{4}{3} + C_2$.
(c) The answers to parts (a) and (b) are the same with $C_1 = \frac{4}{3} + C_2$.

57. (a) $\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}dx = \int (x - 2x^{1/2} + 1)x^{-1/2}dx = \int (x^{1/2} - 2 + x^{-1/2})dx = \frac{2}{3}x^{3/2} - 2x + 2x^{1/2} + C_1$
(b) Let $u = \sqrt{x} - 1$; $du = \frac{1}{2}x^{-1/2}dx$. $\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}dx = 2\int u^2 du = \frac{2}{3}u^3 + C_2 = \frac{2}{3}(\sqrt{x}-1)^3 + C_2$
$= \frac{2}{3}(x^{3/2} - 3x + 3x^{1/2} - 1) + C_2 = \frac{2}{3}x^{3/2} - 2x + 2x^{1/2} - \frac{2}{3} + C_2$
(c) The answers in (a) and (b) are the same with $C_1 = -\frac{2}{3} + C_2$.

58. (a) If $u = x - 1$, then $du = dx$ and $x = u + 1$, so $\int \sqrt{x-1}\,x^2\,dx = \int \sqrt{u}\,(u+1)^2\,du = \int u^{1/2}(u^2 + 2u + 1)\,du$
$= \int (u^{5/2} + 2u^{3/2} + u^{1/2})\,du = \frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C = \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$
(b) If $v = \sqrt{x-1}$, then $x = v^2 + 1$ and $dx = 2v$, so $\int \sqrt{x-1}\,x^2\,dx = \int v(v^2+1)^2(2v)\,dv$
$= \int (2v^6 + 4v^4 + 2v^2)\,dv = \frac{2}{7}v^7 + \frac{4}{5}v^5 + \frac{2}{3}v^3 + C = \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$

59. $\int 2\sin x\cos x\,dx$ (a) Let $u = \sin x$; $du = \cos x\,dx$. $\int 2\sin x\cos x\,dx = 2\int u\,du = u^2 + C_1 = \sin^2 x + C_1$
(b) Let $u = \cos x$; $du = -\sin x\,dx$. $\int 2\sin x\cos x\,dx = -2\int u\,du = -u^2 + C_2 = -\cos^2 x + C_2 = \sin^2 x - 1 + C_2$
(c) $\int 2\sin x\cos x\,dx = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C_3 = -\frac{1}{2}(1 - 2\sin^2 x) + C_3 = \sin^2 x - \frac{1}{2} + C_3$
(d) The answers in (a), (b) and (c) are the same with $C_2 = C_1 + 1$ and $C_3 = C_1 + \frac{1}{2}$.

60. Evaluate $\int \csc^2 x\cot x\,dx$ by two methods: (a) Let $u = \cot x$; (b) let $v = \csc x$. (c) Explain the difference in appearance of the answers obtained in (a) and (b).
(a) If $u = \cot x$ then $du = -\csc^2 x\,dx$ and so $\int \cot x\csc^2 x\,dx = \int u(-du) = -\frac{1}{2}u^2 + C_1 = -\frac{1}{2}\cot^2 x + C_1$.
(b) If $v = \csc x$ then $dv = -\csc x\cot x\,dx$ so $\int \csc x(\csc x\cot x\,dx) = \int v(-dv) = -\frac{1}{2}v^2 + C_2 = -\frac{1}{2}\csc^2 x + C_2$
(c) The difference between the solutions is $\frac{1}{2}(\csc^2 x - \cot^2 x) + C_1 - C_2 = \frac{1}{2} + C_1 - C_2$, a constant.

4.3 DIFFERENTIAL EQUATIONS AND RECTILINEAR MOTION

A *first-order differential equation* in the two variables x and y is an equation involving x, y, and dy/dx. A solution of this differential equation is a function of x which when substituted for y in the differential equation yields an identity. The following steps are used to find a solution of a first-order differential equation if we can separate the variables.

1. *Separate the variables.* That is, write the differential equation in the form

   $$g(y)\,dy = f(x)\,dx$$

2. Antidifferentiate on both sides of the differential equation. This results in an equation of the form

   $$G(y) = F(x) + C$$

   where $G'(y) = g(y)$, $F'(x) = f(x)$, and c is an arbitrary constant. This equation is called the *complete solution* of the differential equation.
3. If *initial conditions* are given (that is, replacements for x and y are given), substitute the given values into the complete solution, solve for c and substitute this value for c in the complete solution. This results in the *particular solution* of the differential equation.

If feasible, the equation should be solved for y. If the differential equation has the simple form $dy/dx = f(x)$, step 1 is not necessary. The steps for finding the complete solution are illustrated in Exercises 4 and 8, and the the steps for finding the particular solution are illustrated in Exercise 16.

A *second-order differential equation* in x and y is an equation involving x, y, dy/dx, and d^2x/dy^2. The only type we consider is of the form

$$\frac{d^2y}{dx^2} = f(x)$$

This is a first-order differential equation in x and y' whose solution may be found by steps 2 and 3 above. The solution is an equation of the form $y = F(x)$. We antidifferentiate to get y as a function of x. There will be two arbitrary constants in the complete solution of a second-order differential equation. The steps for finding the complete solution are illustrated in Exercise 12, and the steps for finding the particular solution are illustrated in Exercise 20.

If a particle moves in a straight line and if its directed distance from the origin is s units of distance at t units of time, then the units of velocity v and the units of acceleration a are given by

$$v = \frac{ds}{dt} \tag{1}$$

$$a = \frac{d^2s}{dt^2} = \frac{dv}{dt} \tag{2}$$

or

$$a = \frac{dv}{ds}\cdot\frac{ds}{dt} = v\cdot\frac{dv}{ds} \tag{3}$$

We use Equations (1) and (2) when we are given the acceleration and wish to find the time, distance, and velocity, as illustrated in Exercises 24 and 28. We use Eq. (3) when we are not interested in time, as illustrated in Exercise 32. If the initial velocity is v_0, we solve Eq. (3) in Exercise 48 to find that

$$2as = v^2 - {v_0}^2$$

If the only force acting on a particle is the force of gravity, then the acceleration is constant, and is approximately 32 ft/sec^2.

Exercises 4.3

In Exercises 1–14, find the complete solution of the differential equation.

1. $\frac{dy}{dx} = 4x - 5$; $y = \int (4x-5)dx = 2x^2 - 5x + C$
2. $\frac{dy}{dx} = 6 - 3x^2$; $y = \int (6-3x^2)dx = 6x - x^3 + C$
3. $\frac{dy}{dx} = 3x^2 + 2x - 7$; $y = \int (3x^2+2x-7)dx = x^3 + x^2 - 7x + C$

4. $\frac{ds}{dt} = 5\sqrt{s}$

▸ First, we separate the variables.

$$\frac{ds}{\sqrt{s}} = 5\,dt$$

Next, we antidifferentiate on both sides.

$$\int \frac{ds}{\sqrt{s}} = \int 5\,dt$$

$$\int s^{-1/2}\,ds = 5\int dt$$

$$2s^{1/2} = 5t + C$$

$$s = \left(\frac{5t + C}{2}\right)^2 \qquad (1)$$

✓ Note that for any value of c, Eq. (1) yields

$$\frac{ds}{dt} = 2\left(\frac{5t + C}{2}\right)\frac{5}{2} = 5\left(\frac{5t + C}{2}\right) = 5\sqrt{s}$$

Thus Eq. (1) is the complete solution of the differential equation.

5. $\frac{dy}{dx} = 3xy^2$; $\frac{dy}{y^2} = 3x\,dx$; $\int y^{-2}dy = 3\int x\,dx$; $-\frac{1}{y} = \frac{3}{2}x^2 + \frac{1}{2}C$; $y = \frac{-2}{3x^2 + C}$

6. $\frac{dy}{dx} = \frac{\sqrt{x} + x}{\sqrt{y} - y}$; $(\sqrt{y} - y)dy = (\sqrt{x} + x)dx$; $\int (y^{1/2} - y)dy = \int (x^{1/2} + x)dx$; $\frac{2}{3}y^{3/2} - \frac{1}{2}y^2 = \frac{2}{3}x^{3/2} + \frac{1}{2}x^2 + C$

7. $\frac{du}{dv} = \frac{3v\sqrt{1 + u^2}}{u}$; $\frac{u\,du}{\sqrt{1 + u^2}} = 3v\,dv$; $-\int (1 + u^2)^{-1/2}(2u\,du) = 3\int v\,dv$; $(1 + u^2)^{1/2} = \frac{3}{2}v^2 + \frac{1}{2}C$;
$2\sqrt{1 + u^2} = 3v^2 + C$

8. $\frac{dy}{dx} = \frac{x^2\sqrt{x^3 - 3}}{y^2}$

▸ We separate the variables and then antidifferentiate on both sides.

$$y^2\,dy = x^2\sqrt{x^3 - 3}\,dx$$

$$\int y^2\,dy = \int x^2\sqrt{x^3 - 3}\,dx$$

Because $d(x^3 - 3) = 3x^2\,dx$, then

$$\frac{1}{3}y^3 = \frac{1}{3}\int (x^3 - 3)^{1/2}(3x^2\,dx) = \frac{1}{3}\left(\frac{2}{3}\right)(x^3 - 3)^{3/2} + C_1$$

$$y^3 = \frac{2}{3}(x^3 - 3)^{3/2} + 3C_1$$

Replacing $3C_1$ by C, since C is any constant, we have

$$y^3 = \frac{2}{3}(x^3 - 3)^{3/2} + C$$

$$y = [\frac{2}{3}(x^3 - 3)^{3/2} + C]^{1/3}$$

which is the general solution.

9. $\frac{dy}{dx} = \frac{\sec^2 x}{\tan^2 y}$; $\tan^2 y\,dy = \sec^2 x\,dx$; $\int (\sec^2 y - 1)dy = \int \sec^2 x\,dx$; $\tan y - y = \tan x + C$

10. $\frac{du}{dv} = \frac{\cos 2v}{\sin 3u}$; $\sin 3u\,du = \cos 2v\,dv$; $\int \sin 3u\,du = \int \cos 2v\,dv$; $-\frac{1}{3}\cos 3u = \frac{1}{2}\sin 2v + C$

11. $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = 5x^2 + 1$; $dy' = (5x^2 + 1)dx$; $\int dy' = \int (5x^2 + 1)dx$; $y' = \frac{dy}{dx} = \frac{5}{3}x^3 + x + C_1$;
$\int dy = \int (\frac{5}{3}x^3 + x + C_1)dx$; $y = \frac{5}{12}x^4 + \frac{1}{2}x^2 + C_1x + C_2$

12. $\dfrac{d^2y}{dx^2} = \sqrt{2x-3}$

Let $y' = dy/dx$. Then $d^2y/dx^2 = dy'/dx$ and we have

$$\frac{dy'}{dx} = \sqrt{2x-3}$$

$$y' = \int \sqrt{2x-3}\, dx$$

Because $d(2x-3) = 2\, dx$, then

$$y' = \frac{1}{2}\int (2x-3)^{1/2}(2\, dx) = \frac{1}{2}\cdot\frac{2}{3}(2x-3)^{3/2} + C_1$$

We antidifferentiate again to get

$$y = \frac{1}{3}\int (2x-3)^{3/2}\, dx + C_1 \int dx$$

As before, $d(2x-3) = 2\, dx$, and so

$$y = \frac{1}{3}\cdot\frac{1}{2}\int (2x-3)^{3/2}(2\, dx) + C_1\int dx = \tfrac{1}{6}\cdot\tfrac{2}{5}(2x-3)^{5/2} + C_1x + C_2 = \tfrac{1}{15}(2x-3)^{5/2} + C_1x + C_2$$

13. $\dfrac{d^2s}{dt^2} = \dfrac{ds'}{dt} = \sin 3t + \cos 3t$; $ds' = (\sin 3t + \cos 3t)dt$; $\int ds' = \int (\sin 3t + \cos 3t)dt$

$s' = \dfrac{ds}{dt} = -\frac{1}{3}\cos 3t + \frac{1}{3}\sin 3t + C_1$; $ds = (-\frac{1}{3}\cos 3t + \frac{1}{3}\sin 3t + C_1)dt$;

$\int ds = \int (-\frac{1}{3}\cos 3t + \frac{1}{3}\sin 3t + C_1)dt$; $s = -\frac{1}{9}\sin 3t - \frac{1}{9}\cos 3t + C_1t + C_2$

14. $\dfrac{d^2u}{dv^2} = \dfrac{du'}{dv} = \tan v \sec^2 v$; $u' = \int \tan v(\sec^2 v\, dv) = \frac{1}{2}\tan^2 v + C = \frac{1}{2}\sec^2 v + C_1$ where $C_1 = C - \frac{1}{2}$.

$u = \int (\frac{1}{2}\sec^2 v + C_1)dv = \frac{1}{2}\tan v + C_1v + C_2$

Exercises 15–20, find the particular solution of the differential equation determined by the initial conditions.

15. $\dfrac{dy}{dx} = x^2 - 2x - 4$; $\int dy = \int (x^2 - 2x - 4)dx$; $y = \frac{1}{3}x^3 - x^2 - 4x + C$

Because $y = -6$ when $x = 3$, $-6 = 9 - 9 - 12 + C$; $C = 6$. Therefore, $y = \frac{1}{3}x^3 - x^2 - 4x + 6$.

16. $\dfrac{dy}{dx} = (x+1)(x+2)$; $y = -\frac{3}{2}$ when $x = -3$

$$y = \int (x^2 + 3x + 2)\, dx = \tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 2x + C \tag{1}$$

We substitute the given replacements for x and y into Eq. (1).

$$-\tfrac{3}{2} = \tfrac{1}{3}(-3)^3 + \tfrac{3}{2}(-3)^2 + 2(-3) + C; \quad C = 0$$

Replacing C by 0 in Eq. (1), we obtain the particular solution

$$y = \tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 2x$$

17. $\dfrac{dy}{dx} = \dfrac{\cos 3x}{\sin 2y}$; $\int \sin 2y\, dy = \int \cos 3x\, dx$; $-\frac{1}{2}\cos 2y = \frac{1}{3}\sin 3x + C$. Because $y = \frac{1}{3}\pi$ when $x = \frac{1}{2}\pi$,

$-\frac{1}{2}\cos\frac{2}{3}\pi = \frac{1}{3}\sin\frac{3}{2}\pi + C$; $-\frac{1}{2}(-\frac{1}{2}) = \frac{1}{3}(-1) + C$; $C = \frac{7}{12} \Rightarrow -\frac{1}{2}\cos 2y = \frac{1}{3}\sin 3x + \frac{7}{12}$; $\cos 2y = -\frac{2}{3}\sin 3x - \frac{7}{6}$

18. $\dfrac{ds}{dt} = \cos\frac{1}{2}t$; $s = \int \cos\frac{1}{2}t\, dt = 2\sin\frac{1}{2}t + C$. Because $s = 3$ when $t = \frac{1}{3}\pi$ then $3 = 2\sin\frac{1}{6}\pi + C = 1 + C$; $C = 2$.

Therefore $s = 2\sin\frac{1}{2}t + 2$.

19. $\dfrac{d^2u}{dv^2} = 4(1+3v)^2$; $u = -1$ and $\dfrac{du}{dv} = -2$ when $v = -1$.

$\dfrac{du'}{dv} = 4(1+3v)^2 = 4(9v^2 + 6v + 1)$; $u' = 4\int (9v^2 + 6v + 1)dv = 4(3v^3 + 3v^2 + v) + C_1$.

Because $u' = -2$ when $v = -1$, $-2 = 4(-1) + C_1$; $C_1 = 2$. Therefore,

$\dfrac{du}{dv} = 12v^3 + 12v^2 + 4v + 2$; $u = \int (12v^3 + 12v^2 + 4v + 2)dv = 3v^4 + 4v^3 + 2v^2 + 2v + C_2$.

Because $u = -1$ when $v = -1$, $-1 = 3 - 4 + 2 - 2 + C_2$; $C_2 = 0$. Therefore, $u = 3v^4 + 4v^3 + 2v^2 + 2v$.

20. $\dfrac{d^2y}{dx^2} = -\dfrac{3}{x^4}$; $y = \frac{1}{2}$ and $\dfrac{dy}{dx} = -1$ when $x = 1$

▸ $y' = dy/dx$, then $dy'/dx = d^2y/dx^2$, and we have

$$\frac{dy'}{dx} = -\frac{3}{x^4}$$
$$y' = \int -3x^{-4}\,dx = x^{-3} + C_1$$

Because $y' = -1$ when $x = 1$, then

$$-1 = 1 + C_1$$

Thus $c_1 = -2$ and

$$y' = x^{-3} - 2$$
$$y = \int (x^{-3} - 2)\,dx = -\tfrac{1}{2}x^{-2} - 2x + C_2$$

Because $y = \frac{1}{2}$ when $x = 1$, then

$$\tfrac{1}{2} = -\tfrac{1}{2}(1^{-2}) - 2(1) + C_2$$

Thus $c_2 = 3$ and

$$y = -\tfrac{1}{2}x^{-2} - 2x + 3$$

In Exercises 21–32, a particle is moving on a line; at t seconds, s feet is its directed distance from the origin, v feet per second is its velocity, and a feet per second per second is its acceleration. In Exercise 29–32, relate v and s.

21. $v = \dfrac{ds}{dt} = \sqrt{2t+4}$; $s = \frac{1}{2}\displaystyle\int (2t+4)^{1/2}(2\,dt)$; $s = \frac{1}{3}(2t+4)^{3/2} + C$.
Because $s = 0$ when $t = 0$, $0 = \frac{1}{3}(4)^{3/2} + C$; $C = -\frac{8}{3}$. Therefore, $s = \frac{1}{3}(2t+4)^{3/2} - \frac{8}{3}$.

22. $v = \dfrac{ds}{dt} = 4 - t$; $s = \displaystyle\int (4-t)dt = 4t - \tfrac{1}{2}t^2 + C$. Because $s = 0$ when $t = 2$, $0 = 4 \cdot 2 - \frac{1}{2} \cdot 2^2 + C = 6 + C$, $C = -6$.
Therefore $s = 4t - \frac{1}{2}t^2 - 6$

23. $a = 5 - 2t$; $v = 2$ and $s = 0$ when $t = 0$.

▸ $\dfrac{dv}{dt} = 5 - 2t$; $dv = (5-2t)dt$; $v = \displaystyle\int (5-2t)dt = 5t - t^2 + C_1$. Because $v = 2$ when $t = 0$, $2 = C_1$.
Therefore, $v = \dfrac{ds}{dt} = 5t - t^2 + 2$; $\displaystyle\int ds = \int (5t - t^2 + 2)dt$; $s = \frac{5}{2}t^2 - \frac{1}{3}t^3 + 2t + C_2$.
Because $s = 0$ when $t = 0$, we obtain $C_2 = 0$. Therefore, $s = 2t + \frac{5}{2}t^2 - \frac{1}{3}t^3$.

24. $a = 17$; $v = 0$ and $s = 0$ when $t = 0$. Express v and s in terms of t.

▸ Because $a = dv/dt$, we are given

$$\frac{dv}{dt} = 17$$
$$v = \int 17\,dt = 17t + C_1$$

Because $v = 0$ when $t = 0$, then $c_1 = 0$, and thus

$$v = 17t \qquad (1)$$

Because $v = ds/dt$, then

$$\frac{ds}{dt} = 17t$$
$$s = \int 17t\,dt = \tfrac{17}{2}t^2 + C_2$$

Because $s = 0$ when $t = 0$, then $c_2 = 0$, and

$$s = \tfrac{17}{2}t^2 \qquad (2)$$

In Equations (1) and (2), v and s are expressed in terms of t.

25. $a = t^2 + 2t$; $s = 1$ when $t = 0$ and $s = -3$ when $t = 2$.

► $\frac{dv}{dt} = t^2 + 2t$; $v = \int (t^2 + 2t)dt$; $\frac{ds}{dt} = \frac{1}{3}t^3 + t^2 + C_1$; $s = \int \left(\frac{1}{3}t^3 + t^2 + C_1\right)dt = \frac{1}{12}t^4 + \frac{1}{3}t^3 + C_1 t + C_2$.
Because $s = 1$ when $t = 0$, $C_2 = 1$. Because $s = -3$ when $t = 2$, $-3 = \frac{4}{3} + \frac{8}{3} + 2C_1 + 1$; $C_1 = -4$.
Therefore, $s = \frac{1}{12}t^4 + \frac{1}{3}t^3 - 4t + 1$ and $v = \frac{1}{3}t^3 + t^2 - 4$.

26. $a = 3t - t^2$; $v = \frac{7}{6}$ and $s = 1$ when $t = 1$.

► $\frac{dv}{dt} = 3t - t^2$; $v = \int (3t - t^2)dt = \frac{3}{2}t^2 - \frac{1}{3}t^3 + C_1$. Because $v = \frac{7}{6}$ when $s = 1$, $\frac{7}{6} = \frac{3}{2} - \frac{1}{3} + C_1$, $C_1 = 0$.
$v = \frac{ds}{dt} = \frac{3}{2}t^2 - \frac{1}{3}t^3$; $s = \int (\frac{3}{2}t^2 - \frac{1}{3}t^3)dt = \frac{1}{2}t^3 - \frac{1}{12}t^4 + C_2$. Because $s = 1$ when $t = 1$, $1 = \frac{1}{2} - \frac{1}{12} + C_2$; $C_2 = \frac{7}{12}$.
$s = \frac{1}{2}t^3 - \frac{1}{12}t^4 + \frac{7}{12}$

27. $a = -4\sqrt{2}\cos(2t - \frac{1}{4}\pi)$; $v = 2$ and $s = 1$ when $t = 0$.

$\frac{dv}{dt} = -4\sqrt{2}\cos(2t - \frac{1}{4}\pi)$; $v = -2\sqrt{2}\int \cos(2t - \frac{1}{4}\pi)(2\,dt) = -2\sqrt{2}\sin(2t - \frac{1}{4}\pi) + C_1$
Because $v = 2$ when $t = 0$, $2 = -2\sqrt{2}\sin(-\frac{1}{4}\pi) + C_1$; $2 = -2\sqrt{2}(-\frac{1}{2}\sqrt{2}) + C_1$; $C_1 = 0$.
Therefore, $v = \frac{ds}{dt} = -2\sqrt{2}\sin(2t - \frac{1}{4}\pi)$; $s = -\sqrt{2}\int \sin(2t - \frac{1}{4}\pi)(2\,dt) = \sqrt{2}\cos(2t - \frac{1}{4}\pi) + C_2$.
Since $s = 1$ when $t = 0$, $1 = \sqrt{2}\cos(-\frac{1}{4}\pi) + C_2$; $1 = \sqrt{2}(\frac{1}{2}\sqrt{2}) + C_2$; $C_2 = 0$. Thus $s = \sqrt{2}\cos(2t - \frac{1}{4}\pi)$.

28. $a = 18\sin 3t$; $v = -6$ and $s = 4$ when $t = 0$. Express v and s in terms of t.

► Because $a = dv/dt$, we are given

$$\frac{dv}{dt} = 18\sin 3t$$
$$v = \int 18\sin 3t\,dt = -6\cos 3t + C_1$$

Because $v = -6$ when $t = 0$, then

$$-6 = -6\cos 0 + C_1$$

Thus, $c_1 = 0$, and v is expressed terms of t by

$$v = -6\cos 3t$$

Because $v = ds/dt$, then

$$\frac{ds}{dt} = -6\cos 3t$$
$$s = \int -6\cos 3t\,dt = -2\sin 3t + C_2$$

Because $s = 4$ when $t = 0$, then $C_2 = 4$. Thus s is expressed in terms of t by

$$s = -2\sin 3t + 4$$

29. $a = 800$; $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$; $v\frac{dv}{ds} = 800$; $\int v\,dv = 800\int ds$; $\frac{1}{2}v^2 = 800s + C$. Because $v = 20$ when $s = 1$,
$\frac{1}{2}(400) = 800 + C$; $C = -600$. Therefore $\frac{1}{2}v^2 = 800s - 600$; $1600s = v^2 + 1200$.

30. $a = 500$; $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$; $v\frac{dv}{ds} = 500$; $\int v\,dv = 500\int ds$; $\frac{1}{2}v^2 = 500s + C$. Because $v = 10$ when $s = 5$,
$\frac{1}{2}(100) = 2500 + C$; $C = -2450$. Therefore $\frac{1}{2}v^2 = 500s - 2450$; $v^2 = 1000s - 4900$.

31. $a = 5s + 2$; $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$; $v\frac{dv}{ds} = 5s + 2$; $\int v\,dv = \int (5s + 2)ds$; $\frac{1}{2}v^2 = \frac{5}{2}s^2 + 2s + \frac{1}{2}C$; $v^2 = 5s^2 + 4s + C$
Because $v = 4$ when $s = 2$, $16 = 20 + 8 + C$; $C = -12$. Therefore, $v^2 = 5s^2 + 4s - 12$.

32. $a = 2s + 1$; $v = 2$ when $s = 1$. Find an equation involving v and s.

► Because $a = dv/dt$ and $v = ds/dt$, then

$$a = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = \frac{dv}{ds}\cdot v \qquad (1)$$

We substitute $a = 2s + 1$ in Eq. (1) and separate the variables.

$$\frac{dv}{ds}\cdot v = 2s + 1$$
$$v\,dv = (2s + 1)\,ds$$

$$\int v\,dv = \int (2s+1)\,ds$$
$$\tfrac{1}{2}v^2 = s^2 + s + C$$

Because $v = 2$ when $s = 1$, then

$$\tfrac{1}{2}\cdot 2^2 = 1^2 + 1 + C$$

Hence $C = 0$ and

$$\tfrac{1}{2}v^2 = s^2 + s$$
$$v = \sqrt{2s^2 + 2s}$$

In Exercises 35–43, the only force considered is that due to the acceleration of gravity, which we take as 32 ft/sec^2 or 9.8 m/sec^2 in the downward direction. See §2.7 for exact values of the sine and cosine.

33. $v = \frac{ds}{dt} = 9\sin 3\pi t$; $s = \int 9\sin 3\pi t\,dt = -\frac{3}{\pi}\cos 3\pi t + C$. Because $s = 0$ when $t = 0$, $0 = -\frac{3}{\pi} + C$; $C = \frac{3}{\pi}$. Hence $s = \frac{3}{\pi}(1 - \cos 3\pi t)$. (a) $s(0.6) = \frac{3}{\pi}(1 - \cos\frac{9}{5}\pi) = \frac{3}{4\pi}(3 - \sqrt{5}) \approx 0.1824$; (b) $s(2.5) = \frac{3}{\pi}(1 - \cos\frac{15}{2}\pi) = \frac{3}{\pi} \approx 0.9550$ (c) $s(4.8) = \frac{3}{\pi}[1 - \cos(14\pi + \frac{2}{5}\pi)] = \frac{3}{4\pi}(5 - \sqrt{5}) \approx 0.6598$; (d) $s(7.2) = \frac{3}{\pi}[1 - \cos(22\pi - \frac{2}{5}\pi)] = s(4.8)$

34. $v = \frac{ds}{dt} = 2\cos\frac{1}{2}\pi t$; $s = \int 2\cos\frac{1}{2}\pi t\,dt = \frac{4}{\pi}\sin\frac{1}{2}\pi t + C$. Because $s = 0$ when $t = 0$, $0 = C$. Hence $s = \frac{4}{\pi}\sin\frac{1}{2}\pi t$.
(a) $s(0.6) = \frac{4}{\pi}\sin\frac{3}{10}\pi = \frac{1}{\pi}(1 + \sqrt{5}) \approx 1.0301$ (b) $s(2.5) = \frac{4}{\pi}\sin\frac{5}{4}\pi = -\frac{2}{\pi}\sqrt{2} \approx -0.9003$
(c) $s(4.8) = \frac{4}{\pi}\sin(2\pi + \frac{2}{5}\pi) = \frac{1}{\pi}\sqrt{10 + 2\sqrt{5}} \approx 1.2109$ (d) $s(7.2) = \frac{4}{\pi}\sin(4\pi - \frac{2}{5}\pi) = -\frac{1}{\pi}\sqrt{10 + 2\sqrt{5}} \approx -1.2109$

35. At t sec, let s ft be the directed distance of the ball from the ground, and let v ft/sec be its velocity. The positive direction is upward. We have a table of boundary conditions where $\bar{t}$ sec and and $\bar{v}$ ft/sec are the time and velocity when the ball strikes the ground and T sec and S ft are the time and distance when the stone reaches its highest point. The acceleration of the ball is that due to gravity. Therefore $a = -32$. Thus

t	s	v
0	0	20
$\bar{t}$	0	$\bar{v}$
T	S	0

$\frac{dv}{dt} = -32$, $\int dv = -32\int dt$; $v = -32t + C_1$. Because $v = 20$ when $t = 0$, we obtain $C_1 = 20$. Hence

$$v = -32t + 20 \qquad (1)$$

$\frac{ds}{dt} = -32t + 20$; $\int ds = \int(-32t + 20)dt$; $s = -16t^2 + 20t + C_2$. Since $s = 0$ when $t = 0$, $C_2 = 0$. Thus

$$s = -16t^2 + 20t \qquad (2)$$

From (1) with $v = 0$ and $t = T$, we have $0 = -32T + 20$; $T = \frac{5}{8}$. (a) The ball will be going upward for $\frac{5}{8}$ sec.

From (2) with $t = \frac{5}{8}$ and $s = S$, we obtain $S = -16\cdot\frac{25}{64} + 20\cdot\frac{5}{8} = \frac{25}{4}$. (b) The ball will reach a height of $\frac{25}{4}$ ft.

From (2) with $s = 0$ and $t = \bar{t}$, we have $-16\bar{t}^2 + 20\bar{t} = 0$; $-4\bar{t}(4\bar{t} - 5) = 0$; $\bar{t} = \frac{5}{4}$.

(c) Therefore it will take $\frac{5}{4}$ sec for the ball to strike the ground.

From (1) with $\bar{t} = \frac{5}{4}$ and $v = \bar{v}$ we have $\bar{v} = -32(\frac{5}{4}) + 20 = -20$.

(e) Therefore the ball will strike the ground with a speed of 20 ft/sec.

36. At t sec, let s ft be the directed distance of the ball from the ground, and let v ft/sec be its velocity. The positive direction is upward. We have a table of boundary conditions where $\bar{t}$ sec and and $\bar{v}$ ft/sec are the time and velocity when the ball strikes the ground and T sec and S ft are the time and distance when the stone reaches its highest point. The acceleration of the ball is that due to gravity. Therefore $a = -32$. Thus

t	s	v
0	0	5
$\bar{t}$	0	$\bar{v}$
T	S	0

$\frac{dv}{dt} = -32$, $\int dv = -32\int dt$; $v = -32t + C_1$. Because $v = 5$ when $t = 0$, we obtain $C_1 = 5$. Hence

$$v = -32t + 5 \qquad (1)$$

$\frac{ds}{dt} = -32t + 5$; $\int ds = \int(-32t + 5)dt$; $s = -16t^2 + 5t + C_2$. Since $s = 0$ when $t = 0$, $C_2 = 0$. Thus

$$s = -16t^2 + 5t \qquad (2)$$

From (1) with $v = 0$ and $t = T$, we have $0 = -32T + 5$; $T = \frac{5}{32}$. (a) The ball will be going upward for $\frac{5}{32}$ sec.

From (2) with $t = \frac{5}{32}$ and $s = S$, we obtain $S = -16 \cdot \frac{25}{1024} + 20 \cdot \frac{5}{32} = \frac{175}{64}$.

(b) The ball will reach a height of $\frac{175}{64}$ ft.

From (2) with $s = 0$ and $t = \bar{t}$, we have $-16\bar{t}^2 + 5\bar{t} = 0$; $-\bar{t}(16\bar{t} - 5) = 0$; $\bar{t} = \frac{5}{16}$.

(c) Therefore it will take $\frac{5}{16}$ sec for the ball to strike the ground.

From (1) with $t = \frac{5}{16}$ and $v = \bar{v}$ we have $v = -32(\frac{5}{16}) + 5 = -5$.

(e) Therefore the ball will strike the ground with a speed of 5 ft/sec.

At t sec, let s ft be the directed distance of the ball from the top of the Washington Monument and let v ft/sec be its velocity. The positive direction is upward. We have a table of boundary conditions where $\bar{t}$ sec and $\bar{v}$ ft/sec are the time and velocity when the ball strikes the ground. The acceleration of the ball is that due to gravity. Thus $a = -32$. Therefore

t	s	v
0	0	0
$\bar{t}$	-555	$\bar{v}$

$\frac{dv}{dt} = -32$; $\int dv = -32 \int dt$; $v = -32t + C_1$

Because $v = 0$ when $t = 0$ then $C_1 = 0$. Hence

$v = -32t$; $\frac{ds}{dt} = -32t$; $\int ds = -32 \int t\, dt$; $s = -16t^2 + C_2$

Because $s = 0$ when $t = 0$ then $C_2 = 0$. Therefore $s = -16t^2$. With $s = -555$ and $t = \bar{t}$ we have

$16\bar{t}^2 = -555$; $\bar{t}^2 = \frac{555}{16}$; $t = \frac{1}{4}\sqrt{555} \approx 5.9$

(a) It will take $\frac{1}{4}\sqrt{555} \approx 5.9$ sec for the ball to reach the ground. In $v = -32t$, let

$t = \frac{1}{4}\sqrt{555}$ and $v = \bar{v}$. We obtain $v = -\frac{32}{4}\sqrt{555} = -8\sqrt{555} \approx -188.5$.

(b) The ball will strike the ground with a speed of $8\sqrt{555} \approx 188.5$ ft/sec.

At t sec, the ball is s ft above the ground and is falling at v ft/sec.
$v = -32t + v_0 = -32t - 64$. $s = -16t^2 + v_0t + s_0 = -16t^2 - 64t + 80$. $s = 0$ when
$0 = 16(t^2 + 4t - 5) = 16(t + 5)(t - 1)$; $t = 1$. $v(1) = -96$.

The ball takes (a) 1 second to reach the ground and (b) hit the ground at a speed of 96 ft/sec.

At t sec, let s ft be the directed distance of the binoculars from the ground, and let v ft/sec be its velocity. The positive direction is upward. We have a table of boundary conditions where $\bar{t}$ sec and $\bar{v}$ ft/sec are the time and velocity when the binoculars strike the ground. The acceleration of the binoculars is that due to gravity. Hence

t	s	v
0	150	10
$\bar{t}$	0	$\bar{v}$

$a = -32$; $\frac{dv}{dt} = -32$; $\int dv = -23 \int dt$; $v = -32t + C_1$. Because $v = 10$ when $t = 0$, $C_1 = 10$. Therefore

$$v = -32t + 10 \tag{1}$$

$\frac{ds}{dt} = -32t + 10$; $\int ds = \int (-32t + 10)dt$; $s = -16t^2 + 10t + C_2$. Because $s = 150$ when $t = 0$, $C_2 = 150$.

Thus $s = -16t^2 + 10t + 150$. With $s = 0$ and $t = \bar{t}$ we have $-16\bar{t}^2 + 10\bar{t} + 150 = 0$; $\bar{t} = \frac{5}{16}(1 + \sqrt{97}) \approx 3.4$.

(a) It will take $\frac{5}{16}(1 + \sqrt{97}) \approx 3.4$ sec for the binoculars to strike the ground.

In (1) let $t = \frac{5}{16}(1 + \sqrt{97})$ and $v = \bar{v}$. We have $v = -10(1 + \sqrt{97}) + 10 = -10\sqrt{97} \approx -98.5$.

(b) Thus the speed of the binoculars on impact is $10\sqrt{97} \approx 98.5$ ft/sec.

A stone is thrown vertically upward from the top of a house 60 ft above the ground with an initial velocity of 40 ft/sec. (a) How long will it take the stone to reach its greatest height, and (b) what is its greatest height? (c) How long will it take the stone to pass the top of the house on its way down, and (d) what is its velocity at that instant? (e) How long will it take the stone to strike the ground, and (f) with what velocity does it strike the ground?

t seconds after the stone was thrown, let

s feet be the distance of the stone above the ground

v feet per second be the velocity of the stone

Because the positive direction is chosen as upward, the acceleration due to gravity is $a = -32$. Because $a = dv/dt$, then

$$\frac{dv}{dt} = -32$$
$$v = \int -32\, dt = -32t + C_1$$

Because $v = 40$ when $t = 0$, then $c_1 = 40$ and

$$v = -32t + 40 \tag{1}$$

Because $v = ds/dt$, then

$$\frac{ds}{dt} = -32t + 40$$
$$s = \int (-32t + 40)\, dt = -16t^2 + 40t + C_2$$

Because the top of the house is 60 ft above the ground, then $s = 60$ when $t = 0$. Thus $c_2 = 60$ and

$$s = -16t^2 + 40t + 60 = -16[t^2 - \tfrac{5}{2}t + (\tfrac{5}{4})^2] + 60 + 25 = -16(t - \tfrac{5}{4})^2 + 85 \tag{2}$$

Therefore (a) the stone takes $\frac{5}{4}$ sec to reach its greatest height and (b) its greatest height is 85 ft.
(c) The stone passes the top of the house on its way down when $s = 60$. Then

$$60 = -16t^2 + 40t + 60$$
$$t(2t - 5) = 0$$

The only positive solution is $t = \frac{5}{2}$ so we conclude that it takes the stone $\frac{5}{2}$ sec to pass the top of the house on its way down.
(d) Replacing t by $\frac{5}{2}$ in Eq. (1), we have

$$v = -32(\tfrac{5}{2}) + 40 = -40$$

Thus, the velocity of the stone is -40 ft/sec at the instant it passes the top of the house on its way down. Its speed is the same as when it was thrown.
(e) The stone strikes the ground when $s = 2$. From Eq. (2) we obtain

$$0 = -16t^2 + 40t + 60$$
$$4t^2 - 10t - 15 = 0$$

The only positive root is

$$t = \frac{10 + \sqrt{340}}{8} = \frac{5 + \sqrt{85}}{4} \approx 3.55$$

Hence the stone takes about 3.55 sec to strike the ground. From Eq. (1),

$$v = -32\left(\frac{5 + \sqrt{85}}{4}\right) + 40 = -8\sqrt{85} \approx -73.8$$

(f) The velocity of the stone is about -73.8 ft/sec when it strikes the ground.

41. At t sec the boulder is s ft above your head and is falling at v ft/sec.
$v = -32t + v_0 = -32$, $s = -16t^2 + v_0 t + s_0 = -16t^2 + 200$. $s = 0$ when $-16t^2 + 200 = 0$; $t^2 = \frac{50}{4}$, $t = \frac{5}{2}\sqrt{2}$.
$v(\frac{5}{2}\sqrt{2}) = -80\sqrt{2}$. (a) You have about 3.54 seconds to act or (b) the boulder strikes at about 113.14 ft/sec.

42. At t sec, let s ft be the directed distance of the ball from the starting point, and let v ft/sec be its velocity. The positive direction is upward. We have a table of boundary conditions where $\bar{v}$ ft/sec is the velocity of the ball when it is 36 ft above the ground (or, equivalently, 16 ft above the starting point). The acceleration of the ball is that due to gravity. Hence

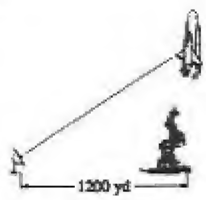
t	s	v
0	0	40
	16	$\bar{v}$

$a = -32$; $v\,\frac{dv}{ds} = -32$; $\int v\, dv = -32\int ds$; $\frac{1}{2}v^2 = -32s + C_1$.
(a) Because $v = 40$ when $s = 0$, we get $C_1 = 800$. Therefore $v^2 = -64s + 1600$.
Let $s = 16$ and $v = \bar{v}$. Because the ball is rising $\bar{v} > 0$. We have $\bar{v}^2 = -64(16) + 1600 = 576$; $\bar{v} = 24$. Therefore,
(b) the velocity of the ball is 24 ft/sec when it is 36 ft from the ground and rising.

43. $v = -9.8t + v_0 = -9.8t + 150$ (a) $s = -4.9t^2 + v_0 t + s_0 = -4.9t^2 + 150t + 2$ (b) $s(2) = 282.4$. The missile is 282.4 ft high. $-4.9t^2 + 150t + 2 = 500$; $4.9t^2 - 150t + 498 = 0$; $t_1 = \frac{1}{49}(750 - 4\sqrt{19905})$, $t_2 = \frac{1}{49}(750 - 4\sqrt{19905})$. (c) It will be 500 ft up after about 3.79 sec going up and 26.82 sec coming down.

44. (a) $v = 22t + v_0 = 22t$; $v(30) = 22(30) = 660$ m/sec (b) $s = 11t^2 + v_0 t + s_0 = 11t^2$; $s(30) = 11(30)^2 = 9900$ m

45. After t sec the shuttle climbs at v yd/sec, is s yd high and the angle of elevation of the dish is θ. $a = \frac{dv}{dt} = 10$; $v = \int 10\,dt = 10t + C_1$. Because $v = 0$ when $t = 0$, $0 = C_1$, $v = 10t$. $s = \int v\,dt = \int 10t\,dt = 5t^2 + C_2$. Because $s = 0$ when $t = 0$, $0 = C_2$, $s = 5t^2$. Because $s = 1200 \tan\theta$,
$v = \frac{ds}{dt} = 1200 \sec^2\theta \frac{d\theta}{dt} = 1200(\tan^2\theta + 1)\frac{d\theta}{dt}$. When $t = 8$, $v = 80$, $s = 320$, and $\tan\theta = \frac{320}{1200} = \frac{4}{15}$. Thus $80 = 1200[(\frac{4}{15})^2 + 1]\frac{d\theta}{dt}$; $\frac{d\theta}{dt} = \frac{15}{241}$.



- The dish is revolving at about 0.0622 rad/sec ≈ 3.57° deg/sec.

46. $a = \frac{dv}{dt} = -6$. $v = \int -6\,dt = -6t + C_1$. Because $v = 20$ when $t = 0$, $20 = C_1$; $v = \frac{ds}{dt} = -6t + 20$;
$s = \int (-6t + 20)dt = -3t^2 + 20t + C_2$. Because $s = 0$ when $t = 0$, $0 = C_2$; $s = -3t^2 + 20t$. The ball stops when $0 = -6t + 20$, $t = \frac{10}{3}$. Then $s = -3(\frac{10}{3})^2 + 20(\frac{10}{3}) = \frac{100}{3}$. The ball will roll $\frac{100}{3}$ feet.

47. $40\,\frac{\text{km}}{\text{hr}} = 40\,\frac{\text{km}}{\text{hr}} \cdot \frac{1000\text{m}}{1\text{km}} \cdot \frac{1\text{hr}}{3600\text{sec}} = \frac{100}{9}\,\frac{\text{m}}{\text{sec}}$; $100\,\frac{\text{km}}{\text{hr}} = 100\,\frac{\text{km}}{\text{hr}} \cdot \frac{1000\text{m}}{1\text{km}} \cdot \frac{1\text{hr}}{3600\text{sec}} = \frac{250}{9}\,\frac{\text{m}}{\text{sec}}$

Consider the starting point at the position of the automobile when its velocity is $\frac{100}{9}$ m/sec. At t sec, let s meters be its distance from the starting point, and let v m/sec be its velocity. Let the positive direction be the direction in which the automobile is traveling. We have a table of boundary conditions. Let A m/sec^2 be the constant acceleration.

t	s	v
0	0	$\frac{100}{9}$
	200	$\frac{250}{9}$

Then $A = v\frac{dv}{ds}$; $A\int ds = \int v\,dv$; $As = \frac{1}{2}v^2 + C$.
Because $v = \frac{100}{9}$ when $s = 0$, we have $0 = \frac{1}{2} \cdot \frac{10000}{81} + C$; $C = -\frac{5000}{81}$. Therefore $As = \frac{1}{2}v^2 - \frac{5000}{81}$.
Because $v = \frac{250}{9}$ when $s = 200$, we have $200A = \frac{1}{2}(\frac{250}{9})^2 - \frac{5000}{81} = \frac{26250}{81}$. $A = \frac{175}{108} \approx 1.62$

- The automobile should maintain a constant acceleration of 1.62 m/sec^2.

48. What constant negative acceleration will enable a driver to decrease the speed from 120 km/hr to 60 km/hr while traveling a distance of 100 meters?

- Let the driver be traveling east. Let s meters be the distance east of the origin t seconds after braking begins, and let v meters/sec be the velocity at this time. v_0 m/sec is his stating velocity and a m/sec^2 is the constant acceleration. As in Exercise 32, we have

$$a = v\frac{dv}{ds}$$

Thus,

$$a\,ds = v\,dv$$
$$\int a\,ds = \int v\,dv$$

Because a is a constant, the above is equivalent to

$$a\int ds = \int v\,dv$$
$$as = \frac{1}{2}v^2 + C$$

Because $v = v_0$ when $s = 0$,

$$0 = \frac{1}{2}{v_0}^2 + C$$
$$C = -\frac{1}{2}{v_0}^2$$
$$as = \frac{1}{2}v^2 - \frac{1}{2}{v_0}^2$$
$$2as = v^2 - {v_0}^2$$

Because $120\,\frac{\text{km}}{\text{hr}} = \frac{120\text{ km}}{\text{hr}} \cdot \frac{1\text{ hr}}{3600\text{ sec}} \cdot \frac{1000\text{ m}}{1\text{ km}} = \frac{100}{3}\,\frac{\text{m}}{\text{sec}}$, we are given that $v_0 = \frac{100}{3}$.

Because 60 km/hr $= \frac{50}{3}$ m/sec, we are given that $v = \frac{50}{3}$ when $s = 100$. Therefore,

$$2a(200) = (\tfrac{50}{3})^2 - (\tfrac{100}{3})^2 = -\tfrac{7500}{9}$$
$$a = -\tfrac{25}{6}$$

• The constant acceleration is $-\frac{25}{6}$ m/sec^2.

49. Consider the starting point at the position of the car when its velocity is $100\,\frac{\text{km}}{\text{hr}} = \frac{250}{9}\,\frac{\text{m}}{\text{sec}}$ (see Exercise 47). At t sec, let s meters be the distance of the car from the starting point, and let v m/sec be its velocity. Let the positive direction be the direction in which the car is traveling. In the table of boundary conditions $\bar{t}$ sec is the time and $\bar{s}$ meters is the distance of the car from the starting point when the car comes to a stop. The acceleration is -8 m/sec^2.

t	s	v
0	0	$\frac{250}{9}$
$\bar{t}$	$\bar{s}$	0

Hence $\frac{dv}{dt} = -8$; $\int dv = -8\int dt$; $v = -8t + C_1$. Because $v = \frac{250}{9}$ when $t = 0$, then $C_1 = \frac{250}{9}$. Therefore,

$$v = -8t + \tfrac{250}{9} \tag{1}$$

$\frac{ds}{dt} = -8t + \frac{250}{9}$; $\int ds = \int(-8t + \frac{250}{9})dt$; $s = -4t^2 + \frac{250}{9}t + C_2$. Since $s = 0$ when $t = 0$, $C_2 = 0$. Thus

$$s = -4t^2 + \tfrac{250}{9}t \tag{2}$$

In (1) let $v = 0$ and $t = \bar{t}$. We have $0 = -8\bar{t} + \frac{250}{9}$; $\bar{t} = \frac{125}{36} \approx 3.47$.

(a) Therefore it will take 3.47 sec for the car to come to a stop.

In (2) let $t = \frac{125}{36}$ and $s = \bar{s}$. Then $s = -4(\frac{125}{36})^2 + \frac{250}{9}\cdot\frac{125}{36} = \frac{15625}{324} \approx 48.22$

(b) Hence the car will travel 48.22 m before stopping.

50. $s = \frac{1}{2}at^2 + v_0t + s_0 = -2t^2 + 6t = -2[t^2 - 3t + (\frac{3}{2})^2] + \frac{9}{2} = \frac{9}{2} - 2(t - \frac{3}{2})^2$. The ball rises $\frac{9}{2}$ ft (in $\frac{3}{2}$ sec).

51. From $2as = v^2 - {v_0}^2$ we have $2(-8)25 = 0^2 - {v_0}^2$, ${v_0}^2 = 400$, $v_0 = 20$. Because

$20\,\frac{\text{m}}{\text{sec}} = 20\,\frac{\text{m}}{\text{sec}}\cdot\frac{3600\text{ sec}}{1\text{ hr}}\cdot\frac{1\text{ km}}{1000\text{ m}} = 72\,\frac{\text{km}}{\text{hr}}$, the greatest speed is 20 m/sec or, equivalently, 72 km/hr.

52. A block of ice slides down a chute with constant acceleration of 3 m/sec^2. The chute is 36 m long and it takes 4 sec for the ice to reach the bottom. (a) What is the initial velocity of the ice? (b) What is the speed of the ice after it has traveled 12 m? (c) How long does it take the ice to go the 12 m?

▸ After t sec, let s m be the distance the block slides down the chute and let v m/sec be the velocity of the ice. Because the acceleration is 3 m/sec^2, then

$$\frac{dv}{dt} = 3$$
$$v = \int 3\,dt = 3t + C_1 \tag{1}$$
$$\frac{ds}{dt} = 3t + C_1$$
$$s = \int (3t + C_1)\,dt = \tfrac{3}{2}t^2 + C_1t + C_2 \tag{2}$$

Because $s = 0$ when $t = 0$, then $C_2 = 0$. Furthermore, when $t = 4$, then $s = 36$. Substituting these values into Eq. (2), we get

$$36 = 24 + 4C_1; \qquad C_1 = 3$$

If we replace the values of C_1 and C_2 in Equations (1) and (2), we have

$$v = 3t + 3 \tag{3}$$
$$s = \tfrac{3}{2}t^2 + 3t \tag{4}$$

(a) Because $v = 3$ when $t = 0$, the initial velocity is 3 m/sec.

(c) To find the time it takes for the ice to travel 12 m, we let $s = 12$ in Eq. (4) and solve for t. Thus,

$$12 = \tfrac{3}{2}t^2 + 3t$$
$$0 = t^2 + 2t - 8 = (t - 2)(t + 4)$$

The only positive root is 2, so it takes the ice 2 sec to go 12 m.

(b) Substituting $t = 2$ in Eq. (3), we get $v = 9$, so the speed of the ice is 9 m/sec after it has traveled 12 m.

In Exercises 53 and 54, find the *orthogonal trajectory* of the family of curves, that is another family of curves such that at any point (x, y) there is a curve of each family through it and the the tangent lines to the two curves at this point are perpendicular.

53. The family of parabolas $x^2 = 4ay$; $2x = 4a\,\frac{dy}{dx}$; $\frac{dy}{dx} = \frac{x}{2}\frac{1}{a} = \frac{x}{2}\frac{4y}{x^2}$; $\frac{dy}{dx} = \frac{2y}{x}$. Hence at any point $P = (x, y)$ not on an axis, the slope of the tangent line to the parabola of the given family through P is $\frac{2y}{x}$ so that the slope of the tangent line to the curve of the family we seek through P must be $-\frac{x}{2y}$. Thus
$\frac{dy}{dx} = -\frac{x}{2y}$; $2y\,dy = -x\,dx$; $\int 2y\,dy = -\int x\,dx$; $y^2 = -\frac{1}{2}x^2 + C$; $x^2 + 2y^2 = C$, a family of ellipses.

54. The family $x^3 + y^3 = a^3$; $3x^2 + 3y^2\frac{dy}{dx} = 0$; $\frac{dy}{dx} = -\frac{x^2}{y^2}$. Hence at any point on the orthogonal trajectory $\frac{dy}{dx} = \frac{y^2}{x^2}$; $y^{-2}dy = x^{-2}dx$; $\int y^{-2}dy = \int x^{-2}dx$; $-\frac{1}{y} = -\frac{1}{x} + C$; $y - x = Cxy$, a family of hyperbolas.

4.4 AREA

4.4.1 Definition Sigma Notation If m and n are integers with $m \le n$ and F is a function defined for all integers in $[m, n]$, then

$$\sum_{i=m}^{n} F(i) = F(m) + F(m+1) + F(m+2) + \cdots + F(n-1) + F(n)$$

The general properties given by the following theorems should be memorized.

4.4.2 Theorem $\sum_{i=1}^{n} c = cn$, where c is any constant

4.4.3 Theorem $\sum_{i=1}^{n} c \cdot F(i) = c\sum_{i=1}^{n} F(i)$, where c is any constant

4.4.4 Theorem $\sum_{i=1}^{n} [F(i) + G(i)] = \sum_{i=1}^{n} F(i) + \sum_{i=1}^{n} G(i)$

4.4.5 Theorem $\sum_{i=a}^{b} F(i) = \sum_{i=a+c}^{b+c} F(i-c)$

$\sum_{i=a}^{b} F(i) = \sum_{i=a-c}^{b-c} F(i+c)$

4.4.6 Theorem Telescoping Sum $\sum_{i=1}^{n} [F(i) - F(i-1)] = F(n) - F(0)$

4.4.7 Theorem If n is a positive integer, then

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{(Formula 1)}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \text{(Formula 2)}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \qquad \text{(Formula 3)}$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \qquad \text{(Formula 4)}$$

Note. $\sum_{i=1}^{n} (i-1)^p = 0^p + 1^p + \cdots + (n-1)^p = 1^p + \cdots + (n-1)^p + n^p - n^p = \sum_{i=1}^{n} i^p - n^p$

4.4.8 Definition Suppose that the function f is continuous on the closed interval $[a, b]$, with $f(x) \ge 0$ for all x in $[a, b]$, and that R is the region bounded by the curve $y = f(x)$, the x axis, and the lines $x = a$ and $x = b$. Divide the interval $[a, b]$ into n subintervals, each of length $\Delta x = (b-a)/n$, and let c_i be a number in the ith subinterval $[x_{i-1}, x_i]$. Then the measure of the area of the region R is given by

$$A = \lim_{n \to +\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

where $f(c_i)$ is the altitude of the ith rectangle. In the method of *inscribed rectangles*, $f(c_i)$ is the absolute minimum function value on the ith subinterval, and in the method of *circumscribed rectangles* $f(c_i)$ is the absolute maximum function value on the ith subinterval. Actually, the limit is the same no matter where c_i is in the ith interval.

Exercises 4.4

In Exercises 1–20, find the sum. Use Theorems 4.4.2–4.4.7 when appropriate.

1. $\sum_{i=1}^{6} (3i - 2) = 3 \sum_{i=1}^{6} i - \sum_{i=1}^{6} 2 = 3 \cdot \frac{6 \cdot 7}{2} - 6 \cdot 2 = 63 - 12 = 51$

2. $\sum_{i=1}^{20} (5i + 4) = 5 \sum_{i=1}^{20} i + \sum_{i=1}^{20} 4 = 5 \cdot \frac{20 \cdot 21}{2} + 20 \cdot 4 = 1130$

3. $\sum_{i=1}^{7} (i^2 + 1) = \sum_{i=1}^{7} i^2 + \sum_{i=1}^{7} 1 = \frac{7 \cdot 8(2 \cdot 7 + 1)}{6} + 7 \cdot 1 = 140 + 7 = 147$

4. $\sum_{i=1}^{7} (i + 1)^2$

▹ We replace i successively with 1, 2, …, 7 and add. Thus

$$\sum_{i=1}^{7} (i + 1)^2 = (1 + 1)^2 + (2 + 1)^2 + (3 + 1)^2 + (4 + 1)^2 + (5 + 1)^2 + (6 + 1)^2 + (7 + 1)^2$$
$$= 4 + 9 + 36 + 49 + 64 = 203$$

5. $\sum_{i=1}^{10} (i - 1)^3 = \sum_{i=0}^{9} i^3 = 0 + \sum_{i=0}^{9} i^3 = \frac{9^2 \cdot 10^2}{4} = \frac{81 \cdot 100}{4} = 2025$

6. $\sum_{i=1}^{10} (i^3 - 1) = \sum_{i=1}^{10} i^3 - \sum_{i=1}^{10} 1 = \frac{10^2 \cdot 11^2}{4} - 10 \cdot 1 = 3015$

7. $\sum_{i=2}^{5} \frac{i}{i - 1} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} = \frac{73}{12}$

8. $\sum_{j=3}^{6} \frac{2}{j(j - 2)}$

▹ We replace j successively with 3, 4, 5, 6 and add. Thus

$$\sum_{j=3}^{6} \frac{2}{j(j - 2)} = \frac{2}{3(3 - 2)} + \frac{2}{4(4 - 2)} + \frac{2}{5(5 - 2)} + \frac{2}{6(6 - 2)} = \frac{2}{3} + \frac{2}{8} + \frac{2}{15} + \frac{2}{24} = \frac{17}{15}$$

9. $\sum_{i=-2}^{3} 2^i = 2^{-2} + 2^{-1} + 2^0 + 2^1 + 2^2 + 2^3 = \frac{1}{4} + \frac{1}{2} + 1 + 2 + 4 + 8 = \frac{63}{4}$

10. $\sum_{i=0}^{3} \frac{1}{1 + i^2} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} = \frac{9}{5}$

11. $\sum_{k=1}^{4} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$

12. $\sum_{k=-2}^{3} \frac{k}{k + 3}$

▹ $$\sum_{k=-2}^{3} \frac{k}{k + 3} = \frac{-2}{-2 + 3} + \frac{-1}{-1 + 3} + \frac{0}{0 + 3} + \frac{1}{1 + 3} + \frac{2}{2 + 3} + \frac{3}{3 + 3} = -2 - \frac{1}{2} + 0 + \frac{1}{4} + \frac{2}{5} + \frac{1}{2} = -\frac{27}{20}$$

13. $\sum_{i=1}^{25} 2i(i - 1) = 2 \sum_{i=1}^{25} i^2 - 2 \sum_{i=1}^{25} i = 2 \cdot \frac{25 \cdot 26(2 \cdot 25 + 1)}{6} - 2 \cdot \frac{25 \cdot 26}{2} = 10{,}400.$

14. $\sum_{i=1}^{20} 3i(i^2 + 2) = \sum_{i=1}^{20} (3i^3 + 6i) = 3 \sum_{i=1}^{20} i^3 + 6 \sum_{i=1}^{20} = 3\left(\frac{20 \cdot 21}{2}\right)^2 + 6 \cdot \frac{20 \cdot 21}{2} = 133{,}560$

15. $\sum_{k=1}^{n} (2^k - 2^{k-1}) = 2^n - 2^0 = 2^n - 1$, using Theorem 4.4.6 with $F(k) = 2^k$.

16. $\sum_{i=1}^{n} (10^{i+1} - 10^i)$

▹ We apply Theorem 4.4.6 with $F(i) = 10^{i+1}$. Thus,

$$\sum_{i=1}^{n} (10^{i+1} - 10^i) = 10^{n+1} - 10$$

17. $\sum_{k=1}^{100} \left[\frac{1}{k} - \frac{1}{k + 1}\right] = -\sum_{k=1}^{100} \left[\frac{1}{k + 1} - \frac{1}{k}\right] = -\left[\frac{1}{101} - \frac{1}{1}\right] = \frac{100}{101}$ using Theorem 4.4.6 with $F(k) = \frac{1}{k + 1}$.

18. $\sum_{i=1}^{n} 2i(1+i^2) = \sum_{i=1}^{n}(2i+2i^3) = 2\sum_{i=1}^{n} i + 2\sum_{i=1}^{n} i^3 = 2\cdot\frac{n(n+1)}{2} + 2\cdot\frac{n^2(n+1)^2}{4} = \frac{1}{2}n(n+1)[2+n(n+1)]$
$= \frac{1}{2}n(n+1)(n^2+n+2)$

19. $\sum_{i=1}^{n} 4i^2(i-2) = \sum_{i=1}^{n}(4i^3-8i^2) = 4\sum_{i=1}^{n} i^3 - 8\sum_{i=1}^{n} i^2 = 4\cdot\frac{n^2(n+1)^2}{4} - 8\cdot\frac{n(n+1)(2n+1)}{6}$

$= \frac{1}{3}n(n+1)[3n(n+1)-4(2n+1)] = \frac{1}{3}n(n+1)(3n^2-5n-4) = n^4 - \frac{2}{3}n^3 - 3n^2 - \frac{4}{3}n$

20. $\sum_{k=1}^{n}[(3^{-k}-3^{k})^2 - (3^{k-1}-3^{-k+1})^2]$

- Because $(3^{-k}-3^k)^2 = (3^k-3^{-k})^2$, we may apply Theorem 4.4.6 with $F(k) = (3^k - 3^{-k})^2$. Thus,

$$\sum_{k=1}^{n}[(3^{-k}-3^k)^2 - (3^{k-1}-3^{-k+1})^2] = (3^n-3^{-n})^2 - (3^0-3^0)^2 = (3^n-3^{-n})^2$$

In Exercises 21–30, use the method of this section to find the area, A square units, of the region; use inscribed or circumscribed rectangles as indicated. For each exercise draw a figure showing the region and the ith rectangle. In Exercises 33-36, take $f(m_i)$ as the measure of the altitude of the ith rectangle, where $m_i = \frac{1}{2}(x_{i-1}+x_1)$ is the midpoint of the ith subinterval.

In Exercises 21, 22 and 34, the region is bounded by $y = x^2$, the x axis, and the line $x = 2$.

21. inscribed rectangles. Let $f(x) = x^2$. The closed interval $[0,2]$ is divided into n subintervals each of length $\Delta x = \frac{2}{n}$. The ith subinterval is $[x_{i-1}, x_i]$, where $i = 1, 2, 3, \ldots, n$. Because f is increasing on $[0,2]$, the absolute minimum value of f on $[x_{i-1}, x_i]$ is $f(x_{i-1})$.

$f(x_{i-1}) = (x_{i-1})^2 = [(i-1)\Delta x]^2 = (i-1)^2(\Delta x)^2 = \frac{4(i-1)^2}{n^2}$

$A = \lim_{n\to+\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\frac{4(i-1)^2}{n^2}\cdot\frac{2}{n} \overset{\text{Note}}{=} \lim_{n\to+\infty}\frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - n^2\right)$

$= \lim_{n\to+\infty}\frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - n^2\right) = \lim_{n\to+\infty}\left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{8}{n}\right] = \frac{8}{3}$

- The area of the region is $\frac{8}{3}$ square units.

22. circumscribed rectangles. Let $f(x) = x^2$. The closed interval $[0,2]$ is divided into n subintervals each of length $\Delta x = \frac{2}{n}$. The ith subinterval is $[x_{i-1}, x_i]$, where $i = 1, 2, 3, \ldots, n$. Because f is increasing on $[0,2]$, the absolute maximum value of f on $[x_{i-1}, x_i]$ is $f(x_i)$.

$f(x_i) = (x_i)^2 = (i\Delta x)^2 = i^2(\Delta x)^2 = \frac{4i^2}{n^2}$. $A = \lim_{n\to+\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\frac{4i^2}{n^2}\cdot\frac{2}{n} = \lim_{n\to+\infty}\frac{8}{n^3}\cdot\sum_{i=1}^{n} i^2$

$= \lim_{n\to+\infty}\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \lim_{n\to+\infty}\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{8}{3}$

- The area of the region is $\frac{8}{3}$ square units.

34. midpoint. Let $f(x) = x^2$. The closed interval $[0,2]$ is divided into n subintervals of length $\Delta x = 2/n$.

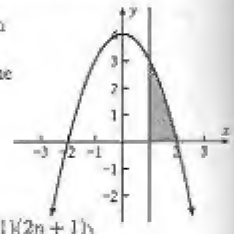
$m_i = \frac{1}{2}(x_i + x_{i-1}) = \frac{1}{2}[i\,\Delta x + (i-1)\Delta x] = \frac{1}{2}(2i-1)\frac{2}{n} = \frac{2i-1}{n}$. $f(m_i) = \left(\frac{2i-1}{n}\right)^2$

$A = \lim_{n\to+\infty}\sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\left(\frac{2i-1}{n}\right)^2\frac{2}{n} = \lim_{n\to+\infty}\left[\frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{8}{n^3}\sum_{i=1}^{n} i + \frac{2}{n^3}\sum_{i=1}^{n} 1\right]$

$= \lim_{n\to+\infty}\left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^3}\cdot\frac{n(n+1)}{2} + \frac{2}{n^3}\cdot n\right] = \lim_{n\to+\infty}\left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 4\left(\frac{1}{n}+\frac{1}{n^2}\right) + \frac{2}{n^2}\right]$

$= \frac{4}{3}\cdot 1\cdot 2 + 0 + 0 = \frac{8}{3}$

In Exercises 23, 24 and 35, the region is above the x axis and to the right of the line $x = 1$ bounded by the x axis, the line $x = 1$, and the curve $y = 4 - x^2$.



- ▸ Let $f(x) = 4 - x^2$. The closed interval $[1, 2]$ is divided into n subintervals of length $\Delta x = \frac{1}{n}$. The ith subinterval is $[x_{i-1}, x_i]$, where $x_i = 1 + i\Delta x$, $i = 1, 2, 3, \ldots, n$.

23. inscribed rectangles. Because f is decreasing on $[1, 2]$, the absolute minimum value of f on $[x_{i-1}, x_i]$ is $f(x_i)$.

$f(x_i) = f(1 + i\Delta x) = f\left(1 + \frac{i}{n}\right) = 4 - \left(1 + \frac{i}{n}\right)^2 = 3 - \frac{2i}{n} - \frac{i^2}{n^2}$

$A = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n \to +\infty} \sum_{i=1}^{n} \left(3 - \frac{2i}{n} - \frac{i^2}{n^2}\right)\frac{1}{n}$

$= \lim_{n \to +\infty}\left(\sum_{i=1}^{n} \frac{3}{n} - \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2\right) = \lim_{n \to +\infty}\left(3 - \frac{2}{n^2}\cdot\frac{n(n+1)}{2} - \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)$

$= \lim_{n \to +\infty}\left[3 - \left(1 + \frac{1}{n}\right) - \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] = 3 - 1 - \frac{1}{3} = \frac{5}{3}$

- ▪ The area of the region is $\frac{5}{3}$ square units.

24. circumscribed rectangles. Let $f(x) = 4 - x^2$. Because f is decreasing on $[1, 2]$, the absolute maximum value of f on $[x_{i-1}, x_i]$ is $f(x_{i-1})$.

$f(x_{i-1}) = f(1 + (i-1)\Delta x) = f\left(1 + \frac{i-1}{n}\right) = 4 - \left(1 + \frac{i-1}{n}\right)^2 = 3 - \frac{2(i-1)}{n} - \frac{(i-1)^2}{n^2}$

$A = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n \to +\infty} \sum_{i=1}^{n} \left(3 - \frac{2(i-1)}{n} - \frac{(i-1)^2}{n^2}\right)\frac{1}{n}$

$= \lim_{n \to +\infty}\left(\sum_{i=1}^{n} \frac{3}{n} - \frac{2}{n^2}\sum_{i=1}^{n} (i-1) - \frac{1}{n^3}\sum_{i=1}^{n} (i-1)^2\right) \overset{\text{Note}}{=} \lim_{n \to +\infty}\left[\sum_{i=1}^{n} \frac{3}{n} - \frac{2}{n^2}\left(\sum_{i=1}^{n} i - n\right) - \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 - n^2\right)\right]$

$= \lim_{n \to +\infty}\left(3 - \frac{2}{n^2}\cdot\frac{n(n+1)}{2} + \frac{2}{n} - \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{n}\right) = \lim_{n \to +\infty}\left[3 - \left(1 + \frac{1}{n}\right) + \frac{2}{n} - \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \frac{1}{n}\right]$

$= 3 - 1 + 0 - \frac{1}{3} + 0 = \frac{5}{3}$. Therefore the area of the region is $\frac{5}{3}$ square units.

35. midpoint. $m_i = \frac{1}{2}(x_i + x_{i-1}) = \frac{1}{2}[(1 + i\,\Delta x) + (1 + (i-1)\Delta x)] = \frac{1}{2}[2 + (2i-1)\frac{1}{n}] = 1 + \frac{2i-1}{2n}$.

$f(m_i) = 4 - \left(1 + \frac{2i-1}{2n}\right)^2 = 3 - \frac{2i-1}{n} - \frac{(2i-1)^2}{4n^2}$

$A = \lim_{n \to +\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n \to +\infty} \sum_{i=1}^{n}\left[3 - \frac{2i-1}{n} - \frac{(2i-1)^2}{4n^2}\right]\frac{1}{n}$

$= \lim_{n \to +\infty}\left[\frac{3}{n}\sum_{i=1}^{n} 1 - \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^2}\sum_{i=1}^{n} 1 - \frac{1}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n^3}\sum_{i=1}^{n} i - \frac{1}{4n^3}\sum_{i=1}^{n} 1\right]$

$= \lim_{n \to +\infty}\left[\frac{3}{n}\cdot n - \frac{2}{n^2}\cdot\frac{n(n+1)}{2} + \frac{1}{n^2}\cdot n - \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3}\cdot\frac{n(n+1)}{2} - \frac{1}{4n^3}\cdot n\right]$

$= \lim_{n \to +\infty}\left[3 - \left(1 + \frac{1}{n}\right) + \frac{1}{n} - \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n^2}\right) - \frac{1}{4n^2}\right]$

$= 3 - 1 + 0 - \frac{1}{3} + 0 + 0 = \frac{5}{3}$

In Exercises 25, 26, and 36, the region is above the x axis and to the left of the line $x = 1$ bounded by the x axis, the line $x = 1$, and the curve $y = 4 - x^2$.

25. circumscribed rectangles. Let $f(x) = 4 - x^2$. We wish to find the area of the region bounded by the curve $y = f(x)$, the x axis and the line $x = 1$. Because f is increasing on $[-2, 0]$ and decreasing on $[0, 1]$ and we wish to use circumscribed rectangles we consider the two separate intervals $[-2, 0]$ and $[0, 1]$. Let A_1 square units be the area of the region bounded by the curve $y = f(x)$, the x axis and the y axis. The closed interval $[-2, 0]$ is divided into n subintervals each of length $\Delta x = \frac{2}{n}$. The ith subinterval is $[x_{i-1}, x_i]$ where $i = 1, 2, 3, \ldots, n$. Because f is increasing on $[-2, 0]$ the absolute maximum value of f on $[x_{i-1}, x_i]$ is $f(x_i)$.

$f(x_i) = f(-2 + i\Delta x) = 4 - \left(-2 + \frac{2}{n}i\right)^2 = \frac{8}{n}i - \frac{4}{n^2}i^2$

$A_1 = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n \to +\infty} \sum_{i=1}^{n}\left(\frac{8}{n}i - \frac{4}{n^2}i^2\right)\cdot\frac{2}{n} = \lim_{n \to +\infty}\left(\frac{16}{n^2}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^2\right)$

$$= \lim_{n\to+\infty}\left[\frac{16}{n^2}\cdot\frac{n(n+1)}{2} - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to+\infty}\left[8\left(1+\frac{1}{n}\right) - \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 8 - \frac{8}{3} = \frac{16}{3}$$

Let A_2 square units be the area of the region bounded by the curve $y = f(x)$, the x axis, the y axis, and the line $x = 1$. The closed interval $[0,1]$ is divided into n subintervals each of length $\Delta x = \frac{1}{n}$. The ith subinterval is $[x_{i-1}, x_i]$ where $i = 1, 2, 3, \ldots, n$. Because f is decreasing on $[0,1]$, the absolute maximum value of f on $[x_{i-1}, x_i]$ is $f(x_{i-1})$.

$$f(x_{i-1}) = f(0 + (i-1)\Delta x) = 4 - \left[(i-1)\frac{1}{n}\right]^2$$

$$A_2 = \lim_{n\to+\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\left[4 - \frac{(i-1)^2}{n^2}\right]\cdot\frac{1}{n} = \lim_{n\to+\infty}\left[\sum_{i=1}^{n}\frac{4}{n} - \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 - n^2\right)\right] \text{(see Note)}$$

$$= \lim_{n\to+\infty}\left[4 - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - n^2\right)\right] = \lim_{n\to+\infty}\left(4 - \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{1}{n}\right) = 4 - \frac{1}{3} = \frac{11}{3}$$

$A = A_1 + A_2 = \frac{16}{3} + \frac{11}{3} = 9$. Thus the required area is 9 square units.

inscribed rectangles. Because $f(x) = 4 - x^2$ is not monotonic on the interval $[-2, 1]$, we divide the region into two parts. Let A_1 square units be the area of the region R_1 to the left of the y axis bounded by the curve, the x axis, and the y axis. The closed interval $[-2, 0]$ is divided into n subintervals, starting from the origin. Thus $\Delta x = \frac{0-(-2)}{n} = \frac{2}{n}$ and $x_i = -i\Delta x$. Because f is increasing on $[-2, 0]$, the absolute minimum value of f on $[x_i, x_{i-1}]$ is $f(x_i)$ where $f(x_i) = 4 - (x_i)^2 = 4 - (-i\,\Delta x)^2 = 4 - \left(-\frac{2i}{n}\right)^2 = 4 - \frac{4i^2}{n^2}$

$$A_1 = \lim_{n\to+\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\left(4 - \frac{4i^2}{n^2}\right)\frac{2}{n} = \lim_{n\to+\infty}\left[\frac{8}{n}\sum_{i=1}^{n} 1 - \frac{8}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to+\infty}\left[\frac{8}{n}\cdot n - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to+\infty}\left[8 - \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 8 - \frac{4}{3}\cdot 1\cdot 2 = \frac{16}{3}$$

Let A_2 square units be the area of the region bounded by the curve, the x axis, the y axis and the line $x = 1$. The closed interval $[0, 1]$ is divided into n subintervals. Thus $\Delta x = 1/n$ and $x_i = i\Delta x$. Because f is decreasing on $[0, 1]$, the absolute minimum value of f in $[x_{i-1}, x_i]$ is $f(x_i)$ where

$$f(x_i) = 4 - (x_i)^2 = 4 - \frac{i^2}{n^2}$$

$$A_2 = \lim_{n\to+\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\left[4 - \frac{i^2}{n^2}\right]\frac{1}{n} = \lim_{n\to+\infty}\left[\frac{4}{n}\sum_{i=1}^{n} 1 - \frac{1}{n^3}\sum_{i=1}^{n} i^2\right]$$

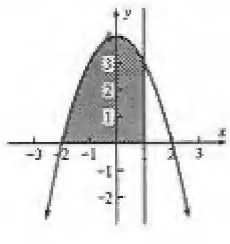
$$= \lim_{n\to+\infty}\left[\frac{4}{n}\cdot n - \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to+\infty}\left[4 - \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 4 - \frac{1}{6}\cdot 2 = \frac{11}{3}$$

- The number of square units in the area of the region is $A_1 + A_2 = \frac{16}{3} + \frac{11}{3} = 9$.

midpoint. Let $f(x) = 4 - x^2$. The closed interval $[-2, 1]$ is divided into n subintervals, each of length $\Delta x = 3/n$. The ith subinterval is $[x_i + x_{i-1}]$ where $i = 1, 2, 3, \ldots, n$. Let the altitude of the ith rectangle be $f(m_i)$ where $m_i = x_i - \frac{1}{2}\Delta x = -2 + \frac{3}{n}i - \frac{3}{2n}$.

$$f(m_i) = 4 - \left(\left(-2 - \frac{3}{2n}\right) + \frac{3}{n}i\right)^2 = 4 - \left(2 + \frac{3}{2n}\right)^2 + \left(4 + \frac{3}{n}\right) - i - \frac{9}{n^2}i^2 = -\frac{6}{n} - \frac{9}{4n^2} + \left(4 + \frac{3}{n}\right)\frac{3}{n}i - \frac{9}{n^2}i^2$$

If A square units is the area of the region above the x axis and to the left of the line $x = 1$ bounded by the x axis, the line $x = 1$, and the curve $y = f(x)$, then



$$A = \lim_{n\to+\infty}\sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\left(-\frac{6}{n} - \frac{9}{4n^2} + \left(4 + \frac{3}{n}\right)\frac{3}{n}i - \frac{9}{n^2}i^2\right)\cdot\frac{3}{n}$$

$$= \lim_{n\to+\infty}\left(\left(-\frac{18}{n^2} - \frac{27}{4n^3}\right)\sum_{i=1}^{n} 1 + \left(4 + \frac{3}{n}\right)\frac{9}{n^2}\sum_{i=1}^{n} i - \frac{27}{n^3}\sum_{i=1}^{n} i^2\right)$$

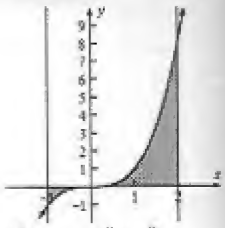
$$= \lim_{n\to+\infty}\left(\left(-\frac{18}{n^2} - \frac{27}{4n^3}\right)\cdot n + \left(4 + \frac{3}{n}\right)\frac{9}{n^2}\cdot\frac{n(n+1)}{2} - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to+\infty}\left[-\frac{18}{n} - \frac{27}{4n^2} + \frac{9}{2}\left(4 + \frac{3}{n}\right)\left(1 + \frac{1}{n}\right) - \frac{9}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] = 18 - 9 = 9$$

- Therefore, the area of the region is 9 square units.

In Exercises 27 and 28, the region is bounded by $y = x^3$, the x axis, and the lines $x = -1$ and $x = 2$.

27. inscribed rectangles. Let $f(x) = x^3$. Because $f(x)$ is nonpositive on $[-1, 0]$ and nonnegative on $[0, 2]$ we consider the two separate intervals $[-1, 0]$ and $[0, 2]$. Let A_1 square units be the area of the region bounded by the curve $y = f(x)$, the line $x = -1$ and the x axis. The closed interval $[-1, 0]$ is divided into n subintervals each of length $\Delta x = \frac{1}{n}$. The ith subinterval is $[x_{i-1}, x_i]$ where $i = 1, 2, 3, \ldots, n$. The function f is increasing on $[-1, 0]$, and in order to have inscribed rectangles we take $-f(x_i)$ as the altitude of the ith rectangle.



$$-f(x_i) = -f(-1 + i\Delta x) = -\left(-1 + \frac{i}{n}\right)^3 = \frac{(n-i)^3}{n^3}$$

$$A_1 = \lim_{n \to +\infty} \sum_{i=1}^{n} -f(x_i)\Delta x = \lim_{n \to +\infty} \sum_{i=1}^{n} \frac{(n-i)^3}{n^3} \cdot \frac{1}{n} = \lim_{n \to +\infty} \frac{1}{n^4}[(n-1)^3 + (n-2)^3 + \cdots + 1^3 + 0^3]$$

$$= \lim_{n \to +\infty} \frac{1}{n^4}\left(\sum_{i=1}^{n} i^3 - n^3\right) = \lim_{n \to +\infty} \frac{1}{n^4}\left(\frac{n^2(n+1)^2}{4} - n^3\right) = \lim_{n \to +\infty} \sum_{i=1}^{n} \left(\frac{1}{4}\left(1 + \frac{1}{n}\right)^2 - \frac{1}{n}\right) = \frac{1}{4}$$

Let A_2 square units be the area of the region bounded by the curve $y = f(x)$, the line $x = 2$ and the x axis. The closed interval $[0, 2]$ is divided into n subintervals each of length $\Delta x = \frac{2}{n}$. The ith subinterval is $[x_{i-1}, x_i]$ where $i = 1, 2, 3, \ldots, n$. The function f is increasing on $[0, 2]$ and in order to have inscribed rectangles we take $f(x_{i-1})$ as the altitude of the ith rectangle.

$$f(x_{i-1}) = f(0 + (i-1)\Delta x) = \left[(i-1)\frac{2}{n}\right]^3$$

$$A_2 = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x = \lim_{n \to +\infty} \sum_{i=1}^{n} \left[(i-1)\frac{2}{n}\right]^3 \frac{2}{n} = \lim_{n \to +\infty} \frac{16}{n^4} \sum_{i=1}^{n} (i-1)^3 \overset{\text{Note}}{=} \lim_{n \to +\infty} \frac{16}{n^4}\left(\sum_{i=1}^{n} i^3 - n^3\right)$$

$$= \lim_{n \to +\infty} \frac{16}{n^4}\left(\frac{n^2(n+1)^2}{4} - n^3\right) = \lim_{n \to +\infty}\left(4\left(1 + \frac{1}{n}\right)^2 - \frac{1}{n}\right) = 4$$

$A = A_1 + A_2 = \frac{1}{4} + 4 = \frac{17}{4}$. Thus the required area is $\frac{17}{4}$ square units.

28. circumscribed rectangles.

▷ Let $f(x) = x^3$. See the figure. Because $f(x)$ is nonpositive on $[-1, 0]$ and nonnegative on $[0, 2]$, we consider these intervals separately. Let A_1 square units be the area of the region R_1 bounded by the curve $y = x^3$, the line $x = -1$, and the x axis. The closed interval $[-1, 0]$ is divided into n subintervals, starting from the origin. Thus $\Delta x = 1/n$ and $x_i = -i\Delta x$. f is increasing on $[-1, 0]$ and in order to have circumscribed rectangles we take $-f(x_i)$ as the altitude of the ith rectangle, where

$$-f(x) = -(x_i)^3 = -(-i\,\Delta x)^3 = -\left(-i \cdot \frac{1}{n}\right)^3 = \frac{i^3}{n^3}$$

$$A_1 = \lim_{n \to +\infty} \sum_{i=1}^{n} -f(x_i)\Delta x = \lim_{n \to +\infty} \sum_{i=1}^{n} \frac{i^3}{n^3} \cdot \frac{1}{n} = \lim_{n \to +\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n \to +\infty} \frac{1}{n^4} \frac{n^2(n+1)^2}{4} = \lim_{n \to +\infty} \frac{1}{4}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{4}$$

Let A_2 square units be the area of the region R_2 bounded by the curve $y = x^3$, the line $x = 2$, and the x axis. The closed interval $[0, 2]$ is divided into n subintervals. Thus $\Delta x = 2/n$ and $x_i = i\Delta x$. f is increasing on $[0, 2]$ and in order to have circumscribed rectangles we take $f(x_i)$ as the altitude of the ith rectangle, where

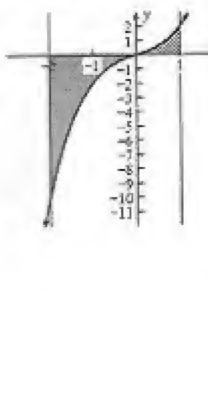
$$f(x_i) = (x_i)^3 = (i\,\Delta x)^3 = \left(i\,\frac{2}{n}\right)^3 = \frac{8i^3}{n^3}$$

$$A_2 = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n \to +\infty} \sum_{i=1}^{n} \frac{8i^3}{n^3} \cdot \frac{2}{n} = 16A_1 = 16 \cdot \frac{1}{4} = 4$$

Thus, the number of square units in the required area is $A_1 + A_2 = \frac{1}{4} + 4 = \frac{17}{4}$.

In Exercises 29 and 30, the region is bounded by $y = x^3 + x$, the x axis, and the lines $x = -2$ and $x = 1$.

29. circumscribed rectangles. Let $f(x) = x^3 + x$. Because $f(x)$ is nonpositive on $[-2, 0]$ and nonnegative on $[0, 1]$ we consider the two separate intervals $[-2, 0]$ and $[0, 1]$. See the figure. Let A_1 square units be the area of the region bounded by the curve $y = f(x)$, the line $x = -1$ and the x axis. The closed interval $[-2, 0]$ is divided into n subintervals each of length $\Delta x = \frac{2}{n}$. The ith subinterval is $[x_{i-1}, x_i]$ where $i = 1, 2, 3, \ldots, n$. The function f is increasing on $[-2, 0]$, and in order to have circumscribed rectangles we take $-f(x_{i-1})$ as the altitude of the ith rectangle.



$$-f(x_{i-1}) = -f(-2 + (i-1)\Delta x) = -\left(-2 + (i-1)\frac{2}{n}\right)^3 - \left(-2 + (i-1)\frac{2}{n}\right)$$

$$= 8\left(1-\frac{i-1}{n}\right)^3 + 2\left(1-\frac{i-1}{n}\right) = 8\left(\frac{n+1-i}{n}\right)^3 + 2\left(\frac{n+1-i}{n}\right)$$

$$A_1 = \lim_{n\to+\infty} \sum_{i=1}^{n} -f(x_{i-1})\Delta x = \lim_{n\to+\infty} \sum_{i=1}^{n} \left(8\frac{(n+1-i)^3}{n^3} + 2\left(\frac{n+1-i}{n}\right)\right)\cdot\frac{2}{n}$$

$$= \lim_{n\to+\infty}\left(\frac{16}{n^4}\sum_{i=1}^{n}(n+1-i)^3 + \frac{4}{n^2}\sum_{i=1}^{n}(n+1-i)\right) = \lim_{n\to+\infty}\left(\frac{16}{n^4}\sum_{i=1}^{n} i^3 + \frac{4}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to+\infty}\left(\frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4} + \frac{4}{n^2}\cdot\frac{n(n+1)}{2}\right) = \lim_{n\to+\infty}\left(4\left(1+\frac{1}{n}\right)^2 + 2\left(1+\frac{1}{n}\right)\right) = 4+2 = 6$$

Let A_2 square units be the area of the region bounded by the curve $y = f(x)$, the line $x = 1$ and the x axis. The closed interval $[0,1]$ is divided into n subintervals each of length $\Delta x = \frac{1}{n}$. The ith subinterval is $[x_{i-1}, x_i]$ where $i = 1, 2, 3, \ldots, n$. The function f is increasing on $[0,1]$ and in order to have circumscribed rectangles we take $f(x_i)$ as the altitude of the ith rectangle.

$$f(x_i) = f(0 + i\Delta x) = \left(\frac{i}{n}\right)^3 + \frac{i}{n}$$

$$A_2 = \lim_{n\to+\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\left(\frac{i^3}{n^3}+\frac{i}{n}\right)\cdot\frac{1}{n} = \lim_{n\to+\infty}\left(\frac{1}{n^4}\sum_{i=1}^{n} i^3 + \frac{1}{n^2}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to+\infty}\left(\frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4} + \frac{1}{n^2}\cdot\frac{n(n+1)}{2}\right) = \lim_{n\to+\infty}\left(\frac{1}{4}\left(1+\frac{1}{n}\right) + \frac{1}{2}\left(1+\frac{1}{n}\right)\right) = \frac{1}{4}+\frac{1}{2} = \frac{3}{4}$$

$A = A_1 + A_2 = 6 + \frac{3}{4}$. Thus the required area is $\frac{27}{4}$ square units.

30. inscribed rectangles. Let $f(x) = x^3 + x$. Because $f(x)$ is nonpositive on $[-2,0]$ and nonnegative on $[0,1]$ we consider the two separate intervals $[-2,0]$ and $[0,1]$. See the figure above. Let A_1 square units be the area of the region bounded by the curve $y = f(x)$, the line $x = -1$ and the x axis. The closed interval $[-2,0]$ is divided into n subintervals each of length $\Delta x = \frac{2}{n}$. The ith subinterval is $[x_{i-1}, x_i]$ where $i = 1, 2, 3, \ldots, n$. The function f is increasing on $[-2,0]$, and in order to have inscribed rectangles we take $-f(x_i)$ as the altitude of the ith rectangle.

$$-f(x_i) = -f(-2+i\Delta x) = -\left(-2+i\frac{2}{n}\right)^3 - \left(-2+i\frac{2}{n}\right) = 8\left(1-\frac{i}{n}\right)^3 + 2\left(1-\frac{i}{n}\right) = 8\left(\frac{n-i}{n}\right)^3 + 2\left(\frac{n-i}{n}\right)$$

$$A_1 = \lim_{n\to+\infty}\sum_{i=1}^{n} -f(x_{i-1})\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\left(8\frac{(n-i)^3}{n^3} + 2\left(\frac{n-i}{n}\right)\right)\cdot\frac{2}{n}$$

$$= \lim_{n\to+\infty}\left(\frac{16}{n^4}\sum_{i=1}^{n}(n-i)^3 + \frac{4}{n^2}\sum_{i=1}^{n}(n-i)\right) = \lim_{n\to+\infty}\left(\frac{16}{n^4}\left(\sum_{i=1}^{n} i^3 - n^3\right) + \frac{4}{n^2}\left(\sum_{i=1}^{n} i - n\right)\right)$$

$$= \lim_{n\to+\infty}\left(\frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{16}{n} + \frac{4}{n^2}\cdot\frac{n(n+1)}{2} - \frac{4}{n}\right) = \lim_{n\to+\infty}\left(4\left(1+\frac{1}{n}\right)^2 + 2\left(1+\frac{1}{n}\right) - \frac{20}{n}\right) = 4+2-0 = 6$$

Let A_2 square units be the area of the region bounded by the curve $y = f(x)$, the line $x = 1$ and the x axis. The closed interval $[0,1]$ is divided into n subintervals each of length $\Delta x = \frac{1}{n}$. The ith subinterval is $[x_{i-1}, x_i]$ where $i = 1, 2, 3, \ldots, n$. The function f is increasing on $[0,1]$ and in order to have inscribed rectangles we take $f(x_{i-1})$ as the altitude of the ith rectangle. $f(x_{i-1}) = f(0 + (i-1)\Delta x) = \left(\frac{i-1}{n}\right)^3 + \frac{i-1}{n}$

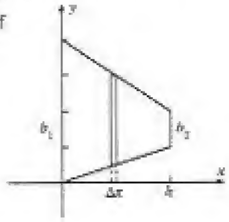
$$A_2 = \lim_{n\to+\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\left(\frac{(i-1)^3}{n^3} + \frac{i-1}{n}\right)\cdot\frac{1}{n} = \lim_{n\to+\infty}\left(\frac{1}{n^4}\left(\sum_{i=1}^{n} i^3 - n^3\right) + \frac{1}{n^2}\left(\sum_{i=1}^{n} i - n\right)\right)$$

$$= \lim_{n\to+\infty}\left(\frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{1}{n} + \frac{1}{n^2}\cdot\frac{n(n+1)}{2} - \frac{1}{n}\right) = \lim_{n\to+\infty}\left(\frac{1}{4}\left(1+\frac{1}{n}\right) + \frac{1}{2}\left(1+\frac{1}{n}\right) - \frac{2}{n}\right) = \frac{1}{4}+\frac{1}{2}-0 = \frac{3}{4}$$

$A = A_1 + A_2 = 6 + \frac{3}{4}$. Thus the required area is $\frac{27}{4}$ square units.

31. Find the area of a trapezoid whose bases have measures b_1 and b_2 and whose altitude has measure h.

The region and the ith inscribed rectangle are shown in the figure. We divide the interval $[0,h]$ into n subintervals of length $\Delta x = h/n$ and $x_i = i\Delta x$. The height of the ith rectangle is $b_1 - \frac{x_i}{h}(b_1 - b_2)$.



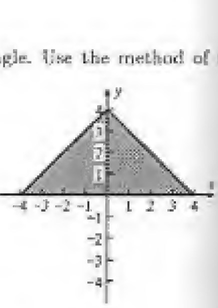
$$A = \lim_{n\to+\infty}\sum_{i=1}^{n}\left[b_1 - \frac{x_i}{h}(b_1-b_2)\right]\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n}\left[b_1 - \frac{i}{n}(b_1-b_2)\right]\frac{h}{n}$$

$$= \lim_{n\to+\infty} h\left[\frac{b_1}{n}\sum_{i=1}^{n} 1 - \frac{b_1-b_2}{n^2}\sum_{i=1}^{n} i\right] = \lim_{n\to+\infty} h\left[\frac{b_1}{n}\cdot n - \frac{b_1-b_2}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to+\infty} h\left[b_1 - \frac{1}{2}(b_1-b_2)\left(1+\frac{1}{n}\right)\right] = h[b_1 - \tfrac{1}{2}(b_1-b_2)] = \tfrac{1}{2}h(b_1+b_2)$$

32. The graph of $y = 4 - |x|$ and the x axis from $x = -4$ to $x = 4$ form the a triangle. Use the method of this section to find the area of this triangle.

▷ A sketch of the region is shown at the right. We utilize symmetry.



Let A_1 square units be the area of the part of the region to the left of the x axis. The closed interval $[-4, 0]$ is divided into n subintervals of length $\Delta x = 4/n$ starting from 0 so that $|x_i| = 4i/n$. The height of the ith inscribed rectangle is $4 - |x_i| = 4 - \frac{4i}{n}$.

$$A_1 = \lim_{n\to+\infty} \sum_{i=1}^{n} (4 - |x_i|)\Delta x = \lim_{n\to+\infty} \sum_{i=1}^{n} \left(4 - \frac{4i}{n}\right)\frac{4}{n} = \lim_{n\to+\infty} 16\left[\frac{1}{n}\sum_{i=1}^{n} 1 - \frac{1}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to+\infty} 16\left[\frac{1}{n}\cdot n - \frac{1}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to+\infty} 16\left[1 - \frac{1}{2}\left(1 + \frac{1}{n}\right)\right] = 8$$

Let A_2 square units be the area of the part of the region to the right of the x axis. The closed interval $[0, 4]$ is divided into n subintervals of length $\Delta x = 4/n$ starting from 0 so that again $|x_i| = 4i/n$. The height of the ith inscribed rectangle is $4 - |x_i| = 4 - \frac{4i}{n}$ and so $A_2 = A_1 = 8$. Therefore $A = A_1 + A_2 = 8$.

33. Let $f(x) = x^2$. The closed interval $[0, 3]$ is divided into n subintervals, each of length $\Delta x = \frac{3}{n}$. The ith subinterval is $[x_{i-1}, x_i]$ where $i = 1, 2, 3, \ldots, n$. Let the altitude of the ith rectangle be $f(m_i)$ where $m_i = \frac{1}{2}(x_i + x_{i-1})$.

$$f(m_i) = [\tfrac{1}{2}(x_i + x_{i-1})]^2 = \frac{1}{4}\left(\frac{3}{n}i + \frac{3}{n}(i-1)\right)^2 = \frac{9}{4n^2}(2i-1)^2 = \frac{9}{4n^2}(4i^2 - 4i + 1)$$

If A square units is the area of the region bounded by the curve $y = f(x)$, the x axis, and the line $x = 3$, then

$$A = \lim_{n\to+\infty} \sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{9}{4n^2}(4i^2 - 4i + 1)\frac{3}{n} = \lim_{n\to+\infty}\left(\frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{27}{n^3}\sum_{i=1}^{n} i + \frac{27}{4n^3}\sum_{i=1}^{n} 1\right)$$

$$= \lim_{n\to+\infty}\left(\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{27}{n^3}\cdot\frac{n(n+1)}{2} + \frac{27}{4n^3}\cdot n\right) = \lim_{n\to+\infty}\left(\frac{9}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \frac{27}{2n}\left(1 + \frac{1}{n}\right) + \frac{27}{4n^2}\right) = 9$$

▪ The area of the region is 9 square units.

In Exercises 37–42, a function f and numbers n, a, and b are given. Approximate the area, A square units, of the region bounded by the curve $y = f(x)$, the x axis, and the lines $x = a$ and $x = b$ by doing the following: Divide the interval $[a, b]$ into n subintervals of equal length Δx units and use a calculator to compute to four decimal places the sum of the areas of n inscribed or circumscribed (as indicated) rectangles each having a width of Δx units.

37. $f(x) = \frac{1}{x}$. The interval $[1, 3]$ is divided into 10 subintervals of length $\frac{3-1}{10} = 0.2$. We wish to use inscribed rectangles. Because $\frac{1}{x}$ is decreasing in $[1, 3]$, the absolute minimum value of $\frac{1}{x}$ on $[x_{i-1}, x_i]$ is $\frac{1}{x_i}$.

$$A > [f(1.2) + f(1.4) + f(1.6) + f(1.8) + f(2.0) + f(2.2) + f(2.4) + f(2.6) + f(2.8) + f(3)]0.2$$

$$= \left(\frac{1}{1.2} + \frac{1}{1.4} + \frac{1}{1.6} + \frac{1}{1.8} + \frac{1}{2.0} + \frac{1}{2.2} + \frac{1}{2.4} + \frac{1}{2.6} + \frac{1}{2.8} + \frac{1}{3}\right)0.2 = 1.0349$$

38. $f(x) = \frac{1}{x^2}$, $a = 1$, $b = 2$, $n = 12$, circumscribed rectangles. The closed interval $[1, 2]$ is divided into 12 subintervals of length $\frac{1}{12}$. Because f is decreasing on $[1, 2]$, the absolute maximum value of f on $[x_{i-1}, x_i]$ is

$$f(x_i) = f\left(1 + \frac{i-1}{12}\right) = \left(\frac{12}{11+i}\right)^2$$

$$A < \left\{1 + \left(\frac{12}{13}\right)^2 + \left(\frac{12}{14}\right)^2 + \left(\frac{12}{15}\right)^2 + \left(\frac{12}{16}\right)^2 + \left(\frac{12}{17}\right)^2 + \left(\frac{12}{18}\right)^2 + \left(\frac{12}{19}\right)^2 + \left(\frac{12}{20}\right)^2 + \left(\frac{12}{21}\right)^2 + \left(\frac{12}{22}\right)^2 + \left(\frac{12}{23}\right)^2\right\}\frac{1}{12}$$

$$= 12\left\{\frac{1}{12^2} + \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \frac{1}{16^2} + \frac{1}{17^2} + \frac{1}{18^2} + \frac{1}{19^2} + \frac{1}{20^2} + \frac{1}{21^2} + \frac{1}{22^2} + \frac{1}{23^2}\right\} = .5323$$

In Exercises 39–42, put your calculator into radian mode.

39. $f(x) = \sin x$. The interval $[\frac{1}{6}\pi, \frac{5}{6}\pi]$ is divided into 8 subintervals of length $\frac{1}{8}[\frac{5}{6}\pi - \frac{1}{6}\pi] = \frac{1}{12}\pi$. We wish to use circumscribed rectangles. In $[\frac{1}{6}\pi, \frac{1}{2}\pi]$, $\sin x$ is increasing so its absolute maximum value is at the right end of a subinterval; in $[\frac{1}{2}\pi, \frac{5}{6}\pi]$, $\sin x$ is decreasing so its absolute maximum value is at the left end of a subinterval.

$A < [\sin \frac{3}{12}\pi + \sin \frac{4}{12}\pi + \sin \frac{5}{12}\pi + \sin \frac{6}{12}\pi + \sin \frac{6}{12}\pi + \sin \frac{7}{12}\pi + \sin \frac{8}{12}\pi + \sin \frac{9}{12}\pi]\frac{1}{12}\pi \approx 1.8530$

40. $f(x) = \cos x$, $a = 0$, $b = \frac{1}{2}\pi$, $n = 6$, inscribed rectangles. The closed interval $[0, \frac{1}{2}\pi]$ is divided into 6 subintervals of length $\frac{1}{12}\pi$. Because f is decreasing on $[0, \frac{1}{2}\pi]$, the absolute minimum value of f on $[x_{i-1}, x_i]$ is $f(x_i) = \cos x_i = \cos\left(\frac{i}{12}\pi\right)$.

$$A > \left[\cos \tfrac{1}{12}\pi + \cos \tfrac{2}{12}\pi + \cos \tfrac{3}{12}\pi + \cos \tfrac{4}{12}\pi + \cos \tfrac{5}{12}\pi + \tfrac{1}{2}\pi\right]\tfrac{1}{12}\pi = 0.8634$$

41. $f(x) = \sin x$. The interval $[\frac{1}{6}\pi, \frac{5}{6}\pi]$ is divided into 8 subintervals of length $\frac{1}{8}[\frac{5}{6}\pi - \frac{1}{6}\pi] = \frac{1}{12}\pi$. We wish to use inscribed rectangles. In $[\frac{1}{6}\pi, \frac{1}{2}\pi]$, $\sin x$ is increasing so its absolute minimum value is at the left end of a subinterval; in $[\frac{1}{2}\pi, \frac{5}{6}\pi]$, $\sin x$ is decreasing so its absolute minimum value is at the right end of a subinterval.

$A > \left(\sin \frac{2}{12}\pi + \sin \frac{3}{12}\pi + \sin \frac{4}{12}\pi + \sin \frac{5}{12}\pi + \sin \frac{7}{12}\pi + \sin \frac{8}{12}\pi + \sin \frac{9}{12}\pi + \sin \frac{10}{12}\pi\right)\frac{1}{12}\pi \approx 1.5912$

42. $f(x) = \cos x$, $a = 0$, $b = \frac{1}{2}\pi$, $n = 6$, circumscribed rectangles. The closed interval $[0, \frac{1}{2}\pi]$ is divided into 6 subintervals of length $\frac{1}{12}\pi$. Because f is decreasing on $[0, \frac{1}{2}\pi]$, the absolute maximum value of f on $[x_{i-1}, x_i]$ is $f(x_{i-1}) = \cos x_{i-1} = \cos\left(\frac{i-1}{12}\pi\right)$

$$A < \left[\cos 0 + \cos \tfrac{1}{12}\pi + \cos \tfrac{2}{12}\pi + \cos \tfrac{3}{12}\pi + \cos \tfrac{4}{12}\pi + \cos \tfrac{5}{12}\pi\right]\tfrac{\pi}{12} = 1.1252$$

43. We wish to prove that $\sum_{i=1}^{n} [F(i) + G(i)] = \sum_{i=1}^{n} F(i) + \sum_{i=1}^{n} G(i)$. Now

$\sum_{i=1}^{n} [F(i) + G(i)] = [F(1) + G(1)] + [F(2) + G(2)] + [F(3) + G(3)] + \cdots + [F(n) + G(n)]$

By applying the associative and commutative laws of addition to the right side of the above equation we get

$\sum_{i=1}^{n} [F(i) + G(i)] = [F(1) + F(2) + F(3) + \cdots + F(n)] + [G(1) + G(2) + G(3) + \cdots + G(n)]$

$\sum_{i=1}^{n} [F(i) + G(i)] = \sum_{i=1}^{n} F(i) + \sum_{i=1}^{n} G(i)$

44. Prove Theorem 4.4.5

By Definition 4.4.1

$\sum_{i=a+c}^{b+c} F(i-c) = F((a+c)-c) + \cdots + F((b+c)-c) = F(a) + \cdots + F(b) = \sum_{i=a}^{b} F(i)$

and

$\sum_{i=a-c}^{b-c} F(i+c) = F((a-c)+c) + \cdots + F((b-c)+c) = F(a) + \cdots + F(b) = \sum_{i=a}^{b} F(i)$

Thus both parts of the theorem are proved.

45. $\sum_{i=1}^{n} i = 1 + 2 + \cdots + (n-1) + n$ and

$\sum_{i=1}^{n} i = n + (n-1) + \cdots + 2 + 1$

Adding the above equations, we obtain

$2\sum_{i=1}^{n} i = (n+1) + (n+1) + \cdots + (n+1) + (n+1) = n(n+1)$

$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$

46. We wish to prove $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Because $(i-1)^3 = i^3 - 3i^2 + 3i - 1$ then $i^2 = \frac{1}{3}[i^3 - (i-1)^3] + i - \frac{1}{3}$. Using Theorem 4.4.6 (telescoping sum) with $F(i) = i^3$ on the first term and Formula 1 on the second term we have

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}\sum_{i=1}^{n}[i^3 - (i-1)^3] + \sum_{i=1}^{n} i - \frac{1}{3}\sum_{i=1}^{n} 1 = \frac{1}{3}n^3 + \frac{n(n+1)}{2} - \frac{1}{3}n = \frac{1}{6}n[2n^2 + 3(n+1) - 2]$$
$$= \frac{1}{6}n[2(n^2-1) + 3(n+1)] = \frac{1}{6}n(n+1)[2(n-1)+3] = \frac{1}{6}n(n+1)(2n+1)$$

47. We wish to prove $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$.

Because $(i-1)^4 = i^4 - 4i^3 + 6i^2 - 4i + 1$ then $i^3 = \frac{1}{4}[i^4 - (i-1)^4 + 6i^2 - 4i + 1]$. Using Theorem 4.4.6 (telescoping sum) with $F(i) = i^4$ on the first term and Formulas 2 and 1, we have

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\left[\sum_{i=1}^{n}[i^4 - (i-1)^4] + 6\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right] = \frac{1}{4}\left[n^4 + 6\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n\right]$$
$$= \frac{1}{4}n[n^3 + (n+1)(2n+1) - 2(n+1) + 1] = \frac{1}{4}n(n^3 + 2n^2 + n) = \frac{1}{4}n^2(n+1)^2$$

48. We wish to prove $\sum_{i=1}^{n} i^4 = \frac{1}{30}n(n+1)(6n^3 + 9n^2 + n - 1)$.

Because $(i-1)^5 = i^5 - 5i^4 + 10i^3 - 10i^2 + 5i - 1$ then $i^4 = \frac{1}{5}[i^5 - (i-1)^5] + 2i^3 - 2i^2 + i - \frac{1}{5}$.

Using Theorem 4.4.6 with $F(i) = i^5$ on the first term, and Formulas 3, 2 and 1 on succeeding terms we have

$$\sum_{i=1}^{n} i^4 = \frac{1}{5}\sum_{i=1}^{n}[i^5 - (i-1)^5] + 2\sum_{i=1}^{n} i^3 - 2\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i - \frac{1}{5}\sum_{i=1}^{n} 1$$
$$= \frac{1}{5}n^5 + 2\cdot\frac{1}{4}n^2(n+1)^2 - 2\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - \frac{1}{5}$$
$$= \frac{1}{5}n(n^4 - 1) + \frac{1}{2}n^2(n+1)^2 - \frac{1}{3}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$
$$= \frac{1}{30}n(n+1)[6(n-1)(n^2+1) + 15n(n+1) - 10(2n+1) + 15]$$
$$= \frac{1}{30}n(n+1)(6n^3 + 9n^2 + n - 1) = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1)$$

4.5 THE DEFINITE INTEGRAL

A *Riemann sum* for the function f on the closed interval $[a, b]$ is any sum of the form

$$\sum_{i=1}^{n} f(w_i)\Delta_i x$$

where

$a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_n = b$
w_i is any number in the closed interval $[x_{i-1}, x_i]$
$\Delta_i x = x_i - x_{i-1}$

The set $\Delta = \{x_0, x_1, x_2, \ldots, x_n\}$ is called a *partition* of the interval $[a, b]$. The largest of the numbers $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$ is called the *norm* of the partition and is represented by $\|\Delta\|$. If the subintervals have equal length, the partition is *regular*. The sum that appears in Definition 4.4.8 is a Riemann sum for a regular partition. Riemann sums approximate better when the w_i are chosen to be the midpoints rather than the left or right endpoints of the interval.

4.5.1 Definition Let f be a function whose domain includes the closed interval $[a, b]$. Then f is said to be *integrable* on $[a, b]$ if there is a number L satisfying the condition that, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every partition Δ for which $\|\Delta\| < \delta$, and for any w_i in the closed interval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, then

$$\left|\sum_{i=1}^{n} f(w_i)\Delta_i x - L\right| < \epsilon$$

For such a situation we write

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x = L$$

4.5.2 Definition If f is a function defined on the closed interval $[a,b]$, then the *definite integral* of f (integrand) from a (lower limit) to b (upper limit), $\int_a^b f(x)\,dx$, is given by

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x$$

if the limit exists. If the partition is regular, we have

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} f(w_i)\Delta x$$

4.5.3 Theorem If a function is continuous on the closed interval $[a,b]$, then it is integrable on $[a,b]$.

The following extension of Theorem 4.5.3 is sometimes useful:

4.5.3′ Theorem If a function is defined and bounded on the closed interval $[a,b]$, then it is integrable on $[a,b]$ if and only if it is continuous there except for a set of points which, for any $\epsilon > 0$, is contained in a union of intervals the sum of the measures of whose length is less than ϵ.

4.5.4 Theorem Let the function f be continuous on $[a,b]$ and $f(x) \geq 0$ for all x in $[a,b]$. Let R be the region bounded by the curve $y = f(x)$, the x axis, and the lines $x = a$ and $x = b$. Then the measure A of the area of the region R is given by

$$A = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x \iff A = \int_a^b f(x)\,dx$$

If the condition $f(x) \geq 0$ is not satisfied, we replace $f(x)$ with $|f(x)|$.

4.5.5 Definition If $a > b$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$, if $\int_b^a f(x)\,dx$ exists.

4.5.6 Definition If $f(a)$ exists, then $\int_a^a f(x)\,dx = 0$.

4.5.9 Theorem If k is any constant, then $\int_a^b k\,dx = k(b-a)$

4.5.10 Theorem If the function f is integrable on the closed interval $[a,b]$ and if k is any constant, then

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

4.5.11 Theorem If the functions f and g are integrable on $[a,b]$, then $f+g$ is integrable on $[a,b]$ and

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

4.5.13 Theorem If f is integrable on a closed interval containing the three numbers a, b, and c, in any order,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Theorem C (Exercise 56) If f is integrable on the closed interval $[-r,r]$, then

(a) if f is an even function, $\int_{-r}^{r} f(x)\,dx = 2\int_0^r f(x)\,dx$

(b) if f is an odd function, $\int_{-r}^{r} f(x)\,dx = 0$

Exercises 4.5

In Exercises 4–8, find the Riemann sum for the function on the interval, using the partition Δ and the given values of w_i. Sketch its graph and show the rectangles, the measure of whose areas are the terms of the sum.

1. $f(x) = x^2$, $0 \leq x \leq 3$. $\sum_{i=1}^{4} f(w_i)\Delta_i x = f(w_1)\Delta_1 x + f(w_2)\Delta_2 x + f(w_3)\Delta_3 x + f(w_4)\Delta_4 x$

   $= f(.25)(.5-0) + f(1)(1.25-.5) + f(1.5)(2.25-1.25) + f(2.5)(3-2.25)$
   $= .25^2 \times .5 + 1^2 \times .75 + 1.5^2 \times 1 + 2.5^2 \times .75 = 7.7188$

2. $f(x) = x^2$, $0 \leq x \leq 3$. $\sum_{i=1}^{5} f(w_i)\Delta_i x = f(w_1)\Delta_1 x + f(w_2)\Delta_2 x + f(w_3)\Delta_3 x + f(w_4)\Delta_4 x + f(w_5)\Delta_5 x$

   $= .5^2 \times (.75-0) + 1^2 \times (1.25-.75) + 1.75^2 \times (2-1.25) + 2.25^2 \times (2.75-2) + 2.75^2 \times (3-2.75) = 8.6719$

3. $f(x) = \frac{1}{x}$, $1 \leq x \leq 3$. $\sum_{i=1}^{4} f(w_i)\Delta_i x = f(w_1)\Delta_1 x + f(w_2)\Delta_2 x + f(w_3)\Delta_3 x + f(w_4)\Delta_4 x$

   $= \frac{1}{1.25} \times (1.67-1) + \frac{1}{2} \times (2.25-1.67) + \frac{1}{2.5} \times (2.67-2.25) + \frac{1}{2.75} \times (3-2.67) = 1.114$

4. $f(x) = \frac{1}{x+2}$; $-1 \le x \le 3$; $x_0 = -1$, $x_1 = -0.25$, $x_2 = 0$, $x_3 = 0.5$, $x_4 = 1.25$, $x_5 = 2$, $x_6 = 2.25$, $x_7 = 2.75$, $x_8 = 3$; $w_1 = -0.75$, $w_2 = 0$, $w_3 = 0.25$, $w_4 = 1$, $w_5 = 1.5$, $w_6 = 2$, $w_7 = 2.5$, $w_8 = 3$

▷ $\sum_{i=1}^{8} f(w_i)\Delta_i x$

$= f(w_1)\Delta_1 x + f(w_2)\Delta_2 x + f(w_3)\Delta_3 x + f(w_4)\Delta_4 x + f(w_5)\Delta_5 x + f(w_6)\Delta_6 x + f(w_7)\Delta_7 x + f(w_8)\Delta_8 x$

$= \frac{1}{-0.75+2} \times (-0.25+1) + \frac{1}{0+2} \times (0+0.25) + \frac{1}{0.25+2} \times (0.5-0) + \frac{1}{1+2} \times (1.25-0.5) + \frac{1}{1.5+2} \times (2-1.25)$

$+ \frac{1}{2+2} \times (2.25-2) + \frac{1}{2.5+2} \times (2.75-2.25) + \frac{1}{3+2} \times (3-2.75) = 1.6351$

5. $f(x) = \sin x$, $0 \le x \le \pi$. $\sum_{i=1}^{5} f(w_i)\Delta_i x = f(w_1)\Delta_1 x + f(w_2)\Delta_2 x + f(w_3)\Delta_3 x + f(w_4)\Delta_4 x + f(w_5)\Delta_5 x$

$= \sin \frac{1}{6}\pi \times (\frac{1}{4}\pi - 0) + \sin \frac{1}{3}\pi \times (\frac{1}{2}\pi - \frac{1}{4}\pi) + \sin \frac{1}{2}\pi \times (\frac{2}{3}\pi - \frac{1}{2}\pi) + \sin \frac{3}{4}\pi \times (\frac{3}{4}\pi - \frac{2}{3}\pi) + \sin \frac{5}{6}\pi \times (\pi - \frac{3}{4}\pi) = 2.1743$

6. $f(x) = 3 \cos \frac{1}{2}x$, $-\pi \le x \le \pi$. $\sum_{i=1}^{5} f(w_i)\Delta_i x = f(w_1)\Delta_1 x + f(w_2)\Delta_2 x + f(w_3)\Delta_3 x + f(w_4)\Delta_4 x + f(w_5)\Delta_5 x$

$= 3[\cos(-\frac{1}{3}\pi) \times (-\frac{1}{3}\pi + \pi) + \cos(-\frac{1}{6}\pi) \times (-\frac{1}{3}\pi + \frac{1}{2}\pi) + \cos 0 \times (\frac{1}{3}\pi + \frac{1}{3}\pi) + \cos \frac{1}{4}\pi \times (\frac{7}{12}\pi - \frac{1}{3}\pi) + \cos \frac{1}{3}\pi \times (\pi - \frac{7}{12}\pi)]$

$= 13.6293$

In Exercises 7–10, approximate the value of the definite integral two ways: (a) use a calculator to compute to four decimal places the corresponding Riemann sum with a regular partition of n subintervals and w_i as the indicated endpoint of each subinterval; (b) use the NINT capability of your calculator.

7. $f(x) = \frac{1}{x^2}$, $n = 9$, w_i is the right endpoint. $\Delta x = \frac{5-2}{9} = \frac{1}{3}$.

(a) $\int_2^5 f(x)dx \approx [f(\frac{7}{3}) + f(\frac{8}{3}) + f(\frac{9}{3}) + f(\frac{10}{3}) + f(\frac{11}{3}) + f(\frac{12}{3}) + f(\frac{13}{3}) + f(\frac{14}{3}) + f(\frac{15}{3})]\frac{1}{3}$

$= [\frac{9}{49} + \frac{9}{64} + \frac{9}{81} + \frac{9}{100} + \frac{9}{121} + \frac{9}{144} + \frac{9}{169} + \frac{9}{196} + \frac{9}{225}]\frac{1}{3} \approx 0.2672$; (b) 0.3000

8. $\int_3^4 \frac{1}{x}\,dx$, $n = 10$, w_i is the left endpoint.

▷ (a) $\Delta x = \frac{4-3}{10} = 0.1$ $\int_3^4 \frac{1}{x}\,dx \approx \left[\frac{1}{3} + \frac{1}{3.1} + \frac{1}{3.2} + \frac{1}{3.3} + \frac{1}{3.4} + \frac{1}{3.5} + \frac{1}{3.6} + \frac{1}{3.7} + \frac{1}{3.8} + \frac{1}{3.9}\right](0.1) = 0.2919$

(b) Using NINT, we find the value to be 0.2877

9. (a) $n = 8$, w_i is the left endpoint. $\Delta x = \frac{\frac{1}{3}\pi - (-\frac{1}{3}\pi)}{8} = \frac{1}{12}\pi$. $\int_{-\pi/3}^{\pi/3} \sec x\,dx \approx$

$\left[\sec(-\frac{4}{12}\pi) + \sec(-\frac{3}{12}\pi) + \sec(-\frac{2}{12}\pi) + \sec(-\frac{1}{12}\pi) + \sec 0 + \sec \frac{1}{12}\pi + \sec \frac{2}{12}\pi + \sec \frac{3}{12}\pi\right]\frac{1}{12}\pi \approx 2.6725$ (b) 2.6339

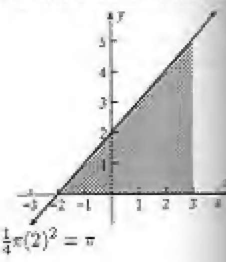
10. $f(x) = \tan x\,dx$; $n = 6$, w_i is the right endpoint. (a) $\Delta x = \frac{\frac{1}{3}\pi - \frac{1}{6}\pi}{6} = \frac{1}{36}\pi$.

$\int_{\pi/6}^{\pi/3} \csc x\,dx \approx [\csc \frac{7}{36}\pi + \csc \frac{8}{36}\pi + \csc \frac{9}{36}\pi + \csc \frac{10}{36}\pi + \csc \frac{11}{36}\pi + \csc \frac{12}{36}\pi]\frac{1}{36}\pi = 0.7325$ (b) 0.7677

In Exercises 11–26, determine the exact value of the definite integral by interpeting it as the measure of the area of a plane region. Check using NINT.

11. $\int_1^3 (x-1)dx = \text{area(triangle } (1,0), (3,0), (3,2)) = \frac{1}{2}(2)2 = 2$

12. $\int_{-2}^{3} (x+2)dx$



▷ The region, show in the figure at the right, is a triangle of base $3 - (-2) = 5$ and height 5. Therefore

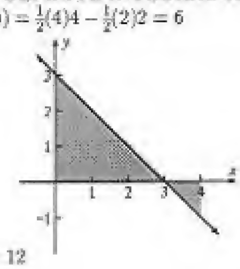
$\int_{-2}^{3} (x+2)dx = \frac{1}{2}(5)5 = \frac{25}{2}$

13. $\int_0^2 \sqrt{4-x^2}\,dx = \text{area(first quadrant part of circle centered at } (0,0) \text{ of radius } 2) = \frac{1}{4}\pi(2)^2 = \pi$

14. $\int_{-4}^{4} \sqrt{16-x^2}\,dx = \text{area(upper half of circle centered at } (0,0) \text{ of radius } 4) = \frac{1}{2}\pi(4)^2 = 8\pi$

15. $\int_{-2}^{4} x\,dx =$ area(triangle $(0,0)$, $(4,0)$, $(4,4)$) − area(triangle $(0,0)$, $(-2,0)$, $(-2,-2)$) $= \frac{1}{2}(4)4 - \frac{1}{2}(2)2 = 6$

16. $\int_{0}^{4}(3-x)dx$



▸ Let $f(x) = 3 - x$. Then $f(0) = 3$, $f(3) = 0$ and $f(4) = -1$. Hence the value of the integral is the measure of the area of the triangle $(0,0)$, $(0,3)$, $(3,0)$ above the x axis minus the measure of the area of the triangle $(3,0)$, $(4,0)$, $(4,-1)$ below the x axis. Therefore

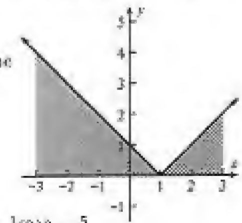
$\int_{0}^{4}(3-x) = \frac{1}{2}(3)3 - \frac{1}{2}(1)1 = \frac{9}{2} - \frac{1}{2} = 4$

17. $\int_{-1}^{2}(5-2x)dx =$ area(trapezoid $(-1,0),(-1,7),(2,1),(2,0)$) $= \frac{1}{2}(7+1)[2-(-1)] = 12$

18. $\int_{-1}^{2}(2x+5)dx =$ area(trapezoid $(-1,0),(-1,3),(2,9),(2,0)$) $= \frac{1}{2}(3+9)[2-(-1)] = 18$

19. $\int_{-2}^{4}|x|dx =$ area(triangle $(0,0)$, $(-2,0)$, $(-2,2)$) + area(triangle $(0,0)$, $(4,0)$, $(4,4)$) $= \frac{1}{2}(2)2 + \frac{1}{2}(4)4 = 10$

20. $\int_{-3}^{3}|1-x|\,dx$



▸ Let $f(x) = |1-x|$. Then $f(-3) = 4$, $f(1) = 0$ and $f(3) = 2$. Hence the value of the integral is the measure of the area of triangle $(1,0)$, $(-3,0)$, $(-3,4)$ plus the measure of the area of triangle $(1,0)$, $(3,0)$, $(3,2)$. Therefore

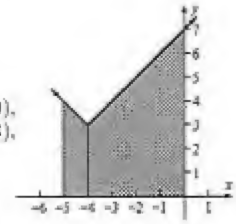
$\int_{-3}^{3}|1-x|dx = \frac{1}{2}(4)4 + \frac{1}{2}(2)2 = 10$

21. $\int_{0}^{5}(|x+3|-5)dx = \int_{0}^{5}[(x+3)-5]dx = \int_{0}^{5}(x-2)dx$
$=$ area(triangle $(2,0)$, $(5,0)$, $(5,3)$) − area(triangle $(2,0)$, $(0,0)$, $(0,-2)$) $= \frac{1}{2}(3)3 - \frac{1}{2}(2)2 = \frac{5}{2}$

22. Let $f(x) = |x-2| - 3$. Then $f(-1) = 3 - 3 = 0$, $f(2) = 0 - 3 = -3$, $f(5) = 3 - 3 = 0$, $f(7) = 5 - 3 = 2$.
$\int_{-1}^{7}(|x-2|-3)dx =$ area(triangle $(5,0)$, $(7,0)$, $(7,2)$) − area(triangle $(5,0)$, $(2,-3)$, $(-1,0)$)
$= \frac{1}{2}(2)(2) - \frac{1}{2}(6)3 = -7$

23. Let $f(x) = 6 - |x-2|$. Then $f(0) = 6 - 2 = 4$, $f(2) = 6 - 0 = 6$, $f(8) = 6 - 6 = 0$.
$\int_{0}^{8}(6-|x-2|)dx =$ area(trapezoid $(0,0)$, $(0,4)$, $(2,6)$, $(2,0)$) + area(triangle $(2,6)$, $(2,0)$, $(8,0)$)
$= \frac{1}{2}(4+6)(2) + \frac{1}{2}(6)6 = 28$

24. $\int_{-5}^{0}(3+|x+4|)dx$



▸ Let $f(x) = 3 + |x+4|$, $f(-5) = 3 + 1 = 4$, $f(-4) = 3 + 0 = 3$, $f(0) = 3 + 4 = 7$. Hence the value of the integral is the measure of the area of trapezoid $(-5,0)$, $(-5,4)$, $(-4,3)$, $(-4,0)$ plus the measure of the area of trapezoid $(-4,0)$, $(-4,3)$, $(0,7)$, $(0,0)$. Therefore

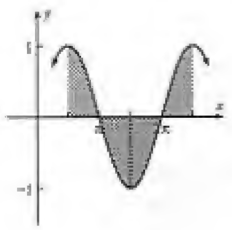
$\int_{-5}^{0}(3+|x+4|)dx = \frac{1}{2}(4+3)(1) + \frac{1}{2}(3+7)4 = \frac{47}{2}$

25. Because $y^2 = 2x - x^2 = 1 - (x^2 - 2x + 1) = 1 - (x-1)^2$; $(x-1)^2 + y^2 = 1$, then
$\int_{0}^{2}\sqrt{2x-x^2}dx =$ area(semicircle centered at $(1,0)$ of radius 1) $= \frac{1}{2}\pi(1)^2 = \frac{1}{2}\pi$

26. Because $y^2 = 5 + 4x - x^2 = 9 - (x^2 - 4x + 4) = 9 - (x-2)^2$; $(x-2)^2 + y^2 = 9$,
$\int_{-1}^{5}\sqrt{5+4x-x^2}\,dx =$ area(semicircle centered at $(2,0)$ radius 3) $= \frac{1}{2}\pi(3)^2 = \frac{9}{2}\pi$

27. $\int_{-\pi}^{\pi}\cos x\,dx = 0$ because each region below the x axis is congruent to one above.

28. $\int_{\pi/2}^{5\pi/2}\sin x\,dx$



▸ Because each region above the x axis is congruent to one below, the integral is 0.

In Exercises 29 and 30, apply Theorem 4.5.9 to determine the exact value of the integral.

29. (a) $\int_{2}^{5} 4\,dx = 4(5-2) = 12$ (b) $\int_{-3}^{4} 7\,dx = 7(4-(-3)) = 49$ (c) $\int_{-5}^{-10} dx = 1[-10-(-5)]$

30. (a) $\int_{5}^{-1} 6\,dx = 6(-1-5) = -36$ (b) $\int_{-2}^{2} \sqrt{5}\,dx = \sqrt{5}[2-(-2)] = 4\sqrt{5}$ (c) $\int_{3}^{3} dx = 1(3-3) = 0$

In Exercises 31–42, use the results below to to find the value of the integral. Check by NINT.

$$\int_{-1}^{2} x^2 dx = 3,\ \int_{-1}^{2} x\,dx = \tfrac{3}{2},\ \int_{0}^{\pi} \sin\,dx = 2,\ \int_{0}^{\pi} \cos x\,dx = 0,\ \int_{0}^{\pi} \sin^2 dx = \tfrac{1}{2}\pi$$

31. $\int_{-1}^{2}(2x^2-4x+5)dx = 2\int_{-1}^{2} x^2\,dx - 4\int_{-1}^{2} x\,dx + 5\int_{-1}^{2} dx = 2(3) - 4(\frac{3}{2}) + 5[2-(-1)] = 6 - 6 + 15 = 15$

32. $\int_{-1}^{2} (8-x^2)\,dx$

▷
$$\begin{aligned}
\int_{-1}^{2} (8-x^2)\,dx &= \int_{-1}^{2} [8+(-x^2)]\,dx \\
&= \int_{-1}^{2} 8\,dx + \int_{-1}^{2} (-x^2)\,dx && \text{(Theorem 4.5.} \\
&= \int_{-1}^{2} 8\,dx - \int_{-1}^{2} x^2\,dx && \text{(Theorem 4.5.} \\
&= 8[2-(-1)] - 3 && \text{(Theorem 4.5.9 and given va} \\
&= 21
\end{aligned}$$

33. $\int_{-1}^{2}(2-5x+\frac{1}{2}x^2)dx = 2\int_{-1}^{2} dx - 5\int_{-1}^{2} x\,dx + \frac{1}{2}\int_{-1}^{2} x^2 dx = 2[2-(-1)] - 5(\frac{3}{2}) + \frac{1}{2}(3) = 6 - \frac{15}{2} + \frac{3}{2} = 0$

34. $\int_{-1}^{2}(3x^2-4x-1)dx = 3\int_{-1}^{2} x^2 dx - 4\int_{-1}^{2} x\,dx - \int_{-1}^{2} dx = 3(3) - 4(\frac{3}{2}) - 1[2-(-1)] = 9 - 6 - 3 = 0$

35. $\int_{2}^{-1}(2x+1)^2 dx = -\int_{-1}^{2}(4x^2+4x+1)dx = -4\int_{-1}^{2} x^2 dx - 4\int_{-1}^{2} x\,dx - \int_{-1}^{2} dx$
$= -4(3) - 4(\frac{3}{2}) - 1\cdot[2-(-1)] = -12 - 6 - 3 = -21$

36. $\int_{-1}^{2} (5x^2 + \frac{1}{3}x - \frac{1}{2})\,dx$

▷ $\int_{-1}^{2} (5x^2 + \frac{1}{3}x - \frac{1}{2})\,dx$
$$\begin{aligned}
&= \int_{-1}^{2} 5x^2\,dx + \int_{-1}^{2} \tfrac{1}{3}x\,dx + \int_{-1}^{2} (-\tfrac{1}{2})\,dx && \text{(Theorem 4.5.} \\
&= 5\int_{-1}^{2} x^2\,dx + \tfrac{1}{3}\int_{-1}^{2} x\,dx - \int_{-1}^{2} \tfrac{1}{2}\,dx && \text{(Theorem 4.5.} \\
&= 5(3) + \tfrac{1}{3}(\tfrac{3}{2}) - \tfrac{1}{2}(3) && \text{(given values and 4.5.} \\
&= 14
\end{aligned}$$

37. $\int_{-1}^{2}(x-1)(2x+3)dx = \int_{-1}^{2}(2x^2+x-3)dx = 2\int_{-1}^{2} x^2 dx + \int_{-1}^{2} x\,dx - 3\int_{-1}^{2} dx$
$= 2(3) + \frac{2}{3} - 3[2-(-1)] = 6 + \frac{2}{3} - 9 = -\frac{2}{3}$

38. $\int_{2}^{-1} 3x(x-4)dx = -\int_{-1}^{2}(3x^2-12x)dx = -\left[3\int_{-1}^{2} x^2 dx - 12\int_{-1}^{2} x\,dx\right] = -[3(3) - 12(\frac{3}{2})] = 9$

39. $\int_{0}^{\pi}(2\sin x + 3\cos x + 1)dx = 2\int_{0}^{\pi} \sin x\,dx + 3\int_{0}^{\pi} \cos x\,dx + \int_{0}^{\pi} dx = 2(2) + 3(0) + 1\cdot[\pi - 0] = 4 + \pi$

40. $\int_{0}^{\pi} 3\cos^2 x\,dx$

▷ $\int_{0}^{\pi} 3\cos^2 x\,dx = 3\int_{0}^{\pi} \cos^2 x\,dx = 3\int_{0}^{\pi} (1-\sin^2 x)\,dx = 3\left[\int_{0}^{\pi} dx - \int_{0}^{\pi} \sin^2 x\,dx\right] = 3[\pi - \frac{1}{2}\pi] = \frac{3}{2}\pi$

41. $\int_{0}^{\pi}(\cos x + 4)^2 dx = \int_{0}^{\pi}(\cos^2 x + 8\cos x + 16)dx = \int_{0}^{\pi}(1 - \sin^2 x + 8\cos x + 16)dx$
$= 17\int_{0}^{\pi} dx - \int_{0}^{\pi} \sin^2 x\,dx + 8\int_{0}^{\pi} \cos x\,dx = 17\cdot[\pi - 0] - \frac{1}{2}\pi + 8\cdot 0 = \frac{33}{2}\pi$

42. $\int_{\pi}^{0}(\sin x-2)^2dx = -\int_{0}^{\pi}(\sin^2 x - 4\sin x + 4)dx = -\left[\int_0^{\pi}\sin^2 x\, dx - 4\int_0^{\pi}\sin x\, dx + \int_0^{\pi} 4\, dx\right]$
$= -[\frac{1}{2}\pi - 4(2) + 4(\pi - 0)] = 8 - \frac{9}{2}\pi$

In Exercises 43–48, use Theorem 4.5.12 to prove $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (11) for the given ordering.

43. $b < a < c$. From Theorem 4.5.12
$\int_b^c f(x)dx = \int_b^a f(x)dx + \int_a^c f(x)dx$
Applying Definition 4.5.5 twice,
$-\int_c^b f(x)dx = -\int_a^b f(x)dx + \int_a^c f(x)dz$
which is equivalent to (11).

44. $c < a < b$. From Theorem 4.5.12
$\int_c^b f(x)dx = \int_c^a f(x)dx + \int_a^b f(x)dx$
Applying Definition 4.5.5,
$\int_c^b f(x)dx = -\int_a^c f(x)dx + \int_a^b f(x)dx$
which is equivalent to (11).

45. $b < c < a$. From Theorem 4.5.12
$\int_b^a f(x)dx = \int_b^c f(x)dx + \int_c^a f(x)dx$
Applying Definition 4.5.5 three times
$-\int_a^b f(x)dx = -\int_c^b f(x)dx - \int_a^c f(x)dx$
which is equivalent to (11).

46. $c < b < a$. From Theorem 4.5.12
$\int_c^a f(x)dx = \int_c^b f(x)dx + \int_b^a f(x)dx$
Applying Definition 5.5.5 twice
$-\int_a^c f(x)dx = \int_c^b f(x)dx - \int_a^b f(x)dx$
which is equivalent to (11).

47. $a = b < c$. From Definition 4.5.5, because $a = b$, $\int_a^c f(x)dx = -\int_c^b f(x)dx$ and from Definition 4.5.6 because $a = b$, $\int_a^b f(x)dx = 0$. Thus $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

48. $a < c = b$

▶ Because $b = c$, then $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx$

and $\int_c^b f(x)\, dx = \int_b^b f(x)\, dx = 0$

Thus, $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$

49. For each positive integer n, consider the regular partition of $[0,2]$ for which $\Delta x = \frac{2}{n}$. Let $w_i = \frac{2i}{n}$ for $1 \le i \le n$. If f is the function for which $f(x) = x^2$, then the corresponding Riemann sum is
$\sum_{i=1}^{n} f(w_i)\Delta x = \sum_{i=1}^{n}\frac{4i^2}{n^2}\cdot\frac{2}{n} = \sum_{i=1}^{n}\frac{8i^2}{n^3}$. Hence $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{8i^2}{n^3} = \int_0^2 x^2 dx$.

50. For each positive integer n, consider the regular partition of $[1,2]$ for which $\Delta x = \frac{1}{n}$. Let $w_i = 1 + \frac{i}{n} = \frac{n+i}{n}$ for $1 \le i \le n$. If f is the function for which $f(x) = 1/x$, then the corresponding Riemann sum is
$\sum_{i=1}^{n} f(w_i)\Delta x = \sum_{i=1}^{n}\frac{n}{n+i}\cdot\frac{1}{n} = \sum_{i=1}^{n}\frac{1}{n+i}$. Hence $\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{1}{n+i} = \int_1^2 \frac{1}{x}\, dx$

51. For each positive integer n, consider the regular partition of $[1,2]$ for which $\Delta x = \frac{1}{n}$. Let $w_i = 1 + \frac{i}{n} = \frac{i+n}{n}$ for $1 \le i \le n$. If f is the function for which $f(x) = 1/x^2$, then the corresponding Riemann sum is
$\sum_{i=1}^{n} f(w_i)\Delta x = \sum_{i=1}^{n}\frac{n^2}{(i+n)^2}\cdot\frac{1}{n} = \sum_{i=1}^{n}\frac{n}{(i+n)^2}$. Therefore $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{(i+n)^2} = \int_1^2\frac{1}{x^2}\, dx$.

52. Show that if f is continuous on $[-1,2]$, then $\int_{-1}^{2} f(x)\, dx + \int_2^0 f(x)\, dx + \int_0^1 f(x)\, dx + \int_1^{-1} f(x)\, dx = 0$

▶ Because f is continuous on $[-1,2]$, then f is integrable on $[-1,2]$. Thus,

$$\begin{aligned}
&\int_{-1}^{2} f(x)\, dx + \int_2^0 f(x)\, dx + \int_0^1 f(x)\, dx + \int_1^{-1} f(x)\, dx \\
&= \int_{-1}^{0} f(x)\, dx + \int_0^1 f(x)\, dx + \int_1^{-1} f(x)\, dx && \text{(Theorem 4.5.11)} \\
&= \int_{-1}^{1} f(x)\, dx + \int_1^{-1} f(x)\, dx && \text{(Theorem 4.5.11)} \\
&= \int_{-1}^{-1} f(x)\, dx && \text{(Theorem 4.5.11)} \\
&= 0 && \text{(Definition 4.5.6)}
\end{aligned}$$

53. Because f is continuous on $[-3,4]$, then from Theorem 4.5.3, f is integrable on $[-3,4]$. Applying Theorem 4.5.11 and then Definition 4.5.5 we get

$$\int_3^{-1} f(x)dx + \int_4^3 f(x)dx + \int_{-3}^4 f(x)dx + \int_{-1}^{-3} f(x)dx = \left(\int_{-3}^4 f(x)dx + \int_4^3 f(x)dx\right) + \left(\int_3^{-1} f(x)dx + \int_{-1}^{-3} f(x)dx\right)$$
$$= \int_{-3}^3 f(x)dx + \int_3^{-3} f(x)dx = \int_{-3}^3 f(x)dx - \int_{-3}^3 f(x)dx = 0$$

54. From Theorem 5.4.3, $\sum\limits_{i=1}^{n} kf(w_i)\Delta_i x = k\sum\limits_{i=1}^{n} f(w_i)\Delta_i x$. Therefore

$$\lim_{\|\Delta\|\to 0} \sum kf(w_i)\Delta_i x = \lim_{\|\Delta\|\to 0} k\sum f(w_i)\Delta_i x = k \lim_{\|\Delta\|\to 0} \sum f(w_i)\Delta_i x$$

55. If $a = b$, by Definition 4.5.6, $\int_a^b k\,dx = 0 = k(b-a)$.

If $a > b$, by Definition 4.5.5 and Theorem 4.5.9, $\int_a^b k\,dx = -\int_b^a k\,dx = -[k(a-b)] = k(b-a)$

56. Prove Theorem C.

▸ By Theorem 4.5.13, $\int_{-r}^{r} f(x)dx = \int_{-r}^{0} f(x)dx + \int_0^r f(x)dx = \text{I} + \text{J}$. Let $\text{T} = \sum\limits_{i=1}^{n} f(w_i)\Delta_i x$ be any Riemann sum for J. Then $\text{S} = \sum\limits_{i=1}^{n} f(-w_i)\Delta_i x$ is a Riemann sum for I and $\int_{-r}^{r} f(x)dx = \lim\limits_{n\to+\infty} \text{S} + \lim\limits_{n\to+\infty} \text{T} = \lim\limits_{n\to+\infty} (\text{S}+\text{T})$.

(a) Suppose f is an even function. Then $\text{S} = \sum\limits_{i=1}^{n} f(w_i)\Delta_i x = \text{T}$ and so $\text{I} + \text{J} = \lim\limits_{n\to+\infty} 2\text{T} = 2\text{J}$.

(b) Suppose f is an odd function. Then $\text{S} = \sum\limits_{i=1}^{n} -f(w_i)\Delta_i x = -\text{T}$ and so $\text{I} + \text{J} = \lim\limits_{n\to+\infty} (-\text{T}+\text{T}) = 0$.

4.6 THE MEAN-VALUE THEOREM FOR INTEGRALS

4.6.1 Theorem If the functions f and g are integrable on the closed interval $[a,b]$ and if $f(x) \geq g(x)$ for all x in $[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

If, in addition, f and g are continuous and not identical on $[a,b]$, then

$$\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$$

4.6.2 Theorem Suppose that the function f is continuous on the closed interval $[a,b]$. If m and M are, respectively, the absolute minimum and absolute maximum function values of f on $[a,b]$ so that $m \leq f(x) \leq \text{M}$ for $a \leq x \leq b$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq \text{M}(b-a)$$

Triangle Inequality (Exercise 50) If $f(x)$ is integrable on $[a,b]$ then (a) $|f(x)|$ is integrable on $[a,b]$ and (b)

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

4.6.3 Mean-Value Theorem for Integrals If the function f is continuous on the closed interval $[a,b]$, then there exists a number c in (a,b) such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

4.6.4 Definition If the function f is integrable on the closed interval $[a,b]$, then the *average value* of f on $[a,b]$ is

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a}$$

Second Mean-Value Theorem for Integrals (Exercise 52) If f and g are two functions continuous on the closed interval $[a,b]$ and $g(x) > 0$ for all x in the open interval (a,b), then there exists a number c in (a,b) such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

Exercises 4.6

In Exercises 1–4, apply Theorem 4.6.1 to determine which of the symbols $\geq$ or $\leq$ should be inserted in the blank to make the following inequality correct. Check using NINT.

1. Because $2x^2 - 4 \geq x^2 - 6$ for all x, it follows from Theorem 4.6.1 that $\int_{-1}^{3} (2x^2 - 4)dx \geq \int_{-1}^{3} (x^2 - 6)dx$.

2. Let ® be the required symbol. The following inequalities are equivalent: $\sqrt{6-x}$ ® $\sqrt{x-2}$, $6 - x$ ® $x - 2$, 8 ® $2x$, 4 ® x. Because $4 \leq x$ on the interval $[4,5]$, then ® is $\leq$ and we conclude that $\sqrt{6-x} \leq \sqrt{x-2}$ on $[4,5]$. Hence, by Theorem 4.6.1, $\int_4^5 \sqrt{6-x}\,dx \leq \int_4^5 \sqrt{x-2}\,dx$

3. If x is in $[\frac{5}{4}\pi, \frac{3}{4}\pi]$ then $\sin^2 x \leq \cos^2 x$. Therefore, from Theorem 4.6.1 $\int_{3\pi/4}^{5\pi/4} \sin^2 x\,dx \leq \int_{3\pi/4}^{5\pi/4} \cos^2 x\,dx$

4. If x is in $[0, \frac{1}{4}\pi]$ then $\cos x \geq \sin x$. Therefore, from Theorem 4.6.1 $\int_0^{\pi/4} \cos x\,dx \geq \int_0^{\pi/4} \sin x\,dx$

In Exercises 5–20, apply Theorem 4.6.2 to find an interval containing the value of the integral. Check by NINT.

5. If x is in $0 \leq x \leq 0.5$ then $0 \leq x^2 \leq 0.5^2 = 0.25$. Thus from Theorem 4.6.2
$0(0.5 - 0) \leq \int_0^{0.5} x^2 dx \leq 0.25(0.5 - 0)$; $0 \leq \int_0^{0.5} x^2 dx \leq 0.125$

6. If $-0.5 \leq x \leq 1$ then $(-0.5)^3 = -0.125 \leq x^3 \leq 1$. Thus from Theorem 4.6.2
$-0.125[1 - (-0.5)] \leq \int_{-0.5}^{1} x^3 dx \leq 1[1 - (-0.5)]$; $-0.1875 \leq \int_{-0.5}^{1} x^3 dx \leq 1.5$

7. If $-1 \leq x \leq 1$ then $\sqrt{2-1} = 1 \leq \sqrt{2+x} \leq \sqrt{3} = \sqrt{2+1}$. Thus from Theorem 4.6.2
$1[1 - (-1)] \leq \int_{-1}^{1} \sqrt{2+x}\,dx \leq \sqrt{3}[1 - (-1)]$; $2 \leq \int_{-1}^{1} \sqrt{2+x}\,dx \leq 2\sqrt{3}$

8. $\int_{-2}^{1} (x+1)^{2/3}\,dx$

▸ Let $f(x) = (x+1)^{2/3}$. Because $f'(x) = \frac{2}{3}(x+1)^{-1/3}$, the only critical number of f is -1. Because

$f(-1) = 0$ $\qquad$ $f(1) = 2^{2/3}$ $\qquad$ $f(-2) = 1$

then the absolute minimum and maximum values of f on the closed interval $[a,b] = [-2,1]$ are $m = 0$ and $M = 2^{2/3}$. Because $b - a = 3$, by Theorem 4.6.2 we conclude that

$$0 \leq \int_{-2}^{1} (x+1)^{2/3}\,dx \leq 3 \cdot 2^{2/3}$$

9. If x is in $[\frac{1}{6}\pi, \frac{1}{3}\pi]$, then $\frac{1}{2} \leq \sin x \leq \frac{1}{2}\sqrt{3}$. Thus from Theorem 4.6.2
$\frac{1}{2}(\frac{1}{3}\pi - \frac{1}{6}\pi) \leq \int_{\pi/6}^{\pi/3} \sin x\,dx \leq \frac{1}{2}\sqrt{3}(\frac{1}{3}\pi - \frac{1}{6}\pi)$; $\frac{1}{12}\pi \leq \int_{\pi/6}^{\pi/3} \sin x\,dx \leq \frac{1}{12}\sqrt{3}\pi$

10. Let $f(x) = \cos x$. Because $f'(x) = -\sin x$, the only critical number of f in $[-\frac{1}{3}\pi, \frac{2}{3}\pi]$ is 0. Now

$f(-\frac{1}{3}\pi) = \cos(-\frac{1}{3}\pi) = \frac{1}{2}$ $\qquad$ $f(0) = \cos 0 = 1$ $\qquad$ $f(\frac{2}{3}\pi) = \cos \frac{2}{3}\pi = -\frac{1}{2}$

Thus, $m = -\frac{1}{2}$ and $M = 1$ are the absolute minimum and maximum values of f on the interval $[-\frac{1}{3}\pi, \frac{2}{3}\pi]$.
Because $\frac{2}{3}\pi - (-\frac{1}{3}\pi) = \pi$, by Theorem 4.6.2 we have $-\frac{1}{2}\pi \leq \int_{-\pi/3}^{2\pi/3} \cos x\,dx \leq \pi$

11. If x is in $[1.5,3]$, then $0 \leq |x-2| \leq 1$. Hence from Theorem 4.6.2,
$0(3 - 1.5) \leq \int_{1.5}^{3} |x-2|dx \leq 1(3 - 1.5)$; $0 \leq \int_{1.5}^{3} |x-2|dx \leq 1.5$

12. $\int_{-1}^{2} \sqrt{x^2+5}\,dx$

▸ Let $f(x) = \sqrt{x^2+5}$. Because $f'(x) = x/\sqrt{x^2+5}$ the only critical number of f in $[-1,2]$ is 0. Now

$f(-1) = \sqrt{6}$ $\qquad$ $f(0) = \sqrt{5}$ $\qquad$ $f(2) = \sqrt{9} = 3$

Thus, $m = \sqrt{5}$ and $M = 3$ are the absolute minimum and maximum values of f on the interval $[-1,2]$.
Because $2 - (-1) = 3$, by Theorem 4.6.2 we have $3\sqrt{5} \leq \int_{-1}^{2} \sqrt{x^2+5}\,dx \leq 3 \cdot 3 = 9$

13. Let $f(x) = \frac{1}{4}x^4 - x^3 + x^2$. Because $f'(x) = x^3 - 3x^2 + 2x = x(x-1)(x-2)$, the critical numbers are 0 and 1. $f(-\frac{1}{2}) = \frac{25}{64}$, $f(0) = 0$, $f(1) = \frac{1}{4}$, $f(\frac{3}{2}) = \frac{9}{64}$. Thus $m = 0$ and $M = \frac{25}{64}$ are the absolute minimum and maximum. Because $1.5 - (-0.5) = 2$, by Theorem 4.6.2, $0 = 2 \cdot 0 \le \int_{-0.5}^{1.5} (\frac{1}{4}x^4 - x^3 + x^2)dx \le 2 \cdot \frac{25}{64} = \frac{25}{32}$

14. Let $f(x) = x - 3x^{1/3}$. Because $f'(x) = 1 - x^{-2/3}$ the only critical number of f in $[0, 1.5]$ is 1. Now
$f(0) = 0$ $\quad f(1) = -2 \quad$ $f(1.5) = -1.93$
Thus, $m = -2$ and $M = 0$ are the absolute minimum and maximum values of f on the interval $[0, 1.5]$.
Because $1.5 - 0 = 1.5$, by Theorem 4.6.2 we have $-3 = 1.5(-2) \le \int_0^{1.5} (x - 3x^{1/3})dx \le 1.5 \cdot 0 = 0$

15. Let $f(x) = x^2(5 - x^2) = 5x^2 - x^4$. Because $f'(x) = 10x - 4x^3 = 4x(2.5 - x^2)$ the only critical number of f in $[1,2]$ is $\sqrt{2.5}$. Now $f(1) = 4$, $f(\sqrt{2.5}) = 2.5^2$, $f(2) = 4$. Thus, $m = 4$ and $M = 2.5^2$ are the absolute minimum and maximum values of f on the interval $[1,2]$. Because $2 - 1 = 1$, by Theorem 4.6.2 we have

$$2 \le \int_1^2 x\sqrt{5 - x^2}\, dx = \int_1^2 \sqrt{f(x)}\, dx \le 2.5$$

16. $\int_{1.5}^{2.5} x\sqrt{3 - x}\, dx$

▸ Let $f(x) = x^2(3 - x) = 3x^2 - x^3$. Because $f'(x) = 6x - 3x^2 = 3x(2 - x)$, the only critical number of f in $[1.5, 2.5]$ is 2. Now $f(1.5) = 3.375$, $f(2) = 4$, $f(2.5) = 3.125$. Thus, $m = 3.125$ and $M = 4$ are the absolute minimum and maximum values of f on the interval $[1.5, 2.5]$. Because $2.5 - 1.5 = 1$, by Theorem 4.6.2 we have

$$\sqrt{3.125} \le \int_{1.5}^{2.5} x\sqrt{3 - x}\, dx = \int_{1.5}^{2.5} \sqrt{f(x)}\, dx \le \sqrt{4} = 2$$

17. If $f(x) = \frac{x}{x+2}$, $f'(x) = \frac{2}{(x+2)^2}$. Because $f'(x) > 0$ for all $x \ne -2$, f is increasing on $[-1, 1]$. Hence if x is in $[-1, 1]$, then $-1 \le f(x) \le \frac{1}{3}$. Because $1 - (-1) = 2$, by Theorem 4.6.2, $2(-1) = -2 \le \int_{-1}^{1} \frac{x}{x+2}\, dx \le 2 \cdot \frac{1}{3} = \frac{2}{3}$

18. Let $f(x) = \frac{x+5}{x-3} = 1 + \frac{8}{x-3}$. Because f is decreasing on $[0, 1]$, then $m = f(1) = -3$ and $M = f(0) = -\frac{5}{3}$ are the absolute minimum and maximum values of f on $[0, 1]$. Because $1 - 0 = 1$, by Theorem 4.6.2 we have

$$-3 \le \int_0^1 \frac{x+5}{x-3}\, dx \le -\frac{5}{3}$$

19. If $f(x) = 4\cos^3 x - 9\cos x$, then $f'(x) = -12\cos^2 x \sin x + 9\sin x = 12\sin x(\frac{3}{4} - \cos^2 x) \ge 0$ on $[\frac{1}{3}\pi, \frac{1}{2}\pi]$. Hence $m = f(\frac{1}{3}\pi) = -4$ and $M = f(\frac{1}{2}) = 0$ are the absolute minimum and maximum values of f on $[\frac{1}{3}\pi, \frac{1}{2}\pi]$. Because $\frac{1}{2}\pi - \frac{1}{3}\pi = \frac{1}{6}\pi$, by Theorem 4.6.2 we have $\frac{1}{6}\pi(-4) = -\frac{2}{3}\pi \le \int_{\pi/3}^{\pi/2} (4\cos^3 x - 9\cos x)dx \le 0$

20. $\int_{-\pi/6}^{\pi/6} \sin^3 x\, dx$

▸ Because $\sin^3 x$ is an odd function and the interval $[-\frac{1}{6}\pi, \frac{1}{6}\pi]$ is symmetrical, the value of the integral is 0 by Theorem 4.5.C.

In Exercises 21–30, find to the nearest one-hundredth the value of c satisfying the mean-value theorem for integrals. Use NINT to approximate the value of the integral to four digits.

21. $\int_0^2 x^2\, dx = \frac{8}{3}$. By the mean-value theorem for integrals there exists a number c in $[0,2]$ such that $\int_0^1 x^2 dx = c^2(2)$; $\frac{8}{3} = 2c^2$; $c^2 = \frac{4}{3}$; $c = \pm\frac{2}{3}\sqrt{3}$; $\frac{2}{3}\sqrt{3} \approx 1.1547 \approx 1.15$ is in $[0, 2]$.

22. $\int_2^4 x^2 dx = \frac{56}{3}$. By the mean-value theorem for integrals there exists a number c in $[2, 4]$ such that $c^2 = \frac{1}{2}\int_2^4 x^2 dx = \frac{1}{2} \cdot \frac{56}{3} = \frac{4}{9} \cdot 21$; $c = \pm\frac{2}{3}\sqrt{21}$; $\frac{2}{3}\sqrt{21} \approx 3.055 \approx 3.06$ is in $[2, 4]$.

23. $\int_1^2 x^3 dx = \frac{15}{4}$. By the mean-value theorem for integrals there exists a number c in $[1, 2]$ such that $\int_1^2 x^3 dx = c^3(1)$; $c^3 = \frac{15}{4} = \frac{30}{8}$; $c = \frac{1}{2}\sqrt[3]{30} \approx 1.554 \approx 1.56$

24. $\int_0^5 (x^3 - 1)\, dx$

► We have $f(x) = x^3 - 1$ and $[a, b] = [0, 5]$. We wish to find a number c with $0 < c < 5$, such that

$$\int_0^5 (x^3 - 1)\, dx = f(c)(5) \tag{1}$$

Because the value of the definite integral in Eq. (1) is $\frac{605}{4}$, Eq. (1) is equivalent to

$\frac{605}{4} = (c^3 - 1)5; \qquad c^3 = \frac{125}{4} = \frac{125}{8} \cdot 2; \qquad c = \frac{5}{2}\sqrt[3]{2} \approx 3.15$

25. $\int_1^4 (x^2 + 4x + 5)dx = 66$. By the mean-value theorem for integrals there exists a number c in $[1, 4]$ such that $\int_1^4 (x^2 + 4x + 5)dx = (c^2 + 4c + 5)(3)$; $66 = (c^2 + 4c + 5)(3)$; $c^2 + 4c - 17 = 0$

$c = \frac{1}{2}(-4 \pm \sqrt{16 + 68}) = -2 \pm \sqrt{21}$; $-2 + \sqrt{21} \approx 2.6$ is in $[1, 4]$.

26. $\int_0^4 (x^2 + x - 6)dx = \frac{16}{3}$. By the mean-value theorem for integrals there exists a number c in $[0, 4]$ such that $\int_0^4 (x^2 + 4x + 5)dx = (c^2 + c - 6)(4)$; $c^2 + c - 6 = \frac{4}{3}$; $c = \frac{1}{6}(-3 \pm \sqrt{273})$; $\frac{1}{6}(-3 + \sqrt{273}) \approx 2.254$ is in $[0, 4]$.

27. $\int_{-2}^2 (x^3 + 1)dx = 4$. By the mean-value theorem for integrals there exists a number c in $[-2, 2]$ such that $\int_{-2}^2 (x^3 + 1)dx = (c^3 + 1)(4)$; $4 = (c^3 + 1)(4)$; $c^3 = 0$; $c = 0$

28. $\int_{-2}^1 x^4\, dx$

► Because the value of the given integral is $\frac{33}{5}$, we wish to find a number c with $-2 < c < 1$, such that

$\frac{33}{5} = 3c^4; \qquad c^4 = \frac{11}{5}; \qquad c = \pm\sqrt[4]{\frac{11}{5}} \approx \pm 1.22$

Since 1.22 is not in the interval $(-2, 1)$, the only suitable value of c is -1.22

29. $\int_2^4 \frac{1}{x^2 - 3}\, dx \approx 0.4927$. By the mean-value theorem for integrals there exists a number c in $[-2, 2]$ such that $\int_2^4 \frac{1}{x^2 - 3}\, dx = \frac{1}{c^2 - 3}(2)$; $c^2 - 3 = \frac{2}{0.4927}$; $c^2 = 7.0593$; $c = \pm 2.657$; 2.657 is in $[2, 4]$.

31. $\int_{\pi/6}^{\pi/4} \tan x\, dx \approx 0.2027$. By the mean-value theorem for integrals there exists a number c in $[\frac{1}{6}\pi, \frac{1}{4}\pi]$ such that $\int_{\pi/6}^{\pi/4} \tan x\, dx = (\tan c)\frac{\pi}{12}$; $\tan c = \frac{12 \times 0.2027}{\pi} = 0.774$; $0.658 \approx 0.66$

32. $\int_{2\pi/3}^{5\pi/6} \cot x\, dx$

► Because the value of the given integral is -0.5493, we wish to find a number c with $\frac{2}{3}\pi < c < \frac{5}{6}\pi$, such that

$-0.5493 = \frac{\pi}{6}\cot c; \qquad \tan c = -\frac{\pi}{0.5493} = -5.719; \qquad c = \tan^{-1}(-5.719) + \pi = 1.744 \approx 1.74$

Exercises 33–40, use the mean-value theorem for integrals to prove the inequality.

33. By the mean-value theorem for integrals there exists a number c in $[0, 2]$ such that $\int_0^2 \frac{1}{x^2 + 4}\, dx = \frac{1}{c^2 + 4}(2)$. But $\frac{1}{c^2 + 4} \leq \frac{1}{4}$ because c is in $[0, 2]$. Therefore $\int_0^2 \frac{1}{x^2 + 4}\, dx \leq \frac{1}{4} \cdot 2 = \frac{1}{2}$.

34. By the mean-value theorem for integrals there exists a number c in $[0, 2]$ such that $\int_{-3}^3 \frac{1}{x^2 + 6}\, dx = \frac{1}{c^2 + 6}(6)$. But $\frac{6}{c^2 + 6} \leq 1$ because $c^2 \geq 0$. Therefore $\int_{-3}^3 \frac{1}{x^2 + 6}\, dx \leq 1$.

35. By the mean-value theorem for integrals there exists a number c in $[-\frac{1}{6}\pi, \frac{1}{6}\pi]$ such that

$\int_{-\pi/6}^{\pi/6} \cos x^2 dx = (\cos c^2)\frac{1}{3}\pi$. But $\cos c^2 \leq 1$. Therefore $\int_{-\pi/6}^{\pi/6} \cos x^2 dx \leq \frac{1}{3}\pi$.

36. $\displaystyle\int_0^{\pi} \sin\sqrt{x}\,dx \le \pi$

▷ Because $\sin\sqrt{x}$ is continuous on the closed interval $[0,\pi]$, by the mean-value theorem for integrals there exists a number c such that $0 < c < \pi$, and

$$\int_0^{\pi} \sin\sqrt{x}\,dx = (\sin\sqrt{c})\pi$$

Because $\sin t \le 1$ for all t, then $\sin\sqrt{c} \le 1$. Therefore,

$$\int_0^{\pi} \sin\sqrt{x}\,dx \le \pi$$

37. By the mean-value theorem for integrals there exists a number c in $[2,5]$ such that $\displaystyle\int_2^5 \frac{1}{x^3+1}\,dx = \frac{1}{c^3+1}(3)$.

But $0 \le \dfrac{1}{c^3+1} \le \dfrac{1}{2^3+1} = \frac{1}{9}$ if c is in $[2,5]$. Therefore $0 \le \displaystyle\int_2^5 \frac{1}{x^3+1}\,dx \le \frac{1}{3}$.

38. By the mean-value theorem for integrals there is a number c in $[5,9]$ such that $\displaystyle\int_5^9 \frac{1}{\sqrt{x}-1}\,dx = \frac{1}{\sqrt{c}-1}(4)$.

But $0 \le \dfrac{4}{\sqrt{c}-1} \le \dfrac{4}{\sqrt{5}-1} = 2$ if c is in $[5,9]$. Therefore $0 \le \displaystyle\int_5^9 \frac{1}{\sqrt{x}-1}\,dx \le 2$

39. By the mean-value theorem for integrals there exists a number c in $[0,2]$ such that

$\displaystyle\int_0^2 \sin\tfrac{1}{2}\pi x\,dx = (\sin\tfrac{1}{2}\pi c)(2)$. But $0 \le \sin\frac{1}{2}\pi c \le 1$ if c is in $[0,2]$. Therefore $0 \le \displaystyle\int_0^2 \sin\tfrac{1}{2}\pi x\,dx \le 2$.

40. $0 \le \displaystyle\int_{-1/2}^{1/2} \cos\pi x\,dx \le 1$

▷ Because $\cos\pi x$ is continuous on the closed interval $[-\frac{1}{2},\frac{1}{2}]$, by the mean-value theorem for integrals there is a number c in $(-\frac{1}{2},\frac{1}{2})$, such that

$$\int_{-1/2}^{1/2} \cos\pi x\,dx = (\cos\pi c)[\tfrac{1}{2} - (-\tfrac{1}{2})] = \cos\pi x$$

Because πc is in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, then $0 < \cos\pi c \le 1$. Therefore

$$0 < \int_{-1/2}^{1/2} \cos\pi x\,dx \le 1$$

41. Given $\displaystyle\int_{-1}^{2} x\,dx = \frac{3}{2}$, the average value of x on $[-1,2]$ is $\dfrac{1}{2-(-1)}\displaystyle\int_{-1}^{2} x\,dx = \frac{1}{3}\cdot\frac{3}{2} = \frac{1}{2}$ and occurs at $x = \frac{1}{2}$.

42. Given $\displaystyle\int_{-1}^{2} x^2 dx = 3$, the average value of x^2 on $[-1,2]$ is $\dfrac{1}{2-(-1)}\displaystyle\int_{-1}^{2} x^2 dx = \frac{1}{3}\cdot 3 = 1$ and occurs at $x = 1$

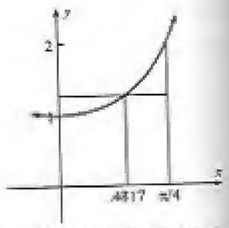
43. Given $\displaystyle\int_0^{\pi} \sin x\,dx = 2$, the average value of the sine function on $[0,\pi]$ is $\dfrac{1}{\pi-0}\displaystyle\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}$

44. Find the average value of the function f defined by $f(x) = \sec^2 x$ on the interval $[0,\frac{1}{4}\pi]$ given the $\int_0^{\pi/4} \sec^2 x\,dx = 1$. Also find the value of x at which the average value occurs. Describe the geometric interpretation of the results.

▷ We apply Definition 4.6.4. The average value of f on $[0,\frac{1}{4}\pi]$ is

$$\frac{1}{\pi/4}\int_0^{\pi/4} \sec^2 x\,dx = \frac{1}{\pi/4} = \frac{4}{\pi}$$

To find the number x at which the average value occurs, we let $f(x) = 4/\pi$ and solve for x. Note that $\sec x > 0$ in $(0,\frac{\pi}{4})$.



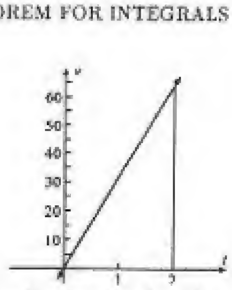
$$\sec^2 x = \frac{4}{\pi}$$
$$\sec x = \sqrt{\frac{4}{\pi}}$$
$$\cos x = \sqrt{\frac{\pi}{4}}$$
$$x = \cos^{-1}\sqrt{\pi/4} \approx 0.4817$$

In the figure at the right we show the graph of the function f. The average value of the function is the height of a rectangle of width $\frac{1}{4}\pi$ such that the area of the rectangle is the same as the area of the region between the x axis and the graph of the function f. The value at which the average value occurs is the x coordinate of the point at which the rectangle intersects the curve.

45. If the positive direction is downward, $\frac{dv}{dt} = 32$; $\int dv = 32 \int dt$; $v = 32t + C$
Because $v = 0$ when $t = 0$, then $C = 0$. Thus $v = 32t$. Let $f(t) = 32t$. Let A.V. be the average value of f on $[0, 2]$. Then

A.V. $= \frac{1}{2}\int_0^2 32t\, dt = 16\int_0^2 t\, dt$

Observe from the figure that $\int_0^2 t\, dt$ is the area of the region enclosed by the triangle. Thus $\int_0^2 t\, dt = 2$. Hence A.V. $= 32$.



46. $f(x) = \sqrt{49 - x^2}$. The graph of f is the part in the first quadrant of the circle of radius 7 centered at O. Therefore $\int_0^7 f(x)dx = \frac{1}{4}\pi(7)^2 = \frac{49}{4}\pi$, the measure of the area of the quarter-circle.
Hence the average value of f on $[0, 7]$ is $\frac{1}{7-0}\int_0^7 \sqrt{49 - x^2}\, dx = \frac{1}{7}\cdot\frac{49}{4}\pi = \frac{7}{4}\pi$.

47. $f(x) = \sqrt{16 - x^2}$. The graph of f is the upper semicircle of radius 4 and center at the origin. Hence $\int_{-4}^4 f(x)dx$ is the measure of the area of the semicircle. Therefore $\int_{-4}^4 \sqrt{16 - x^2}\, dx = \frac{1}{2}\pi(4^2) = 8\pi$.
Thus the average value of f on $[-4, 4]$ is $\frac{1}{4-(-4)}\int_{-4}^4 \sqrt{16 - x^2}\, dx = \frac{1}{8}\cdot 8\pi = \pi$.

48. Suppose that f is integrable on $[-4, 7]$. If the average value of f on the interval $[-4, 7]$ is $\frac{17}{4}$, find $\int_{-4}^7 f(x)\, dx$.
▶ By Definition 4.6.4 we are given that $\frac{1}{7-(-4)}\int_{-4}^7 f(x)\, dx = \frac{17}{4}$. Thus, $\int_{-4}^7 f(x)\, dx = \frac{187}{4}$.

49. If x is in $[0, 1]$ then $x \geq x^2$.
Therefore from Theorem 4.6.1
$\int_0^1 x\, dx \geq \int_0^1 x^2 dx$

If x is in $[1, 2]$ then $x \leq x^2$.
Therefore from Theorem 4.6.1
$\int_0^1 x\, dx \leq \int_0^1 x^2\, dx$

50. Prove the triangle inequality for integrals.
▶ (a) We apply Theorem 4.5.3'. Because f is integrable, its set S of discontinuities is contained in a certain small set. Because the discontinuities of $|f|$ are a subset of S, then $|f|$ is integrable. To prove (b), because $-|f(x)| \leq f(x) \leq |f(x)|$, then by Th.4.6.1, $-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$, which is equivalent to (b).

51. We are given that $\int_a^b f(x)dx = 0$. Because f is continuous on $[a, b]$ it follows from the mean-value theorem for integrals that there is a number c in $[a, b]$ such that $\int_a^b f(x)dx = f(c)(b - a)$; $0 = f(c)(b - a)$; $f(c) = 0$

52. Prove the second mean-value theorem for integrals.
▶ If f is a constant, the result is trivial; assume f is not a constant. Because f is continuous on $[a, b]$, there exist numbers x_m and x_M in $[a, b]$ such that

$$f(x_m) \leq f(x) \leq f(x_M)$$

Multiplying by $g(x) > 0$ we obtain

$$f(x_m)g(x) \leq f(x)g(x) \leq f(x_M)g(x)$$

Therefore, by Theorem 4.6.2

$$f(x_m)\int_a^b g(x)dx < \int_a^b f(x)g(x)dx < f(x_M)\int_a^b g(x)dx$$

Dividing by $\int_a^b g(x)dx > 0$ yields

$$f(x_m) < \int_a^b f(x)g(x)dx \Big/ \int_a^b g(x)dx < f(x_M)$$

By the intermediate-value theorem, there is a number c in (a, b) such that

$$f(c) = \int_a^b f(x)g(x)dx \Big/ \int_a^b g(x)dx$$

Therefore,

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

53. If $g(x) = 1$, the equation of Ex. 52 becomes $\int_a^b f(x)dx = f(c)\int_a^b dx$. But from Theorem 4.5.9, $\int_a^b dx = b - a$. Therefore if f is continuous on $[a, b]$ there is a number c in (a, b) such that $\int_a^b f(x)dx = f(c)(b - a)$.

In Exercises 54–58, use the second mean-value theorem to prove the inequality.

54. Let $f(x) = \dfrac{1}{x^3+2}$ and $g(x) = x$. There is a number c in $(0,4)$ such that $\displaystyle\int_0^4 \frac{x\,dx}{x^3+2} = \frac{1}{c^3+2}\int_0^4 x\,dx$.

Because $0 < c < 4$, then $\dfrac{1}{c^3+2} < 1$. Therefore, $\displaystyle\int_0^4 \frac{x\,dx}{x^3+2} < \int_0^4 x\,dx$

55. Let $f(x) = \dfrac{1}{\sqrt{x^2+4}}$ and $g(x) = x^2$. There is a number c in $(-1,1)$ such that $\displaystyle\int_{-1}^1 \frac{x^2dx}{\sqrt{x^2+4}} = \frac{1}{\sqrt{c^2+4}}\int_{-1}^1 x^2dx$.

Because $\dfrac{1}{\sqrt{c^2+4}} < 1$ if c is in $[-1,1]$ then $\displaystyle\int_{-1}^1 \frac{x^2dx}{\sqrt{c^2+4}} < \int_{-1}^1 x^2dx$.

56. $\displaystyle\int_0^\pi x\sin x\,dx \le \int_0^\pi x\,dx$

▷ Let f and g be the functions defined by

$$f(x) = \sin x \quad\text{and}\quad g(x) = x$$

Then f and g are continuous on $[0,\pi]$ and $g(x) > 0$ for all x in $(0,\pi)$. Thus there is a number c in $(0,1)$ such that

$$\int_0^\pi x\sin x\,dx = \sin c\int_0^\pi x\,dx$$

Because c is in $(0,\pi)$, then $\sin c < 1$, and therefore

$$\int_0^\pi x\sin x\,dx < \int_0^\pi x\,dx$$

57. Let $f(x) = \sin^2\pi x$ and $g(x) = \cos\pi x$. Then there is a number c in $(-\frac{1}{2},\frac{1}{2})$ such that

$$\int_{-1/2}^{1/2} \sin^2\pi x\cos\pi x\,dx = \sin^2\pi c\int_{-1/2}^{1/2}\cos\pi x\,dx$$

But $\sin^2\pi c \le 1$ for all c. Therefore $\displaystyle\int_{-1/2}^{1/2} \sin^2\pi x\cos\pi x\,dx \le \int_{-1/2}^{1/2}\cos\pi x\,dx$.

58. Let $f(x) = \dfrac{\cos x}{x^2+1}$ and $g(x) = x$. Thus there exists a number c in $(0,1)$ such that

$$\int_0^1 \frac{x\cos x}{x^2+1}\,dx = \frac{\cos c}{c^2+1}\int_0^1 x\,dx$$

Because c is in $(0,1)$, then $\cos c < 1$, and therefore $\dfrac{\cos c}{c^2+1} < \dfrac{1}{c^2+1} < 1$. Thus $\displaystyle\int_0^1 \frac{x\cos x}{x^2+1}\,dx < \int_0^1 x\,dx$.

4.7 THE FUNDAMENTAL THEOREMS OF THE CALCULUS

4.7.1 First Fundamental Theorem of the Calculus Let the function f be integrable on the closed interval $[a,b]$ and let x be any number in $[a,b]$. If F is the function defined by

$$F(x) = \int_a^x f(t)\,dt$$

then $F(x)$ is continuous on $[a,b]$. If f is continuous at x, then F is differentiable at x and

$$F'(x) = f(x) \tag{1}$$

(If $x = a$, the derivative in (1) may be a derivative from the right, and if $x = b$, the derivative in (1) may be a derivative from the left.) Eq. (1) can be written as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Initial Condition If $\dfrac{dy}{dx} = f(x)$ and $y(a)$ is given, then $y(x) = y(0) + \displaystyle\int_a^x f(t)dt$. See Exercise 4 Misc. 28.

7.2 Second Fundamental Theorem of the Calculus Let the function f be continuous on the closed interval $[a,b]$ and let g be a function such that

$$g'(x) = f(x) \tag{2}$$

for all x in $[a,b]$. Then

$$\int_a^b f(t)\,dt = g(b) - g(a)$$

(If $x = a$, the derivative in (2) may be a derivative from the right, and if $x = b$, the

derivative in (2) may be a derivative from the left.)

Note that g may be any function that is an antiderivative of f. Usually we choose the antiderivative whose constant term is zero. We denote

$$[g(b) - g(a)] \text{ by } \Big[g(x)\Big]_a^b \text{ or } g(x)\Big]_a^b$$

Furthermore, if c and d are constants, then

$$\Big[c\, f(x) + d\, g(x)\Big]_a^b = c\big[f(b) - f(a)\big] + d\big[g(b) - g(a)\big]$$

Note the distinction between the *indefinite integral*, $\int f(x)\,dx$, which is defined to be a function g such that $g'(x) = f(x)$, and the *definite integral*, $\int_a^b f(x)\,dx$, which is a *number* defined to be the limit of a Riemann sum. The second fundamental theorem of the calculus states that we may use the indefinite integral to calculate the number represented by the definite integral. If we change variables in a definite integral, we must also change the limits. Thus, if $g(x) = u$, $g(a) = A$ and $g(b) = B$, then $g'(x)\,dx = du$ and

$$\int_a^b f(g(x))g'(x)\,dx = \int_A^B f(u)\,du$$

Exercises 4.7

In Exercises 1–34, evaluate the definite integral. In Exercises 1–6 and 29–34, support your answer with NINT.

1. $\int_0^3 (3x^2 - 4x + 1)dx = x^3 - 2x^2 + x\Big|_0^3 = (27 - 18 + 3) - 0 = 12$

2. $\int_0^4 (x^3 - x^2 + 1)dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 + x\Big]_0^4 = (64 - \frac{64}{3} + 4) - 0 = \frac{140}{3}$

3. $\int_3^6 (x^2 - 2x)dx = \frac{1}{3}x^3 - x^2\Big|_3^6 = (72 - 36) - (9 - 9) = 36$

4. $\int_{-1}^3 (3x^2 + 5x - 1)\,dx$

▸ $\int_{-1}^3 (3x^2 + 5x - 1)\,dx = x^3 + \frac{5}{2}x^2 - x\Big]_{-1}^3 = \big[3^3 - (-1)^3\big] + \frac{5}{2}\big[3^2 - (-1)^2\big] + \big[3 - (-1)\big] = 28 + 20 - 4 = 44$

5. $\int_1^2 \frac{x^2 + 1}{x^2}\,dx = \int_1^2 (1 + x^{-2})dx = x - \frac{1}{x}\Big|_1^2 = (2 - \frac{1}{2}) - (1 - 1) = \frac{3}{2}$

6. $\int_{-3}^5 (y^3 - 4y)dy = \frac{1}{4}y^4 - 2y^2\Big]_{-3}^5 = \frac{1}{4}(5^4 - 3^4) - 2(5^2 - 3^2) = 104$

7. $\int_0^1 \frac{z}{(z^2 + 1)^3}\,dz = \frac{1}{2}\int_0^1 (z^2 + 1)^{-3}(2z\,dz) = \frac{1}{2} \cdot \frac{(z^2 + 1)^{-2}}{-2}\Big]_0^1 = -\frac{1}{4(z^2 + 1)^2}\Big]_0^1 = -\frac{1}{16} - (-\frac{1}{4}) = \frac{3}{16}$

8. $\int_1^4 \sqrt{x}(2 + x)\,dx$

▸ $\int_1^4 \sqrt{x}(2 + x)\,dx = \int_1^4 (2x^{1/2} + x^{3/2})\,dx = \frac{4}{3}x^{3/2} + \frac{2}{5}x^{5/2}\Big]_1^4 = \frac{4}{3}(4^{3/2} - 1^{3/2}) + \frac{2}{5}(4^{5/2} - 1^{5/2})$

$= \frac{4}{3}(8 - 1) + \frac{2}{5}(32 - 1) = \frac{28}{3} + \frac{62}{5} = \frac{326}{15}$

9. $\int_0^{10} \sqrt{5x - 1}\,dx = \frac{1}{5}\int_0^{10} (5x - 1)^{1/2}(5\,dx) = \frac{1}{5} \cdot \frac{2}{3}(5x - 1)^{3/2}\Big]_0^{10} = \frac{2}{15}(343 - 8) = \frac{134}{3}$

10. $\int_0^{\sqrt{5}} t\sqrt{t^2 + 1}\,dt = \frac{1}{2}\int_0^{\sqrt{5}} (t^2 + 1)^{1/2}(2t\,dt) = \frac{1}{2} \cdot \frac{2}{3}(t^2 + 1)^{3/2}\Big]_0^{\sqrt{5}} = \frac{1}{3}(6^{3/2} - 1)$

11. $\int_{-2}^0 3w\sqrt{4 - w^2}\,dw = -\frac{3}{2}\int_{-2}^0 (4 - w^2)^{1/2}(-2w\,dw) = -(4 - w^2)^{3/2}\Big]_{-2}^0 = -8$

12. $\int_{-1}^3 \frac{dy}{(y + 2)^3}$

▸ $\int_{-1}^3 \frac{dy}{(y + 2)^3} = \int_{-1}^3 (y + 2)^{-3}\,dy = \frac{(y + 2)^{-2}}{-2}\Big]_{-1}^3 = -\frac{1}{2}[5^{-2} - 1^{-2}] = \frac{12}{25}$

13. $\int_0^{\pi/2} \sin 2x\,dx = \frac{1}{2}\int_0^{\pi/2} \sin 2x(2\,dx) = -\frac{1}{2}\cos 2x\Big]_0^{\pi/2} = \frac{1}{2} - (-\frac{1}{2}) = 1$

14. $\int_0^{\pi} \cos \frac{1}{2}x\,dx = 2\int_0^{\pi} \cos \frac{1}{2}x(\frac{1}{2}\,dx) = 2\sin \frac{1}{2}x\Big]_0^{\pi} = 2(1 - 0) = 2$

15. $\int_1^2 t^2\sqrt{t^3+1}\,dt = \frac{1}{3}\int_1^2 (t^3+1)^{1/2}(3t^2dt) = \frac{1}{3}\cdot\frac{2}{3}(t^3+1)^{3/2}\Big]_1^2 = \frac{2}{9}(27-2\sqrt{2})$

16. $\int_1^3 \frac{x\,dx}{(3x^2-1)^3}$

▹ Let $u = 3x^2 - 1$. Then $du = 6x\,dx$. When $x = 1$, then $u = 2$; when $x = 3$, then $u = 26$. Therefore

$$\int_1^3 \frac{x\,dx}{(3x^2-1)^3} = \int_2^{26} \frac{\frac{1}{6}du}{u^3} = \frac{1}{6}\int_2^{26} u^{-3}\,du = -\frac{1}{12}u^{-2}\Big]_2^{26} = -\frac{1}{12}\Big(\frac{1}{26^2}-\frac{1}{2^2}\Big) = \frac{7}{338}$$

17. $\int_0^1 \frac{(y^2+2y)dy}{\sqrt[3]{y^3+3y^2+4}2} = \frac{1}{3}\int_0^1 (y^3+3y^2+4)^{-1/3}(3y^2+6y)dy = \frac{1}{3}\cdot\frac{3}{2}(y^3+3y^2+4)^{2/3}\Big]_0^1 = -(4-2\sqrt{2}) = 2-\sqrt[3]{2}$

18. $\int_2^4 \frac{w^4-w}{w^3}dw = \int_2^4 (w-w^{-2})dw = \frac{1}{2}w^2 + w^{-1}\Big]_2^4 = \frac{1}{2}(16-4) + (\frac{1}{4}-\frac{1}{2}) = \frac{23}{4}$

19. Let $z = (1+w)^{3/4}$. Then $w = z^{4/3} - 1$; $dw = \frac{4}{3}z^{1/3}dz$. When $w = 0$, $z = 1$; when $w = 15$, $z = 8$.

$\int_0^{15} \frac{w\,dw}{(1+w)^{3/4}} = \int_1^8 \frac{(z^{4/3}-1)(\frac{4}{3}z^{1/3}dz)}{z} = \frac{4}{3}\int_1^8 (z^{2/3}-z^{-2/3})dz = \frac{4}{3}\Big[\frac{3}{5}z^{5/3} - 3z^{1/3}\Big]_1^8 = \frac{4}{3}[(\frac{3}{5}\cdot 32 - 3\cdot 2) - \frac{3}{5} + 3]$
$= \frac{104}{5}$

20. $\int_4^5 x^2\sqrt{x-4}\,dx$

▹ Let $u = \sqrt{x-4}$. Then $u^2 = x - 4$; $x = u^2 + 4$; $dx = 2u\,du$. When $x = 4$, $u = 2$; when $x = 5$, $u = 1$. Therefore

$$\int_4^5 x^2\sqrt{x-4}\,dx = \int_0^1 (u^2+4)^2 u(2u\,du) = 2\int_0^1 (u^6+8u^4+16u^2)\,du = 2\Big[\frac{1}{7}u^7 + \frac{8}{5}u^5 + \frac{16}{3}u^3\Big]_0^1$$
$$= 2[\tfrac{1}{7} + \tfrac{8}{5} + \tfrac{16}{3}] = \tfrac{1486}{105}$$

21. $\int_{-2}^5 |x-3|dx = \int_{-2}^3 |x-3|dx + \int_3^5 |x-3|dx = \int_{-2}^3 (3-x)dx + \int_3^5 (x-3)dx = \Big[3x - \frac{1}{2}x^2\Big]_{-2}^3 + \Big[\frac{1}{2}x^2 - 3x\Big]_3^5$
$= [(9-\frac{9}{2}) - (-6-2)] + [(\frac{25}{2}-15) - (\frac{9}{2}-9)] = \frac{29}{2}$

22. $\int_{-4}^4 |x-2|dx = \int_{-4}^2 (2-x)dx + \int_2^4 (x-2)dx = -\frac{1}{2}(2-x)^2\Big]_{-4}^2 + \frac{1}{2}(x-2)^2\Big]_2^4 = -\frac{1}{2}(0-36) + \frac{1}{2}(4-0) = 20$

23. $\int_{-1}^1 \sqrt{|x|-x}\,dx = \int_{-1}^0 \sqrt{|x|-x}\,dx + \int_0^1 \sqrt{|x|-x}\,dx = \int_{-1}^0 \sqrt{-x-x}\,dx + \int_0^1 \sqrt{x-x}\,dx$
$= -\sqrt{2}\int_{-1}^0 \sqrt{-x}(-dx) + \int_0^1 0\,dx = -\frac{2}{3}\sqrt{2}(-x)^{3/2}\Big]_{-1}^0 = \frac{2}{3}\sqrt{2}$

24. $\int_{-3}^3 \sqrt{3+|x|}\,dx$

▹ Because $\sqrt{3+|x|}$ is an even function, then

$$\int_{-3}^3 \sqrt{3+|x|}\,dx = 2\int_0^3 \sqrt{3+|x|}\,dx = 2\int_0^3 \sqrt{3+x}\,dx = 2\cdot\frac{2}{3}(3+x)^{3/2}\Big]_0^3 = \frac{4}{3}[6^{3/2} - 3^{3/2})$$
$$= \frac{4}{3}[6\sqrt{6} - 3\sqrt{3}] = 4[2\sqrt{6} - \sqrt{3}]$$

25. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1$; $dx = 2u\,du$. When $x = 0$, $u = 1$; when $x = 3$, $u = 2$.
$\int_0^3 (x+2)\sqrt{x+1}\,dx = \int_1^2 (u^2+1)u(2u\,du) = 2\int_1^2 (u^4+u^2)du = \frac{2}{5}u^5 + \frac{2}{3}u^3\Big]_1^2 = (\frac{64}{5} + \frac{16}{3}) - (\frac{2}{5} + \frac{2}{3}) = \frac{62}{5} + \frac{14}{3} = \frac{256}{15}$

26. Let $u = \sqrt{x+3}$, $x = u^2 - 3$, $dx = 2u\,du$. $\int_{x=-2}^1 (x+1)\sqrt{x+3}\,dx = \int_{u=1}^2 (u^2-2)u(2u\,du) = 2\int_1^2 (u^4 - 2u^2)du$

$= 2\Big[\frac{1}{5}u^5 - \frac{2}{3}u^3\Big]_1^2 = 2\Big[\frac{1}{5}(32-1) - \frac{2}{3}(8-1)\Big] = \frac{46}{15}$

27. $\int_0^1 \frac{x^3+1}{x+1}\,dx = \int_0^1 \frac{(x+1)(x^2-x+1)}{x+1}\,dx = \int_0^1 (x^2-x+1)dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 + x\Big]_0^1 = \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}$

28. $\int_1^4 \frac{x^5 - x}{3x^3}\, dx$

▹ $$\int_1^4 \frac{x^5 - x}{3x^3}\, dx = \frac{1}{3}\int_1^4 (x^2 + x^{-2})\, dx = \frac{1}{3}\Big[\frac{1}{3}x^3 + x^{-1}\Big]_1^4 = \frac{1}{9}(4^3 - 1) + \frac{1}{3}(\frac{1}{4} - 1) = 7 - \frac{1}{4} = \frac{27}{4}$$

29. $\int_1^{64} \left(\sqrt{t} - \frac{1}{\sqrt{t}} + \sqrt[3]{t}\right) dt = \int_1^{64} (t^{1/2} - t^{-1/2} + t^{1/3})\,dt = \frac{2}{3}t^{3/2} - 2t^{1/2} + \frac{3}{4}t^{4/3}\Big]_1^{64}$

$= (\frac{1024}{3} - 16 + 192) - (\frac{2}{3} - 2 + \frac{3}{4}) = \frac{6215}{12}$

30. Let $u = x^{3/2}$, $du = \frac{3}{2}x^{1/2}dx$.

$\int_0^1 \sqrt{x}\sqrt{1 + x\sqrt{x}}\, dx = \int_{x=0}^1 (1 + x^{3/2})^{1/2}(x^{1/2}dx) = \int_{u=0}^1 (1+u)^{1/2}(\frac{2}{3}du) = \frac{2}{3}\cdot\frac{2}{3}(1+u)^{3/2}\Big]_0^1 = \frac{4}{9}(2^{3/2} - 1)$

31. $\int_0^1 \sin \pi x \cos \pi x\, dx = \frac{1}{2}(\sin \pi x)^2\Big]_0^1 = \frac{1}{2\pi}(0 - 0) = 0$

32. $\int_0^{\pi/6} (\sin 2x + \cos 3x)\, dx$

▹ $$\int_0^{\pi/6} (\sin 2x + \cos 3x)\, dx = -\frac{1}{2}\cos 2x + \frac{1}{3}\sin 3x\Big]_0^{\pi/6} = -\frac{1}{2}(\cos\frac{1}{3}\pi - \cos 0) + \frac{1}{3}(\sin\frac{1}{2}\pi - \sin 0)$$

$$= -\frac{1}{2}(\frac{1}{2} - 1) + \frac{1}{3}(1 - 0) = \frac{7}{12}$$

33. $\int_{\pi/8}^{\pi/4} 3\csc^2 2x\, dx = -\frac{3}{2}\cot 2x\Big]_{\pi/8}^{\pi/4} = -\frac{3}{2}(0 - 1) = \frac{3}{2}$

34. $\int_0^{1/2} \sec^2\frac{1}{2}\pi t\tan\frac{1}{2}\pi t\, dt = \frac{2}{\pi}\int_0^{1/2}\tan\frac{1}{2}\pi t(\sec^2\frac{1}{2}\pi t\cdot\frac{1}{2}\pi dt) = \frac{2}{\pi}\cdot\frac{1}{2}\tan^2\frac{1}{2}\pi t\Big]_0^{1/2} = \frac{1}{\pi}(1 - 0) = \frac{1}{\pi}$

In Exercises 35–44, compute the derivative using Theorem 4.7.1.

35. $\frac{d}{dx}\int_0^x \sqrt{4 + t^6} = \sqrt{4 + x^6}$

36. $\frac{d}{dx}\int_x^5 \sqrt{1 + t^4}\, dt$

▹ Because Theorem 4.7.1 requires the integral to have a constant lower limit, we first use Definition 4.5.5. Thus

$$\frac{d}{dx}\int_x^5 \sqrt{1 + t^4}\, dt = -\frac{d}{dx}\int_5^x \sqrt{1 + t^4}\, dt = -\sqrt{1 + x^4}$$

37. $\frac{d}{dx}\int_x^3 \sqrt{\sin t}\, dt = \frac{d}{dx}\Big(-\int_3^x \sqrt{\sin t}\, dt\Big) = -\sqrt{\sin x}$

38. $\frac{d}{dx}\int_2^x \frac{1}{t^4 + 4}\, dt = \frac{1}{x^4 + 4}$

39. $\frac{d}{dx}\int_{-x}^x \frac{dt}{3 + t^2} = \frac{d}{dx}\Big(\int_{-x}^0 \frac{dt}{3 + t^2} + \int_0^x \frac{dt}{3 + t^2}\Big) = \frac{d}{dx}\int_0^{-x}\frac{-dt}{3 + t^2} + \frac{d}{dx}\int_0^x \frac{dt}{3 + t^2}$

$= \frac{d}{dx}\int_0^x \frac{d(-t)1}{3 + (-t)^2} + \frac{d}{dx}\int_0^x \frac{dt}{3 + t^2} = \frac{1}{3 + x^2} + \frac{1}{3 + x^2} = \frac{2}{3 + x^2}$

40. $\frac{d}{dx}\int_{-x}^x \cos(t^2 + 1)dt$

▹ Because $\cos(t^2 + 1)$ is an even function,

$\frac{d}{dx}\int_{-x}^x \cos(t^2 + 1)dt = \frac{d}{dx}2\int_0^x \cos(t^2 + 1)dt = 2\cos(x^2 + 1)$

41. Let $u = x^3$. $\frac{d}{dx}\int_2^{x^3} \sqrt[3]{t^2 + 1}\, dt = \frac{d}{du}\int_0^{x^3}\sqrt[3]{t^2 + 1}\, dt\cdot\frac{du}{dx} = \sqrt[3]{u^2 + 1}(3x^2) = 3x^2\sqrt[3]{x^6 + 1}$

42. Let $u = x^2$ and use the chain rule. Thus, $\frac{d}{dx}\int_0^{x^2}\frac{1}{\sqrt{t^2 + 1}}\, dt = \frac{d}{du}\Big(\int_0^u \frac{1}{\sqrt{t^2 + 1}}\, dt\Big)\frac{du}{dx} = \frac{1}{\sqrt{u^2 + 1}}\cdot 2x = \frac{2x}{\sqrt{x^4 + 1}}$

43. Let $u = \tan x$. $\frac{d}{dx}\int_2^{\tan x}\frac{1}{1 + t^2}dt = \frac{d}{du}\int_2^u \frac{1}{1 + t^2}dt\cdot\frac{du}{dx} = \frac{1}{1 + u^2}\cdot\sec^2 x = \frac{\sec^2 x}{1 + \tan^2 x} = 1$

44. $\dfrac{d}{dx}\displaystyle\int_3^{\sin x} \frac{1}{1-t^2}dt$

▹ Let $u = \sin x$ and use the chain rule. Thus,

$$\frac{d}{dx}\int_3^{\sin x} \frac{1}{1-t^2}dt = \frac{d}{du}\int_3^{u} \frac{1}{1-t^2}dt \cdot \frac{du}{dx} = \frac{1}{1-u^2}\cdot \cos x = \frac{1}{1-\sin^2 x}\cdot \cos x = \frac{\cos x}{\cos^2 x} = \sec x$$

In Exercises 45–48, find the average value of f on the interval. In Exercises 45 and 46, find the value of x.

45. The average value of $9 - x^2$ on $[0,3]$ is $\frac{1}{3-0}\int_0^3 (9-x^2)dx = \frac{1}{3}\left[9x - \frac{1}{3}x^3\right]_0^3 = \frac{1}{3}(18-0) = 6$ and $9 - x^2 = 6$; $x^2 = 3$ when $x = \sqrt{3} \in [0,3]$.

46. The average value of $8x - x^2$ on $[0,4]$ is $\frac{1}{4-0}\int_0^4 (8x-x^2)dx = \frac{1}{4}\left[4x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{1}{4}(64 - \frac{64}{3}) = \frac{32}{3}$ and $8x - x^2 = \frac{32}{3}$ when $x^2 - 8x + 16 = \frac{16}{3}$; $(x-4)^2 = \frac{16}{9}\cdot 3$; $x = 4 \pm \frac{4}{3}\sqrt{3}$

47. The average value of $3x\sqrt{x^2-16}$ on $[4,5]$ is

$\frac{1}{5-4}\int_4^5 3x\sqrt{x^2-16}\,dx = 1\cdot\frac{3}{2}\int_4^5 (x^2-16)^{1/2}(2x\,dx) = (x^2-16)^{3/2}\Big|_4^5 = 9^{3/2} - 0 = 27$

48. $f(x) = x^2\sqrt{x-3}$; $[7,12]$

▹ The average value of $x^2\sqrt{x-3}$ on $[7,12]$ is A.V. $= \frac{1}{12-7}\int_{x=7}^{12} x^2\sqrt{x-3}\,dx$.

Let $u = \sqrt{x-3}$; $x = u^2 + 3$; $dx = 2u\,du$. Thus,

$$\text{A.V.} = \frac{1}{5}\int_{u=2}^{3} (u^2+3)^2 u(2u\,du) = \frac{2}{5}\int_2^3 (u^6 + 6u^4 + 9u^2)du = \frac{2}{5}\left[\frac{1}{7}u^7 + \frac{6}{5}u^5 + 3u^3\right]_2^3$$
$$= \frac{2}{5}\left[\frac{1}{7}(3^7 - 2^7) + \frac{6}{5}(3^5 - 2^5) + 3(3^3 - 2^3)\right] = \frac{42{,}304}{175}$$

49. If $E = 2\sin\frac{2}{3}\pi t$, find the square root of the average value of E^2 on $[0,4]$ (the RMS voltage).

▹ A.V. $= \frac{1}{4-0}\int_0^4 4\sin^2\frac{2}{3}\pi t\,dt = \frac{1}{2}\int_0^4 (1 - \cos\frac{4}{3}\pi t)dt = \frac{1}{2}\left[t - \frac{3}{4\pi}\sin\frac{4}{3}\pi t\right]_0^4 = \frac{1}{2}(4 - \frac{3}{4\pi}\sin\frac{16}{3}\pi)$. $\sqrt{\text{A.V.}} \approx 1.4502$

50. A.V. $= \frac{1}{\frac{1}{4}\pi - (-\frac{1}{4}\pi)}\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx = \frac{2}{\pi}\tan x\Big]_{-\pi/4}^{\pi/4} = \frac{4}{\pi}$

51. Take the positive direction downward. Then $\frac{dv}{dt} = 32$; $\int dv = 32\int dt$; $v = 32t + C$. Because $v = 0$ when $t = 0$, then $C = 0$. Therefore $v = 32t$. The average value of v from $t = 0$ to $t = \frac{1}{2}T$ is given by

$V_1 = \frac{1}{\frac{1}{2}T - 0}\int_0^{T/2} 32t\,dt = \frac{2}{T}\,16t^2\Big|_0^{T/2} = 8T$. The average value of v from $t = \frac{1}{2}T$ to $t = T$ is given by

$V_2 = \frac{1}{T - \frac{1}{2}T}\int_{T/2}^{T} 32t\,dt = \frac{2}{T}\,16t^2\Big|_{T/2}^{T} = \frac{2}{T}(16T^2 - 4T^2) = 24T$. Therefore $V_1 = \frac{1}{3}V_2$.

52. A stone is thrown downward with an initial velocity of v_0 feet per second. Neglect air resistance. (a) Show that if v feet per second is the velocity of the stone after falling s feet, then $v = \sqrt{v_0^2 + 2gs}$. (b) Find the average velocity during the first 100 ft of fall if the initial velocity is 60 ft/sec. (Take $g = 32$ and downward as the positive direction.)

▹ (a) From Exercise 4.2.48 with $a = g$ we have $\quad 2gs = v^2 - v_0^2$; $\quad v^2 = v_0^2 + 2gs$; $\quad v = \sqrt{v_0^2 + 2gs}$

(b) With $v_0 = 60$ and $g = 32$ we have $v = \sqrt{3600 + 64s}$. Thus,

$$\text{A.V.} = \frac{1}{100}\int_0^{100}(3600+64s)^{1/2}ds = \frac{1}{100}\cdot\frac{1}{64}\cdot\frac{2}{3}(3600+64s)^{3/2}\Big]_0^{100} = \frac{1}{9600}(10{,}000^{3/2} - 3600^{3/2}) = \frac{245}{3}$$

The average velocity is $\frac{245}{3}$ ft/sec.

53. Applying the quotient rule, $\frac{d}{dT}\left(\int_0^T r(t)dt \Big/ T\right) = \left(r(T)\cdot T - \int_0^T r(t)dt\cdot 1\right)\Big/ T^2 = [r(T) - R(T)]/T$

54. (a) $I = \int_{x=0}^{k}\frac{f(x)}{f(x)+f(k-x)}dx$. Let $u = k - x$, $du = -dx$, reverse limits, then change the dummy back to x.

$I = \int_{u=k}^{0}\frac{f(k-u)}{f(k-u)+f(u)}(-du) = \int_0^k \frac{f(k-u)}{f(k-u)+f(u)}du = \int_0^k\frac{f(k-x)}{f(k-x)+f(x)}dx$

Adding the last integral to the first, we get $2I = \int_0^k \frac{f(x)+f(k-x)}{f(x)+f(k-x)}dx = \int_0^k dx = k$ and so $I = \frac{1}{2}k$.

(b) If $f(x) = \sin x$ and $k = \frac{1}{2}\pi$, then $f(\frac{1}{2}\pi - x) = \sin(\frac{1}{2}\pi - x) = \cos x$ and so $\int_0^{\pi/2}\frac{\sin x}{\sin x + \cos x}dx = \frac{1}{2}(\frac{1}{2}\pi) = \frac{1}{4}\pi$

55. $F(x) = \int_0^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt$. Let $u = \frac{1}{x}$. Then

$F'(x) = \frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt + \frac{d}{du}\int_0^{1/x} \frac{1}{1+t^2}\,dt \cdot \frac{du}{dx} = \frac{1}{1+x^2} + \frac{1}{1+(1/x)^2}\left(-\frac{1}{x^2}\right) = 0$ and so F is constant.

56. Find a function f such that for any real number x, $\int_0^x f(t)dt = \frac{\cos x}{1+x^2} - 1$.

▷ Differentiating on both sides of the equation we get $f(x) = \frac{(-\sin x)(1+x^2) - \cos x)(2x)}{(1+x^2)^2}$.

57. Let $u = 1 - x$, $du = -dx$, reverse the limits, then change the dummy back to x. $\beta(m+1, n+1) =$
$\int_{x=0}^1 x^n(1-x)^m dx = \int_{u=1}^0 (1-u)^n u^m(-du) = \int_0^1 u^m(1-u)^n du = \int_0^1 x^m(1-x)^n dx = \frac{m!n!}{(m+n+1)!}$ by parts

58. The slope of the tangent line to the graph of $y = f(x)$ at the point $(x, f(x))$ is $f'(x)$. Because f' is continuous on $[a,b]$, then the slope is integrable on $[a,b]$. Let A.V. be the average value of the slope on $[a,b]$. Then

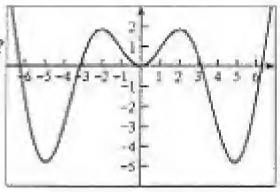
$$\text{A.V.} = \frac{1}{b-a}\int_a^b f'(x)\,dx = \frac{1}{b-a}\Big[f(x)\Big]_a^b = \frac{f(b)-f(a)}{b-a}$$

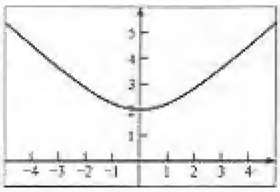
Thus, the average value of the slope of the tangent line of the graph of f on $[a,b]$ equals the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.

59. $\int_4^{16}\left[D_x\int_5^x (2\sqrt{t}-1)dt\right]dx = \int_4^{16}(2\sqrt{x}-1)dx = \frac{4}{3}x^{3/2} - x\Big|_4^{16} = (\frac{256}{3} - 16) - (\frac{32}{3} - 4) = \frac{224}{3} - 12 = \frac{188}{3}$

60. (a) Given $f(x) = x\sin x$. Plot the graphs of f and NDER(NINT$(f(t), 0, x), x)$ in the same window and show that the graphs appear the same.
(b) Repeat part (a) if $f(x) = \sqrt{4+x^2}$.
(c) What theorem do parts (a) and (b) support?

▷ The graphs for (a) and (b) appear at the right. (c) The graphs support the first fundamental theorem of the calculus.





4.8 AREA OF A PLANE REGION

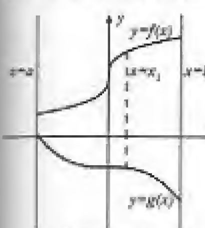


Figure (a)

Let R be a region in the xy plane. R is bounded on the left by the line $x = a$ and bounded on the right by the line $x = b$ if and only if every line of the form $x = x_1$ with $a \le x_1 \le b$ intersects the region R in one or more points, and no other vertical line intersects R. See Fig (a). Furthermore, R is bounded above by the curve $y = f(x)$ and bounded below by the curve $y = g(x)$ if and only if every nonempty intersection of R and a vertical line is a segment that has its upper endpoint on the curve $y = f(x)$ and its lower endpoint on the curve $y = g(x)$, and thus $f(x) \ge g(x)$ for all x in $[a,b]$. The area of any such region is A square units where

$$A = \int_a^b [f(x) - g(x)]\,dx \tag{1}$$

If R is bounded below by the x axis, then $g(x) = 0$ in Formula (1).

We may interchange the roles of x and y in the above discussion. Figure (b) illustrates a region R that is bounded below by the line $y = \alpha$ and bounded above by the line $y = \beta$. Every line of the form $y = y_1$ intersects R if $\alpha \le y_1 \le \beta$ and no other horizontal line intersects R. Furthermore, R is bounded on the right by the curve $x = \lambda(y)$ and bounded on the left by the curve $x = \phi(y)$ if and only if every nonempty intersection of R and a horizontal line is a line segment that has its right endpoint on the curve $x = \lambda(y)$ and its left endpoint on the curve $x = \phi(y)$, and thus $\lambda(y) \ge \phi(y)$ for all y in $[\alpha, \beta]$. The area of any such region is A square units, where

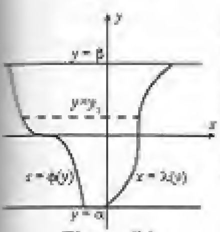


Figure (b)

$$A = \int_\alpha^\beta [\lambda(y) - \phi(y)]\,dy \tag{2}$$

If a region satisfies the conditions for both of these types, then we may use either Formula (1) or Formula (2) to find its area, whichever is easier.

Because the area A of a region was defined to be the limit of Riemann sums for vertical

rectangular elements of area, we must prove that the same number A is obtained when we calculate the limit of Riemann sums for horizontal rectangular elements of area. We state and prove this result for a special case; the general result follows by cutting the region into parts which resemble the special case. Refer to Fig. (c).

Theorem Let $f(x)$ be a continuous function which decreases on $[0,a]$ from $f(0)=\alpha$ to $f(a)=0$ and let the curve $y=f(x)$ also be described by $x=\phi(y)$, where $\phi(0)=a$ and $\phi(\alpha)=0$. If $f'(x)$ is continuous on $[0,a]$, then

$$\int_0^a f(x)\,dx = \int_0^\alpha \phi(y)\,dy$$

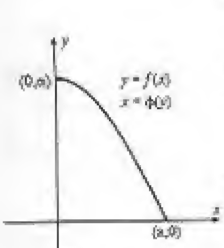


Figure (c)

PROOF: We evaluate the second integral with the substitution $y=f(x)$. Then $\phi(y)=x$ and $dy=f'(x)\,dx$. When $y=0$, $x=a$; when $y=\alpha$, $x=0$. Thus,

$$\int_0^\alpha \phi(y)\,dy = \int_a^0 x\,f'(x)\,dx$$

Because $\frac{d}{dx}[x\,f(x)] = f(x) + x\,f'(x)$, then $x\,f'(x) = \frac{d}{dx}[x\,f(x)] - f(x)$. Thus

$$\begin{aligned}\int_0^\alpha \phi(y)\,dy &= \int_a^0 \frac{d}{dx}[x\,f(x)]\,dx - \int_a^0 f(x)\,dx\\ &= x\,f(x)\Big]_a^0 + \int_0^a f(x)\,dx\\ &= 0\cdot\alpha - a\cdot 0 + \int_0^a f(x)\,dx\\ &= \int_0^a f(x)\,dx\end{aligned}$$

Exercises 4.8

In Exercises 1–38, find the area of the region bounded by the curves. In each exercise do the following:

(a) Draw a figure showing the region and a rectangular element of area.
(b) Express the area of the region as the limit of a Riemann sum.
(c) Find the limit in part (b) by the second fundamental theorem of the calculus.

1. A square units is the area of the region bounded by $y=4-x^2$; x axis.
   Δ is a partition of the interval $[0,2]$ on the x axis.
   $$A = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(4-w_i^2)\Delta_i x = 2\int_0^1(4-x^2)dx = 2\left[4x-\tfrac{1}{3}x^3\right]_0^2 = 2\cdot\tfrac{16}{3} = \tfrac{32}{3}$$

2. A square units is the area of the region bounded by $y=x^2-2x+3$; x axis; $x=-2$; $x=1$.
   Δ is a partition of the interval $[-2,1]$ on the x axis.
   $$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(w_i^2-2w_i+3)\Delta_i x = \int_{-2}^{1}(x^2-2x+3)dx = \tfrac{1}{3}x^3-x^2+3x\Big]_{-2}^{1} = \tfrac{7}{3}-(-\tfrac{38}{3}) = 15$$

3. A square units is the area of the region bounded by $y=4x-x^2$; x axis; $x=1$; $x=3$.
   Δ is a partition of the interval $[1,3]$ on the x axis.
   $$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(4w_i-w_i^2)\Delta_i x = \int_1^3(4x-x^2)dx = 2x^2-\tfrac{1}{3}x^3\Big|_1^3 = 9-\tfrac{5}{3} = \tfrac{22}{3}$$

4. $y=6-x-x^2$; x axis

▸ Let $f(x)=6-x-x^2=(3+x)(2-x)$. The region R is shown in the figure below. R is bounded above by the curve $y=f(x)$ and bounded below by the x axis. If $f(x)=0$, then $x=-3$ or $x=2$. Thus R is bounded on the left by the line $x=-3$ and on the right by $x=2$. The elements of area are vertical rectangles of width $\Delta_i x$ and height $f(w_i)$ units. If A square units is the area of R, then

$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(w_i)\Delta_i x$$

and the limit of the Riemann sums is a definite integral. Hence

$$\begin{aligned}A &= \int_{-3}^{2}(6-x-x^2)\,dx = 6x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3\Big]_{-3}^{2}\\ &= 6[2-(-3)]-\tfrac{1}{2}(4-9)-\tfrac{1}{3}(8+27) = \tfrac{125}{6}\end{aligned}$$

• The area of the region is $\frac{125}{6}$ square units.

5. A square units is the area of the region bounded by $y = \sqrt{x+1}$; x axis; y axis; $x = 8$.
Δ is a partition of the interval $[0, 8]$ of the x axis.
$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{w_i + 1}\, \Delta_i x = \int_0^8 \sqrt{x+1}\, dx = \frac{2}{3}(x+1)^{3/2}\Big]_0^8 = \frac{2}{3}(27-1) = \frac{52}{3}$

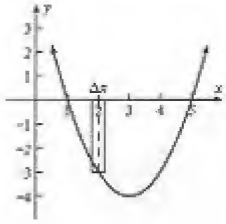
6. A square units is the area of the region bounded by $y = \frac{1}{x^2} - x$; x axis; $x = 2$; $x = 3$.
Δ is a partition of the interval $[2, 3]$ of the x axis. Because $\frac{1}{x^2} < x$ on $[2, 3]$.
$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(w_i - \frac{1}{w_i^2}\right) \Delta_i x = \int_2^3 (x - x^{-2}) dx = \frac{1}{2}x^2 + x^{-1}\Big]_2^3 = \frac{29}{6} - \frac{5}{2}) = \frac{7}{3}$

7. A square units is the area bounded by $x^2 + x - 12$; x axis. Set $x^2 + x - 12 = 0$.
Δ is a partition of the interval $[-4, 3]$ of the x axis.
$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (w_i^2 + w_i - 12)\Delta_i x = \int_{-4}^{3} (x^2 + x - 12) dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 12x\Big]_{-4}^{3} = \frac{45}{2} - (-\frac{104}{3}) = \frac{343}{6}$

8. $y = x^2 - 6x + 5$; x axis

▷ Let $f(x) = x^2 - 6x + 5 = (x-1)(x-5)$. The region R is shown in the figure below. R is bounded above by the x axis and bounded below by the curve $y = f(x)$. If $f(x) = 0$, then $x = 1$ or $x = 5$. The elements of area are vertical rectangles of width $\Delta_i x$ and height $-f(w_i)$. The area of R is A square units, where

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} -f(w_i)\Delta_i x$$
$$= \int_1^5 -(x^2 - 6x + 5)\, dx = -\frac{1}{3}x^3 + 3x^2 - 5x\Big]_1^5$$
$$= -\frac{1}{3}(125 - 1) + 3(25 - 1) - 5(5 - 1) = \frac{32}{3}$$



- The area of R is $\frac{32}{3}$ square units.

9. A square units is the area of the region bounded by $y = \sin x$; x axis; $x = \frac{1}{3}\pi$; $x = \frac{2}{3}\pi$. Δ is a partition of the interval $[\frac{1}{3}\pi, \frac{2}{3}\pi]$ of the x axis.
$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sin w_i\, \Delta_i x = \int_{\pi/3}^{2\pi/3} \sin x\, dx = -\cos x\Big|_{\pi/3}^{2\pi/3} = \frac{1}{2} - (-\frac{1}{2}) = 1$

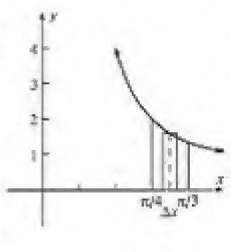
10. A square units is the area of the region bounded by $y = \cos x$; x axis; y axis; $x = \frac{1}{6}\pi$.
Δ is a partition of the interval $[0, \frac{1}{6}\pi]$ of the x axis.
$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \cos w_i\, \Delta_i x = \int_0^{\pi/6} \cos x\, dx = \sin x\Big|_0^{\pi/6} = \frac{1}{2} - 0 = \frac{1}{2}$

11. A square units is the area of the region bounded by $y = \sec^2 x$; x axis; y axis; $x = \frac{1}{4}\pi$.
Δ is a partition of the interval $[0, \frac{1}{4}\pi]$ of the x axis.
$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sec^2 w_i\, \Delta_i x = \int_0^{\pi/4} \sec^2 x\, dx = \tan x\Big|_0^{\pi/4} = 1 - 0 = 1$

12. $y = \csc^2 x$; $x = \frac{1}{4}\pi$; $x = \frac{1}{3}\pi$

▷ The region R is shown in the figure below. If $f(x) = \csc^2 x$, then the elements of area are vertical rectangles of width $\Delta_i x$ and height $f(w_i)$. We are given that R is bounded on the left by $x = \frac{1}{4}\pi$ and on the right by $x = \frac{1}{3}\pi$. Thus

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x = \int_{\pi/4}^{\pi/3} \csc^2 x\, dx = -\cot x\Big]_{\pi/4}^{\pi/3}$$
$$= -\cot \tfrac{1}{3}\pi + \cot \tfrac{1}{4}\pi = -\tfrac{1}{3}\sqrt{3} + 1$$



- The area of R is $-\frac{1}{3}\sqrt{3} + 1$ square units.

13. A square units is the area bounded by $x^2 = -y$; $y = -4$. The region is symmetric with respect to the y axis. Δ is a partition of the interval $[0, 2]$ on the x axis.
$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [-w_i^2 - (-4)]\Delta_i x = 2\int_0^2 (-x^2 + 4) dx = 2\Big[-\frac{1}{3}x^3 + 4x\Big]_0^2 = 2(-\frac{8}{3} + 8) = 2 \cdot \frac{16}{3} = \frac{32}{3}$

14. A square units is the area bounded by $y^2 = -x$; $x = -2$; $x = -4$. The region is symmetric.
Δ is a partition of the interval $[-4, -2]$ on the x axis.

$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\sqrt{-w_i}]\Delta_i x = 2\int_{-4}^{-2}(-x)^{1/2}dx = 2\Big[-\tfrac{2}{3}(-x)^{3/2}\Big]_{-4}^{-2} = \tfrac{4}{3}(4^{3/2} - 2^{3/2}) = \tfrac{1}{3}(32 - 8\sqrt{2})$$

15. A square units is the area bounded by $x^2 + y + 4 = 0$; $y = -8$. The region is symmetric.
Δ is a partition of the interval $[-8, -4]$ on the y axis.

$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{-w_i - 4}\,\Delta_i y = 2\int_{-8}^{-4}\sqrt{-y-4}\,dy = -2\int_{-8}^{-4}(-y-4)^{1/2}(-dy) = -2\cdot\tfrac{2}{3}(-y-4)^{3/2}\Big|_{-8}^{-4}$$
$$= -\tfrac{4}{3}[0 - 4^{3/2}] = \tfrac{32}{3}$$

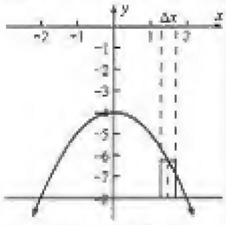
16. $x^2 + y + 4 = 0$; $y = -8$

▹ The region R is shown in the figure below. R is bounded above by the curve $x^2 + y + 4 = 0$. Solving for y we obtain $y = f(x) = -x^2 - 4$. R is bounded below by $y = g(x) = -8$. The elements of area are vertical rectangles of width $\Delta_i x$ and height $[f(w_i) - g(w_i)]$. Because

$$f(x) - g(x) = -x^2 + 4 = -(x-2)(x+2)$$

then R is bounded on the left by $x = -2$ and on the right by $x = 2$.
If the area of R is A square units, then

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [f(w_i) - g(w_i)]\Delta_i x = \int_{-2}^{2} [f(x) - g(x)]\,dx$$
$$= \int_{-2}^{2} (-x^2 + 4)\,dx = -\tfrac{1}{3}x^3 + 4x\Big]_{-2}^{2} = -\tfrac{1}{3}(8 + 8) + 4(2 + 2) = \tfrac{32}{3}$$



■ The area of region R is $\frac{32}{3}$ square units.

In Exercises 17 and 18, the region is bounded by $x^2 - y + 1 = 0$; $x - y + 1 = 0$. Set $x^2 + 1 = x + 1$. $x = 0, 1$.

17. Δ is a partition of the interval $[0, 1]$ on the x axis. Because $x^2 \le x$ on $[0, 1]$,

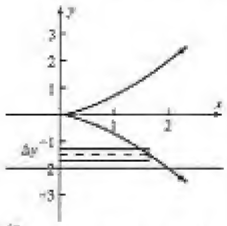
$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(w_i + 1) - (w_i^2 + 1)]\Delta_i x = \int_0^1 [(x+1) - (x^2+1)]dx = \int_0^1 (-x^2 + x)dx = -\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\Big|_0^1 = \tfrac{1}{6}$$

18. Δ is a partition of the interval $[1, 2]$ on the y axis. Because $y - 1 \le \sqrt{y-1}$ on $[1, 2]$,

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\sqrt{w_i - 1} - (w_i - 1)]\Delta_i y = \int_1^2 [(y-1)^{1/2} - (y-1)]dy = \Big[\tfrac{2}{3}(y-1)^{3/2} - \tfrac{1}{2}(y-1)^2\Big]_1^2 = \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{1}{6}$$

19. A square units is the area bounded by $x^3 = 2y^2$; $x = 0$; $y = -2$
in the fourth quadrant. See the figure.
Δ is a partition of the interval $[-2, 0]$ on the y axis.

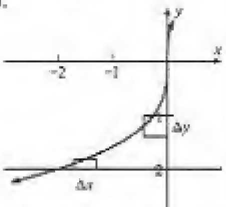
$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt[3]{2w_i^2}\,\Delta_i y = \int_{-2}^{0} \sqrt[3]{2y^2}\,dy$$
$$= \sqrt[3]{2}\int_{-2}^{0} y^{2/3}dy = \sqrt[3]{2}\cdot\tfrac{3}{5}y^{5/3}\Big|_{-2}^{0}$$
$$= \tfrac{3}{5}\sqrt[3]{2}[0 - (-2)^{5/3}] = \tfrac{3}{5}\sqrt[3]{2}(\sqrt[3]{32}) = \tfrac{3}{5}\sqrt[3]{64} = \tfrac{12}{5}$$



20. $y^3 = 4x$; $x = 0$; $y = -2$

▹ R is bounded above by the curve $y^3 = 4x$. Solving for y we obtain $y = (4x)^{1/3}$. R is bounded below by $y = g(x) = -2$. The elements of area are vertical rectangles of width $\Delta_i x$ and height $[f(w_i) - g(w_i)]$. The figure shows vertical and horizontal rectangles. Substituting $y = -2$ into $y^3 = 4x$, we get $-8 = 4x$; $x = -2$. Thus R is bounded on the left by $x = -2$ and on the right by the given line $x = 0$.
Hence,

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [f(w_i) - g(w_i)]\Delta_i x$$
$$= \int_{-2}^{0} [(4x)^{1/3} - (-2)]\,dx = \tfrac{1}{4}\cdot\tfrac{3}{4}(4x)^{4/3} + 2x\Big]_{-2}^{0}$$
$$= \tfrac{3}{16}(0 - 16) + 2(0 + 2) = 1$$



■ The area of region R is 1 square unit.

ALTERNATE SOLUTION: Because the intersection of R with any horizontal line is a line segment that has its right endpoint on the line $x = 0$ and its left endpoint on the curve $y^3 = 4x$, then R is bounded on the right by

$x = \lambda(y) = 0$ and bounded on the left by $x = \phi(y) = \frac{1}{4}y^3$. R is bounded below by $y = -2$ and above by $y = 0$. The elements of area are horizontal rectangles of length $[\phi(w_i) - \lambda(w_i)]$ and height $\Delta_i y$ as illustrated in the figure. Thus

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\lambda(w_i) - \phi(w_i)]\Delta_i y = \int_{-2}^{0} -\tfrac{1}{4}y^3\, dy = -\tfrac{1}{16}y^4 \Big]_{-2}^{0} = -\tfrac{1}{16}(0 - 16) = 1$$

21. A square units is the area of the region bounded by $y = 2 - x^2$; $y = -x$. Set $2 - x^2 = -x$; $0 = x^2 - x - 2 = (x-2)(x+1)$. The two curves intersect at the points$(-1, 1)$ and$(2, -2)$.
Δ is a partition of the interval $[-1, 2]$ on the x axis.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(2 - w_i^2) - (-w_i)]\Delta_i x = \int_{-1}^{2} [(2 - x^2) - (-x)]dx = \int_{-1}^{2} (-x^2 + x + 2)dx$$
$$= -\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 2x \Big|_{-1}^{2} (-\tfrac{8}{3} + 2 + 4) - (\tfrac{1}{3} + \tfrac{1}{2} - 2) = \tfrac{9}{2}$$

22. A square units is the area bounded by $y = x^2$; $y = x^4$. The region is symmetric.
Δ is a partition of the interval $[0, 1]$ on the x axis.

$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (w_i^2 - w_i^4)\, \Delta_i x = 2\int_{0}^{1} (x^2 - x^4)\, dx = 2\Big[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\Big]_{0}^{1} = 2(\tfrac{1}{3} - \tfrac{1}{5}) = \tfrac{4}{15}$$

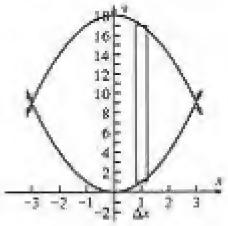
23. A square units is the area bounded by $y^2 = x - 1$; $x = 3$. The region is symmetric.
Δ is a partition of the interval $[1, 3]$ on the x axis.

$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{w_i - 1}\, \Delta_i x = 2\int_{1}^{3} \sqrt{x - 1}\, dx = 2 \cdot \tfrac{2}{3}(x-1)^{3/2}\Big|_{1}^{3} = \tfrac{4}{3}(2)^{3/2} = \tfrac{8}{3}\sqrt{2}$$

24. $y = x^2$; $x^2 = 18 - y$

▸ The region R is shown in the figure below. R is bounded above by the curve $y = f(x) = 18 - x^2$ and below by the curve $y = g(x) = x^2$. The elements of area are vertical rectangles of width $\Delta_i x$ and height $[f(w_i) - g(w_i)]$. Because $f(x) - g(x) = 18 - 2x^2 = 2(9 - x^2) = 2(3 + x)(3 - x)$, then R is bounded on the left by $x = -3$ and on the right by $x = 3$. The area of R is A square units, where

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [f(w_i) - g(w_i)]\Delta_i x$$
$$= \int_{-3}^{3} (18 - 2x^2)\, dx = 18x - \tfrac{2}{3}x^3 \Big]_{-3}^{3}$$
$$= 18(3 + 3) - \tfrac{2}{3}(27 + 27) = 72$$



The area of R is 72 square units.

25. A square units is the area bounded by $y = \sqrt{x}$ and x^3 meeting at $(0, 0)$ and$(1, 1)$.
Δ is a partition of the interval $[0, 1]$ on the x axis.

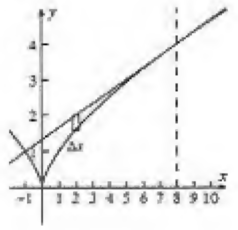
$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (\sqrt{w_i} - w_i^3)\Delta_i x = \int_{0}^{1} (\sqrt{x} - x^3)dx = \tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^4\Big|_{0}^{1} = \tfrac{2}{3} - \tfrac{1}{4} = \tfrac{5}{12}$$

26. A square units is the area bounded by $x = 4 - y^2$; $x = 4 - 4y$. Set $4 - y^2 = 4 - 4y$; $y^2 - 4y = 0$; $y = 0, 4$. Δ is a partition of the interval $[0, 4]$ on the y axis.

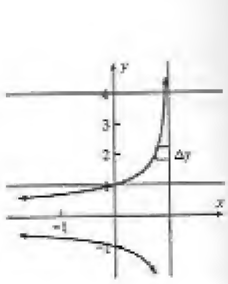
$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(4 - w_i^2) - (4 - 4w_i)]\Delta_i x = \int_{0}^{4} (4y - y^2)dy = 2y^2 - \tfrac{1}{3}y^3\Big]_{0}^{4} = 32 - \tfrac{64}{3} = \tfrac{32}{3}$$

27. A square units is the area bounded by $y^3 = x^2$; $x = 3y - 4$. When they intersect, $y^3 = (3y - 4)^2$; $0 = y^3 - 9y^2 + 2y - 16 = (y - 1)(y^2 - 8y + 16) = (y - 1)(y - 4)^2$.
The two curves cross at $(-1, 1)$ and are tangent at $(8, 4)$.
See the figure. Δ is a partition of the interval $[-1, 8]$ on the x axis.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\tfrac{1}{3}(w_i + 4) - w_i^{2/3}]\Delta_i x = \int_{-1}^{8} [\tfrac{1}{3}(x + 4) - x^{2/3}]dx$$
$$= \tfrac{1}{6}x^2 + \tfrac{4}{3}x - \tfrac{3}{5}x^{5/3}\Big|_{-1}^{8} = (\tfrac{32}{3} + \tfrac{32}{3} - \tfrac{96}{5}) - (\tfrac{1}{6} - \tfrac{4}{3} + \tfrac{3}{5}) = \tfrac{32}{15} - (-\tfrac{17}{30}) = \tfrac{27}{10}$$



28. $xy^2 = y^2 - 1$; $x = 1$; $y = 1$; $y = 4$



▸ The figure shows the region R. Because some upper endpoints of vertical lines meeting R lie on the curve and some lie on the line $y = 4$, while all horizontal lines meeting R are alike, we take horizontal rectangles as elements of area. Because R is bounded on the left by the curve $xy^2 = y^2 - 1$, we solve for x and obtain $x = \phi(y) = 1 - y^{-2}$. R is bounded on the right by $x = \lambda(y) = 1$. R is bounded below by $y = 1$ and above by $y = 4$. Hence

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\lambda(w_i) - \phi(w_i)]\Delta_i y$$
$$= \int_1^4 [1 - (1 - y^{-2})]\, dy = \int_1^4 y^{-2}\, dy = -\frac{1}{y}\Big]_1^4 = -\tfrac{1}{4} - (-1) = \tfrac{3}{4}$$

▪ The area of region R is $\frac{3}{4}$ square units.

29. A square units is the area bounded by $x = y^2 - 2$; $x = 6 - y^2$. The region is symmetric. Set $y^2 - 2 = 6 - y^2$; $2y^2 = 8$; $y^2 = 4$; $y = \pm 2$. The two curves intersect at the points $(2, 2)$ and $(2, -2)$. Δ is a partition of the interval $[0, 2]$ on the y axis.

$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(6 - w_i^2) - (w_i^2 - 2)]\Delta_i y = 2\int_0^2 [(6 - y^2) - (y^2 - 2)]dy = 2\int_0^2 (8 - 2y^2)dy = 16y - \tfrac{4}{3}\Big|_0^2$$
$$= 32 - \tfrac{32}{3} = \tfrac{64}{3}$$

30. A square units is the area bounded by $x = y^2 - y$; $x = y - y^2$. The region is symmetric. Set, $y^2 - y = y - y^2$; $2(y^2 - y) = 0$; $y = 0, 1$. Δ is a partition of the interval $[0, 1]$ on the y axis.

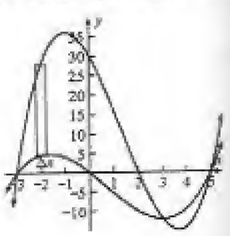
$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (w_i - w_i^2)\Delta_i x = 2\int_0^1 (y - y^2)dy = 2\Big[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\Big]_0^1 = 2(\tfrac{1}{2} - \tfrac{1}{3}) = \tfrac{1}{3}$$

31. Let $f(x) = 2x^3 - 3x^2 - 9x$ and $g(x) = x^3 - 2x^2 - 3x$. Then $f(x) - g(x) = x^3 - x^2 - 6x = x(x - 3)(x + 2)$ is nonnegative in the interval $[-2, 0]$ and nonpositive in the interval $[0, 3]$. Let A_1 and A_2 square units be the areas of the regions bounded by the two curves when x is in the intervals $[-2, 0]$ and $[0, 3]$ respectively. The number of square units in the required area is

$$A_1 + A_2 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [f(w_i) - g(w_i)]\Delta_i x + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [g(w_i) - f(w_i)]\Delta_i x$$
$$= \int_{-2}^{0} (x^3 - x^2 - 6x)dx + \int_0^3 (-x^3 + x^2 + 6x)dx = \Big[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - 3x^2\Big]_{-2}^{0} + \Big[-\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 + 3x^2\Big]_0^3$$
$$= [0 - (4 + \tfrac{8}{3} - 12)] + [(-\tfrac{81}{4} + 9 + 27) - 0] = \tfrac{16}{3} + \tfrac{63}{4} = \tfrac{253}{12}$$

32. $3y = x^3 - 2x^2 - 15x$; $y = x^3 - 4x^2 - 11x + 30$

▸ A sketch of the region R is shown below. Solving the equation $3y = x^3 - 2x^2 - 15x$ for y, we obtain $y = f(x) = \frac{1}{3}x^3 - \frac{2}{3}x^2 - 5x$. Let $g(x) = x^3 - 4x^2 - 11x + 30$. An element of area is a vertical rectangle of width $\Delta_i x$ and height $|f(w_i) - g(w_i)|$. Because



$$f(x) - g(x) = (\tfrac{1}{3}x^3 - \tfrac{2}{3}x^2 - 5x) - (x^3 - 4x^2 - 11x + 30)$$
$$= -\tfrac{2}{3}x^3 + \tfrac{10}{3}x^2 + 6x - 30 = -\tfrac{2}{3}(x^3 - 5x^2 - 9x + 45)$$
$$= -\tfrac{2}{3}[x^2(x - 5) - 9(x - 5)] = -\tfrac{2}{3}(x + 3)(x - 3)(x - 5)$$

then $f(x) - g(x) \le 0$ on $[-3, 3]$ and $f(x) - g(x) \ge 0$ on $[3, 5]$. Thus

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} |f(w_i) - g(w_i)|\Delta_i x$$
$$= \int_{-3}^{3} -[f(x) - g(x)]\, dx + \int_3^5 [f(x) - g(x)]\, dx$$
$$= \int_{-3}^{3} [\tfrac{2}{3}x^3 - \tfrac{10}{3}x^2 - 6x + 30]\, dx + \int_3^5 [-\tfrac{2}{3}x^3 + \tfrac{10}{3}x^2 + 6x - 30]\, dx$$
$$= \Big[\tfrac{1}{6}x^4 - \tfrac{10}{9}x^3 - 3x^2 + 30x\Big]_{-3}^{3} + \Big[-\tfrac{1}{6}x^4 + \tfrac{10}{9}x^3 + 3x^2 - 30x\Big]_3^5$$
$$= [\tfrac{1}{6}(0) - \tfrac{10}{9}(27 + 27) - 3(0) + 30(3 + 3)] + [-\tfrac{1}{6}(625 - 81) + \tfrac{10}{9}(125 - 27) + 3(25 - 9) - 30(5 - 3)]$$
$$= 120 + \tfrac{56}{9} = 126\tfrac{2}{9}$$

▪ The area of region R is $126\frac{2}{9}$ square units.

33. Let $f(x) = x^3 + 3x^2 + 2x$ and $g(x) = 2x^2 + 4x$. Then $f(x) - g(x) = x^3 + x^2 - 2x = x(x+2)(x-1)$ is nonnegative in the interval $[-2, 0]$ and nonpositive in the interval $[0, 1]$. Let A_1 and A_2 square units be the areas of the regions bounded by the two curves when x is in the intervals $[-2, 0]$ and $[0, 1]$ respectively. Then the number of square units in the required area is

$$A_1 + A_2 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [f(w_i) - g(w_i)]\Delta_i x + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [g(w_i) - f(w_i)]\Delta_i x$$
$$= \int_{-2}^{0} (x^3 + x^2 - 2x)dx + \int_0^1 (-x^3 - x^2 + 2x)dx = \left[\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - x^2\right]_{-2}^{0} + \left[-\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 + x^2\right]_0^1$$
$$= [0 - (4 - \tfrac{8}{3} - 4)] + [(-\tfrac{1}{4} - \tfrac{1}{3} + 1) - 0] = \tfrac{8}{3} + \tfrac{5}{12} = \tfrac{37}{12}$$

34. A square units is the area bounded by $y = |x - 1| + 3$; $y = 0$; $x = -2$; $x = 4$. The region is symmetric. Δ is a partition of the interval $[1, 4]$ on the x axis.

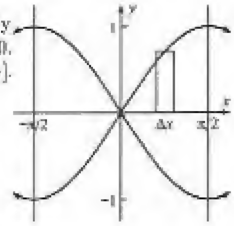
$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (|w_i - 1| + 3)\,\Delta_i x = 2\int_1^4 [(x - 1) + 3]\,dx = 2\left[\tfrac{1}{2}x^2 + 2x\right]_1^4 = 2(16 - \tfrac{5}{2}) = 27$$

35. A square units is the area of the region below $y = \cos x$ and above $y = \sin x$.

Δ is a partition of the interval $[0, \frac{1}{4}\pi]$ on the x axis. $A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (\cos w_i - \sin w_i)\Delta_i x$

$$= \int_0^{\pi/4} (\cos x - \sin x)dx = \sin x + \cos x\Big]_0^{\pi/4} = (\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}) - (0 + 1) = \sqrt{2} - 1$$

36. $y = \sin x$; $y = -\sin x$; $x = -\frac{1}{2}\pi$; $x = \frac{1}{2}\pi$



▷ The figure shows the region R. Let R_1 be the part of R in the first quadrant. By symmetry, the area of R is 4 times the area of R_1. Let $f(x) = \sin x$ and $g(x) = 0$. Elements of area are vertical rectangles of width $\Delta_i x$ and height $[f(w_i) - g(w_i)]$. Because R_1 is bounded on the left by $x = 0$ and on the right by $x = \frac{1}{2}\pi$, then

$$A = 4 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [f(w_i) - g(w_i)]\Delta_i x = 4\int_0^{\pi/2} \sin x\, dx$$
$$= -4\cos x\Big]_0^{\pi/2} = -4(0 - 1) = 4$$

▪ The area of region R is 4 square units.

37. A square units is the area of the region bounded by $y = |x|$; $y = x^2 - 1$; $x = -1$; $x = 1$.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [|w_i| - (w_i^2 - 1)]\Delta_i x = \int_{-1}^{1} [|x| - (x^2 - 1)]dx = \int_{-1}^{0} [-x - (x^2 - 1)]dx + \int_0^1 [x - (x^2 - 1)]dx$$
$$= \int_{-1}^{0} (-x^2 - x + 1)dx + \int_0^1 (-x^2 + x + 1)dx = \left[-\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + x\right]_{-1}^{0} + \left[-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + x\right]_0^1$$
$$= [0 - (\tfrac{1}{3} - \tfrac{1}{2} - 1)] + [(-\tfrac{1}{3} + \tfrac{1}{2} + 1) - 0] = \tfrac{7}{6} + \tfrac{7}{6} = \tfrac{7}{3}$$

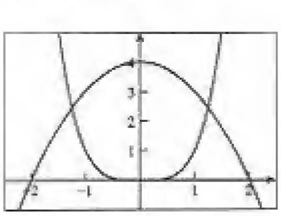
38. A square units is the area of the region bounded by $y = |x + 1| + |x|$; $y = 0$; $x = -2$; $x = 3$.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (|w_i + 1| + |w_i|)\Delta_i x = \int_{-2}^{-1} (-2x - 1)dx + \int_{-1}^{0} 1\, dx + \int_0^3 (2x + 1)dx$$
$$= \left[-x^2 - x\right]_{-2}^{-1} + 1 \cdot 1 + \left[x^2 + x\right]_0^3 = 2 + 1 + 12 = 15$$

In Exercises 39–48, approximate the area, A sq units, of the region bounded by the graphs of the given equations by doing the following: (a) Plot the graphs in a convenient window and find the x coordinates a and b of the points of intersection to 5 significant digits by using the intersection or trace and zoom-in; (b) express the area of the region as the limit of a Riemann sum; (c) approximate the limit in part (b) to 4 s.d. by using NINT.

39. $y = x^4 - 2$; $y = x^2$. $a, b = \pm\sqrt{2} \approx \pm 1.4142$. The region is symmetrical. Δ is a partition of the interval $[0, \sqrt{2}]$ on the x axis. $A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (w_i^2 - (w_i^4 - 2))\Delta_i x = 2\int_0^{\sqrt{2}} (x^2 - x^4 + 2)dx = \frac{56}{15}\sqrt{2} \approx 5.2797$

40. $y = x^4$; $y = 4 - x^2$



▷ (a) A plot is shown at the right. Using zoom-in, we find $a, b = \pm 1.24962$. The region is symmetrical.

(b) Δ is a partition of the interval $[0, b]$ on the x axis.

$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(4 - w_i^2) - w_i^4]\Delta_i x = 2\int_0^{1.24962} (4 - x^2 - x^4)dx$$

(c) Using NINT, we find the area to be 7.4772 sq units.

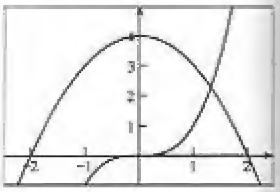
41. $y = x^2 - 1$; $y = \sin^2 x$. $a, b = \pm 1.40449$. The region is symmetrical. Δ is a partition of the interval $[0, b]$ on the x axis. $A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\sin^2 w_i - (w_i^2 - 1)]\Delta_i x = 2\int_0^{1.40449} (\sin^2 x - x^2 + 1)dx = 2.20322$

42. $y = x^2$; $y = \cos x$. $a, b = \pm 0.82413$. The region is symmetrical. Δ is a partition of the interval $[0, b]$ on the x axis. $A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (\cos w_i - w_i^2)\Delta_i x = 2\int_0^{0.82413} (\cos x - x^2)dx = 1.09475$

43. $y = x^3$; $y = 4 - x^2$; the y axis. $a = 0$, $b = 1.31460$. Δ is a partition of the interval $[0, b]$ on the x axis.
$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(4 - w_i^2) - w_i^3]\Delta_i x = \int_0^{1.31460} (4 - x^2 - x^3)dx = 3.7545$

44. $y = x^3$; $y = 4 - x^2$; the x axis

▷ (a) A plot is shown at the right. We see that we need two intervals, $[0, b]$ and $[b, 2]$. Using zoom-in, we find $b = 1.31460$.



(b) Δ is a partition of an interval on the x axis.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} w_i^3 \Delta_i x + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - w_i^2)\Delta_i x$$
$$= \int_0^{1.31460} x^3 dx + \int_{1.31460}^{2} (4 - x^2)dx$$

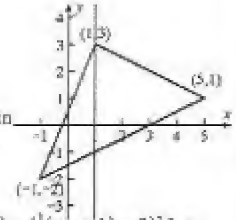
(c) Using NINT, we find the area to be $0.74665 + 0.83222 = 1.5789$ sq units.

45. $y = x^3$; $y = \tan^2 x - 3$; $0 \le x \le \frac{1}{2}\pi$. $a = 0$, $b = 1.12738$. Δ is a partition of the interval $[0, b]$ on the x axis.
$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [w_i^3 - (\tan^2 w_i - 3)]\Delta_i x = \int_0^{1.12738} (x^3 - \tan^2 x + 3)dx = 2.8079$

46. $y = 2 - x^4$; $y = \sec^2 x$. $a, b = \pm 0.71660$. The region is symmetrical. Δ is a partition of the interval $[0, b]$ on the x axis. $A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\sec^2 w_i - (2 - w_i^4)]\Delta_i x = 2\int_0^{0.71660} (\sec^2 x - 2 + x^4)dx = 1.04878$

In Problems 47 and 48, find by integration the area of a triangle having the given vertices.

47. $(5, 1)$, $(1, 3)$, and $(-1, -2)$. See the figure.



An equation of the line L_1 through $(5, 1)$ and $(1, 3)$ is $y = -\frac{1}{2}(x - 1) + 3$.

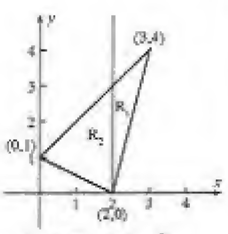
An equation of the line L_2 through $(-1, -2)$ and $(5, 1)$ is $y = \frac{1}{2}(x + 1) - 2$

An equation of the line L_3 through $(-1, -2)$ and $(1, 3)$ is $y = \frac{5}{2}(x - 1) + 3$.
A_1 square units is the area of the region bounded above by L_3 and below by L_2 in the interval $[-1, 1]$ and A_2 square units is the area of the region bounded above by L_1 and below by L_2 in the interval $[1, 5]$.
The number of square units in the total area of the triangle is $A_1 + A_2$

$$= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(\tfrac{5}{2}(w_i - 1) + 3) - (\tfrac{1}{2}(w_i + 1) - 2)]\Delta_i x + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(-\tfrac{1}{2}(w_i - 1) + 3) - (\tfrac{1}{2}(w_i + 1) - 2)]\Delta_i x$$
$$= \int_{-1}^{1} [(\tfrac{5}{2}(x - 1) + 3) - (\tfrac{1}{2}(x + 1) - 2)]dx + \int_1^5 [(-\tfrac{1}{2}(x - 1) + 3) - (\tfrac{1}{2}(x + 1) - 2)]dx$$
$$= \int_{-1}^{1} (2x + 2)dx + \int_1^5 (-x + 5)dx = \Big[x^2 + 2x\Big]_{-1}^{1} + \Big[-\tfrac{1}{2}x^2 + 5x\Big]_1^5 = [3 - (-1)] + (\tfrac{25}{2} - \tfrac{9}{2}) = 4 + 8 = 12$$

48. $(3, 4)$, $(2, 0)$, and $(0, 1)$.

▷ A sketch of the triangle is shown at the right.



An equation of the line that contains $(0, 1)$ and $(3, 4)$ is $y = f(x) = x + 1$.
An equation of the line that contains $(3, 4)$ and $(2, 0)$ is $y = g(x) = 4x - 8$.
An equation of the line that contains $(0, 1)$ and $(2, 0)$ is $y = h(x) = -\frac{1}{2}x + 1$.
R_1 is the part of the triangle to right of $x = 2$ and R_2 is the part to the left.

R_1 is bounded above by the line $y = f(x)$, below by the line $y = g(x)$, on the left by $x = 2$, and on the right by $x = 2$. If A_1 units is the area of R_1, then

$$A_1 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [f(w_i) - g(w_i)]\Delta_i x$$
$$= \int_2^3 [(x + 1) - (4x - 8)]\, dx = -3\int_2^3 (x - 3)\, dx = -3\Big[\tfrac{1}{2}x^2 - 3x\Big]_2^3 = -3[\tfrac{1}{2}(9 - 4) - 3(3 - 2)] = \tfrac{3}{2}$$

R_2 is bounded above by the line $y = f(x)$, below by the line $y = h(x)$, on the left by $y = 0$, and on the right by $x = 2$. If A_2 units is the area of R_2, then

$$A_2 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [f(w_i) - h(_i)]\Delta_i x = \int_0^2 [(x+1) - (-\tfrac{1}{2}x + 1)]\, dx = \int_0^2 \tfrac{3}{2}x\, dx = \tfrac{3}{4}x^2\Big]_0^2 = 3$$

$A = A_1 + A_2 = \frac{3}{2} + 3 = \frac{9}{2}$. The area of the triangle is $\frac{9}{2}$ square units.

We verify this result by using a formula from analytic geometry. If the vertices of a polygon traversed in the counterclockwise direction are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, then the area is given by

$$A = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)]$$

In this exercise the vertices taken counterclockwise are $(0,1)$, $(2,0)$, $(3,4)$. Therefore,

$$A = \tfrac{1}{2}[(0\cdot 0 - 1\cdot 2) + (2\cdot 4 - 0\cdot 3) + (3\cdot 1 - 4\cdot 0)] = \tfrac{1}{2}(-2 + 8 + 3) = \tfrac{9}{2}$$

In Exercises 49–57, find the exact area of the region.

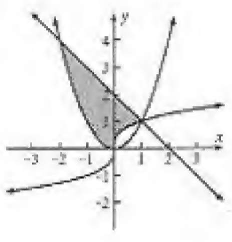
49. Solve for y: $x^3 - x^2 + 2xy - y^2 = 0$; $y^2 - 2xy + x^2 = x^3$; $(y-x)^2 = x^3$; $y - x = \pm x^{3/2}$; $y = x \pm x^{3/2}$

A square units is the area of the region below $y = x + x^{3/2}$ and above $y = x - x^{3/2}$.
Δ is a partition of the interval $[0,4]$ on the x axis. Then

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(w_i + w_i^{3/2}) - (w_i - w_i^{3/2})]\Delta_i x = \int_0^4 [(x + x^{3/2}) - (x - x^{3/2})]dx = 2\int_0^4 x^{3/2}dx$$

$$= 2 \cdot \tfrac{2}{5}x^{5/2}\Big|_0^4 = \tfrac{4}{5}\cdot 4^{5/2} = \tfrac{128}{5}$$

50. See the figure. Solving $y = x^2$ and $y = 2 - x$ we find $x^2 = 2 - x$; $x^2 + x - 2 = 0$; $(x+2)(x-1) = 0$; $x = -2, 1$. Solving $x = y^3$ and $x = 2 - y$ we find $y^3 = 2 - y$; $y^3 + y - 2 = 0$; $(y-1)(y^2 + y + 2) = 0$; $y = 1$, $x = 1$. Solving $y = x^2$ and $x = y^3$ we find $x = 0, 1$. Using partitions of $[-2,0]$ and $[0,1]$ on the x axis we have

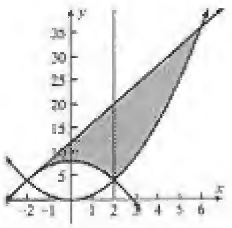


$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(2 - w_i) - w_i^2]\Delta_i x + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(2 - w_i) - w_i^{1/3}]\Delta_i x$$

$$= \int_{-2}^{0} (2 - x - x^2)dx + \int_0^1 (2 - x - x^{1/3})dx$$

$$= \left[2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_{-2}^{0} + \left[2x - \tfrac{1}{2}x - \tfrac{3}{4}x^{4/3}\right]_0^1 = \tfrac{10}{3} + \tfrac{3}{4} = \tfrac{49}{12}$$

51. A square units is the area of the region bounded by the three curves $y = f(x) = x^2$, $y = g(x) = 8 - x^2$, and $y = h(x) = 4x + 12$. See the figure.
$(8 - x^2) - x^2 = 8 - 2x^2 = 2(4 - x^2)$ is nonnegative in $[-2, 2]$.
$(4x + 12) - (8 - x^2) = x^2 + 4x + 4 = (x+2)^2$ is nonnegative in $[-2, +\infty)$ and the graphs are tangent at $(-2, 4)$. A_1 square units is the area of the region below $y = 4x + 12$ and above $y = 8 - x^2$ in $[-2, 2]$. $(4x + 12) - x^2 = -(x^2 - 4x - 12) = -(x - 6)(x + 2)$ is nonnegative in $[-2, 6]$; A_2 square units is the area of the region below $y = 4x + 12$ and above $y = x^2$ in $[2, 6]$. Then



$A = A_1 + A_2$

$$= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(4w_i + 12) - (8 - w_i^2)]\Delta_i x + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(4w_i + 12) - w_i^2]\Delta_i x$$

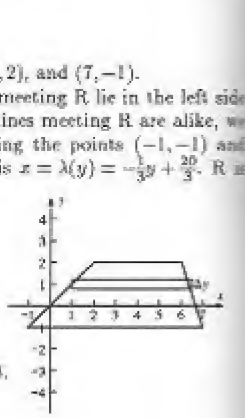
$$= \int_{-2}^{2} [(4x + 12) - (8 - x^2)]dx + \int_2^6 [(4x + 12) - x^2]dx = \int_{-2}^{2} (x^2 + 4x + 4)dx + \int_2^6 (-x^2 + 4x + 12)dx$$

$$= \left[\tfrac{1}{3}x^3 + 2x^2 + 4x\right]_{-2}^{2} + \left[-\tfrac{1}{3}x^3 + 2x^2 + 12x\right]_2^6 = [(\tfrac{8}{3} + 8 + 8) - (-\tfrac{8}{3} + 8 - 8)][(-72 + 72 + 72) - (-\tfrac{8}{3} + 8 + 24)]$$

$$= \tfrac{64}{3} + \tfrac{128}{3} = 64$$

52. Find by integration the area of the trapezoid having vertices at $(-1,-1)$, $(2,2)$, $(6,2)$, and $(7,-1)$.

▷ The region R is shown in the figure below. Because some upper endpoints of lines meeting R lie in the left side of the trapezoid, some in the top, and some in the right side, while all horizontal lines meeting R are alike, we take horizontal rectangles as elements of area. An equation of the line containing the points $(-1,-1)$ and $(2,2)$ is $x=\phi(y)=y$ and an equation of the line containing $(6,2)$ and $(7,-1)$ is $x=\lambda(y)=-\frac{1}{3}y+\frac{20}{3}$. R is bounded below by $y=-1$ and above by $y=2$. Thus



$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}[\lambda(w_i)-\phi(w_i)]\Delta_i y=\int_{-1}^{2}[(-\tfrac{1}{3}y+\tfrac{20}{3})-y]\,dy$$

$$=\int_{-1}^{2}[-\tfrac{4}{3}y+\tfrac{20}{3}]\,dy=-\tfrac{2}{3}y^2+\tfrac{20}{3}y\Big]_{-1}^{2}$$

$$=-\tfrac{2}{3}(4-1)+\tfrac{20}{3}(2+1)=18$$

The area of the trapezoid is 18 square units. The result can be verified as in Ex. 48.

53. A square units is the area of the region below $y=1$ and above $y=\sin x$.
Δ is a partition of the interval $[0,\frac{1}{2}\pi]$ on the x axis.

$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(1-\sin w_i)\Delta_i x=\int_0^{\pi/2}(1-\sin x)dx=x+\cos x\Big]_0^{\pi/2}=(\tfrac{1}{2}\pi+0)-(0+1)=\tfrac{1}{2}\pi-1$$

54. $\sin x=\cos x$ when $\tan x=1$; $x=\frac{1}{4}\pi,\frac{5}{4}\pi$. Δ is a partition of the interval $[\frac{1}{4}\pi,\frac{5}{4}\pi]$ on the x axis.

$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\sin w_i-\cos w_i)\Delta_i x=\int_{\pi/4}^{5\pi/4}(\sin x-\cos x)dx=-\cos x-\sin x\Big]_{\pi/4}^{5\pi/4}=2\sqrt{2}$$

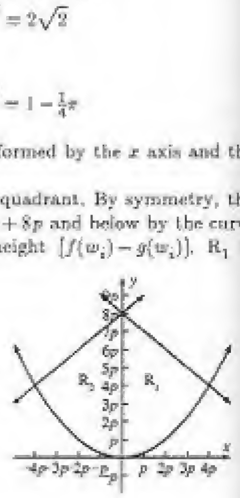
55. A square units is the area of the region bounded by $\tan x$ in the interval $[0,\frac{1}{4}\pi]$.

$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\tan^2 w_i)\Delta_i x=\int_0^{\pi/4}\tan^2 x\,dx=\int_0^{\pi/4}(\sec^2 x-1)dx=\tan x-x\Big|_0^{\pi/4}=1-\tfrac{1}{4}\pi$$

56. Find the area of the region above the parabola $x^2=4py$ and inside the triangle formed by the x axis and the lines $y=x+8p$ and $y=-x+8p$, where $p>0$.

▷ The region R is shown in the figure below. Let R_1 be the part of R in the first quadrant. By symmetry, the area of R is twice the area of R_1. R_1 is bounded above by the line $y=f(x)=-x+8p$ and below by the curve $y=x^2/4p$. The elements of area are vertical rectangles of width $\Delta_i x$ and height $[f(w_i)-g(w_i)]$. R_1 is bounded on the left by $x=0$. Because

$$f(x)-g(x)=(-x+8p)-\frac{x^2}{4p}=-\frac{1}{4p}(x^2+4px-32p^2)=-\frac{1}{4p}(x+8p)(x-4p)$$



then R_1 is bounded on the right by $x=4p$. Thus

$$A=2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}[f(w_i)-g(w_i)]\Delta_i x=2\int_0^{4p}[-x+8p-\frac{x^2}{4p}]\,dx$$

$$=2\Big[-\frac{1}{2}x^2+8px-\frac{x^3}{12p}\Big]_0^{4p}=2\Big(-8p^2+32p^2-\frac{16}{3}p^2\Big)=\frac{112}{3}p^2$$

The area of R is $\frac{112}{3}p^2$ square units.

57. A square units is the area of the region bounded by $y^2=4px$ and $x^2=4py$.

Solving for y, we find $2p^{1/2}x^{1/2}-\frac{1}{4p}x^2=\frac{1}{4p}x^{1/2}(8p^{3/2}-x^{3/2})$ is positive in $[0,4p]$.

$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(2p^{1/2}w_i^{1/2}-\frac{1}{4\pi}w_i^2)\Delta_i x=\int_0^{4p}(2p^{1/2}x^{1/2}-\frac{1}{4p}x^2)dx=2p^{1/2}\cdot\tfrac{2}{3}x^{3/2}-\frac{1}{4p}\cdot\frac{x^3}{3}\Big|_0^{4p}$$

$$=\tfrac{32}{3}p^2-\tfrac{16}{3}p^2=\tfrac{16}{3}p^2$$

58. From Ex. 56, $A=\frac{112}{3}p^2$. The rate of change of A with respect to p at $p=\frac{3}{8}$ is $D_pA\Big|_{p=3/8}=\frac{224}{3}p\Big|_{p=3/8}=28$

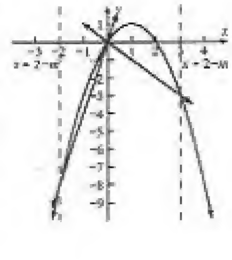
59. From Exercise 57, $A=\frac{16}{3}p^2$. The rate of change of A with respect to p when $p=3$ is $D_pA\Big|_{p=3}=\frac{32}{3}p\Big|_{p=3}=32$

60. Determine m so that the region above the line $y = mx$ and below the parabola $y = 2x - x^2$ has an area of 36 square units.

▷ R is bounded above by the curve $y = f(x) = 2x - x^2$ and below by the line $y = g(x) = mx$. The elements of area are vertical rectangles of width $\Delta_i x$ and height $[f(w_i) - g(w_i)]$. Because

$$f(x) - g(x) = (2x - x^2) - mx = x(2 - m - x)$$

then R is bounded on one side by $x = 0$ and on the other side by $x = 2 - m$.
If $m < 2$, as shown on the right side of the figure, then the area of R is given by



$$\begin{aligned} A &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [f(w_i) - g(w_i)]\Delta_i x = \int_0^{2-m} [(2x - x^2) - mx]\, dx \qquad (1) \\ &= (2 - m)\int_0^{2-m} x\, dx - \int_0^{2-m} x^2\, dx = \tfrac{1}{2}(2-m)\Big]_0^{2-m} - \tfrac{1}{3}x^3\Big]_0^{2-m} \\ &= \tfrac{1}{2}(2-m)(2-m)^2 - \tfrac{1}{3}(2-m)^3 = \tfrac{1}{6}(2-m)^3 \qquad (2) \end{aligned}$$

Because we are given that $A = 36$, from Eq. (2) we have

$$36 = \tfrac{1}{6}(2-m)^3 \qquad (3)$$

$$6^3 = (2-m)^3; \qquad 6 = 2 - m; \quad m = -4$$

If $m > 2$, as shown on the left side of the figure, then the limits in Eq. (1) must be reversed and so $A = \frac{1}{6}(m-2)^3$. Eq. (3) becomes

$$36 = \tfrac{1}{6}(m-2)^3; \qquad 6 = m - 2; \qquad m = 8$$

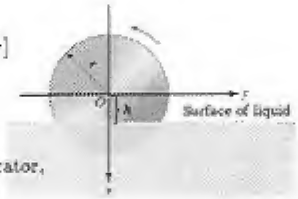
61. Because $x \geq 0$ and $m > 0$, $m - mx^2 = m(1 - x^2)$ is positive in $[0, 1]$.

$K = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} (m - mw_i^{\,2})\Delta_i x = \int_0^1 (m - mx^2)dx = mx - \frac{1}{3}mx^3\Big]_0^1 = m - \frac{1}{3}m = \frac{2}{3}m$. Hence $m = \frac{3}{2}K$.

62. Because $m > 0$, $x_2 - x_1 = \frac{1}{m}y - \frac{1}{4}y^2 = \frac{y}{4}\left(\frac{4}{m} - y\right)$ is positive on $\left[0, \frac{4}{m}\right]$.

$A = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n}\left(\frac{1}{m}w_i - \frac{1}{4}w_i^{\,2}\right)\Delta_i y = \int_0^{4/m}\left(\frac{y}{m} - \frac{1}{4}y^2\right)dy = \frac{1}{2m}y^2 - \frac{1}{12}y^3\Big]_0^{4/m} = \frac{8}{3m^3}$. $A'(m) = -\frac{8}{m^4}$

63. See the figure. The exposed wetted area is the area between the circles of radii r and h minus the area below the line $x = h$ which is symmetrical. Δ is a partition of $[h, r]$ on the x axis. From the Pythagorean theorem, on $[0, r]$



$$\begin{aligned} A &= \pi r^2 - \pi h^2 - 2\lim_{\|\Delta\| \to 0}\sum_{i=1}^{n}\sqrt{r^2 - w_i^{\,2}}\,\Delta_i x = \pi r^2 - \pi h^2 - 2\int_h^r \sqrt{r^2 - x^2}\,dx \\ &= \pi r^2 - \pi h^2 + 2\int_r^h \sqrt{r^2 - x^2}\,dx \end{aligned}$$

Using the first fundamental theorem of calculus and rationalizing the numerator,

$$A'(h) = -2\pi h + 2\sqrt{r^2 - h^2} = 2\frac{(r^2 - h^2) - \pi^2 h^2}{\sqrt{r^2 - h^2} + \pi h} = 2\frac{r^2 - (1 + \pi^2)h^2}{\sqrt{r^2 - h^2} + \pi h}$$

Because $A'(h) > 0$ if $h < \dfrac{r}{\sqrt{1 + \pi^2}} \approx 0.30r = h_1$ and $A'(h) < 0$ if $h > h_1$, then $h = h_1$ gives the absolute maximum value of A.

4.9 VOLUMES OF SOLIDS BY SLICING, DISKS, AND WASHERS

4.9.1 Definition Method of Slicing Let S be a solid such that S lies between planes drawn perpendicular to the x axis at a and b. If the measure of the area of the plane section of S drawn perpendicular to the x axis at x is given by $A(x)$, where A is continuous on $[a, b]$, then the measure of the volume of S is given by

$$V = \lim_{\|\Delta\| \to 0}\sum_{i=1}^{n} A(w_i)\Delta_i x = \int_a^b A(x)\, dx$$

We call $A(w_i)\Delta_i x$ the volume of a slice when applying Definition 4.9.1 to find the volume of a solid. In Exercises 4.8 we showed that the measure of the area of a region in a plane is the same for the methods of vertical and horizontal rectangles, if the bounding curve is smooth. Similarly, the measure of volume of a solid is the same for slices perpendicular to any axis if the boundary of the solid is smooth.

4.9.2 Theorem Method of Disks Let the function f be continuous on the closed interval $[a,b]$, and assume that $f(x) \geq 0$ for all x in $[a,b]$. If S is the solid of revolution obtained by revolving about the x axis the region bounded by the curve $y = f(x)$, the x axis, and the lines $x = a$ and $x = b$, and if V cubic units is the volume of S, then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[f(w_i)]^2 \Delta_i x = \pi \int_a^b [f(x)]^2\, dx$$

We call $\pi[f(w_i)]^2 \Delta_i x$ the volume of an elementary disk when applying Theorem 4.9.2 to find the volume of a solid. If the axis of revolution is not a boundary of the region, we have the following result:

4.9.3 Theorem Method of Washers Let the functions f and g be continuous on the closed interval $[a,b]$, and assume that $f(x) \geq g(x) \geq 0$ for all x in $[a,b]$. If V cubic units is the volume of the solid of revolution generated by revolving about the x axis the region bounded by the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$, then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\{[f(w_i)]^2 - [g(w_i)]^2\} \Delta_i x = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\}\, dx$$

To apply Theorem 4.9.3 we note that $f(x)$ and $g(x)$ represent the distances of the farther and nearer boundary from the axis of revolution and we are assuming that the axis of revolution does not cross the boundary. We call $\pi\{[f(w_i)]^2 - [g(w_i)]^2\}\Delta_i x$ the volume of an elementary disk. If the axis of revolution is an axis of symmetry, we revolve only one side of the axis.

Exercises 4.9

In these Exercises, V cubic units is the required volume and Δ is a partition of the specified interval.

1. The solid is a sphere of radius r centered at the origin. An element of volume is a circular disk x units right of center, $x \in [-r, r]$, of radius $\sqrt{r^2 - w^2}$ and area $\pi(r^2 - w^2)$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(r^2 - w_i^{\,2})\Delta_i x = \pi \int_{-r}^{r} (r^2 - x^2)dx = \pi\Big[r^2 x - \tfrac{1}{3}x^3\Big]_{-r}^{r} = \tfrac{4}{3}\pi r^3$$

2. *The solid is any circular cone whose base of radius a is perpendicular to the x axis. An element of volume is a* circular disk x units right of center, $x \in [0, h]$, of radius aw/h and area $\pi a^2 w^2/h^2$.

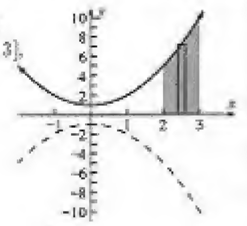
$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{\pi a^2 w_i^{\,2}}{h^2}\Delta_i x = \pi \int_0^h \frac{\pi a^2 x^2}{h^2} dx = \frac{\pi a^2 x^3}{3h^2}\Big]_0^h = \tfrac{1}{3}\pi a^2 h$$

3. The region is bounded by $y = x^3$, the x axis, $x = 1$ and $x = 2$. An element of volume is a circular disk centered on the x axis, $x \in [1, 2]$, of radius $w_i^{\,3}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(w_i^{\,3})^2 \Delta_i x = \pi \int_1^2 x^6 dx = \tfrac{1}{7}\pi x^7\Big|_1^2 = \tfrac{127}{7}\pi$$

4. Find the volume of the solid of revolution generated when the region bounded by the curve $y = x^2 + 1$, the x axis, and the lines $x = 2$ and $x = 3$ is revolved about the x axis.

▸ Let $f(x) = x^2 + 1$. The figure shows the region and a plane section of the solid of revolution. An element of volume is a circular disk centered on the x axis, $x \in [2, 3]$, of radius $f(w_i)$. If V cubic units is the volume of the solid, by Theorem 4.9.2,

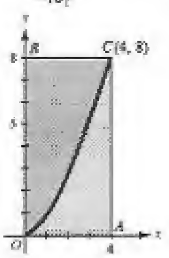


$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[f(w_i)]^2 \Delta_i x = \pi \int_2^3 (x^2 + 1)^2\, dx = \pi \int_2^3 (x^4 + 2x^2 + 1)\, dx$$

$$= \pi\Big[\tfrac{1}{5}x^5 + \tfrac{2}{3}x^3 + x\Big]_2^3 = \pi[\tfrac{1}{5}(243 - 32) + \tfrac{2}{3}(27 - 8) + (3 - 2)] = \tfrac{838}{15}\pi$$

▪ The volume of the solid is $\frac{838}{15}\pi$ cubic units.

In Exercises 5–12, find the volume of the solid of revolution when the given region of the figure is revolved about the indicated axis. An equation of the curve is $y^2 = x^3$.



5. The region is bounded by $y^2 = x^3$, the x axis and $x = 4$. An element of volume is a circular disk centered on the x axis, $x \in [0, 4]$, of radius $w_i^{\,3/2}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(w_i^{\,3/2})^2 \Delta_i x = \pi \int_0^4 x^3\, dx = \tfrac{1}{4}\pi\, x^4\Big|_0^4 = 64\pi$$

6. The region is bounded by $y^2 = x^3$, the x axis and $x = 4$. An element of volume is a circular disk centered on $x = 4$, $y \in [0, 8]$, of radius $4 - w_i^{2/3}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(4 - w_i^{2/3})^2 \Delta_i y = \pi \int_0^8 (4 - x^{2/3})^2 dx = \pi \int_0^8 (16 - 8x^{2/3} + x^{4/3}) dx = \pi\Big[16x - \tfrac{24}{5}x^{5/3} + \tfrac{3}{7}x^{7/3}\Big]_0^8$$

$$= \pi(16 \cdot 8 - \tfrac{24}{5} \cdot 32 + \tfrac{3}{7} \cdot 128) = \tfrac{1024}{35}\pi$$

7. The region is bounded by $y^2 = x^3$, the x axis and $x = 4$. An element of volume is a circular ring centered on the line $y = 8$, $x \in [0, 4]$, of radii 8 and $8 - w_i^{3/2}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\pi(8)^2 - \pi(8 - w_i^{3/2})^2] \Delta_i x = \pi \int_0^4 [64 - (8 - x^{3/2})^2] dx = \pi \int_0^4 (16x^{3/2} - x^3) dx$$

$$= \pi\Big[16 \cdot \tfrac{2}{5}x^{5/2} - \tfrac{1}{4}x^4\Big]_0^4 = \pi\Big[\tfrac{32}{5}(32) - 64\Big] = \tfrac{704}{5}\pi$$

8. OAC about the y axis.

▸ Because the y axis is the axis of revolution, the farther boundary is $x = 4$, and so $f(y) = 4$, and the nearer boundary is the curve $y^2 = x^3$. Solving for x, we have $x = g(y) = y^{2/3}$. An element of volume is a washer centered on the y axis, $y \in [0, 8]$, of radii $f(w_i)$ and $g(w_i)$. By Theorem 4.9.3,

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[f(w_i)^2 - g(w_i)^2] \Delta_i y = \pi \int_0^8 [4^2 - (y^{2/3})^2]\, dy$$

$$= \pi \int_0^8 (16 - y^{4/3})\, dy = \pi\Big[16y - \tfrac{3}{7}y^{7/3}\Big]_0^8 = \tfrac{512}{7}\pi$$

▪ The volume of the solid of revolution is $\frac{512}{7}\pi$ cubic units.

9. The region is bounded by $y^2 = x^3$, the y axis and $y = 8$. An element of volume is a circular disk centered on the y axis, $y \in [0, 8]$, of radius $w_i^{2/3}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(w_i^{3/2})^2 \Delta_i y = \pi \int_0^8 y^{4/3} dy = \tfrac{3}{7}\pi\, y^{7/3}\Big|_0^8 = \pi(\tfrac{3}{7} \cdot 128) = \tfrac{384}{7}\pi$$

10. The region is bounded by $y^2 = x^3$, the y axis and $y = 8$. An element of volume is a circular disk centered on $y = 8$, $x \in [0, 4]$, of radius $8 - w_i^{2/3}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(8 - w_i^{3/2})^2 \Delta_i x = \pi \int_0^4 (8 - x^{3/2})^2 dx = \pi \int_0^4 (64 - 16x^{3/2} + x^3) dx = \pi\Big[64x - \tfrac{32}{5}x^{5/2} + \tfrac{1}{4}x^4\Big]_0^4$$

$$= \pi(64 \cdot 4 - \tfrac{32}{5} \cdot 32 + 64) = \tfrac{576}{5}\pi$$

11. The region is bounded by $y^2 = x^3$, the y axis and $y = 8$. An element of volume is a circular ring centered on the line $x = 4$, $y \in [0, 8]$, of radii 4 and $4 - w_i^{2/3}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\pi(4)^2 - \pi(4 - w_i^{2/3})^2] \Delta_i y = \pi \int_0^8 [16 - (4 - y^{2/3})^2] dy = \pi \int_0^8 (8y^{2/3} - y^{4/3}) dy$$

$$= \pi\Big[8 \cdot \tfrac{3}{5}y^{5/3} - \tfrac{3}{7}y^{7/3}\Big] = \pi(\tfrac{24}{5} \cdot 32 - \tfrac{3}{7} \cdot 128) = \tfrac{3456}{35}\pi$$

12. OBC about the x axis.

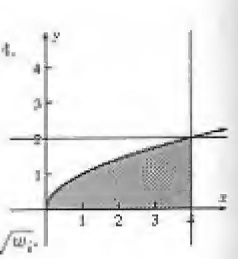
▸ Because the x axis is the axis of revolution, the farther boundary is $y = f(x) = 8$ and the nearer boundary is the curve $y^2 = x^3$. Solving for y, we have $y = g(x) = x^{3/2}$. An element of volume is a washer centered on the x axis, $x \in [0, 4]$, of radii $f(w_i)$ and $g(w_i)$ and height $\Delta_i x$. By Theorem 4.9.3,

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[f(w_i)^2 - g(w_i)^2] \Delta_i x = \pi \int_0^4 [8^2 - (x^{3/2})^2]\, dx$$

$$= \pi \int_0^4 (64 - x^3)\, dx = \pi\Big[64x - \tfrac{1}{4}x^4\Big]_0^4 = \pi[64(4) - \tfrac{1}{4}(256)] = 192\pi$$

▪ The volume of the solid of revolution is 192π cubic units.

In Exercises 13–16, find the volume of the solid of revolution generated by revolving around the indicated line the region bounded by the curve $y = \sqrt{x}$, the x axis, and $x = 4$.



▷ A figure for these exercises is shown at the right. Set $x = 4$: $y = \sqrt{4} = 2$.

13. An element of volume is a circular disk centered on the line $x = 4$, $y \in [0, 2]$, of radius $4 - w_i^2$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(4 - w_i^2)^2 \Delta_i y = \pi \int_0^2 (4 - y^2)^2 dy = \pi \int_0^2 (16 - 8y^2 + y^4) dy$$
$$= \pi\left[16y - \tfrac{8}{3}y^3 + \tfrac{1}{5}y^5\right]_0^2 = \pi(32 - \tfrac{64}{3} + \tfrac{32}{5}) = \tfrac{256}{15}\pi$$

14. An element of volume is a circular disk centered on the x axis, $x \in [0, 4]$, of radius $\sqrt{w_i}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\sqrt{w_i})^2 \Delta_i x = \pi \int_0^4 x\, dx = \tfrac{1}{2}\pi x^2\Big]_0^4 = 8\pi$$

15. An element of volume is a circular ring centered on the y axis, $y \in [0, 2]$, of radii 4 and w_i^2.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\pi(4)^2 - \pi(w_i^2)^2]\Delta_i y = \pi \int_0^2 (16 - y^4) dy = \pi\left[16y - \tfrac{1}{5}y^5\right]_0^2 = \pi(32 - \tfrac{32}{5}) = \tfrac{128}{5}\pi$$

16. the line $y = 2$.

▷ Because the axis of revolution is the line $y = 2$, the farther boundary is the x axis, and so $f(x) = 2$. The nearer boundary is the curve $y = \sqrt{x}$, whose distance from $y = 2$ is $g(x) = 2 - \sqrt{x}$. An element of volume is a washer centered on $y = 2$, $x \in [0, 4]$, of radii $f(w_i)$ and $g(w_i)$. By Theorem 4.9.3,

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[f(w_i)^2 - g(w_i)^2]\Delta_i x = \pi \int_0^4 [4 - (2 - \sqrt{x})^2]\, dx$$
$$= \pi \int_0^4 (4x^{1/2} - x)\, dx = \pi\left[\tfrac{8}{3}x^{3/2} - \tfrac{1}{2}x^2\right]_0^4 = \pi[\tfrac{8}{3}(8) - \tfrac{1}{2}(16)] = \tfrac{40}{3}\pi$$

▪ The volume of the solid of revolution is $\frac{40}{3}\pi$ cubic units.

17. Revolve about the x axis the first quadrant region bounded by the circle $x^2 + y^2 = r^2$ and the coordinate axes. The volume of the sphere will be twice the volume generated. An element of volume is a circular disk centered on the x axis, $x \in [0, r]$, of radius $\sqrt{r^2 - w_i^2}$.

$$V = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\left(\sqrt{r^2 - w_i^2}\right)^2 \Delta_i x = 2\pi \int_0^r (r^2 - x^2) dx = 2\pi\left[r^2 x - \tfrac{1}{3}x^3\right]_0^r = 2\pi(r^3 - \tfrac{1}{3}r^2) = \tfrac{4}{3}\pi r^3$$

18. Revolve about the x axis region bounded by the line $x/h + y/a = 1$ and the coordinate axes. An element of volume of the right-circular cone is a circular disk centered on the x axis, $x \in [0, h]$, of radius $a(1 - w_i/h)$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\left[a(1 - \tfrac{w_i}{h})\right]^2 \Delta_i x = \pi a^2 \int_0^h \left(1 - \frac{2x}{h} + \frac{x^2}{h^2}\right) dx = \pi a^2\left[x - \frac{x^2}{h} + \frac{x^3}{3h^2}\right]_0^h = \tfrac{1}{3}\pi a^2 h$$

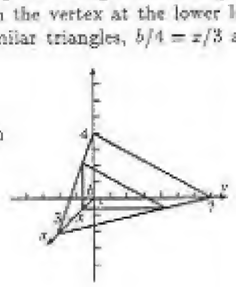
19. The region is bounded by $y = b - \frac{b-a}{h}x$, the x axis and $r = \mathrm{h}$. An element of volume is a circular disk centered on the x axis, $x \in [0, \mathrm{h}]$, of radius $b - \frac{b-a}{h}w_i$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\left(b - \frac{b-a}{h}w_i\right)^2 \Delta_i x = \pi \int_0^h \left(b - \frac{b-a}{h}x\right) dx = -\frac{\pi}{3}\frac{h}{b-a}\left(b - \frac{b-a}{h}x\right)\Big|_0^h = -\frac{\pi}{3}\frac{h}{b-a}(a^3 - b^3)$$
$$= \tfrac{1}{3}\pi \mathrm{h}(a^2 + \mathrm{ab} + b^2)$$

20. Find by slicing the volume of a tetrahedron having three mutually perpendicular faces and three mutually perpendicular edges whose lengths are 3 in., 4 in., and 7 in.

▷ The figure shows the tetrahedron and the x axis chosen along the 3 in. edge, the y axis along the 7 in. edge. A plane section of the tetrahedron drawn perpendicular to the x axis x units from the vertex at the lower left, not from the origin, $x \in [0, 3]$, is a right triangle of leg lengths b and c. By similar triangles, $b/4 = x/3$ and $c/7 = x/3$. Thus the leg lengths are $4(x/3)$ and $7(x/3)$ and the area is

$$A(x) = \frac{1}{2} \cdot \frac{4x}{3} \cdot \frac{7x}{3} = \frac{14}{9}x^2$$



An element of volume is a triangular cylinder of volume $A(w_i)\Delta_i x$. By Definition 4.9.1, if V cubic units is the volume of the solid, then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} A(w_i)\Delta_i x = \int_0^3 \frac{14}{9}x^2\, dx = \frac{14}{27}x^3\Big]_0^3 = \frac{14}{27}(27) = 14$$

▪ The volume of the tetrahedron is 14 cubic inches.

21. The region is bounded by $y = \sec x$, the x axis, the y axis, and $x = \frac{1}{4}\pi$. An element of volume is a circular disk centered on the x axis, $x \in [0, \frac{1}{4}\pi]$, of radius $\sec w_i$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\sec^2 w_i)\Delta_i x = \pi \int_0^{\pi/4} \sec^2 x \, dx = \pi \tan x \Big]_0^{\pi/4} = \pi$$

22. The region is bounded by $y = \csc x$ and the x axis, $x \in [\frac{1}{6}\pi, \frac{1}{3}\pi]$. An element of volume is a circular disk, centered on the x axis, of radius $\csc w_i$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\csc w_i)^2\Delta_i x = \pi \int_{\pi/6}^{\pi/3} \csc^2 x \, dx = -\pi \cot x \Big]_{\pi/6}^{\pi/3} = -\pi(\tfrac{1}{3}\sqrt{3} - \sqrt{3}) = \tfrac{2}{3}\sqrt{3}\pi$$

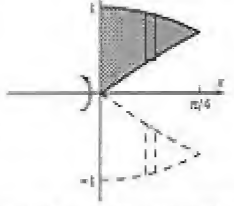
23. The region is bounded by $y = \sin x$ and the x axis, $x \in [0, \pi]$. An element of volume is a circular disk, centered on the x axis, of radius $\sin w_i$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\sin w_i)^2\Delta_i x = \pi \int_0^{\pi} \sin^2 x \, dx = \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx = \frac{\pi}{2}\int_0^{\pi} dx - \frac{\pi}{2}\int_0^{\pi} \cos 2x \, dx =$$

$$= \frac{\pi}{2}\Big[x - \tfrac{1}{2}\sin 2x\Big]_0^{\pi} = \frac{\pi}{2}[(\pi - 0) - (0 - 0)] = \tfrac{1}{2}\pi^2$$

24. The region bounded by the y axis and the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le \frac{1}{4}\pi$ is revolved about the x axis. Find the volume of the solid of revolution generated. (*Hint:* Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.)

▷ The figure shows the region and a plane section of the solid of revolution. Because the x axis is the axis of revolution, the farther boundary is $y = f(x) = \cos x$ and the nearer boundary is $y = g(x) = \sin x$. An element of volume is a washer centered on the x axis, $x \in [0, \frac{1}{4}\pi]$, of radii $f(w_i)$ and $g(w_i)$. By Theorem 4.9.3,



$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[f(w_i)^2 - g(w_i)^2]\Delta_i x = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x) \, dx$$

$$= \pi \int_0^{\pi/4} \cos 2x \, dx = \tfrac{1}{2}\pi \sin 2x \Big]_0^{\pi/4} = \tfrac{1}{2}\pi(\sin \tfrac{1}{2}\pi - \sin 0) = \tfrac{1}{2}\pi(1 - 0) = \tfrac{1}{2}\pi$$

■ The volume of the solid of revolution is $\frac{1}{2}\pi$ cubic units.

25. The region is bounded by $y = \sin x$ and the x axis, $x \in [0, \pi]$. An element of volume is a circular disk, centered on the line $y = 1$, of radius $1 - \sin w_i$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(1 - \sin w_i)^2\Delta_i x = \pi \int_0^{\pi} (1 - 2\sin x + \sin^2 x)dx = \pi \int_0^{\pi} [1 - 2\sin x + \tfrac{1}{2}(1 - \cos 2x)]dx$$

$$= \pi \int_0^{\pi} \left(\tfrac{3}{2} - 2\sin x - \tfrac{1}{2}\cos 2x\right)dx = \pi\Big[\tfrac{3}{2}x + 2\cos x - \tfrac{1}{4}\sin 2x\Big] = \pi\Big[(\tfrac{3}{2}\pi - 2) - 2\Big] = \tfrac{3}{2}\pi^2 - 4\pi$$

If you select the arch from $[\pi, 2\pi]$ you get $V = 4\pi - \frac{1}{2}\pi^2$.

26. By symmetry, the volume of the solid generated if the region of Exercise 24 is revolved about $y = 1$ is also $\frac{1}{2}\pi$.

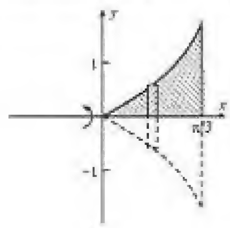
27. The region is bounded by $y = \cot x$, $x = \frac{1}{6}\pi$, and the x axis. An element of volume is a circular disk centered on the x axis, $x \in [\frac{1}{6}\pi, \frac{1}{2}\pi]$, of radius $\cot w_i$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\cot w_i)^2\Delta_i x = \pi \int_{\pi/6}^{\pi/2} \cot^2 x \, dx = \pi \int_{\pi/6}^{\pi/2} (\csc^2 x - 1)dx = \pi(-\cot x - x)\Big|_{\pi/6}^{\pi/2}$$

$$= \pi(-\tfrac{1}{2}\pi + \sqrt{3} + \tfrac{1}{6}\pi) = \sqrt{3}\pi - \tfrac{1}{3}\pi^2$$

28. The region bounded by the curve $y = \tan x$, the line $x = \frac{1}{3}\pi$, and the x axis is revolved about the x axis. Find the volume of the solid generated.

▷ The figure shows the region and a plane section of the solid of revolution. Let $f(x) = \tan x$. An element of volume is a disk centered on the x axis, $x \in [0, \frac{1}{3}\pi]$, of radius $f(w_i)$. By Theorem 4.9.2,



$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi f(w_i)^2\Delta_i x = \pi \int_0^{\pi/3} \tan^2 x \, dx = \pi \int_0^{\pi/3} (\sec^2 x - 1) \, dx$$

$$= \pi\Big[\tan x - x\Big]_0^{\pi/3} = \pi[\tan \tfrac{1}{3}\pi - \tfrac{1}{3}\pi] = \pi[\sqrt{3} - \tfrac{1}{3}\pi]$$

■ The volume of the solid is $\pi[\sqrt{3} - \frac{1}{3}\pi]$ cubic units.

29. The region is bounded by $x = 4 + 6y - 2y^2$ and $x = -4$. $(4 + 6y - 2y^2) - (-4) = -2(y^2 - 3y - 4) = -2(y+1)(y-4)$ is nonnegative in $[-1, 4]$. An element of volume is a circular disk centered on the line $x = -4$, $y \in [-1, 4]$, of radius $(4 + 6w_i - 2w_i^2) + 4$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(8 + 6w_i - 2w_i^2)^2 \Delta_i y = \pi \int_{-1}^{4} (8 + 6y - 2y^2)^2 dy$$
$$= \pi \int_{-1}^{4} (64 + 96y + 4y^2 - 24y^3 + 4y^4) dy = \pi\left[64y + 48y^2 + \tfrac{4}{3}y^3 - 6y^4 + \tfrac{4}{5}y^5\right]$$
$$= \pi\left[(64 \cdot 4 + 48 \cdot 16 + \tfrac{4}{3} \cdot 64 - 6 \cdot 256 + \tfrac{4}{5} \cdot 1024) - (-64 + 48 - \tfrac{4}{3} - 6 - \tfrac{4}{5})\right] = \tfrac{1250}{3}\pi$$

30. The region is bounded by $y = 2\sqrt{x}$ and $y = x$. An element of volume is a circular washer centered on the x axis, $x \in [0, 4]$, of radii $2\sqrt{w_i}$ and w_i.

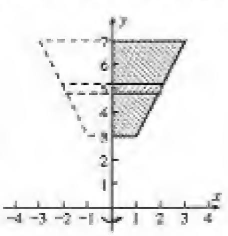
$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi[(2\sqrt{w_i})^2 - w_i^2]\Delta_i x = \pi \int_0^4 (4x - x^2) dx = \pi\left[2x^2 - \tfrac{1}{3}x^3\right]_0^4 = \pi(32 - \tfrac{64}{3}) = \tfrac{32}{3}\pi$$

31. The region is bounded by $y^2 = 4x$ and $y = x$. $y - \frac{1}{4}y^2 = \frac{1}{4}y(4 - y)$ is nonnegative in $[0, 4]$. An element of volume is a circular ring, centered on $x = 4$, of radii $4 - \frac{1}{4}w_i^2$ and $4 - w_i$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi\left[\left(4 - \frac{w_i^2}{4}\right)^2 - (4 - w_i)^2\right]\Delta_i y = \pi \int_0^4 \left[\left(4 - \frac{y^2}{4}\right)^2 - (4 - y)^2\right] dy$$
$$= \pi \int_0^4 \left(16 - 2y^2 + \frac{y^4}{16} - 16 + 8y - y^2\right) dy = \pi \int_0^4 \left(8y - 3y^2 + \frac{y^4}{16}\right) dy = \pi\left[4y^2 - y^3 + \frac{y^5}{80}\right]_0^4$$
$$= \pi(4 \cdot 16 - 64 + \tfrac{64}{5}) = \tfrac{64}{5}\pi$$

32. Find the volume of the solid generated by revolving about the y axis the region bounded by the line through $(1, 3)$ and $(3, 7)$, and the lines $y = 3$, $y = 7$, and $x = 0$.

▷ The figure shows the region and a plane section of the solid of revolution. An equation of the line containing the points $(1, 3)$ and $(3, 7)$ is



$$\frac{x - 1}{y - 3} = \frac{3 - 1}{7 - 3} = \frac{1}{2}$$

Thus $x = f(y) = \frac{1}{2}(y - 3) + 1 = \frac{1}{2}(y - 1)$. An element of volume is a disk centered on the y axis, $y \in [3, 7]$, of radius $f(w_i)$. By Theorem 4.9.2,

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi f(w_i)^2 \Delta_i y = \pi \int_3^7 [\tfrac{1}{2}(y - 1)]^2\, dy = \frac{\pi}{4} \int_3^7 (y - 1)^2\, dy$$
$$= \frac{\pi}{12}(y - 1)^3\Big]_3^7 = \frac{\pi}{12}(6^3 - 2^3) = \tfrac{52}{3}\pi$$

■ The volume of the solid of revolution is $\frac{52}{3}\pi$ cubic units.

33. The region is bounded by $y = x^2$ and $y = 1 + x - x^2$. $(1 + x - x^2) - x^2 = -(2x^2 - x - 1) = -(2x + 1)(x - 1)$ is nonnegative in $[-\frac{1}{2}, 1]$. An element of volume is a circular ring centered on the line $y = -3$, of radii $(1 + w_i - w_i^2) + 3$ and $w_i^2 + 3$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi[(4 + w_i - w_i^2)^2 - (w_i^2 + 3)^2]\Delta_i x = \pi \int_{-1/2}^{1} [(4 + x - x^2)^2 - (x^2 + 3)^2] dx$$
$$= \pi \int_{-1/2}^{1} [(x^4 - 2x^3 - 7x^2 + 8x + 16) - (x^4 + 6x^2 + 9)] dx = \pi \int_{-1/2}^{1} (-2x^3 - 13x^2 + 8x + 7) dx$$
$$= \pi\left[-\tfrac{1}{2}x^4 - \tfrac{13}{3}x^3 + 4x^2 + 7x\right]_{-1/2}^{1} = \pi\left[(-\tfrac{1}{2} - \tfrac{13}{3} + 4 + 7) - (-\tfrac{1}{32} + \tfrac{13}{24} + 1 - \tfrac{7}{2})\right] = \tfrac{261}{32}\pi$$

34. $y^2 = \frac{1}{2}x(x^2 - 4)$ is positive in $[-2, 0]$. An element of volume is a circular disk centered on the x axis, $x \in [-2, 0]$, of radius $y(w_i)$.

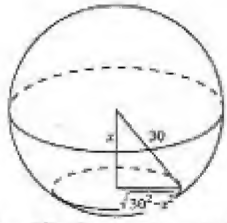
$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi y(w_i)^2 \Delta_i x = \tfrac{1}{2}\pi \int_{-2}^{0} x(x^2 - 4) dx = \tfrac{1}{2}\pi \int_{-2}^{0} (x^3 - 4x) dx = \tfrac{1}{2}\pi\left[\tfrac{1}{4}x^4 - 2x^2\right]_{-2}^{0} = \tfrac{1}{2}\pi(-4 + 8) = 2\pi$$

35. $y^2 = \dfrac{(x^2-9)(1-x^2)}{x^2}$ is positive in $[-3,-1]$ and $[1,3]$. Hence the curve consists of two closed loops, each bisected by the x axis; choose the arc in the first quadrant. An element of volume is a circular disk centered

on the x axis, $x \in [1,3]$, of radius $y(w_i)^2$

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi y(w_i)^2 \Delta_i x = \pi \int_1^3 \frac{(x^2-9)(1-x^2)}{x^2}\,dx = \pi \int_1^3 (-x^2+10-9x^{-2})dx$$
$$= \pi\Big[-\tfrac{1}{3}x^3 + 10x + 9x^{-1}\Big]_1^3 = \pi\Big[(-9+30+3) - (-\tfrac{1}{3}+10+9)\Big] = \tfrac{16}{3}\pi$$

36. An oil tank in the shape of a sphere has a diameter of 60 ft. How much oil does the tank contain if the depth of the oil is 25 ft?



▷ We apply Definition 4.9.1. The figure shows the tank. Because the radius of the tank is 30 ft, by the Pythagorean theorem, a horizontal plane section of the tank x feet below its center, $x \in [5,30]$, is a circle of radius $\sqrt{30^2 - x^2}$ ft and area $\pi(900 - x^2)$ ft^2. Thus

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} A(w_i)\Delta_i x = \int_5^{30} \pi(900-x^2)\,dx = \pi\Big[900x - \tfrac{1}{3}x^3\Big]_5^{30}$$
$$= \pi[900(30-5) - \tfrac{1}{3}(27{,}000 - 125)] = \tfrac{40{,}625}{3}\pi \approx 42{,}542.4$$

▪ The amount of oil in the tank is approximately 42,542 ft^3 or 318,239 gal (231 in^3 = $\frac{231}{1728}$ft^3 = 1 gal exactly)

37. The region is bounded by $y = \csc x$, $y = 2$, $x = \frac{1}{6}\pi$, and $x = \frac{5}{6}\pi$. An element of volume is a circular ring centered on the x axis, $x \in [\frac{1}{6}\pi, \frac{5}{6}\pi]$, of radii 2 and $\csc w_i$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(4-\csc^2 w_i)\Delta_i x = \pi\int_{\pi/6}^{5\pi/6} (4-\csc^2 x)dx = \pi\Big[4x + \cot x\Big]_{\pi/6}^{5\pi/6} = \pi(\tfrac{10}{3}\pi - \sqrt{3} - \tfrac{2}{3}\pi - \sqrt{3})$$
$$= \tfrac{8}{3}\pi^2 - 2\sqrt{3}\pi$$

38. The region is bounded by $y = \sec x$, the y axis, and $y = 2$. An element of volume is a circular ring centered on the x axis, $x \in [0, \frac{1}{3}\pi]$, of radii 2 and $\sec w_i$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(4-\sec^2 w_i)\Delta_i x = \pi\int_0^{\pi/3} (4-\sec^2 x)dx = \pi\Big[4x - \tan x\Big]_0^{\pi/3} = \pi(\tfrac{4}{3}\pi - \sqrt{3})$$

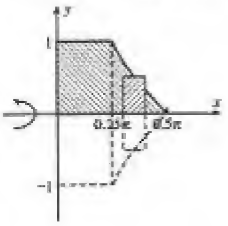
39. The region is bounded by $y = \sqrt{2x+4}$, the x axis, the y axis, and $x = c$. An element of volume is a circular disk centered on the x axis, $x \in [0,c]$, of radius $\sqrt{2w_i + 4}$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(2w_i+4)\Delta_i x = \pi\int_0^c (2x+4)dx = \pi\Big[x^2 + 4x\Big]_0^c = \pi(c^2+4c)$$

Therefore $\pi(c^2+4c) = 12\pi$; $c^2 + 4c - 12 = 0$; $(c+6)(c-2) = 0$. Because $c > 0$, $c = 2$.

40. The region in the first quadrant bounded by the coordinate axes, the line $y = 1$, and the curve $y = \cot x$ is revolved about the x axis. Find the volume of the solid generated.

▷ The figure shows the region and a plane section of the solid. Let



$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{4}\pi \\ \cot x & \text{if } \frac{1}{4}\pi \le x \le \frac{1}{2}\pi \end{cases}$$

An element of volume is a disk centered on the x axis of radius $f(w_i)$, $x \in [0, \frac{1}{2}\pi]$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi f(w_i)^2 \Delta_i x = \pi\int_0^{\pi/2} f(x)^2\,dx$$
$$= \pi\Big[\int_0^{\pi/4} 1^2\,dx + \int_{\pi/4}^{\pi/2} \cot^2 x\,dx\Big] = \pi\Big[\int_0^{\pi/4} dx + \int_{\pi/4}^{\pi/2} (\csc^2 x - 1)\,dx\Big]$$
$$= \pi\Big\{x\Big]_0^{\pi/4} + \Big[-\cot x - x\Big]_{\pi/4}^{\pi/2}\Big\} = \pi\Big\{\tfrac{1}{4}\pi - \cot\tfrac{1}{2}\pi - \tfrac{1}{2}\pi + \cot\tfrac{1}{4}\pi + \tfrac{1}{4}\pi\Big\} = \pi$$

▪ The volume of the solid of revolution is π cubic units.

In Exercises 41–50, use NINT to find the volume, V cubic units, to four significant digits, of the solid generated by revolving the region about the indicated axis.

41. The region bounded by $y = \sqrt[4]{x^3+4}$, the x axis, the y axis, and $x = 2$ about the x axis.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi y^2(w_i)\Delta_i x = \int_0^2 \pi\sqrt{x^3+4}\,dx = 15.146 \approx 15.15$

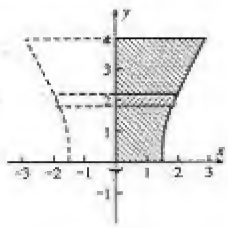
42. The region bounded by $y = \sqrt[3]{x^4-5}$, the x axis, $x = 2$ and $x = 3$ about the x axis.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi y^2(w_i)\Delta_i x = \int_2^3 \pi(x^4-5)^{2/3}dx = 33.961 \approx 33.96$

43. The region bounded by $y = \sqrt[4]{x^3+4}$, the y axis, and $y = 3$ about the y axis. $x = (y^4-4)^{1/3}$

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi x^2(w_i)\Delta_i y = \int_{\sqrt{2}}^3 \pi(y^4-4)^{2/3}dy = 39.691 \approx 39.69$

44. The region bounded by the graph of $y = \sqrt[3]{x^4-5}$, the x axis, the y axis, and the line $y = 4$ about the y axis.



▹ See the figure at the right. An element of volume is a disk centered on the y axis of radius $x(w_i) = \sqrt[4]{w_i^3+5}$, $y \in [0, 4]$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi x^2(w_i)\Delta_i y = \int_0^4 \pi\sqrt{y^3+5}\,dy = 52.5919 \approx 52.59$

▪ The volume of the solid of revolution is 52.59 cubic units.

45. The region bounded by $y = \sin x^3$, the y axis, and $y = 1$, $x \in [0, \sqrt[3]{\pi/2}]$ about the x axis.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[1^2 - y^2(w_i)]\Delta_i x = \int_0^{\sqrt[3]{\pi/2}} \pi[1-(\sin x^3)^2]dx = 2.8222 \approx 2.822$

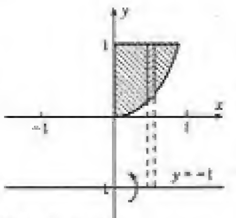
46. The region bounded by $y = \tan x^2$, the y axis, and $y = 1$, $x \in [0, \sqrt{\pi}/2]$ about $y = 1$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[1^2 - y^2(w_i)]\Delta_i x = \int_0^{\sqrt{\pi}/2} \pi[1-(\tan x^2)^2]dx = 1.6203 \approx 1.620$

47. The region bounded by $y = \sin x^3$, the y axis, and $y = 1$, $x \in [0, \sqrt[3]{\pi/2}]$ about $y = 2$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\{(2-y(w_i))^2 - 1^2\} = \int_0^{\sqrt[3]{\pi/2}} \pi[(2-\sin^3 x)^2 - 1]dx = 6.9230$

48. The region bounded by the graph of $y = \tan x^2$, the y axis, and the line $y = 1$, if $x \in [0, \sqrt{\pi}/2]$ about the line $y = -1$.



▹ See the figure at the right. An element of volume is a washer centered on the line $y = -1$ of radii 2 and $1 + \tan w_i^2$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[2^2 - (1+\tan w_i^2)^2]\Delta_i x \approx \int_0^{0.8862} \pi[4-(1+\tan x^2)^2]dx = 6.294$

▪ The volume of the solid of revolution is 6.294 cubic units.

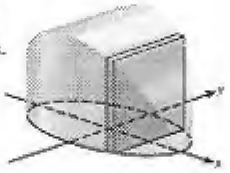
49. The region bounded by the graph of $y = \sin x + 2$, $y = \tan x$, and the y axis about the x axis.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[(\sin w_i + 2)^2 - \tan^2 w_i]\Delta_i x = \int_0^{1.2437} \pi[(\sin x+2)^2 - \tan^2 x]dx = 20.281 \approx 20.28$

50. The region bounded by the graph of $y = \cos(x^2+2)$ and $y = x^2 - 1$ axis about the x axis. The region is symmetrical and $|x^2-1| > |\cos(x^2+2)|$

$y = 2\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[(w_i^2-1)^2 - \cos^2(w_i^2+2)]\Delta_i x = \int_0^{0.5651} 2\pi[(x^2-1)^2 - \cos^2(x^2+2)]dx = 1.928$

51. The base of the solid is the region enclosed by the ellipse $3x^2 + y^2 = 6$. Let $f(x) = y = \sqrt{6-3x^2}$, $x \in [-\sqrt{2}, \sqrt{2}]$. We get half the solid if x is in $[0, \sqrt{2}]$. An element of volume is a right cylinder of altitude $\Delta_i x$ units and base area $[2f(w_i)]^2$ sq. units.

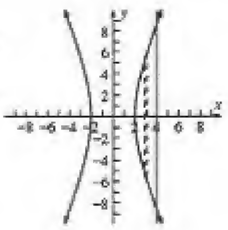


$V = 2\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [2f(w_i)]^2\Delta_i x = 8\int_0^{\sqrt{2}} (6-3x^2)dx = 8\Big[6x - x^3\Big]_0^{\sqrt{2}}$

$= 8(6\sqrt{2} - 2\sqrt{2}) = 32\sqrt{2}$

In Exercises 52 and 54, the base of a solid is the region enclosed by the hyperbola $25x^2 - 4y^2 = 100$ and the line $x = 4$. Find the volume of the solid if all plane sections perpendicular to the x axis are:

52. squares



▷ The figure shows the base of the solid S. Let ϕ be any plane section of S perpendicular to the x axis at x, and let $A(x)$ square units be the area of ϕ. We are given that ϕ is a square. Solving the equation of the hyperbola for y, we get $y = \pm\frac{5}{2}\sqrt{x^2-4}$. We let $f(x) = \frac{5}{2}\sqrt{x^2-4}$. Because $2f(w_i)$ is the measure of the base of the square, $A(x) = [2f(w_i)]^2$. If V cubic units is the volume of the solid, then

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} A(w_i)\Delta_i x = \int_2^4 (5\sqrt{x^2-4})^2 dx = 25\int_2^4 (x^2-4)dx$$
$$= 25\Big[\tfrac{1}{3}x^3 - 4x\Big]_2^4 = 25[\tfrac{1}{3}\cdot 56 - 8] = \tfrac{800}{3}$$

▪ The volume of the solid is $\frac{800}{3}$ cubic units.

54. equilateral triangles. Because the area of an equilateral triangle of side s is $\frac{1}{4}\sqrt{3}$ times the area of a square of side s, the volume of the solid is $\frac{1}{4}\sqrt{3}\cdot\frac{880}{3} = \frac{200}{3}\sqrt{3}$ cubic units.

53. Take the xy plane in the base of the solid and the origin at the center of the circular base. Therefore the base is the region enclosed by the circle $x^2 + y^2 = 49$. Let $f(x) = y = \sqrt{49 - x^2}$, $x \in [-7, 7]$. We get half the solid if x is in $[0, 7]$. An element of volume is a right cylinder of altitude $\Delta_i x$ cm and whose base is an equilateral triangle of side $2f(w_i)$ cm. Because an equilateral triangle of side s units has area $\frac{1}{4}\sqrt{3}\,s^2$ square units, then the element of volume has base area $\frac{1}{4}\sqrt{3}[2f(w_i)]^2$ cm^2.

$$V = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{4}\sqrt{3}[2f(w_i)]^2\Delta_i x = 2\sqrt{3}\int_0^7 (49 - x^2)dx = 2\sqrt{3}\Big[49x - \tfrac{1}{3}x^3\Big]_0^7 = 2\sqrt{3}(343 - \tfrac{343}{3}) = \tfrac{1372}{3}\sqrt{3}$$

55. Refer to the solution of Exercise 53. An element of volume is a right cylinder of altitude $\Delta_i x$ cm whose base is an isosceles triangle of base $2f(w_i)$cm and height w_i cm, and hence of area $\frac{1}{2}[2f(w_i)]w_i$ cm^2.

$$V = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} w_i f(w_i)\Delta_i x = 2\int_0^7 x\sqrt{49-x^2}\,dx = -\int_0^7 (49-x^2)^{1/2}(-2x\,dx) = -\tfrac{2}{3}(49-x^2)^{3/2}\Big|_0^7$$
$$= -\tfrac{2}{3}\cdot 0 + \tfrac{2}{3}\cdot 343 = \tfrac{686}{3}$$

56. The base of a solid is the region enclosed by a circle with a radius of r units, and all plane sections perpendicular to a fixed diameter of the base are isosceles right triangles having the hypotenuse in the plane of the base. Find the volume of the solid.

▷ Take the xy plane in the base of the solid and the origin at the center of the circular base. Therefore the base is the region enclosed by the circle $x^2 + y^2 = r^2$. Let $f(x) = y = \sqrt{r^2 - x^2}$, $x \in [-r, r]$. We obtain half the solid if x is in $[0, r]$. An element of volume is a right cylinder of altitude $\Delta_i x$ units and whose base is an isosceles right triangle having one leg, of length $2f(w_i)$ units, in the xy plane. Because an isosceles right triangle of hypotenuse s units has area $\frac{1}{4}s^2$ square units, then the base of the element of volume has area $\frac{1}{4}[2f(w_i)]^2$ square units.

$$V = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}[f(w_i)]^2\Delta_i x = 2\int_0^r (r^2 - x^2)dx = 2\Big[r^2x - \tfrac{1}{3}x^3\Big]_0^r = 2(r^3 - \tfrac{1}{3}r^3) = \tfrac{4}{3}r^3$$

▪ The volume of the solid is $\frac{4}{3}r^3$ cubic units.

57. Because the area of a an isosceles right triangle with leg of length s is twice the area of one with hypotenuse of length s, the volume of the solid is $\frac{8}{3}r^3$.

58. The base of a solid is the region enclosed by a circle having a radius of 4 in., and each plane section perpendicular to a fixed diameter of the base is an isosceles triangle having an altitude of 10 in. and a chord of the circle as a base. Find the volume of the solid.

▷ The figure shows a section of the solid, where the origin has been chosen at the center of the circle and the x axis as the fixed diameter. A section of the solid perpendicular to the x axis at x is a triangle of base b in. Because the radius is 4 in., by the Pythagorean theorem we have $\frac{1}{2}b = \sqrt{4^2 - x^2}$. Because the altitude is 10 in., the measure of the area of the section is

$$A(x) = \tfrac{1}{2}bh = \sqrt{4^2 - x^2}(10) = 10\sqrt{4^2 - x^2}$$

If V cubic inches is the volume of the solid, then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} A(w_i)\Delta_i x = \int_{-4}^{4} 10\sqrt{4^2 - x^2}\, dx$$

At this time we cannot use the fundamental theorem of the calculus to evaluate this integral, because we cannot find an antiderivative for the integral. However, the graph of the function $f(x) = \sqrt{4^2 - x^2}$ when x is in $[-4, 4]$ is the top half of the circle $x^2 + y^2 = 4^2$. Because $\int_{-4}^{4} \sqrt{4^2 - x^2}\, dx$ gives the area of the region bounded by this arc and the x axis, by the formula for the area of a circle, $A = \pi r^2$,

$$\int_{-4}^{4} \sqrt{4^2 - x^2}\, dx = \tfrac{1}{2}\pi 4^2 = 8\pi$$

Thus $V = 10(8\pi) = 80\pi$ and so the volume of the solid is 80π in.^{3}.

59. The base of the solid is the region enclosed by $x = 2\sqrt{y}$, $y \in [0, 9]$, $x + y = 0$, and $y = 9$. An element of volume is a right cylinder of altitude $\Delta_i y$ units and whose base is a square with diagonal of length $[2\sqrt{w_i} - (-w_i)]$ units. Because a square with diagonal of length d units has area $\frac{1}{2}d^2$ square units, then the base of the element of volume has area $\frac{1}{2}[2\sqrt{w_i} + w_i]^2$ square units.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(2\sqrt{w_i} + w_i)^2 \Delta_i y = \tfrac{1}{2}\int_0^9 (2\sqrt{y} + y)^2 dy = \tfrac{1}{2}\int_0^9 (4y + 4y^{3/2} + y^2) dy = \tfrac{1}{2}\Big[2y^2 + \tfrac{8}{5}y^{5/2} + \tfrac{1}{3}y^3\Big]_0^9$$

$$= \tfrac{1}{2}\Big(2 \cdot 3^4 + \tfrac{8}{5} \cdot 3^5 + 3^5\Big) = \frac{3969}{10}$$

60. Two right-circular cylinders, each having a radius of r units, have axes that intersect at right angles. Find the volume of the solid (groin) common to the two cylinders.

▸ Let S be a plane section of the solid parallel to the xy plane at height h, $-r \le h \le r$, as shown in the small figure. Then S is a square of side $2s$. By the Pythagorean theorem, $s^2 + h^2 = r^2$.

Thus $s = \sqrt{r^2 - h^2}$ and the area of S is $A(h) = 4(r^2 - h^2)$. By Definition 4.8.1 we have

Two cylinders Intersection solid

$$V = 4 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (r^2 - w_i^2)\Delta_i h$$

$$= 4\int_{-r}^{r} (r^2 - h^2)\, dh$$

$$= 8\Big[r^2 h - \tfrac{1}{3}h^3\Big]_0^r = 8[r^2(r) - \tfrac{1}{3}r^3] = \tfrac{16}{3}r^3$$

▪ The volume of the solid common to the two cylinders is $\frac{16}{3}r^3$ cubic units.

61. Take the xy plane in the base of the cylinder and the origin at the center of the circular base. Take the plane, that forms the wedge, through the diameter along the x axis. The base of the wedge is the region enclosed by the top half of the circle $x^2 + y^2 = r^2$ and the x axis for x in $[-r, r]$. Let $f(x) = y = \sqrt{r^2 - x^2}$. We obtain half the wedge if x is in $[0, r]$. An element of volume is a right cylinder having altitude $\Delta_i x$ cm and whose base is an isosceles right triangle of side length $f(w_i)$ cm, and hence area $\frac{1}{2}[f(w_i)]^2$ cm^2.

$$V = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[f(w_i)]^2 \Delta_i x = \int_0^r (r^2 - x^2) dx = r^2 x - \tfrac{1}{3}x^3\Big|_0^r = r^3 - \tfrac{1}{3}r^3 = \tfrac{2}{3}r^3$$

62. The volume of the cone is $\frac{1}{3}\pi(5^2)20 = \frac{500}{3}\pi$. The volume of the wedge is $\frac{30^\circ}{360^\circ} \cdot \frac{500}{3}\pi = \frac{125}{9}\pi$ square units.

63. The point $(10, 6)$ is on the parabola $y^2 = 4px$. Thus $36 = 40p$; $4p = \frac{18}{5}$. An equation of the parabola is $y^2 = \frac{18}{5}x$. An element of volume is a a circular disk centered on the x axis, of radius $(\frac{18}{5}w_i)^{1/2}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\tfrac{18}{5}w_i)\Delta_i x = \tfrac{18}{5}\pi \int_0^{10} x\, dx = \tfrac{9}{5}\pi x^2\Big|_0^{10} = \tfrac{9}{5}\pi(100) = 180\pi$$

4.10 VOLUMES OF SOLIDS BY CYLINDRICAL SHELLS

4.10.1 Theorem Let the function f be continuous on the closed interval $[a,b]$, where $a \geq 0$. Assume that $f(x) \geq 0$ for all x in $[a,b]$. If R is the region bounded by the curve $y = f(x)$, the x axis and the lines $x = a$ and $x = b$, if S is the solid of revolution obtained by revolving R about the y axis, and if V cubic units is the volume of S, then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i f(m_i)\Delta_i x = 2\pi \int_a^b x f(x)\, dx$$

We must show that the volume is the same as that obtained by the disk method of Theorem 4.10.1. We prove the following special case.

Theorem Let $f(x)$ be a continuous function which decreases on $[0,a]$ from $f(0) = \alpha$ to $f(a) = 0$ and let the curve $y = f(x)$ also be described by $x = \phi(y)$, where $\phi(0) = a$ and $\phi(\alpha) = 0$. If $f'(x)$ is continuous on $[0,a]$, then

$$2\pi \int_0^a x\, f(x)\, dx = \pi \int_0^\alpha \phi(y)^2\, dy$$

PROOF: We evaluate the second integral with the substitution $y = f(x)$. Then $\phi(y) = x$ and $dy = f'(x)\, dx$. When $y = 0$, $x = a$; when $y = \alpha$, $x = 0$. Thus,

$$\pi \int_0^\alpha \phi(y)^2\, dy = \int_a^0 x^2 f'(x)\, dx$$

Because $\frac{d}{dx}[x^2 f(x)] = 2xf(x) + x^2 f'(x)$, then $x^2 f'(x) = \frac{d}{dx}[x^2 f(x)] - 2xf(x)$. Thus

$$\pi \int_0^\alpha \phi(y)^2\, dy = \pi\left[\int_a^0 \frac{d}{dx}[x^2 f(x)]\, dx - 2\int_a^0 xf(x)\, dx\right]$$

$$= \pi\left[x^2 f(x)\right]_a^0 + 2\pi \int_0^a xf(x)\, dx = \pi[0 \cdot \alpha - a^2 \cdot 0] + 2\pi \int_0^a xf(x)\, dx$$

$$= 2\pi \int_0^a x\, f(x)\, dx$$

If the region is on one side of the axis of revolution, we have the following:

Theorem Method of Shells Let the function f and g be continuous on the closed interval $[a,b]$. Assume that $f(x) \geq g(x)$ for all x in $[a,b]$. If R is the region bounded by the curve $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$, if S is the solid of revolution obtained by revolving R about the line $x = x_1$ where x_1 is not in (a,b), and if V cubic units is the volume of S, then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi |m_i - x_1| [f(m_i) - g(m_i)]\Delta_i x = 2\pi \int_a^b |x - x_1| [f(x) - g(x)]\, dx$$

To apply the Method of Shells, we note that $f(m_i) - g(m_i)$ is the height of a rectangular element parallel to the axis of revolution, $|m_i - x_1|$ is the mean distance of the rectangle from the axis, and that the region does not cross the axis of revolution. We call $2\pi m_i f(m_i)\Delta_i x$ the volume of a cylindrical shell.

Exercises 4.10

In Exercises 1–12, solve Exercises 5–16 in Section 4.9 by the cylindrical-shell method.

▸ In these Exercises V cubic units is the required volume and Δ is a partition of the specified interval.

1. (Ex. 5) The region bounded by $x = y^{2/3}$, the x axis, $x = 4$. The rectangular elements are horizontal, $y \in [0,8]$. An element of volume is a cylindrical shell centered on the x axis, of mean radius m_i and altitude $4 - m_i^{2/3}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i(4 - m_i^{2/3})\Delta_i y = 2\pi \int_0^8 y(4 - y^{2/3})dy = 2\pi \int_0^8 (4y - y^{5/3})dy = 2\pi\left[2y^2 - \tfrac{3}{8}y^{8/3}\right]_0^8$$

$$= 2\pi(128 - 96) = 64\pi$$

2. (Ex. 6) The region bounded by $x = y^{2/3}$, the x axis, $x = 4$. The rectangular elements are vertical, $x \in [0,4]$. An element of volume is a cylindrical shell centered on $x = 4$, of mean radius $4 - m_i$ and altitude $m_i^{3/2}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(4 - m_i)m_i^{3/2} = 2\pi \int_0^4 (4 - x)x^{3/2}dx = 2\pi \int_0^4 (4x^{3/2} - x^{5/2})dx = 2\pi\left[4 \cdot \tfrac{2}{5}x^{5/2} - \tfrac{2}{7}x^{7/2}\right]_0^4$$

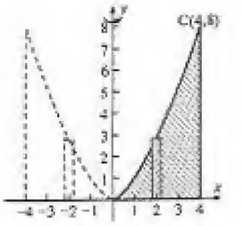
$$= 2\pi(\tfrac{8}{5} \cdot 32 - \tfrac{2}{7} \cdot 128) = \tfrac{1024}{35}\pi$$

3. (Ex. 7) The region is bounded by $x = y^{2/3}$, the x axis, and $x = 4$. The rectangular elements are horizontal, $y \in [0, 8]$. An element of volume is a cylindrical shell centered on the line $y = 8$, of mean radius $8 - m_i$ and altitude $4 - {m_i}^{2/3}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(8 - m_i)(4 - {m_i}^{2/3})\Delta_i y = 2\pi \int_0^8 (8 - y)(4 - y^{2/3})dy = 2\pi \int_0^8 (32 - 4y - 8y^{2/3} + y^{5/3})dy$$
$$= 2\pi\left[32y - 2y^2 - 8 \cdot \tfrac{3}{5}y^{5/3} + \tfrac{3}{8}y^{8/3}\right] = 2\pi(32 \cdot 8 - 2 \cdot 64 - \tfrac{24}{5} \cdot 32 + 3 \cdot 32) = \tfrac{704}{5}\pi$$

4. (From Exercise 8.) OAC about the y axis.

▷ The figure shows the region OAC, which is bounded by the curve $y^2 = x^3$, the x axis, and the line $x = 4$, and a section of the solid generated. Because the rectangular elements are parallel to the y axis, $x \in [0, 4]$, we solve the equation of the curve for y to get $y = f(x) = x^{3/2}$. An element of volume is a cylindrical shell centered on the y axis of mean radius m_i and height $f(m_i)$. If V cubic units is the volume of the solid, then



$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i f(m_i)\Delta_i x = 2\pi \int_0^4 x(x^{3/2})\, dx = 2\pi \int_0^4 x^{5/2}\, dx$$
$$= 2\pi(\tfrac{2}{7})x^{7/2}\Big]_0^4 = \tfrac{4}{7}\pi(128) = \tfrac{512}{7}\pi$$

▪ The volume of the solid of revolution is $\frac{512}{7}\pi$ cubic units.

5. (Ex. 9) The region bounded by $y = x^{3/2}$, the y axis, $y = 8$. The rectangular elements are vertical, $x = [0, 4]$. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude $8 - m_i$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i(8 - {m_i}^{3/2})\Delta_i x = 2\pi \int_0^4 x(8 - x^{3/2})dx = 2\pi \int_0^4 (8x - x^{5/2})dx = 2\pi\left[4x^2 - \tfrac{2}{7}x^{7/2}\right]_0^4$$
$$= 2\pi(4 \cdot 16 - \tfrac{2}{7} \cdot 128) = \tfrac{384}{7}\pi$$

6. (Ex. 10) The region bounded by $x = x^{2/3}$, the y axis, $y = 8$. The rectangular elements are vertical, $y \in [0, 0]$. An element of volume is a cylindrical shell centered on line $y = 8$, of mean radius $8 - m_i$, and altitude ${m_i}^{2/3}$.

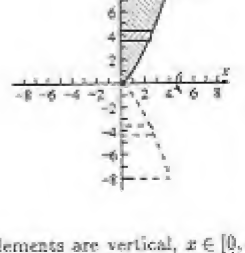
$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(8 - m_i){m_i}^{2/3}\Delta_i y = 2\pi \int_0^8 (8 - y)y^{2/3}dy = 2\pi \int_0^8 (8y^{2/3} - y^{5/3})dy$$
$$= 2\pi\left[32x - 4x^2 - \tfrac{8}{5}x^{5/2} + \tfrac{2}{7}x^{7/2}\right] = 2\pi\left(32 \cdot 4 - 4 \cdot 16 - \tfrac{8}{5} \cdot 32 + \tfrac{2}{7} \cdot 128\right) = \tfrac{3456}{35}\pi$$

7. (Ex. 11) The region bounded by $y = x^{3/2}$, the y axis, $y = 8$. The rectangular elements are vertical, $x \in [0, 4]$. An element of volume is a cylindrical shell centered on $x = 4$, of mean radius $4 - m_i$ and altitude $8 - {m_i}^{3/2}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(4 - m_i)(8 - {m_i}^{3/2})\Delta_i x = 2\pi \int_0^4 (4 - x)(8 - x^{3/2})dx = 2\pi \int_0^4 (32 - 8x - 4x^{3/2} + x^{5/2})dx$$
$$= 2\pi\left[32x - 4x^2 - \tfrac{8}{5}x^{5/2} + \tfrac{2}{7}x^{7/2}\right] = 2\pi\left(32 \cdot 4 - 4 \cdot 16 - \tfrac{8}{5} \cdot 32 + \tfrac{2}{7} \cdot 128\right) = \tfrac{3456}{35}\pi$$

8. (From Exercise 12.) OBC about the x axis.

▷ The figure shows the region OBC and a plane section of the solid generated. Because the rectangular elements are parallel to the x axis, we solve the equation of the curve for x to get $x = f(y) = y^{2/3}$. An element of volume is a circular shell centered on the x axis, $y \in [0, 8]$, of mean radius m_i and height $f(m_i)$. If V cubic units is the volume of the solid, then by Theorem 4.10.1



$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i f(m_i)\Delta_i y = 2\pi \int_0^8 y(y^{2/3})\, dy = 2\pi \int_0^8 y^{5/3}\, dy$$
$$= 2\pi(\tfrac{3}{8})y^{8/3}\Big]_0^8 = \tfrac{3}{4}\pi(256) = 192\pi$$

▪ The volume of the solid of revolution is 192π cubic units.

9. (Ex. 13) The region bounded by $y = x^{1/2}$, the x axis, $x = 4$. The rectangular elements are vertical, $x \in [0, 4]$. An element of volume is a cylindrical shell centered on line $x = 4$, of mean radius $4 - m_i$ and altitude ${m_i}^{1/2}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(4 - m_i){m_i}^{1/2}\Delta_i x = 2\pi \int_0^4 (4 - x)x^{1/2}dx = 2\pi \int_0^4 (4x^{1/2} - x^{3/2})dx$$
$$= 2\pi\left[\tfrac{8}{3}x^{3/2} - \tfrac{2}{5}x^{5/2}\right]_0^4 = 2\pi(\tfrac{8}{3} \cdot 8 - \tfrac{2}{5} \cdot 32) = \tfrac{256}{15}$$

10. (Ex. 14) The region bounded by $x = y^2$, the x axis, $x = 4$. The rectangular elements are horizontal $y \in [0, 2]$.

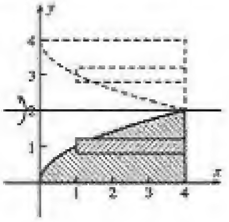
An element of volume is a cylindrical shell centered on the x axis, of mean radius m_i and altitude $4-m_i^2$.

$$V=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\pi m_i(4-m_i^2)\Delta_i y = 2\pi\int_0^2 y(4-y^2)dy = 2\pi\int_0^2(4y-y^3)dy = 2\pi\Big[2y^2-\tfrac{1}{4}y^4\Big]_0^2 = 2\pi\cdot 4 = 8\pi$$

11. (Ex. 15) The region bounded by $y=x^{1/2}$, the x axis, $x=4$. The rectangular elements are vertical, $x\in[0,4]$. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude $m_i^{1/2}$.

$$V=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\pi(m_i)m_i^{1/2}\Delta_i x = 2\pi\int_0^4 x\cdot x^{1/2}dx = 2\pi\int_0^4 x^{3/2}dx = 2\pi\Big[\tfrac{2}{5}x^{5/2}\Big]_0^4 = 2\pi(\tfrac{2}{5}\cdot 32) = \tfrac{128}{5}\pi$$

12. (From Exercise 16.) Find the volume of the solid of revolution generated by revolving about the line $y=2$ the region bounded by the curve $y=\sqrt{x}$, the x axis, and the line $x=4$.



▹ The figure shows the region and a plane section of the solid of revolution. The rectangular elements are parallel to $y=2$ at a mean distance of $2-m_i$. Solving $y=\sqrt{x}$ for x, we find the region bounded on the left by $x=y^2$ and on the right by $x=4$, a height of $4-y^2$, $y\in[0,2]$. Therefore,

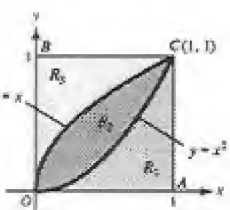
$$V=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\pi(2-m_i)(4-m_i^2) = 2\pi\int_0^2 (2-y)(4-y^2)\,dy$$

$$= 2\pi\int_0^2 (8-4y+2y^2+y^3)\,dy = 2\pi\Big[8y-2y^2-\tfrac{2}{3}y^3+\tfrac{1}{4}y^4\Big]_0^2$$

$$= 2\pi(16-8-\tfrac{16}{3}+4) = \tfrac{40}{3}\pi$$

• The volume of the solid of revolution is $\frac{40}{3}\pi$ cubic units.

In Exercises 13–20, find the volume of the solid generated when the indicated region is revolved about the given line.



13. The region is bounded by $y=x^2$, the x axis, and $x=1$. The rectangular elements are vertical, $x\in[0,1]$. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude m_i^2.

$$V=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\pi(m_i)m_i^2\Delta_i x = 2\pi\int_0^1 x\cdot x^2dx = 2\pi\int_0^1 x^3dx = 2\pi\Big[\tfrac{1}{4}x^4\Big]_0^1 = 2\pi\cdot\tfrac{1}{4} = \tfrac{1}{2}\pi$$

14. The region is bounded by $y=x^2$, the x axis, and $x=1$. The rectangular elements are horizontal, $y\in[0,1]$. An element of volume is a washer centered on the y axis, of radii 1 and $\sqrt{w_i}$

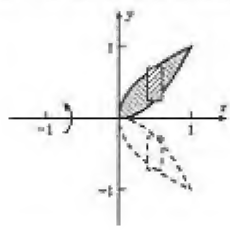
$$V=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} \pi[1^2-(\sqrt{w_i})^2]\Delta_i y = \pi\int_0^1(1-y)dy = -\tfrac{1}{2}\pi(1-y)^2\Big]_0^1 = \tfrac{1}{2}\pi$$

15. The region is bounded by $x=\sqrt{y}$ and $x=y^2$. The rectangular elements are horizontal, $y\in[0,1]$. An element of volume is a cylindrical shell centered on the x axis, of mean radius m_i and altitude $\sqrt{m_i}-m_i^2$.

$$V=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\pi m_i(\sqrt{m_i}-m_i^2)\Delta_i y = 2\pi\int_0^1 y(y^{1/2}-y^2)dy = 2\pi\int_0^1 (y^{3/2}-y^3)dy = 2\pi\Big[\tfrac{2}{5}y^{5/2}-\tfrac{1}{4}\Big]_0^1$$

$$= 2\pi(\tfrac{2}{5}-\tfrac{1}{4}) = \tfrac{3}{10}\pi$$

16. The region bounded by the curves $y^2=x$ and $y=x^2$ is revolved about the x axis. The rectangular elements are perpendicular to the axis of revolution.



▹ The figure shows the region and a plane section of the solid generated. Solving $y^2=x$ for y, we find the farther boundary is $y=f(x)=\sqrt{x}$ and the nearer boundary is $y=g(x)=x^2$. An element of volume is a washer centered on the x axis, $x\in[0,1]$, of radii $f(w_i)$ and $g(w_i)$. Thus,

$$V=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} \pi[f(w_i)^2-g(w_i)^2]\Delta_i x = \pi\int_0^1 [(\sqrt{x})^2-(x^2)^2]\,dx$$

$$= \pi\int_0^1 (x-x^4)\,dx = \pi\Big[\tfrac{1}{2}x-\tfrac{1}{5}x^5\Big]_0^1 = \pi(\tfrac{1}{2}-\tfrac{1}{5}) = \tfrac{3}{10}\pi$$

• The volume of the solid of revolution is $\frac{3}{10}\pi$ cubic units.

17. The region is bounded by $x=y^2$, the y axis, and $y=1$. The rectangular elements are horizontal, $y\in[0,1]$. An element of volume is a cylindrical shell centered on the line $y=2$, of mean radius $2-m_i$ and altitude m_i^2.

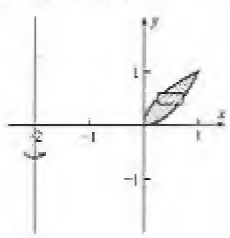
$$V=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\pi(2-m_i)m_i^2\Delta_i y = 2\pi\int_0^1(2-y)y^2dy = 2\pi\int_0^1(2y^2-y^3)dy = 2\pi\Big[\tfrac{2}{3}y^3-\tfrac{1}{4}y^4\Big]_0^1 = 2\pi(\tfrac{2}{3}-\tfrac{1}{4}) = \tfrac{5}{6}\pi$$

18. The region is bounded by $y = \sqrt{x}$, the y axis, and $y = 1$. The rectangular elements are vertical, $x \in [0,1]$. An element of volume is a washer centered on the line $y = 2$, of radii $2 - \sqrt{w_i}$ and 1. V =
$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi[(2-\sqrt{w_i})^2 - 1^2]\Delta_i x = \pi\int_0^1 [(2-\sqrt{x})^2 - 1]dx = \pi\int_0^1 (3 - 4x^{1/2} + x)dx = \pi\Big[3x - \tfrac{8}{3}x^{3/2} + \tfrac{1}{2}x^2\Big]_0^1 = \tfrac{5}{6}\pi$$

19. The region is bounded by $x = \sqrt{y}$ and $x = y^2$. The rectangular elements are vertical, $x \in [0,1]$. An element of volume is a cylindrical shell centered on the line $x = -2$, of mean radius $m_i + 2$ and altitude $m_i^{1/2} - m_i^2$.
$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi(m_i+2)(m_i^{1/2} - m_i^2)\Delta_i x = 2\pi\int_0^1 (x+2)(x^{1/2} - x^2)dx = 2\pi\int_0^1 (2x^{1/2} + x^{3/2} - 2x^2 - x^3)dx$$
$$= 2\pi\Big[\tfrac{4}{3}x^{3/2} + \tfrac{2}{5}x^{5/2} - \tfrac{2}{3}x^3 - \tfrac{1}{4}x^4\Big]_0^1 = 2\pi(\tfrac{4}{3} + \tfrac{2}{5} - \tfrac{2}{3} - \tfrac{1}{4}) = \tfrac{49}{30}\pi$$

20. Find the volume of the solid generated if the region bounded by the curves $y^2 = x$ and $y = x^2$ is revolved about the line $x = -2$. Take rectangular elements that are perpendicular to the axis of revolution.

▸ The figure shows the region and a plane section of the solid generated. The farther boundary is $y = x^2$ and its distance from $x = -2$ is $f(y) = \sqrt{y} + 2$. The nearer boundary is $y^2 = x$ and its distance from $x = -2$ is $g(y) = x + 2 = y^2 + 2$. An element of volume is a washer centered on $x = -2$, $y \in [0,1]$, of radii $f(w_i)$ and $g(w_i)$. Therefore,

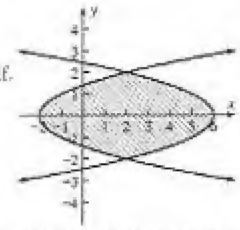


$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi[f(w_i)^2 - g(w_i)^2]\Delta_i y = \pi\int_0^1 [(\sqrt{y}+2)^2 - (y^2+2)^2]\,dy$$
$$= \pi\int_0^1 (y + 4\sqrt{y} + 4 - y^4 - 4y^2 - 4)\,dy = \pi\Big[\tfrac{1}{2}y^2 + \tfrac{8}{3}y^{3/2} - \tfrac{1}{5}y^5 - \tfrac{4}{3}y^3\Big]_0^1$$
$$= \pi(\tfrac{1}{2} + \tfrac{8}{3} - \tfrac{1}{5} - \tfrac{4}{3}) = \tfrac{49}{30}\pi$$

- The volume of the solid of revolution is $\frac{49}{30}\pi$ cubic units.

In Exercises 21–24, find the volume of the solid generated by revolving around the indicated line the region bounded by the curves $x = y^2 - 2$ and $x = 6 - y^2$.

▸ The region is shown at the right. Set $y^2 - 2 = 6 - y^2$; $2y^2 = 8$, $y = \pm 2$



21. the x-axis. The region is symmetric with respect to the x axis; use the upper half. The rectangular elements are horizontal. An element of volume is a cylindrical shell centered on the x axis, of mean radius m_i and altitude $8 - 2m_i^2$.
$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i(8 - 2m_i^2)\Delta_i y = 2\pi\int_0^2 y(8-2y^2)dy = 2\pi\int_0^2 (8y - 2y^3)dy$$
$$= 2\pi\Big[4y^2 - \tfrac{1}{2}y^4\Big]_0^2 = 2\pi(16-8) = 16\pi$$

22. the y axis. The region is symmetric with respect to the x axis; use the upper half and double. The rectangular elements are horizontal. An element of volume is a washer centered on the y axis, of radii $6 - w_i^2$ and $w_i^2 - 2$.
$$V = \lim_{\|\Delta\|\to 0} 2\sum_{i=1}^{n} \pi[(6-w_i^2)^2 - (w_i^2-2)^2]\Delta_i y = 2\pi\int_0^2 [(36 - 12y^2 + y^4) - (y^4 - 4y^2 + 4)]dy$$
$$= 2\pi\int_0^2 (32 - 8y^2)dy = 2\pi\Big[32y - \tfrac{8}{3}y^3\Big]_0^2 = 2\pi\cdot\tfrac{128}{3} = \tfrac{256}{3}\pi$$

23. the line $x = 2$. The region is symmetric with respect to the line $x = 2$; use the right half.

Method 1. The rectangular elements are vertical, $x \in [2,6]$. An element of volume is a cylindrical shell of mean radius $m_i - 2$ and altitude $2\sqrt{6 - m_i}$. Let $\sqrt{6-x} = v$, $x = 6 - v^2$, $dx = -2v\,dv$. When $x = 2$, $v = 2$; when $x = 6$, $v = 0$.
$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi(m_i - 2)2\sqrt{6-m_i}\,\Delta_i x = 4\pi\int_2^6 (x-2)\sqrt{6-x}\,dx = 4\pi\int_2^0 (4 - v^2)v(-2v\,dv)$$
$$= 8\pi\int_0^2 (4v^2 - v^4)dv = 8\pi\Big[\tfrac{4}{3}v^3 - \tfrac{1}{5}v^5\Big]_0^2 = 8(\tfrac{32}{3} - \tfrac{32}{5}) = \tfrac{512}{15}\pi$$

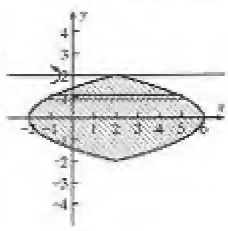
Method 2. The rectangular elements are horizontal, $y \in [-2,2]$. An element of volume is a circular disk centered on the line $x = 2$ of radius $(6 - w_i^2) - 2$.
$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(4 - w_i^2)^2\Delta_i y = \pi\int_{-2}^2 (4-y^2)^2 dy = \pi\int_{-2}^2 (16 - 8y^2 + y^4)dy = \pi\Big[16y - \tfrac{8}{3}y^3 + \tfrac{1}{5}y^5\Big]_{-2}^2$$
$$= \pi\Big[(32 - \tfrac{64}{3} + \tfrac{32}{5}) - (-32 + \tfrac{64}{3} - \tfrac{32}{5})\Big] = \tfrac{512}{15}\pi$$

24. the line $y = 2$.

▸ The figure shows the region R. Because we are given x as a function of y, we take rectangular elements parallel to $y = 2$ at a mean distance of $2 - m_i$. R is bounded on the right by $x = 6 - y^2$ and on the left by $x = y^2 - 2$, a spread of

$$(6 - y^2) - (y^2 - 2) = 8 - 2y^2 = 2(4 - y^2)$$

An element of volume is a cylindrical shell centered on $y = 2$, $y \in [-2, 2]$, of mean radius $2 - m_i$ and height $8 - 2m_i^2$. Therefore,



$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(2 - m_i)2(4 - m_i^2)\Delta_i y = 4\pi \int_{-2}^{2} (2 - y)(4 - y^2)\, dy$$

$$= 4\pi \int_{-2}^{2} (8 - 4y - 2y^2 + y^3)\, dy = 4\pi\Big[8y - 2y^2 - \tfrac{2}{3}y^3 + \tfrac{1}{4}y^4\Big]_{-2}^{2}$$

$$= 4\pi[8(2 + 2) - 2(4 - 4) - \tfrac{2}{3}(8 + 8) + \tfrac{1}{4}(16 - 16)] = 4\pi(32 - \tfrac{32}{3}) = \tfrac{256}{3}\pi$$

▪ The volume of the solid of revolution is $\frac{256}{3}\pi$ cubic units.

In Exercises 25 and 26, find the volume generated if the region bounded by $y^2 = 4px$ and $x = p$ $(p > 0)$ is revolved about the indicated axis.

25. $x = p$. The rectangular elements are vertical, $x \in [0, p]$. An element of volume is a cylindrical shell centered on the line $x = p$, of mean radius $a - m_i$ and altitude $2\sqrt{4am_i} = 4a^{1/2}m_i^{1/2}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(p - m_i)(4p^{1/2}m_i^{1/2})\Delta_i x = 2\pi \int_0^p (p - x)(4p^{1/2}x^{1/2})dx = 8\pi p^{1/2} \int_0^p (px^{1/2} - x^{3/2})dx$$

$$= 8\pi p^{1/2}\Big[\tfrac{2}{3}px^{3/2} - \tfrac{2}{5}x^{5/2}\Big] = 8\pi p^{1/2}(\tfrac{2}{3}p^{5/2} - \tfrac{2}{5}p^{5/2}) = 16\pi p^3(\tfrac{1}{3} - \tfrac{1}{5}) = \tfrac{32}{15}\pi p^3$$

26. the y axis. The rectangular elements are vertical, $x \in [0, p]$. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude $2\sqrt{4am_i} = 4a^{1/2}m_i^{1/2}$.

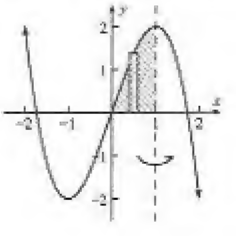
$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i(4p^{1/2}m_i^{1/2})\Delta_i x = 2\pi \int_0^p x(4p^{1/2}x^{1/2})dx = 8\pi p^{1/2} \int_0^p x^{3/2}dx = 8\pi p^{1/2} \cdot \tfrac{2}{5}x^{5/2}\Big]_0^p = \tfrac{16}{5}\pi p^3$$

27. The region is bounded by $y = 3x - x^3$, the y axis and $x = 1$. The rectangular elements are vertical, $x \in [0, 1]$. An element of volume is a cylindrical shell centered on the y axis of mean radius m_i and altitude $3m_i - m_i^3$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i(3m_i - m_i^3)\Delta_i x = 2\pi \int_0^1 x(3x - x^3)dx = 2\pi \int_0^1 (3x^2 - x^4)dx = 2\pi\Big[x^3 - \tfrac{1}{5}x^5\Big]_0^1 = 2\pi \cdot \tfrac{4}{5} = \tfrac{8}{5}\pi$$

28. Find the volume of the solid generated by revolving the region bounded by the graph of $y = 3x - x^3$, the x axis, and the line $x = 1$, about the line $x = 1$.

▸ The figure shows the region. Because y is given as a function of x, the rectangular elements are parallel to $x = 1$, $x \in [0, 1]$, at a mean distance of $1 - m_i$. The elements of volume are cylindrical shells of mean radius $1 - m_i$ and height $3m_i - m_i^3$. Therefore,



$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(1 - m_i)(3m_i - m_i^3)\Delta_i x = 2\pi \int_0^1 (1 - x)(3x - x^3)\, dx$$

$$= 2\pi \int_0^1 (3x - 3x^2 - x^3 + x^4)\, dx = 2\pi\Big[\tfrac{3}{2}x^2 - x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}\Big]_0^1$$

$$= 2\pi(\tfrac{3}{2} - 1 - \tfrac{1}{4} + \tfrac{1}{5}) = 2\pi(\tfrac{9}{20}) = \tfrac{9}{10}\pi$$

▪ The volume of the solid of revolution is $\frac{9}{10}\pi$ cubic units.

29. The region is bounded by $y = 3x - x^3$, the y axis and $y = 2$. The rectangular elements are vertical, $x \in [0, 1]$. An element of volume is a cylindrical shell centered on the line $x = 1$ of mean radius $1 - m_i$ and altitude $2 - (3m_i - m_i^3)$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(1 - m_i)(2 - 3m_i + m_i^3)\Delta_i x = 2\pi \int_0^1 (1 - x)(2 - 3x + x^3)dx$$

$$= 2\pi \int_0^1 (-x^4 + x^3 + 3x^2 - 5x + 2)dx = 2\pi\Big[-\tfrac{1}{5}x^5 + \tfrac{1}{4}x^4 + x^3 - \tfrac{5}{2}x^2 + 2x\Big]_0^1 = 2\pi \cdot \tfrac{11}{20} = \tfrac{11}{10}\pi$$

In Exercises 30 and 31, the region bounded by $y = 4x - \frac{1}{8}x^4$, the axes and $x = 2$ is revolved about the given axis.

30. the y axis. The rectangular elements are vertical, $x \in [0, 2]$. An element of volume is a cylindrical shell centered on the y axis of mean radius m_i and altitude $4m_i - \frac{1}{8}m_i^4$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i(4m_i - \tfrac{1}{8}m_i^4)\Delta_i x = 2\pi \int_0^2 x(4x - \tfrac{1}{8}x^4)dx = 2\pi \int_0^2 (4x^2 - \tfrac{1}{8}x^5)dx = 2\pi\Big[\tfrac{4}{3}x^3 - \tfrac{1}{48}x^6\Big]_0^2$$

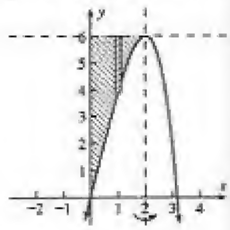
$$= 2\pi(\tfrac{32}{3} - \tfrac{4}{3}) = \tfrac{56}{3}\pi$$

31. $x = 2$. The rectangular elements are vertical, $x \in [0, 2]$. An element of volume is a cylindrical shell centered on the line $x = 2$ of mean radius $2 - m_i$ and altitude $4m_i - \frac{1}{8}m_i^4$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(2 - m_i)(4m_i - \tfrac{1}{8}m_i^4)\Delta_i x = 2\pi \int_0^2 (2 - x)(4x - \tfrac{1}{8}x^4)dx$$

$$= 2\pi \int_0^2 (\tfrac{1}{8}x^5 - \tfrac{1}{4}x^4 - 4x^2 + 8x)dx = 2\pi\Big[\tfrac{1}{48}x^6 - \tfrac{1}{20}x^5 - \tfrac{4}{3}x^3 + 4x^2\Big]_0^2 = 2\pi \cdot \tfrac{76}{15} = \tfrac{152}{15}\pi$$

32. Find the volume generated by revolving the region bounded by the graph of $y = 4x - \frac{1}{8}x^4$, the y axis and the line $y = 6$, about the line $x = 2$.



▹ The figure shows the region. Because y is given as a function of x, we choose rectangular elements parallel to $x = 2$, $x \in [0, 2]$, at a mean distance of $2 - m_i$. The region is bounded above $y = 6$ and below by $y = 4x - \frac{1}{8}x^4$, a spread of $6 - 4x + \frac{1}{8}x^4$. The elements of volume are cylindrical shells of mean radius $2 - m_i$ and height $6 - 4m_i + \frac{1}{8}m_i^4$. Therefore,

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(2 - m_i)(6 - 4m_i + \tfrac{1}{8}m_i^4)\Delta_i x = 2\pi \int_0^2 (2 - x)(6 - 4x + \tfrac{1}{8}x^4)dx$$

$$= 2\pi \int_0^2 (12 - 14x + 4x^2 + \tfrac{1}{4}x^4 - \tfrac{1}{8}x^5)\,dx = 2\pi\Big[12x - 7x^2 + \tfrac{4}{3}x^3 + \tfrac{1}{20}x^5 - \tfrac{1}{48}x^6\Big]_0^2$$

$$= 2\pi[12(2) - 7(4) + \tfrac{4}{3}(8) + \tfrac{1}{20}(32) - \tfrac{1}{48}(64)] = 2\pi(\tfrac{104}{15}) = \tfrac{208}{15}\pi$$

▪ The volume of the solid of revolution is $\frac{208}{15}\pi$ cubic units.

33. The region is bounded by $y = 4x - \frac{1}{8}x^4$, the y axis and $y = 6$. The rectangular elements are vertical $x \in [0, 2]$. An element of volume is a cylindrical shell centered on the y axis of mean radius m_i and altitude $6 - (4m_i - \frac{1}{8}m_i^4) = \frac{1}{8}m_i^4 - 4m_i + 6$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i(\tfrac{1}{8}m_i^4 - 4m_i + 6)\Delta_i x = 2\pi \int_0^2 (\tfrac{1}{8}x^5 - 4x^2 + 6x)dx = 2\pi\Big[\tfrac{1}{48}x^6 - \tfrac{4}{3}x^3 + 3x^2\Big] = 2\pi \cdot \tfrac{8}{3} = \tfrac{16}{3}\pi$$

34. The region bounded by $x = y^3$ and $x = y^{1/3}$ is symmetrical with respect to the origin; take the right half and double. The rectangular elements are horizontal, $y \in [0, 1]$. An element of volume is a cylindrical shell centered on the x axis, of mean radius m_i and altitude $m_i^{1/3} - m_i^3$.

$$V = \lim_{\|\Delta\| \to 0} 2\sum_{i=1}^{n} 2\pi m_i(m_i^{1/3} - m_i^3)\Delta_i y = 4\pi \int_0^1 y(y^{1/3} - y^3)dy = 4\pi \int_0^1 (y^{4/3} - y^4)dy = 4\pi\Big[\tfrac{3}{7}y^{7/3} - \tfrac{1}{5}y^5\Big]_0^1$$

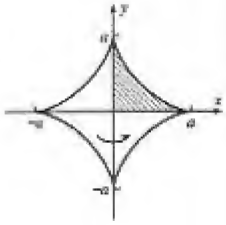
$$= 4\pi \cdot \tfrac{8}{35} = \tfrac{32}{35}\pi$$

35. The region is bounded by $x^2 = 4y$ and $y = 1$. The rectangular elements are horizontal, $y \in [0, 1]$. An element of volume is a cylindrical shell centered on the line $y = 1$, of mean radius $1 - m_i$ and altitude $2\sqrt{4m_i} = 4m_i^{1/2}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(1 - m_i)(4m_i^{1/2})\Delta_i y = 8\pi \int_0^1 (1 - y)y^{1/2}dy = 8\pi \int_0^1 (y^{1/2} - y^{3/2})dy$$

$$= 8\pi\Big[\tfrac{2}{3}y^{3/2} - \tfrac{2}{5}y^{5/2}\Big]_0^1 = 8\pi(\tfrac{2}{3} - \tfrac{2}{5}) = \tfrac{32}{15}\pi$$

36. Find the volume of the solid generated by revolving about the x axis the region bounded by the curve $x^{2/3} + y^{2/3} = a^{2/3}$.



▹ The curve, called an astroid and shown at the right, is symmetric with respect to the coordinate axes. Because the y axis is the axis of revolution, the region must lie on one side of this axis. We use only the part of the region in the first quadrant and double the resulting volume.

If we take rectangular elements perpendicular to the y axis, we solve the equation $x^{2/3} + y^{2/3} = a^{2/3}$ for x to get $x = f(y) = (a^{2/3} - y^{2/3})^{3/2}$,

$y \in [0, a]$. Thus,

$$V = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[f(w_i)]^2 \Delta_i y = 2\pi \int_0^a [(a^{2/3} - y^{2/3})^{3/2}]^2 \, dy = 2\pi \int_0^a (a^{2/3} - y^{2/3})^3 \, dy$$

$$= 2\pi \int_0^a (a^2 - 3a^{4/3}y^{2/3} + 3a^{2/3}y^{4/3} - y^2) \, dy = 2\pi \Big[a^2 y - \tfrac{9}{5}a^{4/3}y^{5/3} + \tfrac{9}{7}a^{2/3}y^{7/3} - \tfrac{1}{3}y^3 \Big]_0^a$$

$$= 2\pi(a^3 - \tfrac{9}{5}a^{4/3}a^{5/3} + \tfrac{9}{7}a^{2/3}a^{7/3} - \tfrac{1}{3}a^3) = \tfrac{32}{105}\pi a^3$$

If we take rectangular elements parallel to the y axis, we solve the equation $x^{2/3} + y^{2/3} = a^{2/3}$ for y to get $y = f(x) = (a^{2/3} - x^{2/3})^{3/2}$, $x \in [0, a]$. Thus,

$$V = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i f(m_i) \Delta_i x = 4\pi \int_0^a x(a^{2/3} - x^{2/3})^{3/2} \, dx = 4\pi \int_0^a x^{4/3}(a^{2/3} - x^{2/3})^{3/2} x^{-1/3} \, dx$$

Let $x^{2/3} = a^{2/3} - y^{2/3}$, $\frac{2}{3}x^{-1/3}dx = -\frac{2}{3}y^{-1/3}dy$. When $x = 0$, $y = a$; when $x = a$, $y = 0$. Therefore,

$$V = 4\pi \int_a^0 (a^{2/3} - y^{2/3})^2 y(-y^{-1/3} \, dy) = 4\pi \int_0^a (a^{4/3} - 2a^{2/3}y^{2/3} + y^{4/3}) y^{2/3} \, dy$$

$$= 4\pi \int_0^a (a^{4/3}y^{2/3} - 2a^{2/3}y^{4/3} + y^2) \, dy = 4\pi \Big[\tfrac{3}{5}a^{4/3}y^{5/3} - \tfrac{6}{7}a^{2/3}y^{7/3} + \tfrac{1}{3}y^3 \Big]_0^a$$

$$= 4\pi(\tfrac{3}{5}a^{4/3}a^{5/3} - \tfrac{6}{7}a^{2/3}a^{7/3} + \tfrac{1}{3}a^3) = \tfrac{32}{105}\pi$$

■ Thus the volume of the solid of revolution is $\frac{32}{105}\pi$ cubic units.

37. The region is bounded by $y = \sin x^2$, the x axis, $x = \frac{1}{2}\sqrt{\pi}$ and $x = \sqrt{\pi}$. The rectangular elements are vertical, $x \in [\frac{1}{2}\sqrt{\pi}, \sqrt{\pi}]$. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude $\sin m_i^2$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i (\sin m_i^2) \Delta_i x = \pi \int_{\sqrt{\pi}/2}^{\sqrt{\pi}} (\sin x^2)(2x \, dx) = -\pi \cos x^2 \Big|_{\sqrt{\pi}/2}^{\sqrt{\pi}} = -\pi(\cos \pi - \cos \tfrac{1}{4}\pi)$$

$$= -\pi(-1 - \tfrac{1}{2}\sqrt{2}) = \tfrac{1}{2}(2 + \sqrt{2})\pi$$

38. The region is bounded by $y = \sin x^2$, the x and y axes. The rectangular elements are vertical, $x \in [0, \sqrt{\frac{1}{2}\pi}]$. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude $\cos m_i^2$.

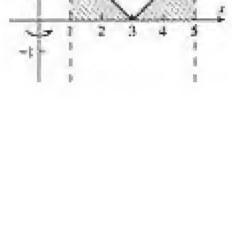
$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i (\cos m_i^2) \Delta_i x = \pi \int_0^{\sqrt{\pi/2}} (\cos x^2)(2x \, dx) = \pi \sin x^2 \Big|_0^{\sqrt{\pi/2}} = \pi(\sin \tfrac{1}{2}\pi - \sin 0) = \pi$$

39. The region is in the first quadrant, bounded by $x = \cos y^2$, the y axis, and the x axis. The rectangular elements are horizontal, $y \in [0, \sqrt{\frac{1}{2}\pi}]$. An element of volume is a cylindrical shell centered on the x axis, of mean radius m_i and altitude $\cos m_i^2$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i (\cos m_i^2) \Delta_i y = \pi \int_0^{\sqrt{\pi/2}} \cos y^2 (2y \, dy) = \pi \Big[\sin y^2\Big]_0^{\sqrt{\pi/2}} = \pi(\sin \tfrac{1}{2}\pi - \sin 0) = \pi$$

40. Find the volume of the solid generated by revolving about the y axis the region bounded by the graph of $y = |x - 3|$ and the lines $x = 1$, $x = 5$, and $y = 0$. Take the rectangular elements of area parallel to the axis of revolution.

▸ The figure shows the region. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude $|m_i - 3|$ on $[1, 5]$. Therefore,



$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i |m_i - 3| = 2\pi \int_1^5 x|x - 3| dx$$

Because $x - 3 \le 0$ on $[1, 3]$ and $x - 3 \ge 0$ on $[3, 5]$, we obtain

$$V = 2\pi \Big[\int_1^3 x(3 - x) dx + \int_3^5 x(x - 3) dx \Big] = 2\pi \Big[\int_1^3 (3x - x^2) dx + \int_3^5 (x^2 - 3x) dx \Big]$$

$$= 2\pi \Big(\Big[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3 \Big]_1^3 + \Big[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 \Big]_3^5 \Big) = 2\pi \Big(\tfrac{3}{2} \cdot 8 - \tfrac{1}{3} \cdot 26 + \tfrac{1}{3} \cdot 98 - \tfrac{3}{2} \cdot 16 \Big) = 24\pi$$

■ Thus the volume of the solid of revolution is 24π cubic units.

In Exercises 41–50, use NINT to find the volume, V cubic units, to four significant digits, of the solid generated by revolving the region about the indicated axis.

41. The region bounded by $y = \sqrt[4]{x^3+4}$, the x axis, the y axis, and $x = 2$ about the y axis.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i y(m_i)\Delta_i x = \int_0^2 2\pi x\sqrt[4]{x^3+4}\,dx = 20.3689 \approx 20.37$$

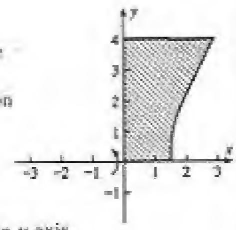
42. The region bounded by $y = \sqrt[3]{x^4-5}$, the x axis, $x = 2$ and $x = 3$ about the y axis.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i y(m_i)\Delta_i x = \int_2^3 2\pi(x^4-5)^{1/3}dx = 51.9178 \approx 51.92$$

43. The region bounded by $y = \sqrt[4]{x^3+4}$, the y axis, and $y = 3$ about the x axis.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i x(m_i)\Delta_i y = \int_{\sqrt{2}}^3 2\pi y(y^4-4)^{1/3}dy = 62.6749 \approx 62.67$$

44. The region bounded by the graph of $y = \sqrt[3]{x^4-5}$, the x axis, the y axis, and the line $y = 4$ about the x axis.



- See the figure at the right. An element of volume is a cylindrical shell centered on the x axis of mean radius m_i and altitude $x(m_i) = \sqrt[4]{m_i^3+5}$, $y \in [0, 4]$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i x(m_i)\Delta_i y = \int_0^4 2\pi y(y^3+5)^{1/4}dy = 112.969 \approx 113.0$$

- The volume of the solid of revolution is 113.0 cubic units.

45. The region bounded by $y = \sin x^3$, the y axis, and $y = 1$, $x \in [0, \sqrt[3]{\pi/2}]$ about the y axis.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i[1 - y(m_i)]\Delta_i x = \int_0^{\sqrt[3]{\pi/2}} 2\pi x[1-\sin x^3]dx = 2.0383 \approx 2.038$$

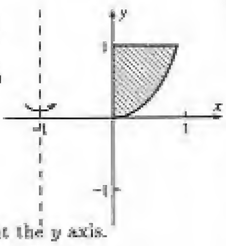
46. The region bounded by $y = \tan x^2$, the y axis, and $y = 1$, $x \in [0, \sqrt{\pi/2}]$ about $y = 1$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i[1 - y(m_i)]\Delta_i x = \int_0^{\sqrt{\pi/2}} 2\pi x(1-\tan x^2)dx = 1.37861$$

47. The region bounded by $y = \sin x^3$, the y axis, and $y = 1$, $x \in [0, \sqrt[3]{\pi/2}]$ about $x = 2$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi(2 - m_i)[1 - y(m_i)] = \int_0^{\sqrt[3]{\pi/2}} 2\pi(2-x)(1-\sin x^3)dx = 7.707$$

48. The region bounded by the graph of $y = \tan x^2$, the y axis, and the line $y = 1$, if $x \in [0, \sqrt{\pi/2}]$ about the line $x = -1$



- See the figure at the right. An element of volume is a cylindrical shell centered on the line $x = -1$ of mean radius $m_i + 1$ and altitude $1 - y(m_i) = 1 - \tan m_i^2$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi(m_i+1)(1-\tan m_i^2)]\Delta_i x = \int_0^{\sqrt{\pi/2}} 2\pi(x+1)(1-\tan x^2)dx$$
$$= 5.3359 \approx 5.336$$

- The volume of the solid of revolution is 5.336 cubic units.

49. The region bounded by the graph of $y = \sin x + 2$, $y = \tan x$, and the y axis about the y axis.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i(\sin m_i + 2 - \tan m_i)\Delta_i x = \int_0^{1.2437} 2\pi x(\sin x + 2 - \tan x)dx = 6.76259 \approx 6.763$$

50. The region bounded by $y = \cos(x^2+2)$ and $y = x^2 - 1$ axis about $x = 1$. The region is symmetrical.

$$V = 2\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi(1-m_i)[\cos(m_i^2+2) - (m_i^2-1)]\Delta_i x = \int_0^{0.5651} 4\pi(1-x)[\cos(x^2+2) - x^2 + 1]dx = 2.738$$

51. The hole can be obtained by revolving about the y axis the region bounded by the circle $x^2 + y^2 = 16$, the y axis, and $x = 2\sqrt{3}$. The rectangular elements are vertical, $x \in [0, 2\sqrt{3}]$. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude $2\sqrt{16 - m_i^2}$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i(2\sqrt{16-m_i^2})\Delta_i x = -2\pi\int_0^{2\sqrt{3}}(16-x^2)^{1/2}(-2x\,dx) = -2\pi\cdot\tfrac{2}{3}(16-x^2)^{3/2}\Big|_0^{2\sqrt{3}}$$
$$= -\tfrac{4}{3}\pi(8-64) = \tfrac{224}{3}\pi$$

52. A hole of radius 2 cm is bored through a spherical shaped solid of radius 6 cm, and the axis of the hole is a diameter of the sphere. Find the volume of the part of the solid that remains.

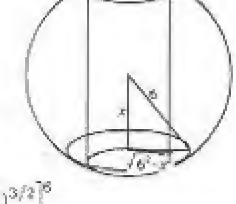
▹ The figure shows the x axis as the axis of the hole, with the origin at the center of the sphere. A plane section perpendicular to the x axis at x is a washer of outer radius $\sqrt{36-x^2}$ cm, inner radius 2 cm, and area measure

$$A(x) = \pi(\sqrt{36-x^2})^2 - \pi(2^2) = \pi(32-x^2)$$

We get half the volume if $0 \le x \le \sqrt{36-4} = 4\sqrt{2}$. By Definition 4.9.1,

$$V = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} A(w_i)\Delta_i x = 2\int_0^{4\sqrt{2}} \pi(32-x^2)\,dx = 2\pi\Big[32x - \tfrac{1}{3}x^3\Big]_0^{4\sqrt{2}}$$

$$= 2\pi[32(4\sqrt{2}) - \tfrac{1}{3}(128\sqrt{2})] = \tfrac{512}{3}\pi\sqrt{2}$$



Suppose we consider the sphere to be the solid generated by revolving a segment of a circle about a diameter. If we take rectangular elements perpendicular to the axis, we get the same integral as above. If we choose rectangular elements parallel to the axis, we get

$$V = 2\pi\int_2^6 2x\sqrt{36-x^2}\,dx = -2\pi\int_2^6 (36-x^2)^{1/2}(-2x\,dx) = -2\pi\Big[\tfrac{2}{3}(36-x^2)^{3/2}\Big]_2^6$$

$$= -\tfrac{4}{3}\pi(0 - 32^{3/2}) = \tfrac{512}{3}\pi\sqrt{2}$$

▪ The volume of the solid that remains is $\frac{512}{3}\pi\sqrt{2}$ cm^3.

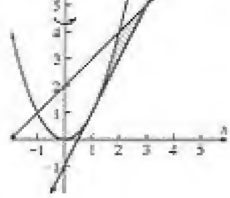
53. The region is bounded by $y = \sqrt[3]{x}$, the x axis, and $x = c$. The rectangular elements are vertical, $x \in [0,c]$. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude $m_i^{\,1/3}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i(m_i^{\,1/3})\Delta_i x = 2\pi\int_0^c x^{4/3}dx = 2\pi \cdot \tfrac{3}{7}x^{7/3}\Big|_0^c = \tfrac{6}{7}c^{7/3}\pi$$

Therefore $12\pi = \frac{6}{7}c^{7/3}\pi$; $c^{7/3} = 14$; $c = \sqrt[7]{14^3} = \sqrt[7]{2744}$.

54. Find the volume of the solid generated by revolving about the y axis the region bounded by the curve $y = x^2$ and the lines $y = 2x - 1$ and $y = x + 2$.

▹ Because $x^2 - (2x-1) = x^2 - 2x + 1 = (x-1)^2$ the line $y = 2x - 1$ is tangent to the curve at $(1,1)$. Because $x^2 - (x+2) = x^2 - x - 2 = (x+1)(x-2)$ the line $y = x + 2$ meets the curve at $(-1,1)$ and $(2,4)$. Because $2x - 1 - (x+2) = x - 3$ the lines meet at $(3,5)$. Only the region R shaded in the figure is bounded by all three. Neither horizontal nor vertical lines meet R in just one kind of segment, but we are given y as a function of x, and so we use vertical rectangular elements at a mean distance of m_i from the y axis. Let



$$f(x) = \begin{cases} x^2 & \text{if } 1 \le x \le 2 \\ x+2 & \text{if } 2 \le x \le 3 \end{cases}$$

R is bounded above by $y = f(x)$ and below by $y = g(x) = 2x - 1$. By the method of shells,

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i[f(m_i) - g(m_i)]\Delta_i x = 2\pi\int_1^2 x[x^2 - (2x-1)]\,dx + 2\pi\int_2^3 x[x + 2 - (2x-1)]\,dx$$

$$= 2\pi\int_1^2 (x^3 + 2x^2 + x)\,dx + 2\pi\int_2^3 (3x - x^2)\,dx = 2\pi\Big[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 + \tfrac{1}{2}x^2\Big]_1^2 + 2\pi\Big[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3\Big]_2^3$$

$$= 2\pi[\tfrac{1}{4}(16-1) - \tfrac{2}{3}(8-1) + \tfrac{1}{2}(4-1)] + 2\pi[\tfrac{3}{2}(9-4) - \tfrac{1}{3}(27-8)] = \tfrac{7}{6}\pi + \tfrac{7}{3}\pi = \tfrac{7}{2}\pi$$

▪ The volume of the solid of revolution is $\frac{7}{2}\pi$ cubic units.

Miscellaneous Exercises for Chapter 4

In Exercises 1–10, perform the antidifferentiation; that is, evaluate the indefinite integral.

1. $\int (2x^3 - x^2 + 3)dx = \frac{1}{2}x^4 - \frac{1}{3}x^3 + 3x + C$

2. $\int 5x(2 + 3x^2)^8 dx = \frac{5}{6}\int (2 + 3x^2)^8(6x\,dx) = \frac{5}{6}[\frac{1}{9}(2 + 3x^2)^9] + C = \frac{5}{54}(2 + 3x^2)^9 + C$

3. $\int x^4\sqrt{x^5-1}\,dx = \frac{1}{5}\int (x^5-1)^{1/2}(5x^4dx) = \frac{1}{5}\cdot\frac{2}{3}(x^5-1)^{3/2} + C = \frac{2}{15}(x^5-1)^{3/2} + C$

4. $\int \sqrt{x}(1+x^2)\,dx$

▷ $\int \sqrt{x}(1+x^2)\,dx = \int (x^{1/2}+x^{5/2})\,dx = \frac{2}{3}x^{3/2} + \frac{2}{7}x^{7/2} + C$

5. Let $v = \sqrt{2s+3}$. Then $v^2 = 2s+3$; $s = \frac{1}{2}(v^2-3)$; $ds = v\,dv$.

$\int \frac{s}{\sqrt{2s+3}}\,ds = \int \frac{\frac{1}{2}(v^2-3)}{v}(v\,dv) = \frac{1}{2}\int (v^2-3)dv = \frac{1}{6}v^3 - \frac{3}{2}v + C = \frac{1}{6}(2s+3)^{3/2} - \frac{3}{2}(2s+3)^{1/2} + C$

6. Let $v = x^2+3$, $x^2 = v-3$, $dv = 2x\,dx$. Then $\int x^3\sqrt{x^2+3}\,dx = \int x^2(x^2+3)^{1/2}(x\,dx)$

$= \int (v-3)v^{1/2}\cdot\frac{1}{2}dv = \frac{1}{2}\int (v^{3/2}-3v^{1/2})dv = \frac{1}{2}(\frac{2}{5}v^{5/2} - 2v^{3/2}) + C = \frac{1}{5}(x^2+3)^{5/2} - (x^2+3)^{3/2} + C$

7. $\int \tan^2 3\theta\,d\theta = \int (\sec^2 3\theta - 1)d\theta = \frac{1}{3}\int \sec^2 3\theta(3\,d\theta) - \int d\theta = \frac{1}{3}\tan 3\theta - \theta + C$

8. $\int t\csc^2 t^2\,dt$

▷ $\int t\csc^2 t^2\,dt = \frac{1}{2}\int \csc^2 t^2(2t\,dt) = -\frac{1}{2}\cot t^2 + C$

9. $\int \frac{5\cos^2 x - 3\tan x}{\cos x}\,dx = 5\int \cos x\,dx - 3\int \tan x\sec x\,dx = 5\sin x - 3\sec x + C$

10. $\int \sin^3 2\theta\cot 2\theta\,d\theta = \frac{1}{2}\int \sin^2 2\theta(2\cos 2\theta\,d\theta) = \frac{1}{2}\cdot\frac{1}{3}\sin^3 2\theta + C = \frac{1}{6}\sin^3 2\theta + C$

In Exercises 11–14, determine the exact value of the integral by using inscribed or circumscribed rectangles. Draw a figure showing the region and the ith rectangle. Check by using the second fundamental theorem of the calculus.

11. inscribed rectangles. Let $f(x) = x^2$. The closed interval $[2,4]$ is divided into n subintervals each of length $\Delta x = \frac{2}{n}$. The ith subinterval is $[x_{i-1}, x_i]$, where $i = 1, 2, 3, \ldots, n$. Because f is increasing on $[2,4]$, the absolute minimum value of f on $[x_{i-1}, x_i]$ is $f(x_{i-2})$.

$f(x_{i-1}) = (x_{i-1})^2 = [2+(i-1)\Delta x]^2 = [2+(i-1)\frac{2}{n}]^2 = 4\left[1 + 2\frac{i-1}{n} + \frac{(i-1)^2}{n^2}\right]$

$A = \lim_{n\to+\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n} 4\left[1 + 2\frac{i-1}{n} + \frac{(i-1)^2}{n^2}\right]\cdot\frac{2}{n}$

$\overset{\text{Note}}{=} \lim_{n\to+\infty} 8\left[\frac{1}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\left(\sum_{i=1}^{n} i - n\right) + \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 - n^2\right)\right]$

$= \lim_{n\to+\infty} 8\left[\frac{1}{n}\cdot n + \frac{2}{n^2}\left(\frac{n(n+1)}{2} - n\right) + \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - n^2\right)\right] = 8(1+1+\frac{1}{3}) = \frac{56}{3}$

▪ The area of the region is $\frac{8}{3}$ square units.

12. $\int_0^3 x^3 dx$; inscribed rectangles

▷ Let $f(x) = x^3$. The closed interval $[0,3]$ is divided into n subintervals each of length $\Delta x = \frac{3}{n}$. The ith subinterval is $[x_{i-1}, x_i]$, where $i = 1, 2, 3, \ldots, n$. Because f is increasing on $[0,3]$, the absolute minimum value of f on $[x_{i-1}, x_i]$ is $f(x_{i-1})$.

$f(x_{i-1}) = (x_{i-1})^3 = [(i-1)\Delta x]^3 = (i-1)^3(\Delta x)^3 = \frac{27(i-1)^3}{n^3}$

$A = \lim_{n\to+\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x = \lim_{n\to+\infty}\sum_{i=1}^{n} \frac{27(i-1)^3}{n^3}\cdot\frac{3}{n} \overset{\text{Note}}{=} \lim_{n\to+\infty}\frac{81}{n^4}\left(\sum_{i=1}^{n} i^3 - n^3\right)$

$= \lim_{n\to+\infty}\frac{81}{n^4}\left(\frac{n^2(n+1)^2}{4} - n^3\right) = \lim_{n\to+\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - \frac{81}{n}\right] = \frac{81}{4}$

By the second fundamental theorem of the calculus, $\int_0^3 x^3 dx = \frac{1}{4}x^4\Big]_0^3 = \frac{1}{4}(3^4 - 0^4) = \frac{81}{4}$

▪ The area of the region is $\frac{81}{4}$ square units.

13. circumscribed rectangles. Let $f(x) = x^3 - 1$. The closed interval $[1,2]$ is divided into n subintervals each of length $\Delta x = \frac{1}{n}$. The ith subinterval is $[x_{i-1}, x_i]$, where $i = 1, 2, 3, \ldots, n$. Because f is increasing on $[1,2]$, the absolute maximum value of f on $[x_{i-1}, x_i]$ is $f(x_i)$.

$f(x_i) = (x_i)^3 - 1 = (1 + i\Delta x)^3 - 1 = 3i\Delta x + 3i^2(\Delta x)^2 + i^3(\Delta x)^3 = \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3}$

$$A = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to+\infty}\left[\sum_{i=1}^{n}\frac{3i}{n}\cdot\frac{1}{n} + \sum_{i=1}^{n}\frac{3i^2}{n^2}\cdot\frac{1}{n} + \sum_{i=1}^{n}\frac{i^3}{n^3}\cdot\frac{1}{n}\right] = \lim_{n\to+\infty}\left[\frac{3}{n^2}\sum_{i=1}^{n} i + \frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n^4}\sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to+\infty}\left[\frac{3}{n^2}\cdot\frac{n(n+1)}{2} + \frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right] = \lim_{n\to+\infty}\left[\frac{3}{2}\left(1+\frac{1}{n}\right) + \frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{1}{4}\left(1+\frac{1}{n}\right)^2\right]$$

$= \frac{3}{2} + 1 + \frac{1}{4} = \frac{11}{4}$. Also $\int_1^2 (x^3 - 1)dx = \frac{1}{4}x^4 - x\Big]_1^2 = \frac{1}{4}\cdot 15 - 1 = \frac{11}{4}$

14. circumscribed rectangles. Let $f(x) = x^2 + 2$. Because f is increasing as we move away from 0, we divide each of the intervals $I_1 = [-3,0]$ and $I_2 = [0,2]$ into n subintervals, starting at 0. In A_1, $\Delta x = 3/n$, $x_i = -3i/n$ and in A_2, $\Delta x = 2/n$, $x_i = 2i/n$.

$$A_1 = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to+\infty}\left[\sum_{i=1}^{n}\left(-\frac{3i}{n}\right)^2\cdot\frac{3}{n} + \sum_{i=1}^{n} 2\cdot\frac{3}{n}\right] = \lim_{n\to+\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{6}{n}\sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to+\infty}\left[\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{6}{n}\cdot n\right] = \lim_{n\to+\infty}\left[\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 6\right] = 9 + 6 = 15$$

$$A_2 = \lim_{n\to+\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to+\infty}\left[\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2\cdot\frac{2}{n} + \sum_{i=1}^{n} 2\cdot\frac{2}{n}\right] = \lim_{n\to+\infty}\left[\frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{4}{n}\sum_{i=1}^{n} 1\right]$$

$= \lim_{n\to+\infty}\left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{4}{n}\cdot n\right] = \lim_{n\to+\infty}\left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 4\right] = \frac{8}{3} + 4 = \frac{20}{3}$. $A = A_1 + A_2 = 15 + \frac{20}{3} = \frac{65}{3}$.

Also, $\int_{-3}^{2} (x^2 + 2)dx = \frac{1}{3}x^3 + 2x\Big]_{-3}^{2} = \frac{1}{3}[8 - (-27)] + 2[2 - (-3)] = \frac{65}{3}$

In Exercises 15-22, evaluate the definite integral by the second fundamental theorem of calculus. Check by NINT.

15. $\int_2^3 \frac{12x\,dx}{(x^2-1)^2} = 6\int_2^3 (x^2-1)^{-2}(2x\,dx) = 6\Big[-(x^2-1)^{-1}\Big]_2^3 = 6[-\frac{1}{8} - (-\frac{1}{3})] = \frac{5}{4}$

16. $\int_{-5}^{5} 2x\sqrt[3]{x^2+2}\,dx$

▹ Because $2x\sqrt[3]{x^2+2}$ is an odd function, then $\int_{-5}^{5} 2x\sqrt[3]{x^2+2}\,dx = 0$

17. $\int_0^{\pi/6} \frac{\sin 2\theta}{\cos^2 2\theta}\,d\theta = -\frac{1}{2}\int_0^{\pi/6} \cos^{-2}2\theta(-2\sin 2\theta\,d\theta) = \frac{1}{2}\Big[\frac{1}{\cos 2\theta}\Big]_0^{\pi/6} = \frac{1}{2}(2-1) = \frac{1}{2}$

18. $\int_{\pi/3}^{\pi} \sin^2\frac{1}{2}t\cos\frac{1}{2}t\,dt = 2\int_{\pi/3}^{\pi}\sin^2\frac{1}{2}t(\cos\frac{1}{2}t\,\frac{1}{2}dt) = 2\cdot\frac{1}{3}\sin^3\frac{1}{2}t\Big]_{\pi/3}^{\pi} = \frac{2}{3}[1^3 - (\frac{1}{2})^3] = \frac{7}{12}$

19. Let $v = \sqrt{x+2}$; then $x = v^2 - 2$ and $dx = 2v\,dv$. When $x = -1$, $v = 1$, and when $x = 7$, $v = 3$. Thus

$$\int_{-1}^{7}\frac{x^2dx}{\sqrt{x+2}} = \int_1^3\frac{(v^2-2)^2(2v\,dv)}{v} = \int_1^3(2v^4 - 8v^2 + 8)dv = \frac{2}{5}v^5 - \frac{8}{3}v^3 + 8v\Big|_1^3 = \frac{2}{5}\cdot 242 - \frac{8}{3}\cdot 26 + 8\cdot 2 = \frac{652}{15}$$

20. $\int_1^2 \frac{y\,dy}{\sqrt{5-y}}$

▹ Let $u = \sqrt{5-y}$. Then $u^2 = 5 - y$, $y = 5 - u^2$, and $dy = -2u\,du$. When $y = 1$, $u = 2$; when $y = 2$, $u = \sqrt{3}$. Thus,

$$\int_1^2\frac{y\,dy}{\sqrt{5-y}} = \int_2^{\sqrt{3}}\frac{(5-u^2)(-2u\,du)}{u} = 2\int_{\sqrt{3}}^{2}(5-u^2)\,du = 2\Big[5u - \frac{1}{3}u^3\Big]_{\sqrt{3}}^{2} = 2[(10-\frac{8}{3}) - (5\sqrt{3} - \sqrt{3})]$$
$$= \frac{44}{3} - 8\sqrt{3}$$

21. $\int_0^{\pi/2}(\tan^2\frac{1}{2}x + \sec^2\frac{1}{2}x)dx = \int_0^{\pi/2}(\sec^2\frac{1}{2}x - 1 + \sec^2\frac{1}{2}x)dx = 4\int_0^{\pi/6}\sec^2\frac{1}{2}x(\frac{1}{2}dx) - \int_0^{\pi/6}dx$

$= \Big[4\tan\frac{1}{2}x - x\Big]_0^{\pi/2} = 4 - \frac{1}{2}\pi$

22. $\int_{\pi/6}^{\pi/3}(1-\cos\theta)\csc^2\theta\, d\theta = \int_{\pi/6}^{\pi/3}(\csc^2\theta - \csc\theta\cot\theta)d\theta = -\cot\theta + \csc\theta\Big]_{\pi/6}^{\pi/3} = (-\frac{1}{3}\sqrt{3}+\frac{2}{3}\sqrt{3}) - (-\sqrt{3}+2)$
$= \frac{4}{3}\sqrt{3}-2$

In Exercises 23–26, find the complete solution of the differential equation.

23. $x^2y\frac{dy}{dx} = (y^2-1)^2$; $\int \frac{y\,dy}{(y^2-1)^2} = \int \frac{dx}{x^2}$; $\frac{1}{2}\int (y^2-1)^{-2}(2y\,dy) = \int x^{-2}\,dx$; $\frac{1}{2}\cdot\frac{(y^2-1)^{-1}}{-1} = \frac{x^{-1}}{-1} - C$
$\frac{1}{2}(y^2-1)^{-1} = x^{-1} + C$; $y^2 = \frac{1}{2x^{-1}+C} + 1 = \frac{x}{2+Cx} + 1$

24. $\frac{d^2y}{dx^2} = 12x^2 - 30x$

▸ Because $dy/dx = y'$, then $d^2y/dx^2 = dy'/dx$. Thus we have

$$\frac{dy'}{dx} = 12x^2 - 30x$$
$$y' = \int (12x^2-30x)dx = 4x^3 - 15x^2 + C_1$$
$$y = \int (4x^3 - 15x^2 + C_1)dx = x^4 - 5x^3 + C_1x + C_2$$

25. $\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \sqrt{2x-1}$; $y' = \frac{1}{2}\int (2x-1)^{1/2}(2\,dx) = \frac{1}{2}\cdot\frac{2}{3}(2x-1)^{3/2} + C_1$
$y = \frac{1}{3}\int (2x-1)^{3/2}dx + C_1\int dx = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{5}(2x-1)^{5/2} + C_1x + C_2 = \frac{1}{15}(2x-1)^{5/2} + C_1x + C_2$

26. $\frac{dy}{dx} = \frac{x\sqrt{1-y^2}}{y\sqrt{2x^2+1}}$; $\frac{y}{\sqrt{1-y^2}}dy = \frac{x}{\sqrt{2x^2+1}}dx$; $-\frac{1}{2}\int (1-y^2)^{-1/2}(-2y\,dy) = \frac{1}{4}\int (2x^2+1)^{-1/2}(4x\,dx)$
$-\frac{1}{2}\cdot 2(1-y^2)^{1/2} = \frac{1}{4}\cdot 2(2x^2+1)^{1/2} - C$; $(1-y^2)^{1/2} = C - \frac{1}{2}(2x^2+1)^{1/2}$; $1-y^2 = [C - \frac{1}{2}(2x^2+1)^{1/2}]^2$

27. $\frac{dy}{dx} = 10 - 4x$; $y = \int (10-4x)dx = 10x - 2x^2 + C$. Because $y = -1$ when $x = 1$, $-1 = 8 + C$; $C = -9$.
Therefore an equation of the curve is $y = 10x - 2x^2 - 9$.

28. The marginal cost function for a particular commodity is given by $C'(x) = 6x - 17$. If the cost of producing 2 units is \$25, find the total cost function.

▸ The cost function is the antiderivative of the marginal cost. By the second fundamental theorem of calculus,
$$C(x) = C(2) + \int_2^x C'(t)dt = 25 + \int_2^x (6t-17)dt = 25 + \Big[3t^2 - 17t\Big]_2^x = 25 + (3x^2-17x) - (-22)$$
$$= 3x^2 - 17x + 47$$

29. $R'(x) = \frac{3}{4}x^2 - 10x + 12$(a) $R(x) = \int (\frac{3}{4}x^2 - 10x + 12)dx = \frac{1}{4}x^3 - 5x^2 + 12x + K$.
Because $R(0) = 0$ then $0 = K$ and $R(x) = \frac{1}{4}x^3 - 5x^2 + 12x$.
(b) $R = px$ hence $px = \frac{1}{4}x^3 - 5x^2 + 12x$ and the demand equation is $p = \frac{1}{4}x^2 - 5x + 12$.

30. $S(2) = S(0) + \int_0^2 S'(t)dt = 8 + \int_0^2 2(t-1)^{2/3}dt = 8 + 2\cdot\frac{3}{5}(t-1)^{5/3}\Big]_0^2 = 8 + \frac{6}{5}\cdot 2 = 10.4$ million dollars

31. (a) $\frac{dV}{dt} = \sqrt{t+1} + \frac{2}{3}t$; $V = \int [(t+1)^{1/2} + \frac{2}{3}t]dt$; $V = \frac{2}{3}(t+1)^{3/2} + \frac{1}{3}t^2 + C$
Because $V = 33$ when $t = 3$, $33 = \frac{2}{3}\cdot 8 + \frac{1}{3}\cdot 9 + C$; $C = \frac{74}{3}$. Hence $V = \frac{2}{3}(t+1)^{3/2} + \frac{1}{3}t^2 + \frac{74}{3}$.
(b) When $t = 8$, $V = \frac{2}{3}\cdot 27 + \frac{1}{3}\cdot 64 + \frac{74}{3} = 64$; so the volume of the balloon is 64 m^3.

32. The enrollment at a certain college has been increasing at the rate of $1{,}000(t+1)^{-1/2}$ students per year since 1985. If the enrollment in 1988 was 10,000, (a) what was the enrollment in 1985 and (b) what is the anticipated enrollment in 1993 if it is expected to be increasing at the same rate?

▸ If $E(t)$ students is the enrollment t years after 1985 then
$$E'(t) = 1{,}000(t+1)^{-1/2}$$
$$E(t) = \int 1{,}000(t+1)^{-1/2}\,dt = 2000(t+1)^{1/2} + C$$

In 1988, $t = 3$ and the enrollment was 10,000. Therefore

$$\begin{aligned} E(3) &= 10{,}000 \\ 2{,}000(3+1)^{1/2} + C &= 10{,}000 \\ 4{,}000 + C &= 10{,}000 \\ C &= 6{,}000 \end{aligned}$$

Therefore,

$$E(t) = 2{,}000(t+1)^{1/2} + 6{,}000$$

(a) Because $t = 0$ in 1985, and $E(0) = 2{,}000 + 6{,}000 = 8{,}000$ the enrollment was 8,000 in 1985.

(b) Because $t = 8$ in 1993, and $E(8) = 2{,}000(9^{1/2}) + 6{,}000 = 12{,}000$
the enrollment will be 12,000 in 1993 if it continues to increase at the same rate.

33. If V cm^3 is the volume of the tumor t days after July 1, then
$\frac{dV}{dt} = \frac{1}{100}(t+6)^{1/2}$; $V = \frac{1}{100}\int (t+6)^{1/2}dt$; $V = \frac{1}{150}[(t+6)^{3/2} + C]$

Because $V = 0.20$ when $t = 3$, $\frac{1}{5} = \frac{27 + C}{150}$; $30 = 27 + C$; $C = 3$. Hence $V = \frac{1}{150}[(t+6)^{3/2} + 3]$.

On July 31, $t = 30$, so $V = \frac{219}{150} = \frac{73}{50}$ and the volume of the tumor is 1.46 cm^3.

34. After experimentation, a certain manufacturer determined that if x units of a certain article of merchandise are produced per day, the marginal cost is given by $c'(x) = 0.3x - 11$ where $c(x)$ dollars is the total cost of producing x units. If the selling price of the article is fixed at \$19 per unit and the overhead cost is \$100 per day, find the maximum daily profit that can be obtained.

▹ The profit is a maximum when

$$\begin{aligned} \text{marginal cost} &= \text{marginal revenue} \\ 0.3x - 11 &= 19 \\ 0.3x &= 30 \\ x &= 100 \end{aligned}$$

The cost of producing 100 units is

$$\begin{aligned} c(100) &= c(0) + \int_0^{100} c'(x)\,dx = 100 + \int_0^{100} (0.3x - 11)\,dx = 100 + \left[0.15x^2 - 11x\right]_0^{100} \\ &= 100 + 0.15(10{,}000) - 11(100) = 500 \end{aligned}$$

The number of dollars of revenue from selling 100 units at \$19 per unit is $19(100) = 1900$. Because

$$\text{profit} = \text{revenue} - \text{cost} = 1900 - 500 = 1400$$

the maximum daily profit is \$1400.

35. If x toys are demanded when p dollars is the price per toy, then
$\frac{dp}{dx} = -\frac{p^2}{30{,}000}$; $-30{,}000\int \frac{dp}{p^2} = \int dx$; $\frac{30{,}000}{p} = x + K$.

Because $x = 1800$ when $p = 10$, $K = 1200$. Hence $\frac{30{,}000}{p} = x + 1200$; $p = \frac{30{,}000}{x + 1200}$.

If $R(x)$ dollars is the total revenue from the sale of x units, $R(x) = px = \frac{30{,}000x}{x + 1200}$.

Let $S(x)$ dollars be the total profit from the sale of x units. Then $S(x) = R(x) - C(x) = \frac{30{,}000x}{x + 1200} - (x + 7500)$, $x > 0$. We wish to find the value of x for which S has an absolute maximum value.

$S'(x) = \frac{30{,}000(x + 1200) - 30{,}000x}{(x+1200)^2} - 1 = \frac{36{,}000{,}000}{(x+1200)^2} - 1$; $S''(x) = -\frac{72{,}000{,}000}{(x+1200)^2}$

Set $S'(x) = 0$: $\frac{36{,}000{,}000}{(x+1200)^2} = 1$; $(x+1200)^2 = 36{,}000{,}000$; $x + 1200 = \pm 6000$; $x = 4800$, $x = -7200$

$x = 4800$ is the only critical number. Because $S''(4800) < 0$, S has a relative maximum value when $x = 4800$ and $p = \frac{30{,}000}{4800 + 1200} = 5$; it is also an absolute maximum value by Theorem 3.9.1(i).

▪ The manufacturer's profit will be a maximum when 4800 toys are sold at \$5 per toy.

In Exercises 36–39, a particle is moving on a line. At t seconds, s ft is the directed distance of the particle from the origin, v ft/sec is its velocity and a ft/sec^2 is its acceleration.

36. $a = 3t + 4$; $v = 5$ and $s = 0$, when $t = 0$. Express v and s in terms of t.

▷ $a = v'(t)$. $v(t) = v(0) + \int_0^t v'(u)du = 5 + \int_0^t (3u + 4)du = 5 + \left[\frac{3}{2}u^2 + 4u\right]_0^t = \frac{3}{2}t^2 + 4t + 5$

$v = s'(t)$. $s(t) = s(0) + \int_0^t s'(u)du = 0 + \int_0^t (\frac{3}{2}u^2 - 4u + 5)du = \left[\frac{1}{2}u^3 - 2u^2 + 5u\right]_0^t = \frac{1}{2}t^3 - 2t^2 + 5t$

37. $a = 6 \cos 2t$; $v = 3$ and $s = 4$ when $t = \frac{1}{2}\pi$. $a = \frac{dv}{dt} = 6 \cos 2t$; $v = 3\int \cos 2t(2\,dt)$; $v = 3 \sin 2t + C_1$

Because $v = 3$ when $t = \frac{1}{2}\pi$, then $C_1 = 3$. Hence $v = \frac{ds}{dt} = 3 \sin 2t + 3$.

$s = \frac{3}{2}\int \sin 2t(2\,dt) + 3\int dt = -\frac{3}{2}\cos 2t + 3t + C_2$. Because $s = 4$ when $t = \frac{1}{2}\pi$,

$4 = -\frac{3}{2}(-1) + \frac{3}{2}\pi + C_2$; $C_2 = \frac{1}{2}(5 - 3\pi)$. Therefore $s = -\frac{3}{2}\cos 2t + 3t + \frac{1}{2}(5 - 3\pi)$.

38. $a = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = v\frac{dv}{ds} = 3s + 4$; $v\,dv = (3s + 4)ds$; $\int v\,dv = \int (3s + 4)ds$; $\frac{1}{2}v^2 = \frac{3}{2}s^2 + 4s + C$;

$v^2 = 3s^2 + 8s + C$. Because $v = 5$ when $s = 1$, then $25 = 11 + C$; $C = 14$. Therefore $v^2 = 3s^2 + 8s + 14$.

39. $s = \int_0^t 3 \cos 2\pi u\,du = \frac{3}{2\pi}\sin 2\pi t$. (a) $s(0.2) \approx 0.4541$; (b-c) $s(0.8) = s(1.7) \approx -0.4541$; (d) $s(2.25) \approx 0.4775$

In Exercises 40–44 and 46, acceleration is due only to gravity at 32 ft/sec^2 or 9.8 m/sec^2 acting downward.

40. A stone is thrown vertically upward from the ground with an initial velocity of 25 ft/sec. (a) How long will the stone be going up, (b) how high will the stone go, and (c) how long will it take the stone to reach the ground? (d) Simulate the motion on your graphics calculator and support your answers in parts (a)-(c). (e) With what speed will the stone strike the ground?

▷ At t seconds, s ft is the height of the stone and v ft/sec is its velocity. $s'(t) = v(t)$ and because the positive direction is upward, $v'(t) = -32$.

$$v(t) = v_0 + \int_0^t v'(u)du = 25 - \int_0^t 32\,du = 15 - 32u\Big]_0^t = 25 - 32t$$

$$s(t) = s_0 + \int_0^t s'(u)du = 0 + \int_0^t (25 - 32u)du = 25u - 16u^2\Big]_0^t = 25t - 16t^2$$

(a) $v(t) \geq 0$ when $25 - 32t \geq 0$; $t \leq \frac{25}{32}$. The stone is rising for $\frac{25}{32}$ sec.

(b) $s(\frac{25}{32}) = 25 \cdot \frac{25}{32} - 16(\frac{25}{32})^2 = \frac{625}{64}$. The stone will go about 9.77 ft high.

(c) The stone will reach the ground in $2 \cdot \frac{25}{32}$ sec $= \frac{25}{16}$ sec ≈ 1.56 sec.

(d) The stone strikes the ground at the same speed at which is was thrown, that is 25 ft/sec.

41. Neglecting air resistance, if an object is dropped from an airplane flying horizontally at a height of 30,000 ft above the ocean, (a) how long would it take the object to strike the water, and (b) with what speed will it strike the water?

▷ Although the motion of the object is not in a straight line, we can apply the principles of rectilinear motion to the vertical component of the velocity. t seconds after the object was dropped, let s feet be the distance it falls, v ft/sec the downward component of its velocity, a ft/sec^2 the downward component of its acceleration. Because we have taken the downward direction as positive, then $a = 32$. Thus

$$\frac{dv}{dt} = 32; \quad v = \int 32\,dt = 32t + C_1$$

Because the plane is flying horizontally, $v = 0$ when $t = 0$. Thus $c_1 = 0$ and $v = 32t$; $\frac{ds}{dt} = 32t$

$$s = \int 32t\,dt = 16t^2 + C_2$$

Because $s = 0$ when $t = 0$, we have $c_2 = 0$ and so $s = 16t^2$.

Now $s = 30{,}000$ when the object strikes the water. Thus,

$30{,}000 = 16t^2$; $t^2 = 1875$; $t = \sqrt{1875} \approx 43.3$ and $v(\sqrt{1875}) = 32\sqrt{1875} \approx 1385.64$

It takes 43.3 sec object to strike the water, at about 1385 ft/sec. (Air resistance limits the speed to 225 ft/sec.)

42. At t sec, let s ft be the directed distance of the bullet from the starting point, v ft/sec be its velocity and $\bar{t}$ sec is the time it takes the bullet to reach the ocean. The positive direction is upward. We have $s(0) = 0$, $v(0) = -2500$, and $s(\bar{t}) = -30{,}000$. The acceleration of the bullet is that due to gravity. Therefore, $a = -32$. Hence $\frac{dv}{dt} = -32$; $v = -32\int dt = -32 + C_1$. Because $v = -2500$ when $t = 0$, $C_1 = -2500$. Therefore

$v = \frac{ds}{dt} = -32t - 2500$; $s = \int (-32t - 2500)dt = -16t^2 - 2500t + C_2$. Because $s = 0$ when $t = 0$, $C_2 = 0$.
Hence $s = -16t^2 - 2500t$. Because $t = \bar{t}$ when $s = -30{,}000$,

$-30{,}000 = -16\bar{t}^2 - 2500t$; $4\bar{t}^2 + 625\bar{t} - 7500 = 0$; $\bar{t} = \frac{1}{8}[-625 \pm \sqrt{625^2 - 16(-7500)}] = \frac{1}{8}[-625 \pm 25\sqrt{817}]$.

The time it takes the bullet to reach the ocean is $\frac{1}{8}(25\sqrt{817} - 625)$ sec ≈ 11.2 sec. The velocity is $-32 \cdot \frac{1}{8}(25\sqrt{817} - 625) - 2500 \approx 2858$ ft/sec. (Air resistance will slow the bullet to 225 ft/sec.)

43. At t sec, let s ft be the directed distance of the ball from the ground, v ft/sec be its velocity. The positive direction is upward. We have $s(0) = 64$, $v(0) = 48$.

$v(t) = v(0) + \int_0^t a(u)du = 48 - \int_0^t 32\, du = 48 - 32t$

$s(t) = s(0) + \int_0^t v(u)du = 64 + \int_0^t (48 - 32u)du = 64 + 48t - 16t^2$

Because $48 - 32t \geq 0$ if $0 \leq t \leq 1.5$, it takes 1.5 sec to reach the greatest height of $s(1.5) = 64 + 48(1.5) - 16(1.5)^2 = 100$ feet. It strikes the ground when $0 = s(t) = 64 + 48t - 16t^2$; $0 = t^2 - 3t - 4 = (t - 4)(t + 1)$; $t = 4$ sec, with velocity $v(4) = 48 - 32(4) = -80$ ft/sec.

44. A ball is dropped from the top of a house 64 ft above the ground. (a) How long will it take the ball to strike the ground, and (b) with what velocity will it strike the ground?

▸ Let the origin be at ground level with positive direction upward. t sec after the ball is dropped, let s ft be its directed distance from the origin, v ft/sec its velocity and a ft/sec^2 its acceleration. Because the acceleration is due to gravity, which acts in the negative direction, we have $a = -32$. Thus

$$\frac{dv}{dt} = -32; \qquad v = \int -32\, dt = -32t + C_1$$

Because the ball is dropped from rest, $v = 0$ when $t = 0$. Thus $c_1 = 0$, and

$$v = -32t; \qquad \frac{ds}{dt} = -32t \tag{1}$$

$$s = \int -32t\, dt = -16t^2 + C_2$$

Because the top of the house is 64 ft above the ground, $s = 64$ when $t = 0$. Thus, $c_2 = 64$, and
$s = -16t^2 + 64$

When the ball strikes the ground, $s = 0$. Hence,
$0 = -16t^2 + 64; \quad t^2 = 4;\ t = 2$

When $t = 2$, from Eq. (1) we get $v = -32(2) = -64$. Thus the ball strikes the ground (a) after 2 sec (b) with a velocity of -64 ft/sec.

45. At t sec, let s ft be the directed distance of the rocket from the ground, v ft/sec be its velocity. The positive direction is upward. We have $s(0) = 0$, $v(0) = 0$, $a = 25$.

$v(t) = v(0) + \int_0^t a\, du = 0 + \int_0^t 25\, du = 25t$. $v(60) = 25(60) = 1500$ m/sec.

$s(t) = s(0) + \int_0^t v(u)du = 0 + \int_0^t 25u\, du = \frac{25}{2}t^2$. $s(60) = \frac{25}{2}(60)^2 = 45{,}000$ m $= 4.5$ km

46. At t sec, let s ft be the directed distance of the projectile from the ground, v ft/sec be its velocity. The positive direction is upward. We have $s(0) = 2.5$, $v(0) = 200$, $a = -9.8$.

$v(t) = v(0) + \int_0^t a\, du = 200 - \int_0^t 9.8\, du = 200 - 9.8t$.

$s(t) = s(0) + \int_0^t v(u)du = 2.5 + \int_0^t (200 - 9.8u)du = 2.5 + 200t - 4.9t^2$. $s(3) = 2.5 + 200(3) - 4.9(3)^2 = 558.4$ m

$s(t) = 2.5 + 200t - 4.9t^2 = 600$; $4.9t^2 - 200t + 602.5 = 0$; $t = \frac{200 \pm \sqrt{28191}}{9.8}$.

The projectile is 600 m above ground after 3.28 sec going up and 37.54 sec going down.

47. t sec after the patrol car starts, let s ft be its directed distance, v ft/sec be its velocity. $v(0) = 0$, $s(0) = 0$, $a = 8$. Because 60 mi/hr = 88 ft/sec, the distance of the auto is $88(t + 3)$.

$v(t) = v(0) + \int_0^t a\, du = 0 + \int_0^t 8\, du = 8t$. $s(t) = s(0) + \int_0^t v(u)du = 0 + \int_0^t 8u\, du = 4t^2$. $4t^2 = 88(t + 3)$ when

$t^2 - 22t - 66 = 0$; $t = 11 \pm \sqrt{187}$. It takes about 24.67 sec. The distance is $4(11 + \sqrt{187})^2 \approx 608.8$ ft.

In Exercises 48 and 49, find the sum.

48. $\sum_{i=1}^{n} 2i(i^3-1)$

▶ Because $\sum_{i=1}^{n} i^4 = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ and $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$,

$\sum_{i=1}^{100} 2i(i^3-1) = 2\sum_{i=1}^{100} i^4 - 2\sum_{i=1}^{100} i = 2\cdot\dfrac{100\cdot 101\cdot 201\cdot 30299}{30} - 2\cdot\dfrac{100\cdot 101}{2} = 4{,}100{,}656{,}560$

49. Let $f(i) = \sqrt[3]{3i+2}$. Then $f(i-1) = \sqrt[3]{3(i-1)+2} = \sqrt[3]{3i-1}$. Therefore

$\sum_{i=1}^{41}(\sqrt[3]{3i-1} - \sqrt[3]{3i+2}) = \sum_{i=1}^{41}[f(i-1)-f(i)] = f(0) - f(41) = \sqrt[3]{2} - \sqrt[3]{125} = \sqrt[3]{3} - 5$

50. $\left(\sum_{i=1}^{n} i\right)^2 = \left[\dfrac{n(n+1)}{2}\right]^2 = \dfrac{n^2(n+1)^2}{4} = \sum_{i=1}^{n} i^3$

If $n=1$, we have $\sum_{i=1}^{1} i^3 = 1^3 = 1$ and $\left(\sum_{i=1}^{1} i\right)^2 = 1^2 = 1$.

If $n=2$, we have $\sum_{i=1}^{2} i^3 = 1^3 + 2^3 = 9$ and $\left(\sum_{i=1}^{2} i\right)^2 = (1+2)^2 = 9$.

If $n=3$, we have $\sum_{i=1}^{3} i^3 = 1^3 + 2^3 + 3^3 = 36$ and $\left(\sum_{i=1}^{3} i\right)^2 = (1+2+3)^2 = 6^2 = 36$.

51. We apply Theorem 4.6.1.

(a) If x is in $[-2,-1]$, $x-3 < x < 0$, so $\frac{1}{x-3} > \frac{1}{x}$. Thus $\int_{-2}^{-1} \frac{dx}{x-3} \geq \int_{-2}^{-1} \frac{dx}{x}$.

(b) If x is in $[1,2]$, $x > 0 > x-3$, so $\frac{1}{x} > \frac{1}{x-3}$. Thus $\int_{1}^{2} \frac{dx}{x} \geq \int_{1}^{2} \frac{dx}{x-3}$.

(c) If x is in $[4,5]$, $0 < x-3 < x$, so $\frac{1}{x-3} > \frac{1}{x}$. Thus $\int_{4}^{5} \frac{dx}{x-3} \geq \int_{4}^{5} \frac{dx}{x}$.

52. Express as a definite integral and evaluate the definite integral: $\lim_{n\to+\infty} \sum_{i=1}^{n} \dfrac{8\sqrt{i}}{n^{3/2}}$

(*Hint:* Consider the function f for which $f(x) = \sqrt{x}$.)

▶ Ignoring the factor 8, we have a Riemann sum for the function f on the interval $[0,1]$ with $\Delta x = 1/n$ and $w_i = i/n$. Thus, by Definition 4.5.2

$$\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{8\sqrt{i}}{n^{3/2}} = 8 \lim_{n\to+\infty} \sum_{i=1}^{n} \sqrt{\tfrac{i}{n}}\,\tfrac{1}{n} = 8\int_0^1 \sqrt{x}\,dx = 8\cdot\tfrac{2}{3}x^{3/2}\Big]_0^1 = 8\cdot\tfrac{2}{3}(1-0) = \tfrac{16}{3}$$

In Exercises 53 and 54, apply Theorem 4.6.2 to find a closed interval containing the value of the definite integral.

53. If t is in $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, $0 \leq \cos t \leq 1$, so $0 \leq \sqrt{\cos t} \leq 1$. Hence from Theorem 4.6.2

$0[\frac{1}{2}\pi - (-\frac{1}{2}\pi)] \leq \int_{-\pi/2}^{\pi/2} \sqrt{\cos t}\,dt \leq 1[\frac{1}{2}\pi - (-\frac{1}{2}\pi)]$; $0 \leq \int_{-\pi/2}^{\pi/2} \sqrt{\cos t}\,dt \leq \pi$

Thus the value of the given definite integral is in $[0,\pi]$.

54. Let $f(x) = x^2 - 2x + 6 = (x-1)^2 + 5$. If $x \in [0,3]$ then $f(1) = 5 \leq f(x) \leq f(3) = 9$. Hence from Theorem 4.6.2 $5(3-0) \leq \int_0^3 (x^2 - 2x + 6)dx \leq 9(3-0)$. The value of the definite integral is in $[15, 27]$.

In Exercises 55 and 56, evaluate the integral by the second fundamental theorem of the calculus. Check by NINT.

55. $\int_{-3}^{3} |x-2|^3 dx = \int_{-3}^{2} |x-2|^3 dx + \int_{2}^{3} |x-2|^3 dx = \int_{-3}^{2} (2-x)^3 dx + \int_{2}^{3} (x-2)^3 dx$

$= -\dfrac{(2-x)^4}{4}\Big|_{-3}^{2} + \dfrac{(x-2)^4}{4}\Big|_{2}^{3} = (0 + \frac{625}{4}) + (\frac{1}{4} - 0) = \frac{313}{2}$

56. $\int_{-2}^{2} x|x-3|dx$

▶ Because $x - 3 \leq 0$ if $x \leq 3$, then

$$\int_{-2}^{2} x|x-3|dx = \int_{-2}^{2} x(3-x)dx = \int_{-2}^{2} (3x - x^2)dx = \tfrac{3}{2}x^2 - \tfrac{1}{3}x^3\Big]_{-2}^{2} = -\tfrac{16}{3}$$

In Exercises 57–60, compute the derivative.

57. $\dfrac{d}{dx}\int_x^4 (3t^2-4)^{3/2}dt = \dfrac{d}{dx}\left[-\int_4^x (3t^2-4)^{3/2}dt\right] = -(3x^2-4)^{3/2}$

58. Because $4/(1+t^2)$ is an even function, then

$$\frac{d}{dx}\int_{-x}^{x}\frac{4}{1+t^2}\,dt = 2\,\frac{d}{dx}\int_{0}^{x}\frac{4}{1+t^2}\,dt = 2\cdot\frac{4}{1+x^2} = \frac{8}{1+x^2}$$

59. Let $u = x^2$ and let a be between x and x^2.

$$\frac{d}{dx}\int_{x}^{x^2}\frac{1}{t}dt = \frac{d}{dx}\left(\int_{a}^{x^2}\frac{1}{t}dt - \int_{a}^{x}\frac{1}{t}dt\right) = \frac{d}{du}\int_{a}^{u}\frac{1}{t}dt\,\frac{du}{dx} - \frac{d}{dx}\int_{a}^{x}\frac{1}{t}dt = \frac{1}{u}(2x) - \frac{1}{x} = \frac{1}{x^2}(2x) - \frac{1}{x} = \frac{1}{x}$$

60. $\dfrac{d}{dx}\displaystyle\int_{1}^{\sec x}\sqrt{t^2-1}\,dt,\ 0 < x < \tfrac{1}{2}\pi$

▸ Let $u = \sec x$ and apply the chain rule.

$$\frac{d}{dx}\int_{1}^{\sec x}\sqrt{t^2-1}\,dt = \frac{d}{du}\int_{1}^{u}\sqrt{t^2-1}\,dt\cdot\frac{d}{dx}(\sec x) = \sqrt{u^2-1}\cdot\sec x\tan x = \sqrt{\sec^2 x-1}\,\sec x\tan x$$
$$= \sec x\tan^2 x$$

61. The average value of the cosine function on $[a, a+2\pi]$ is

$$\frac{1}{(a+2\pi)-a}\int_{a}^{a+2\pi}\cos x\,dx = \frac{1}{2\pi}\sin x\Big|_{a}^{a+2\pi} = \frac{\sin(a+2\pi)-\sin a}{2\pi} = \frac{\sin a - \sin a}{2\pi} = 0$$

62. Interpret the mean-value theorem for integrals in terms of an average function value.

▸ If f is continuous on the closed interval $[a,b]$, then by the mean-value theorem there exists a number c such that $a < c < b$, and $\int_a^b f(x)\,dx = f(c)(b-a)$ or, equivalently, $\dfrac{\int_a^b f(x)\,dx}{b-a} = f(c)$ (1)

Because the fraction on the left side of Eq. (1) is by Definition 4.6.2 the average value of f on $[a,b]$, then $f(c)$ is the average value of f on $[a,b]$. That is, the number c in the conclusion of the mean-value theorem for integrals is the number at which the function value of f is the average function value of f on the closed interval $[a,b]$.

63. $f(x) = x^2\sqrt{x-3}$. Let $v = \sqrt{x-3}$. Then $x = v^2+3$ and $dx = 2v\,dv$. When $x = 7$, $v = 2$ and when $x = 2$, $v = 3$. Therefore the average value of f on $[7,12]$ is

$$\frac{1}{12-7}\int_{7}^{12}x^2\sqrt{x-3}\,dx = \frac{1}{5}\int_{2}^{3}(v^2+3)^2 v(2v\,dv) = \frac{2}{5}\int_{2}^{3}(v^6+6v^4+9v^2)dv = \frac{2}{5}\left[\frac{1}{7}v^7+\frac{6}{5}v^5+3v^3\right]_2^3 = \frac{42{,}304}{175}$$

64. (a) Find the average value A of the function f defined by $f(x) = 1/x^2$ on the interval $[1,r]$. (b) Find $\lim\limits_{r\to+\infty} A$.

▸ (a) $A = \dfrac{1}{r-1}\displaystyle\int_1^r \frac{1}{x^2}\,dx = \frac{1}{r-1}\left[-\frac{1}{x}\right]_1^r = \frac{1}{r-1}\left(1-\frac{1}{r}\right) = \frac{1}{r-1}\cdot\frac{r-1}{r} = \frac{1}{r}$ (b) $\lim\limits_{r\to+\infty} A = \lim\limits_{r\to+\infty}\dfrac{1}{r} = 0$

65. Consider the starting point as the point from which the body falls. At t sec, let x ft be the distance of the body from the starting point and v ft/sec its velocity. Take the positive direction downward. Let $\bar{v}$ ft/sec be the final velocity.

$g = \dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = v\,\dfrac{dv}{dt}$, and so $\frac{1}{2}v^2 = gx + C$. Because $v = 0$ when $x = 0$, we get $C = 0$ so $v^2 = 2gx$.

Because $v > 0$, $v = \sqrt{2gx}$. Because $v = \bar{v}$ when $x = s$, we have $\bar{v} = \sqrt{2gs}$.

Let A.V. be the average value of v for x in $[0,s]$. Then

$$\text{A.V.} = \frac{1}{s-0}\int_0^s\sqrt{2gx}\,dx = \frac{1}{s}\sqrt{2g}\int_0^s x^{1/2}dx = \frac{2}{3s}\sqrt{2g}\,x^{3/2}\Big|_0^s = \frac{2}{3s}\sqrt{2g}\,s^{3/2} = \frac{2}{3}\sqrt{2gs}$$

Because $\bar{v} = \sqrt{2gs}$, we see that A.V. $= \frac{2}{3}\bar{v}$.

66. Suppose a ball is dropped from rest and after t seconds its directed distance from the starting point is s ft and its velocity is v ft/sec. Neglect air resistance. When $t = t_1$, then $s = s_1$ and $v = v_1$.
(a) Express v as a function of t as $v = f(t)$, and find the average value of f on $[0, t_1]$.
(b) Express v as a function of s as $v = h(s)$, and find the average value of h on $[0, s_1]$.
(c) Write the results of parts (a) and (b) in terms of t_1, and determine which average velocity is larger.

▸ Let the origin be at the point where the ball is dropped and let the positive direction be downward. If a ft/sec^2 is the acceleration of the ball, then $a = 32$, because the acceleration is due to the force of gravity.

(a) Because $a = dv/dt$, we have

$$\frac{dv}{dt} = 32$$
$$v = \int 32\,dt = 32 + C_1$$

Because the ball is dropped from rest, $v = 0$ when $t = 0$. Thus $c_1 = 0$, and hence

$$v = f(t) = 32t \tag{1}$$

Let $\bar{f}[0, t_1]$ be the average value of f on $[0, t_1]$. Then

$$\bar{f}[0, t_1] = \frac{1}{t_1}\int_0^{t_1} f(t)\,dt = \frac{1}{t_1}\int_0^{t_1} 32t\,dt = \frac{1}{t_1}\Big[16t^2\Big]_0^{t_1} = \frac{1}{t_1}\cdot 16{t_1}^2 = 16t_1$$

(b) Because $a = \frac{dv}{ds}\cdot\frac{ds}{dt} = \frac{dv}{ds}v$ and $a = 32$, we have

$$32\,ds = v\,dv$$
$$\int 32\,ds = \int v\,dv$$
$$32s = \tfrac{1}{2}v^2 + C_2$$

Because $v = 0$ when $s = 0$, we have $c_2 = 0$, and thus

$$32s = \tfrac{1}{2}v^2$$
$$v = h(s) = 8\sqrt{s} \tag{2}$$

Let $\bar{h}[0, s_1]$ be the average value of h on $[0, s_1]$. Then

$$\bar{h}[0, s_1] = \frac{1}{s_1}\int_0^{s_1} h(s)\,ds = \frac{1}{s_1}\int_0^{s_1} 8\sqrt{s}\,ds = \frac{8}{s_1}\left[\tfrac{2}{3}s^{3/2}\right]_0^{s_1} = \frac{16}{3}\sqrt{s_1}$$

(c) Substituting from Eq. (1) into Eq. (2), we have

$$32t = 8\sqrt{s_1}$$
$$\sqrt{s_1} = 4t_1$$
$$\bar{h}[0, s_1] = \tfrac{16}{3}(4t_1) = \tfrac{64}{3}t_1$$

Since $\frac{64}{3}t_1 > 16t_1$, then the average of h is larger than the average of f.

In Exercises 67–76, find the area of the region bounded by the curve and the lines: (a) Draw a figure showing the region and a rectangular element of area; (b) express the measure of the area of the region as the limit of a Riemann sum. (c) Find the limit in part (b) by evaluating a definite integral: in Exercises 67–72, use the second fundamental theorem of calculus and in Exercises 73–76, approximate to 4 significant digits by zoom and NINT.

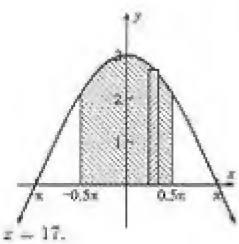
67. A square units is the area of the region bounded by $y = 9 - x^2$; x axis; y axis; $x = 2$.
Δ is a partition of the interval $[0, 2]$ on the x axis. Then

$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(9 - {w_i}^2)\Delta_i x = \int_0^2 (9 - x^2)\,dx = 9x - \tfrac{1}{3}x^3\Big]_0^2 = 18 - \tfrac{8}{3} = \tfrac{46}{3}$$

68. $y = 3\cos\frac{1}{2}x$; x axis; $x = -\frac{1}{2}\pi$; $x = \frac{1}{2}\pi$

▹ The region is shown at the right.

An element of area is a vertical rectangle of width $\Delta_i x$ and height $3\cos\frac{1}{2}x$.



$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 3\cos\tfrac{1}{2}x\,\Delta_i x = \int_{-\pi/2}^{\pi/2} 3\cos\tfrac{1}{2}x\,dx = 6\int_{-\pi/2}^{\pi/2}\cos\tfrac{1}{2}x(\tfrac{1}{2}\,dx)$$
$$= 6\sin\tfrac{1}{2}x\Big]_{-\pi/2}^{\pi/2} = 6[\tfrac{1}{2}\sqrt{2} - (-\tfrac{1}{2}\sqrt{2})] = 6\sqrt{2}$$

69. A square units is the area of the region bounded by $y = 2\sqrt{x-1}$; x axis; $x = 5$; $x = 17$.
Let Δ be a partition of the interval $[5, 17]$ on the x axis. Then

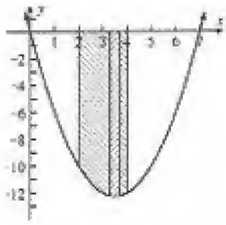
$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\sqrt{w_i - 1}\,\Delta_i x = 2\int_5^{17}\sqrt{x-1}\,dx = \tfrac{4}{3}(x-1)^{3/2}\Big|_5^{17} = \tfrac{4}{3}(64 - 8) = \tfrac{224}{3}$$

70. A square units is the area of the region bounded by $y = 4x^{-2} - x$; x axis; $x = -2$; $x = -1$.
Let Δ be a partition of the interval $[-2, -17]$ on the x axis. Then

$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(4{w_i}^{-2} - w_i)\Delta_i x = \int_{-2}^{-1}(4x^{-2} - x)\,dx = -4x^{-1} - \tfrac{1}{2}x^2\Big]_{-2}^{-1} = -4(-1 + \tfrac{1}{2}) - \tfrac{1}{2}(1 - 4) = \tfrac{7}{2}$$

71. $(5-x^2)-(-4)=9-x^2$ is nonnegative in $[-3,3]$. A square units is the area of the region below $y=5-x^2$ and above $y=-4$ in $[-3,3]$. Because the region is symmetric,

$$A=2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(9-w_i^{\,2})\Delta_i x=2\int_0^3(9-x^2)dx=18x-\tfrac{2}{3}x^3\Big|_0^3=36-0=36$$



72. $y=x^2-7x$; x axis; $x=2$; $x=4$

▸ The region R is shown at the right. Let $f(x)=x^2-7$. An element of area is a vertical rectangle of width $\Delta_i x$ and height $-f(w_i)$. Then

$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}-(w_i^{\,2}-7w_i)\Delta_i x=\int_2^4(-x^2+7x)\,dx$$
$$=-\tfrac{1}{3}x^3+\tfrac{7}{2}x^2\Big]_2^4=-\tfrac{1}{3}(64-8)+\tfrac{7}{2}(16-4)=\tfrac{70}{3}$$

• The area of the region is $\frac{70}{3}$ square units.

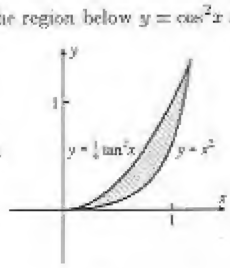
73. $(9-x^2)-x^4$ is nonnegative in $[-b,b]$, $b\approx 1.59417$. A square units is the area of the region below $y=9-x^2$ and above $y=x^4$ in $[-b,b]$. Because the region is symmetric,

$$A=2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(9-w_i^{\,2}-w_i^{\,4})\Delta_i x=2\int_0^b(9-x^2-x^4)dx\approx 21.8757\approx 21.88$$

74. $(16-x^2)-x^3$ is nonnegative in $[0,b]$, $b\approx 2.22677$. A square units is the area of the region below $y=16-x^2$ and above $y=x^3$ in $[0,b]$. $A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(16-w_i^{\,2}-w_i^{\,3})\Delta_i x=\int_0^b(16-x^2-x^3)dx\approx 25.8011\approx 25.80$

75. $\cos^2 x-x^2$ is nonnegative in $[-b,b]$, $b\approx 0.739085$. A square units is the area of the region below $y=\cos^2 x$ and above $y=x^2$ in $[-b,b]$. Because the region is symmetric,

$$A=2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\cos^2 w_i-w_i^{\,2})\Delta_i x=2\int_0^b(\cos^2 x-x^2)dx\approx 0.967793\approx 0.9678$$



76. $y=x^2$; $y=\frac{1}{4}\tan^2 x$, $0\le x\le\frac{1}{2}\pi$

▸ The region is shown at the right. Using zoom, we determine that $x^2\ge\frac{1}{4}\tan^2 x$ on the interval $[-b,b]$, $b=1.16556$. Therefore,

$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(w_i^{\,2}-\tfrac{1}{4}\tan^2 w_i)\Delta_i x=\int_0^b(x^2-\tfrac{1}{4}\tan^2 x)dx\approx 0.091492$$

with the help of NINT. The area of the region is about 0.0915 square units.

In Exercises 77–81, find the exact area of the region.

77. $y^2-y^3=y^2(1-y)\ge 0$ in $[0,1]$. A square units is the area of the region to the left of $x=y^2$ and to the right of $x=y^3$. Δ is a partition of the interval $[0,1]$ on the y axis.

$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(w_i^{\,2}-w_i^{\,3})\Delta_i y=\int_0^1(y^2-y^3)dy=\tfrac{1}{3}y^3-\tfrac{1}{4}y^4\Big|_0^1=\tfrac{1}{3}-\tfrac{1}{4}=\tfrac{1}{12}$$

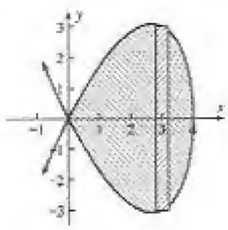
78. A square units is the area of the region below $y=\sin 2x$ and above $y=\sin x$ in $[0,\frac{1}{3}\pi]$.

$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\sin 2w_i-\sin w_i)\Delta_i x=\int_0^{\pi/3}(\sin 2x-\sin x)dx=-\tfrac{1}{2}\cos 2x+\cos x\Big]_0^{\pi/3}=\tfrac{1}{4}$$

79. A square units is the area of the region below $y=\sin x$ and above $y=\cos x$ in $[\frac{1}{4}\pi,\frac{5}{4}\pi]$.

$$A=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\sin w_i-\cos w_i)\Delta_i x=\int_{\pi/4}^{5\pi/4}(\sin x-\cos x)dx=-\cos x-\sin x\Big|_{\pi/4}^{5\pi/4}=\sqrt{2}-(-\sqrt{2})=2\sqrt{2}$$

80. The region bounded by the loop of the curve $y^2=x^2(4-x)$.



▸ The figure shows a sketch of the curve. (Parametric equations for plotting are $x=4(1-t^2)$, $y=8(1-t^2)t$.) Solving the given equation for y, we obtain $y=\pm x\sqrt{4-x}$. Let $f(x)=x\sqrt{4-x}$. By symmetry, the area of the region R bounded by the loop is twice the area bounded above by the curve $y=f(x)$, below by the x axis, on the left by $x=0$, and on the right by $x=4$. The elements of area are vertical rectangles of width $\Delta_i x$ and height $f(w_i)$. If A square units is the area of R, then

$$A=2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}f(w_i)\,\Delta_i x=2\int_0^4 x\sqrt{4-x}\,dx$$

Let $u=4-x$ and $du=-dx$. When $x=0$, $u=4$; when $x=4$, $u=0$. Thus,

$$A = 2\int_4^0 (4-u)\sqrt{u}(-du) = 2\int_0^4 (4u^{1/2} - u^{3/2})\,du = 2\left[\tfrac{8}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right]_0^4 = 2[\tfrac{8}{3}(8) - \tfrac{2}{5}(32)] = \tfrac{256}{15}$$

■ The area of the loop is $\frac{256}{15}$ square units.

81. $\sec^2 x - 2\tan^2 x = \sec^2 x - 2(\sec^2 x - 1) = 2 - \sec^2 x$ is nonnegative in $[0, \frac{1}{4}\pi]$. A square units is the area of the region below $y = \sec^2 x$ and above $y = 2\tan^2 x$ in $[0, \frac{1}{4}\pi]$. Then

$$A = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (2 - \sec^2 w_i)\Delta_i x = \int_0^{\pi/4} (2 - \sec^2 x)dx = 2x - \tan x\Big|_0^{\pi/4} = \tfrac{1}{2}\pi - 1.$$

82. At t hours after midnight the Fahrenheit temperature is $f(t)$ degrees where $f(t) = 60 - 15\sin\frac{1}{12}\pi(8-t)$, $0 \le t \le 24$. (b) At 12 midnight, $t = 0$ and $f(0) = 60 - 15(\frac{1}{2}\sqrt{3}) \approx 47.0$. (c) At 8 A.M., $t = 8$ and $f(8) = 60$. (d) At 12 noon, $t = 12$ and $f(12) = 60 + 15(\frac{1}{2}\sqrt{3}) \approx 73.0$. (e) At 2 P.M., $t = 14$ and $f(14) = 60 + 15(1) = 75$. (f) At 6 P.M., $t = 18$ and $f(18) = 60 + 15(\frac{1}{2}) = 67.5$ (g) The average temperature between 8 A.M. and 6 P.M. is

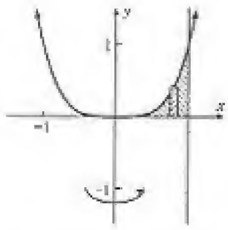
$$\frac{1}{18-8}\int_0^{18} [60 - 15\sin\tfrac{1}{12}\pi(8-t)dt = \tfrac{1}{10}\left[60t - \frac{180}{\pi}\cos\tfrac{1}{12}\pi(8-t)\right]_0^{18}$$

$$= \tfrac{1}{10}[60(18-8) - \tfrac{180}{\pi}(-\tfrac{1}{2}\sqrt{3} - 1)]60 + \tfrac{9}{\pi}(\sqrt{3} + 2) \approx 70.7$$

▸ In Exercises 83–98, V cubic units is the required volume and Δ is a partition of the specified interval.

83. The region is bounded by $y = x^4$, $x = 1$, and the x axis. An element of volume is a circular disk centered on the x axis, $x \in [0, 1]$, of radius w_i^4.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(w_i^4)^2\Delta_i x = \pi\int_0^1 x^8 dx = \pi\left[\tfrac{1}{9}x^9\right]_0^1 = \tfrac{1}{9}\pi$$



84. Find the volume of the solid generated if the region bounded by $y = x^4$, $x = 1$, and the x axis is revolved about the y axis.

▸ The region is shown at the right. A rectangular element is parallel to the y axis. An element of volume is a cylindrical shell centered on the y axis of mean radius m_i and height m_i^4, $x \in [0, 1]$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i \cdot m_i^4\Delta_i x = 2\pi\int_0^1 x^5 dx = 2\pi\cdot\tfrac{1}{6}x^6 = \tfrac{1}{3}\pi$$

■ The volume of the solid of revolution is $\frac{1}{3}\pi$ cubic units.

85. The region is bounded by $y = \sqrt{\sin x}$, the x axis, and $x = \frac{1}{2}\pi$. An element of volume is a circular disk centered on the x axis, $x \in [0, \frac{1}{2}\pi]$, of radius $\sqrt{\sin w_i}$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(\sqrt{\sin w_i})^2\Delta_i x = \pi\int_0^{\pi/2} \sin x\,dx = \pi\left[-\cos x\right]_0^{\pi/2} = \pi[0 - (-1)] = \pi$$

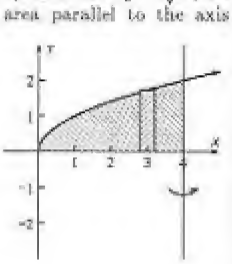
86. The region is bounded by the curve $x = \sqrt{\cos y}$, the line $y = \frac{1}{6}\pi$, and the y axis, where $\frac{1}{6}\pi \le y \le \frac{1}{2}\pi$, is revolved about the y axis. An element of volume is a circular disk of radius $\sqrt{\cos w_i}$ units, $y \in [\frac{1}{6}\pi, \frac{1}{2}\pi]$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(\sqrt{\cos w_i})^2\Delta_i y = \pi\int_{\pi/6}^{\pi/2} \cos y\,dy = \pi\sin y\Big]_{\pi/6}^{\pi/2} = \pi(\sin\tfrac{1}{2}\pi - \sin\tfrac{1}{6}\pi) = \pi(1 - \tfrac{1}{2}) = \tfrac{1}{2}\pi$$

87. The region is bounded by $y = \csc x$, the x axis, and $x = \frac{1}{4}\pi$ and $x = \frac{1}{2}\pi$. An element of volume is a circular disk centered on the x axis, $x \in [\frac{1}{4}\pi, \frac{1}{2}\pi]$, of radius $\csc w_i$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(\csc^2 w_i)\Delta_i x = \pi\int_{\pi/4}^{\pi/2} \csc^2 x\,dx = \pi\left[-\cot x\right]_{\pi/4}^{\pi/2} = \pi[0 - (-1)] = \pi$$

88. Find the volume of the solid of revolution generated when the region bounded by the curve $y = \sqrt{x}$, the x axis, and the line $x = 4$ is revolved about the line $x = 4$. Take elements of area parallel to the axis of revolution.



▸ The figure shows the region and a plane section of the solid of revolution. A rectangular element is parallel to the line $x = 4$. An element of volume is a cylindrical shell centered on $x = 4$ of mean radius $4 - m_i$ and height $\sqrt{m_i}$, $x \in [0, 4]$. If V cubic units is the volume of the solid, then

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi(4 - m_i)\sqrt{m_i}\Delta_i x = 2\pi\int_0^4 (4 - x)\sqrt{x}\,dx$$

$$= 2\pi\int_0^4 (4x^{1/2} - x^{3/2})\,dx = 2\pi\left[\tfrac{8}{3}x^{3/2} - \tfrac{2}{5}x^{5/2}\right]_0^4 = 2\pi[\tfrac{8}{3}(8) - \tfrac{2}{5}(32)] = \tfrac{256}{15}\pi$$

■ The volume of the solid of revolution is $\frac{256}{15}\pi$ cubic units.

89. The region is bounded by $x = y^2 + 2$ and $x = y + 8$. $(y+8) - (y^2+2) = -(y^2 - y - 6) = -(y-3)(y+2)$ is nonnegative in $[-2, 3]$. An element of volume is a circular ring of radii $w_i + 8$ and $w_i^2 + 2$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [\pi(w_i + 8)^2 - \pi(w_i^2 + 2)]\Delta_i y = \pi \int_{-2}^{3} [(y^2 + 16y + 64) - (y^4 + 4y^2 + 4)]dy$$

$$= \pi \int_{-2}^{3} (-y^4 - 3y^2 + 16y + 60)dy = \pi\left[-\tfrac{1}{5}y^5 - y^3 + 8y^2 + 60y\right]_{-2}^{3}$$

$$= \pi\left[(-\tfrac{243}{5} - 27 + 72 + 180) - (\tfrac{32}{5} + 8 + 32 - 120)\right] = 250\pi$$

90. The region is in the first quadrant bounded by $x = y^2$ and $x = y^4$. An element of volume is circular ring of radii w_i^2 and w_i^4, $y \in [0, 1]$.

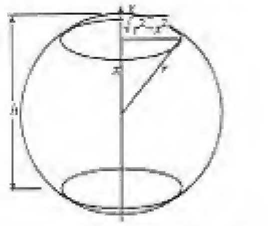
$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[(w_i^2)^2 - (w_i^4)^2]\Delta_i y = \pi \int_{0}^{1} (y^4 - y^8)dy = \pi\left[\tfrac{1}{5}y^5 - \tfrac{1}{9}y^9\right]_0^1 = \tfrac{4}{45}\pi$$

91. The base of the solid is the region in the xy plane bounded by $y^2 = 8x$ and $x = 8$. Let $f(x) = y = \sqrt{8x}$, $x \in [0, 8]$. An element of volume is a right cylinder of altitude $\Delta_i x$ units whose base is a square of side length $2\sqrt{8w_i}$ units and hence of area $32w_i$ square units.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (32w_i)\Delta_i x = 32 \int_{0}^{8} x\,dx = 16x^2\Big|_0^8 = 1024$$

92. Find the volume of the portion (segment) of a sphere of radius r units cut off by a plane h units from a pole.

▷ We apply Definition 4.9.1. The figure shows the sphere with the origin at the center of the sphere and the x axis passing through the pole. If $0 \le h \le 2r$, then by the Pythagorean theorem a plane section of the sphere perpendicular to the x axis at $x \in [r - h, r]$ is a circle of radius $\sqrt{r^2 - x^2}$, and area $A(x) = \pi(r^2 - x^2)$. Therefore



$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} A(w_i)\Delta_i x = \pi \int_{r-h}^{r} (r^2 - y^2)\,dy = \pi\left[r^2 y - \tfrac{1}{3}y^3\right]_{r-h}^{r}$$

$$= \pi\{r^2 h - \tfrac{1}{3}[r^3 - (r-h)^3]\} = \tfrac{1}{3}\pi[3r^2 h - (3r^2 h - 3rh^2 + h^3)]$$

$$= \tfrac{1}{3}\pi h^2(3r - h)$$

The volume of the solid is $\frac{1}{3}\pi h^2(3r - h)$ cubic units. Note that if $h = 2r$, the volume is $\frac{1}{3}\pi(2r)^2 r = \frac{4}{3}\pi r^3$, as expected.

93. The region is bounded by $y = |x - 2|$, the x axis, and $x = 1$ and $x = 4$. An element of volume is a circular disk centered on the x axis, $x \in [1, 4]$, of radius $|w_i - 2|$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi|w_i - 2|^2\Delta_i x = \pi \int_{0}^{4} (x-2)^2 dx = \tfrac{1}{3}\pi(x-2)^3\Big|_0^4 = \tfrac{1}{3}\pi[8 - (-1)] = 3\pi$$

94. The base of the solid is the region in the xy plane bounded by $x^2 + y^2 = r^2$. An element of volume is a right cylinder of altitude $\Delta_i x$ units whose base is a square of diagonal length $d = 2\sqrt{r^2 - w_i^2}$, $x \in [-r, r]$, units and hence of area $\frac{1}{2}d^2 = 2(r^2 - w_i)^2$ square units.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2(r^2 - w_i^2)\Delta_i x = 2\int_{-r}^{r} (r^2 - x^2)dx = 2\left[r^2 x - \tfrac{1}{3}x^3\right]_{-r}^{r} = 2 \cdot \tfrac{4}{3}r^3 = \tfrac{8}{3}r^3$$

95. The curve $y^2 + x = 0$ intersects the line $2y = x + 3$ at the points $(-9, 3)$ and $(-1, 1)$. The curve $y^2 - 4x = 0$ intersects the line $2y = x + 3$ at the points $(1, 2)$ and $(9, 6)$. Let V_1 cubic units be the volume of the solid generated by revolving about the line $y = -1$ the region below the line $y = \frac{1}{2}x + \frac{3}{2}$ and above the curve $y = (-x)^{1/2}$, $x \in [-1, 0]$. An element of volume is a circular ring of radii $(\frac{1}{2}w_i + \frac{3}{2}) + 1$ and $(-w_i)^{1/2} + 1$.

$$V_1 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left\{\pi\left[(\tfrac{1}{2}w_i + \tfrac{3}{2}) + 1\right]^2 - \pi[(-w_i)^{1/2} + 1]^2\right\}\Delta_i x = \pi \int_{-1}^{0} \left\{\left[\tfrac{1}{2}x + \tfrac{5}{2}\right]^2 - [(-x)^{1/2} + 1]^2\right\}dx$$

$$= \pi \int_{-1}^{0} \left[\tfrac{1}{4}(x+5)^2 + x - 2(-x)^{1/2} - 1\right]dx = \pi\left[\tfrac{1}{12}(x+5)^3 + \tfrac{1}{2}x^2 + \tfrac{4}{3}(-x)^{3/2} - x\right]_{-1}^{0}$$

$$= \pi\left[\tfrac{125}{12} - (\tfrac{64}{12} + \tfrac{1}{2} + \tfrac{4}{3} + 1)\right]_{-1}^{0} = \tfrac{9}{4}\pi$$

Let V_2 cubic units be the volume of the solid generated by revolving about the line $y = -1$ the region below the line $y = \frac{1}{2}x + \frac{3}{2}$ and above the curve $y = 2x^{1/2}$, $x \in [0, 1]$. An element of volume is a circular ring of radii $(\frac{1}{2}w_i + \frac{3}{2}) + 1$ and $2w_i^{1/2} + 1$.

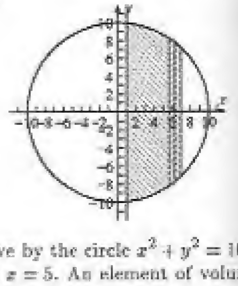
$$V_2 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left\{\pi\left[(\tfrac{1}{2}w_i + \tfrac{5}{2}) + 1\right]^2 - \pi[2w_i^{1/2} + 1]^2\right\}\Delta_i x = \pi\int_0^1 \left[(\tfrac{1}{2}x + \tfrac{5}{2})^2 - (2x^{1/2} + 1)^2\right]dx$$

$$= \pi\int_0^1 \left[\tfrac{1}{4}(x+5)^2 - 4x - 4x^{1/2} - 1\right]dx = \pi\left[\tfrac{1}{12}(x+5)^3 - 2x^2 - \tfrac{8}{3}x^{3/2} - x\right]_0^1 = \pi[(\tfrac{216}{12} - 2 - \tfrac{8}{3} - 1) - \tfrac{125}{12}] = \tfrac{23}{12}\pi$$

$$V = V_1 + V_2 = \tfrac{9}{4}\pi + \tfrac{23}{12}\pi = \tfrac{25}{6}\pi$$

96. A sphere of radius 10 cm is intersected by two parallel planes on the same side of the center of the sphere. The distance from the center of the sphere to one of the planes is 1 cm and the distance between the two planes is 6 cm. Find the volume of the solid portion of the sphere between the two planes.

▹ We apply Definition 4.9.1. The figure shows the x axis taken perpendicular to the parallel planes with the origin at the center of the sphere. A plane section perpendicular to the x axis at $x \in [1,7]$, is a circle of radius $\sqrt{100 - x^2}$ and area $A(x) = \pi(100 - x^2)$. Therefore,



$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} A(w_i)\Delta_i x = \pi\int_1^7 (100 - x^2)\,dx = \pi\left[100x - \tfrac{1}{3}x^3\right]_1^7$$

$$= \pi[100(6) - \tfrac{1}{3}(343 - 1)] = 486\pi$$

▪ The volume of the solid is 486π cubic centimeters.

97. The solid can be obtained by revolving about the x axis the region bounded above by the circle $x^2 + y^2 = 100$, below by the x axis, on the left by the line $x = -1$, and on the right by the line $x = 5$. An element of volume is a circular disk, centered on the x axis, $x \in [-1, 5]$, of radius $\sqrt{100 - w_i^2}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(100 - w_i^2)\Delta_i x = \pi\int_{-1}^{5} (100 - x^2)dx = \pi\left[100x - \tfrac{1}{3}x^3\right]_{-1}^{5} = \pi\left[(500 - \tfrac{125}{3}) - (-100 + \tfrac{1}{3})\right] = 558\pi$$

98. The region is bounded by $y = x^{1/3}$, the x axis, $x = c$. The rectangular elements are vertical, $x \in [0, c]$. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude $m_i^{1/3}$.

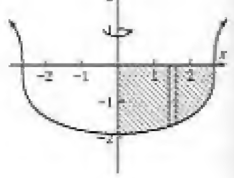
$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i \cdot m_i^{1/3})\Delta_i x = 2\pi\int_0^c x^{4/3}dx = 2\pi \cdot \tfrac{3}{7}x^{7/3}\Big]_0^c = \tfrac{6}{7}\pi c^{7/3} = 12\pi \text{ when } c^{7/3} = 14;\ c = 14^{3/7}$$

In Exercises 99–106, approximate to 4 digits the volume of the solid generated by revolving the region about the indicated axes. Take the rectangular elements perpendicular or parallel to the axis of revolution as indicated.

In Exercises 99–102, the region is bounded by $y = \sqrt[3]{x^2 - 7}$ and the x axis and is symmetrical about the y axis.

99. about the x axis; elements perpendicular.

▹ $V = 2\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi y(w_i)^2\Delta_i x = 2\pi\int_0^{\sqrt{7}} (x^2 - 7)^{2/3}dx = 44.96498 \approx 44.96$



100. about the y axis; elements parallel.

▹ The figure shows the region and a rectangular element. Because the axis of revolution is an axis of symmetry, we revolve only the part in the fourth quadrant. An element of volume is a cylindrical shell of mean radius m_i and height $-y(m_i)$. We are able find the exact volume.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i[-y(m_i)]\Delta_i x = 2\pi\int_0^{\sqrt{7}} x(7 - x^2)^{1/3}dx$$

$$= -\pi\int_0^{\sqrt{7}} (7 - x^2)^{1/3}(-2x\,dx) = -\tfrac{3}{4}\pi(7 - x^2)^{4/3}\Big]_0^{\sqrt{7}} = \tfrac{3}{4}\pi \cdot 7^{4/3} = \tfrac{21}{4}\sqrt[3]{7}\pi \approx 31.55$$

▪ The volume of the solid is 31.55 cubic centimeters.

101. about the x axis; elements parallel.

▹ $V = 2\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i x(m_i)\Delta_i y = 4\pi\int_{-7^{1/3}}^{0} y\sqrt{y^3 + 7}\,dy \approx 44.96$

102. about y axis; elements perpendicular.

▹ $V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi x(w_i)^2\Delta_i y = \pi\int_{-7^{1/3}}^{0} (y^3 + 7)dy = \pi\left[\tfrac{1}{4}y^4 + 7y\right]_{-7^{1/3}}^{0} = \pi[-\tfrac{1}{4} \cdot 7^{4/3} + 7^{4/3}] = \tfrac{21}{4}\sqrt[3]{7}\pi$

103. The region bounded by $y = \cos x^2$, $y = x^3$, y axis; about the y axis; elements parallel.

▹ $\cos x^2 \ge x^3$ on $[0, b]$, $b \approx 0.88928$

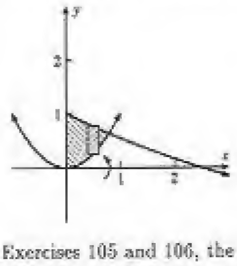
$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i(\cos m_i^2 - m_i^3)\Delta_i x = 2\pi\int_0^b x(\cos x^2 - x^3)dx \approx 1.5346 \approx 1.535$$

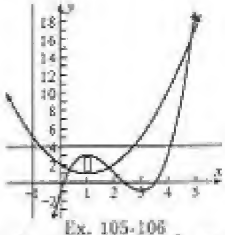
104. The region bounded by the graphs of $y = \cos\sqrt{x}$ and $y = x^2$, and the y axis; about the x axis; elements perpendicular.

▷ The figure shows the region and a rectangular element. Using zoom, we find $\cos\sqrt{x} \geq x^2$ on $[0, b]$, $b \approx 0.79308$. An element of volume is washer of radii $\cos\sqrt{w_i}$ and w_i^2.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[(\cos\sqrt{w_i})^2 - (w_i^2)^2]\Delta_i x = \pi\int_0^b (\cos^2\sqrt{x} - x^4)dx \approx 0.4671$$

▪ The volume of the solid is 0.4671 cubic units.





Ex. 105-106

In Exercises 105 and 106, the region bounded by $y_1 = x^3 - 6x^2 + 9x - 1$, $y_2 = x^2 - 2x + 2$, not met by $y = 4$;

▷ The figure shows two regions determined by the curves and a rectangular element in the one not met by $y = 4$. $y_1 \geq y_2$ on $[a, b]$ where $a \approx 0.34456$, $b \approx 1.78924$

105. about $y = 4$; elements perpendicular

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\{[4 - y_2(w_i)]^2 - [4 - y_1(w_i)]\}\Delta_i x = \pi\int_a^b \{[4 - y_2(x)]^2 - [4 - y_1(x)]^2\}dx \approx 25.166 \approx 25.17$$

106. about $x = -1$; elements parallel

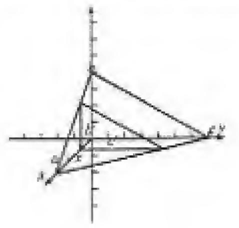
$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi(1 + m_i)[y_1(m_i) - y_2(m_i)]\Delta_i x = 2\pi\int_a^b (1 + x)[y_1(x) - y_2(x)]dx \approx 24.466 \approx 24.47$$

107. Choose the positive x axis downward with the origin at the top of the steeple. An element of volume is a right cylinder of altitude $\Delta_i x$ ft whose base is a square of side $\frac{1}{10} w_i$ ft and area $\frac{1}{100} w_i^2$ ft^2, $x \in [0, 30]$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{100} w_i^2 \Delta_i x = \frac{1}{100}\int_0^{30} x^2 dx = \frac{1}{300}x^3\Big|_0^{30} = 90$$

108. Find by slicing the volume of a tetrahedron having three mutually perpendicular faces and three mutually perpendicular edges whose lengths are a, b, and c units.

▷ The figure shows the tetrahedron and the x axis along the a unit edge. The plane section perpendicular to the x axis at x units from the vertex at the lower left, $x \in [0, a]$ is a right triangle of leg lengths b' and c'. By similar triangles,



$$\frac{b'}{b} = \frac{x}{a} \quad \text{and} \quad \frac{c'}{c} = \frac{x}{a}$$

Thus the leg lengths are $b' = \frac{b}{a}x$ and $c' = \frac{c}{a}x$ and the area measure is

$$A(x) = \frac{1}{2}b'c' = \frac{1}{2}\left(\frac{b}{a}x\right)\left(\frac{c}{a}x\right) = \frac{bc}{2a^2}x^2$$

By Definition 4.9.1,

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} A(w_i)\Delta_i x = \int_0^a \frac{bc}{2a^2}x^2\, dx = \frac{bc}{6a^2}x^3\Big]_0^a = \frac{bc}{6a^2}a^3 = \frac{1}{6}abc$$

▪ The volume of the tetrahedron is $\frac{1}{6}abc$ cubic units.

109. We have A(−4, 4), B(−2, 0), C(0, 8), D(2, 0), E(4, 4). Method 1. The rectangular elements are horizontal and run from AB to ED, $y \in [0, 4]$, and from AC to EC, $y \in [4, 8]$. An element of volume is a cylindrical shell centered on the x axis of mean radius m_i and altitude $2(2 + \frac{1}{2}m_i)$, $y \in [0, 4]$, and altitude $2(8 - m_i)$, $y \in [4, 8]$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i(4 + m_i)\Delta_i y + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi m_i(16 - 2m_i)\Delta_i y$$

$$= 2\pi\int_0^4 y(4 + y)dy + 2\pi\int_4^8 y(16 - 2y)dy = 2\pi\int_0^4 (4y + y^2)dy + 2\pi\int_4^8 (16y - 2y^2)dy$$

$= 2\pi\left[2y^2 + \frac{1}{3}y^3\right]_0^4 + 2\pi\left[8y^2 - \frac{2}{3}y^3\right]_4^8 = 2\pi \cdot \frac{160}{3} + 2\pi \cdot \frac{256}{3} = \frac{832}{3}\pi$

Method 2. Triangle CEO has area $\frac{1}{2}(4)8 = 16$, $\bar{y} = \frac{1}{3}(8+4+0) = 4$; triangle DEO has area $\frac{1}{2}(2) = 4$ and $\bar{y} = \frac{1}{3}(4+0+0)\frac{4}{3}$. By symmetry and the Theorem of Pappus (Theorem 6.3.4)

$V = 2(V_{CEO} + V_{DEO}) = 2[16 \cdot 2\pi(4) + 4 \cdot 2\pi(\frac{4}{3})] = \frac{832}{3}\pi$

110. The region is bounded above by $\tan x$ on $[0, \frac{1}{4}\pi]$ and by $\cot x$ on $[\frac{1}{4}\pi, \frac{1}{2}\pi]$. Using perpendicular elements,

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi \tan^2 w_i \Delta_i x + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi \cot^2 w_i \Delta_i x = \int_0^{\pi/4} \tan^2 x\, dx + \int_{\pi/4}^{\pi/2} \cot^2 x\, dx$$

$$= \int_0^{\pi/4} (\sec^2 x - 1)dx + \int_{\pi/4}^{\pi/2} (\csc^2 x - 1)dx = \tan x - x\Big]_0^{\pi/4} - \cot x - x\Big]_{\pi/4}^{\pi/2} = 2 - \frac{1}{2}\pi$$

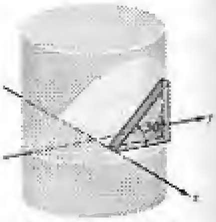
111. The region is bounded by $y = \sin x$, $y = 1$, and the y axis. An element of volume is a circular ring centered on the x axis, $x \in [0, \frac{1}{2}\pi]$, of radii 1 and $\sin w_i$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (\pi \cdot 1^2 - \pi \cdot \sin^2 w_i)\Delta_i x = \pi \int_0^{\pi/2} (1 - \sin^2 x)dx = \pi \int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2x)dx = \pi\left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$= \pi(\tfrac{1}{4}\pi) = \tfrac{1}{4}\pi^2$$

112. A wedge is cut from a right-circular cylinder with a radius of r units by two planes, one perpendicular to the axis of the cylinder and the other intersecting the first alond a diameter of the circular plane section at an angle of 30°. Find the volume of the wedge.

▶ Take the xy plane in the base of the cylinder and the origin at the center of the circular base. Take the plane, that forms the wedge, through the diameter along the x axis. The base of the wedge is the region enclosed by the top half of the circle $x^2 + y^2 = r^2$ and the x axis for x in $[-r, r]$. Let $f(x) = y = \sqrt{r^2 - x^2}$. We obtain half the wedge if x is in $[0, r]$. An element of volume is a right cylinder having altitude $\Delta_i x$ units and whose base is a right triangle of measures $\sqrt{r^2 - w_i^2}$ and $\sqrt{r^2 - w_i^2}\tan 30° = \frac{1}{3}\sqrt{3}\sqrt{r^2 - w_i^2}$ and hence area $\frac{1}{3}\sqrt{3}(r^2 - w_i^2)$



$$V = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{3}\sqrt{3}(r^2 - w_i^2)\Delta_i x = \tfrac{2}{3}\sqrt{3}\int_0^r (r^2 - x^2)dx = \tfrac{2}{3}\sqrt{3}\left[r^2 x - \tfrac{1}{3}x^3\right]_0^r = \tfrac{2}{3}\sqrt{3}(r^3 - \tfrac{1}{3}r^3) = \tfrac{2}{9}\sqrt{3}r^3$$

In Exercises 113 and 114, apply the second fundamental theorem of the calculus to evaluate the definite integral. Then find the value of c satisfying the mean-value theorem for integrals.

113. $\int_0^3 (x^2 + 1)dx = \frac{1}{3}x^3 + x\Big|_0^3 = (9 + 3) - 0 = 12$

By the mean-value theorem for integrals there is a number c in $[0, 3]$ such that

$\int_0^3 (x^2 + 1)dx = (c^2 + 1)(3 - 0)$; $12 = 3c^2 + 3$; $c^2 = 3$; $c = \pm\sqrt{3}$

Because c is in $[0, 3]$, $c = \sqrt{3}$.

114. $\int_1^4 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\Big]_1^4 = \frac{2}{3}(8 - 1) = \frac{14}{3}$

By the mean-value theorem for integrals there is a number c in $[1, 4]$ such that

$\int_1^4 \sqrt{x}\, dx = \sqrt{c}(4 - 1)$; $\frac{14}{3} = 3\sqrt{c}$; $\sqrt{c} = \frac{14}{9}$; $c = \frac{196}{81}$ and $1 < \frac{196}{81} < 4$

115. Let $F(x) = \int_a^x f(t)dt \Big/ \int_a^b f(t)dt$

Then $F(a) = 0$ and $F(b) = 1$. Because f is continuous on [a,b] it follows from the first fundamental theorem that F is differentiable on [a,b] and hence continuous on [a,b]. Therefore, by the intermediate-value theorem for any number k in$(0, 1)$ there is a number c in(a,b) such that $F(c) = k$. From(1) we get

$k = \int_a^c f(t) \Big/ \int_a^b f(t)dt$; $\int_a^c f(t)dt = k\int_a^b f(t)dt$

116. Given $F(x) = \int_x^{2x} \frac{1}{t}\,dt$ and $x > 0$. Prove that F is a constant by showing that $F'(x) = 0$. (*Hint*: Use the first fundamental theorem of the calculus after writing the given integral as the difference of two integrals.)

▶ By Theorem 4.5.13, for any positive constant a we have

$$F(x) = \int_x^{2x} \frac{1}{t}\,dt = \int_x^{a} \frac{1}{t}\,dt + \int_a^{2x} \frac{1}{t}\,dt = \int_a^{2x} \frac{1}{t}\,dt - \int_a^{x} \frac{1}{t}\,dt$$

Let $u = 2x$. By Theorem 4.7.1, it follows that for all $x > 0$

$$F'(x) = \frac{d}{du}\int_a^{u} \frac{1}{t}\,dt \cdot \frac{du}{dt} - \frac{d}{dx}\int_a^{x} \frac{1}{t}\,dt = \frac{1}{u}\cdot 2 - \frac{1}{x} = \frac{1}{2x}\cdot 2 - \frac{1}{x} = 0$$

Therefore, by Theorem 3.3.3, $F(x)$ is a constant on the interval $(0, +\infty)$.

117. We are given $f(x) = x + |x - 1|$ and $F(x) = \begin{cases} x & \text{if } x < 1 \\ x^2 - x + 1 & \text{if } x \geq 1 \end{cases}$. We wish to show that $F'(x) = f(x)$ for all x.

$F'(x) = 1$ if $x < 1$ and $F'(x) = 2x - 1$ if $x > 1$.

$F'_-(1) = \lim\limits_{x \to 1^-} \frac{x - 1}{x - 1} = \lim\limits_{x \to 1^-} 1 = 1$; $F'_+(1) = \lim\limits_{x \to 1^+} \frac{(x^2 - x + 1) - 1}{x - 1} = \lim\limits_{x \to 1^+} \frac{x(x - 1)}{x - 1} = \lim\limits_{x \to 1^+} x = 1$

Therefore $F'(1) = 1$; so $F'(x) = \begin{cases} 1 & \text{if } x < 1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$

Also, $f(x) = x + |x - 1| = \begin{cases} x + (1 - x) & \text{if } x < 1 \\ x + (x - 1) & \text{if } x \geq 1 \end{cases} = \begin{cases} 1 & \text{if } x < 1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$. Hence $F'(x) = f(x)$ for all x.

118. We are given $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x, $f(0) = 0$, and $g(0) = 1$. Let $F(x) = [f(x)]^2 + [g(x)]^2$. We wish to show that $F(x) = 1$. Because $F'(x) = 2f(x)f'(x) + 2g(x)g'(x) = 2f(x)g(x) + 2g(x)[-f(x)] = 0$, then $F(x)$ is a constant. Because $F(0) = [f(0)]^2 + [g(0)]^2 = 0 + 1 = 1$, that constant is 1.

119. $$\int_0^{\pi} \left|\cos x + \tfrac{1}{2}\right| dx = \int_0^{2\pi/3} \left|\cos x + \tfrac{1}{2}\right| dx + \int_{2\pi/3}^{\pi} \left|\cos x + \tfrac{1}{2}\right| dx$$

$$= \int_0^{2\pi/3} (\cos x + \tfrac{1}{2})\,dx - \int_{2\pi/3}^{\pi} (\cos x + \tfrac{1}{2})\,dx = \left[\sin x + \tfrac{1}{2}x\right]_0^{2\pi/3} + \left[-\sin x - \tfrac{1}{2}x\right]_{2\pi/3}^{\pi}$$

$$= \tfrac{1}{2}\sqrt{3} + \tfrac{1}{3}\pi + (-\tfrac{1}{2}\pi + \tfrac{1}{2}\sqrt{3} + \tfrac{1}{3}\pi) = \sqrt{3} + \tfrac{1}{6}\pi.$$

120. Make up an example of a discontinuous function for which the conclusion of the mean-value theorem for integrals (a) is not true and (b) is true.

▶ The conclusion of the mean-value theorem is: there is a number c in $[a, b]$ such that

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\,dx$$

(a) Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

Because f is bounded and has only one point of discontinuity, f is integrable on any interval. Then

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_0^{2} f(x)\,dx = \int_{-2}^{0} 0\,dx + \int_0^{2} 1\,dx = 0 + 2 = 2$$

Therefore,

$$\frac{1}{2 - (-2)}\int_{-2}^{2} f(x)\,dx = \tfrac{1}{4}(2) = \tfrac{1}{2}$$

However, there is no c such that $f(c) = \frac{1}{2}$.

(b) Let $g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$

Changing the value of a function at a single point does not change the value of a definite integral. Hence, as in part (a),

$$\frac{1}{2 - (-2)}\int_{-2}^{2} g(x)\,dx = \tfrac{1}{4}(2) = \tfrac{1}{2}$$

Furthermore, if $c = 0$, then $g(c) = \frac{1}{2}$, and so the conclusion is true.

F I V E

LOGARITHMIC, EXPONENTIAL, INVERSE TRIGONOMETRIC, AND HYPERBOLIC FUNCTIONS

5.1 THE INVERSE OF A FUNCTION

5.1.1 Definition A function f is said to be one-to-one if every number in its range corresponds to exactly one number in its domain; that is, for all x_1 and x_2 in the domain of f

if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

$\Leftrightarrow f(x_1) = f(x_2)$ only when $x_1 = x_2$

If $y = f(x)$ and the function f is one-to-one, then for each replacement of x from the domain of f there corresponds exactly one value of y, and for each replacement of y from the range of f there corresponds exactly one value of x.

Horizontal-Line Test For the graph of a one-to-one function each vertical line intersects the graph in at most one point, and each horizontal line intersects the graph in at most one point.

If a function is either increasing on an interval or decreasing on an interval, then it is said to be monotonic on the interval.

5.1.2 Theorem A function that is monotonic on an interval is one-to-one on the interval.

Reciprocal If f is one-to-one of $(-\infty, +\infty)$, it does not repeat any values. Hence its reciprocal does not repeat any values and is one-to-one on its domain.

5.1.3 Definition If f is a one-to-one function, then there is a function f^{-1}, called the inverse of f, such that

$x = f^{-1}(y)$ if and only if $y = f(x)$

The domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

5.1.4 Theorem If f is a one-to-one function having f^{-1} as its inverse, then f^{-1} is a one-to-one function having f as its inverse. Furthermore,

$f^{-1}(f(x)) = x$ for x in the domain of f and $f(f^{-1}(x)) = x$ for x in the domain of f^{-1}

Graphs The graphs of the functions f and f^{-1} are reflections of each other with respect to the line $y = x$, the line that bisects the first and third quadrants, and their inflection points correspond. See Exercise 67. In parametric mode we plot the function by setting $x_1(t) = t$ $y_1(t) = f(t)$ and the inverse as $x_2(t) = f(t)$, $y_2(t) = t$.

5.1.5–6 Theorem Suppose the continuous function f has the interval $\mathrm{I} = [a, b]$ as its domain. If f is

(i) increasing on I, then f has an inverse f^{-1} that is continuous and increasing on $[f(a), f(b)]$.

(ii) decreasing on I, then f has an inverse f^{-1} that is continuous and decreasing on $[f(b), f(a)]$.

If endpoint a or b is not in the domain we delete endpoint $f(a)$ or $f(b)$ from the conclusion.

5.1.7 Theorem Suppose that the function f is continuous and monotonic on the interval (a, b), and let $y = f(x)$. If f is differentiable on (a, b) and $f'(x) \neq 0$ for any x in (a, b), then the derivative of the inverse function f^{-1}, defined by $x = f^{-1}(y)$ is given by

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

5.1.8 Theorem Suppose the function f is continuous and monotonic on an open interval (a, b) containing the number c, and let $f(c) = d$. If $f'(c) \neq 0$, then $(f^{-1})'(d)$ exists and

$$(f^{-1})'(d) = \frac{1}{f'(c)}$$

Exercises 5.1

In Exercises 1–6, use the horizontal-line test to determine if the functions is 1–1. Sketch or plot its graph.

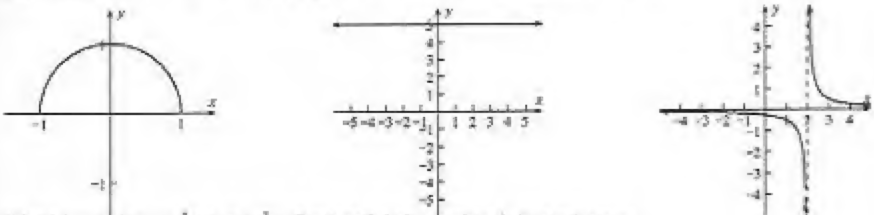
1. (a) $f(x) = 2x + 3$. Because f is increasing, f is one-to-one.
   (b) $f(x) = \frac{1}{2}x^2 - 2$. Because $f(2) = 0$ and $f(-2) = 0$, f is not one-to-one.
   (c) $g(x) = 4 - x^3$. Because g is decreasing, g is one-to-one.

2. (a) $g(x) = 8 - 4x$. Because g is decreasing, g is one-to-one.
   (b) $f(x) = 3 - x^2$. Because $f(2) = -1$ and $f(-2) = -1$, f is not one-to-one.
   (c) $h(x) = \frac{1}{2}x^3 + 1$. Because h is increasing, h is one-to-one.

3. (a) $f(x) = \sqrt{x+3}$. Because f is increasing, f is one-to-one.
   (b) $f(x) = \dfrac{2}{x+3}$. In $(-\infty, -3)$ f is decreasing and negative; in $(-3, +\infty)$ f is decreasing and positive. Hence f is one-to-one.
   (c) $g(x) = |x - 2|$. Because $g(1) = 1$ and $g(3) = 1$, g is not one-to-one.

4. (a) $f(x) = \sqrt{1 - x^2}$ (b) $g(x) = 5$ (c) $f(x) = \dfrac{1}{2x-4}$

▸ (a) Because $f(-1) = f(1) = 0$, by Definition 5.1.1 the function g is not one-to-one. The graph of g is the semicircle shown in Fig. (a). There are horizontal lines that intersect the graph in two points.
(b) Because $g(-x) = g(x) = 5$, by Definition 5.1.1 the function g is not one-to one. The graph of g is the horizontal line shown in Fig. (b).
(c) $(-\infty, 2)$ f is decreasing and negative; in $(2, +\infty)$ f is decreasing and positive. Because f does not repeat any values, f is one-to-one. The graph of f is the hyperbola shown in Fig. (c).



5. (a) $h(x) = 2\sin x$, $-\frac{1}{2}\pi \le x \le \frac{1}{2}\pi$. Because h is increasing, h is one-to-one.
   (b) $f(x) = \frac{1}{2}\tan x$, $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. Because f is increasing, f is one-to-one.
   (c) $G(x) = \sec x$, $x \in [0, \frac{1}{2}\pi) \cup [\pi, \frac{3}{2}\pi)$. In $[0, \frac{1}{2}\pi)$ G is increasing and positive; in $[\pi, \frac{3}{2}\pi)$ G is decreasing and negative. Hence G is one-to-one.

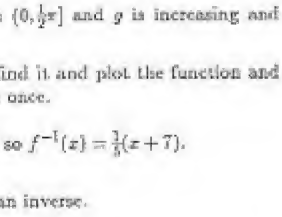
6. (a) $f(x) = 1 - \cos x$, $0 \le x \le \pi$. Because f is decreasing, f is one-to-one.
   (b) $F(x) = \cot \frac{1}{2}x, 0 < x < 2\pi$. Because F is decreasing, F is one-to-one.
   (c) $g(x) = \csc x$, $x \in (0, \frac{1}{2}\pi] \cup (\pi, \frac{3}{2}\pi]$. Because g is decreasing and positive in $(0, \frac{1}{2}\pi]$ and g is increasing and negative in $(\pi, \frac{3}{2}\pi)$, g cannot repeat any values and so g is one-to-one.

In Exercises 7–18, determine if the function has an inverse. If the inverse exists, find it and plot the function and its inverse. Otherwise, plot its graph and a horizontal line that meets it more than once.

7. (a) $f(x) = 5x - 7$. f is increasing $\Rightarrow$ f is one-to-one $\Rightarrow$ f has an inverse.
   Let $y = f(x)$. Then $y = 5x - 7$; $x = \frac{1}{5}(y + 7)$. Therefore $f^{-1}(y) = \frac{1}{5}(y + 7)$, and so $f^{-1}(x) = \frac{1}{5}(x + 7)$.
   The domain of f^{-1} is $(-\infty, +\infty)$ and the range of f^{-1} is $(-\infty, +\infty)$.
   (b) $g(x) = 1 - x^2$. $g(1) = 0 = g(-1)$ $\Rightarrow$ g is not one-to-one $\Rightarrow$ g does not have an inverse.

8. (a) $f(x) = 3x + 6$

▸ Because $f'(x) = 3 > 0$, f is increasing on $(-\infty, +\infty)$. By Theorem 5.1.2, f is therefore one-to-one and thus f has an inverse. To find the inverse of f we let $y = f(x)$ and solve the equation for x in terms of y.



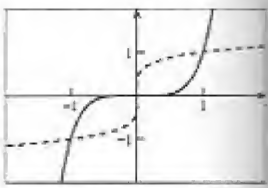
$$y = 3x + 6; \qquad y - 6 = 3x \qquad x = \frac{y-6}{3}$$

We have $\qquad f^{-1}(y) = \dfrac{y-6}{3}$

and replacing y with x we obtain $\qquad f^{-1}(x) = \dfrac{x-6}{3}$

which defines the inverse function f^{-1}. Both the domain and range of f^{-1} consist of the set of all real numbers. The plot shows the graphs of f and f^{-1} (dashed). The graphs are reflections of each other with respect to the line $y = x$.

(b) $f(x) = x^5$



▷ Because $g'(x) = 5x^4 \geq 0$, g is increasing on $(-\infty, +\infty)$. By Theorem 5.1.2, g is therefore one-to-one and thus g has an inverse. To find the inverse of g we let $y = g(x)$ and solve the equation for x in terms of y.

$$y = x^5; \qquad x = \sqrt[5]{y}$$

We have $\qquad g^{-1}(y) = \sqrt[5]{y}$

and replacing y with x we obtain $\qquad g^{-1}(x) = \sqrt[5]{x}$

which defines the inverse function g^{-1}. Both the domain and range of g^{-1} consist of the set of all real numbers. The plot shows the graphs of g and g^{-1} (dashed). Note that the graphs are reflections of each other with respect to the line $y = x$.

9. (a) $f(x) = (4-x)^3$. f is decreasing $\Rightarrow$ f is one-to-one $\Rightarrow$ f has an inverse.
Let $y = f(x)$. Then $y = (4-x)^3$; $y^{1/3} = 4 - x$; $x = 4 - \sqrt[3]{y}$.
Hence $f^{-1}(y) = 4 - \sqrt[3]{y}$, and so $f^{-1}(x) = 4 - \sqrt[3]{x}$.
The domain of f^{-1} is $(-\infty, +\infty)$ and the range of f^{-1} is $(-\infty, +\infty)$.

(b) $h(x) = \sqrt{2x-6}$. h is increasing $\Rightarrow$ h is one-to-one $\Rightarrow$ h has an inverse.
Let $y = h(x)$. Then $y = \sqrt{2}x - 6$; $y^2 = 2x - 6$; $x = \frac{1}{2}y^2 + 3$.
Therefore $h^{-1}(y) = \frac{1}{2}y^2 + 3$, and so $h^{-1}(x) = \frac{1}{2}x^2 + 3$. The domain of h^{-1} is the range of h which is $[0, +\infty)$.
The range of h^{-1} is the domain of h which is $[3, +\infty)$.

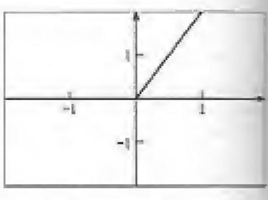
10. (a) $F(x) = 3(x^2 + 1)$
Because $F(-x) = F(x)$, the function F is not one-to-one and thus F does not have an inverse function.

(b) $g(x) = \sqrt{1-x^2}$
Because $g(-x) = g(x)$, the function g is not one-to-one and thus g does not have an inverse function.

11. (a) $F(x) = \sqrt[3]{x+1}$. F is increasing $\Rightarrow$ F is one-to-one $\Rightarrow$ F has an inverse.
Let $y = F(x)$. Then $y = \sqrt[3]{x+1}$; $y^3 = x + 1$; $x = y^3 - 1$. Therefore $F^{-1}(y) = y^3 - 1$, and so $F^{-1}(x) = x^3 - 1$.
The domain of F^{-1} is $(-\infty, +\infty)$ and the range of F^{-1} is $(-\infty, +\infty)$.

(b) $f(x) = (x+2)^4$. $f(0) = 2^4 = f(-4) \Rightarrow f$ is not one-to-one $\Rightarrow f$ does not have an inverse.

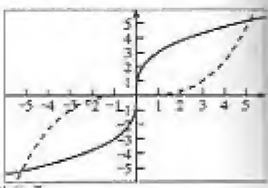
12. (a) $f(x) = |x| + x$



▷ If $x < 0$, then $|x| = -x$. Thus, when $x < 0, f(x) = -x + x = 0$, and so f is not one-to-one and f does not have an inverse function. The plot shows the graph of f. The x axis, which is a horizontal line, intersects the graph in an unlimited number of points.

(b) $g(x) = 3\sqrt[3]{x}$

▷ g is increasing on $(-\infty, +\infty)$. By Theorem 5.1.2, g is therefore one-to-one and thus g has an inverse. To find the inverse of g we let $y = g(x)$ and solve the equation for x in terms of y.



$$y = 3\sqrt[3]{x}; \quad \tfrac{1}{3}y = \sqrt[3]{x}; \qquad x = \tfrac{1}{27}y^3$$

We have $\qquad g^{-1}(y) = \frac{1}{27}y^3$

and replacing y with x we obtain $\qquad g^{-1}(x) = \frac{1}{27}x^3$

which defines the inverse function g^{-1}. Both the domain and range of g^{-1} consist of the set of all real numbers. The plot shows the graphs of g and g^{-1} (dashed). The graphs are reflections of each other with respect to the line $y = x$.

13. (a) $f(x) = 2\sqrt[5]{x}$. f is increasing $\Rightarrow$ f is one-to-one $\Rightarrow$ f has an inverse.
Let $y = f(x)$. Then $y = 2\sqrt[5]{x}$; $y^5 = 32x$; $x = \frac{1}{32}y^5$. Therefore $f^{-1}(y) = \frac{1}{32}y^5$, and so $f^{-1}(x) = \frac{1}{32}x^5$.
The domain of f^{-1} is $(-\infty, +\infty)$ and the range of f^{-1} is $(-\infty, +\infty)$.

(b) $f(x) = \frac{x-3}{x+1} = 1 - \frac{4}{x+1}$. In $(-\infty, -1)$ f is increasing and $f(x) > 1$; in $(-1, +\infty)$ f is increasing and $f(x) < 1$. Therefore f is one-to-one, and so f has an inverse.
Let $y = f(x)$. Then $y = \frac{x-3}{x+1}$; $xy + y = x - 3$; $x - xy = y + 3$; $x(1-y) = y + 3$; $x = \frac{y+3}{1-y}$.
Thus $f^{-1}(y) = \frac{y+3}{1-y}$, and so $f^{-1}(x) = \frac{x+3}{1-x}$. The domain of f^{-1} is $\{x \mid x \neq 1\}$.
The range of f^{-1} is the domain of f which is $\{y \mid y \neq -1\}$.

14. (a) $f(x) = \frac{2x-1}{x} = 2 - \frac{1}{x}$. In $(-\infty, 0)$ f increasing and $f(x) > 2$; In $(0, +\infty)$ f is increasing and $f(x) < 2$.
Therefore f is one-to-one and has an inverse. Let $y = f(x)$.

$$y = \frac{2x-1}{x}; \quad xy = 2x - 1; \quad xy - 2x = -1; \quad x(y-2) = -1; \quad x = \frac{-1}{y-2}$$

Thus, $\quad f^{-1}(y) = \frac{-1}{y-2}$ and $f^{-1}(y) = \frac{-1}{x-2}$

The domain of f^{-1} is $\{x \mid x \neq 2\}$. Because the domain of f is also the range of f^{-1}, then the range of f^{-1} is $\{y \mid y \neq 0\}$.

(b) $g(x) = \frac{8}{x^3+1}$. Because $x^3 + 1$ is an increasing function on $(-\infty, +\infty)$ and hence one-to-one, its reciprocal g is one-to-one and has an inverse. Let $y = g(x)$. $y = \frac{8}{x^3+1}$; $x^3 + 1 = \frac{8}{y}$; $x^3 = \frac{8}{y} - 1 = \frac{8-y}{y}$; $x = \sqrt[3]{\frac{8-y}{y}}$

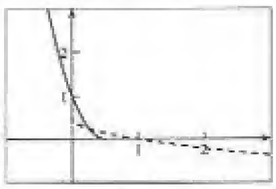
Thus $g^{-1}(y) = \sqrt[3]{\frac{8-y}{y}}$ and $g^{-1}(x) = \sqrt[3]{\frac{8-x}{x}}$.

The domain of g^{-1} is $\{x \mid x \neq 0\}$ and the range of g^{-1} is the domain of g which is $\{x \mid x \neq -1\}$.

15. (a) $g(x) = x^2 + 5$, $x \geq 0$. g is increasing $\Rightarrow$ g is one-to-one $\Rightarrow$ g has an inverse.
Let $y = g(x)$ with $x \geq 0$. Then $y = x^2 + 5$; $x^2 = y - 5$; $x = \sqrt{y-5}$, since $x \geq 0$.
Therefore $g^{-1}(y) = \sqrt{y-5}$, and so $g^{-1}(x) = \sqrt{x-5}$. The domain of g^{-1} is the range of g which is $[5, +\infty)$.
The range of g^{-1} is the domain of g which is $[0, +\infty)$.

(b) $f(x) = (2x+1)^3$, $-\frac{1}{2} \leq x \leq \frac{1}{2}$. f is increasing $\Rightarrow$ f is one-to-one $\Rightarrow$ f has an inverse.
Let $y = (2x+1)^3$ with $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Then $\sqrt[3]{y} = 2x + 1$; $x = \frac{1}{2}(\sqrt[3]{y} - 1)$.
Therefore $f^{-1}(y) = \frac{1}{2}(\sqrt[3]{y} - 1)$, and so $f^{-1}(x) = \frac{1}{2}(\sqrt[3]{x} - 1)$. The domain of f^{-1} is the range of f which is $[0, 8]$. The range of f^{-1} is the domain of f which is $[-\frac{1}{2}, \frac{1}{2}]$.

16. (a) $f(x) = (2x-1)^2$, $x \leq \frac{1}{2}$
▸ The domain of f is given and is $(-\infty, \frac{1}{2}]$. Because $f'(x) = (2x-1)$ and $2x - 1 \leq 0$ if $x \leq \frac{1}{2}$, then f is decreasing on $(-\infty, \frac{1}{2}]$. Thus, f is one-to-one and f has an inverse function. Let $y = f(x)$.



$$y = (2x-1)^2; \quad \sqrt{y} = \sqrt{(2x-1)^2}; \quad \sqrt{y} = |2x - 1|$$

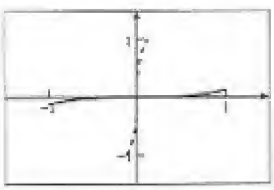
Because $2x - 1 \leq 0$, then $|2x - 1| = -(2x - 1)$. Thus, we have

$$\sqrt{y} = -(2x-1); \quad -\sqrt{y} = 2x - 1; \quad x = \tfrac{1}{2}(1 - \sqrt{y})$$

Hence $\quad f^{-1}(y) = \frac{1}{2}(1 - \sqrt{y})$ and $\quad f^{-1}(x) = \frac{1}{2}(1 - \sqrt{x})$

The domain of f^{-1} is $[0, +\infty)$, and the range of f^{-1} is $(-\infty, \frac{1}{2}]$, because $(-\infty, \frac{1}{2}]$ is the domain of f.

(b) $f(x) = \frac{1}{8}x^3$, $-1 \leq x \leq 1$
▸ The domain of f is given and is $[-\frac{1}{2}, \frac{1}{2}]$. Because $f'(x) = \frac{3}{8}x^2 \geq 0$, then f is increasing. Thus, f is one-to-one and f has an inverse function. Let $y = f(x)$.



$$y = \tfrac{1}{8}x^3; \quad x^3 = 8y; \quad x = 2\sqrt[3]{y}$$

Hence $\quad f^{-1}(y) = 2\sqrt[3]{y}$ and $f^{-1}(x) = 2\sqrt[3]{x}$

The domain of f^{-1} is the range of f which is $[-\frac{1}{8}, \frac{1}{8}]$; the range of f^{-1} is $[-1, 1]$.

17. $F(x) = \sqrt{9-x^2}$, $0 \le x \le 3$. F is decreasing $\Rightarrow$ F is one-to-one $\Rightarrow$ F has an inverse.
Let $y = F(x)$ with $0 \le x \le 3$. Then $y = \sqrt{9-x^2}$; $y^2 = 9 - x^2$; $x^2 = 9 - y^2$; $x = \sqrt{9-x^2}$, since $x \ge 0$. Therefore $F^{-1}(y) = \sqrt{9-y^2}$, and so $F^{-1}(x) = \sqrt{9-x^2}$. The domain of F^{-1} is $[0,3]$ and the range of F^{-1} is $[0,3]$.

18. $G(x) = \sqrt{4x^2-9}$, $x \ge \frac{3}{2}$

$$G'(x) = \tfrac{1}{2}(4x^2-9)^{-1/2}(8x) = \frac{4x}{\sqrt{4x^2-9}}$$

Because $G'(x) > 0$ if $x > \frac{3}{2}$, then G is increasing on $[\frac{3}{2}, +\infty)$. Hence, G is one-to-one and G has an inverse function. Let $y = G(x)$.

$$y = \sqrt{4x^2-9}; \qquad y^2 = 4x^2 - 9; \qquad x^2 = \tfrac{1}{4}(y^2+9); \qquad |x| = \tfrac{1}{2}\sqrt{y^2+9}$$

Because $x \ge \frac{3}{2}$, then $|x| = x$. Thus, $x = \frac{1}{2}\sqrt{y^2+9}$. Hence $G^{-1}(y) = \frac{1}{2}\sqrt{y^2+9}$ and $G^{-1}(x) = \frac{1}{2}\sqrt{x^2+9}$.

The domain of G^{-1} is $[0, +\infty)$ which is the range of G, and the range of G^{-1} is $[\frac{3}{2}, +\infty)$, which is also the domain of G.

In Exercises 19–24 let $y = f(x)$ and $x = f^{-1}(y)$ and verify that $dx/dy = 1/(dy/dx)$.

19. (a) $f(x) = 4x - 3$. Let $y = f(x)$; then $y = 4x - 3$; $\frac{dy}{dx} = 4$ and $x = \frac{1}{4}(y+3)$; $\frac{dy}{dx} = \frac{1}{4}$. Thus $\frac{dy}{dx} = \frac{1}{dx/dy}$.

(b) $f(x) = \sqrt{x+1}$. Let $y = (x)$; then $y = \sqrt{x+1}$; $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}}$ and $y^2 = x+1$; $x = y^2 - 1$; $\frac{dy}{dx} = 2y$
Thus $\frac{1}{dy/dx} = 2\sqrt{x+1} = 2y = \frac{dx}{dy}$.

20. (a) $f(x) = 7 - 2x$.
▸ If $y = 7 - 2x$, then $\frac{dy}{dx} = -2$. Furthermore $x = \frac{1}{2}(7-y)$ and so $\frac{dx}{dy} = -\frac{1}{2} = \frac{1}{dy/dx}$.

(b) $f(x) = 8x^3$
▸ If $y = 8x^3$, then $\frac{dy}{dx} = 24x^2$. Furthermore, $x = \sqrt[3]{\frac{1}{8}y} = \frac{1}{2}y^{1/3}$. Thus $\frac{dx}{dy} = \frac{1}{6}y^{-2/3}$.

Replacing y by $8x^3$, we obtain $\frac{dx}{dy} = \frac{1}{6}(8x^3)^{-2/3} = \frac{1}{6}(\frac{1}{4}x^{-2}) = \frac{1}{24x^2} = \frac{1}{dy/dx}$.

21. (a) $f(x) = \frac{1}{5}x^5$. Let $y = f(x)$; then $y = \frac{1}{5}x^5$; $\frac{dy}{dx} = x^4$ and $x^5 = 5y$; $x = (5y)^{1/5}$; $\frac{dx}{dy} = (5y)^{-4/5}$

$$\frac{dx}{dy} = \frac{1}{(5y)^{4/5}} = \frac{1}{[(5y)^{1/5}]^4} = \frac{1}{x^4} = \frac{1}{dy/dx}$$

(b) $f(x) = \sqrt[3]{x-8}$. Let $y = f(x)$; then $y = \sqrt[3]{x-8}$; $\frac{dy}{dx} = \frac{1}{3}(x-8)^{-2/3}$ and $y^3 = x - 8$; $x = y^3 + 8$; $\frac{dx}{dy} = 3y^2$

$$\frac{1}{dy/dx} = 3(x-8)^{2/3} = 3[(x-8)^{1/3}]^2 = 3y^2 = \frac{dx}{dy}$$

22. (a) $f(x) = \sqrt{4-3x}$. If $y = (4-3x)^{1/2}$ then $\frac{dy}{dx} = \frac{1}{2}(4-3x)^{-1/2}(-3) = -\frac{3}{2}(4-3x)^{-1/2}$ and $y^2 = 4 - 3x$;
$x = \frac{1}{3}(4 - y^2)$; $\frac{dx}{dy} = -\frac{2}{3}y$. $\frac{1}{dy/dx} = -\frac{2}{3}(4-3x)^{1/2} = -\frac{2}{3}y = \frac{dx}{dy}$

(b) $f(x) = \sqrt[5]{x}$. If $y = \sqrt[5]{x}$, then $\frac{dy}{dx} = \frac{1}{5}x^{4/5} = \frac{1}{5x^{4/5}}$ and $x = y^5$; $\frac{dx}{dy} = 5y^4 = 5(\sqrt[5]{x})^4 = 5x^{4/5} = \frac{1}{dy/dx}$

23. $f(x) = \frac{2x-3}{x+2}$. Let $y = f(x)$; then $y = \frac{2x-3}{x+2}$; $\frac{dy}{dx} = \frac{2(x+2)-(2x-3)}{(x+2)^2} = \frac{7}{(x+2)^2}$

$xy + 2y = 2x - 3$; $xy - 2x = -2y - 3$; $x = \frac{2y+3}{2-y}$ $\qquad \frac{dx}{dy} = \frac{2(2-y)-(-1)(2y-3)}{(2-y)^2} = \frac{7}{(2-y)^2}$

$$\frac{1}{dy/dx} = \tfrac{1}{7}(x+2)^2 = \tfrac{1}{7}\left(\frac{2y+3}{2-y}+2\right)^2 = \tfrac{1}{7}\left(\frac{2y+3+4-2y}{2-y}\right)^2 = \tfrac{1}{7}\left(\frac{7}{2-y}\right)^2 = \frac{7}{(2-7)^2} = \frac{dx}{dy}$$

24. $f(x) = \dfrac{3x+4}{2x+6}$

▸ If $y = \dfrac{3x+4}{2x+6}$, then division gives $y = \frac{3}{2} - \dfrac{5}{2x+6}$ and the power rule gives $\dfrac{dy}{dx} = \dfrac{10}{(2x+6)^2}$. Furthermore,

$$y - \tfrac{3}{2} = -\frac{5}{2x+6}; \quad 2x+6 = -\frac{-5}{y-\frac{3}{2}}; \quad x = \frac{1}{2}\left(\frac{-5}{y-\frac{3}{2}} - 6\right); \quad \frac{dx}{dy} = \frac{-5}{2(y-\frac{3}{2})^2}$$

Replacing $y - \frac{3}{2}$ by $-\dfrac{5}{2x+6}$, we obtain $\dfrac{1}{dx/dy} = -\frac{2}{5}(y-\frac{3}{2}) = -\frac{2}{5}\left(\dfrac{-5}{2x+6}\right)^2 = \dfrac{10}{(2x+6)^2} = \dfrac{dy}{dx}$

In Exercises 25–38, find $(f^{-1})'(d)$.

25. (a) $f(x) = \sqrt{3x+1}$; $d = 1$. We wish to find the value of c for which $f(c) = 1$. $\sqrt{3c+1} = 1$; $3c+1 = 1$; $c = 0$
$f'(x) = \dfrac{3}{2\sqrt{3x+1}}$. Thus $f'(x) > 0$ if $x > -\frac{1}{3}$. Hence there is an open interval containing the number 0 on which f is continuous and increasing. Hence Theorem 5.1.8 applies and $(f^{-1})'(1) = \dfrac{1}{f'(0)} = \dfrac{1}{\frac{3}{2}} = \dfrac{2}{3}$

(b) $f(x) = x^2 - 16$, $x \geq 0$; $d = 9$. We seek the value of $c \geq 0$ for which $f(c) = 9$. $c^2 - 16 = 9$; $c^2 = 25$; $c = 5$
$f'(x) = 2x$. Thus $f'(x) > 0$ if $x > 0$. Hence there is an open interval containing 5 on which f is continuous and increasing. Thus Theorem 5.1.8 applies, and $(f^{-1})'(9) = \dfrac{1}{f'(5)} = \dfrac{1}{10}$

26. (a) $f(x) = x^5 + 2$; $d = 1$. We seek the value of c for which $f(c) = 1$. $c^5 + 2 = 1$; $c^5 = -1$; $c = -1$.
$f'(x) = 5x^4 \geq 0$ and so f is continuous and increasing on any interval. Thus Theorem 5.1.8 applies and

$$(f^{-1})'(1) = \frac{1}{f'(-1)} = \frac{1}{5(-1)^4} = \frac{1}{5}$$

(b) $f(x) = \sqrt{4-x}$; $d = 3$. We seek the value of c for which $f(c) = 3$. $\sqrt{4-c} = 3$; $4 - c = 9$; $c = -5$.
$f'(x) = \dfrac{-1}{2\sqrt{4-x}} < 0$ and so f is continuous and decreasing on $(-\infty, -4)$. Thus Theorem 5.1.8 applies, and

$$(f^{-1})'(3) = \frac{1}{f'(-5)} = \frac{-1}{2\sqrt{9}} = -\frac{1}{6}$$

27. (a) $f(x) = x^3 + 5$; $d = -3$. We wish to find the value of c for which $f(c) = -3$. $c^3 + 5 = -3$; $c^3 = -8$; $c = -2$
$f'(x) = 3x^2$. Hence $f'(x) > 0$ for all $x \neq 0$. Hence there is an open interval containing -3 on which f is continuous and increasing. Thus Theorem 5.1.8 applies and $(f^{-1})'(3) = \dfrac{1}{f'(-2)} = \dfrac{1}{3(-2)^2} = \dfrac{1}{12}$

(b) $f(x) = 3x^5 + 2x^3$; $d = 5$. We seek the value of c for which $f(c) = 5$. $3c^5 + 2c^3 = 5$; $3c^5 + 2c^3 - 5 = 0$.
By trial we see that $c = 1$. $f'(x) = 15x^4 + 6x^2$. Thus $f'(x) > 0$ for all $x \neq 0$. Hence there is an open interval containing 1 on which f is continuous and increasing. Hence Theorem 5.1.8 applies and $(f^{-1})'(5) = \dfrac{1}{f'(1)} = \dfrac{1}{21}$

28. (a) $f(x) = 4x^3 + 2x$; $d = 6$

▸ If $f(c) = d$, then $4c^3 + 2c = 6$; $4c^3 + 2c - 6 = 0$; $2c^3 + c - 3 = 0$
By trial we find $c = 1$. Because $f'(x) = 12x^2 + 2$, then $f'(x) > 0$ for all x, and f is monotonic and continuous on any open interval. Therefore, by Theorem 5.1.8 we have

$$(f^{-1})'(6) = \frac{1}{f'(1)} = \frac{1}{12(1)^2 + 2} = \frac{1}{14}$$

(b) $f(x) = \sin x$; $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$; $d = \frac{1}{2}$

▸ If $f(c) = \frac{1}{2}$, then $\sin c = \frac{1}{2}$; $c = \frac{1}{6}\pi$. Because $f'(x) = \cos x > 0$ on the open interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ containing $\frac{1}{6}\pi$, f is continuous and increasing on this interval. By Theorem 5.1.8

$$(f^{-1})'(\tfrac{1}{2}) = \frac{1}{f'(\frac{1}{6}\pi)} = \frac{1}{\cos\frac{1}{6}\pi} = \frac{2}{\sqrt{3}} = \tfrac{2}{3}\sqrt{3}$$

29. (a) $f(x) = \frac{1}{2}\cos^2 x$; $0 \leq x \leq \frac{1}{2}\pi$; $d = \frac{1}{4}$. We seek the value of c in $(0, \frac{1}{2}\pi)$ for which $f(c) = \frac{1}{4}$.

$$\tfrac{1}{2}\cos^2 c = \tfrac{1}{4};\ \cos^2 c = \tfrac{1}{2};\ \cos c = \tfrac{1}{2}\sqrt{2};\ c = \tfrac{1}{4}\pi$$

$f'(x) = \cos x(-\sin x) = -\frac{1}{2}\sin 2x$. Because $f'(x) < 0$ when x is in $(0, \frac{1}{2}\pi)$ there is an open interval containing $\frac{1}{4}\pi$ on which f is continuous and decreasing. By Theorem 5.1.8 $(f^{-1})'(\frac{1}{4}) = \dfrac{1}{f'(\frac{1}{4}\pi)} = \dfrac{1}{-\frac{1}{2}\sin\frac{1}{2}\pi} = \dfrac{1}{-\frac{1}{2}} = -2$

(b) $f(x) = 2\cot x$; $0 < x < \pi$; $d = 2$. We seek the value of c in $(0, \pi)$ for which $f(c) = 2$.

$$2\cot c = 2;\ \cot c = 1;\ c = \tfrac{1}{4}\pi$$

$f'(x) = -2\csc^2 x$. Because $f'(x) < 0$ when x is in $(0, \pi)$, there is an open interval containing $\frac{1}{4}\pi$ on which f is continuous and decreasing. Hence Theorem 5.1.8 applies and $(f^{-1})'(2) = \dfrac{1}{f'(\frac{1}{4}\pi)} = \dfrac{1}{-2(2)} = -\dfrac{1}{4}$

30. (a) $f(x) = \tan 2x$; $-\frac{1}{4}\pi < x < \frac{1}{4}\pi$; $d = 1$. If $f(c) = d$, then $\tan 2c = 1$. Hence, $c = \frac{1}{8}\pi$. Furthermore, $f'(x) = 2\sec^2 2x$. Therefore, $f'(x) > 0$ if $x \in (-\frac{1}{4}\pi, \frac{1}{4}\pi)$ and so f is continuous and increasing on that interval.

By Theorem 5.1.8, we conclude that $(f^{-1})'(1) = \dfrac{1}{f'(\frac{1}{8}\pi)} = \dfrac{1}{2\sec^2 \frac{1}{4}\pi} = \dfrac{1}{4}$.

(b) $f(x) = \sec \frac{1}{2}x$; $0 \le x < \pi$; $d = 2$. If $f(c) = d$, then $\sec \frac{1}{2}c = 2$; $\cos \frac{1}{2}c = \frac{1}{2}$; $\frac{1}{2}c = \frac{1}{3}\pi$; $c = \frac{2}{3}\pi$.

$f'(x) = \frac{1}{2}\sec \frac{1}{2}x \tan \frac{1}{2}x > 0$ if $x \in (0, \pi)$ and so f is continuous and increasing on that interval.

By Theorem 5.1.8, we conclude that $(f^{-1})'(2) = \dfrac{1}{f'(\frac{2}{3}\pi)} = \dfrac{1}{\frac{1}{2}\sec\frac{1}{3}\pi \tan\frac{1}{3}\pi} = \dfrac{1}{\frac{1}{2}(2)\sqrt{3}} = \dfrac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$.

31. (a) $f(x) = \frac{1}{2}\csc x$; $0 < x < \frac{1}{2}\pi$; $d = 1$. We seek the value of c in $(0, \frac{1}{2}\pi)$ for which $f(c) = 1$.

$$\tfrac{1}{2}\csc c = 1;\ \csc c = 2;\ c = \tfrac{1}{6}\pi$$

$f'(x) = -\frac{1}{2}\csc x \cot x$. Because $f'(x) < 0$ when x is in $(0, \frac{1}{2}\pi)$, there is an open interval containing $\frac{1}{6}\pi$ on which f is continuous and decreasing. Hence Theorem 5.1.8 applies and $(f^{-1})'(1) = \dfrac{1}{f'(\frac{1}{6}\pi)} = \dfrac{1}{-\frac{1}{2}(2)\sqrt{3}} = -\dfrac{1}{\sqrt{3}}$

(b) $f(x) = 2x^2 + 8x + 7$; $x \le -2$; $d = 1$. We wish to find the value of $c < -2$ for which $f(c) = 1$.

$$2c^2 + 8c + 7 = 1;\ 2c^2 + 8c + 6 = 0;\ c^2 + 4c + 3 = 0;\ (c+3)(c+1) = 0;\ c = -3$$

$f'(x) = 4x + 8$. Because $f'(x) < 0$ if $x < -2$, There is an open interval containing -3 on which f is decreasing and continuous. Hence Theorem 5.1.8 applies and $(f^{-1})'(1) = \dfrac{1}{f'(-3)} = \dfrac{1}{-4} = -\dfrac{1}{4}$.

32. (a) $f(x) = x^2 - 6x + 7$, $x \le 3$; $d = 0$

▷ If $f(c) = d$, then $c^2 - 6c + 7 = 0$; $c^2 - 6c + 9 = 2$; $(c-3)^2 = 2$; $c - 3 = \pm\sqrt{2}$; $c = 3 \pm \sqrt{2}$

Because we are given that $x \le 3$, we take $c = 3 - \sqrt{2}$. Now $f'(x) = 2x - 6$, and $x \le 3$, so $f'(x) \le 0$. Therefore, f is monotonic and continuous on an open interval that contains c. For example, f is monotonic and continuous on the open interval $(1, 2)$. Furthermore, $f'(c) = f'(3 - \sqrt{2}) = 2(3 - \sqrt{2}) - 6 = -2\sqrt{2}$.

Therefore, by Theorem 5.1.8, $\quad (f^{-1})'(0) = \dfrac{1}{f'(3 - \sqrt{2})} = \dfrac{1}{-2\sqrt{2}} = -\frac{1}{4}\sqrt{2}$

(b) $f(x) = 2x^3 + x + 20$; $d = 2$

▷ If $f(d) = d$, then $2c^3 + c + 20 = 2$; $2c^3 + c + 18 = 0$. By trial, we find $c = -2$. Now $f'(x) = 6x^2 + 1 > 0$. Thus f is continuous and increasing in $(-\infty, +\infty)$. Therefore, by Theorem 5.1.8,

$$(f^{-1})'(2) = \frac{1}{f'(-2)} = \frac{1}{6(-2)^2 + 1} = \frac{1}{25}$$

33. $f(x) = x^3 - \frac{2}{x} - 3$, $x > 0$; $d = 2$. We seek the value of $c > 0$ for which $f(c) = 2$. $c^3 - \frac{2}{c} - 3 = 2$; $c^3 - \frac{2}{c} - 5 = 0$. Using zoom, we find the positive value of $c = 1.82665$. Now $f'(x) = 3x^2 + \dfrac{2}{x^2} > 0$. Thus f is continuous and increasing in $(0, +\infty)$. Therefore, by Theorem 5.1.8, $(f^{-1})'(2) = \dfrac{1}{f'(1.82665)} = 0.09426$.

34. $f(x) = x^3 - \frac{2}{x} - 3$, $x < 0$; $d = -2$. We seek the value of $c < 0$ for which $f(c) = -2$. $c^3 - \frac{2}{c} - 3 = -2$; $c^3 - \frac{2}{c} - 1 = 0$. By trial, we find the negative value of $c = -1$. Now $f'(x) = 3x^2 + \dfrac{2}{x^2} > 0$. Thus f is continuous and increasing in $(-\infty, 0)$. Therefore, by Theorem 5.1.8, $(f^{-1})'(-2) = \dfrac{1}{f'(-1)} = \dfrac{1}{3(-1)^2 + 2/(-1)^2} = \dfrac{1}{5}$

35. $f(x) = x^4 + x^2 - 4$, $x \ge 0$; $d = 1$. We seek the value of $c > 0$ for which $f(c) = 1$. $c^4 + c^2 - 4 = 1$; $c^4 + c^2 + \frac{1}{4} = \frac{21}{4}$; $(c^2 + \frac{1}{2})^2 = \frac{21}{4}$; $c^2 + \frac{1}{2} = \pm\frac{1}{2}\sqrt{21}$. Because c is real, $c^2 + \frac{1}{2} = \frac{1}{2}\sqrt{21}$; $c^2 = \frac{1}{2}(\sqrt{21} - 1)$. Because $c > 0$, $c = \sqrt{\frac{1}{2}(\sqrt{21} - 1)} \approx 1.33839$. Now $f'(x) = 4x^3 + 2x = 2x(2x^2 + 1) > 0$ if $x > 0$. Thus f is continuous and increasing in $(0, +\infty)$. Therefore, by Theorem 5.1.8, $(f^{-1})'(1) = \dfrac{1}{f'(c)} = \sqrt{\frac{1}{840}(1 + \sqrt{21})} = 0.08152$.

36. $f(x) = x^4 + x^2 - 4,\ x \le 0;\ d = -0.5.$

▷ We seek the value of $c > 0$ for which $f(c) = -\frac{1}{2}$. $c^4 + c^2 - 4 = -\frac{1}{2}$; $c^4 + c^2 + \frac{1}{4} = \frac{15}{4}$; $(c^2 + \frac{1}{2})^2 = \frac{15}{4}$; $c^2 + \frac{1}{2} = \pm\frac{1}{2}\sqrt{15}$. Because c is real, $c^2 + \frac{1}{2} = \frac{1}{2}\sqrt{15}$; $c^2 = \frac{1}{2}(\sqrt{15} - 1)$. Because $c < 0$, $c = -\sqrt{\frac{1}{2}(\sqrt{15} - 1)}$ ≈ -1.19859. Now $f'(x) = 4x^3 + 2x = 2x(2x^2 + 1) < 0$ if $x < 0$. Thus f is continuous and decreasing in $(0, +\infty)$. Therefore, by Theorem 5.1.8, $(f^{-1})'(-0.5) = \dfrac{1}{f'(-1.19859)} = -1.0771$.

37. $f(x) = x + \sqrt{\sin x}$; $d = 3$. We seek the value of c for which $f(c) = 3$. $c + \sqrt{\sin c} = 3$. Using zoom, we find $c = 2.0606$. Now $f'(x) = 1 + \dfrac{\cos x}{2\sqrt{\sin x}}$ and so $f'(c) = 0.7496$. Because f' is continuous, $f'(x) > 0$ in some open interval containing c. Therefore, by Theorem 5.1.8, $(f^{-1})'(3) = 1/f'(c) = 1.33409$.

38. $f(x) = \cos^2 x + 2x$; $d = 2$. We seek the value of c for which $f(c) = 2$. $\cos^2 c + 2c = 2$. Using zoom, we find $c = 0.7148$. Now $f'(x) = -2\cos x \sin x + 2 = 2 - \sin 2x > 0$. Thus f is continuous and increasing. Therefore, by Theorem 5.1.8, $(f^{-1})'(2) = \dfrac{1}{f'(0.7148)}$

39. $f(x) = \displaystyle\int_{-3}^{x} \sqrt{t+3}\, dt$, $x > -3$; $d = 18$. We seek the value of $c > -3$ for which $f(c) = 18$.

$$\int_{-3}^{c} \sqrt{t+3}\, dt = \left[\tfrac{2}{3}(t+3)^{3/2}\right]_{-3}^{c} = \tfrac{2}{3}(c+3)^{3/2} = 18;\ (c+3)^{3/2} = 27;\ c+3 = 27^{2/3} = 9;\ c = 6$$

$f'(x) = \sqrt{x+3}$. Because $f'(x) > 0$ if $x > -3$, There is an open interval containing 6 on which f is increasing and continuous. Therefore Theorem 5.1.8 applies and $(f^{-1})'(18) = \dfrac{1}{f'(6)} = \frac{1}{3}$.

40. $f(x) = \displaystyle\int_{x}^{2} t\, dt$, $x < 0$; $d = -6$

▷ $$\int_{x}^{2} t\, dt = \tfrac{1}{2}t^2\Big]_{x}^{2} = 2 - \tfrac{1}{2}x^2$$

Thus, we have $f(x) = 2 - \frac{1}{2}x^2$, $x < 0$; $f'(x) = -x$, $x < 0$

If $f(x) = -6$, and $x < 0$, then $-6 = 2 - \frac{1}{2}x^2$; $-8 = -\frac{1}{2}x^2$; $x^2 = 16$; $x = -4$

Thus, we take $c = -4$. Because $f'(x) = -x > 0$ if $x < 0$, then f is monotonic and continuous on an open interval containing -4. By Theorem 5.1.8 we conclude that $(f^{-1})'(6) = \dfrac{1}{f'(-4)} = \dfrac{1}{4}$

In Exercises 41–46, (a) prove that f has an inverse, (b) find $f^{-1}(x)$ and (c) verify the equations of Theorem 5.1.4.

41. $f(x) = 4x - 3$. (a) f is increasing $\Rightarrow$ f is one-to-one $\Rightarrow$ f has an inverse.

(b) Let $y = f(x)$. Then $y = 4x - 3$; $x = \frac{1}{4}(y + 3)$. Hence $f^{-1}(y) = \frac{1}{4}(y+3)$, and so $f^{-1}(x) = \frac{1}{4}(x+3)$.

(c) $f^{-1}(f(x)) = f^{-1}(4x - 3) = \frac{1}{4}[(4x-3) + 3] = \frac{1}{4}(4x) = x$

$f(f^{-1}(x)) = f(\frac{1}{4}(x+3)) = 4[\frac{1}{4}(x+3)] - 3 = (x+3) - 3 = x$

42. $f(x) = 5x + 2$. (a) f is increasing $\Rightarrow$ f is one-to-one $\Rightarrow$ f has an inverse.

(b) Let $y = f(x)$. Then $y = 5x + 2$; $x = \frac{1}{5}(y - 2)$. Hence $f^{-1}(y) = \frac{1}{5}(y-2)$, and so $f^{-1}(x) = \frac{1}{5}(x-2)$.

(c) $f^{-1}(f(x)) = f^{-1}(5x + 2) = \frac{1}{5}[(5x+2) - 2] = \frac{1}{5}(5x) = x$

$f(f^{-1}(x)) = f(\frac{1}{5}(x-2)) = 5[\frac{1}{5}(x-2)] + 2 = (x-2) + 2 = x$

43. $f(x) = x^3 + 2$. (a) f is increasing $\Rightarrow$ f is one-to-one $\Rightarrow$ f has an inverse.

(b) Let $y = f(x)$. Then $y = x^3 + 2$; $x^3 = y - 2$; $x = \sqrt[3]{y-2}$. Thus $f^{-1}(y) = \sqrt[3]{y-2}$, and so $f^{-1}(x) = \sqrt[3]{x-2}$.

(c) $f^{-1}(f(x)) = f^{-1}(x^3 + 2) = \sqrt[3]{(x^3+2) - 2} = \sqrt[3]{x^3} = x$

$f(f^{-1}(x)) = f(\sqrt[3]{x-2}) = (\sqrt[3]{x-2})^3 + 2 = (x-2) + 2 = x$

44. $f(x) = (x+2)^3$

▷ (a) $f'(x) = 3(x+2)^2 > 0$ if $x \neq -2$. Thus, f is one-to-one and f^{-1} exists.
(b) Let $\quad y = (x+2)^3; \quad \sqrt[3]{y} = x+2; \quad x = -2 + \sqrt[3]{y}; \quad f^{-1}(y) = -2 + \sqrt[3]{y}; \ f^{-1}(x) = -2 + \sqrt[3]{x}$

(c) We have $\quad f^{-1}(f(x)) = f^{-1}((x+2)^3) = -2 + \sqrt[3]{(x+2)^3} = -2 + (x+2) = x$

and $\quad f(f^{-1}(x)) = f(-2 + \sqrt[3]{x}) = [(-2 + \sqrt[3]{x}) + 2]^3 = (\sqrt[3]{x})^3 = x$

45. $f(x) = \dfrac{3x+1}{2x+4} = \dfrac{3}{2} - \dfrac{5}{2x+4}$. (a) In $(-\infty, -2)$ f is increasing and $f(x) > \frac{3}{2}$; in $(-2, +\infty)$ f is increasing and $f(x) < \frac{3}{2}$. Thus f is one-to-one, and so f has an inverse.

(b) Let $y = f(x)$. Then $y = \dfrac{3x+1}{2x+4}$; $2xy + 4y = 3x + 1$; $x(2y-3) = 1 - 4y$; $x = \dfrac{4y-1}{3-2y}$.

Therefore $f^{-1}(y) = \dfrac{4y-1}{3-2y}$, and so $f^{-1}(x) = \dfrac{4x-1}{3-2x}$.

(c) $f^{-1}(f(x)) = f^{-1}\left(\dfrac{3x+1}{2x+4}\right) = \dfrac{4[(3x+1)/(2x+4)] - 1}{3 - 2[(3x+1)/(2x+4)]} = \dfrac{12x+4-2x-4}{6x+12-6x-2} = \dfrac{10x}{10} = x$

$f(f^{-1}(x)) = f\left(\dfrac{4x-1}{3-2x}\right) = \dfrac{3[(4x-1)/(3-2x)]+1}{2[(4x-1)/(3-2x)]+4} = \dfrac{12x-3+3-2x}{8x-2+12-8x} = \dfrac{10x}{10} = x$

46. $f(x) = \dfrac{x-3}{3x-6} = \dfrac{1}{3} - \dfrac{1}{3x-6}$. (a) In $(-\infty, 2)$ f is increasing and $f(x) > \frac{1}{3}$; in $(2, +\infty)$ f is increasing and $f(x) < \frac{1}{3}$. Thus f is one-to-one, and so f has an inverse.

(b) Let $y = f(x)$. Then $y = \dfrac{x-3}{3x-6}$; $3xy - 6y = x - 3$; $3xy - x = x(3y-1) = 6y - 3$; $x = \dfrac{6y-3}{3y-1}$

Therefore $f^{-1}(y) = \dfrac{6y-3}{3y-1}$ and $f^{-1}(x) = \dfrac{6x-3}{3x-1}$.

(c) $f^{-1}(f(x)) = f^{-1}\left(\dfrac{x-3}{3x-6}\right) = \dfrac{6[(x-3)/(3x-6)] - 3}{3[(x-3)/(3x-6)] - 1} = \dfrac{6x-18-9x+18}{3x-9-3x+6} = \dfrac{-3x}{-3} = x$

$f(f^{-1}(x)) = f\left(\dfrac{6x-3}{3x-1}\right) = \dfrac{[(6x-3)/(3x-1)] - 3}{3[(6x-3)/(3x-1)] - 6} = \dfrac{6x-3-9x+3}{18x-9-18x+6} = \dfrac{-3x}{-3} = x$

47. Let $y = f(x)$. Then $y = 32 + \frac{9}{5}x$; $y - 32 = \frac{9}{5}x$; $x = \frac{5}{9}(y-32)$. Hence $f^{-1}(y) = \frac{5}{9}(y-32)$; so $f^{-1}(x) = \frac{5}{9}(x-32)$.

48. If $f(t)$ dollars in the amount in t years of an investment of \$1000 at 12 percent simple interest, then $f(t) = 1000(1 + 0.12t)$. Determine the inverse function f^{-1} that expresses the number of years that \$1000 has been invested at 12 percent interest as a function of the amount of the investment.

▷ Let A be the amount after t years. Then

$$A = 1000(1 + 0.12t); \quad .001A = 1 + 0.12t; \quad t = \frac{.001A - 1}{0.12} = \frac{A - 1000}{120}; \quad f^{-1}(A) = \frac{A - 1000}{120}$$

49. $m = \dfrac{m_0}{\sqrt{1 - (v/c)^2}}$; $\sqrt{1 - (v/c)^2} = \dfrac{m_0}{m}$; $1 - \left(\dfrac{v}{c}\right)^2 = \left(\dfrac{m_0}{m}\right)^2$; $\left(\dfrac{v}{c}\right)^2 = 1 - \left(\dfrac{m_0}{m}\right)^2$; $v = c\sqrt{1 - (m_0/m)^2}$

50. $P = \left(\dfrac{40 + T}{140}\right)^5$; $P^{1/5} = \dfrac{40 + T}{140}$; $40 + T = 140P^{1/5}$; $T = 140P^{1/5} - 40$

51. $f(x) = x^3 + 3x - 1$ (a) $f'(x) = 3x^2 + 3 > 0$. Thus f is increasing and has an inverse.
(b) The slope of the tangent line to the graph of f at the point $(1,3)$ is $f'(1) = 6$.
(c) The slope of the tangent line to the graph of f^{-1} at the point $(3,1)$ is $(f^{-1})'(3)$.

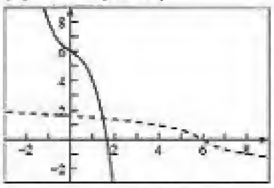
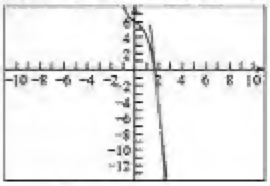
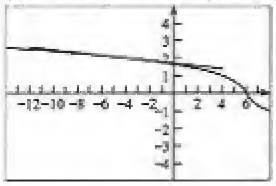
By Theorem 5.1.8, $(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{6}$.

52. Given $f(x) = 6 - x - x^3$. (a) Prove that f has an inverse f^{-1}. (b) Find the slope of the tangent line to the graph of f at the point $(2, -4)$. (c) Find the slope of the tangent line to the graph of f^{-1} at the point $(-4, 2)$. Support your answers in parts (a)-(c) by doing the following: (d) Plot the graphs of f and f^{-1} on the same screen; (e) plot the graphs of f and its tangent line at $(2, -4)$ on the same screen; (f) plot the graphs of f^{-1} and its tangent line at $(-4, 2)$ on the same screen.

▷ (a) $f'(x) = -1 - 3x^2$. Because $f'(x) < 0$ for all x, the f is decreasing on its domain and so it has an inverse.
(b) Because $f^{-1}(2) = -1 - 3(2^2) = -13$, the slope of the tangent line to the graph of f at the point $(2, -4)$ is -13. (c) By Theorem 5.1.8, $(f^{-1})'(-4) = \dfrac{1}{f^{-1}(2)} = -\dfrac{1}{13}$

Thus, $-\frac{1}{13}$ is the slope of the tangent line to the graph of f^{-1} at the point $(-4, 2)$.

(d) Plot of f and f^{-1} (e) Plot of f and tangent at $(2, -4)$ (f) Plot of f^{-1} and tangent at $(-4, 2)$

In Exercises 53 and 54, show that f is its own inverse.

53. $f(x) = y = \sqrt{16 - x^2}$, $0 \le x \le 4$. $y^2 = 16 - x^2$, $x^2 = 16 - y^2$, $f^{-1}(y) = x = \sqrt{16 - y^2}$, $f^{-1}(x) = \sqrt{16 - x^2}$

54. $f(x) = y = \dfrac{x+6}{x-1}$, $xy - y = x + 6$, $xy - x = y + 6$, $x(y-1) = y + 6$, $f^{-1}(y) = x = \dfrac{y+6}{y-1}$, $f^{-1}(x) = \dfrac{x+6}{x-1}$

55. Find k so that $f(x) = \dfrac{x+5}{x+k}$ will be its own inverse.

▸ $f(x) = y = \dfrac{x+5}{x+k} = 1 + \dfrac{5-k}{x+k}$, $y - 1 = \dfrac{5-k}{x+k}$, $(y-1)(x+k) = 5 - k$. x and y are symmetrical if $k = -1$.

In Exercises 56–58, show that the function is its own inverse for any constants k and h.

56. $f(x) = y = \dfrac{x+k}{x-1} = \dfrac{(x-1)+(k+1)}{x-1} = 1 + \dfrac{k+1}{x-1}$, $y - 1 = \dfrac{k+1}{x-1}$, $(y-1)(x-1) = k+1$. symmetric

57. $f(x) = y = \dfrac{kx+1}{x-k} = \dfrac{k(x-k)+k^2+1}{x-k} = k + \dfrac{k^2+1}{x-k}$, $y - k = \dfrac{k^2+1}{x-k}$, $(y-k)(x-k) = k^2+1$. symmetric

58. $f(x) = y = \dfrac{x+h}{kx-1}$, $ky = \dfrac{kx+kh}{kx-1} = \dfrac{(kx-1)+(kh+1)}{kx-1} = 1 + \dfrac{kh+1}{kx-1}$, $(ky-1)(kx-1) = kh+1$. symmetric

In Exercises 59 and 60, (a) show that f is not 1-1; (b) restrict the domain and obtain 1-1 functions f_1 and f_2 each having the same range as f; (c) find ${f_1}^{-1}(x)$ and ${f_2}^{-1}(x)$ and state their domains (d) plot f_1 and ${f_1}^{-1}$ (e) plot f_2 and ${f_2}^{-1}$.

59. $f(x) = x^2 + 4$ (a) $f(-1) = 1 + 4 = f(1)$; (b) $f_1(x) = x^2 + 4$, $x \ge 0$; $f_2(x) = x^2 + 4$, $x \le 0$

(c) $y_1 = x^2 + 4$, $x^2 = y_1 - 4$, $x = \sqrt{y_1 - 4}$, ${f_1}^{-1}(x) = \sqrt{x - 4}$, $x \ge 4$.

$y_2 = x^2 + 4$, $x^2 = y_2 - 4$, $x = -\sqrt{y_2 - 4}$, ${f_2}^{-1}(x) = -\sqrt{x - 4}$, $x \ge 4$.

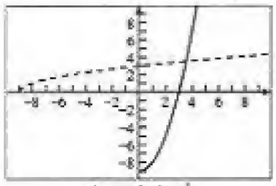
60. $f(x) = x^2 - 9$

▸ (a) $f(-1) = 1 - 9 = f(1)$ (b) $f_1(x) = x^2 - 9$, $x \ge 0$; $f_2(x) = x^2 - 9$, $x \le 0$

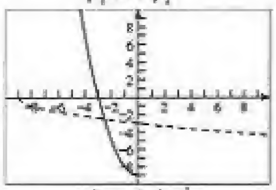
(c) $y_1 = x^2 - 9$, $x^2 = y_1 + 9$, $x = \sqrt{y_1 + 9}$, ${f_1}^{-1}(x) = \sqrt{x + 9}$, $x \ge -9$.

$y_2 = x^2 - 9$, $x^2 = y_2 + 9$, $x = -\sqrt{y_2 + 9}$, ${f_2}^{-1}(x) = -\sqrt{x + 9}$, $x \ge -9$.

The required plots are shown at the right.



f_1 and ${f_1}^{-1}$



f_2 and ${f_2}^{-1}$

61. Prove that $f(x) = y = \begin{cases} x & \text{if } x < 1 \\ x^2 & \text{if } 1 \le x \le 9 \\ 27\sqrt{x} & \text{if } x > 9 \end{cases}$ has an inverse and find it.

▸ Because $1 = 1^2$ and $9^2 = 27\sqrt{9}$, f is increasing and so has an inverse.

$$f^{-1}(y) = x = \begin{cases} y & \text{if } y < 1 \\ \sqrt{y} & \text{if } 1 \le y \le 81 \\ (y/27)^2 & \text{if } y > 81 \end{cases}, \quad f^{-1}(x) = \begin{cases} x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \le x \le 81 \\ (x/27)^2 & \text{if } x > 81 \end{cases}$$

In Exercises 62–65, prove that f has an inverse f^{-1} and compute $(f^{-1})'(0)$.

62. $f(x) = \displaystyle\int_1^x \sqrt{16 - t^4}\,dt$; $-2 \le x \le 2$; $f'(x) = \sqrt{16 - x^4}$. Therefore $f'(x)$ exists if x is in $[-2, 2]$. Hence f is continuous on $[-2, 2]$. Because $f'(x) > 0$ if $-2 < x < 2$, f is increasing on $[-2, 2]$. Therefore f is one-to-one on $[-2, 2]$ and so f has an inverse. Because $\displaystyle\int_1^c \sqrt{16 - t^4}\,dt = 0$ when $c = 1$, it follows from Theorem 5.1.8 that

$$(f^{-1})'(0) = \frac{1}{f'(1)} = \frac{1}{\sqrt{16 - 1}} = \frac{1}{\sqrt{15}}$$

63. $f(x) = \int_2^x \sqrt{9+t^4}\,dt$. $f'(x) = \sqrt{9+x^4} > 0$. Therefore f is increasing and so f has an inverse. Because $\int_2^c \sqrt{9+t^4}\,dt = 0$ when $c = 2$, it follows from Theorem 5.1.8 that $(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{\sqrt{9+2^4}} = \frac{1}{5}$

64. $f(x) = \int_1^{2x} \frac{dt}{\sqrt{1+t^4}}$

▷ Let $u = 2x$. Then $f'(x) = \frac{d}{du}\int_1^u \frac{dt}{\sqrt{1+t^4}} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1+u^4}}(2) = \frac{2}{\sqrt{1+(2x)^4}}$. Hence f is continuous. Because $f'(x) > 0$, f is increasing. By Theorem 5.1.2, f is one-to-one and so f has an inverse. Because $\int_1^{2c} \frac{dt}{\sqrt{1+t^4}} = 0$ when $c = \frac{1}{2}$, it follows from Theorem 5.1.8 that

$$(f^{-1})'(0) = \frac{1}{f'(\frac{1}{2})} = \frac{\sqrt{1+(2\cdot\frac{1}{2})^4}}{2} = \frac{1}{2}\sqrt{2}$$

65. $f(x) = \int_{\pi^3}^{x^3} \cos^2\sqrt[3]{t}\,dt$. Let $u = x^3$. Then $f'(x) = \frac{d}{du}\int_{\pi^3}^{u} \cos^2 t^{1/3} \cdot \frac{du}{dx} = \cos^2 u^{1/3} \cdot 3x^2 = 3x^2\cos^2 x$. Hence f is continuous. Because $f'(x) > 0$, f is increasing. By Theorem 5.1.2, f is one-to-one and so f has an inverse. Because $\int_{\pi^3}^{c^3} \cos^2\sqrt[3]{t}\,dt = 0$ when $c = \pi$, from Theorem 5.1.8 we have $(f^{-1})'(0) = \frac{1}{f'(\pi)} = \frac{1}{3\pi^2\cos^2\pi} = \frac{1}{3\pi^2}$.

66. The formula of Theorem 5.1.8 is $\frac{dx}{dy} = \frac{1}{dy/dx}$ (1) where $y = f(x)$ and $x = f^{-1}(y)$. Because $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ and because $x = f^{-1}(y)$, then $\frac{dx}{dy} = (f^{-1})'(y)$. Substituting into equ. (1) we get $(f^{-1})'(y) = \frac{1}{f'(x)}$.
Now let $x = f^{-1}(y)$ and obtain $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$. Replace y by x and get $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$ (2)

67. We differentiate with respect to x on both sides of Eq. (2), and apply the chain rule on the right side. Thus,

$$(f^{-1})''(x) = \frac{-1}{[f'(f^{-1}(x))]^2}[D_x f'(f^{-1}(x))] = \frac{-1}{[f'(f^{-1}(x))]^2}[f''(f^{-1}(x))D_x f^{-1}(x)] \tag{3}$$

Because $D_x f^{-1}(x) = (f^{-1})'(x)$, we may substitute from Eq. (2) into Eq. (3) and obtain

$$(f^{-1})''(x) = \frac{-1}{[f'(f^{-1}(x))]^2}[f''(f^{-1}(x))]\left[\frac{1}{f'(f^{-1}(x))}\right] = -\frac{f''(f^{-1}(x))}{[f'(f^{-1}(x))]^3}. \; (f^{-1})''(y) = -\frac{f''(x)}{[f'(x)]^3}$$

Exercises 5.1 Supplement

1. Given that the function f is continuous and increasing on the closed interval $[a,b]$, by assuming Theorem 5.1.5 (i) and (ii) prove that f^{-1} is continuous from the right at $f(a)$ and continuous from the left at $f(b)$.

▷ By Theorem 5.1.5(i), $f^{-1}(y)$ is increasing. Let $\epsilon > 0$ be any number less than $b - a$.
Now $f^{-1}(f(a+\epsilon)) = a + \epsilon = f^{-1}(f(a)) + \epsilon$. Thus

$$\text{if } f(a) < y < f(a+\epsilon) \text{ then } f^{-1}(f(a)) < f^{-1}(y) < f^{-1}(f(a+\epsilon))$$
$$\Leftrightarrow \text{if } f(a) < y < f(a) + \delta \text{ then } f^{-1}(f(a)) < f^{-1}(y) < f^{-1}(f(a)) + \epsilon$$

where $\delta = f(a+\epsilon) - f(a) > 0$. This proves that $f^{-1}(y)$ is continuous from the right at $f(a)$. Similarly,

$$\text{if } f(b-\epsilon) < y < f(b) \text{ then } f^{-1}(f(b-\epsilon)) < f^{-1}(y) < f^{-1}(f(b))$$
$$\Leftrightarrow \text{if } f(b) - \delta < y < f(b) \text{ then } f^{-1}(f(b)) - \epsilon < f^{-1}(y) < f^{-1}(f(b))$$

where $\delta = f(b) - f(b-\epsilon) > 0$. This proves that $f^{-1}(y)$ is continuous from the left at $f(b)$.

2. Prove Theorem 5.1.6.

▷ Suppose that the function f is continuous and decreasing on the closed interval $[a,b]$ and let $g = -f$. Then g is continuous and increasing on $[a,b]$ and so we may apply Theorem 5.1.5 to g. Now $y = f^{-1}(x) \Leftrightarrow x = f(y)$ $\Leftrightarrow x = -g(y) \Leftrightarrow g(y) = -x \Leftrightarrow y = g^{-1}(x)$. Therefore, f has an inverse given by $f^{-1}(x) = g^{-1}(-x)$. Graphically, f is the reflection of g in the x axis and f^{-1} is the reflection of g^{-1} in the y axis. Because $g^{-1}(x)$ is defined for $g(a) \le x \le g(b) \Leftrightarrow -g(b) \le -x \le -g(a)$, then $f^{-1}(x) = g^{-1}(-x)$ is defined for $f(b) \le x \le f(a)$, proving (i). Because $g^{-1}(x)$ is increasing by Theorem 5.1.5 (ii), $f^{-1}(x) = g^{-1}(-x)$ is decreasing. Let $h(x) = -x$. Because h continuous for all real numbers and $g^{-1}(x)$ is continuous on its domain by Theorem 5.1.5(iii), the composition $f = g \circ h$ is continuous on its domain.

5.2 THE NATURAL LOGARITHMIC FUNCTION

5.2.1 Definition The natural logarithmic function defined by

$\ln x = \int_1^x \frac{1}{t}\,dt \quad x > 0$

5.2.2 Theorem If u is a differentiable function of x and $u(x) > 0$, then $D_x(\ln u) = \frac{1}{u} \cdot D_x u$

5.3.1 Theorem If u is a differentiable function of x and $u(x) \neq 0$, then $D_x(\ln|u|) = \frac{1}{u} \cdot D_x u$

5.2.3 Theorem $\ln 1 = 0$

5.2.4 Theorem If a and b are any positive numbers, then $\ln(ab) = \ln a + \ln b$

5.2.5 Theorem If a and b are any positive numbers, then $\ln \frac{a}{b} = \ln a - \ln b$

Furthermore, if a and b are both positive or both negative, then $\ln(ab) = \ln|a| + \ln|b|$ and $\ln \frac{a}{b} = \ln|a| - \ln|b|$

5.2.6 Theorem If a is any positive number and r is any rational number, then

$\ln a^r = r \ln a$

Note the distinction between $\ln(x^2)$ and $(\ln x)^2$. By Theorem 5.2.6, $\ln(x^2) = \ln x^2 = 2 \cdot \ln x$. However, $(\ln x)^2 = (\ln x)(\ln x) \neq 2 \cdot \ln x$. Another symbol for $(\ln x)^2$ is $\ln^2 x$. In general, $\ln^r x = (\ln x)^r \neq r \cdot \ln x$. And $\ln x^r = \ln(x^r) = r \cdot \ln x$, if r is a rational number. It is usually better to simplify an expression before doing calculus. See Exercise 12.

We have the following facts about the natural logarithmic function and its graph.

(i) The domain is the set of all positive numbers.
(ii) The range is the set of all real numbers.
(iii) The function is increasing on its entire domain.
(iv) The function is continuous at all numbers in its domain.
(v) The graph of the function is concave downward at all points.
(vi) The graph of the function is asymptotic to the negative side of the y axis through the fourth quadrant.

Moreover, the following limits are important, where n is any positive number. See Exercise 59.

$\lim_{x\to\infty} \ln x = +\infty \qquad \lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to+\infty} \frac{\ln x}{x^n} = 0 \qquad \lim_{x\to 0^+} x^n \ln x = 0$

Exercises 5.2

In Exercises 1–4, verify the property of natural logarithms by applying Definition 5.2.1 and NINT.

1. $\ln 68 = \ln 4 + \ln 17$ ▷ $\int_1^{68} \frac{1}{t}dt = \int_1^4 \frac{1}{t}dt + \int_1^{17} \frac{1}{t}dt$; $4.2195 = 1.3863 + 2.8332$

2. $\ln 1000 = 3 \ln 10$ ▷ $\int_1^{1000} \frac{1}{t}dt = 3\int_1^{10} \frac{1}{t}dt$; $6.9078 = 3(2.3026)$

3. $\ln 13 = \ln 117 - \ln 9$ ▷ $\int_1^{13} \frac{1}{t}dt = \int_1^{117} \frac{1}{t}dt - \int_1^{9} \frac{1}{t}dt$; $2.5649 = 4.7622 - 2.1972$

4. $\ln 81 = 2 \ln 9$

▷ Applying Definition 5.2.1, we need to verify that $\int_1^{81} \frac{1}{t}dt = 2\int_1^{9} \frac{1}{t}dt$.

Using NINT to evaluate the integrals, we have the identity $4.3944 = 2(2.1972)$.

In Exercises 5–30, differentiate the function and simplify the result.

5. $f'(x) = \frac{d}{dx} \ln(4 + 5x) = \frac{1}{4 + 5x} \frac{d}{dx}(4 + 5x) = \frac{5}{4 + 5x}$

6. $g'(x) = \frac{d}{dx} \ln(1 + 4x^2) = \frac{1}{1 + 4x^2} \frac{d}{dx}(1 + 4x^2) = \frac{8x}{1 + 4x^2}$

7. $h'(x) = \frac{d}{dx}(\ln\sqrt{4 + 5x}) = \frac{d}{dx} \frac{1}{2} \ln(4 + 5x) = \frac{1}{2} \cdot \frac{1}{4 + 5x} \cdot 5 = \frac{5}{8 + 10x}$

8. $f(x) = \ln(8 - 2x)$

▷ We apply Theorem 5.2.2 with $u = 8 - 2x$. $f'(x) = \frac{1}{8-2x} D_x(8 - 2x) = \frac{-2}{8-2x} = \frac{1}{x-4}$

9. $f'(t) = \frac{d}{dt}[\ln(3t+1)^2] = \frac{d}{dt}[2\ln(3t+1)] = 2 \cdot \frac{1}{3t+1} \cdot 3 = \frac{6}{3t+1}$

10. $h'(x) = \frac{d}{dx}[\ln(8-2x)^5] = \frac{d}{dx}[5\ln(8-2x)] = 5 \cdot \frac{1}{8-2x}(-2) = \frac{-10}{8-2x} = \frac{5}{x-4}$

11. $g'(t) = \frac{d}{dt}[\ln^2(3t+1)] = 2\ln(3t+1)\frac{d}{dt}[\ln(3t+1)] = 2\ln(3t+1) \cdot \frac{1}{3t+1} \cdot 3 = \frac{6\ln(3t+1)}{3t+1}$

12. $f(x) = \ln\sqrt{1+4x^2}$

▷ First, we use Theorem 5.2.6 to simplify $f(x)$ before differentiating.

$$f(x) = \ln(1+4x^2)^{1/2} = \tfrac{1}{2}\ln(1+4x^2)$$

Thus,

$$f'(x) = \frac{1}{2} \cdot \frac{1}{1+4x^2}(8x) = \frac{4x}{4x^2+1}$$

13. $f'(x) = \frac{d}{dx}[\ln\sqrt[3]{4-x^2}] = \frac{d}{dx}\left[\frac{1}{3}\ln(4-x^2)\right] = \frac{1}{3} \cdot \frac{1}{4-x^2}(-2x) = -\frac{2x}{12-3x^2}$

14. $g'(y) = \frac{d}{dy}\ln(\ln y) = \frac{1}{\ln y} \cdot \frac{d}{dy}\ln y = \frac{1}{y\ln y}$

15. $F'(y) = \frac{d}{dy}[\ln(\sin 5y)] = \frac{1}{\sin 5y}\frac{d}{dy}(\sin 5y) = 5\frac{\cos 5y}{\sin 5y} = 5\cot 5y$

16. $f(x) = x\ln x$

▷ We apply the derivative of a product rule.

$$f'(x) = x \cdot D_x(\ln x) + \ln x \cdot D_x x = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$$

17. $f'(x) = \frac{d}{dx}[\cos(\ln x)] = -\sin(\ln x)\frac{d}{dx}(\ln x) = -\sin(\ln x) \cdot \frac{1}{x} = -\frac{\sin(\ln x)}{x}$

18. $g'(x) = \frac{d}{dx}\ln\cos\sqrt{x} = \frac{1}{\cos\sqrt{x}}\frac{d}{dx}\cos\sqrt{x} = \frac{1}{\cos\sqrt{x}}(-\sin\sqrt{x})\frac{d}{dx}\sqrt{x} = -\frac{\sin\sqrt{x}}{2\sqrt{x}\cos\sqrt{x}}$

19. $G'(x) = \frac{d}{dx}\ln(\sec 2x + \tan 2x) = \frac{1}{\sec 2x + \tan 2x}\frac{d}{dx}(\sec 2x + \tan 2x)$

$= \frac{1}{\sec 2x + \tan 2x}[\sec 2x\tan 2x(2) + \sec^2 2x(2)] = \frac{2\sec 2x(\tan 2x + \sec 2x)}{\sec 2x + \tan 2x} = 2\sec 2x$

20. $h(y) = \csc(\ln y)$

▷ $$h'(y) = -\csc(\ln y)\cot(\ln y)\, D_y(\ln y) = \frac{-\csc(\ln y)\cot(\ln y)}{y}$$

21. $f'(x) = \frac{d}{dx}\ln(\sqrt{\tan x}) = \frac{d}{dx}\left[\frac{1}{2}\ln(\tan x)\right] = \frac{1}{2}\frac{1}{\tan x}\frac{d}{dx}(\tan x) = \frac{1}{2\tan x} \cdot \sec^2 x = \frac{\cos x}{2\sin x} \cdot \frac{1}{\cos^2 x}$

$= \frac{1}{2\sin x\cos x} = \frac{1}{\sin 2x} = \csc 2x$

22. $f'(t) = \frac{d}{dt}\ln\sqrt[4]{\frac{t^2-1}{t^2+1}} = \frac{d}{dt}\frac{1}{4}[\ln(t^2-1) - \ln(t^2+1)] = \frac{1}{4}\left[\frac{2t}{t^2-1} - \frac{2t}{t^2+1}\right] = \frac{t}{t^4-1}$

23. $f'(w) = \frac{d}{dw}\left(\ln\sqrt{\frac{3w+1}{2w-5}}\right) = D_w\frac{1}{2}(\ln|3w+1| - \ln|2w-5|) = \frac{1}{2}\left(\frac{3}{3w+1} - \frac{2}{2w-5}\right)$

$= \frac{1}{2}\left[\frac{3(2w-5) - 2(3w+1)}{(3w+1)(2w-5)}\right] = -\frac{17}{3(3w+1)(2w-5)}$

24. $f(x) = \ln[(5x-3)^4(2x^2+7)^3]$

▸ First, we apply Theorems 5.2.4 and 5.2.6 to express $f(x)$ as a sum.

$$f(x) = \ln(5x-3)^4 + \ln(2x^2+7)^3 = 4\ln|5x-3| + 3\ln(2x^2+7)$$

Then,

$$f'(x) = 4\left(\frac{1}{5x-3}\right)(5) + 3\left(\frac{1}{2x^2+7}\right)(4x) = \frac{20}{5x-3} + \frac{12x}{2x^2+7}$$
$$= \frac{20(2x^2+7)+12x(5x-3)}{(5x-3)(2x^2+7)} = \frac{100x^2-36x+140}{(5x-3)(2x^2+7)}$$

25. $h'(x) = \frac{d}{dx}\left(\frac{x}{\ln x}\right) = \frac{1\cdot\ln x - \frac{1}{x}\cdot x}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$

26. $g'(x) = \frac{d}{dx}\ln(\cos 2x + \sin 2x) = \frac{1}{\cos 2x + \sin 2x}\frac{d}{dx}(\cos 2x + \sin 2x) = 2\frac{-\sin 2x + \cos 2x}{\cos 2x + \sin 2x}$

27. $g'(x) = \frac{d}{dx}\left(\ln\sqrt[3]{\frac{x+1}{x^2+1}}\right) = \frac{1}{3}\frac{d}{dx}[\ln(x+1) - \ln(x^2+1)] = \frac{1}{3}\left[\frac{1}{x+1} - \frac{2x}{x^2+1}\right] = \frac{1}{3}\cdot\frac{(x^2+1)-2x(x+1)}{(x+1)(x^2+1)}$
$= \frac{1-2x-x^2}{3(x+1)(x^2+1)}$

28. $f(x) = \sqrt[3]{\ln x^3}$

▸ By Theorem 5.2.6, $\ln x^3 = 3\ln x$. Thus,

$$f(x) = \sqrt[3]{\ln x^3} = \sqrt[3]{3\ln x} = \sqrt[3]{3}(\ln x)^{1/3}$$

Therefore,

$$f'(x) = \sqrt[3]{3}\left(\tfrac{1}{3}\right)(\ln x)^{-2/3}\, D_x(\ln x) = \frac{1}{3}\sqrt[3]{3}(\ln x)^{-2/3}\left(\frac{1}{x}\right) = \frac{\sqrt[3]{3}}{3x(\ln x)^{2/3}}$$

29. $F'(x) = \frac{d}{dx}[\sqrt{x+1} - \ln(1+\sqrt{x+1})] = \frac{1}{2\sqrt{x+1}} - \frac{1}{1+\sqrt{x+1}}\cdot\frac{1}{2\sqrt{x+1}} = \frac{(1+\sqrt{x+1})-1}{2\sqrt{x+1}(1+\sqrt{x+1})}$
$= \frac{1}{2(1+\sqrt{x+1})}$

30. $G'(x) = \frac{d}{dx}[x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2}]$
$= 1\cdot\ln(x+\sqrt{1+x^2}) + x\cdot\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{d}{dx}(x+\sqrt{1+x^2}) - \frac{1}{2\sqrt{1+x^2}}(2x)$
$= \ln(x+\sqrt{1+x^2}) + \frac{x}{x+\sqrt{1+x^2}}\left[1+\frac{x}{\sqrt{1+x^2}}\right] - \frac{x}{\sqrt{1+x^2}}$
$= \ln(x+\sqrt{1+x^2}) + \frac{x}{x+\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}} = \ln(x+\sqrt{1+x^2})$

In Exercises 31–38, find dy/dx by implicit differentiation.

31. $\ln xy + x + y = 2$; $\ln x + \ln y + x + y = 2$; $\frac{1}{x} + \frac{1}{y}\cdot\frac{dy}{dx} + 1 + \frac{dy}{dx} = 0$; $y + x\frac{dy}{dx} + xy + xy\frac{dy}{dx} = 0$
$(xy+x)\frac{dy}{dx} = -xy - y$; $\frac{dy}{dx} = -\frac{xy+y}{xy+x}$

32. $\ln\left(\frac{y}{x}\right) + xy = 1$

▸ First, we use Theorem 5.2.5 to simplify the equation. Thus, we have
$\ln y - \ln x + xy = 1$
Differentiating on both sides with respect to x, we obtain
$\frac{1}{y}\frac{dy}{dx} - \frac{1}{x} + x\frac{dy}{dx} + y = 0$
Multiplying on both sides by xy, we have

$$x\frac{dy}{dx} - y + x^2y\frac{dy}{dx} + xy^2 = 0; \qquad \frac{dy}{dx}(x + x^2y) = y - xy^2; \qquad \frac{dy}{dx} = \frac{y - xy^2}{x + x^2y}$$

33. $x = \ln(x+y+1)$; $1 = \frac{1}{x+y+1}\left(1+\frac{dy}{dx}\right)$; $x+y+1 = 1+\frac{dy}{dx}$; $\frac{dy}{dx} = x+y$

34. $\ln(x+y) - \ln(x-y) = 4$; $\frac{1}{x+y}(1 + \frac{dy}{dx}) - \frac{1}{x-y}(1 - \frac{dy}{dx}) = 0$; $(\frac{1}{x+y} + \frac{1}{x-y})\frac{dy}{dx} = \frac{1}{x-y} - \frac{1}{x+y}$

$\frac{2x}{x^2-y^2}\frac{dy}{dx} = \frac{2y}{x^2-y^2}$; $\frac{dy}{dx} = \frac{y}{x}$

35. $x + \ln x^2y + 3y^2 = 2x^2 - 1$; $x + 2\ln x + \ln y + 3y^2 = 2x^2 - 1$; $1 + \frac{2}{x} + \frac{1}{y}\frac{dy}{dx} + 6y\frac{dy}{dx} = 4x$

$xy + 2y + x\frac{dy}{dx} + 6xy^2\frac{dy}{dx} = 4x^2y$; $(6xy^2 + x)\frac{dy}{dx} = 4x^2y - xy - 2y$; $\frac{dy}{dx} = \frac{4x^2y - xy - 2y}{6xy^2 + x}$

36. $x \ln y + y \ln x = xy$

▹ We differentiate on both sides with respect to x, and multiply both sides by xy. Thus,

$$\frac{x}{y}\frac{dy}{dx} + \ln y + \frac{y}{x} + (\ln x)\frac{dy}{dx} = x \cdot \frac{dy}{dx} + y$$

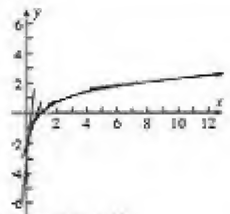
$$x^2\frac{dy}{dx} + xy\ln y + y^2 + xy(\ln x)\frac{dy}{dx} = x^2y\frac{dy}{dx} + xy^2$$

$$(x^2 + xy\ln x - x^2y)\frac{dy}{dx} = xy^2 - y^2 - xy\ln y$$

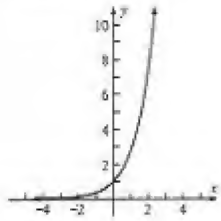
$$\frac{dy}{dx} = \frac{xy^2 - y^2 - xy\ln y}{x^2 + xy\ln x - x^2y} = \frac{y(xy - y - x\ln y)}{x(x - xy + y\ln x)}$$

37. The points plotted are $(\frac{1}{9}, -2\ln 3)$, $(\frac{1}{3}, -\ln 3)$, $(1, 0)$, $(3, \ln 3)$, and $(9, 2\ln 3)$. At each of these points a piece of the tangent line is drawn with respective slopes 9, 3, 1, $\frac{1}{3}$, and $\frac{1}{9}$. See the figure, below left.

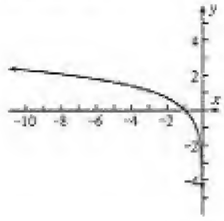
In Exercises 38–45, sketch the graph of the equation.



38. $x = \ln y$

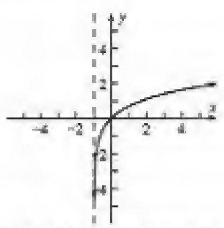


39. $y = \ln(-x)$

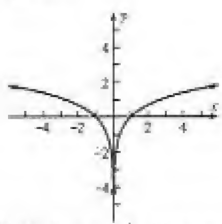


40. $y = \ln(x+1)$

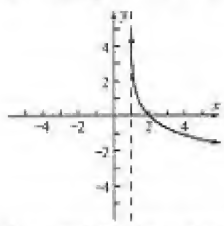
▹ Because the domain of the natural logarithmic function is the set of positive numbers, then $x + 1 > 0$, or $x > -1$. Thus, the domain of the function defined by the given equation is $(-1, +\infty)$. Furthermore, the line $x = -1$ is a vertical asymptote of the graph, and because $\ln 1 = 0$, the graph contains the point $(0, 0)$. A sketch of the graph is shown below, left.



41. $y = \ln|x|$



42. $y = \ln\frac{1}{x-1} = -\ln(x-1)$



43. Let $f(x) = x - \ln x$; $f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$; $f''(x) = \frac{1}{x^2}$. If $0 < x < 1$ then $f'(x) < 0$ and f is decreasing; if $x > 1$ then $f'(x) > 0$ and f is increasing. Hence f has an absolute minimum value at $x = 1$. The graph of f is concave upward. See the figure below.

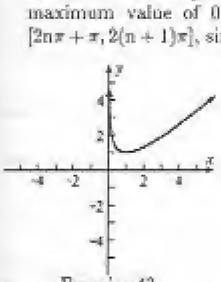
44. $y = x + 2\ln x$

▹ $$\frac{dy}{dx} = 1 + \frac{2}{x}; \qquad \frac{d^2y}{dx^2} = -\frac{2}{x^2}$$

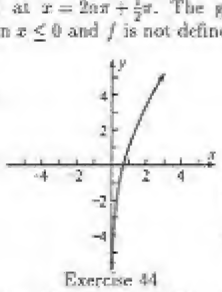
Because the domain of the natural logarithmic function is $x > 0$, then $dy/dx > 0$ and so the graph is

increasing. Because $d^2y/dx^2 < 0$, the graph is concave downward. Because $\lim_{x\to 0^+} y = -\infty$, then the y axis is a vertical asymptote. The graph is shown below.

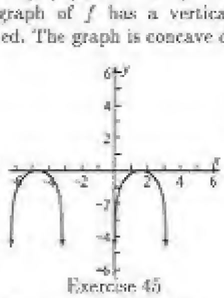
45. Let $f(x) = \ln \sin x$; $f'(x) = \frac{\cos x}{\sin x}$; $f''(x) = -\frac{1}{\sin^2 x}$. Let n be any integer. If x is in $(2n\pi, 2n\pi + \frac{1}{2}\pi)$, $f'(x) > 0$ and f is increasing; if x is in $(2n\pi + \frac{1}{2}\pi, 2n\pi + \pi)$, $f'(x) < 0$ and f is decreasing. Thus f has an absolute maximum value of 0 at $x = 2n\pi + \frac{1}{2}\pi$. The graph of f has a vertical asymptote at $x = n\pi$. If x is in $[2n\pi + \pi, 2(n+1)\pi]$, $\sin x \le 0$ and f is not defined. The graph is concave downward. See the figure below.



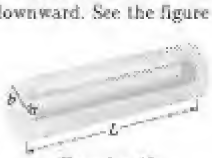
Exercise 43



Exercise 44



Exercise 45



Exercise 48

46. $T = \sqrt{\frac{3L}{g}}\csc\theta$; $\ln T = \frac{1}{2}(\ln\frac{3L}{g} + \ln\csc\theta)$; $\frac{dT}{T} = \frac{1}{2}(0 + \frac{-\csc\theta\cot\theta\, d\theta}{\csc\theta}) = -\frac{d\theta}{2\tan\theta}$

47. $PV = C$; $\ln P + \ln V = \ln C$; $\frac{dP}{P} + \frac{dV}{V} = 0$; $\frac{dP}{P} = -\frac{dV}{V}$

48. The capacitance between two coaxial cylinders of length L cm and radii a and b cm, shown in the figure above, is C farads, where $C = \frac{2\epsilon_0 L}{\ln(b/a)}$ and ϵ_0 is a positive constant. Find $\lim_{a\to b^-} C$.

▷ Because $a < b$, then $b/a > 1$ and $\ln(a/b) > 0$. Thus $\lim_{a\to b^-}\ln(a/b) = 0^+$ and so $\lim_{a\to b^-} C = +\infty$.

49. $y = \ln x$; $\frac{dy}{dx} = \frac{1}{x}$, $y_1 = \ln 2$ and $m = \frac{1}{2}$. The equation of the tangent line is $y = \frac{1}{2}(x-2) + \ln 2$.

50. $y = \ln x$; $\frac{dy}{dx} = \frac{1}{x}$. At the point (x_1, y_1) on the curve, the slope of the normal line is $-x_1$. The given line has the equation $y = -\frac{1}{2}x + \frac{1}{2}$; so its slope is $-\frac{1}{2}$. Because the two lines are parallel, $-x_1 = -\frac{1}{2}$, $x_1 = \frac{1}{2}$. Therefore the required normal line goes through the point $(\frac{1}{2}, -\ln 2)$ on the curve and it has slope $-\frac{1}{2}$. Its equation is

$y + \ln 2 = -\frac{1}{2}(x - \frac{1}{2})$; $y + \ln 2 = -\frac{1}{2}x + \frac{1}{4}$; $2x + 4y + 4\ln 2 - 1 = 0$

51. Find an equation of the normal line to the graph of $y = x\ln x$ that is perpendicular to the line $x - y + 7 = 0$.

▷ If the normal line to the graph of the equation $y = x\ln x$ is perpendicular to the line $x - y + 7 = 0$, then the tangent line to the graph of the equation $y = x\ln x$ is parallel to the line $x - y + 7 = 0$. Because the slope-intercept form of the equation of the given line is $y = x + 7$, we have $m = 1$. Thus, we find the point on the given curve where the tangent line has slope 1. We have

$y = x\ln x$; $\quad \frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$; $\quad$ If $\frac{dy}{dx} = 1$, then $\quad 1 = 1 + \ln x$; $0 = \ln x$; $\quad x = 1$

Substituting $x = 1$ into Eq. (1), we obtain $y = 0$, and thus conclude that the required line intersects the curve at the point (1,0). Furthermore, because the slope of the tangent line at that point is 1, then the slope of the normal line is -1, and an equation of the normal line is

$$y - 0 = (-1)(x-1); \quad x + y - 1 = 0$$

52. A particle is moving on a line according to the equation of motion $s = (t+1)^2\ln(t+1)$, where s feet is its directed distance from the starting point at t seconds. Find the velocity and acceleration when $t = 3$.

▷ At t sec, v ft/sec be the its velocity and a ft/sec^2 be it acceleration. Then, using the product rule,

$v(t) = \frac{ds}{dt} = \frac{d}{dt}[(t+1)^2\ln(t+1)] = 2(t+1)\ln(t+1) + (t+1)^2\frac{1}{t+1} = 2(t+1)\ln(t+1) + (t+1)$

$a(t) = \frac{dv}{dt} = 2\ln(t+1) + 2(t+1)\frac{1}{t+1} + 1 = 2\ln(t+1) + 3$

Hence $v(3) = 8\ln 4 + 4$ and $a(3) = 2\ln 4 + 3$.

▪ The velocity is $(8\ln 4 + 4)$ ft/sec and the acceleration is $(2\ln 4 + 3)$ ft/sec.

53. Let the measure of the speed of the signal be $v(x)$. Then for some $k > 0$,

$v(x) = kx^2 \ln \frac{1}{x} = -kx^2 \ln x$; $v'(x) = -k\left(2x \ln x + x^2 \cdot \frac{1}{x}\right) = -2kx(\ln x + \frac{1}{2})$

When $\ln x < -\frac{1}{2}$, $v'(x) > 0$ and when $\ln x > -\frac{1}{2}$, $v'(x) < 0$.

∎ v has an absolute maximum value when $\ln x = -\frac{1}{2}$.

54. $y = x \ln x$. $y' = 1 \ln x + x \cdot \frac{1}{x} = \ln x + 1$. In 1991, $x = 5$ and $y = \ln 5 + 1 \approx 2.609$; in 1996, $x = 10$ and $y = \ln 10 + 1 \approx 3.303$. Sales were \$2,609,000 and \$3,303,000.

55. If S dollars is the total weekly income from sales when the weekly advertising expense is x dollars then (a) $S = 4000 \ln x$; $\left.\frac{dS}{dx}\right|_{x=800} = \left.\frac{4000}{x}\right|_{x=800} = 5$

Therefore the weekly sales income changes at the rate of \$5 per \$1 change in the weekly advertising budget when the weekly advertising budget is \$800.

(b) When the weekly advertising budget is increased by \$150, an approximate increase in total weekly income from sales is $5 \cdot 150 = \$750$. The actual increase is $400 \ln 950 - 4000 \ln 800 \approx \$688$

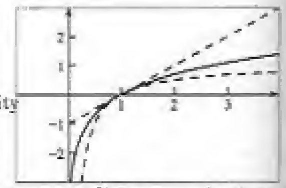
56. (a) Plot the graphs of

$$f(x) = 1 - \frac{1}{x} \quad g(x) = \ln x \quad h(x) = x - 1$$

in the same window and observe that $f(x) < g(x) < h(x)$.

(b) Confirm your observation in part (a) analytically by proving the inequality

$$1 - \frac{1}{x} < \ln x < x - 1, \quad x > 0 \text{ and } x \neq 1$$



▸ (a) A plot is shown at the right.

(b) Let $f(x) = x - 1 - \ln x$ and $g(x) = 1 - \ln x - \frac{1}{x}$. Then $f'(x) = 1 - \frac{1}{x}$. Because $f'(x) < 0$ in $(0, 1)$ and $f'(x) > 0$ in $(1, +\infty)$, f has an absolute minimum value when $x = 1$, and $f(1) = 0$. Hence

if $x \neq 1$, $x - 1 - \ln x > 0$; $\ln x < x - 1$ (1)

Also, $g'(x) = -\frac{1}{x} + \frac{1}{x^2}$. Because $g'(x) > 0$ in $(0, 1)$ and $g'(x) < 0$ in $(1, +\infty)$, g has an absolute maximum value at $x = 1$ and $g(1) = 0$. Hence

if $x \neq 1$, $1 - \ln x - \frac{1}{x} > 0$; $1 - \frac{1}{x} < \ln x$ (2)

Combining (1) and (2) we have shown that for $x > 0$, $x \neq 1$

$1 - \frac{1}{x} < \ln x < x - 1$ (3)

57. Substituting $x + 1$ for x in formula (3) above and dividing all terms by x gives

$1 - \frac{1}{x+1} < \ln(x+1) < x$; $\frac{1}{x+1} < \frac{\ln(x+1)}{x} < 1$

Now $\lim_{x \to 0} \frac{1}{x+1} = 1$, and $\lim_{x \to 0} 1 = 1$; so by the squeeze theorem $\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$. Alternatively, let $f(x) = \ln(1 + x)$; $f(0) = 0$. By the mean-value theorem there is a number c between 0 and x such that $\frac{\ln(1+x)}{x} = \frac{f(x) - f(0)}{x - 0} = f'(c) = \frac{1}{1+c}$. Then $\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{c \to 0} \frac{1}{1+c} = 1$.

58. Let $F(x) = \ln(1 + x)$ and so $F(0) = \ln 1 = 0$. By the definition of derivative,

$F'(0) = \lim_{x \to 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x \to 0} \frac{\ln(1+x)}{x}$ (4). Furthermore, $F'(x) = \frac{1}{1+x}$ and so $F'(0) = 1$ (5). Substituting from (5) into (4), we get $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$.

59. If $0 < t < 1$ then $\frac{1}{t^{3/2}} > \frac{1}{t} > 0$. By Theorem 4.6.1, if $0 < x < 1$ then

$\int_x^1 t^{-3/2} dt > \int_x^1 \frac{1}{t} dt > 0$; $-2t^{-1/2}\Big|_x^1 > \ln|t|\Big|_x^1 > 0$; $2x^{-1/2} - 2 > -\ln x > 0$; $2x^{1/2} - 2x > -x \ln x > 0$

Because $\lim_{x \to 0^+} 2x^{1/2} - 2x = 0$, it follows from the squeeze theorem that $\lim x \ln x = 0$. Now let $n > 0$.

If $y = x^n$, then $\lim_{x \to 0^+} x^n \ln x = \lim_{y \to 0^+} y \ln y^{1/n} = \frac{1}{n} \lim_{y \to 0^+} y \ln y = 0$

If $y = \frac{1}{x^n}$, then $\lim_{x \to +\infty} \frac{\ln x}{x^n} = \lim_{y \to 0^+} y \ln y^{-1/n} = -\frac{1}{n} \lim_{y \to 0^+} y \ln y = 0$

60. If $f(t)$ measures the market share of a substitute technology over t units of times, then
$$\ln\frac{f(t)}{1-f(t)}+\frac{\sigma}{1-f(t)}=c_1+c_2t \tag{1}$$
where σ, c_1, and c_2 are constants. Show that $f'(t)$, the rate of substitution, is given by
$$f'(t)=\frac{c_2f(t)[1-f(t)]^2}{\sigma f(t)+[1-f(t)]} \tag{2}$$

▷ Eq. (1) is equivalent to $c_1+c_2t=\ln f(t)-\ln[1-f(t)]+\sigma[1-f(t)]^{-1}$ (3)
Differentiating on both sides of eq. (3), factoring and adding, we obtain
$$c_2=\frac{f'(t)}{f(t)}+\frac{f'(t)}{1-f(t)}+\frac{\sigma f'(t)}{[1-f(t)]^2}=f'(t)\left[\frac{1}{f(t)}+\frac{1}{1-f(t)}+\frac{\sigma}{[1-f(t)]^2}\right]$$
$$=f'(t)\left[\frac{[1-f(t)]^2+f(t)[1-f(t)]+f(t)\sigma}{f(t)[1-f(t)]^2}\right]=f'(t)\left[\frac{[1-2f(t)+f^2(t)]+[f(t)-f^2(t)]+\sigma f(t)}{f(t)[1-f(t)]^2}\right]$$
and so
$$c_2=f'(t)\frac{\sigma f(t)+[1-f(t)]}{f(t)[1-f(t)]^2}$$ which is equivalent to (2).

5.3 LOGARITHMIC DIFFERENTIATION AND INTEGRALS YIELDING THE NATURAL LOGARITHMIC FUNCTION

5.3.1 Theorem If u is a differentiable function of x, $D_x(\ln|u|)=\frac{1}{u}D_xu$

Logarithmic Differentiation It follows that $D_xu=u\,D_x(\ln|u|)$ which can be used to simplify the work involved in differentiating complicated expressions involving powers products and quotients. The resulting formula is valid when $u=0$ provided it is continuous there.

5.3.2 Theorem $\int\frac{1}{u}\,du=\ln|u|+C$

5.3.3 Theorem $\int\tan u\,du=\ln|\sec u|+C$

5.3.4 Theorem $\int\cot u\,du=\ln|\sin u|+C=-\ln|\csc u|+C$

5.3.5 Theorem $\int\sec u\,du=\ln|\sec u+\tan u|+C$

5.3.6 Theorem $\int\csc u\,du=\ln|\csc u-\cot u|+C=\ln\left|\tan\frac{1}{2}u\right|+C$

Proof $\csc u-\cot u=\frac{1-\cos u}{\sin u}=\frac{2\sin^2\frac{1}{2}u}{2\sin\frac{1}{2}u\cos\frac{1}{2}u}=\frac{\sin\frac{1}{2}u}{\cos\frac{1}{2}u}=\tan\frac{1}{2}u$

Transcendental Function A function which is not algebraic (§ 2.9). $\ln x$ is transcendental. See Exercise 56.

Problems 5.3

In Exercises 1–8, use Theorem 5.3.1 to find dy/dx.

1. $y=\ln|x^3+1|$ ▷ $\frac{dy}{dx}=\frac{1}{x^3+1}\frac{d}{dx}(x^3+1)=\frac{3x^2}{x^3+1}$

2. $y=\ln|x^2-1|$ ▷ $\frac{dy}{dx}=\frac{1}{x^2-1}\frac{d}{dx}(x^2-1)=\frac{2x}{x^2-1}$

3. $y=\ln|\cos 3x|$ ▷ $\frac{dy}{dx}=\frac{1}{\cos 3x}\frac{d}{dx}(\cos 3x)=-\frac{3\sin 3x}{\cos 3x}=-3\tan 3x$

4. $y=\ln|\sec 2x|$ ▷ $\frac{dy}{dx}=\frac{1}{\sec 2x}D_x(\sec 2x)=\frac{2\sec 2x\tan 2x}{\sec 2x}=2\tan 2x$

5. $y=\ln|\tan 4x+\sec 4x|$ ▷ $\frac{dy}{dx}=\frac{1}{\tan 4x+\sec 4x}\frac{d}{dx}(\tan 4x+\sec 4x)=\frac{4\sec^2 4x+4\sec 4x\tan 4x}{\tan 4x+\sec 4x}$
$=\frac{4\sec 4x(\sec 4x+\tan 4x)}{\tan 4x+\sec 4x}=4\sec 4x$

6. $y=\ln|\cot 3x-\csc 3x|$ ▷ $\frac{dy}{dx}=\frac{1}{\cot 3x-\csc 3x}\frac{d}{dx}(\cot 3x-\csc 3x)=\frac{-3\csc^2 3x+3\csc 3x\cot 3x}{\cot 3x-\csc 3x}$
$=3\csc 3x\frac{-\csc 3x+\cot 3x}{\cot 3x-\csc 3x}=3\csc 3x$

7. $y = \ln\left|\dfrac{3x}{x^2+4}\right| = \ln 3 + \ln|x| - \ln(x^2+4)$ ▷ $\dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{2x}{x^2+4} = \dfrac{(x^2+4) - x(2x)}{x(x^2+4)} = \dfrac{4-x^2}{x(x^2+4)}$

8. $y = \sin(\ln|2x+1|)$

▷ $$\frac{dy}{dx} = \cos(\ln|2x+1|)\, D_x \ln|2x+1| = \cos(\ln|2x+1|)\left(\frac{1}{2x+1}\right)(2) = \frac{2\cos(\ln|2x+1|)}{2x+1}$$

In Exercises 9–14, find dy/dx by logarithmic differentiation.

9. $y = x^2(x^2-1)^3(x+1)^4 = x^2(x-1)^3(x+1)^7$; $|y| = |x|^2|x-1|^3|x+1|^7$

$\ln|y| = 2\ln|x| + 3\ln|x-1| + 7\ln|x+1|$; $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2}{x} + \dfrac{3}{x-1} + \dfrac{7}{x+1}$

$$\frac{dy}{dx} = y\left[\frac{2(x-1)(x+1) + 3x(x+1) + 7x(x-1)}{x(x-1)(x+1)}\right] = x^2(x-1)^3(x+1)^7 \cdot \frac{12x^2-4x-2}{x(x-1)(x+1)}$$
$$= 2x(x-1)^2(x+1)^6(6x^2-2x-1)$$

10. $y = (5x-4)(x^2+3)(3x^3-5)$; $|y| = |5x-4||x^2+3||3x^3-5|$; $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{5}{5x-4} + \dfrac{2x}{x^2+3} + \dfrac{9x^2}{3x^3-5}$

$$\frac{dy}{dx} = y\left[\frac{5(x^2+3)(3x^3-5) + 2x(5x-4)(3x^3-5) + 9x^2(5x-4)(x^2+3)}{(5x-4)(x^2+3)(3x^3-5)}\right] =$$
$$(5x-4)(x^2+3)(3x^3-5)\cdot\frac{90x^5-60x^4+180x^3-183x^2+40x-75}{(5x-4)(x^2+3)(3x^3-5)} = 90x^5-60x^4+180x^3-183x^2+40x-75$$

Because this problem does not involve exponents, the easiest method is to multiply out before differentiating.

11. $y = \dfrac{x^2(x-1)^2(x+2)^3}{(x-4)^5}$; $|y| = \dfrac{|x|^2|x-1|^2|x+2|^3}{|x+4|^5}$

$\ln|y| = 2\ln|x| + 2\ln|x-1| + 3\ln|x+2| - 5\ln|x-4|$; $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2}{x} + \dfrac{2}{x-1} + \dfrac{3}{x+2} - \dfrac{5}{x-4}$

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{2}{x-1} + \frac{3}{x+2} - \frac{5}{x-4}\right)$$
$$= \frac{x^2(x-1)^2(x+2)^3}{(x-4)^5}\cdot\frac{2(x-1)(x+2)(x-4) + 2x(x+2)(x-4) + 3x(x-1)(x-4) - 5x(x-1)(x+2)}{x(x-1)(x+2)(x-4)}$$
$$= \frac{x(x-1)(x+2)^2}{(x-4)^6}(2x^3-30x^2-6x+16)$$

12. $y = x^5(x+2)/(x-3)$

▷ Because the domain of the natural logarithmic function is the set of all positive numbers, we first take the absolute value of each side, and then take the natural logarithm of each side and apply the laws of logarithms to simplify the right-hand side.

$$|y| = \left|\frac{x^5(x+2)}{(x-3)}\right|; \qquad \ln|y| = 5\ln|x| + \ln|x+2| - \ln|x-3|$$

We differentiate on both sides implicitly with respect to x, applying Theorem 5.3.1.

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{5}{x} + \frac{1}{x+2} - \frac{1}{x-3} = \frac{5(x+2)(x-3) + x(x-3) - x(x+2)}{x(x+2)(x-3)} = \frac{5x^2-10x-30}{x(x+2)(x-3)}$$

Because $y = x^5(x+2)/(x-3)$, if we multiply on both sides by y we obtain

$$\frac{dy}{dx} = \frac{x^5(x+2)}{x-3}\cdot\frac{5(x^2-2x-6)}{x(x+2)(x-3)} = \frac{5x^4(x^2-2x-6)}{(x-3)^2}$$

13. $y = \dfrac{x^3+2x}{\sqrt[5]{x^7+1}}$; $|y| = \dfrac{|x^3+2x|}{|x^7+1|^{1/5}}$; $\ln|y| = \ln|x^3+2x| - \dfrac{1}{5}\ln|x^7+1|$; $\dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{3x^2+2}{x^3+2x} - \dfrac{7x^6}{5(x^7+1)}$

$$\frac{dy}{dx} = y\cdot\frac{5(3x^2+2)(x^7+1) - 7x^6(x^3+2x)}{5(x^3+2x)(x^7+1)} = \frac{x^3+2x}{(x^7+1)^{1/5}}\cdot\frac{15x^9+10x^7+15x^2+10-7x^9-14x^7}{5(x^3+2x)(x^7+1)}$$
$$= \frac{8x^9-4x^7+15x^2+10}{5(x^7+1)^{6/5}}$$

14. $y = \dfrac{\sqrt{1-x^2}}{(x+1)^{2/3}} = \dfrac{(1-x)^{1/2}(1+x)^{1/2}}{(x+1)^{2/3}} = (1-x)^{1/2}(1+x)^{-1/6}$; $\ln y = \frac{1}{2}\ln(1-x) - \frac{1}{6}\ln(1+x)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{-1}{1-x} - \dfrac{1}{6}\dfrac{1}{1+x}$; $\dfrac{dy}{dx} = y \cdot \dfrac{-3(1+x)-(1-x)}{6(1-x)(1+x)} = (1-x)^{1/2}(1+x)^{-1/6}\dfrac{-2(x+2)}{6(1-x)(1+x)}$

$= -\dfrac{x+2}{3(1-x)^{1/2}(1+x)^{7/6}}$

In Exercises 15–32, evaluate the indefinite integral.

15. $\displaystyle\int \frac{dx}{3-2x} = -\frac{1}{2}\int \frac{-2\,dx}{3-2x} = -\frac{1}{2}\ln|3-2x| + C$

16. $\displaystyle\int \frac{x}{2-x^2}dx$

▹ Let $u = 2 - x^2$. Then $du = -2x\,dx$, and by Theorem 5.3.2 we have

$$\int \frac{x\,dx}{2-x^2} = \int \frac{-\frac{1}{2}\,du}{u} = -\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2-x^2| + C$$

17. $\displaystyle\int \frac{3x^2}{5x^3-1}\,dx = \frac{1}{5}\int \frac{15x^2dx}{5x^3-1} = \frac{1}{5}\ln|5x^3-1| + C$

18. $\displaystyle\int \frac{2x-1}{x(x-1)}dx = \int \frac{(2x-1)dx}{x^2-x} = \ln|x^2-x| + C$

19. Let $u = \ln x$; then $du = \frac{dx}{x}$; $\displaystyle\int \frac{dx}{x\ln x} = \int \frac{1}{\ln x}\cdot\frac{dx}{x} = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln x| + C$

20. $\displaystyle\int \frac{\sin 3t}{\cos 3t - 1}\,dt$

▹ Let $u = \cos 3t - 1$. Then $du = -3\sin 3t\,du$, and by Theorem 5.3.2 we have

$$\int \frac{\sin 3t\,dt}{\cos 3t - 1} = \int \frac{-\frac{1}{3}du}{u} = -\frac{1}{3}\int \frac{du}{u} = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|\cos 3t - 1| + C$$

21. $\displaystyle\int (\cot 5x + \csc 5x)dx = \frac{1}{5}\ln|\sin 5x| + \frac{1}{5}\ln|\csc 5x - \cot 5x| + C$

$= \frac{1}{5}\ln\left|\sin 5x\left(\frac{1}{\sin 5x} - \frac{\cos 5x}{\sin 5x}\right)\right| + C = \frac{1}{5}\ln(1-\cos 5x) + C$

22. $\displaystyle\int \frac{\cos 3x + 3}{\sin 3x}\,dx = \int (\cot 3x + 3\csc 3x)dx = \frac{1}{3}\ln|\sin 3x| + \ln|\csc 3x - \cot 3x| + C$

23. $\displaystyle\int \frac{2 - 3\sin 2x}{\cos 2x}\,dx = \int (2\sec 2x - 3\tan 2x)dx$

$= \ln|\sec 2x + \tan 2x| - \frac{3}{2}\ln|\sec 2x| + C = \ln(1+\sin 2x) + \frac{1}{2}\ln|\cos 2x| + C$

24. $\displaystyle\int (\tan 2x - \sec 2x)dx$

▹ Let $u = 2x$ and $du = 2\,dx$. Thus,

$$\int (\tan 2x - \sec 2x)\,dx = \frac{1}{2}\int (\tan u - \sec u)\,du = \frac{1}{2}(\ln|\sec u| - \ln|\sec u + \tan u|) + C$$

$$= \frac{1}{2}(\ln|\sec 2x| - \ln|\sec 2x + \tan 2x|) + C = -\frac{1}{2}\ln(1+\sin 2x) + C$$

25. $\displaystyle\int \frac{2x^3}{x^2-4}\,dx = \int \left(2x + 4\cdot\frac{2x}{x^2-4}\right)dx = x^2 + 4\ln|x^2-4| + C$

26. By long division, $\displaystyle\int \frac{5-4y^2}{3+2y}dy = \int \left(-2y + 3 - 2\frac{2}{3+2y}\right)dy = -y^2 + 3y - 2\ln|3+2y| + C$

27. Let $u = \ln 3x$; then $du = \frac{dx}{x}$; $\displaystyle\int \frac{\ln^2 3x}{x}\,dx = \int (\ln 3x)^2\cdot\frac{dx}{x} = \int u^2du = \frac{1}{3}u^3 + C = \frac{1}{3}\ln^3 3x + C$

28. $\displaystyle\int \frac{(2+\ln^2 x)dx}{x(1-\ln x)}$

▹ Let $u = \ln x$. Then $du = dx/x$. Thus,

$$\int \frac{(2+\ln^2 x)dx}{x(1-\ln x)} = \int \frac{2+\ln^2 x}{1-\ln x}\cdot\frac{dx}{x} = \int \frac{2+u^2}{1-u}du$$

Dividing the numerator by the denominator, and using the the fact that $1-u$ is linear, we obtain

$$\int \frac{2+u^2}{1-u}du = \int (-u-1)du + 3\int \frac{du}{1-u} = -\tfrac{1}{2}u^2 - u - 3\ln|1-u| + C$$

Replacing u by $\ln x$, we get

$$\int \frac{(2+\ln^2 x)dx}{x(1-\ln x)} = -\tfrac{1}{2}\ln^2 x - \ln x - 3\ln|1-\ln x| + C$$

29. Let $u = \ln x$; then $du = \frac{dx}{x}$.

$$\int \frac{2\ln x + 1}{x[(\ln x)^2 + \ln x]}dx = \int \frac{2\ln x+1}{(\ln x)^2+\ln x}\cdot\frac{dx}{x} = \int \frac{2u+1}{u^2+u}du = \ln|u^2+u| + C = \ln|(\ln x)^2 + \ln x| + C$$

30. By long division,

$$\int \frac{3x^5 - 2x^3 + 5x^2 - 2}{x^3+1}dx = \int \left(3x^2 - 2 + \frac{2x^2}{x^3+1}\right)dx = \int \left(3x^2 - 2 + \tfrac{2}{3}\cdot\frac{3x^2}{x^3+1}\right) = x^3 - 2x + \tfrac{2}{3}\ln|x^3+1| + C$$

31. Let $u = \ln x$; then $du = \frac{dx}{x}$; $\int \frac{\tan(\ln x)}{x}dx = \int \tan u\, du = \ln|\sec u| + C = \ln|\sec(\ln x)| + C$

32. $\int \frac{\cot\sqrt{t}}{\sqrt{t}}dt$

▷ Let $u = \sqrt{t}$. Then $du = \frac{1}{2}t^{-1/2}\,dt$, and so $dt/\sqrt{t} = 2\,du$. Thus

$$\int \frac{\cot\sqrt{t}}{\sqrt{t}} = 2\int \cot u\, du = 2\ln|\sin u| + C = 2\ln|\sin\sqrt{t}| + C$$

In Exercises 33–44, find the exact value of the definite integral and check by NINT.

33. $\int_0^2 \frac{3x}{x^2+4}dx = \frac{3}{2}\int_0^2 \frac{2x\,dx}{x^2+4} = \frac{3}{2}\ln(x^2+4)\Big]_0^2 = \frac{3}{2}(\ln 8 - \ln 4) = \frac{3}{2}\ln 2 \approx 1.0397$

34. $\int_0^2 \frac{1}{7x+10}dx = \frac{1}{7}\ln|7x+10|\Big]_0^2 = \frac{1}{7}(\ln 24 - \ln 10) = \frac{1}{7}\ln 2.4 \approx 0.1251$

35. $\int_4^5 \frac{x}{4-x^2}dx = -\frac{1}{2}\int_4^5 \frac{-2x\,dx}{4-x^2} = -\frac{1}{2}\ln|4-x^2|\Big]_4^5 = -\frac{1}{2}(\ln 21 - \ln 12) = \frac{1}{2}\ln\frac{4}{7} \approx -0.2798$

36. $\int_1^5 \frac{4x^3-1}{2x-1}dx$

▷ We divide the numerator of the fraction by the linear denominator, which yields

$$\int_1^5 \frac{4x^3-1}{2x-1}dx = \int_1^5 \left(2x^2 + x + \frac{1}{2} - \frac{\frac{1}{2}}{2x-1}\right)dx = \left[\frac{2}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4}\ln|2x-1|\right]_1^5 = \frac{290}{3} - \frac{1}{4}\ln 9 \approx 96.117$$

We confirm this value using NINT.

37. $\int_1^3 \frac{2t+3}{t+1}dt = \int_1^3\left(2 + \frac{1}{t+1}\right)dt = 2t + \ln|t+1|\Big|_1^3 = (6+\ln 4) - (2+\ln 2) = 4 + \ln\frac{4}{2} = 4 + \ln 2$

38. $\int_3^5 \frac{2x}{x^2-5}dx = \ln|x^2-5|\Big|_3^5 = \ln 20 - \ln 4 = \ln\frac{20}{4} = \ln 5$

39. $\int_0^{\pi/2} \frac{\cos t}{1+2\sin t}dt = \frac{1}{2}\int_0^{\pi/2}\frac{2\cos t\,dt}{1+2\sin t} = \frac{1}{2}\ln|1+2\sin t|\Big]_0^{\pi/2} = \frac{1}{2}[\ln(1+2\cdot 1) - \ln(1+0)] = \frac{1}{2}\ln 3 \approx 0.5493$

40. $\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})}dx$

▷ Let $u = \sqrt{x}$; $du = \frac{1}{2\sqrt{x}}dx$. when $x = 1$, $u = 1$; when $x = 4$, $u = 2$. Therefore

$$\int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})}dx = 2\int_1^4 \frac{1}{1+\sqrt{x}}\frac{dx}{2\sqrt{x}} = 2\int_1^2 \frac{1}{u}du = 2\ln|u|\Big]_1^2 = 2(\ln 2 - \ln 1) = 2\ln 2 \approx 1.3863$$

We confirm this value using NINT.

41. $\int_0^{\pi/6}(\tan 2x + \sec 2x)dx = \left[\frac{1}{2}\ln|\sec 2x| + \frac{1}{2}\ln|\sec 2x + \tan 2x|\right]_0^{\pi/6}$

$= \frac{1}{2}[\ln 2 + \ln(2+\sqrt{3})] = \frac{1}{2}\ln(4+2\sqrt{3}) = \ln(1+\sqrt{3})$

42. $\int_{\pi/12}^{\pi/6}(\cot 3x + \csc 3x)dx = \frac{1}{3}\Big[\ln|\sin 3x| + \ln|\csc 3x - \cot 3x|\Big]_{\pi/12}^{\pi/6}$

$= \frac{1}{3}[\ln(\sin \frac{1}{2}\pi) - \ln(\sin \frac{1}{4}\pi) + \ln(\csc \frac{1}{2}\pi - \cot \frac{1}{2}\pi) - \ln(\csc \frac{1}{4}\pi - \cot \frac{1}{4}\pi)]$

$= \frac{1}{3}[\ln 1 - \ln(\frac{1}{2}\sqrt{2}) + \ln(1 - 0) - \ln(\sqrt{2} - 1)] = -\frac{1}{3}\ln[\frac{1}{2}(2 - \sqrt{2})] = \frac{1}{3}\ln[2(2 + \sqrt{2})] \approx 0.6404$

43. Let $u = \ln x$; then $du = \frac{dx}{x}$; $\int_2^4 \frac{dx}{x \ln^2 x} = \int_{\ln 2}^{\ln 4} u^{-2}du = -\frac{1}{u}\Big|_{\ln 2}^{\ln 4} = -\frac{1}{\ln 4} + \frac{1}{\ln 2} = -\frac{1}{\ln 4} + \frac{2}{\ln 4} = \frac{1}{\ln 4}$

44. $\int_2^4 \frac{\ln x}{x}\, dx$

▹ Let $u = \ln x$, and $du = (1/x)\, dx$. When $x = 2$, then $u = \ln 2$; when $x = 4$, then $u = \ln 4$.

$$\int_2^4 \frac{\ln x}{x}dx = \int_{\ln 2}^{\ln 4} u\, du = \frac{1}{2}u^2\Big]_{\ln 2}^{\ln 4} = \frac{1}{2}(\ln^2 4 - \ln^2 2) = \frac{1}{2}[(2\ln 2)^2 - \ln^2 2] = \frac{3}{2}\ln^2 2 \approx 0.7207$$

We confirm this value using NINT.

45. (a) $\int \cot u\, du = \int \frac{\cos u}{\sin u}\, du = \ln|\sin u| + C$

(b) Let $v = \csc u - \cot u$; then $dv = (-\csc u \cot u + \csc^2 u)du$;

$\int \csc u\, du = \int \frac{\csc u(\csc u - \cot u)}{\cos u - \cot u}du = \int \frac{dv}{v} = \ln|v| + C = \ln|\csc u - \cot u| + C$

(c) $\int \csc u\, du = \int \sec(u - \frac{1}{2}\pi)du = \ln\left|\sec(u - \frac{1}{2}\pi) + \tan(u - \frac{1}{2}\pi)\right| + C = \ln|\csc u - \cot u| + C$

46. (a) $\int \csc u\, du = \ln|\csc u - \cot u| + C = -\ln\left|\frac{1}{\csc u - \cot u}\cdot\frac{\csc u + \cot u}{\csc u + \cot u}\right| + C = -\ln\left|\frac{\csc u + \cot u}{\csc^2 u - \cot^2 u}\right| + C$

$= -\ln|\csc u + \cot u| + C$

(b) Let $v = \csc u + \cot u$; then $dv = (-\csc^2 u - \csc u \cot u)du$

$\int \csc u\, du = \int \csc u \frac{\csc u + \cot u}{\csc u + \cot u}du = \int \frac{(\csc^2 u + \csc u \cot u)du}{\csc u + \cot u} = \int \frac{-dv}{v} = -\ln|v| + C = -\ln|\csc u + \cot u| + C$

47. $f(x) = \frac{1}{x}$. The average value of f on $[1, 5]$ is $\frac{1}{5 - 1}\int_1^5 \frac{1}{x}\, dx = \frac{1}{4}\ln 5 \approx 0.402$.

48. If $f(x) = (x + 2)/(x - 3)$, find the average value of f on the interval $[4, 6]$.

▹ The average value of f on the interval $[4, 6]$ is given by

$$\text{A.V.} = \frac{1}{6 - 4}\int_4^6 \frac{x + 2}{x - 3}\, dx = \frac{1}{2}\int_4^6\left(1 + \frac{5}{x - 3}\right)dx = \frac{1}{2}\Big[x + 5\ln|x - 3|\Big]_4^6 = \frac{1}{2}[(6 + 5\ln 3) - (4 + 5\ln 1)] = 1 + \frac{5}{2}\ln 3$$

49. Let P lb/ft^2 be the pressure when the volume is V ft^3. Then by Boyle's law, PV = C. Because P = 2000 when V = 4, then C = 8000. Therefore $P = \frac{8000}{V}$. The average value of P as V increases from 4 to 8 is

$\frac{1}{8 - 4}\int_4^8 \frac{8000}{V}\, dV = 2000\ln|V|\Big]_4^8 = 2000(\ln 8 - \ln 4) = 2000\ln\frac{8}{4} = 2000\ln 2 \approx 1386$

Therefore the average pressure is 1386 lb/ft^2.

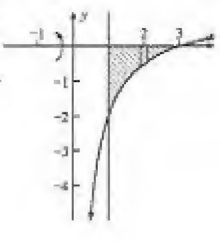
50. $A = \int_0^4 \frac{x}{2x^2 + 4}\, dx = \frac{1}{4}\int_0^4 \frac{4x\, dx}{2x^2 + 4} = \frac{1}{4}\ln(2x^2 + 4)\Big]_0^4 = \frac{1}{4}(\ln 36 - \ln 4) = \frac{1}{4}\ln 9 = \frac{1}{2}\ln 3$

51. A square units is the area bounded by $y = 2/(x - 2)$, the x axis, $x = 4$ and $x = 5$. Then

$A = \lim_{\|\Delta\| \to 0}\sum_{i=1}^{n}\frac{2}{w_i - 3}\Delta_i x = \int_4^5 \frac{2}{x - 3}\, dx = 2\ln|x - 3|\Big|_4^5 = 2\ln 2 - 2\ln 1 = 2\ln 2 = \ln 4 \approx 1.386$

52. Find the volume of the solid of revolution generated when the region bounded by the curve $y = 1 - 3/x$, the x axis, and the line $x = 1$ is revolved about the x axis.

▹ See the figure. Let $f(x) = 1 - 3/x$. The element of volume is a circular disk of thickness $\Delta_i x$ units and radius $|f(w_i)|$ units. Thus, if V cubic units is the volume,



$$V = \lim_{\|\Delta\| \to 0}\sum_{i=1}^{n}\pi[f(w_i)]^2\Delta_i x = \int_1^3 \pi\left(1 - \frac{3}{x}\right)^2 dx$$

$$= \int_1^3 \pi\left(1 - \frac{6}{x} + \frac{9}{x^2}\right)dx = \pi\left[x - 6\ln|x| - \frac{9}{x}\right]_1^3 = (8 - 6\ln 3)\pi$$

53. V cubic units is the volume when $y = 1 + 2/\sqrt{x}$ is revolved about the x axis. An element of volume is a circular disk centered on the x axis, $x \in [1, 4]$, of radius $1 + 2/\sqrt{w_i}$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\left(1 + \frac{2}{w_i^{1/2}}\right)^2 \Delta_i x = \pi \int_1^4 \left(1 + \frac{2}{x^{1/2}}\right)^2 dx = \pi \int_1^4 \left(1 + \frac{4}{x^{1/2}} + \frac{4}{x}\right) dx = \pi\left[x + 8x^{1/2} + 4 \ln|x|\right]_1^4$$
$$= \pi[(4 + 16 + 4 \ln 4) - (1 + 8 + 4 \ln 1)] = \pi(11 + 8 \ln 2) \approx 51.978$$

54. $d > 2a$. $L = \int_a^{d-a} \frac{\mu_0 I}{2\pi}\left(\frac{1}{x} + \frac{1}{d - x}\right) dx = \frac{\mu_0 I}{2\pi}\Big[\ln|x| - \ln|d - x|\Big]_a^{d-a} = \frac{\mu_0 I}{2\pi}[\ln(d - a) - \ln a - \ln a + \ln(d - a)]$
$= \frac{\mu_0 I}{\pi} \ln\left(\frac{d - a}{a}\right)$

55. Prove that $\lim\limits_{x \to +\infty} \frac{\ln x}{x} = 0$ by two methods. (a) Let $x = \frac{1}{t}$ and use the result of Exercise 5.2.57.

(b) First prove that $\int_1^x \frac{1}{\sqrt{t}}\, dt > \int_1^x \frac{1}{t}\, dt$ by applying Theorem 4.6.1. Then use the squeeze theorem.

▸ (a) The given Exercise states that $\lim\limits_{x \to 0^+} x \ln x = 0$ (1)

If $x = \frac{1}{t}$ then $t \to +\infty$ as $x \to 0^+$ and $\ln x = \ln \frac{1}{t} = -\ln t$. Substituting into (1), we have

$\lim\limits_{t \to +\infty} \frac{1}{t}(-\ln t) = 0$ or, equivalently, $\lim\limits_{x \to +\infty} \frac{\ln x}{x} = 0$

(b) Because $x \to +\infty$, we may assume that $x \geq 1$. Because $\ln 1 = 0$ and the ln function is increasing, we have

$\ln x \geq 0 \quad \text{if } x \geq 1$ (2)

Furthermore, if $t \geq 1$, then $\sqrt{t} \leq t$, so $\frac{1}{\sqrt{t}} \geq \frac{1}{t}$ if $t \geq 1$

Therefore, by Theorem 4.6.1,

$$\int_1^x \frac{1}{\sqrt{t}}\, dt \geq \int_1^x \frac{1}{t}\, dt \quad \text{if } x \geq 1 \tag{3}$$

Now

$$\int_1^x \frac{1}{\sqrt{t}}\, dt = \int_1^x t^{-1/2} dt = 2t^{1/2}\Big]_1^x = 2(x^{1/2} - 1) \tag{4}$$

and

$$\int_1^x \frac{1}{t}\, dt = \ln x \tag{5}$$

Substituting from (4) and (5) into (3), we have

$$2(x^{1/2} - 1) \geq \ln x \quad \text{if } x \geq 1 \tag{6}$$

Combining (2) and (6), we have $\quad 0 \leq \ln x \leq 2(x^{1/2} - 1) \quad \text{if } x \geq 1$

Dividing by x if $x \geq 1$, we obtain $\quad 0 \leq \frac{\ln x}{x} \leq 2\left(\frac{1}{x^{1/2}} - \frac{1}{x}\right) \quad \text{if } x \geq 1$

Because $\quad \lim\limits_{x \to +\infty} 2\left(\frac{1}{x^{1/2}} - \frac{1}{x}\right) = 0$

by the squeeze theorem we conclude that $\quad \lim\limits_{x \to +\infty} \frac{\ln x}{x} = 0$

56. Prove that $\ln x$ is transcendental.

▸ Replace x with $x^{1/n}$ in Ex. 55: $0 = \lim\limits_{x \to +\infty} \frac{\ln x^{1/n}}{x^{1/n}} = \frac{1}{n} \lim\limits_{x \to +\infty} \frac{\ln x}{x^{1/n}} = \frac{1}{n}\left[\lim\limits_{x \to +\infty} \frac{(\ln x)^n}{x}\right]^{1/n}$ so $\lim\limits_{x \to +\infty} \frac{(\ln x)^n}{x} = 0$.

Suppose that $y = \ln x$ is algebraic. Then y satisfies a polynomial equation
$0 = b_n(y)x^n + b_{n-1}(y)x^{n-1} + \cdots + b_1(y)x + b_0(y)$ where $b_n(y) \not\equiv 0$. Then

$$0 = \lim_{x \to +\infty}\left[b_n(y) + \frac{b_{n-1}(y)}{x} + \cdots + \frac{b_1(y)}{x^{n-1}} + \frac{b_0(y)}{x^n}\right] = \lim_{x \to +\infty} b_n(\ln x)$$

Because the last limit is infinite, we have a contradiction. Thus $\ln x$ is transcendental.

5.4 THE NATURAL EXPONENTIAL FUNCTION

5.4.1 Definition The *natural exponential function* is the inverse of the natural logarithmic function; thus it is defined by

$$\exp(x) = y \quad \text{if and only if } x = \ln y$$

5.4.2 Definition If a is any positive number and x is any real number, we define

$$a^x = \exp(x \ln a) = e^{x \ln a}$$

Furthermore, $0^x = 0$ if $x > 0$. If $a < 0$ and x is rational in lowest terms with an odd denominator, then $a^x = -|a|^x$.

5.4.3 Theorem If a is any positive number and x is any real number,

$$\ln a^x = x \ln a$$

5.4.4 Definition The number e is defined by the formula

$$e = \exp 1$$

The value of e to seven decimal places is 2.7182818.

5.4.5 Theorem $\ln e = 1$

5.4.6 Theorem For all values of x

$$\exp x = e^x$$

and e^x is always positive.

5.4.7 Theorem If a and b are any real numbers, then

$$e^a \cdot e^b = e^{a+b}$$

5.4.8 Theorem If a and b are any real numbers, then

$$e^a \div e^b = e^{a-b}$$

5.4.9 Theorem If a and b are any real numbers, then

$$(e^a)^b = e^{ab}$$

5.4.10 Theorem If u is a differentiable function of x,

$$D_x(e^u) = e^u \, D_x u$$

5.4.11 Theorem

$$\int e^u \, du = e^u + C$$

5.4.12 Theorem If n is any real number and the function f is defined by

$f(x) = x^n$ where $x > 0$ then $f'(x) = nx^{n-1}$

Distinguish carefully between x^a and a^x, where a is a real number. We have

$D_x(a^x) = a^x \ln a$ if $a > 0$ $\qquad D_x(x^a) = ax^{a-1}$ if $x > 0$

The following limits are important.

$$\lim_{x \to +\infty} e^x = +\infty, \quad \lim_{x \to -\infty} e^x = 0, \quad \lim_{h \to 0}(1+h)^{1/h} = e$$

Because the natural exponential function and the natural logarithmic function are inverse functions, we have

$$\exp(\ln x) = x \quad \text{and } \ln(\exp x) = x \text{ or equivalently } e^{\ln x} = x \text{ and } \ln e^x = x$$

Exercises 5.4

In Exercises 1–4, compute the value of a^x by using $e^{x \ln a}$. Support your answer by computing the value directly.

1. (a) $2^{\sqrt{2}} = e^{\sqrt{2} \ln 2} = e^{0.9803} = 2.6651$ (b) $\sqrt{2}^e = e^{e \ln \sqrt{2}} = e^{0.9421} = 2.5653$
2. (a) $\sqrt{2}^{\sqrt{2}} = e^{\sqrt{2} \ln \sqrt{2}} = e^{0.4901} = 1.6325$ (b) $5^\pi = e^{\pi \ln 5} = e^{5.0562} = 156.9925$
3. (a) $e^e = 15.1543$ (b) $\sqrt{3}^\pi = e^{\pi \ln \sqrt{3}} = e^{3.4514} = 31.5443$
4. (a) $\pi^e = e^{e \ln \pi} = e^{3.1117} = 22.4592$ (b) $\pi^\pi = e^{\pi \ln \pi} = e^{3.5963} = 36.4622$

In Exercises 5–20, find dy/dx, and support your answer by plotting it and NDER in the same window.

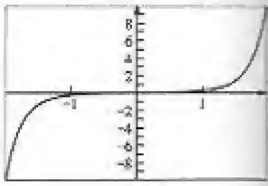
5. $\frac{dy}{dx} = \frac{d}{dx} e^{5x} = e^{5x} \frac{d}{dx}(5x) = 5e^{5x}$
6. $\frac{dy}{dx} = \frac{d}{dx} e^{-7x} = e^{-7x} \frac{d}{dx}(-7x) = -7e^{-7x}$
7. $\frac{dy}{dx} = \frac{d}{dx} e^{-3x} = e^{-3x} \frac{d}{dx}(-3x^2) = -6x\, e^{-3x}$

8. $y = e^{x^2-3}$

▹ We apply Theorem 5.4.10 with $u = x^2 - 3$.

$$\frac{dy}{dx} = e^{x^2-3}\mathrm{D}_x(x^2-3) = 2xe^{x^2-3}$$

The plot shows dy/dx and NDER.



9. $\frac{dy}{dx} = \frac{d}{dx}e^{\cos x} = e^{\cos x}\frac{d}{dx}(\cos x) = -e^{\cos x}\sin x$

10. $\frac{dy}{dx} = \frac{d}{dx}e^{2\sin 3x} = e^{2\sin 3x}\frac{d}{dx}(2\sin 3x) = 6\cos 3x\, e^{2\sin 3x}$

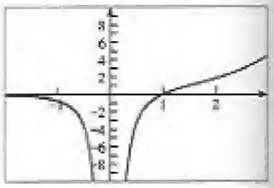
11. $\frac{dy}{dx} = \frac{d}{dx}(e^x\sin e^x) = e^x\frac{d}{dx}(\sin e^x) + \frac{d}{dx}(e^x)\cdot\sin e^x = e^x\cos x(e^x) + e^x\sin e^x = e^{2x}\cos e^x + e^x\sin e^x$

12. $y = \frac{e^x}{x}$

▹ We apply the derivative of a quotient rule.

$$\frac{dy}{dx} = \frac{xe^x - e^x\cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$$

The plot shows dy/dx and NDER.



13. $\frac{dy}{dx} = \frac{d}{dx}\left(\tan e^{\sqrt{x}}\right) = \sec^2 e^{\sqrt{x}}\frac{d}{dx}e^{\sqrt{x}} = \sec^2 e^{\sqrt{x}}\cdot e^{\sqrt{x}}\frac{d}{dx}\sqrt{x} = \frac{e^{\sqrt{x}}\sec^2 e^{\sqrt{x}}}{2\sqrt{x}}$

14. $\frac{dy}{dx} = \frac{d}{dx}(e^{e^x}) = e^{e^x}\frac{d}{dx}e^x = e^x e^{e^x}$

15. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2} = \frac{(e^{2x}+2+e^{-2x}) - (e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}$

$= \frac{4}{(e^x+e^{-x})^2}$

16. $y = \ln\frac{e^{4x}-1}{e^{4x}+1}$

▹ Before differentiating we apply Theorem 5.2.5.

$y = \ln(e^{4x}-1) - \ln(e^{4x}+1)$

We differentiate, applying Theorems 5.2.2 and 5.4.10. Thus

$$\frac{dy}{dx} = \frac{1}{e^{4x}-1}\mathrm{D}_x(e^{4x}-1) - \frac{1}{e^{4x}+1}\mathrm{D}_x(e^{4x}+1) = \frac{4e^{4x}}{e^{4x}-1} - \frac{4e^{4x}}{e^{4x}+1} = \frac{4e^{8x}+4e^{4x}-(4e^{8x}-4e^{4x})}{(e^{4x}-1)(e^{4x}+1)}$$

$$= \frac{8e^{4x}}{e^{8x}-1},\ x > 0$$

17. $\frac{dy}{dx} = \frac{d}{dx}(x^5e^{-3\ln x}) = \frac{d}{dx}(x^5e^{\ln x^{-3}}) = \frac{d}{dx}(x^5\cdot x^{-3}) = \frac{d}{dx}x^2 = 2x,\ x > 0$

18. $\frac{dy}{dx} = \frac{d}{dx}\ln(e^x + e^{-x}) = \frac{1}{e^x+e^{-x}}\frac{d}{dx}(e^x+e^{-x}) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

19. $\frac{dy}{dx} = \frac{d}{dx}(\sec e^{2x} + e^{2\sec x}) = \sec e^{2x}\tan e^{2x}\frac{d}{dx}(e^{2x}) + e^{2\sec x}\frac{d}{dx}(2\sec x)$

$= 2e^{2x}\sec e^{2x}\tan e^{2x} + 2e^{2\sec x}\sec x\tan x$

20. $y = \tan e^{3x} + e^{\tan 3x}$

▹ $$\frac{dy}{dx} = \sec^2(e^{3x})\mathrm{D}_x e^{3x} + e^{\tan 3x}\,\mathrm{D}_x\tan 3x = \sec^2(e^{3x})e^{3x}\,\mathrm{D}_x(3x) + e^{\tan 3x}\sec^2 3x\,\mathrm{D}_x(3x)$$

$$= 3(e^{3x}\sec^2 e^{3x} + e^{\tan 3x}\sec^2 3x)$$

In Exercises 21–24, find dy/dx by implicit differentiation.

21. $e^x + e^y = e^xe^y$; $e^x + e^y\frac{dy}{dx} = e^xe^y\frac{dy}{dx} + e^xe^y$; $(e^y - e^xe^y)\frac{dy}{dx} = e^xe^y - e^x$; $-e^{-x}\frac{dy}{dx} = e^y$; $\frac{dy}{dx} = -e^{y-x}$

22. $e^y = \ln(x^3+3y)$; $e^y\frac{dy}{dx} = \frac{3x^2+3(dy/dx)}{x^3+3y}$; $(x^3e^y+3ye^y)\frac{dy}{dx} = 3x^2 + 3\frac{dy}{dx}$; $(x^3e^y+3ye^y-3)\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{x^3e^y+3ye^y-3}$

23. $y^2e^{2x} + xy^3 = 1$; $y^2e^{2x}(2) + 2y\frac{dy}{dx}e^{2x} + x\left(3y^2\frac{dy}{dx}\right) + y^3 = 0$; $(2e^{2x} + 3xy)\frac{dy}{dx} = -y^2 - 2ye^{2x}$

$\frac{dy}{dx} = -\frac{y^2 + 2ye^{2x}}{2e^{2x} + 3xy}$

24. $ye^{2x} + xe^{2y} = 1$

▷ Differentiating on both sides of the equation with respect to x gives

$$ye^{2x}\cdot 2 + e^{2x}\frac{dy}{dx} + xe^{2y}\cdot 2\frac{dy}{dx} + e^{2y}\cdot 1 = 0; \qquad (e^{2x} + 2xe^{2y})\frac{dy}{dx} = -2ye^{2x} - e^{2y}; \quad \frac{dy}{dx} = -\frac{2ye^{2x} + e^{2y}}{e^{2x} + 2xe^{2y}}$$

In Exercises 25–32, evaluate the indefinite integral.

25. $\int e^{2-5x}dx = -\frac{1}{5}\int e^{2-5x}(-5\,dx) = -\frac{1}{5}e^{2-5x} + C$

26. $\int e^{2x+1}dx = \frac{1}{2}e^{2x+1} + C$

27. $\int \frac{1 + e^{2x}}{e^x}\,dx = \int \left(e^{-x} + e^x\right)dx = -e^{-x} + e^x + C$

28. $\int e^{3x}e^{2x}\,dx$

▷ By Theorems 5.4.7 and 5.4.11 with $u = 5x$ and the fact that $5x$ is linear,

$$\int e^{3x}e^{2x}\,dx = \int e^{5x}\,dx = \tfrac{1}{5}e^{5x} + C$$

29. Let $u = 1 - 2e^{3x}$; then $du = -6e^{3x}dx$.

$$\int \frac{e^{3x}}{(1 - 2e^{3x})^2}\,dx = -\frac{1}{6}\int (1 - 2e^{3x})^{-2}(-6e^{3x}dx) = -\frac{1}{6}\int u^{-2}du = -\frac{1}{6}\left(-\frac{1}{u}\right) + C = \frac{1}{6(1 - 2e^{3x})} + C$$

30. Let $u = 2x^3$; then $du = 6x^2dx$. $\int x^2e^{2x^3}dx = \frac{1}{6}\int e^{2x^3}(6x^2dx) = \frac{1}{6}\int e^u du = \frac{1}{6}e^u + C = \frac{1}{6}e^{2x^3} + C$

31. $\int \frac{e^{2x}}{e^x + 3}\,dx = \int \left(e^x - 3\,\frac{e^x}{e^x + 3}\right)dx = e^x - 3\ln(e^x + 3) + C$

32. $\int \frac{dx}{1 + e^x}$

▷ We multiply the numerator and denominator of the given fraction by e^{-x}.

$$\int \frac{dx}{1 + e^x} = \int \frac{e^{-x}\,dx}{e^{-x} + 1} \tag{1}$$

Let $u = e^{-x} + 1$. Then $du = -e^{-x}dx$. With these substitutions in the right-hand side of (1) we get

$$\int \frac{dx}{1 + e^x} = -\int \frac{du}{u} = -\ln|u| + C = -\ln|e^{-x} + 1| + C = -\ln(e^{-x} + 1) + C$$

Alternatively, $\int \frac{1}{1 + e^x}dx = \int \left(1 - \frac{e^x}{1 + e^x}\right)dx = \int dx - \int \frac{d(1 + e^x)}{1 + e^x} = x - \ln(1 + e^x) + C$

In Exercises 33–40, evaluate the definite integral. Check using NINT.

33. $\int_0^1 e^2dx = e^2\int_0^1 dx = e^2x\Big|_0^1 = e^2 \approx 7.3891$

34. $\int_1^{e^2} \frac{dx}{x} = \ln|x|\Big]_1^{e^2} = \ln e^2 - \ln 1 = 2 - 0 = 2$

35. $\int_e^{e^3} \frac{dx}{x} = \ln x\Big|_e^{e^3} = \ln e^3 - \ln e = 3 - 1 = 2$

36. $\int_1^e \frac{\ln x}{x}\,dx$

▷ Let $u = \ln x$ and $du = dx/x$. When $x = 1$, then $u = 0$; when $x = e$, then $u = 1$. Thus

$$\int_1^e \frac{\ln x}{x}dx = \int_0^1 u\,du = \tfrac{1}{2}u^2\Big]_0^1 = \tfrac{1}{2}$$

We confirm this value using NINT.

37. Let $u = \ln x$; $du = \frac{dx}{x}$. $\int_e^{e^2} \frac{dx}{x(\ln x)^2} = \int_e^{e^2} (\ln x)^{-2}\frac{dx}{x} = \int_1^2 u^{-2}du = -\frac{1}{u}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$

38. $\int_0^3 \frac{e^x + e^{-x}}{2}dx = \frac{1}{2}\left[e^x - e^{-x}\right]_0^3 = \frac{1}{2}(e^3 - e^{-3}) \approx 10.0179$

39. $\int_0^2 xe^{4-x^2}\,dx = -\frac{1}{2}\int_0^2 e^{4-x^2}\,(-2x\,dx) = -\frac{1}{2}e^{4-x^2}\Big|_0^1 = -\frac{1}{2}(e^0 - e^4) = \frac{1}{2}(e^4 - 1) \approx 26.80$

40. $\int_1^2 \frac{e^x}{e^x + e}\,dx$

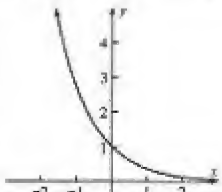
▹ Note that e is a constant, and thus $D_x e = 0$. Let $u = e^x + e$. Then $du = e^x dx$; when $x = 1$, $u = 2e$; when $x = 2$, $u = e^2 + e$. Thus,

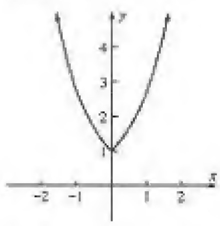
$$\int_1^2 \frac{e^x}{e^x + e}dx = \int_{2e}^{e^2+e}\frac{du}{u} = \ln|u|\Big]_{2e}^{e^2+e} = \ln(e^2+e) - \ln(2e) = \ln\frac{e^2+e}{2e} = \ln\tfrac{1}{2}(e+1) \approx 0.6201$$

We confirm this value using NINT.

41. (a) $\ln x = 0$ when $x = 1$; (b) $e^x = 1$ when $x = 0$; (c) $\ln x = 1$ when $x = e$; (d) e^x is always positive, never 0; the graph has the x axis as an asymptote.

42. Sketch the graphs. (a) $y = e^{-x}$ (b) $y = e^{|x|}$





In Exercises 43 and 44, find the area of the region.

43. A square units is the area of the region bounded by $y = e^x$, the axes, and $x = 2$.

$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} e^{w_i}\Delta_i x = \int_0^2 e^x dx = e^x\Big|_0^2 = e^2 - e^0 = e^2 - 1$$

44. The region bounded by the curve $y = e^x$ and the line through the points $(0,1)$ and $(1,e)$.

▹ The points lie on the curve. Because $y' = e^x$ and $y'' = e^x > 0$, the graph is concave upward and the chord joining the points lies above the curve. A equation of the line is $y = 1 + (e-1)x$. If A square units is the area of the region, then

$$A = \int_0^1 [1 + (e-1)x - e^x]\,dx = \left[x + \tfrac{1}{2}(e-1)x^2 - e^x\right]_0^1 = 1 + \tfrac{1}{2}(e-1) - e + 1 = \tfrac{1}{2}(3-e)$$

▪ The area of the region is $\frac{1}{2}(3-e)$ square units.

45. $y = e^{-x}$; $\frac{dy}{dx} = -e^{-x}$. Therefore the slope of the tangent line to the curve $y = e^{-x}$ at the point (x_1, y_1) is $-e^{x_1}$.

The slope of the line $y = 2x - 5$ is 2. Therefore the slope of a line perpendicular to this line is $-\frac{1}{2}$.

Hence we have the equation $-e^{x_1} = -\frac{1}{2}$; $e^{x_1} = \frac{1}{2}$; $-x_1 = \ln\frac{1}{2}$; $x_1 = \ln 2$

When $x_1 = \ln 2$, $y_1 = e^{-\ln 2} = \frac{1}{2}$. Therefore the required line contains the point $(\ln 2, \frac{1}{2})$ and its slope is $-\frac{1}{2}$.

An equation of this line is $y - \frac{1}{2} = -\frac{1}{2}(x - \ln 2)$; $y = -\frac{1}{2}x + \frac{1}{2} + \frac{1}{2}\ln 2$

46. $y = e^{2x}$; $y' = 2e^{2x}$. $y(\ln 2) = e^{2\ln 2} = 4$; $y'(\ln 2) = 2\cdot 4 = 8$. The slope of the tangent line is 8; the normal line has slope $-\frac{1}{8}$ and equation $y = 4 - \frac{1}{8}(x - \ln 2)$.

47. A particle is moving along a line and at t sec its velocity is v ft/sec where $v = e^3 - e^{2t}$. Set $v = 0$ and get $e^3 = e^{2t}$; $t = \frac{3}{2}$. When $t > \frac{3}{2}$, $v < 0$. If $s(t)$ ft is the distance traveled by the body at t sec, then $\frac{ds}{dt} = v$ and the distance traveled when $0 \le t \le \frac{3}{2}$ is

$$s\left(\tfrac{3}{2}\right) - s(0) = \int_0^{3/2}\frac{ds}{dt}dt = \int_0^{3/2}(e^3 - e^{2t})dt = e^3 t - \frac{1}{2}e^{2t}\Big|_0^{3/2} = \left(\frac{3}{2}e^3 - \frac{1}{2}e^3\right) - \left(-\frac{1}{2}\right) = e^3 + \frac{1}{2} \approx 20.586$$

48. A particle is moving along a line, where s ft is the directed distance of the particle from the origin, v ft/sec is the velocity of the particle, and a ft/sec^2 is the acceleration of the particle at t sec. If $a = e^t + e^{-t}$ and $v = 1$ and $s = 2$ when $t = 0$, find v and s in terms of t.

▹ Because $a = dv/dt$, we are given that

$$\frac{dv}{dt} = e^t + e^{-t}; \qquad \int dv = \int (e^t + e^{-t})dt; \quad v = e^t - e^{-t} + C$$

Because $v = 1$ when $t = 0$, we have $\qquad 1 = e^0 - e^0 + C$

Thus, C = 1 and $\qquad v = e^t - e^{-t} + 1$

Furthermore, $v = ds/dt$, and so

$$\frac{ds}{dt} = e^t - e^{-t} + 1; \qquad \int ds = \int (e^t - e^{-t} + 1)dt; \qquad s = e^t + e^{-t} + t + k$$

Because $s = 2$ when $t = 0$, we have $\qquad 2 = e^0 + e^0 + k$

and thus $k = 0$. Hence $\qquad s = e^t + e^{-t} + t$

In Ex. 49-50, if p lb/ft^2 is the atmospheric pressure at a height of h ft above sea level, then $p = 2116e^{-0.0000318h}$.

49. $\frac{dp}{dt} = 2116(-0.0000318)e^{-0.0000318h}\frac{dh}{dt} = -0.06729e^{-0.0000318h}\frac{dh}{dt}$.
When h = 5000 and $\frac{dh}{dt} = 160$ we get $\frac{dp}{dt} = -0.06729e^{-0.159}(160) \approx -9.17$.

- The atmospheric pressure is decreasing at the rate of 9.17 lb/ft^2/sec.

50. We know p so we solve for h before taking differentials. $\ln p - \ln 2116 = -0.0000318h$; $\frac{dp}{p} = -0.0000318dh$.
When $p = 1500$ and $dp = 1480 - 1500 = -20$, $dh = \frac{dp}{-0.0000318p} = \frac{-20}{-0.0000318(1500)} = 419.3$. Rise 419 ft.

51. If ℓ ft is the length of an iron rod when t degrees is its temperature, then
$\ell = 60e^{0.00001t}$; $d\ell = 0.0006e^{0.00001t}dt$
When $t = 0$ and $dt = \Delta t = 10$, then $d\ell = 0.0006e^0(10) = 0.006$. Therefore $\Delta\ell \approx 0.006$.

52. A simple electric circuit containing no capacitors, a resistance of R ohms, and an inductance of L henrys has the electromotive force cut off when the current is I_0 amperes. The current dies down so that at t sec the current is i amperes and $i = I_0e^{-(R/L)t}$. Show that the rate of change of the current is proportional to the current.

- I_0, e, R, and L are all constants; the variables are t and i. Let $k = -R/L$. Then

$$i = I_0e^{kt}; \qquad \frac{di}{dt} = I_0e^{kt} \cdot k = ki$$

- Therefore, the rate of change of i, the current, is proportional to i.

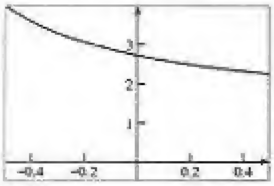
53. When a budget is increased by x thousand dollars, profit increases by P hundred dollars where $P = 25x^2e^{-0.2x}$, $x \geq 0$. $P' = 25(2xe^{-0.2x} - 0.2x^2e^{-0.2x}) = 125x(10 - x)e^{-0.2x}$. Because $P' > 0$ if $0 < x < 10$ and $P' < 0$ if $x > 10$, P has an absolute maximum value when $x = 10$ and $P = 25(10)^2e^{-2} = 338.34$.

- The budget increases by \$10,000 and profit by \$33,834.

54. $f(x) = x^\pi, x > 0$ (a) $f'(x) = \pi x^{\pi-1} > 0$ so f is increasing (b) $f''(x) = \pi(\pi - 1)x^{\pi-2} > 0$ so f is concave upward

55. (a) $\lim\limits_{x\to\infty^+}\left(1 + \frac{1}{x}\right)^x = \lim\limits_{h\to0^+}(1 + h)^{1/h} = e$ and $\lim\limits_{x\to\infty^-}\left(1 + \frac{1}{x}\right)^x = \lim\limits_{h\to0^-}(1 + h)^{1/h} = e$
(b) $\frac{1}{2}(1.001^{10000} + 0.999^{-10000}) = \frac{1}{2}(2.7181459 + 2.7184177) = 2.7182818$

56. Plot the graph of the function defined by $f(x) = (1 + x)^{1/x}$ in the $[-0.5, 0.5]$ by $[2, 3]$ window. $f(0)$ is undefined but $\lim\limits_{x\to0}f(x) = e$. Approximate the value of e to five significant digits by using this graph and the intersect or trace and zoom capabilities of your graphics calculator.



- The plot is shown at the right. Using zoom we obtain the value 2.71828 for the y intercept.

In Exercises 57 and 58, f is a mathematical model describing damped harmonic motion.
(a) Show that $F(t) \leq f(t) \leq G(t)$. (b) Plot the graphs of the 3 functions. (c) Prove that $\lim\limits_{t\to+\infty} f(t) = 0$.

57. $f(t) = e^{-t/2}\cos 4t$; $F(t) = -e^{-t/2}$; $G(t) = e^{-t/2}$ $\qquad$ 58. $f(t) = e^{-t/8}\sin 3t$; $F(t) = -e^{-t/8}$; $G(t) = e^{-t/8}$

- The inequality follows because the sine and cosine function lie between -1 and 1. Because a decreasing exponential function approaches 0 as x approaches $+\infty$, it follows from the squeeze theorem that f approaches 0.

59. We wish to solve $\frac{dy}{dx} = y$. Method 1. See proof of Theorem 5.6.1. Method 2. We have $\frac{dy}{dx}e^{-x} = ye^{-x}$.
Let $g(x) = ye^{-x}$. Then $g'(x) = \frac{dy}{dx}e^{-x} - ye^{-x} = 0$ so that $g(x)$ is a constant. Thus $ye^{-x} = k$; $y = ke^x$.

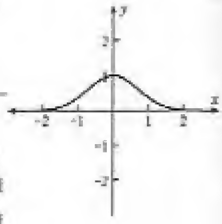
In Exercises 60 and 61, (a) find the relative extrema of f; (b) determine the values of f at which the extrema occur; (c) determine the interval on which f is increasing and (d) decreasing; (e) determine where the graph of f is concave upward and (f) downward; (g) find the slope of any inflectional tangent. Use the information in parts (a) to (f) to sketch the graph.

60. $f(x) = e^{-x^2}$

▷ f is continuous at every real number.

$f'(x) = e^{-x^2}(-2x) = \dfrac{-2x}{e^{x^2}}$ $\qquad$ $f''(x) = e^{-x^2}(-2) + (-2x)e^{-x^2}(-2x) = \dfrac{4(x^2 - \frac{1}{2})}{e^{x^2}}$

Both $f'(x)$ and $f''(x)$ are defined for all x. Because $f'(0) = 0$, then 0 is a critical number. If $f''(x) = 0$, then $x = \pm\frac{1}{2}\sqrt{2}$. The table shows all the requested information and the graph is shown at the right.



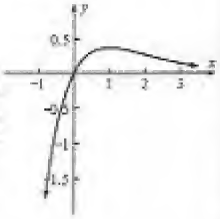
	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < -\frac{1}{2}\sqrt{2}$		$+$	$+$	increasing	concave upward
$x = -\frac{1}{2}\sqrt{2}$	$\sqrt{1/e}$	$\sqrt{2/e}$	0	increasing	point of inflection
$-\frac{1}{2}\sqrt{2} < x < 0$		$+$	$-$	increasing	concave downward
$x = 0$	1	0	$-$	absolute maximum	concave downward
$0 < x < \frac{1}{2}\sqrt{2}$		$-$	$-$	decreasing	concave downward
$x = \frac{1}{2}\sqrt{2}$	$\sqrt{1/e}$	$-\sqrt{2/e}$	0	decreasing	point of inflection
$x > \frac{1}{2}\sqrt{2}$		$-$	$+$	decreasing	concave upward

Note: $\frac{1}{2}\sqrt{2} \approx 0.707$, $\sqrt{1/e} \approx 0.607$, $\sqrt{2/e} \approx 0.858$

61. $f(x) = xe^{-x}$; $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$

$f''(x) = -e^{-x}(1 - x) - e^{-x} = e^{-x}(x - 2)$

Set $f'(x) = 0$ and obtain $x = 1$. Set $f''(x) = 0$ and obtain $x = 2$.



	$f(x)$	$f'(x)$	$f''(x)$	f is/has a	graph is/has a
$x < 1$		$+$	$-$	increasing	concave downward
$x = 1$	e^{-1}	0	$-$	absolute maximum	concave downward
$1 < x < 2$		$-$	$-$	decreasing	concave downward
$x = 2$	$2e^{-2}$	$-e^{-2}$	0	decreasing	point of inflection
$2 < x$		$-$	$+$	decreasing	concave upward

Note: $e^{-1} \approx 0.368$, $e^{-2} \approx 0.135$, $2e^{-2} \approx 0.271$

62. Let $x = \ln u$. Then $\lim\limits_{x \to +\infty} \dfrac{x}{e^x} = \lim\limits_{x \to +\infty} \dfrac{\ln u}{u} = 0$ by Exercise 5.3.54.

63. The t axis is not a horizontal asymptote of $f(t) = e^{-t/4} \sin 4t$ because Definition 3.7.4 does not allow a graph to cross its asymptote for arbitrarily large values.

64. Show that the function $f(x) = x^{\sqrt{2}}$ of Example 5 is continuous from the right at 0 by showing that

$$\lim_{x \to 0^+} x^{\sqrt{2}} = 0$$

▷ Let $u = \ln x$. Because $\lim\limits_{x \to 0^+} \ln x = -\infty$, we have

$$\lim_{x \to 0^+} x^{\sqrt{2}} = \lim_{x \to 0^+} e^{\sqrt{2} \ln x} = \lim_{u \to -\infty} e^{\sqrt{2}u} = 0$$

65. Let $N > 0$ be any number and let $M = \ln N$. Because the exponential function is increasing

if $x \geq M$ then $e^x > e^M = e^{\ln N}$

$\Leftrightarrow$ if $x \geq M$ then $e^x > N$

This proves that $\lim\limits_{x \to +\infty} e^x = +\infty$.

66. Let $\epsilon > 0$. $e^x < \epsilon \Leftrightarrow x < \ln \epsilon = N$. This proves that $\lim\limits_{x \to -\infty} e^x = 0$.

5.5 OTHER EXPONENTIAL AND LOGARITHMIC FUNCTIONS

5.5.1 Definition If a is any positive number and x is any real number, then the function f defined by
$f(x) = a^x$
is called the exponential function to the base a.

Laws of Exponents (See Exercises 37–40.) If a and b are positive numbers and x and y are real numbers,
$a^x a^y = a^{x+y}$, $a^x \div a^y = a^{x-y}$, $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$, $a^0 = 1$

5.5.2 Theorem If a is any positive number and u is a differentiable function of x,
$\mathrm{D}_x(a^u) = a^u \ln a \,\mathrm{D}_x u$
Note that $\mathrm{D}_x(u^v)$ cannot be found by Theorem 5.4.12 or directly by Theorem 5.5.2. It may be found by logarithmic differentiation or by Theorem 5.5.2 applied to $e^{v \ln u}$.

5.5.3 Theorem If a is any positive number other than 1,
$\int a^u \, du = \dfrac{a^u}{\ln a} + C$

5.5.4 Definition If a is any positive number except 1, the logarithmic function to the base a is the inverse of the exponential function to the base a, and we write
$y = \log_a x$ if and only if $a^y = x$

5.5.5 Theorem If u is a differentiable function of x,
$\mathrm{D}_x(\log_a u) = \dfrac{\log_a e}{u} \cdot \mathrm{D}_x u \;\Leftrightarrow\; \mathrm{D}_x(\log_a u) = \dfrac{1}{(\ln a)u} \cdot \mathrm{D}_x u$
The logarithmic function to the base a in terms of the natural logarithmic function:
$\log_a x = \dfrac{\ln x}{\ln a}$
A special case of the above is
$\log_a e = \dfrac{1}{\ln a}$

Exercises 5.5

In Exercises 1–20, find the derivative of the function.

1. $f'(x) = \dfrac{d}{dx} 3^{5x} = 3^{5x}(\ln 3)\dfrac{d}{dx}(5x) = (5 \ln 3)3^{5x}$

2. $f'(x) = \dfrac{d}{dx} 6^{-3x} = 6^{-3x}(\ln 6)\dfrac{d}{dx}(-3x) = -3(\ln 6)6^{-3x}$

3. $f'(t) = \dfrac{d}{dt} 4^{3t^2} = 4^{3t^2}(\ln 4)\dfrac{d}{dt}(3t^2) = 4^{3t^2}(\ln 4)6t$

4. $g(x) = 10^{x^2-2x}$

▶ We apply Theorem 5.5.2 with $u = x^2 - 2x$ and $a = 10$. Thus
$$g'(x) = 10^{x^2-2x}(\ln 10)\mathrm{D}_x(x^2 - 2x) = 10^{x^2-2x}(\ln 10)(2x - 2)$$

5. $f'(x) = \dfrac{d}{dx} 4^{\sin 2x} = 4^{\sin 2x}(\ln 4)\dfrac{d}{dx}(\sin 2x) = 4^{\sin 2x}(2 \ln 4)\cos 2x$

6. $f'(x) = \mathrm{D}_x 2^{\csc 3x} = 2^{\csc 3x}(\ln 2)\mathrm{D}_x(\csc 3x) = -3(\ln 2) \csc 3x \cot 3x \; 2^{\csc 3x}$

7. $g'(x) = \dfrac{d}{dx}(2^{5x}3^{4x^2}) = \left(\dfrac{d}{dx} 2^{5x}\right)3^{4x^2} + 2^{5x}\left(\dfrac{d}{dx} 3^{4x^2}\right) = 2^{5x}3^{4x^2}(\ln 2)5 + 2^{5x}3^{4x^2}(\ln 3)(8x)$
$= 2^{5x}3^{4x^2}(5 \ln 2 + 8x \ln 3)$

8. $f(x) = (x^3 + 3)2^{-7x}$

▶
$$f'(x) = (x^3 + 3)\mathrm{D}_x(2^{-7x}) + 2^{-7x}\mathrm{D}_x(x^3 + 3) = (x^3 + 2)2^{-7x}(\ln 2)(-7) + 2^{-7x}(3x^2)$$
$$= 2^{-7x}[3x^2 - 7(\ln 2)(x^3 + 3)]$$

9. $h'(x) = \dfrac{d}{dx} \dfrac{\log_{10} x}{x} = \dfrac{\dfrac{d}{dx}(\log_{10} x) \cdot x - \log_{10} x \cdot 1}{x^2} = \dfrac{\dfrac{\log_{10} e}{x} \cdot x - \log_{10} x}{x^2} = \dfrac{\log_{10} e - \log_{10} x}{x^2} = \dfrac{1}{x^2} \log_{10} \dfrac{e}{x}$

10. $f'(t) = D_t \log_{10}\dfrac{t}{t+1} = D_t[\log_{10}t - \log_{10}(t+1)] = \dfrac{\log_{10}e}{t} - \dfrac{\log_{10}e}{t+1} = \dfrac{\log_{10}e}{t(t+1)}$

11. $f'(x) = \dfrac{d}{dx}\sqrt{\log_a x} = \dfrac{d}{dx}(\log_a x)^{1/2} = \frac{1}{2}(\log_a x)^{-1/2}\dfrac{d}{dx}(\log_a x) = \dfrac{\log_a e}{2x\sqrt{\log_a x}}$

12. $g(w) = \tan 2^{3w}$

▹ $$g'(w) = \sec^2 2^{3w}\, D_w(2^{3w}) = 3(\ln 2)2^{3w}\sec^2 2^{3w}$$

13. $f'(t) = \dfrac{d}{dt}\sec 3^{t^2} = \sec 3^{t^2}\tan 3^{t^2}\dfrac{d}{dt}(3^{t^2}) = -3^{t^2}\sec 3^{t^2}\tan 3^{t^2}\,(2t\ln 3)$

14. Let $y = x^{\ln x}$. $\ln y = (\ln x)(\ln x) = (\ln x)^2$; $\dfrac{1}{y}\dfrac{dy}{dx} = 2(\ln x)\frac{1}{x}$. $f'(x) = \dfrac{dy}{dx} = y\cdot\dfrac{2\ln x}{x} = 2(\ln x)x^{\ln x - 1}$

15. Let $y = x^{\sqrt{x}}$, $x > 0$. $\ln y = \sqrt{x}\ln x$; $\dfrac{1}{y}\dfrac{dy}{dx} = \sqrt{x}\cdot\frac{1}{x} + \ln x\cdot\dfrac{1}{2\sqrt{x}}$

$$f'(x) = \frac{dy}{dx} = y\left[\frac{2+\ln x}{2\sqrt{x}}\right] = x^{\sqrt{x}}\left[\frac{2+\ln x}{2x^{1/2}}\right] = \tfrac{1}{2}x^{\sqrt{x}-(1/2)}(2+\ln x)$$

16. $f(x) = x^{x^2}$

▹ Let $\quad y = x^{x^2};\quad \ln y = \ln x^{x^2};\quad \ln y = x^2\ln x$

Differentiating on both sides of the above equation with respect to x, we obtain

$$\frac{1}{y}D_x y = x^2\cdot\frac{1}{x} + (\ln x)2x$$
$$D_x y = yx(1+2\ln x) = x^{x^2}\cdot x(1+2\ln x) = x^{x^2+1}(1+2\ln x)$$

17. Let $y = z^{\cos z}$. $\ln y = \cos z\ln z$; $\dfrac{1}{y}\dfrac{dy}{dz} = \cos z\cdot\frac{1}{z} - \sin z\ln z$

$$g'(z) = \frac{dy}{dz} = y(z^{-1}\cos z - \sin z\ln z) = z^{\cos z}(z^{-1}\cos z - \sin z\ln z) = z^{\cos z - 1}(\cos z - z\sin z\ln z)$$

18. Let $y = x^{e^x}$. $\ln y = e^x\ln x$; $\dfrac{1}{y}\dfrac{dy}{dx} = e^x\ln x + e^x\cdot\frac{1}{x}$

$$f'(x) = \frac{dy}{dx} = y\left[\frac{xe^x\ln x + e^x}{x}\right] = x^{e^x}e^x\left[\frac{x\ln x + 1}{x}\right] = x^{e^x-1}e^x(x\ln x + 1)$$

19. Let $y = (\sin x)^{\tan x}$. $\ln y = \tan x\ln(\sin x)$; $\dfrac{1}{y}\dfrac{dy}{dx} = \tan x\cdot\dfrac{1}{\sin x}\cdot\cos x + \sec^2 x\ln(\sin x)$

$$h'(x) = \frac{dy}{dx} = y[1+\sec^2 x\ln(\sin x)] = (\sin x)^{\tan x}[1+\sec^2 x\ln(\sin x)]$$

20. $g(t) = (\cos t)^t$; $\cos t > 0$

▹ Let $\quad s = (\cos t)^t;\quad \ln s = \ln(\cos t)^t;\quad \ln s = t\ln(\cos t)$

Differentiating on both sides with respect to t, we have

$$\frac{1}{s}D_t s = t\,D_t\ln(\cos t) + \ln(\cos t)D_t t$$
$$D_t s = s\left[t\cdot\frac{1}{\cos t}\cdot D_t(\cos t) + \ln(\cos t)\right] = (\cos t)^t\left[\frac{-t\sin t}{\cos t} + \ln(\cos t)\right] = (\cos t)^t[\ln(\cos t) - t\tan t]$$

In Exercises 21–30, evaluate the indefinite integral.

21. $\displaystyle\int 3^{2x}dx = \frac{1}{2}\int 3^{2x}(2\,dx) = \frac{3^{2x}}{2\ln 3} + C$

22. $\displaystyle\int a^{nx}\,dx = \frac{1}{n}\int a^{nx}(n\,dx) = \frac{a^{nx}}{n\ln a} + C$

23. $\displaystyle\int a^t e^t dt = \int (ae)^t dt = \frac{(ae)^t}{\ln ae} + C = \frac{a^t e^t}{\ln a + 1} + C$

24. $\displaystyle\int 5^{x^4+2x}(2x^3+1)dx$

▹ Let $u = x^4 + 2x$. Then $du = (4x^3+2)dx = 2(2x^3+1)dx$. Thus,

$$\int 5^{x^4+2x}(2x^3+1)dx = \frac{1}{2}\int 5^u du = \frac{1}{2}\cdot\frac{5^u}{\ln 5} + C = \frac{5^{x^4+2x}}{2\ln 5} + C$$

25. $\displaystyle\int x^2 10^{x^3}\,dx = \frac{1}{3}\int 10^{x^3}(3x^2dx) = \frac{10^{x^3}}{3\ln 10} + C$

26. Let $u = x \ln x$; $du = \left(\ln x + x \cdot \frac{1}{x}\right)dx = (\ln x + 1)dx$. $\int a^{x \ln x}(\ln x + 1)dx = \int a^u du = \frac{a^u}{\ln a} + C = \frac{a^{x \ln x}}{\ln a} + C$

27. $\int e^y 2^e\, 3^e\, dy = \int 6^e\, (e^y dy) = \frac{6e^y}{\ln 6} + C$

28. $\int \frac{4^{\ln(1/x)}}{x}dx$

▷ Because $4^{\ln(1/x)} = 4^{-\ln x} = e^{-\ln x \ln 4} = x^{-\ln 4}$, then $\int \frac{4^{\ln(1/x)}}{x}dx = \int \frac{x^{-\ln 4}}{x}dx = \int x^{-\ln 4 - 1}dx = \frac{x^{-\ln 4}}{-\ln 4} + C$

29. $\int \frac{\log_2 x^2}{x}dx = \frac{2}{\ln 2}\int \frac{\ln x}{x}dx = \frac{2}{\ln 2}\int \ln x\, d(\ln x) = \frac{(\ln x)^2}{\ln 2} + C$

30. $\int \frac{(\log_3 x)^2}{x}dx = \int \left(\frac{\ln x}{\ln 3}\right)^2 \frac{dx}{x} = \frac{1}{(\ln 3)^2}\int (\ln x)^2 d(\ln x) = \frac{(\ln x)^3}{3(\ln 3)^2} + C$

In Exercises 31–34, compute the value of the logarithm on your calculator to five significant digits.

31. (a) $\log_5 e = 1 \div \ln 5 = 0.62133$ (b) $\log_3 7 = \ln 7 \div \ln 3 = 1.7712$

32. (a) $\log_6 10 = \ln 10 \div \ln 6 = 1.2851$ (b) $\log_2 361 = \ln 361 \div \ln 2 = 8.4959$

33. (a) $\log_2 10 = \ln 10 \div \ln 2 = 3.3219$ (b) $\log_{10} e = 1 \div \ln 10 = 0.43429$

34. (a) $\log_3 2 = \ln 2 \div \ln 3 = 0.63093$ (b) $\log_4 4728 = \ln 4728 \div \ln 4 = 6.1035$

In Exercises 35–36, given $\log_{10} e = 0.4343$, use differentials to approximate the logarithm to 3 digits and check.

35. Let $y = f(x) = \log_{10} x$ so $dy = \frac{\log_{10} e}{x}\, dx$. With $x = 1000$ and $dx = \Delta x = -3$,

$\log_{10} 997 = f(1000 - 3) \approx f(1000) + dy = \log_{10} 1000 + \frac{\log_{10} e}{x}(-3) = 3 + \frac{0.4343}{1000}(-3) = 2.998697$

In fact, $\log_{10} 997 = 2.998695$.

36. $\log_{10} 1.015$

▷ Let $f(x) = \log_{10} x$. Then

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x = \log_{10} x + \frac{\log_{10} e}{x}\Delta x$$

Let $x = 1$, $\Delta x = 0.015$ and $\log_{10} e = 0.4343$ to get

$$\log_{10} 1.015 = \log_{10} 1 + \frac{0.4343}{1} \cdot 0.015 = 0 + 0.4343(0.015) \approx 0.00651$$

In fact, $\log_{10} 1.015 = 0.00647$.

In Exercises 37–40, prove the property if a and b are positive numbers and x any y are real numbers.

37. $a^x \div a^y = e^{x \ln a} \div e^{y \ln a} = e^{x \ln a - y \ln a} = e^{(x-y)\ln a} = a^{x-y}$

38. $(a^x)^y = (e^{x \ln a})^y = e^{(xy)\ln a} = a^{xy}$

39. $(ab)^x = e^{x \ln ab} = e^{x(\ln a + \ln b)} = e^{x \ln a} e^{x \ln b} = a^x b^x$

40. $a^0 = 1$

▷ We apply Definition 5.4.2 and Theorem 5.4.6. Thus,

$$a^0 = \exp(0 \cdot \ln a) = \exp 0 = e^0 = 1$$

In Exercises 41–44, prove the property if a is any positive number except 1, and x and y are any positive numbers.

41. If $\log_a x = M$ and $\log_a y = N$ then $x = a^M$ and $y = a^N$ so that $xy = a^M a^N$; $xy = a^{M+N}$. Hence $\log_a xy = M + N = \log_a x + \log_a y$. An alternative proof using the formula $\log_a x = \frac{\ln x}{\ln a}$ is as follows:

$\log_a xy = \frac{\ln xy}{\ln a} = \frac{\ln x + \ln y}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} = \log_a x + \log_a y$

42. If $\log_a x = M$ and $\log_a y = N$ then $x = a^M$ and $y = a^N$ so that $x \div y = a^M \div a^N$; $x \div y = a^{M-N}$. Hence $\log_a(x \div y) = M - N = \log_a x - \log_a y$

43. If $\log_a 1 = M$ then $a^M = 1$; so $M = 0$. Thus $\log_a 1 = 0$.

44. $\log_a x^y = y \log_a x$

▷ We apply Theorem 5.4.3. Because $\log_a x = \dfrac{\ln x}{\ln a}$ then $\log_a x^y = \dfrac{\ln x^y}{\ln a} = \dfrac{y \ln x}{\ln a} = y \log_a x$

45. If $S(t)$ is the number of extra daily sales as a result of a sales campaign when t days have elapsed since the campaign ended, then
$S(t) = 1000 \cdot 3^{-t/2}$; $S'(t) = 1000 \cdot 3^{-t/2}(-\frac{1}{2})\ln 3 = -500(\ln 3)3^{-t/2}$
(a) $S'(4) = -500(1.0986)3^{-2} = -61.03$. Therefore 4 days after the campaign has ended, the number of extra daily sales is decreasing at the rate of 61 sales per day.
(b) $S'(10) = -500(1.0986)3^{-5} = -2.26$. Therefore 10 days after the campaign has ended, the number of extra daily sales is decreasing at the rate of 2.26 sales per day.

46. In t years the number of employees will be $N(t)$, where $N(t) = 1000(0.8)^{t/2}$.
(a) $N(4) = 1000(0.8)^2 = 640$. (b) $N'(t) = 500(\ln 0.8)(0.8)^{t/2}$; $N'(4) = 500(\ln 0.8)(0.8)^2 = -71.4$.

- After 4 years there will be 640 employees and their number will be decreasing at the rate of 71 per year.

47. If s ft is the directed distance of the particle from the starting point at t sec, then

$$s = A2^{kt} + B2^{-kt};\; v = \frac{ds}{dt} = kA2^{kt}\ln 2 - kB2^{-2kt}\ln 2$$

$$a = \frac{dv}{dt} = k^2A2^{kt}(\ln 2)^2 + k^2B2^{-kt}(\ln 2)^2 = k^2(\ln 2)^2[A2^{kt} + B2^{-kt}] = k^2(\ln 2)^2 s$$

Therefore a is proportional to s. The motion is not simple harmonic because the constant $k^2(\ln 2)^2$ is positive.

48. A particle moves along a line according to the equation of motion $s = t^{1/t}$, where s ft is the directed distance of the particle from the origin at t sec. Find the velocity and acceleration at 2 sec.

▷
$$\ln s = \ln t^{1/t};\quad \ln s = \frac{\ln t}{t};\quad \frac{1}{s}\frac{ds}{dt} = \frac{t(1/t) - \ln t}{t^2};\quad \frac{ds}{dt} = \frac{s(1 - \ln t)}{t^2} \tag{1}$$

When $t = 2$, then $s = 2^{1/2} = \sqrt{2}$. Thus from (1) we have

$$\left.\frac{ds}{dt}\right]_{t=2} = \frac{\sqrt{2}(1 - \ln 2)}{4} = 0.108$$

Hence the velocity at 2 sec is 0.108 ft/sec. Because $ds/dt = v$, from (1) we have

$$vt^2 = s(1 - \ln t);\quad v(2t) + t^2\frac{dv}{dt} = s\left(-\frac{1}{t}\right) + (1 - \ln t)\frac{ds}{dt}$$

$$\frac{dv}{dt} = \frac{-s/t + (1 - \ln t)v - 2tv}{t^2} = -\frac{st + vt^2(2t - 1 + \ln t)}{t^4} = -\frac{st + s(1 - \ln t)(2t - 1 + \ln t)}{t^4}$$

Substituting $t = 2$ and $s = \sqrt{2}$ we get

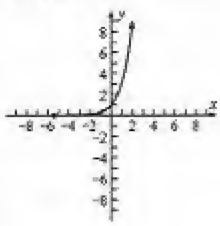
$$a(2) = -\frac{2\sqrt{2} + \sqrt{2}(1 - \ln 2)(3 + \ln 2)}{16} = -0.277$$

Therefore, when $t = 2$ the acceleration is -0.277 ft/sec^2.

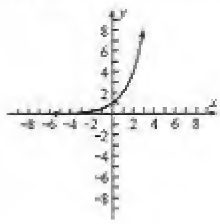
49. (a) If y dollars is the value of the painting t years after its purchase, then $y = 200 \cdot 2^{t/10}$
(b) In 1984, $t = 60$; so $y = 200 \cdot 2^{60/10} = 200 \cdot 2^6 = 12{,}800$.
Therefore in 1984 the value of the painting was \$12,800.
(c) $y'(t) = 200 \cdot 2^{t/10}(\frac{1}{10})\ln 2 = 20(\ln 2)2^{t/10}$ and $y'(60) = 20(0.6931)2^6 = 887.23$.
Thus in 1984 the value of the painting was increasing at the rate of \$887 per year.

In Exercises 50–52, sketch the graph of the equation.

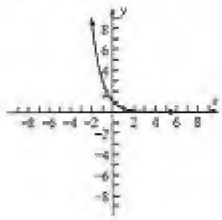
50. (a) $y = 3^x$



51. (a) $y = 2^x$



52. (a) $y = 3^{-x}$



(b) $y = \log_3 x$

(b) $y = \log_2 x$

(b) $y = \log_{1/3} x = -\log_3 x$

53. The region is bounded by the graph of $y = 5^x$ and the lines $x = 1$ and $y = 1$. The width of the element of area is $\Delta_i x$ units and the length is $(5^{w_i} - 1)$ units. Thus, if A square units is the area of the region, then

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (5^{w_i} - 1)\Delta_i x = \int_0^1 (5^x - 1)dx = \frac{5^x}{\ln 5} - x\Big]_0^1 = \left(\frac{5}{\ln 5} - 1\right) - \frac{1}{\ln 5} = \frac{4}{\ln 5} - 1$$

54. The region is in the first quadrant bounded above by the curve $y = e^x$ and below by the curve $y = 2^x$, $x \in [0,2]$. If A square units is the area of the region, then

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (e^{w_i} - 2^{w_i})\Delta_i x = \int_0^2 (e^x - 2^x)dx = e^x - \frac{2^x}{\ln 2}\Big|_0^2 = \left(e^2 - \frac{4}{\ln 2}\right) - \left(1 - \frac{1}{\ln 2}\right) = e^2 - 1 - \frac{3}{\ln 2} \approx 2.061$$

55. The region is in the first quadrant bounded above by $y = 5^x$, below by $y = 1$, $x \in [0,1]$. An element of volume is a circular ring, centered on the x axis, of radii 5^{w_i} and 1. If V cubic units is the volume of the solid of revolution, then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\left[(5^{w_i})^2 - (1)^2\right]\Delta_i x = \pi \int_0^1 (5^{2x} - 1)dx = \pi\left[\frac{5^{2x}}{2 \ln 5} - x\right]_0^1 = \pi\left[\left(\frac{25}{2 \ln 5} - 1\right) - \frac{1}{2 \ln 5}\right] = \left(\frac{12}{\ln 5} - 1\right)\pi$$
$$\approx 6.456$$

56. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = e^x$ and $y = 2^x$ and the line $x = 2$ about the x axis.

▸ The region is in the first quadrant bounded above by $y = e^x$, below by $y = 2^x$, $x \in [0,2]$. An element of volume is a circular ring, centered on the x axis, of radii e^{w_i} and 2^{w_i}. If V cubic units is the volume of the solid of revolution, then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\left[(e^{w_i})^2 - (2^{w_i})^2\right]\Delta_i x = \pi \int_0^2 (e^{2x} - 4^x)dx = \pi\left[\frac{1}{2}e^{2x} - \frac{4^x}{\ln 4}\right]_0^2 = \pi\left[\left(\frac{1}{2}e^4 - \frac{16}{\ln 4}\right) - \left(\frac{1}{2} - \frac{1}{\ln 4}\right)\right]$$
$$= \pi\left(\frac{1}{2}e^4 - \frac{1}{2} - \frac{15}{\ln 4}\right) \approx 50.199$$

• The volume of the solid of revolution is 50.2 cubic units.

57. The region is bounded above by $y = \ln x$ and below by $y = \log_{10} x$, $x \in [1,3]$. If A square units is the area,

$$A = \int_1^3 (\ln x - \log_{10} x)dx = 0.73306$$

58. The region is bounded above by $y = \ln x$ and below by $y = \log_{10} x$, $x \in [1,3]$. If V cubic units is the volume,

$$V = \pi \int_1^3 [(\ln x)^2 - (\log_{10} x)^2]dx = 2.6234$$

In Exercises 59 and 60 (a) plot the graphs of f, f', and f'' in convenient windows and determine (b) any relative extrema of f, where f is (c) increasing and (d) decreasing; (e) where the graph of f is concave upward and downward, and (f) any points of inflection. (g) Confirm your answers analytically.

59. $f(x) = x^{\ln x} = e^{(\ln x)^2}$. $\lim_{x \to 0^+} f(x) = +\infty$. $f'(x) = e^{(\ln x)^2} D_x(\ln x)^2 = x^{\ln x}\left(2 \ln x \cdot \frac{1}{x}\right)$.

Because $\ln x < 0$ if $0 < x < 1$ and $\ln x > 0$ if $x > 1$, then f is decreasing in $(0,1)$, increasing in $(1, +\infty)$ and has an absolute minimum value at 1 of $1^0 = 1$.

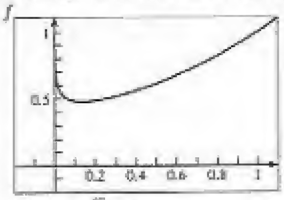
$$f''(x) = x^{\ln x}\left(2 \ln x \cdot \frac{1}{x}\right) + 2x^{\ln x}\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right) = \frac{4x^{\ln x}}{x^2}\left(\ln^2 x + \frac{1}{2} - \frac{1}{2}\ln x\right) = \frac{4x^{\ln x}}{x^2}\left[\left(\ln x + \frac{1}{2}\right)^2 + \frac{1}{4}\right] > 0$$

Hence f is concave upward on $(0, +\infty)$ and has no points of inflection.

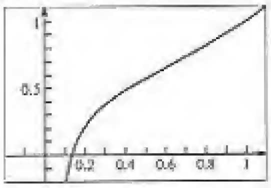
60. $f(x) = x^{\sqrt{x}}$

▷ Below are plots of f, f', and f'' in a $[0,1]\times[0,1]$ window. f appears to be decreasing from 1 to a minimum near 0.15 of about 0.5 and then increasing; its graph appears to be concave upward. This is confirmed by the root of f' at 0.135 and the fact that the graph of f'' is above the x axis.

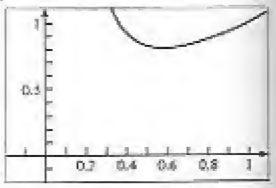
f



f'



f''



$f(x) = e^{\sqrt{x}\ln x}$. $f(0)$ is not defined; because e^x is continuous, $\lim_{x\to 0^+} f(x) = \exp(\lim_{x\to 0^+} \sqrt{x}\ln x) = \exp(0) = 1$

$$f'(x) = e^{\sqrt{x}\ln x} D_x(\sqrt{x}\ln x) = x^{\sqrt{x}}\Big(\frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x}\Big) = x^{\sqrt{x}}\frac{\ln x + 2}{2\sqrt{x}}$$

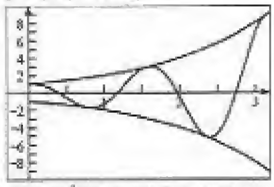
Because $\ln x < -2$ if $0 < x < e^{-2}$) and $\ln x > -2$ if $x > -2$, then f is decreasing on $(0, e^{-2})$, increasing on $(e^{-2}, +\infty)$ and has an absolute minimum value at $e^{-2} \approx 0.135$ of $e^{-2/e} \approx 0.4791$.

$$f''(x) = x^{\sqrt{x}}\Big(\frac{\ln x + 2}{2\sqrt{x}}\Big)^2 + x^{\sqrt{x}}\frac{(1/x)2\sqrt{x} - (\ln x + 2)(1/\sqrt{x})}{(2\sqrt{x})^2} = \frac{x^{\sqrt{x}}}{4x}\Big[(\ln x + 2)^2 - \frac{\ln x}{\sqrt{x}}\Big]$$

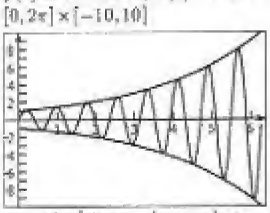
If $0 < x < 1$, $-\ln x/\sqrt{x} > 0$ and if $x \geq 1$, $\ln x/\sqrt{x} \leq \ln x < \ln x + 2 < (\ln x + 2)^2$. Hence $f''(x) > 0$ for all $x > 0$. Thus f is concave upward on $(0, +\infty)$ and has no points of inflection.

In Exercises 61 and 62, plot the graphs of undamped harmonic motion f and its bounding curves F and G.

61. $f(t) = 2^t\cos 4t$, $F(t) = -2^t$, $G(t) = 2^t$, $[0,\pi]\times[-10,10]$



62. $f(t) = 3^{t/3}\sin 8t$, $F(t) = -3^{t/3}$, $G(t) = 3^{t/3}$, $[0,2\pi]\times[-10,10]$



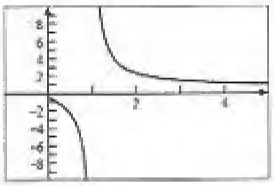
63. $f(x) = \frac{1}{2}(a^x + a^{-x})$. $2f(b)f(c) = 2\cdot\frac{1}{2}(a^b + a^{-b})\cdot\frac{1}{2}(a^c + a^{-c}) = \frac{1}{2}(a^{b+c} + a^{-b-c} + a^{b-c} + a^{-b+c})$
$= f(b+c) + f(b-c)$

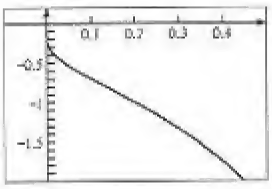
64. By knowing the values of $\log_{10}2$ and $\log_{10}3$, explain why you can compute, without a calculator, $\log_{10}4$, $\log_{10}5$, $\log_{10}6$, $\log_{10}8$, and $\log_{10}9$, but not $\log_{10}7$.

▷ Because we also know $\log_{10}10 = 1$, we can express 4, 5, 6, 8 and 9, but not 7, as products and quotients of 2, 3, and 10. Thus $\log_{10}4 = \log_{10}2^2 = 2\log_{10}2$, $\log_{10}5 = \log_{10}(10/2) = 1 - \log_{10}2$, $\log_{10}6 = \log_{10}(2\cdot 3) = \log_{10}2 + \log_{10}3$, $\log_{10}8 = \log_{10}2^3 = 3\log_{10}2$, and $\log_{10}9 = \log_{10}3^2 = 2\log_{10}3$.

65. $\log_{10}x = \ln x \Leftrightarrow \frac{\ln x}{\ln 10} = \ln x \Leftrightarrow \ln x\Big(\frac{1}{\ln 10} - 1\Big) = 0 \Leftrightarrow \ln x = 0 \Leftrightarrow x = 1$

67. $f(x) = \log_x 5 = \ln 5/\ln x$. f is negative in $(0,1)$ and positive in $(1,+\infty)$. Asymptotes are the x axis and $x = 1$. f appears to be decreasing everywhere; the first branch seems to be concave downward and the second branch concave upward. See the left plot below.





$f'(x) = -\ln 5 \cdot \frac{1}{(\ln x)^2} \cdot \frac{1}{x} < 0$ and so f is decreasing. $f''(x) = -\ln 5\left[\frac{-2}{(\ln x)^3 x} \cdot \frac{1}{x} - \frac{1}{(\ln x)^2} \cdot \frac{-1}{x^2}\right] = \frac{\ln 5(\ln x + 2)}{(\ln x)^3 (x^2)}$

$f''(x) > 0$ if $0 < x < e^{-2} \approx 0.135$ and $f''(x) < 0$ if $x > e^{-2}$. Thus the graph of f is concave upward in $(0, e^{-2})$, downward in $(e^{-2}, 1)$ and $(1, +\infty)$ and has a point of inflection at $x = e^{-2}$. See the right plot above.

5.6 APPLICATIONS OF THE NATURAL EXPONENTIAL FUNCTION

5.6.1 Theorem Suppose y is a continuous function of t with $y > 0$ for all $t \geq 0$. Furthermore

$$\frac{dy}{dt} = ky$$

where k is a constant and $y = y_0$ when $t = 0$. Then

$$y = y_0 e^{kt}$$

If $k < 0$ we have *exponential growth* and if $k < 0$ we have *exponential decay*.
The differential equation in Theorem 5.6.1 means that y' is directly proportional to y.

Newton's Law of Cooling: rate of change of temperature is proportional to temperature difference with ambient.

Bounded Growth If y' is directly proportional to the difference between y and A, where A is a given constant, then the differential equation becomes

$$\frac{dy}{dt} = k(\mathrm{A} - y)$$

and its solution is

$$y = \mathrm{A} + (y_0 - \mathrm{A})e^{-kt}$$

Compound Interest If P dollars is invested at an annual rate of $100i$ percent with interest compounded continuously, then the amount after t years is A dollars with $\mathrm{A} = \mathrm{P}e^{it}$.

Effective Rate The effective annual rate of interest is the interest on \$1 for one year.

Half-life If A_0 units of a substance with half-life h units are present initially, the amount after t units is A with $\mathrm{A} = \mathrm{A}_0(\frac{1}{2})^{t/h}$.

Normal Probability For the standardized normal probability density function, the probability that a random choice of x will be in the closed interval $[a, b]$ is given by $\mathrm{P}([a,b]) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}dx$. Though we cannot find an antiderivative, it follows from Example 13.4.5 that $\mathrm{P}([-\infty, +\infty]) = 1$.

Exercises 5.6

1. Let y be the population t years from 1955.
We have a table of initial conditions.

t	0	40	T
y	40,000	60,000	80,000

(a) $\frac{dy}{dt} = ky$; so $y = y_0 e^{kt} = 40{,}000e^{kt}$. With $t = 40$, $y = 60{,}000$ we have $60{,}000 = 40{,}000e^{40k}$; $e^{40k} = \frac{3}{2}$; $e^k = (\frac{3}{2})^{1/40}$. Thus $y = 40{,}000(\frac{3}{2})^{t/40}$. (b) In 2001, $t = 46$ and $y = 40000(\frac{3}{2})^{46/40} = 63{,}762$ people.

(c) With $y = 80{,}000$, $t = \mathrm{T}$ we have $80{,}000 = 40{,}000(\frac{3}{2})^{T/40}$; $2 = (\frac{3}{2})^{T/40}$; $\ln 2 = \frac{\mathrm{T}}{40}\ln 1.5$; $\mathrm{T} = \frac{40 \ln 2}{\ln 1.5} \approx 68.4$

- Therefore, the population will be 80,000 in 2023.

2. Let y be the number of bacteria present at t hours.
We have a table of initial conditions.

t	0	$\frac{1}{2}$	2
y	1000	2000	y_2

(a) $\frac{dy}{dt} = ky$; so $y = y_0 e^{kt} = 1000e^{kt}$. With $y = 2000$, $t = \frac{1}{2}$ we have $2000 = 1000e^{k/2}$; $e^{k/2} = 2$; $e^k = 4$.

Thus $y = 1000(4)^t$. (b) When $t = 2$ we have $y_2 = 1000(16) = 16{,}000$. 16,000 bacteria present after 2 hours.
(c) With $y = 50{,}000$ and $t = x$ we have $50000 = 1000(4)^x$; $4^x = 50$; $x \ln 4 = \ln 50$; $x = \ln 50 \div \ln 4 = 2.82$

- 50,000 bacteria are present after 2 hr 49 min.

3. i amperes is the current after t sec. $i = 40$ when $t = 0$ and $i = 15$ when $t = 0.01$.

(a) $\frac{di}{dt} = ki$; so $i = i_0 e^{kt} = 40e^{kt}$. With $i = 15$ and $t = 0.01$ we have $15 = 40e^{0.01k}$; $e^{0.01k} = \frac{3}{8}$; $e^k = (\frac{3}{8})^{100}$.

Thus $i = 40(\frac{3}{8})^{100t}$. (b) $i(0.02) = 40(\frac{3}{8})^2 = \frac{45}{8} = 5.625$. After 0.02 sec the current is 5.625 amperes.

4. The population of a town is decreasing at a rate proportional to its size. In 1980 the population was 50,000 and in 1990 it was 44,000. (a) If y is the population t years since 1980, express y as a function of t. (b) Calculate what the population will be in the year 2000.

▷ (a) We have a table of initial conditions.

t	0	10	20
y	50,000	44,000	A_{20}

$\frac{dy}{dt} = ky$; so $y = y_0e^{kt} = 50{,}000e^{kt}$. With $t = 10$, $y = 44{,}000$ we have $44{,}000 = 50{,}000e^{10k}$; $e^{10k} = \frac{22}{25}$; $e^k = (\frac{22}{25})^{1/10}$. Thus $y = 50{,}000(\frac{22}{25})^{t/10}$.

(b) When $t = 20$ we have $y_{20} = 50{,}000(\frac{22}{25})^2 = 38{,}720$. The expected population in 2000 is 38,720.

In Exercises 5–26, define all variables precisely as numbers. Use t to represent time and define the other variables in terms of t. Be sure to write a conclusion.

5. Let y be the attendance t days after opening. $y = 5000$ when $t = 0$ and $y = 2000$ when $t = 3$. $dy/dt = ky$ so $y = y_0e^{kt} = 5000e^{kt}$. With $y = 2000$ and $t = 3$ we have $2000 = 5000e^{3k}$; $e^{3k} = .4$; $e^k = (.4)^{1/3}$. Thus $y = 5000(.4)^{t/3}$. $y(6) = 5000(.4)^2 = 800$. Attendance will be 800 on the sixth day.

6. Let y be the population t years after 1900. $y = 30{,}000$ when $t = 0$ and $y = 60{,}000$ when $t = 60$. $dy/dt = ky$ so $y = y_0e^{kt} = 30{,}000e^{kt}$. With $y = 60{,}000$ and $t = 60$ we have $60{,}000 = 30{,}000e^{60k}$; $2 = e^{60k}$; $e^k = 2^{1/60}$. Thus $y = 30{,}000(2)^{t/60}$. $y(110) = 30{,}000(2)^{110/60} = 106{,}908$. The population will be 107,000 in 2010.

7. Let y dollars be the value t years from now. $y = 35{,}000$ when $t = 0$ and $y = 25{,}000$ when $t = -10$. $dy/dt = ky$ so $y = y_0e^{kt} = 35{,}000e^{kt}$. With $y = 25{,}000$ and $t = -10$ we have $25{,}000 = 35{,}000e^{-10k}$; $e^{10k} = 1.4$; $e^k = 1.4^{1/10}$. Thus $y = 35{,}000(1.4)^{t/10}$. With $y = 50{,}000$ and $t = x$ we have $50{,}000 = 35{,}000(1.4)^{x/10}$; $\frac{10}{7} = 1.4^{x/10}$; $\ln\frac{10}{7} = \frac{x}{10}\ln 1.4$; $x = 10\ln\frac{10}{7} \div \ln 1.4 = 10.6$. The value will be \$50,000 in 10.6 years.

8. After an automobile is 1 year old, its rate of depreciation at any time is proportional to its value at that time. If an automobile was purchased on March 1, 1993 and its values on March 1, 1994 and March 1, 1995, were, respectively, \$7000 and \$5,800, what is its expected value on March 1, 1999?

▷ Let V dollars be the value of the automobile t years after March 1, 1994. We have a table of initial conditions.

t	0	1	5
V	7000	5800	V_5

$\frac{dV}{dt} = kV$; so $V = V_0e^{kt} = 7000e^{kt}$. With $V = 5800$, $t = 1$ we have $5800 = 7000e^k$; $e^k = \frac{29}{35}$.

Thus $V = 7000(\frac{29}{35})^t$. When $t = 5$ we have $V_5 = 7000(\frac{29}{35})^5 \approx 2734$.

The automobile's expected value on March 1, 1999 is \$2734.

9. Because the number triples each hour, the number of bacteria present 4 hours ago is $10{,}000 \div 3^4 = 123$.

10. The half-life formula is $A = A_0(\frac{1}{2})^{t/1690}$.

(a) With $t = 100$, $A_{100} = A_0(\frac{1}{2})^{100/1690} = 0.96A_0$. After 100 years, 96% of the original amount will remain.

(b) With $t = 1000$, $A_{1000} = A_0(\frac{1}{2})^{1000/1690} = 0.66A_0$. After 1000 years, 66% of the original amount will remain.

11. Because 70% of the substance remains after 15 years, we have

$.7A_0 = A_0(\frac{1}{2})^{15/h}$; $.7 = .5^{15/h}$; $\ln .7 = \frac{15}{h}\ln .5$; $h = 15\ln .5 \div \ln .7 = 29.15$. The half-life is 29.15 years.

12. The winter mortality of a certain species of wildlife in a particular geographical region is proportional to the number of the species present at any time. On December 21, the first day of winter, 2400 of the species were present; 2000 were present 30 days later. How many of the species were expected to survive the winter; that is, how many were expected to be alive 90 days after December 21?

▷ Let S be the number of the species present t days after December 21. We have a table of initial conditions.

t	0	30	90
S	2400	2000	S_{90}

$\frac{dS}{dt} = kS$; so $S = S_0e^{kt} = 2400e^{kt}$. With $S = 2000$, $t = 30$ we have $2000 = 2400e^{30t}$; $e^{30t} = 2.5$; $e^t = (2.5)^{1/30}$

Thus $S = 2400(2.5)^{t/30}$. When $t = 90$, $S = 2400(2.5)^3 \approx 1389$.

▪ Therefore 1389 of the species are expected to survive the winter.

13. (a) At 5% compounded continuously, $A = 5000e^{.05} = 5256.36$. (b) Compounded 365 times a year, $A = 5000(1 + .05/365)^{365} = 5256.34$ (c) The effective rate is $(1 + 0.05)^{365} - 1 = 0.0513 = 5.13\%$

14. (a) At 6% compounded continuously, $A = 5000e^{.06} = 5309.18$. (b) Compounded 365 times a year, $A = 5000(1 + .06/365)^{365} = 5309.16$ (c) The effective rate is $(1 + 0.06)^{365} - 1 = 0.0618 = 6.18\%$

15. Let A dollars be the amount in t years of an investment of P dollars at interest compounded continuously. We have a table of initial conditions.

t	0	10	T
A	P	2P	3P

$\frac{dA}{dt} = kA$; so $A = A_0e^{kt} = Pe^{kt}$. With $A = 2P$ and $t = 10$ we have $2P = Pe^{10k}$; $e^k = 2^{1/10}$

Thus $A = P(2)^{t/10}$. With $A = 3P$ and $t = T$ we have $3P = P(2)^{T/10}$; $2^{T/10} = 3$; $\frac{T}{10}\ln 2 = \ln 3$;

$T = \frac{10\ln 3}{\ln 2} \approx 15.9$. Therefore the original amount will triple itself in 15.9 years.

16. If the purchasing power of a dollar is decreasing at the rate of 10 percent annually, compounded continuously, how long will it take for the purchasing power to be \$0.50?

▹ Substituting into $A = Pe^{it}$ with $P = 1$, $A = 0.50$ and $i = -0.10$ we have

$$0.50 = e^{-0.10t}; \quad \ln 0.50 = -0.10t; \quad t = \frac{\ln 0.50}{-0.10} = 6.93$$

▪ The purchasing power is \$0.50 after 7 years.

17. Let A grams be the amount of the substance untransformed at t min. We have a table of initial conditions.

t	0	10	15
A	A_0	$\frac{2}{3}A_0$	$A_0 - 20$

$\frac{dA}{dt} = kA$; so $A = A_0e^{kt}$. With $A = \frac{2}{3}A_0$ and $t = 10$ we have $\frac{2}{3}A_0 = A_0e^{10k}$; $e^k = (\frac{2}{3})^{1/10}$

Thus $A = A_0(\frac{2}{3})^{t/10}$. With $A = A_0 - 20$ and $t = 15$ we have

$A_0 - 20 = A_0(\frac{2}{3})^{15/10}$; $A_0 - A_0(\frac{2}{3})^{3/2} = 20$; $A_0[1 - (\frac{2}{3})^{3/2}] = 20$; $A_0 = \frac{20}{1 - (\frac{2}{3})^{3/2}} \approx 43.9$

18. A tank contains 200 ℓ of brine in which has 3 kg/ℓ of salt. Brine containing 1 kg/ℓ salt flows into the tank at the rate of 4 ℓ/min and the mixture runs out at the same rate. When will the tank contain 1.5 kg/ℓ salt?

▹ Let y kg be the total amount of salt in the tank after t min. Salt is entering the tank at the rate $4\ell/\text{min} \times 1\text{kg}/\ell = 4\text{kg/min}$ and out at $4\ell/\text{min} \times y\ \text{kg}/200\ell = \frac{1}{50}y$ kg/min. When $t = 0$, $y = 200 \times 3 = 600$; when $t = x$, $y = 200 \times 1.5 = 300$. $\frac{dy}{dt} = \text{inflow} - \text{outflow} = 4 - \frac{1}{50}y = \frac{1}{50}(200 - y)$. Substituting $A = 200$, $y_0 = 600$ and $k = \frac{1}{50}$ into $y = A + (y_0 - A)e^{-kt}$ we obtain $y = 200 + (600 - 200)e^{-t/50} = 200 + 400e^{-t/50}$. With $y = 300$ and $t = x$, we have $300 = 200 + 4e^{-x/50}$; $400e^{-t/50} = 100$; $e^{-t/50} = 0.25$, $-\frac{1}{50}x = \ln 0.25$; $x = -50\ln 0.25 = 69.3$.

▪ The concentration will be 1.5 kg/ℓ after 69.3 minutes.

19. Let x kg be the amount of salt in the tank t minutes after fresh water starts to enter. We have a table of initial conditions.

t	0	60
x	70	x_{60}

Brine is running out of the tank at a rate of 3 liters/min; so the tank is losing $3\left(\frac{x}{100}\right)$kg of salt per minute.

$\frac{dx}{dt} = -0.03x$; so $x = x_0e^{-0.03t} = 70e^{-0.03t}$. When $t = 60$ we have $x_{60} = 70e^{-1.8} \approx 11.6$.

▪ Therefore there is 11.6 kg of salt in the tank at the end of 1 hour.

20. Sugar decomposes in water at a rate proportional to the amount still unchanged. If 50 kg of sugar is present initially and at the end of 5 hours this is reduced to 20 kg, how long will it take until 90% of the sugar is decomposed?

▹ Let y be the amount of sugar unchanged after t hours. We are given that $\frac{dy}{dt} = ky$.

Because $y = 50$ when $t = 0$, by Theorem 5.6.1 we have $y = 50\,e^{kt}$. Because $y = 20$ when $t = 5$,

$$20 = 50\,e^{5k}; \quad e^{5k} = 0.4; \quad e^k = (0.4)^{1/5}$$

Therefore, $y = 50(0.4)^{t/5}$.

When 90% of the sugar is decomposed, the amount remaining is 5 kg. Let $t = T$ when $y = 5$. We have

$5 = 50(0.4)^{T/5}$; $\quad 0.1 = (0.4)^{T/5}$; $\quad \ln 0.1 = (T/5)\ln 0.4$; $\quad T = 5\ln 0.1 \div \ln 0.4 = 12.56$

▪ It will take 12.6 hours for 90% of the sugar to decompose.

21 and 22. Let x units be the amount of ^{14}C present t years after the death of the tree. The half-life is 5730 years.

21. In T years the amount is 45% of the original amount x_0. $x = x_0(\frac{1}{2})^{t/5730}$. With $x = 0.45x_0$ and $t = T$ we have

$0.45x_0 = x_0(0.5)^{T/5730}$; $\ln 0.45 = \frac{T}{5730}\ln 0.5$; $T = \frac{5730 \ln 0.45}{\ln 0.5} \approx 6{,}600.98$

- Therefore the tree from which the charcoal came died 6,600 years ago.

22. In T years the amount is 25% of the original amount x_0. $x = x_0(\frac{1}{2})^{t/5730}$. With $x = 0.25x_0$ and $t = T$ we have

$0.25x_0 = x_0(0.5)^{T/5730}$; $\ln 0.25 = \frac{T}{5730}\ln 0.5$; $T = \frac{5730 \ln 0.25}{\ln 0.5} \approx 11{,}460$

- Therefore the tree from which the charcoal came died 11,460 years ago.

23. From Example 6, if y degrees is the temperature of the body after t minutes then

$y = 35 + 85\left(\frac{5}{17}\right)^{t/40}$. If $y = 45$ then $45 = 35 + 85\left(\frac{5}{17}\right)^{t/40}$; $10 = 85\left(\frac{5}{17}\right)^{t/40}$; $\frac{10}{85} = \left(\frac{5}{17}\right)^{t/40}$;

$\ln\frac{10}{85} = \frac{t}{40}\ln\frac{5}{17}$; $t = \frac{40\ln(10/85)}{\ln(5/17)} = 69.95$. The body is at 45° after 70 minutes.

24. A pot of water was initially boiling at 100° and was cooling in air at a temperature of 0°. After 20 min the temperature of the water was 90°. (a) After how many minutes was the temperature of the water 80°? (b) What was the temperature of the water after 1 hr? Use Newton's law of cooling.

▸ If y degrees is the difference between the temperature of the water and the air temperature after t min, then Newton's law of cooling states that

$$\frac{dy}{dt} = ky$$

t	0	20	T	60
y	100	90	80	y_{60}

The initial conditions are given in the table.

(a) We have $y = y_0e^{kt} = 100e^{kt}$. Because $y = 90$ when $t = 20$, then

$$90 = 100e^{20k}; \quad 0.9 = e^{20k}; \quad e^k = (0.9)^{1/20}; \quad y = 100(0.9)^{1/20} \tag{1}$$

Because $y = 80$ when $t = T$, then

$$80 = 100(0.9)^{T/20};\ 0.8 = (0.9)^{T/20};\ \ln 0.8 = (T/20)\ln 0.9;\ T = 20\ln 0.8 \div \ln 0.9 \approx 42.36$$

- The temperature is 80° after 42.36 minutes.

(b) We have from Eq. (1) $\quad y_{60} = 100(0.9)^3 = 72.9$. After 1 hour the temperature is exactly 72.9°.

25. Let y degrees be the temperature, and $Y = y - 35$ degrees the temperature difference, t minutes after the thermometer is removed. We have the boundary conditions:

t	0	$\frac{1}{2}$	a	3
y	75	65	50	y_3

From Newton's law of cooling, $\frac{dY}{dt} = kY$, $Y_0 = 75 - 35 = 40$. $Y = Y_0e^{kt}$; $y - 35 = 40e^{kt}$.

When $t = \frac{1}{2}$ and $y = 65$ we get $30 = 40e^{k/2}$; $e^{k/2} = \frac{3}{4}$. Thus $y = 35 + 40(e^{k/2})^{2t}$; $y = 35 + 40(\frac{3}{4})^{2t}$.

(a) When $t = a$ and $y = 50$ we get $50 = 35 + 40(\frac{3}{4})^{2a}$; $\frac{15}{40} = \left(\frac{3}{4}\right)^{2a}$; $\ln\frac{15}{40} = 2a\ln\frac{3}{4}$;

$a = \frac{\ln(15/40)}{2\ln(3/4)} = 1.70$. The temperature is 50° after 1.70 minutes = 1:42 minutes.

(b) When $t = 3$ we get $y_3 = 35 + 40\left(\frac{3}{4}\right)^6 = 42.1$. After 3 minutes the temperature is 42.1°.

26. We see that the half-life of the temperature difference is 40 minutes. Therefore, in 40 more minutes the difference will decrease from 100° to 50°.

27. Let y be the number of facts the student has memorized at the end of t min. We have a table of initial conditions.

t	0	20	60	180
y	0	15	y_{60}	y_{180}

$\frac{dy}{dt} = k(60 - y)$; $\frac{dy}{60 - y} = k\,dt$; $\int \frac{dy}{60 - y} = k\int dt$; $-\ln|60 - y| = kt + \bar{C}$; $60 - y = Ce^{-kt}$; $y = 60 - Ce^{-kt}$

With $y = 0$, $t = 0$ we have $0 = 60 - C$; $C = 60$. Thus $y = 60 - 60e^{-kt}$. With $y = 15$, $t = 20$ we have

$15 = 60 - 60e^{-20k}$; $60e^{-20k} = 45$; $e^{-20k} = \frac{3}{4}$; $e^{-k} = (\frac{3}{4})^{1/20}$. Therefore $y = 60 - 60(\frac{3}{4})^{t/20}$.

(a) When $t = 60$, $y_{60} = 60 - 60(\frac{3}{4})^3 \approx 34.7$ (b) When $t = 180$, $y_{180} = 60 - 60(\frac{3}{4})^9 \approx 55.5$

28. A new worker on an assembly line can do a particular task in such a way that if y units are completed per day after t days on the assembly line, then

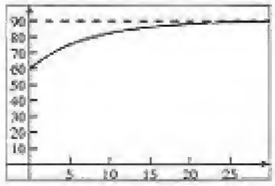
$$\frac{dy}{dt} = k(90 - y)$$

where k is a positive constant. On the day the worker starts, 60 units are completed, and on the fifth day day the worker completes 75 units. (a) Express y as a function of t. (b) How many units per day can the worker eventually be expected to complete. (c) Plot the graph of your function in part (a) and the horizontal asymptote of the graph. (d) How many units does the worker complete on the ninth day? (e) Show that the worker is producing at almost full potential after 30 days.

▷ (a) With $A = 90$ and $y_0 = 60$ in the bounded growth solution, we have $y = A + (y_0 - A)e^{-kt} = 90 - 30e^{-kt}$. With $y = 75$ and $t = 5$ we have

$$75 = 90 - 30e^{-5k}; \qquad 30e^{-5k} = 15; \quad e^{-5k} = \tfrac{1}{2}; \qquad e^{-k} = (\tfrac{1}{2})^{1/5}; \quad y = 90 - 30(\tfrac{1}{2})^{t/5}$$

(b) Because $\lim\limits_{t \to +\infty} [90 - 30(\frac{1}{2})^{t/5}] = 90$, the worker will eventually complete 90 units per day. (c) A plot is shown at the right.



(d) $y(9) = 90 - 30(\frac{1}{2})^{9/5} \approx 81.4$
The worker completes 81 units on the ninth day.

(e) $y(30) = 90 - 30(\frac{1}{2})^{6} = 89.53$.
After 30 days the worker is producing almost 90 units.

In Exercises 29–33, use NINT to estimate the answer to five significant digits.

29. $P([0,1]) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^1 e^{-x^2/2} dx = 0.3413447 \approx 0.34414$

30. $P([0,1]) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-3}^3 e^{-x^2/2} dx = 0.99730$

In Exercises 31 and 32, the error function is defined by $\operatorname{erf}(x) = \dfrac{1}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2} dt$

31. $\operatorname{erf}(1) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^1 e^{-t^2} dt = 0.84270$

32. Find the value of erf(3).

▷ $\operatorname{erf}(3) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^3 e^{-t^2} dt = 0.99998$

33. Find the general solution of the Gompertz growth equation $\dfrac{dy}{dt} = ky \ln \dfrac{a}{y}$.

▷ We have $\dfrac{1}{y}\dfrac{dy}{dt} = k(\ln a - \ln y)$. With $u = \ln y$, $\dfrac{du}{dt} = \dfrac{1}{y}\dfrac{dy}{dt}$ we obtain $\dfrac{du}{dt} = k(\ln a - u)$, a bounded growth equation. Thus $u = \ln a + (u_0 - \ln a)e^{-kt}$. Then $y = \exp(u) = \exp[\ln a + (\ln y_0 - \ln a)e^{-kt}] = a\left(\dfrac{y_0}{a}\right)^{e^{-kt}}$.

34. Prove the half-life formula $y = y_0(\frac{1}{2})^{t/h}$.

▷ The exponential growth formula is $y = y_0 e^{kt}$. with $y = \frac{1}{2}y_0$ and $t = h$ we get
$\frac{1}{2}y_0 = y_0 e^{kh}$; $\frac{1}{2} = e^{kh}$; $e^k = (\frac{1}{2})^{1/h}$; $y = y_0(\frac{1}{2})^{t/h}$

5.7 INVERSE TRIGONOMETRIC FUNCTIONS

5.7.1 Definition The inverse sine function, denoted by $\sin^{-1}$ is defined as follows:
$y = \sin^{-1} x$ if and only if $x = \sin y$ and $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$

5.7.2 Theorem If u is a differentiable function of x $\qquad D_x(\sin^{-1}) = \dfrac{1}{\sqrt{1-u^2}} D_x u$

5.7.3 Definition The inverse cosine function, denoted by $\cos^{-1}$, is defined as follows:
$y = \cos^{-1} x$ if and only if $x = \cos y$ and $0 \le y \le \pi$

5.7.4 Theorem If u is a differentiable function of x $\qquad D_x(\cos^{-1} u) = -\dfrac{1}{\sqrt{1-u^2}} D_x u$

5.7.5 Definition The inverse tangent function, denoted by $\tan^{-1}$, is defined as follows:
$y = \tan^{-1} x$ if and only if $x = \tan y$ and $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$

5.7.6 Theorem If u is a differentiable function of x $\qquad D_x(\tan^{-1} u) = \dfrac{1}{1+u^2} D_x u$

5.7.7 Definition The inverse secant function, denoted by $\sec^{-1}$, is defined as follows:

$$y = \sec^{-1} x \text{ if and only if } x = \sec y \text{ and } \begin{cases} 0 \le y < \frac{1}{2}\pi & \text{if } x \ge 1 \\ \pi \le y < \frac{3}{2}\pi & \text{if } x \le -1 \end{cases}$$

To calculate, use $\sec^{-1} x = \begin{cases} \cos^{-1} \frac{1}{x} & \text{if } x \ge 1 \\ 2\pi - \cos^{-1} \frac{1}{x} & \text{if } x \le -1 \end{cases}$

5.7.8 Theorem If u is a differentiable function of x $\qquad D_x(\sec^{-1} u) = \dfrac{1}{u\sqrt{u^2-1}} D_x u$

5.7.9 Definition The inverse cotangent function, denoted by $\cot^{-1}$, is defined by $\cot^{-1} x = \frac{1}{2}\pi - \tan^{-1} x$, where x is any real number

5.7.10 Theorem If u is a differentiable function of x $\qquad D_x(\cot^{-1} u) = -\dfrac{1}{1+u^2} D_x u$

5.7.11 Definition The inverse cosecant function, denoted by $\csc^{-1}$, is defined by $\csc^{-1} x = \frac{1}{2}\pi - \sec^{-1} x$ for $x \ge 1$

To calculate, use $\csc^{-1} x = \begin{cases} \sin^{-1} \frac{1}{x} & \text{if } x \ge 1 \\ -\pi - \sin^{-1} \frac{1}{x} & \text{if } x \le -1 \end{cases}$

5.7.12 Theorem If u is a differentiable function of x $\qquad D_x(\csc^{-1} u) = -\dfrac{1}{u\sqrt{u^2-1}} D_x$

Summary

$y = f(x)$	Domain	Range	Derivative of $f(u)$
$y = \sin^{-1} x$	$\{x \mid -1 \le x \le 1\}$	$\{y \mid -\frac{1}{2}\pi \le y \le \frac{1}{2}\pi\}$	$1/\sqrt{1-u^2} \cdot D_x u$
$y = \cos^{-1} x$	$\{x \mid -1 \le x \le 1\}$	$\{y \mid 0 \le y \le \pi\}$	$-1/\sqrt{1-u^2} \cdot D_x u$
$y = \tan^{-1} x$	$\{x \mid -\infty < x < +\infty\}$	$\{y \mid -\frac{1}{2} < y < \frac{1}{2}\pi\}$	$1/(1+u^2) \cdot D_x u$
$y = \cot^{-1} x$	$\{x \mid -\infty < x < +\infty\}$	$\{y \mid 0 < y < \pi\}$	$-1/(1+u^2) \cdot D_x u$
$y = \sec^{-1} x$	$\{x \mid x \ge 1 \text{ or } x \le -1\}$	$\{y \mid 0 \le y < \frac{1}{2}\pi \text{ or } \pi \le y < \frac{3}{2}\pi\}$	$1/u\sqrt{u^2-1} \cdot D_x u$
$y = \csc^{-1} x$	$\{x \mid x \ge 1 \text{ or } x \le -1\}$	$\{y \mid 0 < y \le \frac{1}{2}\pi \text{ or } -\pi < y \le -\frac{1}{2}\pi\}$	$-1/u\sqrt{u^2-1} \cdot D_x u$

Solving Equations If $\sin x = y$ then $x = 2k\pi + \sin^{-1} y$ or $x = (2n-1)\pi - \sin^{-1}$ where k and n are any integers.
If $\cos x = y$ then $x = 2k\pi + \cos^{-1} y$ or $x = 2n\pi - \cos^{-1} y$ where k and n are any integers.
If $\tan x = y$ then $x = k\pi + \tan^{-1} x$ where k is any integer.

Sum Formulas $\sin^{-1} x + \cos^{-1} x = \frac{1}{2}\pi$; $\tan^{-1} x + \cot^{-1} x = \frac{1}{2}\pi$; $\sec^{-1} x + \csc^{-1} x = \frac{1}{2}\pi$;

$\tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \dfrac{x \pm y}{1 \mp xy} + k\pi$ where $k = 0$ if $1 \mp xy > 0$ and $k = \operatorname{sgn}(x)$ if $1 \mp xy < 0$

We note that if f is any one of the six trigonometric functions and x is a *positive* number in the domain of f^{-1}, then the number in the range of f^{-1} is between 0 and $\frac{1}{2}\pi$. That is,

$$0 \le f^{-1}(x) \le \tfrac{1}{2}\pi \text{ if } x > 0$$

However, when $-x$ is a *negative* number in the domain of f^{-1} we have the following results.

$-\frac{1}{2}\pi \le \sin^{-1}(-x) < 0 \quad \frac{1}{2}\pi < \cos^{-1}(-x) \le \pi \qquad \sin^{-1}(-x) = -\sin^{-1} x$; $\cos^{-1}(-x) = \pi - \cos^{-1} x$

$-\frac{1}{2}\pi < \tan^{-1}(-x) < 0 \quad \frac{1}{2}\pi < \cot^{-1}(-x) < \pi \qquad \tan^{-1}(-x) = -\tan^{-1} x$; $\cot^{-1}(-x) = \pi - \cot^{-1} x$

$-\pi < \csc^{-1}(x) \le -\frac{1}{2}\pi \quad \pi \le \sec^{-1}(-x) < \frac{3}{2}\pi \qquad \csc^{-1}(-x) = \csc^{-1} x - \pi$; $\sec^{-1}(-x) = \sec^{-1} x + \pi$

To calculate a function value when given the value of an inverse function, express it as a fraction and use it to define a real or artificial right triangle. See Exercises 8 and 12.

Exercises 5.7

In Exercises 1–6, determine the exact function value.

1. (a) $\sin^{-1} \frac{1}{2} = \frac{1}{6}\pi$ (b) $\sin^{-1}(-\frac{1}{2}) = -\frac{1}{6}\pi$ (c) $\cos^{-1} \frac{1}{2} = \frac{1}{3}\pi$ (d) $\cos^{-1}(-\frac{1}{2}) = \frac{1}{3}\pi$
2. (a) $\sin^{-1} \frac{1}{2}\sqrt{3} = \frac{1}{3}\pi$ (b) $\sin^{-1}(-\frac{1}{2}\sqrt{3}) = -\frac{1}{3}\pi$ (c) $\cos^{-1} \frac{1}{2}\sqrt{3} = \frac{1}{6}\pi$ (d) $\cos^{-1}(-\frac{1}{2}\sqrt{3}) = \pi - \frac{1}{6}\pi = \frac{5}{6}\pi$
3. (a) $\tan^{-1} \dfrac{1}{\sqrt{3}} = \frac{1}{6}\pi$ (b) $\tan^{-1}(-\sqrt{3}) = -\frac{1}{3}\pi$ (c) $\sec^{-1} \dfrac{2}{\sqrt{3}} = \frac{1}{6}\pi$ (d) $\sec\left(-\dfrac{2}{\sqrt{3}}\right) = \frac{7}{6}\pi$

4. (a) $\cot^{-1}\dfrac{1}{\sqrt{3}}$ (b) $\cot^{-1}(-\sqrt{3})$ (c) $\csc^{-1}\dfrac{2}{\sqrt{3}}$ (d) $\csc^{-1}(-\dfrac{2}{\sqrt{3}})$

▸ (a) By Definition 5.7.9 $\cot^{-1}\dfrac{1}{\sqrt{3}} = \frac{1}{2}\pi - \tan^{-1}\dfrac{1}{\sqrt{3}} = \frac{1}{2}\pi - \frac{1}{6}\pi = \frac{1}{3}\pi$

(b) By Definition 5.7.9 $\cot^{-1}(-\sqrt{3}) = \frac{1}{2}\pi - \tan^{-1}(-\sqrt{3}) = \frac{1}{2}\pi + \tan^{-1}\sqrt{3} = \frac{1}{2}\pi + \frac{1}{3}\pi$

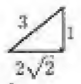
(c) By Definition 5.7.11 $\csc^{-1}\dfrac{2}{\sqrt{3}} = \sin^{-1}\frac{1}{2}\sqrt{3} = \frac{1}{3}\pi$

(d) By Definition 5.7.11 $\csc^{-1}(-\dfrac{2}{\sqrt{3}}) = -\pi - \sin(-\frac{1}{2}\sqrt{3}) = -\pi + \sin\frac{1}{2}\sqrt{3} = -\pi + \frac{1}{3}\pi = -\frac{2}{3}\pi$

5. (a) $\sin^{-1}1 = \frac{1}{2}\pi$ (b) $\sin^{-1}(-1) = -\frac{1}{2}\pi$ (c) $\csc^{-1}1 = \frac{1}{2}\pi$ (d) $\csc^{-1}(-1) = -\frac{1}{2}\pi$ (e) $\sin^{-1}0 = 0$

6. (a) $\cos^{-1}1 = 0$ (b) $\cos^{-1}(-1) = \pi$ (c) $\sec^{-1}1 = 0$ (d) $\sec^{-1}(-1) = \pi$ (e) $\cos^{-1}0 = \frac{1}{2}\pi$

7. Given $\theta = \sin^{-1}\frac{1}{3}$. Then $\sin\theta = \frac{1}{3}$ and $0 < \theta < \frac{1}{2}\pi$.



(a) $\cos\theta = \frac{2}{3}\sqrt{2}$ (b) $\tan\theta = \dfrac{1}{2\sqrt{2}} = \frac{1}{4}\sqrt{2}$

(c) $\cot\theta = 2\sqrt{2}$ (d) $\sec\theta = \dfrac{3}{2\sqrt{2}} = \frac{3}{4}\sqrt{2}$ (e) $\csc\theta = 3$

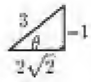
8. Given $\theta = \cos^{-1}\frac{2}{3}$, find the exact value of each of the following:
(a) $\sin\theta$ (b) $\tan\theta$ (c) $\cot\theta$ (d) $\sec\theta$ (e) $\csc\theta$

▸ Because $\cos\theta = \frac{2}{3}$, and $0 < \theta < \frac{1}{2}\pi$, there exists a right triangle containing an acute angle whose measure is θ. Furthermore, $\frac{2}{3}$ is the ratio of the adjacent side divided by the hypotenuse of the right triangle, as illustrated in the figure. The length of the opposite side of the triangle is found by applying the Pythagorean theorem, $a^2 + b^2 = c^2$. From the figure we conclude that

(a) $\sin\theta = \dfrac{\sqrt{5}}{3}$ (b) $\tan\theta = \dfrac{\sqrt{5}}{2}$ (c) $\cot\theta = \dfrac{2}{\sqrt{5}} = \frac{2}{5}\sqrt{5}$

(d) $\sec\theta = \frac{3}{2}$ (e) $\csc\theta = \dfrac{3}{\sqrt{5}} = \frac{3}{5}\sqrt{5}$

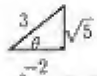
9. Given $\theta = \sin^{-1}(-\frac{1}{3})$. Then $\sin\theta = -\frac{1}{3}$ and $-\frac{1}{2}\pi < \theta < 0$ so $x > 0$ and $y < 0$.



(a) $\cos\theta = \frac{2}{3}\sqrt{2}$ (b) $\tan\theta = -\frac{1}{4}\sqrt{2}$

(c) $\cot\theta = -2\sqrt{2}$ (d) $\sec\theta = \frac{3}{4}\sqrt{2}$ (e) $\csc\theta = -3$

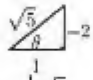
10. Given $\theta = \cos^{-1}(-\frac{2}{3})$. Then $\cos\theta = -\frac{2}{3}$ and $\frac{1}{2}\pi < \theta \le \pi$ so $x < 0$ and $y > 0$.



(a) $\sin\theta = \dfrac{\sqrt{5}}{3}$ (b) $\tan\theta = -\dfrac{\sqrt{5}}{2}$

(c) $\cot\theta = -\dfrac{2}{\sqrt{5}} = -\frac{2}{5}\sqrt{5}$ (d) $\sec\theta = -\frac{3}{2}$ (e) $\csc\theta = \dfrac{3}{\sqrt{5}} = \frac{3}{5}\sqrt{5}$

11. Given $\theta = \tan^{-1}(-2)$. Then $\tan\theta = -2$ and $-\frac{1}{2}\pi < \theta < 0$ so $x > 0$ and $y < 0$.

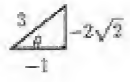


(a) $\sin\theta = -\dfrac{-2}{\sqrt{5}} = -\frac{2}{5}\sqrt{5}$ (b) $\cos\theta = \dfrac{1}{\sqrt{5}} = \frac{1}{5}\sqrt{5}$

(c) $\cot\theta = -\frac{1}{2}$ (d) $\sec\theta = \sqrt{5}$ (e) $\csc\theta = -\frac{1}{2}\sqrt{5}$

12. Given $\theta = \sec^{-1}(-3)$, find the exact value of each of the following:
(a) $\sin\theta$ (b) $\cos\theta$ (c) $\tan\theta$ (d) $\cot\theta$ (e) $\csc\theta$

▸ Because $\sec\theta = \dfrac{3}{-1}$ and $\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$, we make an artificial right triangle whose hypotenuse (which must be positive) is 3 and whose adjacent side is -1. By the Pythagorean theorem, the opposite side is $\pm\sqrt{3^2 - 1^2}$ $= \pm 2\sqrt{2}$. Because we wish $\tan\theta$ to be positive, we use $-2\sqrt{2}$. From the figure we conclude that



(a) $\sin\theta = \dfrac{-2\sqrt{2}}{3}$ (b) $\cos\theta = \dfrac{-1}{3}$

(c) $\tan\theta = \dfrac{-2\sqrt{2}}{-1} = 2\sqrt{2}$ (d) $\cot\theta = \dfrac{-1}{2\sqrt{2}} = -\frac{1}{4}\sqrt{2}$

(e) $\csc\theta = \dfrac{3}{-2\sqrt{2}} = -\frac{3}{4}\sqrt{2}$

In Exercises 13–24, find the exact value of the quantity.

13. (a) $\sin^{-1}(\sin \frac{1}{6}\pi) = \frac{1}{6}\pi$ (b) $\sin^{-1}[\sin(-\frac{1}{6}\pi)] = -\frac{1}{6}\pi$

(c) $\sin^{-1}(\sin \frac{5}{6}\pi) = \sin^{-1}(\sin \frac{1}{6}\pi) = \frac{1}{6}\pi$ (d) $\sin^{-1}(\sin \frac{11}{6}\pi) = \sin^{-1}[\sin(-\frac{1}{6}\pi)] = -\frac{1}{6}\pi$

14. (a) $\sin^{-1}(\sin \frac{1}{3}\pi) = \frac{1}{3}\pi$ (b) $\sin^{-1}[\sin(-\frac{1}{3}\pi)] = -\frac{1}{3}\pi$

(c) $\sin^{-1}(\sin \frac{2}{3}\pi) = \sin^{-1}(\sin \frac{1}{3}\pi) = \frac{1}{3}\pi$ (d) $\sin^{-1}(\sin \frac{5}{3}\pi) = \sin^{-1}[\sin(-\frac{1}{3}\pi)] = -\frac{1}{3}\pi$

15. (a) $\cos^{-1}(\cos \frac{1}{3}\pi) = \frac{1}{3}\pi$ (b) $\cos^{-1}[\cos(-\frac{1}{3}\pi)] = \cos^{-1}(\cos \frac{1}{3}\pi) = \frac{1}{3}\pi$

(c) $\cos^{-1}(\cos \frac{2}{3}\pi) = \frac{2}{3}\pi$ (d) $\cos^{-1}(\cos \frac{4}{3}\pi) = \cos^{-1}(\cos \frac{4}{3}\pi) = \frac{4}{3}\pi$

16. (a) $\cos^{-1}(\cos \frac{1}{4}\pi)$; (b) $\cos^{-1}[\cos(-\frac{1}{4}\pi)]$; (c) $\cos^{-1}(\cos \frac{3}{4}\pi)$; (d) $\cos^{-1}(\cos \frac{5}{4}\pi)$

▷ Because the range of $\cos^{-1}$ is $[0, \pi]$, we find an angle in that range with the same cosine. We make no use of the fact that we know the value of the cosine of the given angle.

(a) Because $\frac{1}{4}\pi \in [0, \pi]$, then $\cos^{-1}(\cos \frac{1}{4}\pi) = \frac{1}{4}\pi$. (c) Because $\frac{3}{4}\pi \in [0, \pi]$, then $\cos^{-1}(\cos \frac{3}{4}\pi) = \frac{3}{4}\pi$.

(b) Because $\cos(-\frac{1}{4}\pi) = \cos(\frac{1}{4}\pi)$ and $\frac{1}{4}\pi \in [0, \pi]$, then $\cos^{-1}[\cos(-\frac{1}{4}\pi)] = \frac{1}{4}\pi$.

(d) Because $\cos \frac{5}{4}\pi = \cos \frac{3}{4}\pi$ and $\frac{3}{4}\pi \in [0, \pi]$, then $\cos^{-1}(\cos \frac{5}{4}\pi) = \frac{4}{4}\pi$.

17. (a) $\tan^{-1}(\tan \frac{1}{6}\pi) = \frac{1}{6}\pi$ (b) $\tan^{-1}[\tan(-\frac{1}{3}\pi)] = -\frac{1}{3}\pi$

(c) $\tan^{-1}(\tan \frac{7}{6}\pi) = \tan^{-1}(\tan \frac{1}{6}\pi) = \frac{1}{6}\pi$ (d) $\tan^{-1}[\tan(-\frac{4}{3}\pi)] = \tan^{-1}[\tan(-\frac{1}{3}\pi)] = -\frac{1}{3}\pi$

18. (a) $\tan^{-1}(\tan \frac{1}{3}\pi) = \frac{1}{3}\pi$ (b) $\tan^{-1}[\tan(-\frac{1}{6}\pi)] = -\frac{1}{6}\pi$

(c) $\tan^{-1}(\tan \frac{4}{3}\pi) = \tan^{-1}(\tan \frac{1}{3}\pi) = \frac{1}{3}\pi$ (d) $\tan^{-1}[\tan(-\frac{7}{6}\pi)] = \tan^{-1}[\tan(-\frac{1}{6}\pi)] = -\frac{1}{6}\pi$

19. (a) $\sec^{-1}(\sec \frac{1}{3}\pi) = \frac{1}{3}\pi$ (b) $\sec^{-1}[\sec(-\frac{1}{3}\pi)] = \sec^{-1}(\sec \frac{1}{3}\pi) = \frac{1}{3}\pi$

(c) $\sec^{-1}(\sec \frac{2}{3}\pi) = \sec^{-1}(\sec \frac{4}{3}\pi) = \frac{4}{3}\pi$ (d) $\sec^{-1}(\sec \frac{4}{3}\pi) = \frac{4}{3}\pi$

20. (a) $\sec^{-1}(\sec \frac{1}{4}\pi)$; (b) $\sec^{-1}[\sec(-\frac{1}{4}\pi)]$; (c) $\sec^{-1}(\sec \frac{3}{4}\pi)$; (d) $\sec^{-1}(\sec \frac{5}{4}\pi)$

▷ (a) Because $\frac{1}{4}\pi \in [0, \frac{1}{2}\pi)$, then $\sec^{-1}(\sec \frac{1}{4}\pi) = \frac{1}{4}\pi$. (d) Because $\frac{5}{4}\pi \in [\pi, \frac{3}{2}\pi)$, then $\sec^{-1}(\sec \frac{5}{4}\pi) = \frac{5}{4}\pi$.

(b) Because $\sec(-\frac{1}{4}\pi) = \sec \frac{1}{4}\pi$ and $\frac{1}{4}\pi \in [0, \frac{1}{2}\pi)$, then $\sec^{-1}[\sec(-\frac{1}{4}\pi)] = \frac{1}{4}\pi$.

(c) Because $\sec \frac{3}{4}\pi = \sec \frac{5}{4}\pi$ and $\frac{5}{4}\pi \in [\pi, \frac{3}{2}\pi)$, then $\sec^{-1}(\sec \frac{3}{4}\pi) = \frac{5}{4}\pi$.

21. (a) Let $\sin^{-1} \frac{1}{2}\sqrt{3} = \alpha$. Then $\sin \alpha = \frac{1}{2}\sqrt{3}$ and $0 < \alpha < \frac{1}{2}\pi$. Therefore, $\tan \alpha = \sqrt{3}$.

(b) Let $\tan^{-1} \frac{1}{2}\sqrt{3} = \beta$. Then $\tan \beta = \frac{1}{2}\sqrt{3}$ and $0 < \beta < \frac{1}{2}\pi$. Therefore, $\sin \beta = \dfrac{\sqrt{3}}{\sqrt{7}} = \frac{1}{7}\sqrt{21}$.

22. (a) Let $\tan^{-1}(-3) = \alpha$. Then $\tan \alpha = -3$ and $-\frac{1}{2}\pi < \alpha < 0$. Therefore, $\cos \alpha = \dfrac{1}{\sqrt{10}} = \frac{1}{10}\sqrt{10}$.

(b) Let $\sec^{-1}(-3) = \beta$. Then $\sec \beta = -3$ and $\pi < \beta < \frac{3}{2}\pi$. Therefore $\tan \beta = 2\sqrt{2}$.

23. (a) Let $\sin^{-1}(-\frac{1}{2}) = \alpha$. Then $\sin \alpha = -\frac{1}{2}$ and $-\frac{1}{2}\pi < \alpha < 0$. Therefore, $\cos \alpha = \frac{1}{2}\sqrt{3}$.

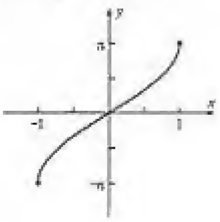
(b) Let $\cos^{-1}(-\frac{1}{2}) = \beta$. Then $\cos \beta = -\frac{1}{2}$ and $\frac{1}{2}\pi < \beta < \pi$. Therefore, $\sin \beta = \frac{1}{2}\sqrt{3}$.

24. (a) $\tan[\cot^{-1}(-1)]$; (b) $\cot[\tan^{-1}(-1)]$

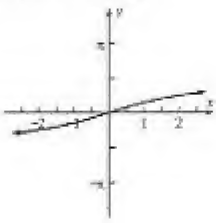
▷ Because $\tan \theta$ and $\cot \theta$ are reciprocals, both answers are -1.

In Exercises 25–30, sketch the graph of the function. Support your graph by plotting.

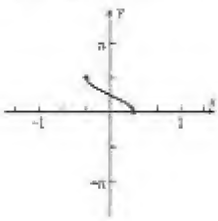
25. $f(x) = 2 \sin^{-1} x$



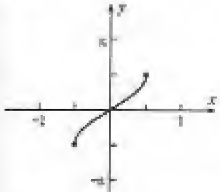
27. $g(x) = \text{arc tan } \frac{1}{2}x$



29. $h(x) = \frac{1}{2} \cos^{-1} 3x$

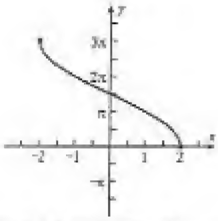


26. $g(x) = \arcsin 2x$



28. $f(x) = \frac{1}{2}\tan^{-1} x$

30. $h(x) = 3\cos^{-1}\frac{1}{2}x$



In Exercises 31 and 32, sketch the graph of the function. Check by plotting. State the domain D and range R.

31. (a) $f(x) = \sin(\sin^{-1} x)$ (b) $f(x) = \sin^{-1}(\sin x)$

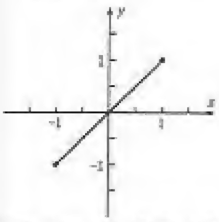
▷ (a) $D = [-1, 1]$, $R = [-1, 1]$

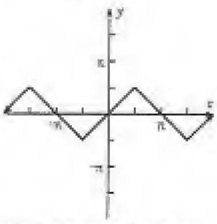
▷ (b) $D = (-\infty, +\infty)$, $R = [-\frac{1}{2}\pi, \frac{1}{2}\pi]$

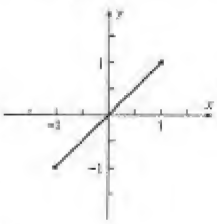
32. (a) $f(x) = \cos(\cos^{-1} x)$ (b) $f(x) = \cos^{-1}(\cos x)$

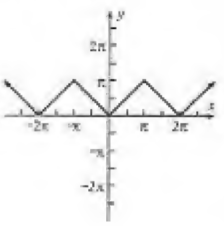
▷ (a) $D = [-1, 1]$, $R = [-1, 1]$

▷ (b) $D = (-\infty, +\infty)$, $R = [0, \pi]$









In Exercises 33–42, find the derivative of the function.

33. (a) $\frac{d}{dx}(\sin^{-1}\frac{1}{2}x) = \frac{1}{\sqrt{1-\frac{1}{4}x^2}}\frac{d}{dx}(\frac{1}{2}x) = \frac{1}{\sqrt{4-x^2}}$ (b) $\frac{d}{dx}(\tan^{-1}2x) = \frac{1}{1+(2x)^2}\frac{d}{dx}(2x) = \frac{2}{1+4x^2}$

34. (a) $f'(x) = \frac{d}{dx}(\cos^{-1}3x) = \frac{1}{\sqrt{1-(3x)^2}}\frac{d}{dx}(3x) = -\frac{3}{\sqrt{1-9x^2}}$

(b) $g'(x) = \frac{d}{dx}(\sec^{-1}2x) = \frac{1}{(2x)\sqrt{(2x)^2-1}}\frac{d}{dx}(2x) = \frac{1}{x\sqrt{4x^2-1}}$

35. (a) $F'(x) = \frac{d}{dx}(2\cos^{-1}\sqrt{x}) = 2\left(-\frac{1}{\sqrt{1-x}}\right)\frac{d}{dx}\sqrt{x} = -\frac{2}{\sqrt{1-x}}\left(\frac{1}{2\sqrt{x}}\right) = -\frac{1}{\sqrt{x-x^2}}$

(b) $g'(t) = \frac{d}{dt}(\sec^{-1}5t + \csc^{-1}5t) = \frac{d}{dt}(\frac{1}{2}\pi) = 0$

36. (a) $g(x) = \frac{1}{2}\sin^{-1}e^x$; (b) $f(y) = \tan^{-1}y^2 + \cot^{-1}y^2$

▷ (a) $g'(x) = \frac{1}{2}\frac{1}{\sqrt{1-(e^x)^2}}D_x(e^x) = \frac{e^x}{2\sqrt{1-e^{2x}}}$ (b) By Definition 5.7.9, $f(y) = \frac{1}{2}\pi$ and so $f'(y) = 0$

37. (a) $f'(x) = \frac{d}{dx}(\sin^{-1}\sqrt{1-x^2}) = \frac{1}{\sqrt{1-(1-x^2)}}\frac{d}{dx}\sqrt{1-x^2} = \frac{1}{\sqrt{x^2}}\cdot\frac{1}{2\sqrt{1-x^2}}(-2x) = -\frac{x}{|x|\sqrt{1-x^2}}$

(b) $G'(x) = \frac{d}{dx}(\cot^{-1}\frac{2}{x}) = \frac{-1}{1+(2/x)^2}\left(-\frac{2}{x^2}\right) = \frac{2}{x^2+4}$

38. (a) $f'(w) = \frac{d}{dw}(2\tan^{-1}\frac{1}{w}) = 2\frac{1}{1+(1/w)^2}\left(-\frac{1}{w^2}\right) = -\frac{2}{w^2+1}$

(b) $F'(x) = \frac{d}{dx}(x\cos^{-1}x) = 1\cdot\cos^{-1}x + x\left(-\frac{1}{\sqrt{1-x^2}}\right) = \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$

39. (a) $f'(x) = \frac{d}{dx}\cos^{-1}(\sin x) = -\frac{1}{\sqrt{1-\sin^2 x}}(\cos x) = -\frac{\cos x}{|\cos x|}$

(b) $f'(x) = \frac{d}{dx}\left(4\sin^{-1}\frac{1}{2}x + x\sqrt{4-x^2}\right) = \frac{4}{\sqrt{1-x^2/4}}\left(\frac{1}{2}\right) + \sqrt{4-x^2} + x\cdot\frac{-2x}{2\sqrt{4-x^2}}$

$= \frac{4}{\sqrt{4-x^2}} + \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = \frac{4-x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2} = \sqrt{4-x^2} + \sqrt{4-x^2} = 2\sqrt{4-x^2}$

40. (a) $h(x) = \tan^{-1} \dfrac{2x}{1-x^2}$; (b) $g(x) = \sec^{-1}\sqrt{x^2+4}$

▹ (a)
$$h'(x) = \frac{1}{1+[2x/(1-x^2)]^2} D_x \frac{2x}{1-x^2} = \frac{1}{1+4x^2/(1-x)^2} \cdot \frac{2(1-x^2)-2x(-2x)}{(1-x^2)^2} = \frac{2+2x^2}{(1-x^2)^2+4x^2}$$
$$= \frac{2(1+x^2)}{(1+x^2)^2} = \frac{2}{1+x^2}$$

Or, because $h(x) = 2\tan^{-1}x + k\pi$, where $k = 0$ if $x^2 < 1$ and $k = \operatorname{sgn}(x)$ if $x^2 > 1$, then $h'(x) = \dfrac{2}{1+x^2}$

(b)
$$g'(x) = \frac{1}{\sqrt{x^2+4}\sqrt{(x^2+4)-1}} \cdot D_x\sqrt{x^2+4} = \frac{1}{\sqrt{x^2+4}\sqrt{x^2+3}} \cdot \frac{x}{\sqrt{x^2+4}} = \frac{x}{(x^2+4)\sqrt{x^2+3}}$$

41. (a) $\dfrac{d}{dx}(x\tan^{-1}x - \ln\sqrt{1+x^2}) = \dfrac{d}{dx}[x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)] = 1\cdot\tan^{-1}x + x\cdot\dfrac{1}{1+x^2} - \dfrac{1}{2}\cdot\dfrac{2x}{1+x^2} = \tan^{-1}x$

(b) $g'(x) = \dfrac{d}{dx}\sec^{-1}(2e^{3x}) = \dfrac{1}{2e^{3x}\sqrt{(2e^{3x})^2-1}} \cdot \dfrac{d}{dx}(2e^{3x}) = \dfrac{6e^{3x}}{2e^{3x}\sqrt{4e^{6x}-1}} = \dfrac{3}{\sqrt{4e^{6x}-1}}$

42. (a) $f'(x) = D_x(x\sin^{-1}x + x\cos^{-1}x) = D_x(\frac{1}{2}\pi x) = \frac{1}{2}\pi$

(b) $f'(x) = D_x\csc^{-1}\sqrt{x} = \dfrac{-1}{\sqrt{x}\sqrt{x-1}} \cdot D_x\sqrt{x} = \dfrac{-1}{\sqrt{x}\sqrt{x-1}} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{-1}{2x\sqrt{x-1}}$

43. $y = 2\sin 4\pi(t+\frac{1}{8})$, $t \geq 0$.

(a) $\frac{1}{2}y = \sin 4\pi(t+\frac{1}{8})$. Let k and n be any integers. Then

$4\pi(t+\frac{1}{8}) = 2k\pi + \sin^{-1}(\frac{1}{2}\pi)$ or $4\pi(t+\frac{1}{8}) = (2n+1)\pi - \sin^{-1}(\frac{1}{2}\pi)$

$t = \dfrac{1}{4\pi}\sin^{-1}\left(\dfrac{1}{2}y\right) - \dfrac{1}{8} + \dfrac{1}{2}k$ or $t = \dfrac{1}{8} - \dfrac{1}{4\pi}\sin^{-1}\left(\dfrac{1}{2}y\right) + \dfrac{1}{2}n$

(b) Setting $y = 1$ with $t > 0$ we obtain

$t = \dfrac{1}{4\pi}\sin^{-1}\dfrac{1}{2} - \dfrac{1}{8} + \dfrac{1}{2}k = \dfrac{1}{24} - \dfrac{1}{8} + \dfrac{1}{2}k = \dfrac{8k-1}{12}$ or $t = \dfrac{1}{8} - \dfrac{1}{4\pi}\sin^{-1}\dfrac{1}{2} + \dfrac{1}{2}n = \dfrac{1}{8} - \dfrac{1}{24} + \dfrac{1}{2}n = \dfrac{6n+1}{12}$

Letting $n = 0$, $k = 1$ and $n = 1$ gives the values $t = \dfrac{1}{12}$, $t = \dfrac{5}{12}$ and $t = \dfrac{7}{12}$.

44. A 60-cycle alternating current is described by the equation $x = 20\sin 120\pi(t - \frac{11}{120})$, where x amperes is the current at t sec. (a) Solve the equation for t. (b) Use the equation in part (a) to determine the smallest three positive values of t for which the current is 10 amperes.

▹ (a) Solving the given equation for t, we have

$$\sin 120\pi\left(t - \frac{11}{720}\right) = \frac{x}{20}$$

$$120\pi\left(t - \frac{11}{720}\right) = 2k\pi + \sin^{-1}\frac{x}{20} \quad \text{or} \quad 120\pi\left(t - \frac{11}{720}\right) = (2k+1)\pi - \sin^{-1}\frac{x}{20}$$

where k is any integer. Thus

$$t - \frac{11}{720} = \frac{1}{120\pi}\sin^{-1}\frac{x}{20} + \frac{k}{60} \qquad \text{or} \qquad t - \frac{11}{720} = \frac{1}{120} - \frac{1}{120\pi}\sin^{-1}\frac{x}{20} + \frac{k}{60}$$

Therefore, the solution set of the given equation is

$\{t \mid t = \frac{11}{720} + \frac{1}{120\pi}\sin^{-1}\frac{x}{20} + \frac{k}{60}\} \cup \{t \mid t = \frac{17}{720} - \frac{1}{120\pi}\sin^{-1}\frac{x}{20} + \frac{k}{60}\}$, k is any integer

(b) when $x = 10$,

$$t = \frac{11}{720} + \frac{1}{120\pi}\sin^{-1}\frac{1}{2} + \frac{k}{20} \qquad \text{or} \qquad t = \frac{17}{720} - \frac{1}{120\pi}\sin^{-1}\frac{1}{2} + \frac{k}{60}$$

$$t = \frac{11}{720} + \frac{1}{120\pi}\cdot\frac{\pi}{6} + \frac{k}{60} \qquad \text{or} \qquad t = \frac{17}{720} - \frac{1}{120\pi}\cdot\frac{\pi}{6} + \frac{k}{60}$$

$$t = \frac{11}{720} + \frac{1}{720} + \frac{k}{60} \qquad \text{or} \qquad t = \frac{17}{720} - \frac{1}{720} + \frac{k}{60}$$

$$t = \frac{1}{60} + \frac{k}{60} \qquad \text{or} \qquad t = \frac{1}{45} + \frac{k}{60}$$

if $k = -1$

$$t = \frac{1}{60} - \frac{1}{60} \qquad \text{or} \qquad t = \frac{1}{45} - \frac{1}{60}$$

$$t = 0 \qquad \text{or} \qquad t = \frac{1}{180}$$

if $k = 0$

$t = \frac{1}{60}$ or $t = \frac{1}{45}$

- The three smallest positive values of t for which the current is 10 amperes are $\frac{1}{180}$, $\frac{1}{60}$ and $\frac{1}{45}$.

45. $y = \sec^{-1}(2x+1)$; $\left.\frac{dy}{dx}\right|_{x=1/2} = \left.\frac{2}{(2x+1)\sqrt{(2x+1)^2-1}}\right|_{x=1/2} = \frac{2}{2\sqrt{2^2-1}} = \frac{1}{\sqrt{3}}$. At $(\frac{1}{2}, \frac{1}{3}\pi)$,

the slope of the tangent line is $\frac{1}{\sqrt{3}}$ and its equation is $y - \frac{1}{3}\pi = \frac{1}{\sqrt{3}}(x - \frac{1}{2})$; $2\sqrt{3}x - 6y + 2\pi - \sqrt{3} = 0$.
The slope of the normal line is $-\sqrt{3}$ and its equation is $y - \frac{1}{3}\pi = -\sqrt{3}(x - \frac{1}{2})$; $6\sqrt{3}x + 6y - 2\pi - 3\sqrt{3} = 0$.

46. Because $\theta = \tan^{-1}\frac{7x}{x^2+4} = \tan^{-1}\frac{7}{x + 144/x} = \tan^{-1}\frac{7}{(\sqrt{x} - 12/\sqrt{x})^2 + 24}$, then θ has an absolute maximum value when $\sqrt{x} - \frac{12}{\sqrt{x}} = 0$; $x = 12$. The observer should stand 12 ft from the wall.

47. A sign 3 ft high has its base 2 ft above the eye level of a woman attempting to read it. Let E be the position of the woman's eye, x ft the distance of E from the wall, and θ the radian measure of the angle subtended at E by the sign. Let A be the top of the sign, B the bottom of the sign, C the point of the wall at the woman's eye level. Then

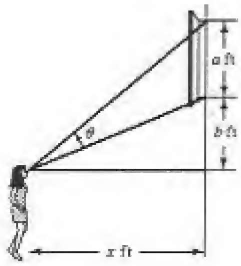
$\theta = m(\angle AEB) = m(\angle AEC) - m(\angle BEC) = \cot^{-1}\frac{x}{5} - \cot^{-1}\frac{x}{2}$, $x \geq 0$

$$\frac{d\theta}{dx} = -\frac{1}{5}\cdot\frac{1}{1+\frac{x^2}{25}} + \frac{1}{2}\cdot\frac{1}{1+\frac{x^2}{4}} = \frac{-5}{25+x^2} + \frac{2}{4+x^2} = \frac{-5(4+x^2)+2(25+x^2)}{(25+x^2)(4+x^2)} = \frac{3(10-x^2)}{(25+x^2)(4+x^2)}$$

Set $\frac{d\theta}{dx} = 0$. $\sqrt{10}$ is the only critical number. If $0 \leq x < \sqrt{10}$ then $\frac{d\theta}{dx} > 0$ and if $x > \sqrt{10}$ then $\frac{d\theta}{dx} < 0$. Hence θ has an absolute maximum value when $x = \sqrt{10}$.

48. Example 8 and Exercise 47 are particular cases of the following more general situation: An object (for instance, a picture or a sign) a feet high is placed on a wall with its base b feet above the eye level of an observer. Show that the observer gets the best view of the object when the distance of the observer from the wall is $\sqrt{b(a+b)}$ feet.

▷ If θ is the radian measure of the angle subtended at the observer's eye, then θ is the difference of angles in two right triangles with adjacent sides x ft and opposite sides $(a+b)$ ft and b feet. See the figure. Therefore



$$\theta = \cot^{-1}\frac{x}{a+b} - \cot^{-1}\frac{x}{b}$$

$$\frac{d\theta}{dx} = -\frac{1}{a+b}\cdot\frac{1}{1+x^2/(a+b)^2} + \frac{1}{b}\cdot\frac{1}{1+x^2/b^2}$$

$$= -\frac{a+b}{(a+b)^2+x^2} + \frac{b}{b^2+x^2} = \frac{a[b(a+b)-x^2]}{[(a+b)^2+x^2](b^2+x^2)}$$

Because $d\theta/dx > 0$ if $0 \leq x < \sqrt{b(a+b)}$ and $d\theta/dx < 0$ if $x > \sqrt{b(a+b)}$ then θ has an absolute maximum value when $x = \sqrt{b(a+b)}$.

49. At t sec, let E be the position of the observer's eye, x cm the distance of E from the wall, and θ the radian measure of the angle subtended at E by the picture. Let A be the top of the picture, B the bottom of the picture, and C the point on the wall at the observer's eye level.

$\theta = m(\angle AEB) = m(\angle AEC) - m(\angle BEC) = \cot^{-1}\frac{x}{70} - \cot^{-1}\frac{x}{30}$

$$\left.\frac{d\theta}{dt}\right|_{x=100} = \left[-\frac{1}{70}\cdot\frac{1}{1+\frac{x^2}{4900}} + \frac{1}{30}\cdot\frac{1}{1+\frac{x^2}{900}}\right]_{x=100}\cdot\frac{dx}{dt} = \left[-\frac{1}{70\left(1+\frac{100}{49}\right)} + \frac{1}{30\left(1+\frac{100}{9}\right)}\right](-40)$$

$$= \left(-\frac{7}{1490} + \frac{3}{1090}\right)(-40) = \frac{1,264}{16,241} \approx 0.078$$

Hence, when the observer is 100 ft from the wall, θ is decreasing at 0.078 rad/sec.

50. At t sec, there are x ft of rope out and the angle of depression is θ. Then $\frac{dx}{dt} = -2$ and $\theta = \csc^{-1}\frac{1}{20}x$. Therefore

$$\left.\frac{d\theta}{dt}\right]_{x=52} = -\frac{1}{20}\frac{1}{\frac{1}{20}x\sqrt{(\frac{1}{20}x)^2-1}}\cdot\left.\frac{dx}{dt}\right]_{x=52} = \frac{2}{52\sqrt{2.6^2-1}} = \frac{1}{26(2.4)} = \frac{5}{312}.$$ The angle is increasing at $\frac{5}{312}$ rad/s.

51. At t min, let x km be the distance of the spot of light at S from the point A on the beach nearest the light and let θ be the radian measure of the angle subtended at the light by line segment AS. Because the light makes 2 revolutions per minute, $\frac{d\theta}{dt} = 4\pi$. When $x = 2$, $\sec\theta = \frac{1}{3}\sqrt{13}$. Therefore

$$x = 3\tan\theta;\ \left.\frac{dx}{dt}\right|_{x=2} = 3\sec^2\theta\cdot\left.\frac{d\theta}{dt}\right|_{x=2} = 3\cdot\frac{13}{9}\cdot 4\pi = \frac{52}{3}\pi$$

- Therefore, the speed of the light along the shore is $\frac{52}{3}\pi$ km/min.

52. A ladder 25 ft long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall so that the top is sliding down at 3 ft/sec, how fast is the measure of the angle between the ladder and the ground changing when the bottom of the ladder is 15 ft from the wall?

▷ t seconds after the ladder began to be pulled away from the wall, let
x feet be the distance between the wall and the bottom of the ladder
y feet be the distance between the floor and the top of the ladder
θ radians be the measure of the angle between the ladder and the ground
Refer to the figure. We are given that $dy/dt = -3$. We want to find $d\theta/dt$ when $x = 15$. Because $x^2 + y^2 = 625$, then $y = 20$ when $x = 15$. We have

$$\theta = \sin^{-1}\frac{y}{25};\qquad \frac{d\theta}{dt} = \frac{1}{\sqrt{1-\left(\frac{y}{25}\right)^2}}\cdot\frac{1}{25}\frac{dy}{dt}$$

Substituting $dy/dt = -3$, we obtain $\quad \frac{d\theta}{dt} = \frac{-3}{\sqrt{625-y^2}}$

Then, $\left.\frac{d\theta}{dt}\right]_{y=20} = \frac{-3}{\sqrt{625-20^2 2}} = -\frac{1}{5}$

- The angle is decreasing at $\frac{1}{5}$ radian per second when the bottom of the ladder is 15 ft from the wall.

In Exercises 53 and 54, a woman is walking at the rate of 5 ft/sec along the diameter of a circular courtyard or radius r feet. A light at one end of a diameter perpendicular to her path casts a shadow on the circular wall.

53. How fast is the shadow moving along the wall her distance to the center of the courtyard is $\frac{1}{2}r$?

54. How far is she from the center of the courtyard when the speed of her shadow along the wall is 9 ft/sec?

▷ Refer to the figure. The light is at point L, the center of the circle is at point O, and line segment LT is a diameter of the circle. At a certain instant the woman is at point W moving away from point O along the diameter perpendicular to LT. At a certain instant the light casts a shadow on point S. Thus, the shadow moves along arc TS away from point T. We define the variables, t seconds since the woman began to walk, let

x feet be the length of line segment OW
s feet be the length of arc TS
θ radians be the measure of angle TLS
We are given that $dx/dt = 5$. We find an equation that expresses the functional relationship between x and s.
If r ft is the radius of the circular courtyard, we have $\qquad 2\theta = \frac{s}{r}$

because 2θ is the radian measure of a central angle in a circle with radius r that intercepts an arc of length s.

Thus $\qquad \theta = \frac{s}{2r} \qquad (1)$

Furthermore, from right triangle LOW, we have $\qquad \theta = \tan^{-1}\left(\frac{x}{r}\right) \qquad (2)$

From (1) and (2) we get $\tan^{-1}\left(\frac{x}{r}\right) = \frac{s}{2r}$. $\qquad (3)$

53. $\left.\frac{dy}{dt}\right|_{x=r/2} = 2r\cdot\frac{1}{1+\left(\frac{x}{r}\right)^2}\cdot\frac{1}{r}\cdot\left.\frac{dx}{dt}\right|_{x=r/2} = 2\cdot\frac{1}{1+(\frac{1}{2})^2}\cdot 5 = 2\cdot\frac{4}{5}\cdot 5 = 8$

- Therefore, at the given instant the shadow is moving along the wall at 8 ft/sec.

54. We want to find x when $ds/dt = 9$.

Differentiating on both sides of (3) with respect to t, we obtain

$$\frac{1}{1+(x/r)^2}\cdot\frac{1}{r}\frac{dx}{dt}=\frac{ds/dt}{2r};\qquad \frac{r^2}{r^2+x^2}\frac{dx}{dt}=\frac{1}{2}\frac{ds}{dt}$$

Substituting $dx/dt = 5$ and $ds/dt = 9$, we get $\quad \dfrac{5r^2}{r^2+x^2}=\dfrac{9}{2}$

Solving for x, we have $\quad x = \frac{1}{3}r$.

- When the speed of the woman's shadow is 9 ft/sec, her distance from the center of the courtyard is one-third of the radius of the courtyard.

55. The length of the rope between the weight and the hook is x ft when the radian measure of the angle between the rope and the floor is θ, and $\dfrac{dx}{dt}=-\dfrac{3}{4}$. Then

$$\csc\theta=\frac{x}{8},\ x>8;\ \theta=\csc^{-1}\frac{x}{8};\ \frac{d\theta}{dt}=\frac{-1}{\frac{x}{8}\sqrt{\frac{x^2}{64}-1}}\cdot\frac{1}{8}\cdot\frac{dx}{dt}=\frac{-8}{x\sqrt{x^2-64}}\left(-\frac{3}{4}\right)=\frac{6}{x\sqrt{x^2-64}}$$

56. Prove that if $x>0$, then $\tan^{-1}x+\tan^{-1}(1/x)=\frac{1}{2}\pi$ two ways: (a) Definition 5.7.9; (b) Show that the left side is a constant and then evaluate the constant.

▶ (a) If $x>0$ then $\quad \tan^{-1}x+\tan^{-1}(1/x)=\tan^{-1}x+\cot^{-1}x=\tan^{-1}x+(\frac{1}{2}\pi-\tan x)=\frac{1}{2}\pi$

(b) Because

$$\frac{d}{dx}\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right)=\frac{1}{1+x^2}+\frac{1}{1+(1/x)^2}\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{1}{1+x^2}+\frac{x^2}{x^2+1}\left(-\frac{1}{x^2}\right)=\frac{1}{1+x^2}-\frac{1}{x^2+1}=0$$

then $\quad \tan^{-1}x+\tan^{-1}\frac{1}{x}=C$

When $x=1$, we have $\quad \tan^{-1}1+\tan^{-1}1=C;\quad \frac{1}{4}\pi+\frac{1}{4}\pi=C;\quad C=\frac{1}{2}\pi$

Therefore, $\quad \tan^{-1}x+\tan^{-1}\frac{1}{x}=\frac{1}{2}\pi$

57. $\csc^{-1}x=\frac{1}{2}\pi-\sec^{-1}x$ for $|x|\geq 1$ and the range of $\sec^{-1}$ is $[0,\frac{1}{2}\pi)\cup[\pi,\frac{3}{2}\pi)$. If $\sec^{-1}x\in[0,\frac{1}{2}\pi)$, then $\csc^{-1}x\in(0,\frac{1}{2}\pi]$; if $\sec^{-1}x\in[\pi,\frac{3}{2}\pi)$, then $\csc^{-1}x\in(-\pi,-\frac{1}{2}\pi]$. Hence the range of $\csc^{-1}$ is $(-\pi,-\frac{1}{2}\pi]\cup(0,\frac{1}{2}\pi]$.

58. Because $D_x\tan^{-1}x=1/(1+x^2)>0$, then the graph of $\tan^{-1}x$ is increasing for all x.

59. $f(x)=\tan^{-1}\frac{1}{x}-\cot^{-1}x$ (a) $f'(x)=\dfrac{1}{1+(1/x)^2}\cdot\dfrac{-1}{x^2}-\left(\dfrac{-1}{1+x^2}\right)=-\dfrac{1}{1+x^2}+\dfrac{1}{1+x^2}=0$

(b) $f(-1)=\tan^{-1}(-1)-\cot(-1)=-\frac{1}{4}\pi-(\frac{3}{4}\pi)=-\pi$ while $f(1)=\tan^{-1}1-\cot^{-1}1=\frac{1}{4}\pi+\frac{1}{4}\pi=\frac{1}{2}\pi$
(c) Theorem 4.1.2 is not contradicted because f is not defined at 0.

In Exercises 60–62, prove the result of the substitution and show how the domain applies.

60. Substituting $\theta=\sin^{-1}(\frac{1}{3}x)$ in the expression $\sqrt{9-x^2}$ yields $3\cos\theta$.

▶ $\theta=\sin^{-1}(\frac{1}{3}x)$, $\frac{1}{3}x=\sin\theta$, $x=3\sin\theta$. $\sqrt{9-x^2}=\sqrt{9-(3\sin\theta)^2}=\sqrt{9(1-\sin^2\theta)}=\sqrt{9\cos^2\theta}$
$=3\cos\theta$ because $\cos\theta\geq 0$ for $\theta\in[-\frac{1}{2}\pi,\frac{1}{2}\pi]$.

61. $\theta=\tan^{-1}(\frac{1}{2}x)$, $\frac{1}{2}x=\tan\theta$, $x=2\tan\theta$. $\sqrt{x^2+4}=\sqrt{(2\tan\theta)^2+4}=\sqrt{4(\tan^2\theta+1)}=\sqrt{4\sec^2\theta}$
$=2\sec\theta$ because $\sec\theta>0$ for $\theta\in(-\frac{1}{2}\pi,\frac{1}{2}\pi)$.

62. $\theta=\sec^{-1}(\frac{1}{5}x)$, $\frac{1}{5}x=\sec\theta$, $x=5\sec\theta$. $\sqrt{x^2-25}=\sqrt{(5\sec\theta)^2-25}=\sqrt{25(\sec^2\theta-1)}=\sqrt{25\tan^2\theta}$
$=5\tan\theta$ because $\tan\theta>0$ for $\theta\in[0,\frac{1}{2}\pi)\cup[\pi,\frac{3}{2}\pi)$.

5.8 INTEGRALS YIELDING INVERSE TRIGONOMETRIC FUNCTIONS

5.8.1 Theorem
$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C \tag{1}$$
$$\int \frac{du}{1+u^2} = \tan^{-1} u + C \tag{2}$$
$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C \tag{3}$$

From the above we may derive the following more general forms which should be memorized.

5.8.2 Theorem
$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1} \tfrac{u}{a} + C \text{ where } a > 0 \tag{4}$$
$$\int \frac{du}{a^2+u^2} = \tfrac{1}{a} \tan^{-1} \tfrac{u}{a} + C \text{ where } a \neq 0 \tag{5}$$
$$\int \frac{du}{u\sqrt{u^2-a^2}} = \tfrac{1}{a} \sec^{-1} \tfrac{u}{a} + C \text{ where } a > 0 \tag{6}$$

If the denominator contains a quadratic trinomial, we complete the square. See Exercise 12.

Exercises 5.8

In Exercises 1–16 evaluate the indefinite integral. Support your answer by showing its derivative is the integrand.

1. $\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{1-(2x)^2}} = \frac{1}{2}\sin^{-1}2x + C$

2. $\int \frac{dx}{x^2+25} = \frac{1}{5}\tan^{-1}\frac{x}{5}$

3. $\int \frac{dx}{9x^2+16} = \frac{1}{3}\int \frac{3\,dx}{(3x)^2+4^2} = \frac{1}{3}\cdot\frac{1}{4}\tan^{-1}\frac{3}{4}x + C = \frac{1}{12}\tan^{-1}\frac{3x}{4} + C$

4. $\int \frac{dt}{\sqrt{1-16t^2}}$

▷ We apply Theorem 5.8.1 with $u = 4t$ and $du = 4dt$

$$\int \frac{dt}{\sqrt{1-16t^2}} = \int \frac{\frac{1}{4}du}{1-u^2} = \tfrac{1}{4}\sin^{-1} u + C = \tfrac{1}{4}\sin^{-1} 4t + C$$

5. $\int \frac{dx}{4x\sqrt{x^2-16}} = \frac{1}{4}\int \frac{dx}{x\sqrt{x^2-4^2}} = \frac{1}{4}\cdot\frac{1}{4}\sec^{-1}\frac{x}{4} + C = \frac{1}{16}\sec^{-1}\frac{x}{4} + C$

6. Let $u = x^2$. Then $du = 2x\,dx$. $\int \frac{x\,dx}{x^4+16} = \frac{1}{2}\int \frac{du}{u^2+4^2} = \frac{1}{2}\cdot\frac{1}{4}\tan^{-1}\frac{1}{4}u + C = \frac{1}{8}\tan^{-1}\frac{1}{4}x^2 + C$

7. $\int \frac{r\,dr}{\sqrt{16-9r^4}} = \frac{1}{6}\int \frac{6r\,dr}{\sqrt{4^2-(3r^2)^2}} = \frac{1}{6}\sin^{-1}\frac{3r^2}{4} + C$

8. $\int \frac{du}{u\sqrt{16u^2-9}}$

▷ Applying Theorem 5.8.2 with $a = \frac{3}{4}$, we obtain

$$\int \frac{du}{u\sqrt{16u^2-9}} = \frac{1}{4}\int \frac{du}{u\sqrt{u^2-\frac{9}{16}}} = \frac{1}{4}\cdot\frac{1}{\frac{3}{4}}\sec^{-1}\frac{u}{\frac{3}{4}} + C = \tfrac{1}{3}\sec^{-1}\tfrac{4}{3}u + C$$

9. $\int \frac{e^x\,dx}{7+e^{2x}} = \int \frac{e^x\,dx}{(\sqrt{7})^2+(e^x)^2} = \frac{1}{\sqrt{7}}\tan^{-1}\frac{e^x}{\sqrt{7}} + C$

10. Let $u = \cos x$. Then $du = -\sin x\,dx$. $\int \frac{\sin x\,dx}{\sqrt{2-\cos^2 x}} = \int \frac{-du}{\sqrt{2-u^2}} = -\sin^{-1}\frac{u}{\sqrt{2}} + C = -\sin^{-1}\left(\frac{\cos x}{\sqrt{2}}\right) + C$

11. Let $x = u^2$, $dx = 2u\,du$. Then $\int \frac{dx}{(1+x)\sqrt{x}} = \int \frac{2u\,du}{(1+u^2)u} = 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C$

12. $\displaystyle\int \frac{ds}{\sqrt{2s-s^2}}$

▸ We complete the square under the radical sign. Because

$$2s-s^2=-(s^2-2s+1)+1=1-(s-1)^2$$

then

$$\int \frac{ds}{\sqrt{2s-s^2}}=\int\frac{ds}{\sqrt{1-(s-1)^2}}=\sin^{-1}(s-1)+C$$

13. $\displaystyle\int \frac{dx}{x^2-x+2}=\int\frac{dx}{(x^2-x+\frac{1}{4})+(2-\frac{1}{4})}=\int\frac{dx}{(x-\frac{1}{4})^2+\frac{7}{4}}=\frac{2}{\sqrt{7}}\tan^{-1}\left[\frac{2}{\sqrt{7}}\left(x-\frac{1}{2}\right)\right]+C$

$\displaystyle=\frac{2}{\sqrt{7}}\tan^{-1}\frac{2x-1}{7}+C$

14. $\displaystyle\int \frac{dx}{\sqrt{3x-x^2-2}}=\int\frac{dx}{\sqrt{\frac{1}{4}-[x^2-3x+(\frac{3}{2})^2}}=\int\frac{dx}{\sqrt{(\frac{1}{2})^2-(x-\frac{3}{2})^2}}=\sin^{-1}2(x-\tfrac{3}{2})+C=\sin^{-1}(2x-3)+C$

15. $\displaystyle\int \frac{dx}{\sqrt{15+2x-x^2}}=\int\frac{dx}{\sqrt{(15+1)-(x^2-2x+1)}}=\int\frac{dx}{\sqrt{4^2-(x-1)^2}}=\sin^{-1}\frac{x-1}{4}+C$

Using Theorem 5.7.2 with $u=\frac{1-x}{4}$ gives the answer in the form $\cos^{-1}\frac{1-x}{4}+C$.

16. $\displaystyle\int \frac{2dt}{(t-3)\sqrt{t^2-6t+5}}$

▸ First we complete the square under the radical sign.

$$\int\frac{2\,dt}{(t-3)\sqrt{t^2-6t+5}}=2\int\frac{dt}{(t-3)\sqrt{(t-3)^2-4}}$$

Then we let $u=t-3$, $du=dt$, and apply Theorem 5.8.2.

$$\int\frac{2dt}{(t-3)\sqrt{t^2-6t+5}}=2\int\frac{du}{u\sqrt{u^2-2^2}}=2(\tfrac{1}{2})\sec^{-1}\tfrac{1}{2}u+C=\sec^{-1}\left(\frac{t-3}{2}\right)+C$$

In Exercises 17–28, compute the exact value of the definite integral. Check using NINT.

17. $\displaystyle\int_0^1\frac{1+x}{1+x^2}dx=\int_0^1\frac{dx}{1+x^2}+\frac{1}{2}\int_0^1\frac{2x\,dx}{1+x^2}=\tan^{-1}x+\tfrac{1}{2}\ln(1+x^2)\Big]_0^1$

$=(\tan^{-1}1+\frac{1}{2}\ln 2)-(\tan^{-1}0+\frac{1}{2}\ln 1)=\frac{1}{4}\pi+\frac{1}{2}\ln 2\approx 1.131972$

18. $\displaystyle\int_2^5\frac{dx}{x^2-4x+13}=\int_2^5\frac{dx}{(x-2)^2+9}=\frac{1}{3}\tan^{-1}\frac{x-2}{3}\Big]_2^5=\tfrac{1}{3}(\tan^{-1}1-\tan^{-1}0)=\tfrac{1}{12}\pi\approx 0.261799$

19. $\displaystyle\int_{-4}^{-2}\frac{dt}{\sqrt{-t^2-6t-5}}=\int_{-4}^{-2}\frac{dt}{\sqrt{4-(t^2+6t+9)}}=\int_{-4}^{-2}\frac{dt}{\sqrt{2^2-(t+3)^2}}=\sin^{-1}\frac{t+3}{2}\Big]_{-4}^{-2}=\sin^{-1}\tfrac{1}{2}-\sin^{-1}\left(-\tfrac{1}{2}\right)$

$=\frac{1}{6}\pi-\left(-\frac{1}{6}\pi\right)=\frac{1}{3}\pi\approx 1.047198$

20. $\displaystyle\int_0^{\sqrt{3}}\frac{x\,dx}{\sqrt{12-x^4}}$

▸ Let $u=x^2$, $du=2x\,dx$. When $x=0$, $u=0$; when $x=\sqrt{3}$, $u=3$.

$$\int_0^{\sqrt{3}}\frac{x\,dx}{\sqrt{12-x^4}}=\int_0^3\frac{\frac{1}{2}du}{\sqrt{12-u^2}}=\frac{1}{2}\sin^{-1}\frac{u}{2\sqrt{3}}\Big]_0^3=\tfrac{1}{2}(\sin^{-1}\tfrac{1}{2}\sqrt{3}-\sin^{-1}0)=\tfrac{1}{6}\pi\approx 0.523599$$

21. $\displaystyle\int_0^1\frac{dx}{e^x+e^{-x}}=\int_0^1\frac{e^x dx}{(e^x)^2+1}=\tan^{-1}e^x\Big]_0^1=\tan^{-1}e-\tan^{-1}1=\tan^{-1}e-\tfrac{1}{4}\pi\approx 0.432885$

22. Let $u=3\tan x$, $dx=3\sec^2x\,dx$. When $x=0$, $u=0$; when $x=\frac{1}{6}\pi$, $u=\sqrt{3}$. Thus

$$\int_0^{\pi/6}\frac{\sec^2x\,dx}{1+9\tan^2x}=\int_0^{\sqrt{3}}\frac{\frac{1}{3}du}{1+u^2}=\frac{1}{3}\tan^{-1}u\Big]_0^{\sqrt{3}}=\tfrac{1}{3}(\tan^{-1}\sqrt{3}-\tan^{-1}0)=\tfrac{1}{9}\pi\approx 0.349066$$

23. Let $u=\ln x$; so $du=\frac{1}{x}\,dx$. Then $\displaystyle\int_1^e\frac{dx}{x[1+(\ln x)^2]}=\int_0^1\frac{du}{1+u^2}=\tan^{-1}u\Big]_0^1=\tan^{-1}1-\tan^{-1}0=\tfrac{1}{4}\pi.$

24. $\displaystyle\int_{1/\sqrt{2}}^{1} \frac{dx}{x\sqrt{4x^2-1}}$

▸ Let $u = 2x$. Then $du = 2dx$. When $x = 1/\sqrt{2}$, $u = \sqrt{2}$; when $x = 1$, $u = 2$. Thus,

$$\int_{1/\sqrt{2}}^{1} \frac{dx}{x\sqrt{4x^2-1}} = \int_{\sqrt{2}}^{2} \frac{\frac{1}{2}\,du}{\frac{1}{2}u\sqrt{u^2-1}} = \sec^{-1} u\Big]_{\sqrt{2}}^{2} = \sec^{-1} 2 - \sec^{-1}\sqrt{2} = \cos^{-1}\tfrac{1}{2} - \cos^{-1}\tfrac{1}{2}\sqrt{2}$$
$$= \tfrac{1}{3}\pi - \tfrac{1}{4}\pi = \tfrac{1}{12}\pi$$

25. $\displaystyle\int_{-1}^{1} \frac{x\,dx}{\sqrt{8-2x-x^2}} = \int_{-1}^{1} \frac{(x+1)-1}{\sqrt{9-(x+1)^2}}dx = \mathrm{I}$. Let $u = 9-(x+1)^2$, $du = -2(x+2)dx$. When $x = -1$, $u = 9$; when $x = 1$, $u = 5$. Thus

$$\mathrm{I} = \int_{9}^{5} \frac{-\frac{1}{2}du}{\sqrt{u}} - \int_{-1}^{1} \frac{dx}{\sqrt{9-(x+1)^2}} = -\sqrt{u}\Big]_{9}^{5} - \sin^{-1}\frac{x+1}{3}\Big]_{-1}^{1} = 3 - \sqrt{5} - \sin^{-1}\tfrac{2}{3} \approx 0.0342044$$

26. Let $v = x+1$; so $x = v-1$ and $dx = dv$. When $x = 0$, $v = 1$; when $x = 1$, $v = 2$. Then

$$\int_{0}^{1} \frac{(2+x)dx}{\sqrt{4-2x-x^2}} = \int_{0}^{1} \frac{(2+x)dx}{\sqrt{5-(x^2+2x+1)}} = \int_{0}^{1} \frac{(2+x)dx}{\sqrt{5-(x+1)^2}} = \int_{1}^{2} \frac{(v+1)dv}{\sqrt{5-v^2}} = \int_{1}^{2} \frac{v\,dv}{\sqrt{5-v^2}} + \int_{1}^{2} \frac{dv}{\sqrt{5-v^2}}$$
$$= -\tfrac{1}{2}\int_{1}^{2}(5-v^2)^{-1/2}(-2v\,dv) + \sin^{-1}\frac{v}{\sqrt{5}}\Big]_{1}^{2} = -\sqrt{5-v^2}\Big]_{1}^{2} + \sin^{-1}\frac{2}{\sqrt{5}} - \sin^{-1}\frac{1}{\sqrt{5}} = 1 + \sin^{-1}\tfrac{3}{5} \approx 1.693501$$

27. $\displaystyle\int_{0}^{3} \frac{2x^3dx}{2x^2-4x+3} = \int_{0}^{3}\left[x+2+\frac{5x-6}{2(x^2-2x+1)}\right]dx = \int_{0}^{3}\left[x+2+\frac{5(x-1)}{2(x-1)^2+1} - \frac{1}{2(x-1)^2+1}\right]dx$

$$= \left[\tfrac{1}{2}x^2 + 2x + \tfrac{5}{4}\ln[2(x-1)^2+1] - \tfrac{1}{2}\sqrt{2}\tan^{-1}[\sqrt{2}(x-1)]\right]_{0}^{3} = \tfrac{21}{2} + \tfrac{5}{4}\ln 3 - \tfrac{1}{2}\sqrt{2}(\tan^{-1}\sqrt{2} - \pi) \approx 10.3273$$

28. $\displaystyle\int_{1}^{4} \frac{x\,dx}{x^2+x+1}$

▸ First we complete the square in the denominator. Let $u = x + \frac{1}{2}$, $du = dx$. When $x = 1$, $u = \frac{3}{2}$; when $x = 4$, $u = \frac{9}{2}$. We note that $d(u^2 + \frac{3}{4}) = 2u\,du$ and separate into two terms as in Example 3.

$$\int_{1}^{4} \frac{x\,dx}{x^2+x+1} = \int_{1}^{4} \frac{x\,dx}{[x^2+x+(\frac{1}{2})^2]+\frac{3}{4}} = \int_{1}^{4} \frac{x\,dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \int_{3/2}^{9/2} \frac{u-\frac{1}{2}}{u^2+\frac{3}{4}}du = \int_{3/2}^{9/2}\left[\frac{1}{2}\frac{2u}{u^2+\frac{3}{4}} - \frac{1}{2}\frac{1}{u^2+\frac{3}{4}}\right]du$$
$$= \left[\tfrac{1}{2}\ln(u^2+\tfrac{3}{4}) - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}/2}\right]_{3/2}^{9/2} = \tfrac{1}{2}(\ln 21 - \ln 7) - \tfrac{1}{3}\sqrt{3}(\tan^{-1} 3\sqrt{3} - \tan^{-1}\sqrt{3})$$
$$= \tfrac{1}{2}\ln 7 - \tfrac{1}{3}\sqrt{3}(\tan^{-1} 3\sqrt{3} - \tfrac{1}{3}\pi) \approx 0.780424$$

29. A square units is the area of the region bounded by $y = 8/(x^2+4)$, the axes, and $x = 2$.

$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} \frac{8}{w_i^{\,2}+4}\Delta_i x = 8\int_{0}^{2}\frac{1}{x^2+2}dx = 8\cdot\tfrac{1}{2}\tan^{-1}\frac{x}{2}\Big]_{0}^{2} = 4(\tan^{-1}1 - \tan^{-1}0) = 4(\tfrac{1}{4}\pi - 0) = \pi$$

30. Find the area of the region bounded by the curves $x^2 = 4ay$ and $y = 8a^3/(x^2+4a^2)$.

▸ Assume $a > 0$. To find the intersection of the two curves, we eliminate y from the given pair of equations.

$$\frac{x^2}{4a} = \frac{8a^3}{x^2+4a^2}; \qquad x^4 + 4a^2x^2 - 32a^4 = 0; \quad (x^2+8a^2)(x^2-4a^2) = 0;\ x^2 - 4a^2 = 0;\ x = \pm 2a$$

We ignore $x^2 + 8a^2 = 0$ because there is no real solution of this equation. The region is bounded above by the curve $y = 8a^3/(x^2+4a^2)$, bounded below by the curve $y = x^2/4a$, bounded on the left by the line $x = -2a$, and bounded on the right by the line $x = 2a$. Thus,

$$A = \int_{-2a}^{2a}\left[\frac{8a^3}{x^2+4a^2} - \frac{x^2}{4a}\right]dx = 8a^3\int_{-2a}^{2a}\frac{dx}{x^2+(2a)^2} - \frac{1}{4a}\int_{-2a}^{2a}x^2\,dx$$
$$= 8a^3\cdot\frac{1}{2a}\left[\tan^{-1}\frac{x}{2a}\right]_{-2a}^{2a} - \frac{1}{12a}\left[x^3\right]_{-2a}^{2a} = 4a^2[\tan^{-1}1 - \tan^{-1}(-1)] - \frac{1}{12a}[8a^3 - (-8a^3)]$$
$$= 4a^2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] - \frac{16a^3}{12a} = 2\pi a^2 - \frac{4}{3}a^2$$

If $a < 0$, we get a congruent region below the x axis. Thus, the area is $2\pi a^2 - \frac{4}{3}a^2$ square units.

31. A square units is the area bounded by $y = 1/\sqrt{5-4x-x^2}$, the x axis, $x = -\frac{7}{2}$, and $x = \frac{1}{2}$.

$$A = \int_{-7/2}^{-1/2} \frac{dx}{\sqrt{5-4x-x^2}} = \int_{-7/2}^{-1/2} \frac{dx}{\sqrt{9-(x^2+4x+4)}} = \int_{-7/2}^{-1/2} \frac{dx}{\sqrt{3^2-(x+2)^2}} = \sin^{-1}\frac{x+2}{3}\Big]_{-7/2}^{-1/2}$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{1}{6}\pi - \left(-\frac{1}{6}\pi\right) = \frac{1}{3}\pi$$

In Exercises 32 and 33, prove the formula by showing that the derivative of the right side equals the integrand.

32. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

▹ $\displaystyle\frac{d}{du}\left(\frac{1}{a}\tan^{-1}\frac{u}{a}\right) = \frac{1}{a}\cdot\frac{1}{1+(u/a)^2}\cdot D_x\left(\frac{u}{a}\right) = \frac{1}{a}\cdot\frac{1}{1+u^2/a^2}\cdot\frac{1}{a} = \frac{1}{a^2+u^2}$

• Because the derivative of the right side is equal to the integrand, we have proved the formula.

33. $\displaystyle\frac{d}{du}\left(\frac{1}{a}\sec^{-1}\frac{u}{a}\right) = \frac{1}{a}\cdot\frac{1}{\frac{u}{a}\sqrt{\frac{u^2}{a^2}-1}}\cdot\frac{1}{a} = \frac{1}{u\sqrt{u^2-a^2}}$ because $a > 0$, so that $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

In Exercises 34–36, prove the formula of Theorem 5.8.2 by making a substitution and then using Theorem 5.8.1

34. Let $u = ax$ and $du = a\,dx$. Then

$$\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{a\,dx}{\sqrt{a^2-a^2x^2}} \overset{a \geq 0}{=} \int \frac{dx}{\sqrt{1-x^2}} \overset{(1)}{=} \sin^{-1}x + C = \sin^{-1}\frac{u}{a} + C$$

35. Let $u = ax$, $du = a\,dx$. Then $\displaystyle\int \frac{du}{a^2+u^2} = \int \frac{a\,dx}{a^2+a^2x^2} = \frac{1}{a}\int \frac{dx}{1+x^2} = \frac{1}{a}\tan^{-1}x + C = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

36. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$, where $a > 0$.

▹ Let $u = ax$ and $du = a\,dx$. Then

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{a\,dx}{ax\sqrt{(ax)^2-a^2}} = \int \frac{dx}{x\sqrt{a^2(x^2-1)}} \overset{a \geq 0}{=} \frac{1}{a}\int \frac{dx}{x\sqrt{x^2-1}} \overset{(3)}{=} \frac{1}{a}\sec^{-1}x + C = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

37. We have $v\frac{dv}{ds} = -k^2s$; $\int v\,dv = -k^2\int s\,ds$; $\frac{1}{2}v^2 = -\frac{1}{2}k^2s^2 + \frac{1}{2}C$; $v^2 = -k^2s^2 + C$.

Because $v^2 \geq 0$ and $-k^2s^2 \leq 0$, then $C \geq 0$. If $C = 0$, then $v = s = 0$ and the particle is not moving.

Therefore, let $c = k^2a^2$ with $a > 0$. Then $v^2 = -k^2s^2 + k^2a^2$; $v = \pm k\sqrt{a^2-s^2}$

(b) Because $v = \frac{ds}{dt}$ we have $\frac{ds}{dt} = \pm k\sqrt{a^2-s^2}$; $\displaystyle\int \frac{ds}{\sqrt{a^2-s^2}} = \pm k\int dt$; $-\cos^{-1}\frac{s}{a} = \pm kt + C$.

When $t = 0$, $s = a$. Thus $C = \cos^{-1}1 = 0$. Hence $-\cos^{-1}\frac{s}{a} = \pm kt$; $\cos^{-1}\frac{s}{a} = \mp kt$; $\frac{s}{a} = \cos(\mp kt)$.

Because $\cos(-kt) = \cos(kt)$, we have $s = a\cos kt$.

(c) Because $a > 0$ and $\cos kt \leq 1$, then the largest value of $|s| = |a\cos t|$ is a.

(d) Let t_1 be the value of t when $s = a$ and let m be any even integer. Then

$a = a\cos kt_1$; $\cos kt_1 = 1$; $kt_1 = m\pi$; $t_1 = \frac{m\pi}{k}$

Let t_2 be the value of t when $s = -a$ and let n be any odd integer. Then

$-a = a\cos kt_2$; $\cos kt_2 = -1$; $kt_2 = n\pi$; $t_2 = \frac{n\pi}{k}$

Hence, the particle is at a when $t = 0$; it is at $-a$ when $t = \frac{\pi}{k}$; it is at a when $t = \frac{2\pi}{k}$.

Therefore if T sec is the time for the particle to go from a to $-a$ and return, $T = \frac{2\pi}{k} - 0 = \frac{2\pi}{k}$

38. Part (b) of Ex. 37 states that if $s = a\cos kt$, then the particle has simple harmonic motion. We are given that

$$s = 5 - 10\sin^2 2t = 5(1 - 2\sin^2 2t) \tag{1}$$

Because $\cos 2x = 1 - 2\sin^2 x$, then with $x = 2t$ we have

$$\cos 4t = 1 - 2\sin^2 2t \tag{2}$$

Substituting from (2) into (1), we have

$$s = 5\cos 4t$$

Therefore, the motion is simple harmonic. The amplitude is $a = 5$ and the period is $2\pi/k = 2\pi/4 = \frac{1}{2}\pi$.

In Exercises 39–41, show that the exact value of the definite integral is π. Then use NINT to approximate π.

39. $\displaystyle\int_0^{0.5} \frac{6\,dx}{\sqrt{1-x^2}} = 6\sin^{-1}x\Big]_0^{0.5} = 6\sin^{-1}0.5 = 6\cdot\frac{1}{6}\pi = \pi$

40. $\int_0^1 \frac{4}{1+x^2}\,dx$

▷ $\int_0^1 \frac{4}{1+x^2}dx = 4\tan^{-1}x\Big]_0^1 = 4\tan^{-1}1 = 4\cdot\frac{1}{4}\pi = \pi$. Using NINT, we find $\pi = 3.14159265$ to 9 digits.

41. $\int_{\sqrt{2}}^2 \frac{12\,dx}{x\sqrt{x^2-1}} = 12\sec^{-1}x\Big]_{\sqrt{2}}^2 = 12(\sec^{-1}2 - \sec^{-1}\sqrt{2}) = 12(\frac{1}{3}\pi - \frac{1}{4}\pi) = \pi$

42. Prove each formula by showing that the derivative of the right side equals the integrand. Relate to Th. 5.8.1

▷ (a) $\frac{d}{du}(-\cos^{-1}u) = -(\frac{1}{\sqrt{1-u^2}}) = \frac{1}{\sqrt{1-u^2}}$ so that $\int \frac{du}{\sqrt{1-u^2}} = -\cos^{-1}u + C = -(\frac{1}{2}\pi - \sin^{-1}x) + C$

$= \sin^{-1}x + \bar{C}$ where $\bar{C} = C - \frac{1}{2}\pi$. Therefore, the formula of this exercise is equivalent to formula (1).

(b) $\frac{d}{du}(-\cot^{-1}u) = -\left(-\frac{1}{1+u^2}\right) = \frac{1}{1+u^2}$ so that $\int \frac{du}{1+u^2} = -\cot^{-1}u + C$. By Definition 5.7.9,

$-\cot^{-1}u + C = -(\frac{1}{2}\pi - \tan^{-1}u) + C = \tan^{-1}u + \bar{C}$ where $\bar{C} = C - \frac{1}{2}\pi$

Therefore, the formula of this exercise is equivalent to formula (2).

(c) $\frac{d}{du}(-\csc^{-1}u) = -\frac{-1}{u\sqrt{u^2-1}} = \frac{1}{u\sqrt{u^2-1}}$ and so the formula is proved. Furthermore, by Definition 5.7.11.

$-\csc^{-1}u + C = \sec^{-1}u + \bar{C}$ where $\bar{C} = C - \frac{1}{2}\pi$. Thus the given formula is equivalent to formula (3).

5.9 HYPERBOLIC FUNCTIONS

5.9.1 Definition The *hyperbolic sine function*, denoted by sinh, the *hyperbolic cosine function*, denoted by cosh are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh 0 = 0 \qquad \cosh 0 = 1$$

Catenary A uniform flexible cable hanging from two points has the shape of $y = a\cosh(x/a)$.

5.9.2 Theorem If u is a differentiable function of x,

$$D_x(\sinh u) = \cosh u\, D_x u \qquad D_x(\cosh u) = \sinh u\, D_x u$$

5.9.3 Definition The *hyperbolic tangent function*, *hyperbolic cotangent function*, *hyperbolic secant function*, and *hyperbolic cosecant function* are defined, respectively, as follows:

$$\tanh x = \frac{\sinh x}{\cosh x},\ \coth x = \frac{1}{\tanh x},\ \operatorname{sech} x = \frac{1}{\cosh x},\ \operatorname{csch} x = \frac{1}{\sinh x}$$

Domain and Range

Function	sinh	cosh	tanh	coth	sech	csch
Domain	$(-\infty,+\infty)$	$(-\infty,+\infty)$	$(-\infty,+\infty)$	$(-\infty,0)\cup(0,+\infty)$	$(-\infty,+\infty)$	$(-\infty,0)\cup(0,+\infty)$
Range	$(-\infty,+\infty)$	$[1,+\infty)$	$(-1,1)$	$(-\infty,-1)\cup(1,+\infty)$	$(0,1]$	$(-\infty,0)\cup(0,+\infty)$

Odd and Even sinh, tanh, coth and csch and their inverses are odd functions; cosh and sech are even.

In Terms of e^x $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, $\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$, $\operatorname{csch} x = \frac{2}{e^x - e^{-x}}$

Identities $\cosh^2 x - \sinh^2 x = 1$, $\quad 1 - \tanh^2 x = \operatorname{sech}^2 x$, $\quad 1 - \coth^2 x = -\operatorname{csch}^2 x$

$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$, $\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$

$\sinh 2x = 2\sinh x\cosh x$, $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$

5.9.4 Theorem If u is a differentiable function of x,

$$D_x(\tanh u) = \operatorname{sech}^2 u\, D_x u \qquad D_x(\coth u) = -\operatorname{csch}^2 u\, D_x u$$

$$D_x(\operatorname{sech} u) = -\operatorname{sech} u \tanh u\, D_x u \qquad D_x(\operatorname{csch} u) = -\operatorname{csch} u \coth u\, D_x u$$

5.9.5 Theorem

$$\int \sinh u\, du = \cosh u + C \qquad \int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

5.9.6 Definition The *inverse hyperbolic sine function* denoted by $\sinh^{-1}$, is defined as follows:

$y = \sinh^{-1} x$ if and only if $x = \sinh y$

5.9.7 Definition The *inverse hyperbolic cosine function*, denoted by $\cosh^{-1}$, is defined as follows:

$y = \cosh^{-1} x$ if and only if $x = \cosh y$ and $y \geq 0$

5.9.8 Definition The *inverse hyperbolic tangent function* and *inverse hyperbolic cotangent function*, denoted respectively by $\tanh^{-1}$ and $\coth^{-1}$, are defined as follows:

$y = \tanh^{-1} x$ if and only if $x = \tanh y$

$y = \coth^{-1} x$ if and only if $x = \coth y$

Each inverse hyperbolic function may be expressed in terms of natural logarithms.

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ $\quad x$ any real number (1) $\quad \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ $\quad x \geq 1$ (2)

$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$ $\quad |x| < 1$ (3) $\quad \coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$ $\quad |x| > 1$ (4)

Following are the differentiation formulas for the inverse hyperbolic functions.

5.9.9 Theorem If u is a differentiable function of x,

$D_x(\sinh^{-1} u) = \frac{1}{\sqrt{u^2+1}} D_x u$ $\qquad D_x(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}} D_x u \quad u > 1$

$D_x(\tanh^{-1} u) = \frac{1}{1-u^2} D_x u \quad |u| < 1$ $\qquad D_x(\coth^{-1} u) = \frac{1}{1-u^2} D_x u \quad |u| > 1$

5.9.10 Theorem

$$\int \frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\tfrac{u}{a} + C = \ln(u + \sqrt{u^2+a^2}) + C \text{ if } a > 0$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\tfrac{u}{a} + C = \ln(u + \sqrt{u^2-a^2}) + C \text{ if } u > a > 0$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\frac{u}{a} + C & \text{if } |u| < a \\ \frac{1}{a}\coth^{-1}\frac{u}{a} + C & \text{if } |u| > a \end{cases} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C \text{ if } u \neq a \text{ and } a \neq 0$$

Exercises 5.9

In Exercises 1–6, determine the exact function value and express irrational values to four significant digits.

1. (a) $\sinh 0 = 0$; (b) $\cosh 0 = 1$; (c) $\sinh 1 = \frac{1}{2}(e - e^{-1}) = 1.175$; (d) $\sinh(-1) = \frac{1}{2}(e^{-1} - e) = -1.175$

2. (a) $\tanh 0 = 0$; (b) $\operatorname{sech} 0 = \frac{1}{\cosh 0} = 1$; (c) $\cosh 1 = \frac{1}{2}(e + e^{-1}) = 1.543$; (d) $\cosh(-1) = \frac{1}{2}(e^{-1} + e) = 1.543$

3. (a) $\tanh 2 = \frac{e^2 - e^{-2}}{e^2 + e^{-2}} = .9640$ $\qquad$ (b) $\tanh(-2) = \frac{e^{-2} - e^2}{e^{-2} + e^2} = -.9460$

   (c) $\cosh(\ln 2) = \frac{1}{2}(2 + \frac{1}{2}) = \frac{5}{4}$ $\qquad$ (d) $\cosh(\ln \frac{1}{2}) = \frac{1}{2}(\frac{1}{2} + 2) = \frac{5}{4}$

4. (a) $\coth(0.5)$; (b) $\coth(-0.5)$; (c) $\sinh(\ln 2)$; (d) $\sinh(\ln 0.5)$

▹ (a) $\coth(0.5) = \frac{e^{0.5} + e^{-0.5}}{e^{0.5} - e^{-0.5}} = \frac{e+1}{e-1} = 1.313$ $\qquad$ (b) $\coth(-0.5) = \frac{e^{-0.5} + e^{0.5}}{e^{-0.5} - e^{0.5}} = \frac{1+e}{1-e} = -1.313$

   (c) $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$ $\qquad$ (d) $\sinh(\ln 0.5) = \frac{e^{\ln 0.5} - e^{-\ln 0.5}}{2} = \frac{\frac{1}{2} - 2}{2} = -\frac{3}{4}$

5. (a) $\operatorname{sech} 2 = \frac{2}{e^2 + e^{-2}} = 0.2658$ $\qquad$ (b) $\operatorname{sech}(-2) = \frac{2}{e^{-2} + e^2} = 0.2658$

   (c) $\coth(-1) = \frac{e^{-1} + e}{e^{-1} - e} = \frac{1 + e^2}{1 - e^2} = -1.313$ $\qquad$ (d) $\operatorname{csch}(\ln 1.5) = \frac{2}{\frac{3}{2} - \frac{2}{3}} = \frac{12}{5}$

6. (a) $\operatorname{csch} 2 = \frac{2}{e^2 - e^{-2}} \approx 0.2757$ $\qquad$ (b) $\operatorname{csch}(-2) = \frac{2}{e^{-2} - e^2} = -0.2757$

   (c) $\tanh 1 = \frac{e - e^{-1}}{e + e^{-1}} = \frac{e^2 - 1}{e^2 + 1} = 0.7616$ $\qquad$ (d) $\operatorname{sech}(\ln 1.5) = \frac{2}{\frac{3}{2} + \frac{2}{3}} = \frac{12}{13}$

In Exercises 7–10, prove the identity.

7. (a) $1-\tanh^2 x = 1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2 = 1-\frac{e^{2x}-2+e^{-2x}}{e^{2x}+2+e^{-2x}} = \frac{4}{e^{2x}+2+e^{-2x}} = \left(\frac{2}{e^x+e^{-x}}\right)^2 = \operatorname{sech}^2 x$

(b) $\sinh x \cosh y + \cosh x \sinh y = \frac{1}{2}(e^x-e^{-x})\cdot\frac{1}{2}(e^y+e^{-y}) + \frac{1}{2}(e^x+e^{-x})\cdot\frac{1}{2}(e^y-e^{-y}) = \frac{1}{2}(e^{x+y}-e^{-x-y})$
$= \sinh(x+y)$

8. (a) $1-\coth^2 x = -\operatorname{csch}^2 x$; (b) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

▸ (a) $1-\coth^2 x = 1-\frac{\cosh^2 x}{\sinh^2 x} = \frac{\sinh^2 x-\cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x$

(b) $\cosh x \cosh y + \sinh x \sinh y = \frac{1}{2}(e^x+e^{-x})\cdot\frac{1}{2}(e^y+e^{-y}) + \frac{1}{2}(e^x-e^{-x})\cdot\frac{1}{2}(e^y-e^{-y}) = \frac{1}{2}(e^{x+y}+e^{-x-y})$

$$= \cosh(x+y)$$

9. $\frac{1+\tanh x}{1-\tanh x} = \frac{1+(e^x-e^{-x})/(e^x+e^{-x})}{1-(e^x-e^{-x})/(e^x+e^{-x})} = \frac{(e^x+e^{-x})+(e^x-e^{-x})}{(e^x+e^{-x})-(e^x-e^{-x})} = \frac{2e^x}{2e^{-x}} = e^{2x}$

10. $\tanh(\ln x) = \frac{e^{\ln x}-e^{-\ln x}}{e^{\ln x}+e^{-\ln x}} = \frac{x-\frac{1}{x}}{x+\frac{1}{x}} = \frac{x^2-1}{x^2+1}$ if $x > 0$

11. $D_x(\coth x) = D\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} = \frac{\sinh \cdot x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x$

12. Prove (a) $D_x(\operatorname{sech} x) = -\operatorname{sech} x \tan x$; (b) $D_x(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

▸ (a)$D_x(\operatorname{sech} x) = D_x\left(\frac{1}{\cosh x}\right) = \frac{-1}{\cosh^2 x}\cdot D_x(\cosh x) = \frac{-\sinh x}{\cosh^2 x} = -\frac{1}{\cosh x}\cdot\frac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$

(b)$D_x(\operatorname{csch} x) = D_x\left(\frac{1}{\sinh x}\right) = \frac{-1}{\sinh^2 x}\cdot D_x(\sinh x) = \frac{-\cosh x}{\sinh^2 x} = -\frac{1}{\sinh x}\cdot\frac{\cosh x}{\sinh x} = -\operatorname{csch} x \coth x$

In Exercises 13–18, find the derivative of the function.

13. (a) $D_x \sinh x^2 = \cosh x^2\, D_x x^2 = 2x \cosh x^2$
(b) $D_w \operatorname{sech}^2 4w = 2 \operatorname{sech} 4w\, D_w \operatorname{sech} 4w = 2 \operatorname{sech} 4w(-\operatorname{sech} 4w \tanh 4w \cdot 4) = -8 \operatorname{sech}^2 4w \tanh 4w$

14. (a) $D_x \tanh^3\sqrt{x} = 3\tanh^2\sqrt{x}\, D_x \tanh\sqrt{x} = 3\tanh^2\sqrt{x}\operatorname{sech}^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$

(b) $D_t \cosh t^3 = \sinh t^3\, D_t t^3 = 3t^2 \sinh t^3$

15. (a) $D_x \coth\left(\frac{1}{x}\right) = -\operatorname{csch}^2\frac{1}{x}\cdot D_x\left(\frac{1}{x}\right) = \frac{1}{x^2}\operatorname{csch}^2\frac{1}{x}$

(b) $D_x \ln(\tanh x) = \frac{1}{\tanh x} D_x \tanh x = \frac{\operatorname{sech}^2 x}{\tanh x} = \frac{1}{\sinh x \cosh x} = \operatorname{csch} 2x,\ x > 0$

16. (a) $f(y) = \coth(\ln y)$; (b) $h(x) = e^x \cosh x$

▸ (a) Applying Theorem 5.9.4 with $u = \ln y$, we obtain

$$f'(y) = -\operatorname{csch}^2(\ln y)\, D_x \ln y = \frac{-\operatorname{csch}^2(\ln y)}{y}$$

(b) $h(x) = e^x\cdot\frac{1}{2}(e^x+e^{-x}) = \frac{1}{2}(e^{2x}+1)$ and so $h'(x) = e^{2x}$.

17. (a) $D_x \tan^{-1}(\sinh 2x) = \frac{1}{1+\sinh^2 2x} D_x \sinh 2x = \frac{2\cosh 2x}{\cosh^2 x} = 2 \operatorname{sech} 2x$

(b) $g = (\cosh x)^x$. $\ln g = x \ln(\cosh x)$; $\frac{g'}{g} = 1\cdot\ln(\cosh x) + x\cdot\frac{\sinh x}{\cosh x}$, $g' = [\ln(\cosh x) + x \tanh x](\cosh x)^x$

18. (a) $g'(x) = \frac{d}{dx}\sin^{-1}(\tanh x^2) = \frac{1}{\sqrt{1-\tanh^2 x^2}}\frac{d}{dx}(\tanh x^2) = \frac{1}{\operatorname{sech} x^2}(\operatorname{sech}^2 x^2)(2x) = 2x \operatorname{sech} x^2$

(b) $f = x^{\sinh x}$. $\ln f = \sinh x \ln x$; $\frac{f'}{f} = \cosh x \ln x + \sinh x\left(\frac{1}{x}\right)$

$f' = x^{\sinh x}\left(\frac{x\cosh x \ln x + \sinh x}{x}\right) = x^{\sinh x - 1}(x \cosh x \ln x + \sinh x)$

In Exercises 19–24, evaluate the indefinite integral.

19. $\int \sinh^4 x \cosh x\, dx = \int \sinh^4 x\, d(\sinh x) = \frac{1}{5}\sinh^5 x + C$

20. $\int x \cosh x^2 \sinh x^2\, dx$

▷ Call the integral I. We first substitute for the nonlinear argument: Let $u = x^2$, $du = 2x\, dx$.

$$I = \int \cosh x^2 \sinh x^2 (x\, dx) = \frac{1}{2}\int \cosh u \sinh u\, du$$

Now let $v = \cosh u$, $dv = \sinh u\, du$. Therefore

$$I = \frac{1}{2}\int v\, dv = \frac{1}{4}v^2 + C$$

Substituting back, we obtain successively

$$I = \frac{1}{4}\cosh^2 u + C = \frac{1}{4}\sinh^2 x^2 + C$$

Alternatively, let $v = \sinh u$, $dv = \cosh u\, du$ to get $I = \frac{1}{4}\cosh^2 x^2 + \bar{C}$.

21. $\int x^2 \operatorname{csch}^2 x^3 dx = \frac{1}{3}\int \operatorname{csch}^2 x^3\, dx^3 = -\frac{1}{3}\coth x^3 + C$

22. $\int \coth^2 3x\, dx = \int (1 + \operatorname{csch}^2 3x) dx = x - \frac{1}{3}\coth 3x + C$

23. Let $u = \cosh 2x$; $du = 2 \sinh 2x\, dx$. Then

$$\int \tanh 2x \ln(\cosh 2x) dx = \frac{1}{2}\int \ln(\cosh 2x)\frac{2\sinh 2x\, dx}{\cosh 2x} = \frac{1}{2}\int \ln u \frac{du}{u} = \frac{1}{4}(\ln u)^2 + C = \frac{1}{4}\ln^2(\cosh 2x) + C$$

24. $\int \operatorname{sech}^2 x \tanh^2 x\, dx$

▷ We make the implicit substitution $u = \tanh x$, $du = \operatorname{sech}^2 x$.

$$\int \operatorname{sech}^2 x \tanh^2 x\, dx = \int \tanh^2 x\, d(\tanh x) = \frac{1}{3}\tanh^3 x + C$$

25. (a) $\int \tanh u\, du = \int \frac{\sinh u\, du}{\cosh u} = \ln|\cosh u| + C = \ln(\cosh u) + C$ because $\cosh u > 0$.

(b) $\int \operatorname{csch} u\, du = \int \frac{du}{\sinh u} = \int \frac{du}{2\sinh\frac{1}{2}u \cosh\frac{1}{2}u} = \int \frac{\frac{1}{\cosh^2\frac{1}{2}u}}{\frac{\sinh\frac{1}{2}u}{\cosh\frac{1}{2}u}}(\frac{1}{2}du) = \int \frac{\operatorname{sech}^2\frac{1}{2}u}{\tanh\frac{1}{2}u}(\frac{1}{2}du) = \ln\left|\tanh\frac{1}{2}u\right| + C$

26. (a) $\int \coth u\, du = \int \frac{\cosh u\, du}{\sinh u} = \ln|\sinh u| + C$

(b) $\int \operatorname{sech} u\, du = \int \frac{2}{e^u + e^{-u}}du = 2\int \frac{e^u\, du}{(e^u)^2 + 1} = 2\tan^{-1}e^u + C$

In Exercises 27–32, find the exact value of the definite integral and approximate to four digits. Check using NINT.

27. $\int_0^{\ln 3} \operatorname{sech}^2 t\, dt = \tanh t\Big]_0^{\ln 3} = \tanh(\ln 3) - \tanh 0 = \frac{e^{\ln 3} - e^{-\ln 3}}{e^{\ln 3} + e^{-\ln 3}} = \frac{3 - \frac{1}{3}}{3 + \frac{1}{3}} = \frac{8}{10} = \frac{4}{5}$

28. $\int_0^{\ln 2} \tanh z\, dz$

▷ $$\int_0^{\ln 2} \tanh z\, dz = \int_0^{\ln 2} \frac{\sinh z}{\cosh z}\, dz \tag{1}$$

We let $u = \cosh z$ and $du = \sinh z\, dz$. When $z = \ln 2$, then $e^z = 2$, and $e^{-z} = \frac{1}{2}$, so

$$\cosh z = \frac{e^z + e^{-z}}{2} = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4}$$

Thus, $u = \frac{5}{4}$ when $z = \ln 2$. When $z = 0$, then $u = \cosh 0 = 1$. With these substitutions in Eq. (1) we obtain

$$\int_0^{\ln 2} \tanh z\, dz = \int_1^{5/4} \frac{du}{u} = \ln|u|\Big]_1^{5/4} = \ln\frac{5}{4} - \ln 1 = \ln\frac{5}{4} \approx 0.2231$$

29. Let $u = \sqrt{x}$; $du = \frac{1}{2\sqrt{x}}$. Then $\int_{x=1}^{4} \frac{\sinh\sqrt{x}}{\sqrt{x}}\, dx = 2\int_{u=1}^{2} \sinh u\, du = 2\cosh u\Big]_1^2 = 2\cosh 2 - 2\cosh 1 \approx 4.438$

30. Let $u = x^2$, $du = 2x\,dx$. $\displaystyle\int_{x=1}^{2} x\,\text{sech}^2 x^2 dx = \frac{1}{2}\int_{u=1}^{4} \text{sech}^2 u\,du = \frac{1}{2}\tanh u\Big]_1^4 = \frac{1}{2}(\tanh 4 - \tanh 1) \approx 0.1189$

31. $\displaystyle\int_2^3 \text{sech}^2 x \tanh^5 x\,dx = \int_2^3 \tanh^5 x\,(\text{sech}^2\,x\,dx) = \frac{1}{6}\tanh^6 x\Big]_2^3 = \frac{1}{6}(\tanh^6 3 - \tanh^6 2) \approx 0.02800$

32. $\displaystyle\int_0^2 \sinh^3 x \cosh x\,dx$

▷ $\displaystyle\int_0^2 \sinh^3 x \cosh x\,dx = \int_0^2 \sinh^3 x\,d(\cosh x)$

$$= \frac{1}{4}\sinh^4 x\Big]_0^2 = \frac{1}{4}\sinh^4 2 \approx 43.26$$

In Exercises 33–36, determine the exact function value.

33. (a) Because $\cosh 0 = 1$, then $\cosh^{-1} 1 = 0$. (b) $\tanh^{-1}\frac{1}{2} = \frac{1}{2}\ln\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{2}\ln 3$

34. (a) $\sinh^{-1} 1 = \ln(1 + \sqrt{1^2+1}) = \ln(1+\sqrt{2})$ (b) $\coth^{-1} 2 = \tanh^{-1}\frac{1}{2} = \frac{1}{2}\ln 3$

35. (a) $\sinh^{-1}\frac{1}{2} = \ln(1 + \sqrt{(\frac{1}{2})^2+1}) = \ln(1+\frac{1}{2}\sqrt{5})$ (b) $\coth^{-1}(-2) = \tanh^{-1}(-\frac{1}{2}) = \frac{1}{2}\ln\dfrac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{2}\ln\frac{1}{3}$

36. (a) $\cosh^{-1} 2$; (b) $\tanh^{-1}(-\frac{1}{2})$

▷ (a) We use Eq. (2).

$$\cosh^{-1} 2 = \ln(2 + \sqrt{2^2-1}) = \ln(2+\sqrt{3})$$

(b) We use Eq.(3)

$$\tanh^{-1}(-\tfrac{1}{2}) = \frac{1}{2}\ln\frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{2}\ln\frac{1}{3} = -\frac{1}{2}\ln 3$$

In Exercises 37–39, prove the formula.

37. Let $y = \cosh^{-1}u$, $u > 1$. Then $y > 0$ and $\cosh y = u$. Differentiating on both sides with respect to x, we obtain $\sinh y\,D_x y = D_x u$. Thus $D_x y = \dfrac{1}{\sinh y}D_x u$. From the identity $\cosh^2 y - \sinh^2 y = 1$ and because $y > 0$, we get $\sinh y = \sqrt{\cosh^2 y - 1}$. Therefore, $D_x y = \dfrac{1}{\sqrt{\cosh^2 y - 1}}D_x u = \dfrac{1}{\sqrt{u^2-1}}D_x u$

38. Let $y = \tanh^{-1}x$ and $|x| < 1$. Then $x = \tanh y$ and $D_x y = \dfrac{1}{D_y x} = \dfrac{1}{\text{sech}^2 y} = \dfrac{1}{1-\tanh^2 y} = \dfrac{1}{1-x^2}$.

From the chain rule, if u is a differentiable function of x, $D_x(\tanh^{-1}u) = \dfrac{1}{1-u^2}D_x u$, $|u| < 1$

39. Let $y = \coth^{-1}u$, $|u| > 1$. Then $u = \coth y$ and $D_x u = -\text{csch}^2 y\,D_x y = (1-\coth^2 y)D_x y = (1-u^2)D_x y$.

Therefore $D_x y = \dfrac{1}{1-u^2}D_x u$.

In Exercises 40–48, find the derivative of the function.

40. (a) $G(x) = \cosh^{-1}\frac{1}{3}x$; (b) $F(x) = \tanh^{-1} x^3$

▷ (a) $G'(x) = \dfrac{1}{\sqrt{\frac{1}{9}x^2-1}}\cdot\dfrac{1}{3} = \dfrac{1}{\sqrt{x^2-9}}$ (b) $F'(x) = \dfrac{1}{1-(x^3)^2}D_x x^3 = \dfrac{3x^2}{1-x^6}$, $|x| < 1$

41. (a) $D_x(\sinh^{-1} 4x) = \dfrac{1}{\sqrt{(4x)^2+1}}D_x(4x) = \dfrac{4}{\sqrt{16x^2+1}}$ (b) $D_x(\coth x^2) = \dfrac{1}{1-(x^2)^2}D_x x^2 = \dfrac{2x}{1-x^4}$, $|x| > 1$

42. (a) $h'(w) = \dfrac{d}{dw}\coth^{-1}(3w+1) = \dfrac{1}{1-(3w+1)^2}(3) = \dfrac{3}{1-(9w^2+6w+1)} = -\dfrac{1}{2w+3w^2}$

(b) $f'(x) = D_x(x^2\sinh^{-1} x^2) = 2x\sinh^{-1} x^2 + x^2\cdot\dfrac{1}{\sqrt{(x^2)^2+1}}\cdot 2x = 2x\sinh^{-1}x^2 + \dfrac{2x^3}{\sqrt{x^4+1}}$

43. (a) $D_x\cosh^{-1}(\tan x) = \dfrac{1}{\sqrt{\tan^2 x - 1}}D_x\tan x = \dfrac{\sec^2 x}{\sqrt{\tan^2 x - 1}}$

(b) $D_x\tanh^{-1}(\cos x) = \dfrac{1}{1-\cos^2 x}D_x\cos x = \dfrac{-\sin x}{\sin^2 x} = -\csc x$

44. (a) $g(x) = \tanh^{-1}(\sin 3x)$; (b) $F(x) = \coth^{-1}(3 \sin x)$

▹ (a) $g'(x) = \dfrac{1}{1 - \sin^2 3x} \cdot 3 \cos 3x = \dfrac{3 \cos 3x}{\cos^2 3x} = 3 \sec 3x$

(b) $F'(x) = \dfrac{1}{1 - (3 \sin x)^2} D_x(3 \sin x) = \dfrac{3 \cos x}{1 - 9 \sin^2 x}$

45. (a) $f'(z) = \dfrac{d}{dz}(\coth^{-1} z^2)^3 = 3(\coth^{-1} z^2)^2 \dfrac{d}{dz}(\coth^{-1} z^2) = 3(\coth^{-1} z^2)^2 \cdot \dfrac{1}{1 - z^4}(2z) = \dfrac{6z(\coth^{-1} z^2)^2}{1 - z^4}$

(b) $g'(x) = D_x \tanh^{-1}(\sin e^x) = \dfrac{1}{1 - \sin^2 e^x} D_x(\sin e^x) = \dfrac{e^x \cos e^x}{\cos^2 e^x} = e^x \sec e^x$

46. (a) $D_t \sinh^{-1} e^{2t} = \dfrac{1}{\sqrt{(e^{2t})^2 + 1}} D_t e^{2t} = \dfrac{2e^{2t}}{\sqrt{e^{4t} + 1}}$ (b) $D_x \cosh^{-1}(\ln x) = \dfrac{1}{\sqrt{\ln^2 x - 1}} D_x \ln x = \dfrac{1}{x\sqrt{\ln^2 x - 1}}$

47. $G'(x) = \dfrac{d}{dx}(x \sinh^{-1} x - \sqrt{1 + x^2}) = \sinh^{-1} x + x \cdot \dfrac{1}{\sqrt{x^2 + 1}} - \dfrac{x}{\sqrt{x^2 + 1}} = \sinh^{-1} x$

48. $H(x) = \ln\sqrt{1 - x^2} - x \tanh^{-1} x$

▹ $H(x) = \frac{1}{2} \ln(1 - x^2) - x \tanh^{-1} x$

Thus $H'(x) = \dfrac{-x}{1 - x^2} - \left[x \cdot \dfrac{1}{1 - x^2} + \tanh^{-1} x\right] = -\dfrac{2x}{1 - x^2} - \tanh^{-1} x$

In Exercises 49–54, express the integral in terms of an inverse hyperbolic function and as a natural logarithm.

49. $\displaystyle\int \frac{dx}{\sqrt{4 + x^2}} = \sinh^{-1} \frac{x}{2} + C = \ln(x + \sqrt{x^2 + 4}) + C$

50. $\displaystyle\int \frac{dx}{25 - x^2} = \begin{cases} \frac{1}{5} \tanh^{-1} \frac{x}{5} + C & \text{if } |x| < 5 \\ \frac{1}{5} \coth^{-1} \frac{x}{5} + C & \text{if } |x| > 5 \end{cases} = \frac{1}{10} \ln\left|\frac{5 + x}{5 - x}\right| + C$

51. $\displaystyle\int \frac{x\,dx}{\sqrt{x^4 - 1}} = \frac{1}{2}\int \frac{2x\,dx}{\sqrt{(x^2)^2 - 12}} = \frac{1}{2} \cosh^{-1} x^2 + C = \frac{1}{2} \ln(x^2 + \sqrt{x^4 - 1}) + C$

52. $\displaystyle\int \frac{dx}{\sqrt{25x^2 + 9}}$

▹ Let $u = 5x$ and $du = 5dx$. Then

$$\int \frac{dx}{\sqrt{25x^2 + 9}} = \frac{1}{5}\int \frac{du}{\sqrt{u^2 + 9}} = \frac{1}{5} \sinh^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{5} \sinh^{-1}\left(\frac{5x}{3}\right) + C = \frac{1}{5} \ln(5x + \sqrt{25x^2 + 9}) + C$$

53. $\displaystyle\int \frac{dt}{4e^{-t} - e^t} = \int \frac{e^t dt}{4 - (e^t)^2} = \begin{cases} \frac{1}{2} \tanh^{-1} \frac{1}{2}e^t + C & \text{if } e^t < 2 \\ \frac{1}{2} \cosh^{-1} \frac{1}{2}e^t + C & \text{if } e^t > 2 \end{cases} = \frac{1}{4} \ln\left|\frac{2 + e^t}{2 - e^t}\right| + C$

54. Let $u = e^t$, $du = e^t dt$. Then $\displaystyle\int \frac{dt}{\sqrt{5 - e^{-2t}}} = \int \frac{e^t dt}{\sqrt{5e^{2t} - 1}} = \frac{1}{\sqrt{5}}\int \frac{\sqrt{5}\,du}{\sqrt{5u^2 - 1}} = \frac{1}{\sqrt{5}} \cosh^{-1} \sqrt{5}u + C$

$= \dfrac{1}{\sqrt{5}} \ln(\sqrt{5}u + \sqrt{5u^2 - 1}) + C = \dfrac{1}{\sqrt{5}} \cosh^{-1}(\sqrt{5}\, e^t) + C = \dfrac{1}{\sqrt{5}} \ln(\sqrt{5}\, e^t + \sqrt{5e^{2t} - 1}) + C$

In Exercises 55–60, find the exact value of the integral in terms of inverse hyperbolic functions. Check by NINT.

55. $\displaystyle\int_3^5 \frac{dx}{\sqrt{x^2 - 4}} = \cosh^{-1} \frac{1}{2}x \Big]_3^5 = \cosh^{-1} \frac{5}{2} - \cosh^{-1} \frac{3}{2} \approx 0.6044$

$\displaystyle= \ln(x + \sqrt{x^2 - 4})\Big]_3^5 = \ln \frac{5 + \sqrt{21}}{3 + \sqrt{5}}$

56. $\displaystyle\int_{-4}^{-3} \frac{dx}{1 - x^2}$

▹ $\displaystyle\int_{-4}^{-3} \frac{dx}{1 - x^2} = \coth^{-1} x \Big]_{-4}^{-3} = \coth^{-1}(-3) - \coth^{-1}(-4) = \tanh^{-1} \frac{1}{4} - \tanh \frac{1}{3} \approx -0.09116$

57. $\displaystyle\int_{-1/2}^{1/2} \frac{dx}{1 - x^2} = 2 \tanh^{-1} x \Big]_0^{1/2} = 2 \tanh^{-1} \frac{1}{2} \approx 1.099$

58. $\displaystyle\int_{-2}^{2} \frac{dx}{\sqrt{16+x^2}} = 2\sinh^{-1}\tfrac14 x\Big]_0^2 = 2\sinh^{-1}\tfrac12 \approx 0.9624$

59. $\displaystyle\int_2^3 \frac{dx}{\sqrt{9x^2-12x-5}} = \frac13\int_2^3 \frac{3\,dx}{\sqrt{(3x-2)^2-9}} = \frac13\cosh^{-1}\frac{3x-2}{3}\Big]_2^3 = \tfrac13(\cosh^{-1}\tfrac73 - \cosh^{-1}\tfrac43) \approx 0.2319$

60. $\displaystyle\int_1^2 \frac{dx}{\sqrt{x^2+2x}}$

▷ $\displaystyle\int_1^2 \frac{dx}{\sqrt{x^2+2x}} = \int_1^2 \frac{dx}{\sqrt{(x+1)^2-1}} = \cosh^{-1}(x+1)\Big]_1^2 = \cosh^{-1}3 - \cosh^{-1}2 \approx 0.4458$

61. $f(x) = a\cosh(x/a)$, $a > 0$. $f'(x) = \sinh(x/a)$, $f''(x) = (1/a)\cosh(x/a)$.
Because $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$, then f is decreasing when $x < 0$, increasing when $x > 0$ and has an absolute minimum value when $x = 0$. Because $f''(x) > 0$, then f is concave upward everywhere.

62. $D_x\sinh x = \cosh x > 0$ and so sinh is increasing everywhere. Because sinh is differentiable, it is continuous.

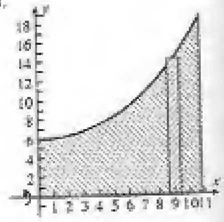
In Exercises 63 and 64, the region is bounded by the catenary $y = 6\cosh\frac{x}{6}$, the axes, and the line $x = 6\ln 6$.

63. A square units is the area of the region.

$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 6\cosh\frac{w_i}{6}\,\Delta_i x = 6\int_0^{6\ln 6}\cosh\frac{x}{6}\,dx = 36\sinh\frac{x}{6}\Big]_0^{6\ln 6} = 36\sinh\ln 6 = 36\cdot\tfrac12(6-\tfrac16) = 105$$

64. Find the volume of the solid of revolution if the region is revolved about the x axis.

▷ The figure shows a sketch of the region. The element of volume is a circular disk with thickness $\Delta_i x$ units and radius $y(w_i)$ units. If V cubic units is the volume,



$$\begin{aligned}
V &= \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\pi[y(w_i)]^2\Delta_i x = \int_0^{6\ln 6}\pi[y(x)]^2dx \\
&= 36\pi\int_0^{6\ln 6}\cosh^2\frac{x}{6}\,dx = 36\pi\int_0^{6\ln 6}[\tfrac12(e^{x/6}+e^{-x/6})]^2dx \\
&= 36\pi\int_0^{6\ln 6}\tfrac14[e^{x/3}+2+e^{-x/3}]dx = 9\pi\Big[3e^{x/3}+2x-3e^{-x/3}\Big]_0^{6\ln 6} \\
&= 9\pi(3e^{2\ln 6}+12\ln 6-3e^{-2\ln 6}) = 9\pi(108+12\ln 6-\tfrac{1}{12}) = 9\pi(\tfrac{1295}{12}+12\ln 6)
\end{aligned}$$

65. If s cm is the directed distance of the particle from the origin at t sec, then

$s = e^{-t/2}(3\sinh t + 4\cosh t) = e^{-t/2}[\frac32(e^t - e^{-t}) + \frac42(e^t + e^{-t})] = \frac12(7e^{t/2} + e^{-3t/2})$

$v = D_t s = \frac14(7e^{t/2} - 3e^{-3t/2})$, $a = D_t v = \frac18(7e^{t/2} + 9e^{-3t/2}) = 3\cdot\frac14 s - 2\cdot\frac12 v$

66. $\displaystyle\frac{dy}{dx} = \sqrt{\frac{y^2}{a^2}-1}$; $\displaystyle dx = \frac{a\,dy}{\sqrt{y^2-a^2}}$; $\displaystyle\int_0^x dt = \int_a^y \frac{a\,du}{\sqrt{u^2-a^2}}$; $\displaystyle x = \cosh^{-1}\frac{u}{a}\Big]_a^y = \cosh^{-1}\frac{y}{a}$; $y = a\cosh x$, a catenary

67. $\displaystyle dt = \frac{324}{g}\cdot\frac{dv}{324-v^2}$; $\displaystyle\int_0^t dx = \int_{200}^{v}\frac{324}{g}\,\frac{du}{324-u^2}$; $\displaystyle t = \frac{18}{g}\coth^{-1}\frac{u}{18}\Big]_{200}^{v} = \frac{18}{g}\Big(\coth^{-1}\frac{v}{18} - \coth^{-1}\frac{100}{9}\Big)$

68. The region bounded by the curve $y = (16-x^2)^{-1/2}$, the x axis, and the line $x = -2$ and $x = 3$ is revolved about the x axis. (a) Show that the exact measure of the volume of the solid generated is $\frac14\pi[\tanh^{-1}\frac34 - \tanh^{-1}(-\frac12)]$. (b) Use your calculator to approximate the volume to four significant digits.

▷ An element of volume is a circular disk centered on the x axis, $x \in [-2, 3]$, of radius $(16 - w_{i2})^{-1/2}$. If V cubic units is the volume of the solid of revolution, then

$$V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\pi\Big(\frac{1}{\sqrt{16-w_{i2}}}\Big)^2\Delta_i x = \pi\int_{-2}^{3}\frac{dx}{16-x^2} = \pi\Big[\frac14\tanh^{-1}\frac{x}{4}\Big]_{-2}^{3} = \frac{\pi}{4}\Big[\tanh^{-1}\frac34 - \tanh^{-1}\Big(-\frac12\Big)\Big] \approx 1.1962$$

69. $\displaystyle V = 2\cdot 100\int_0^{20}\Big(31 - 20\cosh\frac{x}{20}\Big)dx = 200\Big[31x - 400\sinh\frac{x}{20}\Big]_0^{20} = 200(620 - 400\sinh 1) \approx 29{,}983\ \text{ft}^3$

70. (a) $s = A\sin kt + B\cos kt$, $v = k(A\cos kt - B\sin kt)$, $a = k^2(-A\sin kt - B\cos kt) = -k^2 s$ and $-k^2 < 0$
(b) $s = A\sinh kt + B\cosh kt$, $v = k(A\cosh kt + B\sinh kt)$, $a = k^2(A\sinh kt + B\cosh kt) = k^2 s$ and $k^2 > 0$
Thus the motion in (a) is simple harmonic, but the motion in (b) is not.

Miscellaneous Exercises for Chapter 5

In Exercises 1–6, determine if the function f has an inverse. If so, find it and state its domain and range; (b) plot f and its inverse on the same screen. If not, plot a horizontal line that meets the graph of f more than once.

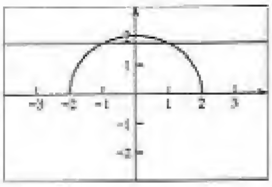
1. $f(x) = x^3 - 4$. f is increasing $\Rightarrow$ f is one-to-one $\Rightarrow$ f has an inverse.
(a) Let $y = f(x)$. Then $y = x^3 - 4$; $x = \sqrt[3]{y+4}$. Hence $f^{-1}(y) = \sqrt[3]{y+4}$, and so $f^{-1}(x) = \sqrt[3]{x+4}$.
Domain of f^{-1} is $(-\infty, +\infty)$ and range of f^{-1} is $(-\infty, +\infty)$.

2. $f(x) = 2\sqrt[3]{x} - 1$. f is increasing $\Rightarrow$ f is one-to-one $\Rightarrow$ f has an inverse. (a) Let $y = f(x)$. Then $y = 2\sqrt[3]{x} - 1$; $\sqrt[3]{x} = \frac{1}{2}(y+1)$; $x = \frac{1}{8}(y+1)^3$. Hence $f^{-1}(y) = \frac{1}{8}(y+1)^3$, and so $f^{-1}(x) = \frac{1}{8}(x+1)^3$. Domain of f^{-1} is $(-\infty, +\infty)$ and range of f^{-1} is $(-\infty, +\infty)$.

3. $f(x) = 9 - x^2$. $f(3) = 0 = f(-3)$ $\Rightarrow$ f is not one-to-one $\Rightarrow$ f does not have an inverse. The line $y = 3$ meets the graph of f twice.

4. $f(x) = \sqrt{4 - x^2}$

▷ Because $f(1) = f(-1) = \sqrt{3}$, then f is not a one-to-one function. Therefore, an inverse function does not exist. The figure shows the semicircle that is the graph of the function f as well as the horizontal line $y = \sqrt{3}$, which intersects the graph of f in two points, $(1, \sqrt{3})$ and $(-1, \sqrt{3})$.



5. $f(x) = \dfrac{3x-4}{x} = 3 - \dfrac{4}{x}$. On $(-\infty, 0)$ f is increasing and $f(x) > 3$; on $(0, +\infty)$ f is increasing and $f(x) < 3$. Therefore f is one-to-one, and so f has an inverse.

(a) Let $y = f(x)$. Then $y = \dfrac{3x-4}{x}$; $xy = 3x - 4$; $3x - xy = 4$; $x = \dfrac{4}{3-y}$. Hence $f^{-1}(y) = \dfrac{4}{3-y}$, and so $f^{-1}(x) = \dfrac{4}{3-y}$. The domain of f^{-1} is $\{x \mid x \neq 3\}$. The range of f^{-1} is the domain of f which is $\{y \mid y \neq 0\}$.

6. $f(x) = |2x - 3|$. $f(1) = |-1| = |1| = f(2)$ $\Rightarrow$ f is not one-to-one $\Rightarrow$ f does not have an inverse. The line $y = 1$ meets the graph of f twice.

In Exercises 7 and 8, (a) prove that f has an inverse, (b) find $f^{-1}(x)$, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ (Theorem 5.1.4).

7. (a) $f(x) = \sqrt[3]{x+1}$. f is increasing $\Rightarrow$ f is one-to-one $\Rightarrow$ f has an inverse. (b) Let $y = f(x)$. Then $y = \sqrt[3]{x+1}$; $y^3 = x + 1$; $x = y^3 - 1$. Thus $f^{-1}(y) = y^3 - 1$, and so $f^{-1}(x) = x^3 - 1$.

(c) $f^{-1}(f(x)) = f^{-1}(\sqrt[3]{x+1}) = (\sqrt[3]{x+1})^3 - 1 = (x+1) - 1 = x$

$$f(f^{-1}(x)) = f(x^3 - 1) = \sqrt[3]{(x^3-1)+1} = \sqrt[3]{x^3} = x$$

8. $f(x) = \dfrac{2x-1}{2x+1}$

▷ $f(x) = 1 - \dfrac{2}{2x+1}$; $f'(x) = \dfrac{4}{(2x+1)^2}$

Because $f'(x) > 0$ for all $x \neq -\frac{1}{2}$, then f is increasing and greater than 1 on $(-\infty, -\frac{1}{2})$ and increasing and less than 1 on $(-\frac{1}{2}, +\infty)$. Thus, f is one-to-one and f has an inverse function.

(b) We find $f^{-1}(x)$. Let $y = f(x)$. Then

$$y = \frac{2x-1}{2x+1}; \quad 2xy + y = 2x - 1; \quad 2xy - 2x = -y - 1; \quad x(2y-2) = -y - 1; \quad x = -\frac{y+1}{2(y-1)}$$

Therefore, $\qquad f^{-1}(y) = -\dfrac{y+1}{2(y-1)}$ and $\qquad f^{-1}(x) = -\dfrac{x+1}{2(x-1)} = \dfrac{1+x}{2(1-x)}$

(c) We verify the equations of Theorem 5.1.4

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2x-1}{2x+1}\right) = \frac{1 + \frac{2x-1}{2x+1}}{2\left(1 - \frac{2x-1}{2x+1}\right)} = \frac{(2x+1)+(2x-1)}{2[(2x+1)-(2x-1)]} = \frac{4x}{2(2)} = x$$

and

$$f(f^{-1}(x)) = f\left(\frac{1+x}{2(1-x)}\right) = \frac{2\left(\frac{1+x}{2(1-x)}\right) - 1}{2\left(\frac{1+x}{2(1-x)}\right) + 1} = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{(1+x)-(1-x)}{(1+x)+(1-x)} = \frac{2x}{2} = x$$

In Exercises 9–12, find $f(f^{-1})'(d)$.

9. $f(x) = x^2 - 6x + 8$, $x \geq 3$; $d = 3$. We seek the value of $c \geq 3$ for which $f(c) = 3$.
$c^2 - 6c + 8 = 3$; $c^2 - 6c + 5 = 0$; $(c-1)(c-5) = 0$; $c = 5$
$f'(x) = 2x - 6$. Thus $f'(x) > 0$ if $x > 3$. Hence there is a closed interval containing 5 on which f is continuous and increasing. Thus Theorem 5.1.8 applies and $(f^{-1})'(3) = \dfrac{1}{f'(5)} = \dfrac{1}{4}$.

10. $f(x) = \sqrt{3x+4}$; $d = 5$. We seek the value of c for which $f(c) = 5$: $\sqrt{3c+4} = 5$; $3c + 4 = 25$; $3c = 21$; $c = 7$
$f'(x) = \dfrac{3}{2\sqrt{3x+4}} > 0$. Hence there is a closed interval containing 7 on which f is continuous and increasing. Thus Theorem 5.1.8 applies and $(f^{-1})'(5) = \dfrac{1}{f'(7)} = \dfrac{2\sqrt{25}}{3} = \dfrac{10}{3}$.

11. $f(x) = 8x^3 + 6x$; $d = 4$. We wish to find the value of c for which $f(c) = 4$.
$8c^3 + 6c = 4$; $4c^3 + 3c - 2 = 0$. Trying $c = \pm 1$, $c = \pm\frac{1}{2}$, $c = \pm\frac{1}{4}$, by synthetic division, we see that $c = \frac{1}{2}$.
$f'(x) = 24x^2 + 6$. Hence $f'(x) > 0$ for all x. Thus there is a closed interval containing $\frac{1}{2}$ on which f is continuous and increasing. Hence Theorem 5.1.8 applies and $(f^{-1})'(4) = \dfrac{1}{f'(\frac{1}{2})} = \dfrac{1}{12}$

12. Find $(f^{-1})'(d)$ if $f(x) = x^5 + x - 22$ and $d = 12$.
▷ Because $f'(x) = 5x^4 + 1$ then $f'(x) > 0$ for all x, so f is monotonic and continuous on every closed interval. Furthermore, because $f(2) = 12$, we may apply Theorem 5.1.8 with $c = 2$ and $d = 12$. Because $f'(2) = 81$, then $(f^{-1})'(12) = \dfrac{1}{f'(2)} = \dfrac{1}{81}$

In Exercises 13–30, differentiate the function and simplify the result.

13. (a) $D_x \ln(\cos 3x) = \dfrac{1}{\cos 3x} D_x \cos 3x = \dfrac{-3 \sin 3x}{\cos 3x} = -3 \tan 3x$

(b) $D_x \ln(x^2+1)^2 = 2D_x \ln(x^2+1) = 2\dfrac{1}{x^2+1} D_x(x^2+1) = \dfrac{4x}{x^2+1}$

14. (a) $D_x \cos(3 \ln x) = -\sin(3 \ln x) D_x(3 \ln x) = -\dfrac{3 \sin(3 \ln x)}{x}$

(b) $\dfrac{d}{dx}(\ln x^2)^2 = 2(\ln x^2)\dfrac{d}{dx}(\ln x^2) = 2(\ln x^2)\dfrac{1}{x^2}(2x) = \dfrac{4 \ln x^2}{x}$ which equals $\dfrac{8 \ln x}{x}$ if $x > 0$

15. (a) $D_t \sin e^{4t} = \cos e^{4t} D_t e^{4t} = 4e^{4t} \cos e^{4t}$ (b) $D_t 2^{\tan t} = 2^{\tan t} \ln 2\, D_t \tan t = 2^{\tan t} \ln 2 \sec^2 t$

16. (a) $f(w) = e^{\sin 4w}$; (b) $F(w) = \tan 2^w$
▷ (a) $f'(w) = e^{\sin 4w} D_w \sin 4w = 4 \cos 4w\, e^{\sin 4w}$ (b) $F'(w) = \sec^2 2^w D_w 2^w = 2^w \ln 2 \sec^2 2^w$

17. (a) $D_x \tan^{-1} e^x = \dfrac{1}{1+(e^x)^2} D_x e^x = \dfrac{e^x}{1+e^{2x}}$ (b) $D_x e^{\cot^{-1} x} = e^{\cot^{-1} x} D_x \cot^{-1} x = -\dfrac{e^{\cot^{-1} x}}{1+x^2}$

18. (a) $D_x \cos^{-1} 3^x = -\dfrac{1}{\sqrt{1-(3^x)^2}} D_x 3^x = -\dfrac{3^x \ln 3}{\sqrt{1-3^{2x}}}$ (b) $D_x 3^{\sin^{-1} x} = 3^{\sin^{-1} x} \ln 3\, D_x \sin^{-1} x = \dfrac{3^{\sin^{-1} x} \ln 3}{\sqrt{1-x^2}}$

19. (a) $D_w \sinh^3 2w = 3 \sinh^2 2w(\cosh 2w)(2) = 6 \sinh^2 2w \cosh 2w$ (b) $D_w \cosh 2w^3 = \sinh 2w^3 D_w w^3 = 3w^2$

20. (a) $g(t) = \tanh(\ln t)$; (b) $G(t) = \ln(\coth t)$
▷ (a) $g'(t) = \operatorname{sech}^2(\ln t) D_t \ln t = \dfrac{\operatorname{sech}^2(\ln t)}{t}$ (b) $G'(t) = \dfrac{1}{\coth t} D_t \coth t = -\dfrac{\operatorname{csch}^2 t}{\coth t}$

21. (a) $D_x \operatorname{sech}(\tan x) = -\operatorname{sech}(\tan x)\tanh(\tan x) D_x \tan x = -\operatorname{sech}(\tan x)\tanh(\tan x)\sec^2 x$
(b) $D_x \tanh(\sec x) = \operatorname{sech}^2(\sec x) D_x \sec x = \operatorname{sech}^2(\sec x) \sec x \tan x$

22. (a) $D_x \sinh^{-1} x^2 = \dfrac{1}{\sqrt{(x^2)^2+1}} D_x x^2 = \dfrac{2x}{\sqrt{x^4+1}}$ (b) $g(x) = \tanh^{-1} 2x = \dfrac{1}{1-(2x)^2} D_x(2x) = \dfrac{2}{1-4x^2}$

23. $D_x \log_{10}\left(\dfrac{1+x}{1-x}\right)^2 = 2D_x[\log_{10}(1+x) - \log_{10}(1-x)] = \dfrac{2}{\ln 10}\left[\dfrac{1}{1+x} + \dfrac{1}{1-x}\right] = \dfrac{4}{\ln 10(1-x^2)}$

24. $g(x) = \ln\sqrt{\dfrac{2x+1}{x-3}}$
▷ First we replace the given expression by an equivalent sum, allowing $2x+1$ and $x-3$ to be both negative.

$$g(x) = \ln\left(\frac{2x+1}{x-3}\right)^{1/2} = \frac{1}{2}\ln\frac{2x+1}{x-3} = \frac{1}{2}\left[|2x+1| - \ln|x-3|\right]$$

Hence,

$$g'(x) = \tfrac{1}{2}\left[\frac{2}{2x+1} - \frac{1}{x-3}\right] = \tfrac{1}{2}\left[\frac{2(x-3)-(2x+1)}{(2x+1)(x-3)}\right] = \frac{-7}{2(2x+1)(x-3)}$$

25. $g(t) = (\sin t)^{2t}$; $\ln g = 2t \ln \sin t$; $\frac{g'}{g} = 2 \ln \sin t + 2t \cdot \frac{\cos t}{\sin t}$; $g'(t) = (2 \ln \sin t + 2t \cot t)(\sin t)^{2t}$

26. $f(t) = t^{3/\ln t}$. Because $t > 0$, then by Definition 5.4.2 $f(t) = e^{(3 \ln t)/\ln t} = e^3$; $f'(t) = 0$.

27. $D_x \cosh^{-1}\sqrt{x} = \frac{1}{\sqrt{x-1}} D_x\sqrt{x} = \frac{1}{\sqrt{x-1}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x^2-x}}$ if $x > 1$

28. $G(x) = \sec^{-1}\sqrt{x^2+1}$

▷ $$G'(x) = \frac{1}{\sqrt{x^2+1}\sqrt{(x^2+1)-1}} \frac{d}{dx}\sqrt{x^2+1} = \frac{1}{\sqrt{x^2+1}\sqrt{x^2}} \cdot \tfrac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{|x|(x^2+1)}$$

29. $D_x \cos^{-1}(\tanh 2x) = -\frac{1}{\sqrt{1-\tanh^2 2x}} D_x \tanh 2x = \frac{2 \operatorname{sech}^2 2x}{\sqrt{\operatorname{sech}^2 2x}} = 2 \operatorname{sech} 2x$

30. $D_x \coth^{-1}(\csc x) = \frac{1}{1-\csc^2 x} D_x \csc x = -\frac{1}{\cot^2 x}(-\csc x \cot x) = \sec x$

In Exercises 31 and 32, find dy/dx by logarithmic differentiation.

31. $y = x^3(x^2+1)^2(x-1)^4$; $\ln|y| = 3 \ln|x| + 2 \ln(x^2+1) + 4 \ln|x-1|$; $\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + \frac{2 \cdot 2x}{x^2+1} + \frac{4}{x-1}$
$= \frac{11x^3 - 7x^2 + 7x - 3}{x(x^2+1)(x-1)}$; $\frac{dy}{dx} = x^2(x^2+1)(x-1)^3(11x^3 - 7x^2 + 7x - 3)$

32. $y = \frac{\sqrt{4-x^2}}{\sqrt[3]{x^6+8}}$

▷ $$\ln y = \tfrac{1}{2}\ln(4-x^2) - \tfrac{1}{3}\ln(x^6+8)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2} \cdot \frac{-2x}{4-x^2} - \frac{1}{3} \cdot \frac{6x^5}{x^6+8} = \frac{2x^7 - x^6 - 8x^5 - 8x}{(4-x^2)(x^6+8)}$$
$$\frac{dy}{dx} = \frac{\sqrt{4-x^2}}{\sqrt[3]{x^6+8}} \cdot \frac{2x^7 - x^6 - 8x^5 - 8x}{(4-x^2)(x^6+8)} = \frac{2x^7 - x^6 - 8x^5 - 8x}{\sqrt{4-x^2}(x^6+8)^{4/3}}$$

In Exercises 33 and 34, compute a^x by using the definition and check by computing directly.

33. (a) $3^{\sqrt{2}} = e^{\sqrt{2}\ln 3} = e^{1.5537} = 4.7288$ (b) $2^\pi = e^{\pi \ln 2} = e^{2.1776} = 8.8250$

34. (a) $7^e = e^{e \ln 7} = e^{5.2895} = 198.25$ (b) $e^\pi = 23.141$

In Exercises 35–48, evaluate the indefinite integral.

35. $\int \frac{3e^{2x}}{1+e^{2x}}\, dx = \frac{3}{2}\int \frac{2e^{2x}dx}{1+e^{2x}} = \frac{3}{2}\ln(1+e^{2x}) + C$

36. $\int e^{x^2-2x}(x-1)dx$

▷ Let $u = x^2 - 2x$ and $du = 2x - 2 = 2(x-1)$. Then
$$\int e^{x^2-2x}(x-1)dx = \int e^u \cdot \tfrac{1}{2}du = \tfrac{1}{2}e^u + C = \tfrac{1}{2}e^{x^2-2x} + C$$

37. $\int (e^{3x} + 2^{3x})dx = \frac{1}{3}\int e^{3x}(3\, dx) + \frac{1}{3}\int 2^{3x}(3\, dx) = \frac{1}{3}e^{3x} + \frac{2^{3x}}{3 \ln 2} + C$

38. Let $u = \ln x^2 = 2 \ln x$, $du = \frac{2\, dx}{x}$. $\int \frac{10^{\ln x^2}}{x}dx = \int 10^u \cdot \tfrac{1}{2}du = \frac{10^u}{2 \ln 10} + C = \frac{10^{\ln x^2}}{2 \ln 10} + C$

39. Let $u = e^x$, $du = e^x dx$. $\int e^x 2^{e^x} dx = \int 2^u du = \frac{2^u}{\ln 2} + C = \frac{2^{e^x}}{\ln 2} + C$

40. $\int \frac{10^x+1}{10^x-1}dx$

▷ We multiply the numerator and denominator by $10^{-x/2}$. Thus,

$$\int \frac{10^x+1}{10^x-1}\, dx = \int \frac{10^{x/2}+10^{-x/2}}{10^{x/2}-10^{-x/2}}\, dx \tag{1}$$

Let $u = 10^{x/2} - 10^{-x/2}$. Then

$$du = [10^{x/2} \cdot \tfrac{1}{2} - 10^{-x/2}(-\tfrac{1}{2})]\ln 10\, dx$$

$$\frac{2\,du}{\ln 10} = (10^{x/2} + 10^{-x/2})\, dx$$

Substituting into the right-hand side of Eq. (1), we have

$$\int \frac{10^x+1}{10^x-1}\,dx = \frac{2}{\ln 10}\int \frac{du}{u} = \frac{2\ln|u|}{\ln 10} + C = \frac{2\ln|10^{x/2}-10^{-x/2}|}{\ln 10} + C = 2\log_{10}|10^{x/2}-10^{-x/2}| + C$$

$$= 2\log_{10}\left|\frac{10^{x/2}-10^{-x/2}}{1}\cdot\frac{10^{x/2}}{10^{x/2}}\right| + C = 2\log_{10}|10^x-1| - 2\log_{10} 10^{x/2} + C$$

$$= 2\log_{10}|10^x - 1| - x + C$$

41. Let $u = x^2$, $du = 2x\,dx$. $\int \frac{4x\,dx}{\sqrt{1-x^4}} = \int \frac{2\,du}{\sqrt{1-u^2}} = 2\sin^{-1}u + C = 2\sin^{-1}x^2 + C$

42. $\int \frac{dy}{9e^y + e^{-y}} = \int \frac{e^y dy}{(3e^y)^2+1} = \frac{1}{3}\tan^{-1}(3e^y) + C$

43. $\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+9} = \frac{1}{3}\tan^{-1}\frac{1}{3}(x+1) + C$

44. $\int \frac{dx}{\sqrt{5+4x-x^2}}$

▷ $\int \frac{dx}{\sqrt{5+4x-x^2}} = \int \frac{dx}{\sqrt{9-(x^2-4x+4)}} = \int \frac{dx}{\sqrt{9-(x-2)^2}} = \sin^{-1}\frac{1}{3}(x-2) + C$

45. Let $u = e^{-x}$, $du = -e^{-x}dx$.

$$\int \frac{dx}{\sqrt{e^{2x}-8}} = \frac{1}{\sqrt{8}}\int \frac{e^{-x}dx}{\sqrt{\frac{1}{8}-e^{-2x}}} = \frac{1}{\sqrt{8}}\int \frac{-du}{\sqrt{\frac{1}{8}-u^2}} = -\frac{1}{\sqrt{8}}\sin^{-1}\frac{u}{1/\sqrt{8}} + C = -\frac{1}{4}\sqrt{2}\sin^{-1}2\sqrt{2}e^{-x} + C$$

46. $\int x\coth\frac{1}{2}x^2\,dx = \int \frac{\cosh\frac{1}{2}x^2\,d(\frac{1}{2}x^2)}{\sinh\frac{1}{2}x^2} = \ln\left|\sinh\frac{1}{2}x^2\right| + C$

47. $\int \tanh^2 3w\,dw = \int (1-\operatorname{sech}^2 3w)dw = w - \frac{1}{3}\tanh 3w + C$

48. $\int \frac{\cosh t\,dt}{\sqrt{\sin t}}$

▷ Let $u = \sinh t$. Then $du = \cosh t\,dt$. Thus,

$$\int \frac{\cosh t\,dt}{\sqrt{\sinh t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}du = 2u^{1/2} + C = 2\sqrt{\sin t} + C$$

In Exercises 49–58, evaluate the definite integral. Check by NINT.

49. $\int_0^2 x^2e^{x^3}dx = \frac{1}{3}\int_0^2 e^{x^3}(3x^2dx) = \frac{1}{3}e^{x^3}\Big|_0^2 = \frac{1}{3}(e^8-1) \approx 993.3$

50. $\int_0^1 (e^{2x}+1)^2dx = \int_0^1 (e^{4x}+2e^{2x}+1)dx = \left[\frac{1}{4}e^{4x}+e^{2x}+x\right]_0^1 = \frac{1}{4}e^4 + e^2 - \frac{1}{4} \approx 20.79$

51. $\int_1^8 \frac{x^{1/3}}{x^{4/3}+4}\,dx = \frac{3}{4}\int_1^8 \frac{\frac{4}{3}x^{1/3}dx}{x^{4/3}+4} = \frac{3}{4}\ln\left|x^{4/3}+4\right|\Big|_1^8 = \frac{3}{4}(\ln 20 - \ln 5) = \frac{3}{4}\ln 4 = \frac{3}{4}\ln 2^2 = \frac{3}{2}\ln 2 \approx 1.040$

52. $\int_{1/3}^{1/2} \frac{4x^{-3}+2}{x^{-2}-x}\,dx$

▷ Let $u = x^{-2} - x$. Then $du = (-2x^{-3}-1)\,dx$ and $-2\,du = (4x^{-3}+2)\,dx$

When $x = \frac{1}{3}$, then $u = \frac{26}{3}$, when $x = \frac{1}{2}$, then $u = \frac{7}{2}$. Thus,

$$\int_{1/3}^{1/2} \frac{4x^{-3}+2}{x^{-2}-x}\,dx = -2\int_{26/3}^{7/2}\frac{du}{u} = -2\ln u\Big]_{26/3}^{7/2} = -2[\ln\tfrac{7}{2} - \ln\tfrac{26}{3}] = -2\ln\tfrac{21}{52} = 2\ln\tfrac{52}{21} \approx 1.813$$

53. Let $u = e^x$; $du = e^x\,dx$. Then $\displaystyle\int_0^{\ln 2} \frac{e^{2x}}{e^x - 5}\,dx = \int_0^{\ln 2} \frac{e^x \cdot e^x dx}{e^x - 5} = \int_1^2 \frac{u}{u-5}\,du = \int_1^2 \Big(1 + \frac{5}{u-5}\Big)du$

$\displaystyle = u + 5\ln|u-5|\Big]_1^2 = (2 + 5\ln 3) - (1 + 5\ln 4) = 1 + 5\ln\tfrac{3}{4} \approx -0.4384$

54. Let $u = \ln x$, $du = \frac{dx}{x}$. $\displaystyle\int_{x=e}^{e^2} \frac{dx}{x\ln x} = \int_{u=1}^{2} \frac{du}{u} = \ln u\Big]_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.6931$

55. $\displaystyle\int_1^2 \frac{(t+2)dt}{\sqrt{4t-t^2}} = \int_1^2 \frac{(t+2)dt}{\sqrt{4-(t^2-4t+4)}} = \int_1^2 \frac{(t+2)dt}{\sqrt{4-(t-2)^2}}$

Let $u = t-2$; then $du = dt$. When $t = 1$, $u = -1$; when $t = 2$, $u = 0$. We have

$$\int_{-1}^0 \frac{u+4}{\sqrt{4-u^2}}\,du = -\frac{1}{2}\int_{-1}^0 \frac{-2u}{\sqrt{4-u^2}} + 4\int_{-1}^0 \frac{du}{\sqrt{4-u^2}} = -\tfrac{1}{2}(2\sqrt{4-u^2}) + 4\sin^{-1}\tfrac{u}{2}\Big]_{-1}^0$$

$$= (-2+0) - (-\sqrt{3} - \tfrac{2}{3}\pi) = \tfrac{2}{3}\pi + \sqrt{3} - 2 \approx 1.826$$

56. $\displaystyle\int_{-1}^1 \frac{2x+6}{x^2+2x+5}\,dx$

▷ First, we complete the square in the denominator.

$$\int_{-1}^1 \frac{2x+6}{x^2+2x+5}\,dx = \int_{-1}^1 \frac{2(x+1)+4}{(x+1)^2+4}\,dx \qquad (1)$$

Next we let $u = x+1$ and $du = dx$. When $x = -1$, $u = 0$; when $x = 1$, $u = 2$. Substituting into (1), we obtain

$$\int_{-1}^1 \frac{2x+6}{x^2+2x+5}\,dx = \int_0^2 \frac{(2u+4)du}{u^2+4} = \int_0^2 \frac{2u\,du}{u^2+4} + \int_0^2 \frac{4du}{u^2+4} = \ln(u^2+4)\Big]_0^2 + 4\Big(\frac{1}{2}\Big)\tan^{-1}\Big(\frac{u}{2}\Big)\Big]_0^2$$

$$= (\ln 8 - \ln 4) + 2(\tan^{-1}1 - \tan^{-1}0) = \ln\frac{8}{4} + 2\left(\frac{\pi}{4}\right) = \ln 2 + \tfrac{1}{2}\pi \approx 2.264$$

57. $\displaystyle\int_0^1 \sqrt{\cosh^2 y - 1}\,dy = \int_0^1 \sqrt{\sinh^2 y}\,dy = \int_0^1 \sinh y\,dy = \cosh y\Big]_0^1 = \cosh 1 - \cosh 0 = \cosh 1 - 1 \approx 0.5431$

58. $\displaystyle\int_0^2 \operatorname{sech}^2 \tfrac{1}{2}x\,dx = 2\tanh\tfrac{1}{2}x\Big]_0^2 = 2\tanh 1 \approx 1.523$

59. $ye^x + xe^y + x + y = 0$; $\Big(ye^x + e^x\frac{dy}{dx}\Big) + \Big(xe^y\frac{dy}{dx} + e^y\Big) + 1 + \frac{dy}{dx} = 0$; $(e^x + xe^y + 1)\frac{dy}{dx} = -ye^x - e^y - 1$

$\displaystyle\frac{dy}{dx} = -\frac{ye^x + e^y + 1}{e^x + xe^y + 1}$

60. Show that $\cosh(\ln x) = \dfrac{x^2+1}{2x}$.

▷ Because $\cosh x = \dfrac{e^x + e^{-x}}{2}$ then $\cosh(\ln x) = \dfrac{e^{\ln x} + e^{-\ln x}}{2} = \dfrac{x + \frac{1}{x}}{2} = \dfrac{x^2+1}{2x}$

61. (a) From formula 5.9.8(2), $\cosh^{-1}2 = \ln(2 + \sqrt{2^2-1}) = \ln(2+\sqrt{3})$

(b) Because $\tanh^{-1}x = \frac{1}{2}\ln\dfrac{1+x}{1-x}$ if $|x| < 1$ then $\tanh^{-1}\frac{1}{4} = \frac{1}{2}\ln\dfrac{1+\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{2}\ln\frac{5}{3}$

62. (a) $\displaystyle\lim_{x\to+\infty}\coth x = \lim_{x\to+\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to+\infty}\frac{(e^x+e^{-x})e^{-x}}{(e^x-e^{-x})e^{-x}} = \lim_{x\to+\infty}\frac{1+e^{-2x}}{1-e^{-2x}} = \frac{1 + \lim_{x\to+\infty} e^{-2x}}{1 - \lim_{x\to+\infty} e^{-2x}} = \frac{1+0}{1-0} = 1$

(b) $\displaystyle\lim_{x\to+\infty}\operatorname{csch} x = \lim_{x\to+\infty}\frac{2}{e^x-e^{-x}} = \lim_{x\to+\infty}\frac{2e^{-x}}{1-e^{-2x}} = \frac{2\cdot 0}{1-0} = 0$

63. Find an equation of the tangent line to $y = x^{x-1}$ at $(2,2)$. Plot the curve and the line.

▷ $\ln y = (x-1)\ln x$, $\frac{y'}{y} = \ln x + (x-1)\frac{1}{x}$; $y' = (\ln x + 1 - \frac{1}{x})y$. $y'(2) = (\ln 2 + \frac{1}{2})2 = 2\ln 2 + 1$.

$y = (2\ln 2 + 1)(x-2) + 2 = (2\ln 2 + 1)x - 4\ln 2$. Because $y'(x) < 0$ if $0 < x < 1$ and $y'(x) > 0$ if $x > 1$, then $x = 1$ is an absolute minimum value. $y'' = (\frac{1}{x} + \frac{1}{x^2})y + (\ln x + 1 - \frac{1}{x})^2 y > 0$; the graph is concave upward.

64. Use differentials to find an approximate value to five decimal places of $\log_{10}100937$. Use the fact that $\log_{10}e = 0.434$ to three decimal places. Check using your calculator.

▸ Let $y = f(x) = \log_{10}x$ so $dy = \frac{\log_{10}e}{x}\,dx$. With $x = 100{,}000$ and $\Delta x = dx = 937$

$\log_{10}100937 = f(100{,}000 + \Delta x) \approx f(100{,}000) + dy = \log_{10}100{,}000 + \frac{\log_{10}e}{x}dx = 5 + \frac{0.434}{100{,}000}(937) = 5.00407.$

In fact, $\log_{10}100{,}937 = 5.00405$.

65. A particle is moving on a line. At t sec, s ft is its directed distance from the origin, v ft/sec is its velocity, and a ft/sec^2 is its acceleration.

$a = \frac{dv}{dt} = e^t + e^{-t}$; $v = \int (e^t + e^{-t})dt = e^t - e^{-t} + C_1$

Because $v = 1$ when $t = 0$, then $1 = 1 - 1 + C_1$; $C_1 = 1$. Therefore

$v = \frac{ds}{dt} = e^t - e^{-t} + 1$; $s = \int (e^t - e^{-t} + 1)dt = e^t + e^{-t} + t + C_2$

Because $s = 2$ when $t = 0$, then $2 = 1 + 1 + 0 + C_2$; $C_2 = 0$. Therefore $s = e^t + e^{-t} + t$.

66. The area of the region bounded by the curve $y = e^{-x}$, the coordinate axes, and the line $x = b$ $(b > 0)$ is a function of b. If f is this function, find $f(b)$. Also find $\lim_{b\to+\infty} f(b)$.

▸ The region is bounded above by the curve $y = e^{-x}$, bounded below by the line $y = 0$, bounded on the left by the line $x = 0$, and bounded on the right by the line $x = b$. Thus, $f(b) = \int_0^b e^{-x}\,dx = -e^{-x}\Big]_0^b = -e^{-b} + 1.$

Furthermore, $\lim\limits_{b\to+\infty} f(b) = \lim\limits_{b\to+\infty}[-e^{-b} + 1] = \lim\limits_{b\to-\infty}[-e^{b} + 1] = 0 + 1 = 1.$

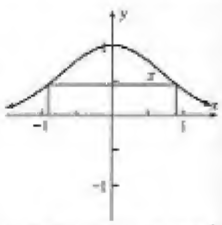
67. An element of volume is a circular disk centered on the x axis, $x \in [0,b]$, of radius e^{-w_i}. If $g(b)$ cubic units is the volume of the solid of revolution, then

$g(b) = \lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} \pi\left(e^{-w_i}\right)^2 \Delta_i x = \pi\int_0^b e^{-2x}dx = -\tfrac{1}{2}\pi\left[e^{-2x}\right]_0^b = -\tfrac{1}{2}\pi[e^{-2b} - e^0] = \tfrac{1}{2}\pi(1 - e^{-2b})$

$\lim\limits_{b\to+\infty} g(b) = \lim\limits_{b\to+\infty} \tfrac{1}{2}\pi(1 - e^{-2b}) = \tfrac{1}{2}\pi$

68. Prove that if a rectangle is to have its base on the x axis and two of its vertices on the curve $y = e^{-x^2}$, then the rectangle will have the largest possible area if the two vertices are at the points of inflection of the graph.

▸ The figure shows the curve $y = e^{-x^2}$ and a rectangle with base on the x axis and two vertices on the curve. Let $f(x) = e^{-x^2}$. Then the first-quadrant vertex that is on the curve is at the point $(x, f(x))$. Hence, the length of the rectangle is $2x$, and the width is $f(x)$. If A square units is the area of the rectangle, then



$$A(x) = 2xf(x) = 2xe^{-x^2} \quad \text{if } x > 0$$

We want to find the absolute maximum value of A. Hence we differentiate A.

$$A'(x) = 2xe^{-x^2}(-2x) + e^{-x^2}\cdot 2 = \frac{2(1 - 2x^2)}{e^{x^2}}$$

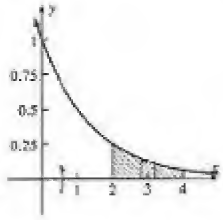
Because $A'(x) > 0$ if $0 < x < \frac{1}{2}\sqrt{2}$ and $A'(x) < 0$ if $x > \frac{1}{2}\sqrt{2}$, A has an absolute maximum value at $x = \frac{1}{2}\sqrt{2}$. Thus, the first quadrant vertex of the rectangle is at the point where $x = \frac{1}{2}\sqrt{2}$. By symmetry, the second quadrant vertex is at the point where $x = -\frac{1}{2}\sqrt{2}$. Furthermore, in Exercise 60 of Exercises 5.4 we showed that the curve $y = e^{-x^2}$ has a point of inflection at the points where $x = \pm\frac{1}{2}\sqrt{2}$. Hence, the rectangle will have the largest possible area if the vertices are at the points of inflection of the curve.

69. $f(x) = \ln|x|$; $x < 0$. $x < 0 \Rightarrow |x|$ is decreasing $\Rightarrow f$ is decreasing $\Rightarrow f$ is one-to-one $\Rightarrow f$ has an inverse.
Let $y = f(x)$. Then $y = \ln|x|$; $|x| = e^y$; $-x = e^y$; $x = -e^y$.
If $g = f^{-1}$, then $g(y) = -e^y$ and so $g(x) = -e^x$. The domain of g is the range of f which is $(-\infty, +\infty)$.

70. Let $f(x) = x - 1 - \ln x$, $f(1) = 0$. By the mean-value theorem there is a number c between x and 1 such that $f(x) = f(x) - f(1) = (x-1)f'(c) = (x-1)(1 - \frac{1}{c})$. If $0 < x < 1$ then $0 < c < 1$ and $x - 1 < 0$ and $1 - \frac{1}{c} < 0$; if $x > 1$ then $c > 1$, $x - 1 > 0$ and $1 - \frac{1}{c} > 0$. In either case, the product is positive. Hence $x - 1 > \ln x$ if $x \neq 1$.

71. The pressure is p lb/in^2 when the volume is v in^3, and the initial pressure and volume are p_0 lb/in^2 and $v_0{}^2$ in^3. We are given $\frac{dp}{dv} = -k\frac{p}{v}$; $\frac{dp}{p} + k\frac{dv}{v} = 0$; $\ln|p| + k\ln|v| = C$; $pv^k = C$. When $v = v_0$, $p = p_0$; so $C = p_0v_0{}^k$. Therefore $pv^k = p_0v_0{}^k$.

72. Find the volume of the solid of revolution generated if the region bounded by the curve $y = 2^{-x}$ and the lines $x = 1$ and $x = 4$ is revolved about the x axis.



▸ The figure shows a sketch of the region and an element of area. The element of volume is a circular disk with thickness $\Delta_i x$ units and radius $f(w_i)$ units, where $f(x) = 2^{-x}$. The volume of the solid of revolution is given by

$$V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\pi[f(w_i)]^2\Delta_i x = \int_1^4 \pi(2^{-x})^2 dx = \int_1^4 \pi 4^{-x} dx$$
$$= \frac{-\pi}{\ln 4}4^{-x}\Big]_1^4 = \frac{63\pi}{256(2\ln 2)}$$

• The volume of revolution is $63\pi/(512\ln 2)$ cubic units.

73. The half-life of cesium 137 is 30 years and 65 mg are present.
$y = y_0(\frac{1}{2})^{t/h} = 65(\frac{1}{2})^{t/30}$. $10 = 65(\frac{1}{2})^{x/30}$; $6.5 = 2^{x/30}$; $\ln 6.5 = (x/30)\ln 2$; $x = 30\ln 6.5 \div \ln 2 = 81.1$ years

74. The doubling time of a population is 60 years. The population is now 120,000. $y = y_0(2)^{t/d} = 120{,}000(2)^{t/60}$.

$150{,}000 = 120{,}000(2)^{x/60}$; $1.25 = (2)^{x/60}$; $\ln 1.25 = (x/60)\ln 2$; $x = 60\ln 1.25 \div \ln 2 = 19.3$ years.

$y(20) = 120{,}000(2)^{1/3} = 151{,}190$

75. y is the number of verbs memorized after t min, $y \le 50$. We have a table of initial conditions

t	0	30	T
y	0	20	49

$\frac{dy}{50-y} = k\,dt$; $\int\frac{dy}{50-y} = k\int dt$; $-\ln(50-y) = kt + \ln C$; $50 - y = Ce^{-kt}$; $y = 50 - Ce^{kt}$
With $y = 0$ and $t = 0$ we have $0 = 50 - C$; $C = 50$. Therefore (a) $y = 50 - 50e^{-kt}$.
With $y = 20$ and $t = 30$ we have $20 = 50 - 50e^{-30k}$; $50e^{-30k} = 30$; $e^{-30k} = 0.6$; $e^{-k} = (0.6)^{1/30}$
Therefore $y = 50 - 50(0.6)^{t/30}$. (c) When $t = 60$, $y = 50 - 50(0.6)^2 = 32$. (d) With $y = 49$ and $t = T$ we have

$49 = 50 - 50(0.6)^{T/30}$; $(0.6)^{T/30} = 0.02$; $\frac{T}{30}\ln 0.6 = \ln 0.02$; $T = \frac{30\ln 0.02}{\ln 0.6} = 229.7$

After 229.7 min $\approx$ 3.83 hr, the student will have only one verb left to memorize.

76. Interest on a savings account is computed at 10% per year compounded continuously. If one wishes to have \$1000 in the account at the end of a year by making a single deposit now, what is the amount of the deposit?

▸ The law of exponential growth applies. If P dollars is invested at the rate of 10% per year with interest compounded continuously, then the amount after t years is A dollars, where $A = Pe^{0.10t}$. We find P if $A = 1000$ when $t = 1$. Thus,

$$1000 = Pe^{0.10};\quad P = 1000e^{-0.10} = 904.84$$

• The deposit should be \$904.84

77. Let A dollars be the amount of the investment after t years if P dollars is the original amount and interest is 100i percent, compounded continuously. Then $A = Pe^{it}$. With $A = 2P$, $i = 0.08$, and $t = T$ we have

$2P = Pe^{0.08T}$; $e^{0.08T} = 2$; $0.08T = \ln 2$; $T = \frac{\ln 2}{0.08} = 8.66$. Therefore the investment will double in 8.66 years.

78. After t min there are y coulombs. $y_0 = 8$ and $\frac{3}{4}$ of the charge remains after 15 min. Thus $y = 8(\frac{3}{4})^{t/15}$.

2 coulombs remain after x min. $2 = 8(\frac{3}{4})^{x/15}$; $\frac{1}{4} = (\frac{3}{4})^{x/15}$; $\ln .25 = (x/15)\ln .75$; $x = 15\ln .25 \div \ln .75 = 83.9$

79. Bacteria are doubling every 20 min. Hence in 1 hour they have doubled 3 time, that they have multiplied by 8. The original number is $150{,}000 \div 8 = 187{,}500$.

80. A paleontologist discovered an insect preserved inside a transparent amber, and the amount of ^{14}C present in the insect was determined to be 2% of its original amount. Use the fact that the half-life of ^{14}C is 5730 years to determine the age of the insect at the time of its discovery.

▸ y percent of ^{14}C is present after t years. The half-life formula yields $\quad y = y_0(\frac{1}{2})^{t/h} = 100(\frac{1}{2})^{t/5730}$
With $y = 2$ and $t = x$ we have

$$2 = 100(\tfrac{1}{2})^{x/5730};\quad 2^{x/5730} = 50;\quad \frac{x}{5730}\ln 2 = \ln 50;\quad x = 5730\ln 50 \div \ln 2 = 32{,}339$$

• The insect is about 32,000 years old.

81. Let x units be the amount of the substance remaining t years after it started to disappear. The half-life of the substance is 1900 years. $x = x_0(\frac{1}{2})^{t/1900}$. With $x = 0.05x_0$ and $t = T$ we have

$0.05x_0 = x_0(0.5)^{T/1900}$; $\ln 0.05 = \frac{T}{1900}\ln 0.5$; $T = \frac{1900 \ln 0.05}{\ln 0.5} \approx 8212$

- Therefore it will take 8212 years for 95% of the deposit to disappear.

82. A population grows at x percent per year and doubles in 25 years.

$(1 + \frac{x}{100})^{25} = 2$, $1 + \frac{x}{100} = 2^{.04}$, $x = 100(2^{.04} - 1) = 2.81$

83. Let x lb be the amount of salt in the tank t minutes after the flow begins.

We have a table of initial conditions. The tank is gaining 6 lb salt per minute and is losing $2(\frac{x}{60})$lb salt per minute. Hence

t	0	T
x	120	135

$\frac{dx}{dt} = 6 - \frac{x}{30} = \frac{180 - x}{30}$; $\frac{dx}{180 - x} = \frac{dt}{30}$; $\int \frac{dx}{180 - x} = \frac{1}{30}\int dt$; $-\ln|180 - x| + \ln C = \frac{t}{30}$; $180 - x = Ce^{-t/30}$

With $x = 120$, $t = 0$, we have $60 = C$. Thus $180 - x = 60e^{-t/30}$. With $x = 135$, $t = T$, we have

$45 = 60e^{-T/30}$; $0.75 = e^{-T/30}$; $\ln 0.75 = -\frac{T}{30}$; $T = 30 \cdot \ln 0.75 = 8.63$

- Therefore there will be 135 lb of salt in the tank 8.63 minutes after the flow begins.

84. A tank contains 100 liters of fresh water, and brine containing 2 kg of salt per liter flows into the tank at the rate of 3 liters/min. If the mixture, kept uniform by stirring, flows out at the same rate, how many kilograms of salt are there in the tank at the end of 30 min?

▸ Let y kg be the amount of salt in the tank after t min. Brine containing 2 kg/liter flows into the tank at the rate of 3 liters/min. Thus, salt is entering the tank at the rate of 6 kg/min. Because the capacity of the tank is 100 liters, after t min the tank contains $y/100$ kg of salt per liter. And because the mixture flows out of the tank at the rate of 3 liters/min, salt is leaving the tank at the rate of $3y/100$ kg/min. Therefore, $\frac{dy}{dt} = 6 - \frac{3y}{100}$

We solve the differential equation with the initial condition $y = 0$ when $t = 0$. Separating the variables, we get

$$\frac{dy}{200 - y} = 0.03\,dt;\quad \int \frac{dy}{200 - y} = \int 0.03\,dt;\quad -\ln|200 - y| = 0.03t + k;\quad 200 - y = C\,e^{-0.03t}$$

Because $y = 0$ when $t = 0$, then $C = 200$ and

$$200 - y = 200\,e^{-0.03t};\qquad y = 200 - 200\,e^{-0.03t}$$

We find y when $t = 30$.

$$y = 200 - 200\,e^{(-0.03)(30)} = 118.7$$

- There are 118.7 kg of salt in the tank after 30 min.

85. Let y degrees be the temperature, and $Y = y - 40$ degrees the temperature difference t minutes ago. We have the boundary conditions:

t	30	10	0
y	150	90	y

From Newton's law of cooling, $\frac{dY}{dt} = kY$, $Y = Y_0e^{kt}$. If $t = 10$ and $Y = 90 - 40 = 50$ we get $50 = Y_0e^{10k}$; $e^{10k} = 50Y_0^{-1}$.

When $t = 30$ and $Y = 150 - 40 = 110$ we get $110 = Y_0e^{30k} = Y_0(50Y_0^{-1})^3 = 125{,}000Y_0^{-2}$. Therefore $Y_0 = \sqrt{125{,}000/110} \approx 33.7$ and $y_0 = 73.7$. The current temperature is 73.7.

86. $f(x) = e^x$; $f'(x) = e^x$. Let (c, e^c) be the point on the graph of f for which the tangent line there passes through the origin. The slope of the line through (c, e^c) and $(0,0)$ is $\frac{e^c}{c}$. The line also has a slope of $f'(c) = e^c$. Hence $\frac{e^c}{c} = e^c$; $c = 1$. Thus the point is $(1, e)$.

87. $E = 20\cos 120\pi t$; $\frac{E}{20} = \cos 120\pi t$. (a) Let k and n be any integers. Then

$120\pi t = 2\pi k + \cos^{-1}\left(\frac{E}{20}\right)$ or $120\pi t = 2\pi n - \cos^{-1}\left(\frac{E}{20}\right)$; $t = \frac{k}{60} + \frac{1}{120\pi}\cos^{-1}\left(\frac{E}{20}\right)$ or $t = \frac{n}{60} - \frac{1}{120\pi}\cos^{-1}\left(\frac{E}{20}\right)$

The smallest positive value of t is $\frac{1}{120\pi}\cos^{-1}\left(\frac{E}{20}\right)$.

(b) When $E = 10$, $t = \frac{1}{120\pi}\cos^{-1}\frac{1}{2} = \frac{1}{120\pi}\cdot\frac{\pi}{3} = \frac{1}{360}$. (c) When $E = 5$, $t = \frac{1}{120\pi}\cos^{-1}\frac{1}{4} \approx 0.0035$.

(d) When $E = -10$, $t = \frac{1}{120\pi}\cos^{-1}\left(-\frac{1}{2}\right) = \frac{1}{120\pi}\cdot\frac{2\pi}{3} = \frac{1}{180}$. (e) When $E = -5$, $t = \frac{1}{120\pi}\cos^{-1}\left(-\frac{1}{4}\right) \approx 0.0048$.

88. A weight is suspended from a spring and vibrating vertically according to the equation $y = 4 \sin 2\pi(t + \frac{1}{6})$, where y centimeters is the directed distance of the weight from its central position t seconds after the start of the motion, and the positive direction is upward. (a) Solve the equation for t. Use the equation in part (a) to determine the smallest positive value of t for which the displacement of the weight above its central position is (b) 2 cm and (c) 3 cm.

▸ (a) We have $\frac{1}{4}y = \sin 2\pi(t + \frac{1}{6})$

Let k and n be any integers. Then

$$2\pi(t + \tfrac{1}{6}) = 2k\pi + \sin^{-1}\tfrac{1}{4}y \qquad \text{or} \qquad 2\pi(t + \tfrac{1}{6}) = (2n+1)\pi - \sin^{-1}\tfrac{1}{4}y$$
$$t = k - \tfrac{1}{6} + \frac{1}{2\pi}\sin^{-1}\tfrac{1}{4}y \qquad \text{or} \qquad t = n + \tfrac{1}{3} - \frac{1}{2\pi}\sin^{-1}\tfrac{1}{4}y$$

(b) With $y = 2$, $\frac{1}{2\pi}\sin^{-1}\frac{1}{4}y = \frac{1}{2\pi}\sin^{-1}\frac{1}{2} = \frac{1}{12}$. We try values of k and n.

$k = 0$: $t = -\frac{1}{6} + \frac{1}{12} = -\frac{1}{12}$ $\qquad$ $k = 1$: $1 - \frac{1}{6} + \frac{1}{12} = \frac{11}{12}$ $\qquad$ $n = 0$: $\frac{1}{3} - \frac{1}{12} = \frac{1}{4}$

▪ The smallest positive value of t is $\frac{1}{4}$.

(c) With $y = 3$, $\frac{1}{2\pi}\sin^{-1}\frac{1}{4}y = \frac{1}{2\pi}\sin^{-1}\frac{3}{4} \approx 0.135$. We try values of k and n.

$k = 0$: $t = -\frac{1}{6} + 0.135 = -0.032$ $\qquad$ $k = 1$: $1 - \frac{1}{6} + 0.135 = 0.968$ $\qquad$ $n = 0$: $\frac{1}{3} - 0.135 = 0.198$

▪ The smallest positive value of t is 0.198

89. A square units is the area of the region bounded by $y = 9/\sqrt{9 - x^2}$, the axes, and $x = 2\sqrt{2}$.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{9}{\sqrt{9 - w_i^2}}\Delta_i x = 9\int_0^{2\sqrt{2}} \frac{1}{\sqrt{9 - x^2}}\,dx = 9\left[\sin^{-1}\frac{x}{3}\right]_0^{2\sqrt{2}} = 9\sin^{-1}\frac{2\sqrt{2}}{3}$$

90. The region is bounded by the curve $y = \sqrt{\sinh x}$, the x axis, and the lines $x = 2$ and $x = \ln 2$. An element of volume is a circular disk centered on the x axis, $x \in [0, \ln 2]$, of radius $\sqrt{\sinh w_i}$. If V cubic units is the volume of the solid of revolution, then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\sqrt{\sinh w_i})^2 \Delta_i x = \int_0^{\ln 2} \pi \sinh x\,dx = \pi\Big[\cosh x\Big]_0^{\ln 2} = \pi[\cosh(\ln 2) - \cosh 0]$$
$$= \pi\left[\tfrac{1}{2}(e^{\ln 2} + e^{-\ln 2}) - 1\right] = \pi[\tfrac{1}{2}(2 + \tfrac{1}{2}) - 1] = \tfrac{1}{4}\pi$$

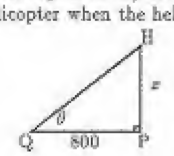
91. At t hours, let x km be the distance from the automobile to the point on the road nearest the searchlight, and let θ be the radian measure of the angle between the light beam and the shortest ray to the road. The car is traveling at 60 km/hr; thus $\frac{dx}{dt} = 60$. Hence

$\tan\theta = 2x$ and $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$; $\theta = \tan^{-1}2x$; $\dfrac{d\theta}{dt} = \dfrac{2}{1 + 4x^2}\cdot\dfrac{dx}{dt} = \dfrac{120}{1 + 4x^2}$

(a) When $x = 0$, $\frac{d\theta}{dt} = 120$. Hence the light beam is changing direction at 120 rad/hr.

(b) When $x = \frac{1}{2}$, $\frac{d\theta}{dt} = \frac{120}{1 + 1} = 60$. Hence the light beam is changing direction at 60 rad/hr.

92. A helicopter leaves the ground at a point 800 ft from an observer and rises vertically at 25 ft/sec. Find the time rate of change of the measure of the observer's angle of elevation of the helicopter when the helicopter is 600 ft above the ground.



▸ Refer to the figure. The observer is at point Q, the helicopter leaves the ground at point P, and after t sec the helicopter is at point H, which is x ft from point P. Thus, x ft is the distance that the helicopter rises in t sec, and we are given that $\quad \dfrac{dx}{dt} = 25$

Let θ be the radian measure of the observer's angle of elevation after t sec.

We want to find $\dfrac{d\theta}{dt}$ when $x = 600$. Because $\tan\theta = \dfrac{x}{800}$, then $\quad \theta = \tan^{-1}\dfrac{x}{800}$

We differentiate on both sides with respect to t. Thus, $\quad \dfrac{d\theta}{dt} = \dfrac{1}{1 + \left(\frac{x}{800}\right)^2}\left(\dfrac{1}{800}\right)\dfrac{dx}{dt}$

Because $dx/dt = 25$, we have $\qquad \dfrac{d\theta}{dt} = \dfrac{\frac{1}{32}}{1 + \left(\frac{x}{800}\right)^2}$

When $x = 600$, we obtain $\qquad \dfrac{d\theta}{dt}\Big]_{x=600} = \dfrac{\frac{1}{32}}{1 + (\frac{3}{4})^2} = \dfrac{1}{50}$

- The observer's angle of elevation is increasing at the rate of $\frac{1}{50}$ radians per second.

93. At t hours, let x miles be the distance from the observer to the point on the ground directly under the airplane, and let θ be the radian measure of the angle of elevation of the airplane at the observer. The airplane is flying at 300 mi/hr; so $dx/dt = 300$. Hence

$$\cot\theta = \frac{x}{4};\ \theta = \cot^{-1}\frac{x}{4};\ \frac{d\theta}{dt} = -\frac{\frac{1}{4}}{1+\frac{1}{16}x^2}\cdot\frac{dx}{dt} = -\frac{4}{x^2+16}\cdot 300 = -\frac{1200}{x^2+16};\ \frac{d\theta}{dt}\bigg]_{x=2} = -60$$

Hence the measure of the angle of elevation is changing at the rate of 60 rad/hr.

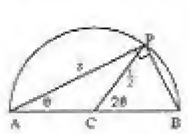
94. When the observer is x ft from the wall, the angle subtended by the picture is $\theta = \cot^{-1}\frac{x}{12} - \cot^{-1}\frac{x}{7}$.

$$\frac{d\theta}{dt} = \left(-\frac{\frac{1}{12}}{1+\frac{1}{144}x^2} + \frac{\frac{1}{7}}{1+\frac{1}{49}x^2}\right)\frac{dx}{dt} = \left(\frac{7}{49+x^2} - \frac{12}{144+x^2}\right)3.\ \theta'(10) = \left(\frac{7}{149} - \frac{12}{244}\right)3 = -\frac{60}{9089} \approx -0.0066 \text{ rad/sec}$$

In Exercises 95 and 96, two points A and B are diametrically opposite each other on the shores of a circular lake 1 km in diameter. Find the least amount of time it takes a man to go from point A to point B if:

95. He can row at the rate of $1\frac{1}{2}$ km/hr and walk at the rate of 5 km/hr.

96. He can row at the rate of 2 km/hr and walk at the rate of 3 km/hr.



▸ See the figure. Suppose the man rows from A to a point P on the circumference and then walks from P to B. Let s km be the distance from A to P and let θ be the radian measure of $\angle$PAB. Then $\theta = \cos^{-1}$s. Arc PB has length $\frac{1}{2}$m($\angle$PCB) $= \frac{1}{2}\cdot 2\theta = \cos^{-1}$s. Let T hr be the total time for the trip.

95. Because the man rows at $\frac{3}{2}$ km/hr and walks at 5 km/hr,

$$\mathrm{T} = \frac{2}{3}s + \frac{1}{5}\cos^{-1}s,\ s\in[0,1];\ \frac{d\mathrm{T}}{ds} = \frac{2}{3} - \frac{1}{5\sqrt{1-s^2}};\ \frac{d^2\mathrm{T}}{ds^2} = \frac{-s}{5(1-s^2)^{3/2}}$$

Because $d^2\mathrm{T}/ds^2 < 0$, any critical number is a relative maximum. Hence the absolute minimum value occurs at an endpoint. Because $\mathrm{T}(0) = \frac{1}{5}\cdot\frac{\pi}{2} = \frac{\pi}{10} < \mathrm{T}(1) = \frac{2}{3}$, the man should walk all the way and the time is $\frac{\pi}{10}$ hr.

96. Because the man rows at 2 km/hr and walks at 3 km/hr

$$\mathrm{T} = \frac{1}{2}s + \frac{1}{3}\cos^{-1}s,\ s\in[0,1];\ \frac{d\mathrm{T}}{ds} = \frac{1}{2} - \frac{1}{3\sqrt{1-s^2}};\ \frac{d^2\mathrm{T}}{ds^2} = \frac{-s}{3(1-s^2)^{3/2}}$$

Because $d^2\mathrm{T}/ds^2 < 0$, any critical number is a relative maximum. Hence the absolute minimum value occurs at an endpoint. Because $\mathrm{T}(0) = \frac{1}{3}\cdot\frac{\pi}{2} = \frac{\pi}{6} > \mathrm{T}(1) = \frac{1}{2}$, the man should row all the way and the time is $\frac{1}{2}$ hr.

97. From the definition of a derivative

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \lim_{x\to 0}\frac{\log_a(1+x) - \log_a 1}{x-0} = \frac{d}{dx}\log_a(1+x)\Big|_{x=0} = \frac{\log_a e}{1+x}\Big|_{x=0} = \log_a e = \frac{1}{\ln a}$$

98. Let $y = a^x - 1$; $a^x = y + 1$; $x = \log_a(y+1)$; $\lim_{y\to 0} x = \log_a 1 = 0$. Thus $\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\frac{y}{\log_a(y+1)} \overset{\text{Ex.97}}{=} \ln a$

99. From the definition of the derivative, $\lim_{x\to 1}\frac{x^b-1}{x-1} = \frac{d}{dx}x^b\Big|_{x=1} = bx^{b-1}\Big|_{x=1} = b$

100. Prove by using the definition of a derivative that $\lim_{x\to 0}\frac{e^{ax}-1}{x} = a$.

▸ Let $f(x) = e^{ax}$. Then, by definition

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{e^{ax}-1}{x} \tag{1}$$

Furthermore, because $f'(x) = ae^{ax}$

we have $$f'(0) = a\cdot e^0 = a \tag{2}$$

From (1) and (2) we conclude that $\lim_{x\to 0}\frac{e^{ax}-1}{x} = a$

101. Let $g(x) = f(x)e^{-cx}$. Then $g'(x) = f'(x)e^{-cx} + f(x)(-ce^{-cx})$. Because $f'(x) = cf(x) < g'(x) = cf(x)e^{-cx} - cf(x)e^{-cx} = 0$. Therefore, $g(x) = k$ where k is a constant. Hence $f(x)e^{-cx} = k$; $f(x) = ke^{cx}$.

102. Let $P(n)$ be the proposition: $D_x{}^n(\ln x) = (-1)^{n-1}(n-1)!x^{-n}$. $P(1)$: $D_x(\ln x) = (-1)^0 0! x^{-1} = \frac{1}{x}$ is true by Theorem 5.2.2. Suppose P(k) for some positive integer k. Then

$$D_x{}^{k+1}(\ln x) = D_x[D_x{}^k(\ln x)] = D_x[(-1)^{k-1}(k-1)!x^{-k}] = (-1)^{k-1}(k-1)!(-k)x^{-k-1} = (-1)^k k! x^{-(k+1)}$$

which is $P(k+1)$. Hence by mathematical induction, $P(n)$ is true for every positive integer n.

103. If $t \geq 0$ then $\int_0^t e^{-|x|}dx = \int_0^t e^{-x}dx = -e^{-x}\Big|_0^t = -e^{-t} + e^0 = 1 - e^{-|t|}$.

If $t < 0$ then $\int_0^t e^{-|x|}dx = \int_0^t e^{x}dx = e^x\Big|_0^t = e^t - e^0 = -(1 - e^{-|t|})$.

We can combine these two cases and write $\int_0^t e^{-|x|}dx = \operatorname{sgn} t(1 - e^{-|t|})$.

104. Prove that if $x > 0$, and $\int_1^x t^{h-1}dt = 1$, then $\lim\limits_{h \to 0} x = \lim\limits_{h \to 0}(1+h)^{1/h}$.

▶ If $h \neq 0$, then

$$\int_1^x t^{h-1}dt = \frac{t^h}{h}\bigg]_{t=1}^x = \frac{x^h}{h} - \frac{1}{h} \tag{1}$$

By the given equation $\int_1^x t^{h-1}dt = 1$ and Eq. (1), we conclude that if $h \neq 0$, and $1 + h > 0$, then

$$\frac{x^h}{h} - \frac{1}{h} = 1; \qquad x^h = 1 + h; \qquad x = (1+h)^{1/h}$$

Thus, $\quad \lim\limits_{h \to 0} x = \lim\limits_{h \to 0}(1+h)^{1/h}$

105. A tractrix has the equation $x = a\sinh^{-1}\sqrt{\frac{a^2}{y^2} - 1} - \sqrt{a^2 - y^2}$, $a > 0$. Differentiating implicitly,

$$1 = \frac{a}{\sqrt{\left(\frac{a^2}{y^2} - 1\right) + 1}} \cdot \frac{d}{dx}\sqrt{\frac{a^2}{y^2} - 1} - \frac{d}{dx}\sqrt{a^2 - y^2} = |y| \cdot \tfrac{1}{2}\left(\frac{a^2}{y^2} - 1\right)^{-1/2}\left(\frac{-2a^2}{y^3}\right)\frac{dy}{dx} - \tfrac{1}{2}(a^2 - y^2)^{-1/2}(-2y)\frac{dy}{dx}$$

$$= |y| \cdot \frac{-a^2|y|}{y^3\sqrt{a^2 - y^2}}\frac{dy}{dx} + \frac{y}{\sqrt{a^2 - y^2}}\frac{dy}{dx} = \frac{-a^2}{y\sqrt{a^2 - y^2}}\frac{dy}{dx} + \frac{y^2}{y\sqrt{a^2 - y^2}}\frac{dy}{dx} = \frac{-(a^2 - y^2)}{y\sqrt{a^2 - y^2}} = -\frac{\sqrt{a^2 - y^2}}{y}\frac{dy}{dx}$$

Thus, the slope of the tractrix at any point (x,y) is $\frac{dy}{dx} = \frac{-y}{\sqrt{a^2 - y^2}}$.

106. (a) $\int_{-2}^{-1}\frac{dx}{x-3} = \ln|x-3|\Big|_{-2}^{-1} = \ln 4 - \ln 5 = \ln\frac{4}{5}$; $\int_{-2}^{-1}\frac{dx}{x} = \ln|x|\Big|_{-2}^{-1} = \ln 1 - \ln 2 = -\ln 2 = \ln\frac{1}{2}$

Because $\ln\frac{4}{5} > \ln\frac{1}{2}$, then $\int_{-2}^{-1}\frac{dx}{x-3} \geq \int_{-2}^{-1}\frac{dx}{x}$.

(b) $\int_1^2\frac{dx}{x} = \ln|x|\Big|_1^2 = \ln 2 - \ln 1 = \ln 2$; $\int_1^2\frac{dx}{x-3} = \ln|x-3|\Big|_1^2 = \ln 1 - \ln 2 = -\ln 2$

Because $\ln 2 > -\ln 2$, then $\int_1^2\frac{dx}{x} \geq \int_1^2\frac{dx}{x-3}$.

(c) $\int_4^5\frac{dx}{x-3} = \ln|x-3|\Big|_4^5 = \ln 2 - \ln 1 = \ln 2$; $\int_4^5\frac{dx}{x} = \ln|x|\Big|_4^5 = \ln 5 - \ln 4 = \ln\frac{5}{4}$

Because $\ln 2 > \ln\frac{5}{4}$, then $\int_4^5\frac{dx}{x-3} \geq \int_4^5\frac{dx}{x}$.

107. The gudermannian, is the function defined by $\operatorname{gd} x = \tan^{-1}(\sinh x)$. Show that $D_x(\operatorname{gd} x) = \operatorname{sech} x$.

▶ $$D_x(\operatorname{gd} x) = \frac{1}{1 + \sinh^2 x}D_x(\sinh x) = \frac{\cosh x}{1 + \sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x} = \operatorname{sech} x$$

S I X

ADDITIONAL APPLICATIONS OF THE DEFINITE INTEGRAL

6.1 LENGTH OF ARC OF THE GRAPH OF A FUNCTION

6.1.2 Theorem If the function f and its derivative f' are continuous on the closed interval $[a, b]$, then the length of arc of the curve $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

6.1.3 Theorem If the function g and its derivative g' are continuous on the closed interval $[c, d]$, then the length of arc of the curve $x = g(y)$ from the point $(g(c), c)$ to the point $(g(d), d)$ is given by

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

The discussion following Theorem 1.10.2 ($\lim_{\theta \to 0} \sin\theta/\theta = 1$) while plausible, used area which is Definition 4.4.8 or 4.5.3 and radian measure of an angle which is the length of arc subtended by the angle on the unit circle. Using only the definitions and theorems of this book, we now establish

2.7.1 Theorem $\frac{d}{d\theta} \sin\theta = \cos\theta$

PROOF: Let (x, y) be a point on the unit circle $x^2 + y^2 = 1$ so that by Definition A.3.2, $x = \cos\theta$ and $y = \sin\theta$. If $x = g(x) = \sqrt{1 - y^2}$, then

$$1 + [g'(y)]^2 = 1 + \left(\frac{y}{\sqrt{1 - y^2}}\right)^2 = \frac{1}{1 - y^2}$$

Thus, by Definition A.3.1 (radian measure) and Theorem 6.1.3, we have

$$\theta = \int_0^y \sqrt{1 + g'(t)^2}\, dt$$

$$= \int_0^y \frac{dt}{\sqrt{1 - t^2}}$$

If $|y| < 1$, we may differentiate with respect to θ using the chain rule and the fundamental theorem of the calculus to get

$$1 = \frac{d}{d\theta} \int_0^y \frac{dt}{\sqrt{1 - t^2}} = \frac{d}{dy} \int_0^y \frac{dt}{\sqrt{1 - t^2}} \cdot \frac{dy}{d\theta} = \frac{1}{\sqrt{1 - y^2}} \frac{dy}{d\theta}$$

and so

$$\frac{d}{d\theta} \sin\theta = \frac{dy}{d\theta} = \sqrt{1 - y^2} = x = \cos\theta$$

By Theorem 3.3.4, this result is valid when $|y| = 1$.

Exercises 6.1

In Exercises 1–24, find the exact length of arc by the second fundamental theorem of the calculus. In Exercises 1–4, use 3 methods: (a) the distance formula or the Pythagorean theorem (b) Theorem 6.1.2; (c) Theorem 6.1.3.

▶ In these Exercises L units is the length of the specified arc.

1. $y = 3x$ from $(1, 3)$ to $(2, 6)$

(a) By the distance formula, $L = \sqrt{(1 - 2)^2 + (3 - 6)^2} = \sqrt{1 + 9} = \sqrt{10}$

(b) $y = f(x) = 3x$; $f'(x) = 3$; $a = 1$, $b = 2$. $L = \int_1^2 \sqrt{1 + 9}\, dx = \sqrt{10} \int_1^2 dx = \sqrt{10}(2 - 1) = \sqrt{10}$

(c) $x = g(y) = \frac{1}{3} y$; $g'(y) = \frac{1}{3}$; $c = 3$, $d = 6$. $L = \int_3^6 \sqrt{1 + (\frac{1}{3})^2}\, dy = \frac{1}{3}\sqrt{103} \int_0^6 dy = \frac{1}{3}\sqrt{10}(6 - 3) = \sqrt{10}$

2. $x + 3y = 4$ from $(-2, 2)$ to $(4, 0)$.

(a) By the distance formula, $L = \sqrt{(4+2)^2 + (0-2)^2} = \sqrt{36+4} = 2\sqrt{10}$

(b) $y = f(x) = \frac{1}{3}(4-x)$, $f' = -\frac{1}{3}$, $a = -2$, $b = 4$. $L = \int_{-2}^{4} \sqrt{1 + (\frac{1}{3})^2}\, dx = \frac{1}{3}\sqrt{10} \int_{-2}^{4} dx = \frac{1}{3}\sqrt{10} \cdot 6 = 2\sqrt{10}$

(c) $x = g(y) = 4 - 3y$, $g' = -3$, $c = 0$, $d = 2$. $L = \int_0^2 \sqrt{1+3^2}\, dx = \sqrt{10} \int_0^2 dx = \sqrt{10}(2-0) = 2\sqrt{10}$

3. $4x + 9y = 36$ between $(9,0)$ and $(0,4)$

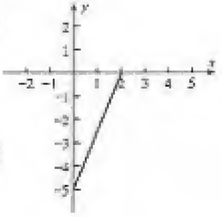
(a) By the Pythagorean theorem, $L = \sqrt{9^2 + 4^2} = \sqrt{81+16} = \sqrt{97}$.

(b) $y = f(x) = 4 - \frac{4}{9}x$; $f'(x) = -\frac{4}{9}$; $a = 0$, $b = 9$. $L = \int_0^9 \sqrt{1 + \frac{16}{81}}\, dx = \frac{1}{9}\sqrt{97} \int_0^9 dx = \frac{1}{9}\sqrt{97}(9-0) = \sqrt{97}$

(c) $g(y) = 9 - \frac{9}{4}y$; $g'(y) = -\frac{9}{4}$; $c = 0$, $d = 4$. $L = \int_0^4 \sqrt{1 + \frac{81}{16}}\, dy = \frac{1}{4}\sqrt{97} \int_0^4 dy = \frac{1}{4}\sqrt{97}(4-0) = \sqrt{97}$

4. the segment of the line $5x - 2y = 10$ between its x and y intercepts

▸ The figure shows a sketch of the line segment.



(a) Because the x and y intercepts are 2 and -5, respectively, by the Pythagorean theorem the length of the segment is given by $\quad L = \sqrt{2^2 + 5^2} = \sqrt{29}$

(b) Solving the equation of the line for y gives $\quad y = f(x) = \frac{5}{2}x - 5$

Therefore $f'(x) = \frac{5}{2}$. By Theorem 6.1.2 we have

$$L = \int_0^2 \sqrt{1 + [f'(x)]^2}\, dx = \int_0^2 \sqrt{1 + (\tfrac{5}{2})^2}\, dx = \tfrac{1}{2}\sqrt{29}\, x\Big]_0^2 = \tfrac{1}{2}\sqrt{29}\,(2-0) = \sqrt{29}$$

(c) Solving the equation of the line for x gives

$$x = g(y) = \tfrac{2}{5}y + 2$$

Thus $g'(y) = \frac{2}{5}$. Applying Theorem 6.1.3, we get

$$L = \int_{-5}^{0} \sqrt{1 + (\tfrac{2}{5})^2}\, dy = \tfrac{1}{5}\sqrt{29}\, y\Big]_{-5}^{0} = \tfrac{1}{5}\sqrt{29}\,(0+5) = \sqrt{29}$$

5. The curve $9y^2 = 4x^3$ from $(0,0)$ to $(3, 2\sqrt{3})$. $y = f(x) = \frac{2}{3}x^{3/2}$; $f'(x) = x^{1/2}$. From Theorem 6.1.2
$L = \int_0^3 \sqrt{1+x}\, dx = \frac{2}{3}(1+x)^{3/2}\Big|_0^3 = \frac{2}{3}(8-1) = \frac{14}{3}$

6. The curve $x^2 = (2y+3)^3$ from $(1,-1)$ to $(7\sqrt{7}, 2)$. $x = g(y) = (2y+3)^{3/2}$; $g'(y) = 3(2y+3)^{1/2}$. From Thm 6.1.3, $L = \int_{-1}^{2} \sqrt{1 + [3(2y+3)^{1/2}]^2}dy = \int_{-1}^{2} \sqrt{18y+28}dy = \frac{2}{3} \cdot \frac{1}{18}(18y+28)^{3/2}\Big]_{-1}^{2} = \frac{1}{27}(64^{3/2} - 10^{3/2})$

7. The curve $8y = x^4 + 2x^{-2}$ from $x = 1$ to $x = 2$. $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$. Then $D_x y = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$. From Thm 6.1.2
$L = \int_1^2 \sqrt{1 + (\frac{1}{2}x^3 - \frac{1}{2}x^{-3})^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4}x^{-6}}\, dx = \int_1^2 \sqrt{\frac{1}{4}x^6 + \frac{1}{2} + \frac{1}{4}x^{-6}}\, dx$
$= \int_1^2 \sqrt{(\frac{1}{2}x^3 + \frac{1}{2}x^{-3})^2}\, dx = \frac{1}{2}\int_1^2 (x^3 + x^{-3})dx = \frac{1}{2}[\frac{1}{4}x^4 - \frac{1}{2}x^{-2}]_1^2 = \frac{1}{2}[(4 - \frac{1}{8}) - (\frac{1}{4} - \frac{1}{2})] = \frac{33}{16}$

In Exercises 8 and 9, find the length of the arc of the curve $y^3 = 8x^2$ from the point $(1,2)$ to the point $(27,18)$.

8. Use Theorem 6.1.2

▸ Solving $y^3 = 8x^2$ for y, we obtain $y = f(x) = 2x^{2/3}$. Then

$$f'(x) = \tfrac{4}{3}x^{-1/3}$$
$$1 + [f'(x)]^2 = 1 + \tfrac{16}{9}x^{-2/3} = \tfrac{1}{9}x^{-2/3}(9x^{2/3} + 16)$$
$$\sqrt{1 + [f'(x)]^2} = \tfrac{1}{3}x^{-1/3}\sqrt{9x^{2/3}+16} \ \text{(because } x > 0\text{)}$$

Therefore, by Theorem 6.1.2, we have $\quad L = \frac{1}{3}\int_1^{27} \sqrt{9x^{2/3}+16} \cdot x^{-1/3}\, dx \qquad (1)$

Let $u = 9x^{2/3} + 16$. Then $du = 6x^{-1/3}\, dx$. When $x = 1, u = 25$; when $x = 27$, $u = 97$. Hence, from (1) we get

$$L = \tfrac{1}{18}\int_{25}^{97} u^{1/2}\, du = \tfrac{1}{27}u^{3/2}\Big]_{25}^{97} = \tfrac{1}{27}(97^{3/2} - 125) \approx 30.8$$

▪ The arc is approximately 30.8 units long.

9. Use Theorem 6.1.3. $x = g(y) = \frac{1}{4}\sqrt{2}y^{3/2}$; $g'(y) = \frac{3}{8}\sqrt{2}y^{1/2}$. From Theorem 6.1.3
$\mathrm{L} = \int_2^{18} \sqrt{1 + \frac{9}{32}y}\, dy = \frac{32}{9} \cdot \frac{2}{3}(1 + \frac{9}{32}y)^{3/2}\Big|_2^{18} = \frac{64}{27}[(1 + \frac{81}{16})^{3/2} - (1 + \frac{9}{16})^{3/2}] = \frac{64}{27}\left[\frac{97^{3/2}}{64} - \frac{25^{3/2}}{64}\right] = \frac{1}{27}(97^{3/2} - 125)$

10. The curve $y = \frac{2}{3}(x-5)^{3/2}$. $\mathrm{D}_x y = (x-5)^{1/2}$ from $x = 6$ to $x = 8$. From Theorem 6.1.2
$\mathrm{L} = \int_6^8 \sqrt{1 + (x-5)}dx = \int_6^8 \sqrt{x-4}dx = \frac{2}{3}(x-4)^{3/2}\Big]_6^8 = \frac{2}{3}(4^{3/2} - 2^{3/2}) = \frac{2}{3}(8 - 2\sqrt{2})$

11. The curve $y = \frac{1}{3}(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$. $\mathrm{D}_x y = x(x^2+2)^{1/2}$. From Theorem 6.1.2
$\mathrm{L} = \int_0^3 \sqrt{1 + x^2(x^2+2)}\, dx = \int_0^3 \sqrt{x^4 + 2x^2 + 1}\, dx = \int_0^3 \sqrt{(x^2+1)^2}\, dx = \int_0^3 (x^2+1)dx = \frac{1}{3}x^3 + x\Big|_0^3 = 9 + 3 = 12$

12. Find the length of the arc of the curve $6xy = y^4 + 3$ from the point where $y = 1$ to the point where $y = 2$.

▹ Solving $6xy = y^4 + 3$ for x, we get $x = g(y) = \frac{1}{6}y^3 + \frac{1}{2}y^{-1}$. Then

$$g'(y) = \tfrac{1}{2}y^2 - \tfrac{1}{2}y^{-2}$$
$$1 + [g'(y)]^2 = 1 + (\tfrac{1}{2}y^2 - \tfrac{1}{2}y^{-2})^2 = \tfrac{1}{4}(y^4 + 2 + y^{-4}) = \tfrac{1}{4}(y^2 + y^{-2})^2$$
$$\sqrt{1 + [g'(y)]^2} = \tfrac{1}{2}(y^2 + y^{-2})$$

Therefore, by Theorem 6.1.3, we have

$$\mathrm{L} = \frac{1}{2}\int_1^2 (y^2 + y^{-2})\, dy = \frac{1}{2}\left[\frac{1}{3}y^3 - y^{-1}\right]_1^2 = \frac{17}{12}$$

▪ Thus the length of the arc is $\frac{17}{12}$ units.

13. The curve $y = \frac{1}{3}\sqrt{x}(3x-1)$ from $x = 1$ to $x = 4$. $y = x^{3/2} - \frac{1}{3}x^{1/2}$; $\mathrm{D}_x y = \frac{3}{2}x^{1/2} - \frac{1}{6}x^{-1/2}$. From Thm 6.1.2
$\mathrm{L} = \int_1^4 \sqrt{1 + \left(\frac{3x^{1/2}}{2} - \frac{1}{6x^{1/2}}\right)^2} dx = \int_1^4 \sqrt{1 + \frac{9x}{4} - \frac{1}{2} + \frac{1}{36x}}\, dx = \int_1^4 \sqrt{\frac{9x}{4} + \frac{1}{2} + \frac{1}{36x}}\, dx$
$= \int_1^4 \sqrt{\left(\frac{3x^{1/2}}{2} + \frac{1}{6x^{1/2}}\right)^2} dx = \int_1^4 (\frac{3}{2}x^{1/2} + \frac{1}{6}x^{-1/2})dx = x^{3/2} + \frac{1}{3}x^{1/2}\Big|_1^4 = (8 + \frac{2}{3}) - (1 + \frac{1}{3}) = \frac{22}{3}$

14. The curve $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$ from $(2, \frac{19}{12})$ to $(5, \frac{314}{15})$. $\mathrm{D}_x y = \frac{1}{2}(x^2 - x^{-2})$. From Theorem 6.1.2
$\mathrm{L} = \int_2^5 \sqrt{1 + [\frac{1}{2}(x^2 - x^{-2})]^2}dx = \int_2^5 \sqrt{1 + \frac{1}{4}(x^4 - 2 + x^{-4})}dx = \int_2^5 \sqrt{\frac{1}{4}(x^4 + 2 + x^{-4})}dx$
$= \int_2^5 \sqrt{[\frac{1}{2}(x^2 + x^{-2})]^2}dx = \int_2^5 \frac{1}{2}(x^2 + x^{-2})dx = \frac{1}{6}x^3 - \frac{1}{2}x^{-1}\Big]_2^5 = \frac{1}{6}(125 - 8) - \frac{1}{2}(\frac{1}{5} - \frac{1}{2}) = \frac{393}{20}$

15. The astroid $x^{2/3} + y^{2/3} = 1$ from $x = \frac{1}{8}$ to $x = 1$. $y = (1 - x^{2/3})^{3/2}$; $\mathrm{D}_x y = \frac{3}{2}(1 - x^{2/3})^{1/2}(-\frac{2}{3}x^{-1/3})$. From Theorem 6.1.2 $\mathrm{L} = \int_{1/8}^1 \sqrt{1 + (1 - x^{2/3})x^{-2/3}}\, dx = \int_{1/8}^1 x^{-1/3}dx = \frac{3}{2}x^{2/3}\Big|_{1/8}^1 = \frac{3}{2}(1 - \frac{1}{4}) = \frac{9}{8}$

Alternatively, differentiating implicitly, $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot \mathrm{D}_x y = 0$; $\mathrm{D}_x y = -\frac{y^{1/3}}{x^{1/3}}$. From Theorem 6.1.2
$\mathrm{L} = \int_{1/8}^1 \sqrt{1 + \left(-\frac{y^{1/3}}{x^{1/3}}\right)^2} dx = \int_{1/8}^1 x^{-1/3}\sqrt{x^{2/3} + y^{2/3}}\, dx = \int_{1/8}^1 x^{-1/3}dx = \frac{3}{2}x^{2/3}\Big|_{1/8}^1 = \frac{3}{2}(1 - \frac{1}{4}) = \frac{9}{8}$

16. The astroid $x^{2/3} + y^{2/3} = a^{2/3}$ (a is a constant, $a > 1$) in the first quadrant from the point where $x = 1$ to the point where $x = a$.

▹ Solving the given equation for y, we obtain

$$y = f(x) = (a^{2/3} - x^{2/3})^{3/2}$$
$$f'(x) = \tfrac{3}{2}(a^{2/3} - x^{2/3})^{1/2}(-\tfrac{2}{3}x^{-1/3}) = -x^{-1/3}(a^{2/3} - x^{2/3})^{1/2}$$
$$[f'(x)]^2 = x^{-2/3}(a^{2/3} - x^{2/3}) = a^{2/3}x^{-2/3} - 1$$
$$1 + [f'(x)]^2 = a^{2/3}x^{-2/3}$$
$$\sqrt{1 + [f'(x)]^2} = a^{1/3}x^{-1/3}$$

Therefore, by Theorem 6.1.2,

$$L = \int_1^a a^{1/3} x^{-1/3}\, dx = \tfrac{3}{2} a^{1/3} x^{2/3}\Big]_1^a = \tfrac{3}{2} a^{1/3}(a^{2/3} - 1) = \tfrac{3}{2}(a - a^{1/3})$$

- The length of arc is $\frac{3}{2}(a - a^{1/3})$ units.

17. The curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ in the first quadrant from $x = 1$ to $x = a > 1$. $y = b\left[1 - \left(\frac{x}{a}\right)^{2/3}\right]^{3/2}$;

$D_x y = -\frac{b}{a}\left[1 - \left(\frac{x}{a}\right)^{2/3}\right]^{1/2}\left(\frac{x}{a}\right)^{-1/3}$. By Theorem 6.1.2

$$L = \int_{a/8}^{a} \sqrt{1 + \frac{b^2}{a^2}\left[1 - \left(\frac{x}{a}\right)^{2/3}\right]\left(\frac{x}{a}\right)^{-2/3}}\, dx = \int_{a/8}^{a} \frac{1}{a}\left(\frac{x}{a}\right)^{-1/3} \sqrt{(a^2 - b^2)\left(\frac{x}{a}\right)^{2/3} + b^2}\, dx$$

If $b = a$, $\int_{a/8}^{a} \left(\frac{x}{a}\right)^{-1/3} dx = \frac{3a}{2}\left(\frac{x}{a}\right)^{2/3}\Big|_{a/8}^{a} = \frac{9}{8}a$. Otherwise, let $u = (a^2 - b^2)\left(\frac{x}{a}\right)^{2/3} + b^2$; $du = \frac{2}{3}\cdot\frac{a^2 - b^2}{a}\left(\frac{x}{a}\right)^{-1/3} dx$.

When $x = \frac{a}{8}$, $u = (a^2 - b^2)\cdot\frac{1}{4} + b^2 = \frac{1}{4}(a^2 + 3b^2)$; when $x = a$, $u = (a^2 - b^2)\cdot 1 + b^2 = a^2$. Hence

$$L = \frac{3}{2(a^2 - b^2)} \int_{(a^2+3b^2)/4}^{a^2} u^{1/2} du = \frac{1}{a^2 - b^2} u^{3/2}\Big|_{(a^2+3b^2)/4}^{a^2} = \frac{1}{a^2 - b^2}\left[a^3 - \frac{(a^2 + 3b^2)^{3/2}}{8}\right]$$

$= \frac{8a^3 - (a^2 + 3b^2)^{3/2}}{8(a^2 - b^2)}$. Rationalizing the numerator, $L = \frac{9}{8}\cdot\frac{7a^4 + 6a^2b^2 + 3b^4}{8a^3 + (a^2 + 3b^2)^{3/2}}$, in both cases.

18. The curve $9y^2 = x^2(2x + 3)$ in the second quadrant from $x = -1$ to $x = 0$. $y = -\frac{1}{3}x\sqrt{2x + 3}$. Let $u = 2x + 3$;

$D_x u = 2$. $y = -\frac{1}{3}\cdot\frac{1}{2}(u - 3)u^{1/2} = -\frac{1}{6}(u^{3/2} - 3u^{1/2})$. $D_x y = -\frac{1}{6}(\frac{3}{2}u^{1/2} - \frac{3}{2}u^{-1/2})\cdot 2 = -\frac{1}{2}(u^{1/2} - u^{-1/2})$

$$L = \int_{u=1}^{3} \sqrt{1 + \tfrac{1}{4}(u^{1/2} - u^{-1/2})^2}\cdot\tfrac{1}{2}du = \tfrac{1}{2}\int_1^3 \sqrt{1 + \tfrac{1}{4}u - \tfrac{1}{2} + \tfrac{1}{4}u^{-1}}\,du = \tfrac{1}{2}\int_1^3 \sqrt{\tfrac{1}{4}(u^{1/2} + u^{-1/2})^2}\,du$$

$$= \frac{1}{4}\int_1^3 (u^{1/2} + u^{-1/2})du = \tfrac{1}{6}\left[u^{3/2} + 3u^{1/2}\right]_1^3 = \tfrac{1}{6}(3^{3/2} + 3\sqrt{3} - 4) = \tfrac{1}{3}(3\sqrt{3} - 2)$$

19. The curve $9y^2 = x(x - 3)^2$ in the first quadrant from $x = 1$ to $x = 3$. $y = \frac{1}{3}x^{3/2} - x^{1/2}$; $D_x y = \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2}$.

$L = \int_1^3 \sqrt{1 + (\frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2})^2}\, dx = \int_1^3 \sqrt{1 + \frac{1}{4}x - \frac{1}{2} + \frac{1}{4}x^{-1}}\, dx = \int_1^3 \sqrt{\frac{1}{4}x + \frac{1}{2} + \frac{1}{4}x^{-1}}\, dx =$

$\int_1^3 \sqrt{(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2})^2}\, dx = \frac{1}{2}\int_1^3 (x^{1/2} + x^{-1/2})dx = \frac{1}{2}\left[\frac{2}{3}x^{3/2} + 2x^{1/2}\right]_1^3 = \frac{1}{2}\left[(2\sqrt{3} + 2\sqrt{3}) - (\frac{2}{3} + 2)\right] = 2\sqrt{3} - \frac{4}{3}$

20. The curve $9y^2 = 4(1 + x^2)^3$ in the first quadrant from the point where $x = 0$ to the point where $x = 2\sqrt{2}$.

▹ Solving $9y^2 = 4(1 + x^2)^3$ for y, we have $y = f(x) = \frac{2}{3}(1 + x^2)^{3/2}$

$$f'(x) = 2x(1 + x^2)^{1/2}$$

$$1 + [f'(x)]^2 = 1 + 4x^2(1 + x^2) = 4x^4 + 4x^2 + 1 = (2x^2 + 1)^2$$

$$\sqrt{1 + [f'(x)]^2} = 2x^2 + 1$$

Therefore, by Theorem 6.1.2, we have

$$L = \int_0^{2\sqrt{2}} (2x^2 + 1)\, dx = \tfrac{2}{3}x^3 + x\Big]_0^{2\sqrt{2}} = \tfrac{38}{3}\sqrt{2}$$

- The arc is $\frac{38}{3}\sqrt{2}$ units long.

21. The curve $y = \ln \sec x$ from $x = 0$ to $x = \frac{1}{4}\pi$. $y' = \frac{1}{\sec x}\cdot\sec x \tan x = \tan x$.

$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx = \int_0^{\pi/4} \sec x\, dx = \ln(\sec x + \tan x)\Big]_0^{\pi/4} = \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(\sqrt{2} + 1)$

22. The curve $y = \ln \csc x$ from $x = \frac{1}{6}\pi$ to $x = \frac{1}{2}\pi$. $y' = \frac{1}{\csc x}(-\csc x \cot x) = -\cot x$. $L = \int_{\pi/6}^{\pi/2} \sqrt{1 + \cot^2 x}\,dx$

$= \int_{\pi/6}^{\pi/2} \csc x\, dx = -\ln(\csc x + \cot x)\Big]_{\pi/6}^{\pi/2} = \ln(\csc \frac{1}{6}\pi + \cot \frac{1}{6}\pi) - \ln(\csc \frac{1}{2}\pi + \cot \frac{1}{2}\pi) = \ln(2 + \sqrt{3})$

23. The curve $f(x) = \int_0^x \sqrt{\cos t}\,dt$ from $x = 0$ to $x = \frac{1}{2}\pi$. By the first fundamental theorem of the calculus,

$f'(x) = \sqrt{\cos x}$. $[f'(x)]^2 = \cos x$. Hence

$L = \int_0^{\pi/2} \sqrt{1 + \cos x}\, dx = \int_0^{\pi/2} \sqrt{2\cos^2 \frac{1}{2}x}\, dx = \sqrt{2}\int_0^{\pi/2} \cos \frac{1}{2}x\, dx = \sqrt{2}\left[2 \sin \frac{1}{2}x\right]_0^{\pi/2} = \sqrt{2}\cdot\sqrt{2} = 2$

24. The curve $f(x) = \int_0^x \sqrt{\sin t}\, dt$ from the point where $x = 0$ to the point where $x = \frac{1}{2}\pi$.

▸ By the first fundamental theorem of calculus,

$$f'(x) = \sqrt{\sin x}; \qquad [f'(x)]^2 = \sin x$$

Therefore, by Theorem 6.1.2, we have

$$L = \int_{x=0}^{\pi/2} \sqrt{1 + \sin x}\, dx$$

We make the substitution $x = \frac{1}{2}\pi - u$, $dx = -du$, $\sin x = \sin(\frac{1}{2}\pi - u) = \cos u$. Hence

$$L = \int_{u=\pi/2}^{0} \sqrt{1 + \cos u}(-du) = \int_0^{\pi/2} \sqrt{2\cos^2 \tfrac{1}{2}u}du = \sqrt{2}\int_0^{\pi/2} \cos \tfrac{1}{2}u\, du = \sqrt{2}\Big[2\sin \tfrac{1}{2}u\Big]_0^{\pi/2} = \sqrt{2}\cdot\sqrt{2} = 2$$

In Exercises 25-34, find the length of arc to 4 significant digits by using NINT to calculate the definite integral.

25. The parabola $y = x^2$ from $(0,0)$ to $(2,4)$. $y' = 2x$. $L = \int_0^2 \sqrt{1 + (2x)^2}dx \approx 4.647$. Exact: $\sqrt{17} + \frac{1}{4}\ln(4 + \sqrt{17})$

26. The parabola $y = \frac{1}{4}(x-2)^2 + 1$ from $(-2,5)$ to $(4,2)$. $y' = \frac{1}{2}(x-2)$. $L = \int_{-2}^{4} \sqrt{1 + \frac{1}{4}(x-2)^2}dx \approx 8.211$
Exact: $2\sqrt{5} + \sqrt{2} - \ln(2 + \sqrt{10} - 2\sqrt{2} - 5)$

27. The sine curve from $(0,0)$ to $(\pi,0)$. $y = \sin x$; $D_x y = \cos x$. By Theorem 6.1.2 $L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx \approx 3.820$

28. The cosine curve from the point $(0,1)$ to the point $(\frac{1}{3}\pi, \frac{1}{2})$.

▸ Let $f(x) = \cos x$. Then $f'(x) = -\sin x$ and

$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + \sin^2 x}$$

By Theorem 6.1.2, we have

$$L = \int_0^{\pi/3} \sqrt{1 + \sin^2 x}\, dx$$

Using NINT to evaluate the integral, we find $L = 1.186$ to four significant digits.

29. The curve $y = \frac{1}{3}x^3$ from $(0,0)$ to $(1, \frac{1}{3})$. $D_x y = x^2$. By Theorem 6.1.2 and NINT $L = \int_0^1 \sqrt{1 + x^4}\, dx \approx 1.089$

30. The curve $y = \tan x$ from $(0,0)$ to $(\frac{1}{4}\pi, 1)$. $y' = \sec^2 x$. $L = \int_0^{\pi/4} \sqrt{1 + \sec^4 x}\, dx \approx 1.278$

31. The curve $y = x^3 - 6x^2 + 9x - 1$ from $(0,-1)$ to $(3,-1)$. $y' = 3x^2 - 12x + 9$.
$L = \int_0^3 \sqrt{1 + (3x^2 - 12x + 9)^2}dx \approx 8.815$

32. The curve $y = x^3 - 2x^2 - 5x + 6$ from the point $(-2,0)$ to the point $(3,0)$.

▸ Let $f(x) = x^3 - 2x^2 - 5x + 6$. Then $f'(x) = 3x^2 - 4x - 5$ and

$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + (3x^2 - 4x - 5)^2}$$

By Theorem 6.1.2, we have

$$L = \int_{-2}^{3} \sqrt{1 + (3x^2 - 4x - 5)^2}dx$$

Using NINT to evaluate the integral, we find $L = 25.40$ to four significant digits.

33. The curve $y = x^3 - 3x^2 + 3$ between the two positive x intercepts. Using zoom we find 1.3473 and 2.5321.
$y' = 3x^2 - 6x$. $L = \int_{1.3473}^{2.5321} \sqrt{1 + (3x^2 - 6x)^2}dx \approx 2.422$

34. The curve $y = 3 - (x-1)^4$ between the x intercepts. $(x-1)^4 = 3$, $x - 1 = \pm 3^{1/4}$, $x = 1 \pm 3^{1/4}$.
$y' = -4(x-1)^3$. $L = \int_{1-3^{1/4}}^{1+3^{1/4}} \sqrt{1 + 16(x-1)^6} \approx 7.253$

35. The catenary $y = 200\cosh(x/200)$ between $x = -150$ and $x = 150$. $y' = \sinh(\frac{1}{200}x)$.
$L = \int_{-150}^{150} \sqrt{1 + \sinh^2(\frac{1}{200}x)}dx = 2\int_0^{150} \sqrt{\cosh^2(\frac{1}{200}x)}dx = 2\int_0^{150} \cosh \frac{1}{200}x\, dx = 400 \sinh \frac{1}{200}x\Big]_0^{150} = 400 \sinh \frac{3}{4}$

36. Explain why you cannot use Theorem 6.1.2 to compute the length of arc of the graph of $y^3 = x^2$ from the origin to the point (1,1). Can you use Theorem 6.1.3 to compute this length of arc? If your answer is no, explain why. If your answer is yes, find the length of arc.

▹ Solving $y^3 = x^2$ for y, we have $y = f(x) = x^{2/3}$, $f'(x) = \frac{2}{3}x^{-1/3}$. Because $f'(x)$ is not continuous at 0, Theorem 6.1.2 does not apply. Solving $y^3 = x^2$ for x, we have $x = g(y) = y^{3/2}$

$$g'(y) = \tfrac{3}{2}y^{1/2}$$
$$1 + [g'(y)]^2 = 1 + \tfrac{9}{4}y$$

Because $g'(y)$ is continuous on $[0,1]$, we may apply Theorem 6.1.3 to get

$$L = \int_0^1 \sqrt{1 + \tfrac{9}{4}y}\,dy = \tfrac{2}{3}\cdot\tfrac{4}{9}(1 + \tfrac{9}{4}y)^{3/2}\Big]_0^1 = \tfrac{8}{27}\Big[(\tfrac{13}{4})^{3/2} - 1\Big] = \tfrac{1}{27}(13\sqrt{13} - 8)$$

▪ The arc is $\frac{1}{27}(13\sqrt{13} - 8)$ units long.

6.2 CENTER OF MASS OF A ROD

Center of Mass Newton's second law of motion states that if a body of mass M moving at a velocity v is acted on by a force F, then $F = d/dt(Mv)$. If M is constant, then $F = Ma$. Thus the *mass* M can be defined as the ratio F/a. If a system of n particles is located on the x axis at the points $x_1, x_2, \ldots, x_n$ and the mass of the ith particle is given by m_i, then the *moment of mass* for the system with respect to the origin is defined to be $\sum_{i=1}^{n} m_i x_i$. The *center of mass* of the system is the point $\bar{x}$ such that if the total mass of the system, $\sum_{i=1}^{n} m_i$, were concentrated there, its moment of mass, $\bar{x}\sum_{i=1}^{n} m_i$, would be equal to the moment of mass of the system. We conclude that

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \tag{1}$$

If a rod of length L meters with its left endpoint at the origin is such that for a continuous function $\rho(x)$ the mass of the segment $[x, x + \Delta x]$ is $\rho(w)\Delta x$ where w is a point of the segment, then $\rho(x)$ kilograms per meter is the *linear density* at x. It follows that the total mass of the rod is M kilograms, where

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho(w_i)\Delta_i x = \int_0^L \rho(x)\,dx \tag{2}$$

6.2.2 Definition A rod of length L meters has its left endpoint at the origin. If $\rho(x)$ kilograms per meter is the linear density at a point x meters from the origin, where ρ is continuous on $[0, L]$. The *moment of mass* of the rod with respect to the origin is M_0 kilogram-meters where

$$M_0 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} w_i \rho(w_i)\Delta_i x = \int_0^L x\rho(x)\,dx \tag{3}$$

The center of mass of the rod is at the point $\bar{x}$ such that if M kilograms is the total mass of the rod, then $\bar{x}M = M_0$, that is

$$\bar{x} = \frac{M_0}{M} \tag{4}$$

The following table summarizes the units used in this book and the value of the gravitational constant g in each. Note that pound is a unit of force (lbf) *and* mass (lb).

System	Force	Mass	Acceleration	g
British	pound (lbf)	slug = 32.2 lb	ft/sec^2	32.2 ft/sec
SI (mks)	newton (N)	kilogram (kg)	m/sec^2	9.81 m/sec^2
CGS	dyne (= 10^{-5} N)	gram (g)	cm/sec^2	981 cm/sec^2

Exercises 6.2

In Exercises 1–4, a particle of given mass and acceleration is moving on a horizontal line. Find the force exerted.

1. Mass is 50 slugs; acceleration is 5 ft/sec^2. If F lbf is the force, $F = (50)(5) = 250$.

2. Mass is 10 kg; acceleration is 6 m/sec^2. If F N is the force, $F = 10(6) = 60$.

3. Mass is 80 g; acceleration is 50 cm/sec^2. If F dynes is the force, $F = (80)(50) = 4000$.

4. Mass is 22 slugs; acceleration of 4 ft/sec^2.

▸ We are given that $M = 22$ and $a = 4$. Then

$$F = Ma = 22(4) = 88$$

• Because we are using British units, the force is 88 pounds.

In Exercises 5–8, a horizontal force acts on a particle and its mass or acceleration is given. Find the other quantity.

5. Force is 6 nt; mass is 4 kg. If a m/sec^2 is the acceleration, $a = \frac{6}{4} = \frac{3}{2}$.

6. Force is 32 lbf; mass is 8 slugs. If a ft/sec^2 is the acceleration, $a = \frac{32}{8} = 4$.

7. Force is 24 lbf; acceleration is 9 ft/sec^2. If m slugs is the mass, $m = \frac{24}{9} = \frac{8}{3}$.

8. Force is 700 dynes; acceleration is 80 cm/sec^2.

▸ We are given that $F = 700$ and $a = 80$. Then

$$M = \frac{F}{a} = \frac{700}{80} = 8.75$$

Because we using CGS units, the mass is 8.75 grams.

In Exercises 9–12, a system of particles on the x axis has the given masses (kg) and position (m). Find the center of mass.

9. $$x = \frac{\sum_{i=1}^{4} m_i x_i}{\sum_{i=1}^{4} m_i} = \frac{5\cdot 2 + 6\cdot 3 + 4\cdot 5 + 3\cdot 8}{5+6+4+3} = \frac{72}{18} = 4$$

10. $$x = \frac{\sum_{i=1}^{4} m_i x_i}{\sum_{i=1}^{4} m_i} = \frac{2(-4) + 8(-1) + 4(2) + 2(3)}{2+8+4+2} = \frac{-2}{16} = -\frac{1}{8}$$

11. $$x = \frac{\sum_{i=1}^{5} m_i x_i}{\sum_{i=1}^{5} m_i} = \frac{2(-3) + 4(-2) + 20(4) + 10(6) + 30(9)}{2+4+20+10+30} = \frac{396}{6}$$

12. $m_1 = 5$ at -7; $m_2 = 3$ at -2; $m_3 = 5$ at 2; $m_4 = 1$ at 2; $m_5 = 8$ at 10

▸ We use Eq. (1) with $x_1 = -7$, $x_2 = -2$, $x_3 = 0$, $x_4 = 2$, $x_5 = 10$. Thus,

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + m_5x_5}{m_1 + m_2 + m_3 + m_4 + m_5} = \frac{5(-7) + 3(-2) + 5(0) + 1(2) + 8(10)}{5+3+5+1+8} = \frac{41}{22}$$

• Thus the center of mass is $\frac{41}{22}$ meters to the right of the origin.

In Exercises 13–21, find the total mass M of the rod and the center of mass $\bar{x}$.

13. $$M = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (2w_i + 3)\Delta_i x = \int_0^6 (2x+3)dx = x^2 + 3x\Big]_0^6 = 54$$

$$\bar{x} = \frac{1}{54}\int_0^6 x(2x+3)dx = \frac{1}{54}\int_0^6 (2x^2+3x)dx = \frac{1}{54}\Big[\tfrac{2}{3}x^3 + \tfrac{3}{2}x^2\Big]_0^6 = \frac{144+54}{54} = \frac{198}{54} = \frac{11}{3}$$

• Therefore the center of mass is $\frac{11}{3}$ m from one end.

14. $M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (3w_i + 2)\Delta_i x = \int_0^{20} (3x+2)dx = \frac{3}{2}x^2 + 2x\Big|_0^{20} = 640$

$\bar{x} = \frac{1}{640}\int_0^{20} x(3x+2)dx = \frac{1}{640}\int_0^{20}(3x^2+2x)dx = \frac{1}{640}\left[x^3 + x^2\right]_0^{20} = \frac{8400}{640} = \frac{105}{8}$

- Therefore the center of mass is $\frac{105}{8}$ cm from one end.

15. $M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4w_i + 1)\Delta_i x = \int_0^{9} (4x+1)dx = 2x^2 + x\Big|_0^{9} = 2 \cdot 81 + 9 = 171$

$\bar{x} = \frac{1}{171}\int_0^{9} x(4x+1)dx = \frac{1}{171}\int_0^{9}(4x^2+x)dx = \frac{1}{171}\left[\frac{4}{3}x^3 + \frac{1}{2}x^2\right]_0^{9} = \frac{1}{171}(\frac{4}{3}\cdot 729 + \frac{1}{2}\cdot 81) = 5.92$

- Therefore the center of mass is 5.92 in. from one end.

16. The length of a rod is 3 ft, and the linear density of the rod at a point x ft from one end is $(5+2x)$ slugs/ft.

▷ We have $\rho(x) = 5 + 2x$. If M slugs is the mass, by Eq. (2)

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (5 + 2w_i)\Delta_i x = \int_0^{3} (5+2x)\, dx = 5x + x^2\Big]_0^{3} = 5(3) + 9 = 24$$

The total mass of the rod is 24 slugs. If the center of mass is at $\bar{x}$, by Eq. (4)

$$\bar{x} = \frac{1}{M}\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (5 + 2w_i)w_i\Delta_i x = \frac{1}{24}\int_0^{3} (5+2x)x\, dx = \frac{1}{24}\int_0^{3} (5x + 2x^2)\, dx$$

$$= \frac{1}{24}\left[\frac{5}{2}x^2 + \frac{2}{3}x^3\right]_0^{3} = \frac{1}{24}(\frac{45}{2} + 18) = \frac{1}{24}\cdot\frac{81}{2} = \frac{27}{16}$$

- The center of mass of the rod is $\frac{27}{16}$ ft from the end of the rod.

17. Let $\rho(x)$ g/cm be the density at the point x cm from the left end of the rod. Let $\rho(x) = ax + b$. Because $\rho(0) = 3$ and $\rho(12) = 4$, we have $b = 3$ and $12a + b = 4$. Therefore $a = \frac{1}{12}$ and $b = 3$; thus $\rho(x) = \frac{1}{12}x + 3$.

$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (\frac{1}{12}w_i + 3)\Delta_i x = \int_0^{12} (\frac{1}{12}x + 3)dx = \frac{1}{24}x^2 + 3x\Big|_0^{12} = 6 + 36 = 42$

$\bar{x} = \frac{1}{42}\int_0^{12} x(\frac{1}{12}x + 3)dx = \frac{1}{42}\int_0^{12}(\frac{1}{12}x^2 + 3x)dx = \frac{1}{42}\left[\frac{1}{36}x^3 + \frac{3}{2}x^2\right]_0^{12} = \frac{48 + 216}{42} = \frac{264}{42} = \frac{44}{7}$

- Therefore the center of mass is $\frac{44}{7}$ cm from the left end.

18. Let $\rho(x)$ g/cm be the density at the point x cm from the left end of the rod. Let $\rho(x) = mx + b$. Because $\rho(0) = 2$ and $\rho(10) = 3$, we have $b = 3$ and $m = \frac{1}{10}$. Thus $\rho(x) = \frac{1}{10}x + 2$.

$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (\frac{1}{10}w_i + 2)\Delta_i x = \int_0^{10} (\frac{1}{10}x + 2)dx = \frac{1}{20}x^2 + 2x\Big|_0^{10} = 5 + 20 = 25$

19. Let $\rho(x)$ kg/m be the density of the rod at the point x meters from the end having the greater density. Then $\rho(x) = c(10 - x)$. Because $\rho(6) = 3$, we have $3 = 4c$; $c = \frac{3}{4}$. Therefore $\rho(x) = \frac{3}{4}(10 - x)$.

$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{3}{4}(10 - w_i)\Delta_i x = \frac{3}{4}\int_0^{6}(10 - x)dx = \frac{3}{4}\left[10x - \frac{1}{2}x^2\right]_0^{6} = \frac{3}{4}(60 - 18) = \frac{3}{4}\cdot 42 = 31.5$

$\bar{x} = \frac{1}{\frac{3}{4}\cdot 42}\cdot\frac{3}{4}\int_0^{6} x(10 - x)dx = \frac{1}{42}\int_0^{6}(10x - x^2)dx = \frac{1}{42}\left[5x^2 - \frac{1}{3}x^3\right]_0^{6} = \frac{180 - 72}{42} = \frac{108}{42} = \frac{18}{7}$

- Therefore the center of mass is $\frac{18}{7}$ m from the end having the greater density.

20. A rod is 10 ft long, and the measure of the linear density at a point is a linear function of the measure of the distance from the center of the rod. The linear density at each end of the rod is 5 slugs/ft and at the center the linear density is $3\frac{1}{2}$ slugs/ft.

▷ By symmetry, the center of mass is at the center of the rod. Put the x axis on the rod with the origin at the center. Because $\rho(0) = \frac{7}{2}$ and $\rho(5) = 5$, then $\frac{\Delta\rho}{\Delta x} = \frac{3/2}{5} = \frac{3}{10}$ and $\rho(x) = \frac{7}{2} + \frac{3}{10}x$, $x \in [0, 5]$. By symmetry and Eq. (2),

$$M = 2\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho(w_i)\Delta_i x = 2\int_0^{5}\left(\frac{7}{2} + \frac{3}{10}x\right) dx = 2\left[\frac{7}{2}x + \frac{3}{20}x^2\right]_0^{5} = 2[\frac{7}{2}(5) + \frac{3}{20}(25)] = \frac{85}{2}$$

- The total mass of the rod is $\frac{85}{2}$ slugs.

21. Let $\rho(x)$ slugs/ft be the density of the rod at the point x ft from one end. Then $\rho(x) = kx^3$. Because $\rho(2) = 2$, we have $2 = 8k$; $k = \frac{1}{4}$. Therefore $\rho(x) = \frac{1}{4}x^3$. M =

$\lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} (\frac{1}{4}w_i^3)\Delta_i x = \frac{1}{4}\int_0^4 x^3 dx = \frac{1}{16}x^4\Big|_0^4 = 16$. $\bar{x} = \frac{1}{16}\int_0^4 x(\frac{1}{4}x^3)dx = \frac{1}{16}\cdot\frac{1}{4}\int_0^4 x^4 dx = \frac{1}{64}\cdot\frac{1}{5}x^5\Big|_0^4 = \frac{2^{10}}{5 \cdot 2^6} = \frac{16}{5}$

Therefore the center of mass is $\frac{16}{5}$ ft from one end.

22. Let $\rho(x)$ kg/m be the density of the rod at the point x meters from the near end. $\rho(0) = k$. Then $(x+2)$m is the distance from the point and $\rho(x) = \frac{1}{2}k(x+2)$.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}k(w_i + 2)\Delta_i x = \int_0^5 \tfrac{1}{2}k(x+2)dx = \tfrac{1}{4}k(x+2)^2\Big]_0^5 = \tfrac{1}{4}k(45) = 135;\ k = 12$$

23. Let $\rho(x)$ kg/m be the density of the rod at the point x meters from the end having the greater density. Then $\rho(x) = c(4-x)$. Because $\rho(3) = 2$, we have $2 = c$. Hence $\rho(x) = 8 - 2x$.

$$\bar{x} = \frac{1}{15}\int_0^3 x(8-2x)dx = \frac{1}{15}\int_0^3 (8x - 2x^2)dx = \frac{1}{15}\Big[4x^2 - \tfrac{2}{3}x^3\Big]_0^3 = \frac{26-18}{15} = \frac{18}{15} = 1.2$$

- Therefore the center of mass is 1.2 m from the end having the greater density.

24. The measure of the linear density at a point on a rod varies directly as the fourth power of the measure of the distance of the point from one end. The length of the rod is 2 m. If the total mass of the rod is $\frac{64}{5}$ kg, find the center of mass of the rod.

▷ We are given that $\rho(x) = kx^4$. Then

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho(w_i)\Delta_i x = \int_0^2 kx^4\, dx = \tfrac{1}{5}kx^5\Big]_0^2 = \tfrac{32}{5}k$$

Because the total mass of the rod is $\frac{64}{5}$ kg, then

$$\tfrac{32}{5}k = \tfrac{64}{5};\quad k = 2 \text{ so } \quad \rho(x) = 2x^4$$

By Eq. (4) we conclude that

$$\bar{x} = \tfrac{5}{64}\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (2w_i^4)w_i\Delta_i x = \tfrac{5}{32}\int_0^2 x^5\, dx = \tfrac{5}{192}x^6\Big]_0^2 = \tfrac{5}{192}\cdot 64 = \tfrac{5}{3}$$

- The center of mass is $\frac{5}{3}$ meters from the end of the rod.

25. Let $\rho(x)$ g/cm be the density at the point x cm from the left end of the rod where $\rho(x) = 2/(1+x)$.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{2}{1+w_i}\Delta_i x = \int_0^{15} \frac{2\, dx}{1+x} = 2\ln(1+x)\Big]_0^{15} = 2\ln 16 = 8\ln 2$$

$$\bar{x} = \frac{1}{M}\int_0^{15} \frac{2x\, dx}{1+x} = \frac{2}{M}\int_0^{15}\Big(1 - \frac{1}{1+x}\Big)dx = \frac{2}{M}\Big[x - \ln(1+x)\Big]_0^{15} = 2\frac{15 - \ln 16}{8\ln 2} = \frac{15 - 4\ln 2}{4\ln 2} = \frac{15}{4\ln 2} - 1$$

26. Let $\rho(x)$ kg/m be the linear density x m from the left end. Then $\rho(x) = k(L - x)$.

$M = \int_0^L k(L-x)dx = -\frac{1}{2}k(L-x)^2\Big]_0^L = \frac{1}{2}kL^2$. Thus $k = 2M/L^2$.

27. Let $\rho(x)$ kg/m be the linear density of the rod at the point x meters from the end. Then $\rho(x) = kx^2$.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k(w_i^2)\Delta_i x = k\int_0^6 x^2 dx = \tfrac{1}{3}kx^3\Big|_0^6 = 72k$$

Hence $24 = 72k$; $k = \frac{1}{3}$. Thus $\rho(x) = \frac{1}{3}x^2$ and the largest value of ρ in $[0, 6]$ is $\rho(6) = \frac{1}{3}(6)^2 = 12$.

- Hence the greatest linear density is 12 kg/m.

28. A rod is L meters long and its center of mass is at the point $\frac{3}{4}$L from the left end. If the measure of the linear density at a point is proportional to a power of the measure of the distance from the left end, and the linear density at the right end is 20 kg/m, find the linear density at a point x meters from the left end.

▷ Let $\rho(x)$ slugs/ft be the density of the rod at the point x ft from the left end. Then

$\rho(x) = kx^n$. Because $\rho(L) = 20$, then $20 = kL^n$; $k = \frac{20}{L^n}$. Thus $\rho(x) = \frac{20}{L^n}x^n$.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{20}{L^n}(w_i)^n\Delta_i x = \frac{20}{L^n}\int_0^L x^n dx = \frac{20}{L^n}\cdot\frac{x^{n+1}}{n+1}\Big|_0^L = \frac{20L}{n+1}$$

$$\bar{x} = \frac{n+1}{20L}\cdot\int_0^L\Big(\frac{20}{L^n}x^n\Big)dx = \frac{n+1}{L^{n+1}}\int_0^L x^{n+1}dx = \frac{n+1}{L^{n+1}}\cdot\frac{x^{n+2}}{n+2}\Big|_0^L = \frac{n+1}{n+2}L$$

Hence $\frac{3}{4}L = \frac{n+1}{n+2}L$; $3n + 6 = 4n + 4$; $n = 2$. Therefore $\rho(x) = \frac{20}{L^2}x^2$.

6.3 CENTROID OF A PLANE REGION

Center of Mass Let the masses of n particles located at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy plane be measured by $m_1, m_2, \ldots, m_n$. Then the total mass of the system is

$$M = \sum_{i=1}^{n} m_i x_i$$

and the center of mass of the system is at (x, y) where

$$\bar{x} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \quad \text{and} \quad y = \frac{1}{M}\sum_{i=1}^{n} m_i y_i$$

Let R be a region in the xy plane that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. If $\rho(x)$ is a continuous function such that the mass of a rectangular element perpendicular to the x axis between x and $x + \Delta x$ is $\rho(m)[f(m) - g(m)]\Delta x$ for some m in $[x, x + \Delta x]$, then its center of mass is at the point $(m, \frac{1}{2}[f(m) + g(m)])$ and $\rho(x)$ is the *area density* at the point (x, y). It follows that the total mass of R is given by

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho(m_i)[f(m_i) - g(m_i)]\Delta_i x$$

$$= \int_a^b \rho(x)[f(x) - g(x)]\, dx$$

Furthermore, the *moments* of R with respect to the y and x axes are

$$M_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho(m_i)[f(m_i) - g(m_i)]m_i \Delta_i x$$

$$= \int_a^b \rho(x)[f(x) - g(x)]x\, dx$$

$$M_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}\rho(m_i)[f(m_i) - g(m_i)][f(m_i) + g(m_i)]\Delta_i x$$

$$= \frac{1}{2}\int_a^b \rho(x)[f(x)^2 - g(x)^2]\, dx$$

The *center of mass* of R is the point $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

If $f(x) = 1$ then M is just A, the area of R, and $(\bar{x}, \bar{y})$ is called the *centroid* of R. Similar definitions hold if the area density and the equations of the boundary are given as functions of y. In Ch. 13 we consider area densities that are functions of both x and y.

6.3.3 Theorem If the plane region R has the line L as an axis of symmetry, the centroid of R lies on L.

6.3.4 Theorem of Pappus (Ex. 40) If a plane region R of area A is revolved about a line L in its plane that does not cut the region, then the measure of the volume of the solid generated is given by

$$V = 2\pi\bar{y}A$$

where $\bar{y}$ is the measure of the distance from the centroid of R to the line L.

The following three results of Exercises of this section are used throughout the book.

Centroid of a Triangle = centroid of equal masses at the vertices = point of intersection of the medians. If the vertices are (a_1, b_1), (a_2, b_2), (a_3, b_3), then the centroid is $(\frac{1}{3}(a_1 + a_2 + a_3), \frac{1}{3}(b_1 + b_2 + b_3))$. See Exercises 6 and 34.

Centroid of a semicircle (Ex.35) The distance from the centroid of a semicircle of radius r to its diameter is $4r/3\pi$.

Exercises 6.3

In these Exercises $(\bar{x}, \bar{y})$ is the center of mass. A square units is the area of the region, V cubic units is the volume of the solid, and Δ is a partition of the specified interval.

1. $m_y = \sum_{i=1}^{3} m_i x_i = 1(-1) + 2(2) + 3(3) = 12$; $m_x = \sum_{i=1}^{3} m_i y_i = 1(3) + 2(1) + 3(-1) = 2$
$M = \sum_{i=1}^{3} m_i = 1 + 2 + 3 = 6$. Therefore $\bar{x} = \frac{m_y}{M} = \frac{12}{6} = 2$ and $\bar{y} = \frac{m_x}{M} = \frac{2}{6} = \frac{1}{3}$

▪ The center of mass is at the point $(2, \frac{1}{3})$.

2. $m_y = \sum_{i=1}^{4} m_i x_i = 2(-1) + 3(1) + 3(0) + 4(2) = 9$; $m_x = \sum_{i=1}^{4} m_i y_i = 2(-2) + 3(3) + 3(5) + 4(1) = 24$

$M = \sum_{i=1}^{4} m_i = 2 + 3 + 3 + 4 = 12$. Therefore $\bar{x} = \frac{m_y}{M} = \frac{9}{12} = \frac{3}{4}$ and $\bar{y} = \frac{m_x}{M} = \frac{24}{12} = 2$

▪ The center of mass is at the point $(\frac{3}{4}, 2)$.

3. $m_x = \sum_{i=1}^{4} m_i y_i = 2(2) + 5(0) + 4(20) + m(-2) = 84 - 2m$; $M = \sum_{i=1}^{4} m_i = 2 + 5 + 4 + m = 11 + m$

Because $\bar{y} = \frac{m_x}{M}$, we have $5 = \frac{84 - 2m}{11 + m}$; $55 + 5m = 84 - 2m$; $7m = 29$; $m = \frac{29}{7}$.

4. Find the center of mass of the three particles having masses of 3, 7, and 2 kg located at the points $(2,3)$, $(-1,4)$, $(0,2)$, respectively.

▸ If M kg is the total mass of the three particles, then

$$M = \sum_{i=1}^{3} m_i = 3 + 7 + 2 = 12$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the center of mass. Then

$$\bar{x} = \frac{1}{M}\sum_{i=1}^{3} m_i x_i = \frac{1}{12}[3(2) + 7(-1) + 2(0)] = -\frac{1}{12} \qquad \bar{y} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i = \frac{1}{12}[3(3) + 7(4) + 2(2)] = \frac{41}{12}$$

▪ The center of mass is $(-\frac{1}{12}, \frac{41}{12})$.

5. Let the measure of the mass of each particle be m. Then

$m_y = \sum_{i=1}^{3} m_i x_i = m(4) + m(-3) + m(1) = 2m$; $m_x = \sum_{i=1}^{3} m_i y_i = m(-2) + m(0) + m(5) = 3m$

$M = \sum_{i=1}^{3} m_i = m + m + m = 3m$; $\bar{x} = \frac{m_y}{M} = \frac{2m}{3m} = \frac{2}{3}$ and $\bar{y} = \frac{m_x}{M} = \frac{3m}{3m} = 1$.

▪ The center of mass is at the point $(\frac{2}{3}, 1)$.

6. The center of mass of 3 equal masses at the vertices of a triangle lies at the point of intersection of its medians.

▸ If a mass m is at each of (a_1, b_1), (a_2, b_2), (a_3, b_3), then

$$\bar{x} = \frac{ma_1 + ma_2 + ma_3}{m + m + m} = \frac{a_1 + a_2 + a_3}{3}; \quad \bar{y} = \frac{mb_1 + mb_2 + mb_3}{m + m + m} = \frac{b_1 + b_2 + b_3}{3}$$

In Exercises 7–14, find the centroid of the region with the indicated boundaries.

7. $\bar{x} = 0$ because the y axis is an axis of symmetry. $4 - x^2$ is nonnegative in $[-2, 2]$.

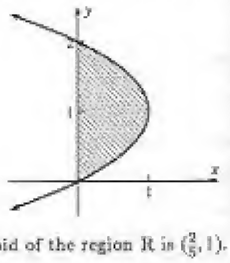
$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - m_i^2)\Delta_i x = \int_{-2}^{2} (4 - x^2)\,dx = 4x - \tfrac{1}{3}x^3\Big|_{-2}^{2} = \frac{32}{3}$$

$$m_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(4 - m_i^2)^2 \Delta_i x = \tfrac{1}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)\,dx = \tfrac{1}{2}\Big[16x - \tfrac{8}{3}x^3 + \tfrac{1}{5}x^5\Big] = \frac{256}{15}$$

$\bar{y} = \frac{1}{A} \cdot m_x = \frac{3}{32} \cdot \frac{256}{15} = \frac{8}{5}$. Thus the centroid is at the point $(0, \frac{8}{5})$.

8. The parabola $x = 2y - y^2$ and the y axis.

▸ The figure shows the region R. R is bounded on the right by the curve $x = f(y) = 2y - y^2$, on the left by the line $x = g(y) = 0$, below by the line $y = 0$ and above by the line $y = 2$. Because $g(y) = 0$, we omit it. The area of R is given by



$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(m_i)\Delta_i y = \int_0^2 (2y - y^2)\,dy = y^2 - \tfrac{1}{3}y^3\Big]_0^2 = 4 - \tfrac{1}{3}(8) = \tfrac{4}{3}$$

If $(\bar{x}, \bar{y})$ is the centroid of R, then

$$\bar{x} = \frac{1}{A}\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2} f(m_i)^2 \Delta_i y = \tfrac{3}{4} \cdot \tfrac{1}{2}\int_0^2 (2y - y^2)^2\,dy$$

$$= \tfrac{3}{8}\int_0^2 (4y^2 - 4y^3 + y^4)\,dy = \tfrac{3}{8}\Big[\tfrac{4}{3}y^3 - y^4 + \tfrac{1}{5}y^5\Big]_0^2$$

$$= \tfrac{3}{8}\Big[\tfrac{4}{3}(8) - 16 + \tfrac{1}{5}(32)\Big] = \tfrac{3}{8} \cdot \tfrac{16}{15} = \tfrac{2}{5}$$

Because the line $y = 1$ is an axis of symmetry for R, then $\bar{y} = 1$. Thus, the centroid of the region R is $(\frac{2}{5}, 1)$.

9. $\bar{x} = 0$ because the y axis is an axis of symmetry. $4 - x^2$ is nonnegative in $[-2, 2]$.

$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - m_i^2)\Delta_i x = \int_{-2}^{2} (4 - x^2)dx = 4x - \frac{1}{3}x^3\Big]_{-2}^{2} = \frac{32}{3}$

$m_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}(4 + m_i^2)(4 - m_i^2)\Delta_i x = \frac{1}{2}\int_{-2}^{2} (16 - x^4)dx = \frac{1}{2}\Big[16x - \frac{1}{5}x^5\Big]_{-2}^{2} = \frac{1}{2}[(32 - \frac{32}{5}) - (-32 + \frac{32}{5})] = \frac{128}{5}$

$y = \frac{1}{A} \cdot m_x = \frac{3}{32} \cdot \frac{128}{5} = \frac{12}{5}$. Hence the centroid is at the point $(0, \frac{12}{5})$.

10. $m_x = \int_0^4 \frac{1}{4}y^2 \cdot y\, dy = \int_0^4 \frac{1}{4}y^3 dy = \frac{1}{16}y^4\Big]_0^4 = 16$. $m_y = \frac{1}{2}\int_0^4 (\frac{1}{4}y^2)^2 dy = \frac{1}{32}\int_0^4 y^4 dy = \frac{1}{160}y^5\Big]_0^4 = \frac{1024}{160} = \frac{32}{5}$.

$A = \int_0^4 \frac{1}{4}y^2 dy = \frac{1}{12}y^3\Big]_0^4 = \frac{16}{3}$. $\bar{x} = \frac{1}{A} \cdot m_y = \frac{3}{16} \cdot \frac{32}{5} = \frac{6}{5}$. $\bar{y} = \frac{1}{A} \cdot m_x = \frac{3}{16} \cdot 16 = 3$

11. Because $x > 0$, $4x - x^3 = x(4 - x^2)$ is nonnegative in $[0, 2]$.

$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4m_i - m_i^3)\Delta_i x = \int_0^2 (4x - x^3)dx = 2x^2 - \frac{1}{4}x^4\Big]_0^2 = 4$

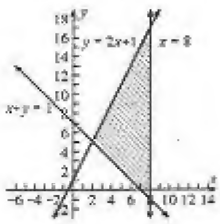
$m_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}(4m_i + m_i^3)(4m_i - m_i^3)\Delta_i x = \frac{1}{2}\int_0^2 (16x^2 - x^6)dx = \frac{1}{2}\Big[\frac{16}{3}x^3 - \frac{1}{7}x^7\Big] = \frac{256}{21}$

$m_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} m_i(4m_i - m_i^3)\Delta_i x = \int_0^2 x(4x - x^3)dx = \int_0^2 (4x^2 - x^4)dx = \frac{4}{3}x^3 - \frac{1}{5}x^5\Big]_0^2 = \frac{32}{3} - \frac{32}{5} = \frac{64}{15}$

$\bar{x} = \frac{1}{A} \cdot m_y = \frac{1}{4} \cdot \frac{64}{15} = \frac{16}{15}$ and $y = \frac{1}{A} \cdot m_x = \frac{1}{4} \cdot \frac{256}{21} = \frac{64}{21}$. Thus the centroid is at the point $(\frac{16}{15}, \frac{64}{21})$.

12. The lines $y = 2x + 1$, $x + y = 7$, and $x = 8$.

▷ The figure shows the region R. R is bounded above by the line $y = f(x) = 2x + 1$ and and below by the line $x + y = 7$, or, equivalently, $y = g(x) = -x + 7$, on the left by the line $x = 2$ and on the right by the line $x = 8$. Furthermore, R is a triangle of base $17 - (-1) = 18$ and height $8 - 2 = 6$ and so the area of R is given by



$A = \frac{1}{2}bh = \frac{1}{2}(18)(6) = 54$

$\bar{x} = \frac{1}{A}\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [f(m_i) - g(m_i)]m_i\Delta_i x = \frac{1}{54}\int_2^8 [(2x + 1) - (-x + 7)]x\, dx$

$= \frac{1}{54}\int_2^8 (3x^2 - 6x)\, dx = \frac{1}{54}\Big[x^3 - 3x^2\Big]_2^8 = \frac{1}{54}[(512 - 8) - 3(64 - 4)]$

$= \frac{324}{54} = 6$

$\bar{y} = \frac{1}{A}\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}[f(m_i) - g(m_i)][f(m_i) + g(m_i)]\Delta_i x = \frac{1}{54}\int_2^8 \frac{1}{2}(3x - 6)(x + 8)dx$

$= \frac{1}{36}\int_2^8 (x^2 + 6x - 16)\, dx = \frac{1}{36}\Big[\frac{1}{3}x^3 + 3x^2 - 16x\Big]_2^8 = \frac{1}{36}[\frac{1}{3}(512 - 8) + 3(64 - 4) - 16(8 - 2)] = \frac{252}{36} = 7$

The centroid of R is $(6, 7)$. Note that the centroid of the triangle is the center of mass of equal masses at the three vertices $(2, 5)$, $(8, -1)$ and $(8, 17)$:

$$\bar{x} = \frac{2 + 8 + 8}{3} = \frac{18}{3} = 6 \qquad \text{and} \qquad \bar{y} = \frac{5 + (-1) + 17}{3} = \frac{21}{3} = 7$$

13. $(2x - x^2) - (x^2 - 4) = -2x^2 + 2x + 4 = -2(x + 1)(x - 2)$ is nonnegative in $[-1, 2]$.

$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} [(2m_i - m_i^2) - (m_i^2 - 4)]\Delta_i x = \int_{-1}^{2} (-2x^2 + 2x + 4)dx = -\frac{2}{3}x^3 + x^2 + 4x\Big]_{-1}^{2}$

$= (-\frac{16}{3} + 4 + 8) - (\frac{2}{3} + 1 - 4) = 9$

$m_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}[m_i - m_i^2) + (m_i^2 - 4)][(2m_i - m_i^2) - (m_i^2 - 4)]\Delta_i x = -\int_{-1}^{2} (2x - 4)(-2x^2 + 2x + 4)dx$

$= 2\int_{-1}^{2} (-x^3 + 3x^2 - 4)dx = 2\Big[-\frac{1}{4}x^4 + x^3 - 4x\Big]_{-1}^{2} = 2\Big[(-4 + 8 - 8) - (-\frac{1}{4} - 1 + 4)\Big] = -\frac{27}{2}$

$m_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} m_i[(2m_i - m_i^2) - (m_i^2 - 4)]\Delta_i x = \int_{-1}^{2} x(-2x^2 + 2x + 4)dx = 2\int_{-1}^{2} (-x^3 + x^2 + 2x)dx$

$= 2\Big[-\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2\Big]_{-1}^{2} = 2\Big[(-4 + \frac{8}{3} + 4) - (-\frac{1}{4} - \frac{1}{3} + 1)\Big] = \frac{9}{2}$

$\bar{x} = \frac{1}{A} \cdot m_y = \frac{1}{9} \cdot \frac{9}{2} = \frac{1}{2}$ and $\bar{y} = \frac{1}{A} \cdot m_x = \frac{1}{9}\Big(-\frac{27}{2}\Big) = -\frac{3}{2}$. Hence the centroid is at the point $(\frac{1}{2}, -\frac{3}{2})$.

14. $x^3 \le x^2$ on $[0,1]$. $A = \int_0^1 (x^2 - x^3)dx = \frac{1}{3}x^3 - \frac{1}{4}x^4\Big]_0^1 = \frac{1}{12}$.

$m_y = \int_0^1 (x^2 - x^3)x\,dx = \int_0^1 (x^3 - x^4)dx = \frac{1}{4}x^4 - \frac{1}{5}x^5\Big]_0^1 = \frac{1}{20}$. $\bar{x} = \frac{1}{A}\cdot m_y = 12\cdot\frac{1}{20} = \frac{3}{5}$

$m_x = \frac{1}{2}\int_0^1 [(x^2)^2 - (x^3)^2]dx = \frac{1}{2}\int_0^1 (x^4 - x^6)dx = \frac{1}{2}\Big[\frac{1}{5}x^5 - \frac{1}{7}x^7\Big]_0^1 = \frac{1}{35}$. $\bar{y} = \frac{1}{A}\cdot m_x = 12\cdot\frac{1}{35} = \frac{12}{35}$

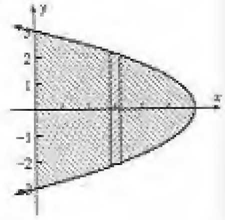
In Exercises 15 and 16, find the center of mass of the lamina bounded by the parabola $2y^2 = 18 - 3x$ and the y axis.

15. $\rho(x,y) = \sqrt{6-x}$. Because the lamina is symmetric with respect to the x axis and the area density is not a function of y, then $\bar{y} = 0$. $y^2 = \frac{3}{2}(6-x)$ is nonnegative in $[0,6]$. $m_y =$

$$\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\cdot m_i\sqrt{6-m_i}\sqrt{\tfrac{3}{2}(6-m_i)}\Delta_i x = \sqrt{6}\int_0^6 x(6-x)dx = \sqrt{6}\Big[3x^2 - \tfrac{1}{3}x^3\Big]_0^6 = \sqrt{6}(3\cdot 36 - 2\cdot 36) = 36\sqrt{6}$$

$$M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\sqrt{6-m_i}\sqrt{\tfrac{3}{2}(6-m_i)}\Delta_i x = \sqrt{6}\int_0^6 (6-x)dx = \sqrt{6}\Big[6x - \tfrac{1}{2}x^2\Big] = \sqrt{6}(36-18) = 18\sqrt{6}$$

$\bar{x} = \dfrac{m_y}{M} = \dfrac{36\sqrt{6}}{18\sqrt{6}} = 2$. Therefore the center of mass is at the point $(2,0)$.



16. The area density at any point (x,y) is x kg/m^2.

▷ The figure shows the region R. We are given that $\rho(x) = x$. Because ρ is a function of x, we must solve the boundary for y as a function of x. Thus $y = \pm\sqrt{9-\frac{3}{2}x}$, $x\in[0,6]$. We use symmetry and let $f(x) = \sqrt{9-\frac{3}{2}x}$ and $g(x) = 0$, which we ignore. The measure of a rectangular element of area perpendicular to the x axis is $2f(m_i)\Delta_i x$ and its mass is $2\rho(m_i)f(m_i)\Delta_i x$. Thus

$$\begin{aligned} M &= \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\rho(m_i)f(m_i)\Delta_i x & \qquad M_y &= \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\rho(m_i)f(m_i)m_i\Delta_i x \\ &= \int_0^6 2\rho(x)f(x)\,dx & &= \int_0^6 2\rho(x)f(x)x\,dx \\ &= \int_0^6 2x\sqrt{9-\tfrac{3}{2}x}\,dx & &= \int_0^6 2x^2\sqrt{9-\tfrac{3}{2}x}\,dx \end{aligned}$$

Let $u = \sqrt{9-\frac{3}{2}x}$. Then $x = \frac{2}{3}(9-u^2)$; $dx = -\frac{4}{3}u\,du$; $u = 3$ when $x = 0$; and $u = 0$ when $x = 6$. Thus

$$\begin{aligned} M &= \int_3^0 2[\tfrac{2}{3}(9-u^2)]u(-\tfrac{4}{3}u\,du) & \qquad M_y &= \int_3^0 2[\tfrac{2}{3}(9-u^2)]^2u(-\tfrac{4}{3}u\,du) \\ &= \tfrac{16}{9}\int_0^3 (9u^2-u^4)\,du & &= \tfrac{32}{27}\int_0^3 (81u^2 - 18u^4 + u^6)\,du \\ &= \tfrac{16}{9}\Big[3u^3 - \tfrac{1}{5}u^5\Big]_0^3 & &= \tfrac{32}{27}\Big[27u^3 - \tfrac{18}{5}u^5 + \tfrac{1}{7}u^7\Big]_0^3 \\ &= \tfrac{16}{9}[3(27) - \tfrac{1}{5}(243)] & &= \tfrac{32}{27}[27(3^3) - \tfrac{18}{5}(3^5) + \tfrac{1}{7}(3^7)] \\ &= \tfrac{16}{9}\cdot\tfrac{162}{5} = \tfrac{288}{5} & &= 32[27 - \tfrac{18}{5}(9) + \tfrac{1}{7}(81)] = \tfrac{6912}{35} \end{aligned}$$

Therefore

$$\bar{x} = \frac{M_y}{M} = \frac{5}{288}\cdot\frac{6912}{35} = \frac{24}{7}$$

By symmetry, $\bar{y} = 0$. Thus the center of mass is $(\frac{24}{7}, 0)$.

In Exercises 17–24, use your graphics calculator to find to four significant digits the centroid of the region bounded by the given curves. The interval and area A were determined in the indicated exercise in Exercises 4.8.

17. Exercise 39. $y = x^4 - 2$; $y = x^2$; $[-\sqrt{2}, \sqrt{2}]$; $A = \frac{56}{15}\sqrt{2}$. By symmetry, $\bar{x} = 0$.

$m_x = \frac{1}{2}\int_{-\sqrt{2}}^{\sqrt{2}} [(x^2)^2 - (x^4-2)^2]dx = \int_0^{\sqrt{2}} (3x^4 - x^8 - 4)dx = -\frac{16}{9}\sqrt{2}$. $\bar{y} = -\frac{16}{9}\sqrt{2}/\frac{56}{15}\sqrt{2} = -\frac{10}{21} \approx -0.4762$

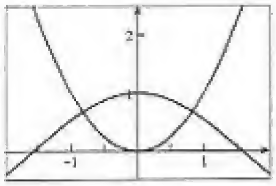
18. Exercise 40. $y = x^4$; $y = 4 - x^2$; $[-1.24962, 1.24962]$; $A = 7.4772$. By symmetry, $\bar{x} = 0$.

$m_x = \frac{1}{2}\int_{-1.24962}^{1.24962} [(4-x^2)^2 - (x^4)^2]dx \approx 14.57418$. $\bar{y} = \dfrac{14.57418}{7.4772} = 1.949$

19. Exercise 41. $y = x^2 - 1$; $y = \sin^2 x$; $[-1.40449, 1.40449]$; A $= 2.20322$. By symmetry, $\bar{x} = 0$

$m_x = \frac{1}{2}\int_{-1.40449}^{1.40449}[(\sin^2 x)^2 - (x^2-1)^2]dx \approx -0.22474$. $\bar{y} = \frac{-0.22474}{2.20322} = -0.1020$

20. Exercise 42. $y = x^2$; $y = \cos x$; $[-0.82413, 0.82413]$; A $= 1.09475$



▸ Because x^2 and $\cos x$ are even functions, the region, shown at the right, is symmetric with respect to $x = 0$ and so $\bar{x} = 0$. Using NINT, we have

$m_x = \frac{1}{2}\int_{-0.84213}^{0.82413}[\cos^2 x - (x^2)^2]dx \approx 0.58528$; $\quad \bar{y} = \frac{m_x}{A} = \frac{0.58528}{1.09475} = 0.5346$

• The centroid is at $(0, 0.5346)$.

21. Exercise 43. $y = x^3$; $y = 4 - x^2$; the y axis; $[0, 1.3146]$; A $= 3.7545$

$m_y = \int_0^{1.3146}[(4 - x^2) - x^3]x\,dx \approx 1.9245$; $\bar{x} = \frac{1.9245}{3.7545} = 0.5126$

$m_x = \frac{1}{2}\int_0^{1.3146}[(4 - x^2)^2 - (x^3)^2]dx \approx 7.3956$; $\bar{y} = \frac{7.3956}{3.7545} = 1.970$

22. Exercise 44. $y = x^3$; $y = 4 - x^2$; the x axis; $[0, 1.31460]$ and $[1.31460, 2]$; A $= 1.57886$

$m_y = \int_0^{1.3146} x^3 \cdot x\,dx + \int_{1.3146}^{2}(4 - x^2)x\,dx \approx 2.07553$; $\bar{x} = \frac{2.07553}{1.57886} = 1.31457$

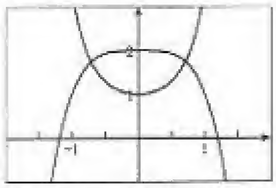
$m_x = \frac{1}{2}\int_0^{1.3146}(x^3)^2 dx + \frac{1}{2}\int_{1.3146}^{2}(4-x^2)^2 dx \approx 1.1377$; $\bar{y} = \frac{1.1377}{1.5789} = 0.7206$

23. Exercise 45. $y = x^3$; $y = \tan^2 x - 3$; $[0, 1.12738]$; A $= 2.8079$

$m_y = \int_0^{1.12738}[x^3 - (\tan^2 x - 3)^2]x\,dx \approx 1.3788$; $\bar{x} = \frac{1.3788}{2.8079} = 0.4910$

$m_x = \frac{1}{2}\int_0^{1.12738}[(x^3)^2 - (\tan^2 x - 3)^2]dx \approx -3.0397$; $\bar{y} = \frac{-3.0397}{2.8079} = -1.083$

24. Exercise 46. $y = 2 - x^4$; $y = \sec^2 x$; $[-0.71660, 0.71660]$; A $= 1.04878$



▸ Because $2 - x^4$ and $\sec x$ are even functions, the region, shown at the right, is symmetric with respect to $x = 0$ and so $\bar{x} = 0$. Using NINT, we have

$m_x = \frac{1}{2}\int_{-0.71660}^{0.71660}[(2 - x^4)^2 - (\sec^2 x)^2]dx = 1.629577$; $\bar{y} = \frac{1.629577}{1.04878} = 1.554$

• The centroid is at $(0, 1.554)$.

In Exercises 25–32, use your graphics calculator to find to four significant digits the centroid of the region bounded by the given curves from the indicated exercise in Exercises 4.9.

25. Exercise 41. The region bounded by $y = \sqrt[4]{x^3 + 4}$, the x axis, the y axis, and $x = 2$.

▸ A $= \int_0^2 \sqrt[4]{x^3 + 4}\,dx \approx 3.093309$. $M_y = \int_0^2 \sqrt[4]{x^3 + 4} \cdot x\,dx = 3.24181$. $\bar{x} = \frac{3.24181}{3.093309} = 1.0480$

$M_x = \frac{1}{2}\int_0^2 \sqrt{x^3 + 4}\,dx \approx 2.41058$. $\bar{y} = \frac{2.41058}{3.093309} = 0.7793$

26. Exercise 42. The region bounded by $y = \sqrt[3]{x^4 - 5}$, the x axis, $x = 2$ and $x = 3$.

▸ A $= \int_2^3 \sqrt[3]{x^4 - 5}\,dx \approx 3.238548$. $M_y = \int_2^3 \sqrt[3]{x^4 - 5} \cdot x\,dx \approx 8.262977$. $\bar{x} = \frac{8.262977}{3.238548} = 2.551$

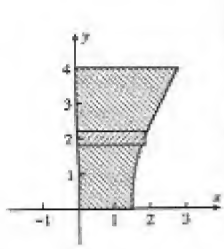
$M_x = \frac{1}{2}\int_2^3 (x^4 - 5)^{2/3}dx \approx 5.41065$. $\bar{y} = \frac{5.41065}{3.238548} = 1.167$

27. Exercise 43. The region bounded by $y = \sqrt[4]{x^3 + 4}$, the y axis, and $y = 3$.

▸ $x = \sqrt[3]{y^4 - 4}$. A $= \int_{\sqrt{2}}^{3} \sqrt[3]{y^4 - 4}\,dy \approx 4.20015$. $M_y = \frac{1}{2}\int_{\sqrt{2}}^{3}(y^4 - 4)^{2/3}dy \approx 20.5674$. $\bar{x} = \frac{6.317056}{4.20015} = 1.504$

$M_x = \int_{\sqrt{2}}^{3} \sqrt[3]{y^4 - 4} \cdot y\,dy \approx 9.975026$. $\bar{y} = \frac{9.975026}{4.20015} = 2.375$

28. Exercise 44. The region bounded by the graph of $y = \sqrt[3]{x^4 - 5}$, the x axis, the y axis, and the line $y = 4$.



▷ See the figure at the right. We solve for x: $x = \sqrt[4]{y^3 + 5}$.

$$A = \int_0^4 \sqrt[4]{y^3+5}\, dy \approx 7.988689$$

$$M_y = \tfrac{1}{2}\int_0^4 \sqrt{y^3+5}\, dy \approx 8.370266.\ \bar{x} = \frac{M_y}{A} = \frac{8.370266}{7.988689} = 1.048$$

$$M_x = \int_0^4 \sqrt[4]{y^3+5}\cdot y\, dy \approx 17.97957.\ \bar{y} = \frac{M_x}{A} = \frac{17.97957}{7.988689} = 2.251$$

■ The centroid is at (1.048, 2.251).

29. Exercise 45. The region bounded by $y = \sin x^3$, the y axis, and $y = 1$, $x \in [0, \sqrt[3]{\pi/2}]$.

▷ $x = \sqrt[3]{\sin^{-1} y}$. $A = \int_0^1 \sqrt[3]{\sin^{-1} y}\, dy \approx 0.7754962$. $M_y = \frac{1}{2}\int_0^1 (\sin^{-1} y)^{2/3} dy \approx 0.3244096$. $\bar{x} = \frac{0.3244096}{0.7754962} = 0.4183$

$M_x = \int_0^1 \sqrt[3]{\sin^{-1} y}\cdot y\, dy \approx 0.4491669$. $\bar{y} = \frac{0.4491669}{0.7754962} = 0.5792$

30. Exercise 46. The region bounded by $y = \tan x^2$, the y axis, and $y = 1$, $x \in [0, \sqrt{\pi}/2]$.

▷ $x = \sqrt{\tan^{-1} y}$. $A = \int_0^1 \sqrt{\tan^{-1} y}\, dy \approx 0.6298233$. $M_y = \frac{1}{2}\int_0^1 \tan^{-1} y\, dy \approx 0.2194123$ (Exact: $\frac{1}{8}\pi - \frac{1}{4}\ln 2$).

$\bar{x} = \frac{0.2194123}{0.6298233} = 0.3484$. $M_x = \int_0^1 \sqrt{\tan^{-1} y}\cdot y\, dy \approx 0.3719476$. $\bar{y} = \frac{0.3719476}{0.6298233} = 0.5906$

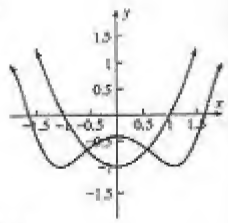
31. Exercise 49. The region bounded by the graph of $y = \sin x + 2$, $y = \tan x$, and the y axis.

▷ $\sin x + 2 = \tan x$ when $x = 1.2437$. $A = \int_0^{1.2437} (\sin x + 2 - \tan x) dx = 2.0307$.

$M_y = \int_0^{1.2437} (\sin x + 2 - \tan x) x\, dx \approx 1.0763$. $\bar{x} = \frac{1.0763}{2.0307} = 0.5300$

$M_x = \frac{1}{2}\int_0^{1.2437} [(\sin x + 2)^2 - \tan^2 x] dx \approx 3.2278$. $\bar{y} = \frac{3.2278}{2.0307} = 1.590$

32. Exercise 50. The region bounded by the graph of $y = \cos(x^2 + 2)$ and $y = x^2 - 1$.



▷ $\cos(x^2+2) = x^2 - 1$ when $x = \pm 0.56515$. By symmetry, $\bar{x} = 0$.

$$A = 2\int_0^{0.56515} [\cos(x^2+2) - (x^2-1)] dx \approx 0.43574.$$

$$M_x = \tfrac{1}{2}\cdot 2\int_0^{0.56515} [\cos^2(x^2+2) - (x^2-1)^2] dx \approx -0.30685$$

$$\bar{y} = \frac{M_x}{A} = \frac{-0.39685}{0.43574} = -0.7042$$

■ The centroid is at (0, −0.7042).

33. $y^2 = 4px$, $x \in [0, a]$. Because the region is symmetric with respect to the x axis, $\bar{y} = 0$.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\cdot 2p^{1/2} m_i^{1/2} \Delta_i x = 4p^{1/2}\int_0^a x^{1/2} dx = 4p^{1/2}\Big[\tfrac{2}{3}x^{3/2}\Big]_0^a = \tfrac{8}{3}a^{3/2}p^{1/2}$$

$$m_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2m_i\cdot 2p^{1/2} m_i^{1/2} \Delta_i x = 4p^{1/2}\int_0^a x^{3/2} dx = 4p^{1/2}\Big[\tfrac{2}{5}x^{5/2}\Big]_0^a = \tfrac{8}{5}a^{5/2}p^{1/2}$$

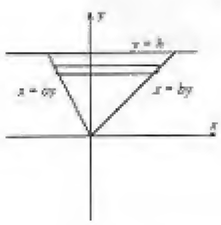
$\bar{x} = \frac{m_y}{A} = \frac{\frac{8}{5}a^{5/2}\pi^{1/2}}{\frac{8}{3}a^{3/2}\pi^{1/2}} = \frac{3}{5}a$. The centroid will be at (p, 0) if $\frac{3}{5}a$ = p. Thus $a = \frac{5}{3}$p.

We have assumed p and a are positive; if p is negative, so is a, with the same result.

34. Prove that the distance from the centroid of a triangle to any side of the triangle is equal to one-third of the length of the altitude to that side.

▷ Choose the x axis parallel to the given side with the origin at the vertex. Refer to the figure. Let the triangle be bounded on the left by the line $x = ay$ and on the right by $x = by, y \in [0, h]$, and above by $y = h$, where h is the length of the altitude. Then

$$\begin{aligned} A &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (am_i - bm_i)\Delta_i y & \quad M_x &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (am_i - bm_i)m_i\Delta_i y \\ &= \int_0^h (a-b)y\,dy & &= \int_0^h (a-b)y^2\,dy \\ &= \tfrac{1}{2}(a-b)y^2\Big]_0^h & &= \tfrac{1}{3}(a-b)y^3\Big]_0^h \\ &= \tfrac{1}{2}(a-b)h^2 & &= \tfrac{1}{3}(a-b)h^3 \end{aligned}$$



Therefore $\quad \bar{y} = \dfrac{M_x}{A} = \dfrac{\frac{1}{3}(a-b)h^3}{\frac{1}{2}(a-b)h^2} = \frac{2}{3}h$

and so the distance of the centroid from the given side is $h - \frac{2}{3}h = \frac{1}{3}h$.

In Exercises 35–39, use the Theorem of Pappus to find the indicated quantity.

35. Let the semicircle be above the x axis with its center at the origin. If its radius is r units then $A = \frac{1}{2}\pi r^2$. If the region is revolved about the axis a sphere is obtained and $V = \frac{4}{3}\pi r^3$. Because the region is symmetric with resect to the y axis, $\bar{x} = 0$. By the theorem of Pappus

$\frac{4}{3}\pi r^3 = 2\pi\bar{y}\cdot\frac{1}{2}\pi r^2$; $\bar{y} = \frac{4r}{3\pi}$

Therefore the centroid is at the point on the bisecting radial line whose distance from the center of the semicircle is $\frac{4}{3\pi}$ times the radius.

36. The volume of a right-circular cone with base radius r units and height h units.

▶ The cone is the solid generated by revolving a right triangle of leg lengths h and r about the side of length h. From Exercise 34, the distance of the centroid from the axis is $\bar{y} = \frac{1}{3}r$.
By the theorem of Pappus, $V = 2\pi\bar{y}A = 2\pi\cdot\frac{1}{3}r\cdot\frac{1}{2}hr = \frac{1}{3}\pi r^2 h$.

37. Let R be the region bounded by the semicircle $y = \sqrt{r^2 - x^2}$ and the x axis. Use the theorem of Pappus to find the moment of R with respect to the line $y = -r$.
The area of the semicircular region R is $A = \frac{1}{2}\pi r^2$. If R is revolved about the x axis, a sphere is generated with volume $V = \frac{4}{3}\pi r^3$. Substituting into the theorem of Pappus and solving for $\bar{y}$, we get

$$V = 2\pi\bar{y}A; \quad \tfrac{4}{3}\pi r^3 = 2\pi\bar{y}(\tfrac{1}{2}\pi r^2); \qquad \bar{y} = \frac{4r}{3\pi}$$

Let $\bar{r}$ be the distance between the line $y = -r$ and the the centroid of R. Then

$$\bar{r} = r + \bar{y} = r + \frac{4r}{3\pi}$$

The moment of R with respect to the line $y = -r$ is given by

$$M = A\bar{r} = (\tfrac{1}{2}\pi r^2)\left(r + \frac{4r}{3\pi}\right) = (\tfrac{1}{2}\pi + \tfrac{2}{3})r^3$$

38. By the theorem of Pappus, $V = 2\pi\bar{s}A$, where $\bar{s}$ is the distance from the centroid of R to the line $x - y - r = 0$. Now $A = \frac{1}{2}\pi r^2$ and by Exercise 21 the centroid of R is at the point $\left(0, \frac{4r}{3\pi}\right)$. Hence by the formula for the distance from a point to a line

$$\bar{s} = \frac{\left|0 - \frac{4r}{3\pi} - r\right|}{\sqrt{1^2 + (-1)^2}} = \tfrac{1}{2}\sqrt{2}\left(\frac{4r}{3\pi} + r\right). \text{ Thus } V = 2\pi\cdot\tfrac{1}{2}\sqrt{2}\left(\frac{4r}{3\pi} + r\right)\cdot\tfrac{1}{2}\pi r^2 = \tfrac{1}{6}\sqrt{2}(4 + 3\pi)\pi r^3.$$

39. Prove the Theorem of Pappus for volumes of of solids of revolution.

▶ Let the x axis be L and let R be bounded above by $y = f(x)$ and below by $y = g(x)$, $x \in [a, b]$. Then

$$\begin{aligned} V &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi[f(m_i)^2 - g(m_i)^2]\Delta_i x = 2\pi\int_a^b \tfrac{1}{2}[f(x)^2 - g(x)^2]\,dx \\ &= 2\pi M_x = 2\pi\cdot\frac{M_x}{A}\cdot A = 2\pi\bar{y}A \end{aligned}$$

6.4 WORK

Constant Force If a constant force of F units acting on a body causes a displacement of D units, where the force and the displacement are in the same direction, then the work done by the force is given by

$$W = F \cdot D$$

The following definition is used to calculate the work done by a variable force. Its use is illustrated in Exercises 4, 8, 12 and 20.

6.4.1 Definition Let the function f be continuous on the closed interval $[a, b]$ and $f(x)$ units be the force acting on an object at the point x on the x axis. If W units is the *work* done by the force as the object moves from a to b, then

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x = \int_a^b f(x)\, dx$$

The following result is used if the object is not a point but a solid of known cross section, as illustrated in Exercises 16 and 24.

Work by slicing If a solid cylinder of weight density ρ, base area measure $A(x)$ and height measure $h(x)$, and hence of weight $\rho A(x)h(x)$, $x \in [a, b]$, is raised a distance Δx, the result is the same as if a slice of area $A(x)$ and height Δx is moved a distance $h(x)$ from the bottom to the top. Hence, by Definition 6.4.1, the total work is

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho A(w_i)h(w_i)\Delta_i x = \int_a^b \rho A(x)h(x)\, dx$$

Hooke's Law If a spring is stretched x units beyond its natural length, it is pulled back with a force equal to kx units, where k is a constant depending on the material and the natural length of the spring.

The following table summarizes the units used and the weight density of water in each.

System	Force	Distance	Work	density
British	pound (lb)	foot (ft)	ft-lb	62.4 lb/ft^3
SI (mks)	newton (N)	meter (m)	joule	9810 N/m^3
CGS	dyne ($= 10^{-5}$ N)	centimeter (cm)	erg	981 dyne/cm^3

Exercises 6.4

In these Exercises Δ is a partition of the specified interval.

In Exercises 1 and 2, find the work done by force f lb moving a particle from a to b feet from the origin.

1. W ft-lb is the work done as the particle moves from $x = 1$ to $x = 3$.

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (2w_i + 1)^2 \Delta_i x = \int_1^3 (2x+1)^2 dx = \tfrac{1}{3} \cdot \tfrac{1}{2}(2x+1)^3 \Big|_1^3 = \tfrac{1}{6}(343 - 27) = \tfrac{158}{3}$$

2. W ft-lb is the work done as the particle moves from $x = 0$ to $x = 2$. $W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} w_i^2 \sqrt{w_i^3 + 1}\,\Delta_i x$

$$= \int_0^2 (x^3 + 1)^{1/2}(x^2 dx) = \tfrac{1}{3}\int_0^2 (x^3+1)^{1/2} d(x^3 + 1) = \tfrac{2}{9}(x^3+1)^{3/2}\Big]_0^2 = \tfrac{2}{9}(27 - 1) = \tfrac{52}{9}$$

In Exercises 3 and 4, find the work done by force f newtons moving a particle from a to b meters from the origin.

3. W joules is the work done as the particle moves from $x = 3$ to $x = 8$. Let $v = \sqrt{x+1}$. Then $v^2 = x + 1$; $x = v^2 - 1$; $dx = 2v\, dv$. When $x = 3$, $v = 2$; when $x = 8$, $v = 3$. Hence

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} w_i\sqrt{w_i + 1}\,\Delta_i x = \int_3^8 x\sqrt{x+1}\, dx = \int_2^3 (v^2 - 1)v(2v\, dv) = 2\int_2^3 (v^4 - v^2)dv$$

$$= 2\Big[\tfrac{1}{5}v^5 - \tfrac{1}{3}v^3\Big]_2^3 = 2\Big[\big(\tfrac{243}{5} - 9\big) - \big(\tfrac{32}{5} - \tfrac{8}{3}\big)\Big] = \tfrac{1076}{15}$$

4. $f(x) = (4x - 1)^2$; $a = 1$; $b = 4$.

▹ We apply Definition 6.4.1. Thus, if W joules is the work done, then

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x = \int_1^4 (4x-1)^2\, dx = \frac{1}{3 \cdot 4}(4x-1)^3\Big]_1^4 = \tfrac{1}{12}(15^3 - 3^3) = 279$$

▪ The work is 279 joules.

5. $96 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (2w_i - 3)\Delta_i x = \int_0^K (2x-3)dx = x^2 - 3x \Big|_0^K = K^2 - 3K$; $K^2 - 3K - 96 = 0$

$K = \frac{1}{2}(3 \pm \sqrt{9+384}) = \frac{1}{2}(3 \pm \sqrt{393})$. Because $K > 0$, $K = \frac{1}{2}(3 + \sqrt{393})$.

6. $90 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4w_i - 3)\Delta_i x = \int_0^K (4x-3)dx = 2x^2 - 3x \Big|_0^K = 2K^2 - 3K$; $2K^2 - 3K - 90 = 0$;
$(K+6)(2K-15) = 0$; $K = -6$ or $K = \frac{15}{2}$. Because $K > 0$, $K = \frac{15}{2}$.

7. Place the spring along the x axis with the origin at the point where the stretching starts. By Hooke's law, $f(x) = kx$. Because $f(\frac{1}{2}) = 20$, $\frac{1}{2}k = 20$; so $k = 40$. Hence $f(x) = 40x$. W in.-lb is the work done in stretching the spring from 8 in. to 11 in.; x is in $[0,3]$. $W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 40 w_i \Delta_i x = 40 \int_0^3 x\,dx = 20x^2 \Big|_0^3 = 180$

8. A spring has a natural length of 10 in., and a 30-lb force stretches it to $11\frac{1}{2}$ in. (a) Find the work done in stretching the spring from 10 in. to 12 in. (b) Find the work done in stretching the spring from 12 in. to 14 in.

▸ Place the x axis on the spring with the origin at the free end of the spring in its natural state. Let $f(x)$ lb be the force when the spring is stretched x inches. By Hooke's law, $f(x) = kx$. Because a 30-lb force stretches the spring by $\frac{3}{2}$ in., then $30 = kx$ and so $k = 20$. Therefore, $f(x) = 20x$.

(a) Let W in.-lb be the work done in stretching the spring from 10 in. to 12 in. When the length is 10 in., the spring is in its natural state and thus $x = 0$. When the length is 12 in., the spring has been stretched by 2 in. and thus $x = 2$. By Definition 6.4.1 we have

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x = \int_0^2 20x\,dx = 10x^2 \Big]_0^2 = 10(4) = 40$$

▪ The work is 40 inch-pounds.

(b) When the spring is stretched from a length of 12 in. to a length of 14 in., then $2 \le x \le 4$, and thus

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x = \int_2^4 20x\,dx = 10x^2 \Big]_2^4 = 10(16-4) = 120$$

▪ The work is 120 inch-pounds.

9. Place the spring along the x axis with the origin at the point where the stretching starts. By Hooke's law, $f(x) = kx$. Because $f(\frac{1}{2}) = 8$, $\frac{1}{2}k = 8$; so $k = 16$. Hence $f(x) = 16x$. W joules is the work done in stretching the spring from 4m to 5m; x is in $[0,1]$. $W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 16 w_i \Delta_i x = 16 \int_0^1 x\,dx = 8x^2 \Big|_0^1 = 8$

10. Place the spring along the x axis with the origin at the point where the stretching starts. By Hooke's law, $f(x) = kx$. Because $f(4) = 500$, $4k = 500$; so $k = 125$. Hence $f(x) = 125x$. W ergs is the work done in stretching the spring from 20cm to 28cm; x is in $[0,8]$.

$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 125 w_i \Delta_i x = 125 \int_0^8 x\,dx = \frac{125}{2}x^2 \Big|_0^8 = 4000$

11. Place the spring along the x axis with the origin at the point where the compressing starts. By Hooke's law, $f(x) = kx$. Because $f(2) = 600$, $2k = 600$; so $k = 300$. Hence $f(x) = 300x$. W ergs is the work done in compressing the spring from 12 cm to 9 cm; x is in $[0,3]$.

$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 300 w_i \Delta_i x = 300 \int_0^3 x\,dx = 150x^2 \Big|_0^3 = 150 \cdot 9 = 1350$

12. A spring has a natural length of 6 in. A 1200-lb force compresses it to $5\frac{1}{2}$ in. Find the work done in compressing it from 6 to $4\frac{1}{2}$ in.

▸ Place the x axis along the spring with the origin at the point where the compression begins. Let $f(x)$ lb be the force when the spring is compressed x in. By Hooke's law, $f(x) = kx$. Because a 1200-lb force compresses the spring $\frac{1}{2}$ in., then $1200 = k \cdot \frac{1}{2}$ and so $k = 2400$. Therefore $f(x) = 2400x$. Let W in-lb. be the work done in compressing the spring from 6 to $4\frac{1}{2}$ in. Then x is in $[0, \frac{3}{2}]$, and by Definition 6.4.1, we have

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2400 w_i \Delta_i x = \int_0^{3/2} 2400x\,dx = 1200x^2 \Big]_0^{3/2} = 1200(\tfrac{9}{4}) = 2700$$

▪ The work done is 2700 inch-pounds.

13. Choose the positive x axis downward and take the origin at the top of the tank; x is in $[0,5]$. Let W ft-lb be the work done in pumping the water up to a level 1 ft above the surface of the tank. An element of volume is a rectangular slab of altitude $\Delta_i x$ ft, length 15 ft and width 25 ft, and is raised $(w_i + 1)$ft.

$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 62.4(15)(25)(w_i + 1)\Delta_i x = 23400 \int_0^5 (x+1)dx = 23400\Big[\frac{1}{2}x^2 + x\Big]_0^5 = 23400(\frac{25}{2} + 5) = 409500$

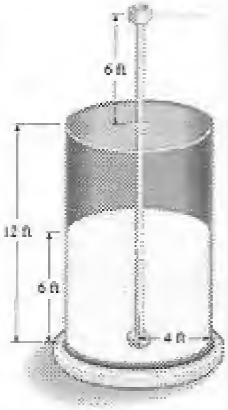
14. Choose the positive x axis upward and take the origin at the bottom of the tank; x is in $[0, 2]$. Let W ft-lb be the work done in pumping the water up to the top of the trough. An element of volume is a rectangular slab of altitude $\Delta_i x$ ft, length 10 ft and width w_i ft, and is raised $(2 - w_i)$ft.

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 62.4(10)w_i(2 - w_i)\Delta_i x = 624 \int_0^2 (2x - x^2)dx = 624\Big[x^2 - \tfrac{1}{3}x^3\Big]_0^2 = 624 \cdot \tfrac{4}{3} = 832$$

15. Choose the positive x axis downward and take the origin at the top of the tank; x is in $[2, 6]$. Let W ft-lb be the work done in pumping the water to the top of the tank. An element of volume is a circular disk of radius $\sqrt{6^2 - w_i^2}$ ft and is raised w_i ft.

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} w\pi(36 - w_i^2)w_i \,\Delta_i x = \pi w \int_2^6 (36 - x^2)x\, dx = \pi w \int_2^6 (36x - x^3)dx = \pi w\Big[18x^2 - \tfrac{1}{4}x^4\Big]_2^6$$
$$= \pi w[(18 \cdot 36 - \tfrac{1}{4} \cdot 6^4) - (18 \cdot 4 - \tfrac{1}{4} \cdot 16)] = 256\pi w$$

16. A right circular cylindrical tank with a depth of 12 ft and a radius of 4 ft is half full of oil weighing 60 lb/ft^3. Find the work done in pumping the oil to a height 6 ft above the tank.



▷ Refer to the figure. Take the origin at the center of the bottom of the tank with positive direction upward. At a height of 6 ft above the tank $x = 18$. An element of volume is a disk of radius 4 ft, height $\Delta_i x$ and density 60 lb/ft^3. Thus the number of pounds in its weight is

$$60 \cdot \pi(4)^2\Delta_i x = 960\pi\Delta_i x$$

and it is raised a mean distance of $(18 - w_i)$ ft, $x \in [0, 6]$. Therefore

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 960\pi(18 - w_i)\Delta_i x = \int_0^6 960\pi(18 - x)\, dx$$
$$= 960\pi\Big[18x - \tfrac{1}{2}x^2\Big]_0^6$$
$$= 960\pi[18(6) - \tfrac{1}{2}(36)] = 86400\pi$$

▪ The work done is approximately 271,400 foot-pounds.

17. Choose the positive x axis downward and take the origin at the top of the well; x is in $[0, 200]$. Let W ft-lb be the work done in pulling the cable and weight to the top of the well. When the cable is raised a distance of $\Delta_i x$ ft its length is w_i ft and the total weight is $(4w_i + 100)$lb.

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4w_i + 100)\Delta_i x = \int_0^{200} (4x + 100)dx = 2x^2 + 100x\Big|_0^{200} = 80{,}000 + 20{,}000 = 100{,}000$$

18. Choose the positive x axis downward and take the origin at the top of the well; x is in $[0, 100]$. Let W ft-lb be the work done in pulling the chain and weight to the top of the well. When the chain is raised a distance of $\Delta_i x$ ft its length is w_i ft and the total weight is $(\frac{1}{10}w_i + 80)$lb.

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (\tfrac{1}{10}w_i + 80)\Delta_i x = \int_0^{100} (\tfrac{1}{10}x + 80)dx = \tfrac{1}{20}x^2 + 80x\Big|_0^{100} = 500 + 8000 = 8{,}500$$

19. Choose the positive x axis downward and take the origin at the top of the well; x is in $[0, 100]$. Let W ft-lb be the work done in raising the bucket to the top of the well. When the chain is raised $\Delta_i x$ ft, its length is w_i ft and its weight is $0.1w_i$ lb. The weight of the sand is $60(w_i/100) = 0.6w_i$.

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (20 + 0.1w_i + 0.6w_i)\Delta_i x = \int_0^{100} (20 + 0.70x)dx = 20x + 0.35x^2\Big|_0^{100} = 2000 + 3500 = 5500$$

20. As a flour sack is being raised a distance of 9 ft, flour leaks out at such a rate that the number of pounds lost is directly proportional to the square root of the distance traveled. If the sack originally contained 60 lb of flour and it loses a total of 12 lb while being raised the 9 ft, find the work done in raising the sack.

▷ Let the x axis be directed upward with the origin at the point where the sack is originally. Let $f(x)$ lb be the weight of the sack when it is x ft above the origin. We are given that the weight lost is $k\sqrt{x}$. Because the sack has lost 12 lb when $x = 9$, then $12 = k\sqrt{9}$ and so $k = 4$. Therefore $f(x) = 60 - 4\sqrt{x}$. By Definition 6.4.1, if W ft-lb is the work, then

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x = \int_0^9 (60 - 4\sqrt{x})\, dx = 60x - \tfrac{8}{3}x^{3/2}\Big]_0^9 = 60(9) - \tfrac{8}{3}(27) = 468$$

▪ The work done is 468 foot-pounds.

21. Choose the positive x axis downward and take the origin at the surface of the water; x is in $[0,5]$. Let W joules be the work done in pumping the water to the top of the tank. An element of volume is a circular disk of thickness $\Delta_i x$ m and radius 5 m and is raised $(5+w_i)$ m.

$W = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 1000(9.81)(25\pi)(5+w_i)\Delta_i x = 245250\pi \int_0^5 (5+x)dx = 245250\pi\left[5x+\frac{1}{2}x^2\right]_0^5$
$= 245250\pi(25+\frac{25}{2}) = 9{,}196{,}875\pi$

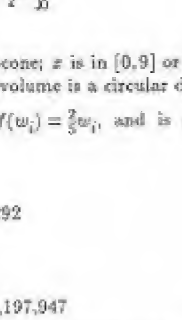
22 and 23. Choose the positive x axis upward and take the origin at the vertex of the cone; x is in $[0,9]$ or $[0,8]$. Let W joules be the work in pumping the oil to the top of the tank. An element of volume is a circular disk of thickness $\Delta_i x$ m and radius $f(w_i)$ m, where by similar triangles, $\frac{f(w_i)}{4} = \frac{w_i}{10}$; $f(w_i) = \frac{2}{5}w_i$, and is raised $(10-w_i)$ m.

22. $W = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (1000)(9.81)\pi(\frac{2}{5}w_i)^2\,(10-w_i)\Delta_i x = 9810\pi \int_0^9 (\frac{2}{5}x)^2(10-x)dx$
$= 9810\pi \int_0^9 (\frac{8}{5}x^2 - \frac{4}{25}x^3)dx = 9810\pi\left[\frac{8}{15}x^3 - \frac{1}{25}x^4\right]_0^9 = 9810\pi\left[\frac{8}{15}(9)^3 - \frac{1}{25}(9)^4\right] = 3{,}894{,}292$

23. $W = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (950)(9.81)\pi(\frac{2}{5}w_i)^2\,(10-w_i)\Delta_i x = 9319.5\pi \int_0^8 (\frac{2}{5}x)^2(10-x)dx$
$= 9319.5\pi \int_0^8 (\frac{8}{5}x^2 - \frac{4}{15}x^3)dx = 9319.5\pi\left[\frac{8}{15}x^3 - \frac{1}{25}x^4\right]_0^8 = 9319.5\pi\left[\frac{8}{15}(8)^3 - \frac{1}{25}(8)^4\right] = 3{,}197{,}947$

24. A tank in the form of an inverted right-circular cone is 8 m across the top and 10 m deep. The tank is filled to a height of 9 m with water. Find the work necessary to pump half of the water to the top of the tank.

▷ Refer to the figure. We take the origin at the bottom of the tank with the positive x axis upward. If r m is the radius of a section at height x m, by similar triangles $r/4 = x/10$ and so



$$r = \frac{2}{5}x$$

From the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a circular cone, we conclude that when the water is x meters deep, the volume of the water is $V(x)$ m^3, where

$$V(x) = \frac{1}{3}\pi(\frac{2}{5}x)^2 x = \frac{4}{75}\pi x^3$$

Let the water be a meters deep when half the water is removed. Then

$$V(a) = \frac{1}{2}V(9);\quad \frac{4}{75}\pi a^3 = \frac{1}{2}\cdot\frac{4}{75}\pi 9^3;\quad a^3 = 364.5;\quad a = (364.5)^{1/3}$$

An element of volume is a circular disk of radius r m, height $\Delta_i x$ m, and weight density 9810 N/m^3. Thus, the number of newtons in its weight is

$$9810\pi r^2\Delta_i x = 9810\pi(\frac{2}{5}x)^2\Delta_i x = 1596.6\pi x^2\Delta_i x$$

and it is raised a mean distance of $(10-w_i)$ meters, $x \in [a,9]$. If W joules is the work, then

$$W = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 1596.6\pi w_i(10-w_i)\Delta_i x = \int_a^9 1596.6\pi x^2(10-x)\,dx = 1596.6\pi \int_a^9 (10x^2 - x^3)\,dx$$

$$= 1596.6\pi\left[\frac{10}{3}x^3 - \frac{1}{4}x^4\right]_a^9 = 1596.6\pi[\frac{10}{3}(729-364.5) - \frac{1}{4}(9^4 - 364.5^{4/3})] = 1{,}132{,}000$$

▪ The work is 1,132,000 joules.

25. Choose the positive x axis downward and take the origin at the top of the tank; x is in $[0,2]$. Let W ft-lb be the work done in pumping the water to a point 5 ft above the top of the tank. An element of volume is a rectangular slab of thickness $\Delta_i x$ ft, width 2 ft, length 6 ft, and is raised $(5+w_i)$ ft.

$W = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} w(2)(6)(5+w_i)\Delta_i x = 12w \int_0^2 (5+x)dx = 12w\left[5x+\frac{1}{2}x^2\right]_0^2 = 12(62.4)(10+2) = 8986$

A 0.1 hp motor can do 55 ft-lb of work per second. Hence the time it takes is 163.4 sec.

26. Choose the positive x axis upward with the origin at the center of the earth. F lb is the force, where $F = (\frac{R}{x})^2 w$. If W mi-lb is the work done in moving the meteorite from a miles to the surface, then

$W = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (\frac{R}{w_i})^2 w\Delta_i x = R^2 w \int_R^a x^{-2}dx = -R^2 w x^{-1}\Big]_R^a = R^2 w\left(\frac{1}{R} - \frac{1}{a}\right)$

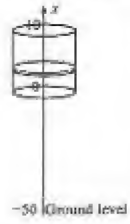
27. Choose the positive x axis downward and take the origin at the top of the tank; x is in $[0, 6]$. Let W ft-lb be the work done in pumping the oil to the top of the tank. An element of volume is a rectangular slab of thickness $\Delta_i x$ ft, width 4 ft, length 12 ft, and is raised w_i ft.

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 50(4)(12) w_i \,\Delta_i x = 2400 \int_0^6 x\,dx = 1200x^2 \Big]_0^6 = 1200(36) = 43{,}200$$

Let a ft be the distance that the surface of the oil is lowered when one-third of the work is done. Then

$$\frac{43{,}200}{3} = 1200x^2 \Big]_0^a = 1200a^2;\ a^2 = 12;\ a = 2\sqrt{3}$$

28. A cylindrical tank 10 ft high and 5 ft in radius is standing on a platform 50 ft high. Find the depth of the water when one half of the work required to fill the tank from the ground level through a pipe in the bottom has been done.



▷ Refer to the figure. We take the origin at the center of the bottom of the tank with the x axis directed upward. An element of volume is a circular disk of radius 5 ft, height $\Delta_i x$ ft and weight density w lb/ft^3. Then its weight is $w\pi(5^2)\Delta_i x$ lb and it is raised a mean distance of $(50 + w_i)$ ft. (Although water enters the tank through the bottom, each element of volume must be raised to the level of water in the tank.) Let b ft be the depth of the tank when half the work is done. Then

$$\int_0^b 25w\pi(50 + x)\,dx = \frac{1}{2}\int_0^{10} 25w\pi(50 + x)\,dx$$

$$2\int_0^b (x + 50)\,dx = \int_0^{10} (x + 50)\,dx$$

$$(x + 50)^2 \Big]_0^b = \tfrac{1}{2}(x + 50)^2 \Big]_0^{10}$$

$$(b + 50)^2 - 50^2 = \tfrac{1}{2}(60^2 - 50^2)$$

$$(b + 50)^2 = 3050$$

$$b = \sqrt{3050} - 50 \approx 5.23$$

▪ The water is approximately 5.23 feet deep when half the work is done.

29. Choose the positive x axis downward with the origin 1 ft above the top of the tank. Let W ft-lb be the work done in pumping all the water to a point 1 ft above the top of the tank. An element of volume is a circular disk centered on the x axis, $x \in [1, 4]$, of radius $\sqrt{e^{-2w_i}/w_i}$ ft, and is raised w_i ft. Then W =

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} w\pi\left(\frac{e^{-2w_i}}{w_i}\right) w_i \Delta_i x = 62.4\pi \int_1^4 e^{-2x}\,dx = -31.2\pi e^{-2x} \Big]_1^4 = -31.2\pi(e^{-8} - e^{-2}) = 31.2\pi(e^{-2} - e^{-8}) \approx 13.2$$

30. An element of volume is a slab of area A and thickness $\Delta_i x$; the force is PA.

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} PA\Delta_i x = \lim_{\|\Delta\| \to 0} P\Delta_i V = \int_{V_1}^{V_2} P\,dV$$

31. Let p lb/in^2 be the pressure when the volume is V in^3. By Boyle's law $PV = C$. When $P = 50$, $V = 60$; thus $C = 3000$. Therefore $P = \frac{3000}{V}$. From the result of Exercise 30, using a minus sign because the piston is compressing the gas, if W in-lb is the work done by the piston, then

$$W = -\int_{60}^{40} P\,dV = \int_{40}^{60} \frac{3000}{V}\,dV = 3000 \ln|V| \Big]_{40}^{60} = 3000(\ln 60 - \ln 40) = 3000 \ln \tfrac{3}{2}$$

6.5 FORCE DUE TO FLUID PRESSURE

If a flat surface of area A square units is submerged horizontally at a depth of h units in a fluid with weight density ρ pounds per cubic unit, then the total force due to fluid pressure on one side of the surface is F pounds, where

$$F = \rho h A$$

Pascal's principle At any point in a fluid, the pressure is the same in all directions.

6.5.1 Definition Suppose that a flat plate is submerged in a fluid for which a measure of its weight density is ρ. x units along the plate, the depth is $h(x)$ units below the surface of the fluid and the length of the plate is $f(x)$ units, where f and h are continuous and nonnegative on the closed interval $[a, b]$. If F is the measure of the force caused by fluid pressure on one side of the plate, then

$$\mathrm{F} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho h(w_i) f(w_i) \Delta_i x = \int_a^b \rho h(x) f(x)\, dx$$

6.5.3 Centroid If the centroid of a plane region of area A is $\bar{x}$ units below the surface of a fluid of mass density ρ, then the total force F due to the fluid pressure against one side of the region is the same as it would be if the region were horizontal at a depth of $\bar{x}$ units below the surface of the fluid, that is

$$\mathrm{F} = \rho \bar{x} \mathrm{A}$$

Exercises 6.5

In these Exercises Δ is a partition of the specified interval, A square units is the area of the given region, and $\bar{x}$ is the abscissa of the centroid of the region.

1. Choose the positive x axis downward with the origin at the top of the plate, $x \in [0, 8]$.
If F lb is the force on one side of the plate, then

$$\mathrm{F} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho w_i (10) \Delta_i x = 10\rho \int_0^8 x\, dx = 5\rho x^2 \Big|_0^8 = 320\rho = 19{,}968$$

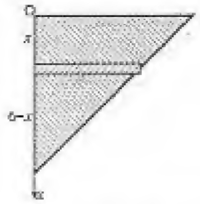
2. If F lb is the force on one side of the plate, then $\mathrm{F} = \rho \bar{x} \mathrm{A} = 62.4 \times 2 \times 4^2 = 1996.8$.

3. Choose the positive x axis downward with the origin at the top of the plate, $x \in [2, 6]$.
If F lb is the force on one side of the plate, then

$$\mathrm{F} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho w_i (4) \Delta_i x = 4\rho \int_2^6 x\, dx = 2\rho x^2 \Big|_2^6 = 2\rho(36 - 4) = 64\rho = 3993.6$$

4. A plate in the shape of an isosceles right triangle is submerged vertically in a tank of water, with one leg lying in the surface. The legs are each 6 ft long. Find the force due to water pressure on one side of the plate.

▶ The figure shows the plate. The origin is at the surface of the water. Because the plate is vertical, at x ft, $x \in [0, 6]$, the depth is x and the length at that depth is $(6 - x)$ ft. By Definition 6.5.1,



$$\mathrm{F} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho w_i (6 - w_i) \Delta_i x = \int_0^6 \rho x (6 - x)\, dx$$

$$= \rho \int_0^6 (6x - x^2)\, dx = \rho \Big[3x^2 - \tfrac{1}{3}x^3\Big]_0^6 = \rho[3(36) - \tfrac{1}{3}(216)]$$

$$= (62.4)(36) = 2246.4$$

▪ The force due to water pressure is 2246.4 pounds.

12. Solve Exercise 4 by using Formula 6.5.3.

▶ Refer to the figure above. If A ft^2 is the area of the triangular region, then $\mathrm{A} = \frac{1}{2}(6)6 = 18$. In Exercise 6.3.34 we proved that the distance from the centroid of a triangle to any side of the triangle is one-third the length of the altitude to that side. Because the length of the altitude to the horizontal side is 6 ft, then the distance from that side to the centroid of the region is 2 ft. Substituting $\bar{x} = 2$ and $\mathrm{A} = 18$ into Formula 6.5.3, we get

$$\mathrm{F} = \rho \bar{x} \mathrm{A} = \rho(2)18 = 62.4(36) = 2246.4$$

▪ The force due to water pressure is 2246.4 pounds.

5. Choose the positive x axis downward with the origin at the top of the tank, $x \in [0, \frac{3}{2}]$.
If F lb is the force on one end of the tank, then

$$\mathrm{F} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho w_i (2) \Delta_i x = 2\rho \int_0^{3/2} x\, dx = \rho x^2 \Big|_0^{3/2} = 2.25\rho = 140.4$$

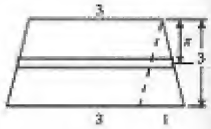
6 and 14. Because the water is 1 ft deep, the width of the water is $\frac{2}{3}\sqrt{3}$ and by Exercise 6.3.34, $\bar{x} = \frac{1}{3}$. If F lb is the force on one side of the plate, then $\mathrm{F} = \rho \bar{x} \mathrm{A} = 62.4 \times \frac{1}{3} \times (\frac{1}{2} \cdot \frac{2}{3}\sqrt{3} \cdot 1) = 12.009$

7. Choose the positive x axis upward with the origin at the vertex of the triangle. An equation of one side is $y = \frac{2}{3}x$, $x \in [0, 3]$. If F nt is the force on the gate, then F =

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2(9810)(18 - w_i)(\tfrac{2}{3}w_i)\Delta_i x = 13080 \int_0^3 (18x - x^2)dx = 13080\Big[9x^2 - \tfrac{1}{3}x^3\Big]_0^3 = 13080[81 - 9] = 941760$$

8. The face of a gate of a dam is vertical and in the shape of an isosceles trapezoid 3 m wide at the top, 4 m wide at the bottom, and 3 m high. If the upper base is 20 m below the surface of the water, find the total force due to liquid pressure on the gate.

▸ The figure shows the face of the gate. The origin is at the center of the upper base of the trapezoid. Because the plate is vertical, at x m, $x \in [0,3]$, the depth is $(x+20)$ m and, by similar triangles, the width is $(\frac{1}{3}x+3)$ m. By Definition 6.5.1,



$$F = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \rho(\tfrac{1}{3}w_i+3)(w_i+20)\Delta_i x = \int_0^3 \rho(\tfrac{1}{3}x+3)(x+20)\,dx$$
$$= \rho\int_0^3 (\tfrac{1}{3}x^2+\tfrac{29}{3}x+60)\,dx = \rho\left[\tfrac{1}{9}x^3+\tfrac{29}{6}x^2+60x\right]_0^3$$
$$= \rho[\tfrac{1}{9}(3^3)+\tfrac{29}{6}(3^2)+60(3)] = 9810(226.5) = 2,221,965$$

▪ The total force is about 2,222,000 newtons.

9. Choose the positive x axis upward with the origin at the bottom of the dam. An equation of one side of the triangle is $y = \frac{5}{4}x$, $x \in [0,10]$. If F nt is the force on the face of the dam, then

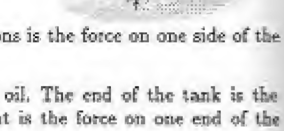
$$F = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2(9810)(10-w_i)(\tfrac{5}{4}w_i)\Delta_i x = 24525\int_0^{10}(10x-x^2)dx$$
$$= 24525\left[5x^2-\tfrac{1}{3}x^3\right]_0^{10} = 24525(500-\tfrac{1000}{3}) = 4,087,500$$

10. The radius of the tank is 2 m. From Exercise 6.3.35, $\bar{x} = \frac{4r}{3\pi} = \frac{8}{3\pi}$. If F newtons is the force on one side of the plate, then $F = \rho\bar{x}A = 7360\cdot\frac{8}{3\pi}(\frac{1}{2}\cdot\pi 2^2) = 39,253$

11. Choose the positive x axis downward with the origin at the surface of the oil. The end of the tank is the region enclosed by the circle $x^2+y^2=r^2$; so $y=\sqrt{r^2-x^2}$, $x\in[0,r]$. If F nt is the force on one end of the tank, then
$$F = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2(7360)w_i\sqrt{r^2-w_i^{\,2}}\Delta_i x = -7360\int_0^r \sqrt{r^2-x^2}(-2x\,dx) = -7360(\tfrac{2}{3})(r^2-x^2)^{3/2}\Big|_0^r = 4906.7r^3$$
Therefore $80,000 = 4906.7r^3$; $r^3 = 16.30$; $r \approx 2.54$. Hence the radius is about 2.54 m.

13. Equation (3) states that if F is the measure of the force due to liquid pressure against a vertical plane region, then $F = \rho\bar{x}A$. In Exercise 5, $A = 2(\frac{3}{2}) = 3$ and $\bar{x} = \frac{3}{4}$. Thus $F = \rho(\frac{3}{4})(3) = 2.25\rho$.

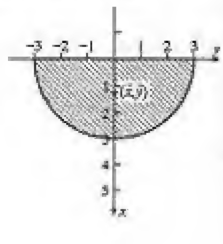
15. Choose the positive x axis downward with the origin at the center of the top of the dam. See the figure. The slope of line AB is $m = \frac{-15}{20} = -\frac{3}{4}$ and $y = 30-\frac{3}{4}(x-20) = 45-\frac{3}{4}x$, $x\in[0,20]$. If m_y is the moment of the trapezoid about the y axis, then



$$m_y = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} w_i(2)(45-\tfrac{3}{4}w_i)\Delta_i x = \int_0^{20}(90x-\tfrac{3}{2}x^2)dx$$
$$= 45x^2-\tfrac{1}{2}x^3\Big|_0^{20} = 45\cdot 400-\tfrac{1}{2}\cdot 8000 = 14,000$$
Therefore $F = \rho\bar{x}A = \rho m_y = 14,000(62.4) = 873,600$

16. A semicircular plate with a radius of 3 ft is submerged vertically in a tank of water, with its diameter lying on the surface. Use Formula 6.5.3 to find the total force due to water pressure on one side of the plate.

▸ The figure shows the plate. The area of the plate is A ft^2, where
$$A = \tfrac{1}{2}\pi(3^2) = \tfrac{9}{2}\pi$$
In Exercise 6.3.35 we proved that the distance from the diameter of a semicircular region of radius r units to its centroid is $(4r)/(3\pi)$ units. Because $r = 3$, the distance from the surface of the water to the centroid of the region is $\bar{x}$ feet, where



$$\bar{x} = \tfrac{4}{\pi}$$
Substituting the values of A and $\bar{x}$ into Formula 6.5.3, we obtain
$$F = \rho\cdot\tfrac{4}{\pi}\cdot\tfrac{9}{2}\pi = 18\rho = 18(62.4) = 1123.2$$

▪ The total force due to water pressure is 1123.2 pounds.

17. See Exercise 15. If m_y ft-lb is the moment of the force about the lower base, then

$$m_y = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \rho(20 - w_i)(w_i)(2)(45 - \tfrac{3}{4}w_i)\Delta_i x = \rho \int_0^{20} (1200x - 30x^2 - \tfrac{3}{2}x^3)dx$$

$$= \rho\left[600x^2 - 10x^3 - \tfrac{3}{8}x^4\right]_0^{20} = 62.4\left[600 \cdot 400 - 10 \cdot 8{,}000 - \tfrac{3}{8} \cdot 160{,}000\right] = 6.24 \times 10^6$$

18. If F newtons is the force on one side of the plate, then

$$F = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \frac{\rho}{2}\left(\frac{3}{2} - \frac{w_i^2}{6}\right)\left(\frac{3}{2} - \frac{w_i^2}{6}\right)\Delta_i x = 9.81 \int_0^3 (\tfrac{9}{4} - \tfrac{1}{2}x^2 + \tfrac{1}{36}x^4)dx = 9.81\left[\tfrac{9}{4}x - \tfrac{1}{6}x^3 + \tfrac{1}{180}x^5\right]_0^3 = 9.81 \cdot \tfrac{18}{5} = 35.316$$

19. Choose the positive x axis downward with the origin at the center of the circle. Then $x^2 + y^2 = 9$ so $y = \sqrt{9 - x^2}$, $x \in [0, 3]$. If F lb is the total force on an end, then

$$F = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \rho w_i(2\sqrt{9 - x^2})\Delta_i x = -\rho \int_0^3 \sqrt{9 - x^2}(-2x\, dx) = -\tfrac{2}{3}\rho(9 - x^2)^{3/2}\Big]_0^3 = -\tfrac{2}{3}\rho(0 - 27) = 18\rho$$

Because $\rho = 42$, the total force is $18 \cdot 42 = 756$ lb.

20. If the end of a water tank is in the shape of a rectangle and the tank is full, show that the measure of the force due to water pressure on one end is the product of the measure of the area of the end and the measure of the force at the geometrical center.

▸ We are asked to prove a special case of Formula 6.5.3. Because the end of the tank is vertical and the tank is a ft deep, then at x, $x \in [0, a]$, the depth is x ft and the length is b ft. Hence, the total force is given by

$$F = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \rho w_i b \Delta_i x = \int_0^a \rho x b\, dx = \tfrac{1}{2}\rho b x^2\Big]_0^a = \tfrac{1}{2}\rho a^2 b \qquad (1)$$

Furthermore, the center of the rectangle is at the point $(\frac{1}{2}a, \frac{1}{2}b)$. Thus, $\frac{1}{2}a\rho$ lb represents the force per square foot of area at this point. Because the area of the rectangle is ab ft^2 and $(\frac{1}{2}a\rho)(ab) = \frac{1}{2}\rho a^2$, by comparison with (1), we conclude that the total force due to liquid pressure on the end of the tank is the product of the measure of the area of the end and the measure of the force at the geometrical center.

21. The base of the pool is a rectangle of length $\sqrt{40^2 + (8 - 2)^2} = 2\sqrt{409}$ ft. Its width is 25 ft and its centroid is $\frac{1}{2}(8 + 2) = 5$ ft below the surface. The the force on the bottom is $2\sqrt{409} \times 25 \times 5\rho = 250\sqrt{409}(64.2) \approx 324{,}591$ lb.

22. If F lb is the force on the dam, $F = \rho\bar{x}A = 62.4(10\sqrt{2})(80 \times 40) = 2{,}823{,}902$

23. Take the xy plane in the face of the dam, the positive x axis downward and the origin at the center at the top of the face. The vertical distance from the point $(x, 0)$ on the x axis to the surface of the water is $\frac{1}{2}\sqrt{3}x$, $x \in [0, 30]$. If F lb is the force on the face of the dam, then

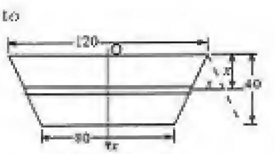
$$F = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \rho \cdot 50(\tfrac{1}{2}\sqrt{3}\, w_i)\Delta_i x = 25\sqrt{3}\rho \int_0^{30} x\, dx = 25\sqrt{3}\rho\left[\tfrac{1}{2}x^2\right]_0^{30} = 11{,}250\sqrt{3}\,\rho$$

24. The face of a dam adjacent to the water is inclined at an angle of 30° from the vertical. The face is an isosceles trapezoid 120 ft wide at the top, 80 ft wide at the bottom, and with a slant height of 40 ft. If the dam is full of water, find the total force due to water pressure on the face.

▸ The figure shows the x axis in the face of the dam with the origin at the center of the top. Because the dam is inclined at 30° from the vertical, at x, $x \in [0, 40]$, the number of feet in the depth is

$$h(x) = x \cos 30° = \tfrac{1}{2}\sqrt{3}x$$

and by similar triangles, the length is $(120 - x)$ ft. If F lb is the total force due to water pressure, then



$$F = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \rho(\tfrac{1}{2}\sqrt{3}\, w_i)(120 - w_i)\Delta_i x = \tfrac{1}{2}\rho\sqrt{3} \int_0^{40} (120x - x^2)\, dx$$

$$= \tfrac{1}{2}\rho\sqrt{3}\left[60x^2 - \tfrac{1}{3}x^3\right]_0^{40} = \tfrac{1}{2}\rho\sqrt{3}[60(1600) - \tfrac{1}{3}(64000)]$$

$$= \tfrac{1}{3}(112{,}000)\sqrt{3}\rho = \tfrac{1}{3}(112{,}000)\sqrt{3}(62.4) = 4{,}035{,}000$$

▪ The force on the dam due to water pressure is approximately 4 million pounds.

Miscellaneous Exercises for Chapter 6

1. An equation of the arc is $y = \frac{1}{\sqrt{6}}x^{1/2}(x-2) = \frac{1}{\sqrt{6}}(x^{3/2} - 2x^{1/2})$; so $y' = \frac{1}{\sqrt{6}}(\frac{3}{2}x^{1/2} - x^{-1/2})$. From Th. 6.1.2

$$L = \int_2^8 \sqrt{1 + \left[\frac{1}{\sqrt{6}}(\tfrac{3}{2}x^{1/2} - x^{-1/2})\right]^2}\, dx = \int_2^8 \sqrt{1 + \tfrac{1}{6}(\tfrac{9}{4}x - 3 + x^{-1})}dx = \frac{1}{\sqrt{6}}\int_2^8 \sqrt{\tfrac{9}{4}x + 3 + x^{-1}}\, dx$$
$$= \frac{1}{\sqrt{6}}\int_2^8 \sqrt{(\tfrac{3}{2}x^{1/2} + x^{-1/2})^2}\, dx = \frac{1}{\sqrt{6}}\int_2^8 (\tfrac{3}{2}x^{1/2} + x^{-1/2})dx = \frac{1}{\sqrt{6}}\Big[x^{3/2} + 2x^{1/2}\Big]_2^8$$
$$= \frac{1}{\sqrt{6}}[(16\sqrt{2} + 4\sqrt{2}) - (2\sqrt{2} + 2\sqrt{2})] = \frac{1}{\sqrt{6}}(16\sqrt{2}) = \tfrac{16}{3}\sqrt{3}$$

2. An equation of the arc is $y = a^{-1/2}x^{3/2}$; $y' = \frac{3}{2}a^{-1/2}x^{1/2}$.

$$L = \int_0^{4a} \sqrt{1 + (\tfrac{3}{2}a^{-1/2}x^{1/2})^2}dx = \int_0^{4a} \sqrt{1 + \tfrac{9}{4a}x}\, dx = \tfrac{2}{3}\cdot\tfrac{4a}{9}(1 + \tfrac{9}{4a}x)^{3/2}\Big]_0^{4a} = \tfrac{8a}{27}(10^{3/2} + 1)$$

3. The arc has equation $y = \frac{27}{8}(4 - x^{2/3})^{3/2}$ with endpoints $(-1, 17.54)$ and $(-.125, 24.51)$.

Then $y' = -\frac{27}{8}(4 - x^{2/3})^{1/2} \cdot x^{-1/3}$. From Theorem 6.1.2, $L =$

$$\int_{-1}^{-1/8} \sqrt{1 + \tfrac{729}{64}(4 - x^{2/3})x^{-2/3}}\, dx = \tfrac{1}{8}\int_{-1}^{-1/8} \sqrt{2916 - 665x^{2/3}}(-x^{-1/3})dx = \tfrac{1}{8}\cdot\tfrac{1}{665}(2916 - 665x^{2/3})^{3/2}\Big|_{-1}^{-1/8}$$
$$= \frac{1}{5320}\Big[\tfrac{1}{8}(10999)^{3/2} - (2251)^{3/2}\Big] \approx 7.03$$

4. Find the length of the arc of the curve $3y = (x^2 - 2)^{3/2}$ from the point where $x = 3$ to the point where $x = 6$.

▹ Because $y = \frac{1}{3}(x^2 - 2)^{3/2}$, we have

$$f(x) = \tfrac{1}{3}(x^2 - 2)^{3/2}$$
$$f'(x) = \tfrac{1}{2}(x^2 - 2)^{1/2}(2x) = x(x^2 - 2)^{1/2}$$
$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + x^2(x^2 - 2)} = \sqrt{x^4 + 2x^2 + 1} = \sqrt{(x^2 - 1)^2} = x^2 - 1$$

If L units is the length of the arc, then

$$L = \int_3^6 \sqrt{1 + [f'(x)]^2}\, dx = \int_3^6 (x^2 - 1)\, dx = \tfrac{1}{3}x^3 - x\Big]_3^6 = \tfrac{1}{3}(216 - 27) - (6 - 3) = 60$$

▪ The length of the arc is 60 units.

5. $\bar{x} = \dfrac{4(-5) + 2(4) + 7(2)}{4 + 2 + 7} = \dfrac{2}{13}$

6. $\bar{x} = \dfrac{5(-1) + 2(2) + 8(5)}{5 + 2 + 8} = \dfrac{39}{15} = \dfrac{13}{5}$; $\bar{y} = \dfrac{5(3) + 2(-1) + 8(2)}{15} = \dfrac{29}{15}$. The centroid is at $(\frac{13}{5}, \frac{29}{15})$.

7. Let the measure of the mass of each particle be m. Then $M = \sum_{i=1}^{4} m = 4m$;

$m_y = \sum_{i=1}^{4} mx_i = m(3) + m(2) + m(-1) = 6m$; $m_x = \sum_{i=1}^{4} my_i = m(0) + m(2) + m(4) + m(2) = 8m$

Therefore $\bar{x} = \dfrac{m_y}{M} = \dfrac{6m}{4m} = \dfrac{3}{2}$ and $\bar{y} = \dfrac{m_x}{M} = \dfrac{8m}{4m} = 2$. The center of mass is at the point $(\frac{3}{2}, 2)$.

8. Three particles, each having the same mass are located on the x axis at the points having coordinates -4, 1, and 5, where the distance is measured in meters. Find the coordinates of the center of mass of the system.

▹ Let m kg be the mass of each particle. Then $3m$ kg is the total mass of the system. If $(\bar{x}, \bar{y})$ is the center of mass, then $\bar{y} = 0$ and

$$\bar{x} = \frac{1}{3m}\sum_{i=1}^{3} m_i x_i = \frac{1}{3m}[m(-4) + m(1) + m(5)] = \frac{1}{3m}(2m) = \frac{2}{3}$$

▪ The center of mass of the system is at $(\frac{2}{3}, 0)$.

9. If M slugs is the total mass of the rod, then

$M = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} 2\sqrt{w_i + 1}\,\Delta_i x = \int_0^8 2\sqrt{x+1}\,dx = \frac{4}{3}(x+1)^{3/2}\Big|_0^8 = \frac{4}{3}(27-1) = \frac{104}{3}$

Let $v = \sqrt{x+1}$; then $x = v^2 - 1$ and $dx = 2v\,dv$. When $x = 0$, $v = 1$ and when $x = 8$, $v = 3$. Hence

$m_y = \int_0^8 x(2\sqrt{x+1})dx = \int_1^3 (v^2-1)(2v)(2v\,dv) = 4\int_1^3 (v^4 - v^2)dv = 4\left[\frac{1}{5}v^5 - \frac{1}{3}v^3\right]_1^3 = 4[(\frac{243}{5} - 9) - (\frac{1}{5} - \frac{1}{3})]$

$= \frac{2384}{15}$. Therefore $\bar{x} = \frac{1}{M} \cdot m_y = \frac{3}{104} \cdot \frac{2384}{15} = \frac{298}{65}$. The center of mass is $\frac{298}{65}$ in. from the left end.

10. If M kg is the total mass of the rod, then $M = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} (3w_i + 1)\Delta_i x = \int_0^4 (3x+1)dx = \frac{3}{2}x^2 + x\Big]_0^4 = 28$

$m_y = \int_0^4 (3x+1)x\,dx = \int_0^4 (3x^2 + x)dx = x^3 + \frac{1}{2}x^2\Big]_0^4 = 72.$

Therefore $\bar{x} = \frac{1}{M} \cdot m_y = \frac{72}{28} = \frac{18}{7}$. The center of mass is $\frac{18}{7}$ m. from the left end.

11. $m_y = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} w_i(9 - w_i^2)\Delta_i x = \int_0^3 (9x - x^3)dx = \left[\frac{9}{2}x^2 - \frac{1}{4}x^4\right]_0^3 = \frac{81}{2} - \frac{81}{4} = \frac{81}{4}$

$m_x = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} \frac{1}{2}(9 - w_i^2)(9 - w_i^2)\Delta_i x = \frac{1}{2}\int_0^3 (81 - 18x^2 + x^4)dx = \frac{1}{2}\left[81x - 6x^3 + \frac{1}{5}x^5\right]_0^3$

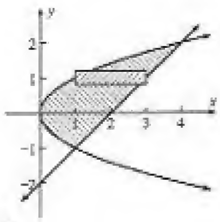
$= \frac{1}{2}\left(81 \cdot 3 - 6 \cdot 27 + \frac{1}{5} \cdot 243\right) = \frac{324}{5}$. $A = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} (9 - w_i^2)\Delta_i x = \int_0^3 (9 - x^2)dx = \left[9x - \frac{1}{3}x^3\right]_0^3 = 18$

Hence $\bar{x} = \frac{1}{A} \cdot m_y = \frac{1}{18} \cdot \frac{81}{4} = \frac{9}{8}$ and $\bar{y} = \frac{1}{A} \cdot m_x = \frac{1}{18} \cdot \frac{324}{5} = \frac{18}{5}$. The centroid is at the point $(\frac{9}{8}, \frac{18}{5})$.

12. Find the centroid of the region bounded by the parabola $y^2 = x$ and the line $y = x - 2$.

▸ The figure shows the region R. R is bounded on the right by the line $x = f(y) = y + 2$ and on the left by the curve $x = g(y) = y^2$. The rectangular elements are parallel to the x axis with length $f(w_i) - g(w_i)$. Because

$$f(y) - g(y) = y + 2 - y^2 = -(y^2 - y - 2) = -(y+1)(y-2)$$



then $y \in [-1, 2]$. If A units is the area of R, then

$A = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} [f(w_i) - g(w_i)]\Delta_i y = \int_{-1}^{2} (y + 2 - y^2)\,dy = \frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\Big]_{-1}^{2}$

$= \frac{1}{2}(4-1) + 2(2+1) - \frac{1}{3}(8+1) = \frac{9}{2}$

The x coordinate of the centroid of the element of area is $\frac{1}{2}[f(w_i) + g(w_i)]$. Thus,

$\bar{x} = \frac{1}{A}\lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} \frac{1}{2}[f(w_i) + g(w_i)][f(w_i) - g(w_i)]\Delta_i y = \frac{2}{9}\int_{-1}^{2} \frac{1}{2}[f(y)^2 - g(y)^2]dy$

$= \frac{1}{9}\int_{-1}^{2} [(y+2)^2 - (y^2)^2]\,dy = \frac{1}{9}\int_{-1}^{2} (y^2 + 4y + 4 - y^4)\,dy = \frac{1}{9}\left[\frac{1}{3}y^3 + 2y^2 + 4y - \frac{1}{5}y^5\right]_{-1}^{2}$

$= \frac{1}{9}[\frac{1}{3}(8+1) + 2(4-1) + 4(2+1) - \frac{1}{5}(32+1)] = \frac{1}{9} \cdot \frac{72}{5} = \frac{8}{5}$

Because the y coordinate of the centroid of the element of area is w_i,

$\bar{y} = \frac{1}{A}\lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} w_i[f(w_i) - g(w_i)]\Delta_i y = \frac{2}{9}\int_{-1}^{2} y(y + 2 - y^2)\,dy = \frac{2}{9}\int_{-1}^{2} (y^2 + 2y - y^3)\,dy$

$= \frac{2}{9}\left[\frac{1}{3}y^3 + y^2 - \frac{1}{4}y^4\right]_{-1}^{2} = \frac{2}{9}[\frac{1}{3}(8+1) + (4-1) - \frac{1}{4}(16-1)] = \frac{2}{9} \cdot \frac{9}{4} = \frac{1}{2}$

▪ The centroid of the region is $(\frac{8}{5}, \frac{1}{2})$.

Let S be a parabolic segment formed by a line meeting a parabola in points A and B and let C be the intersection of the tangents to the parabola at A and B. Using the same method, we can show that the area of S is $\frac{2}{3}$ the area of triangle ABC and that centroid of S is the center of mass of weights 2 at A, 2 at B and 1 at C.

13. $\sqrt{x} - x^2 = \sqrt{x}(1 - x^{3/2})$ is nonnegative in $[0, 1]$.

$A = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} (\sqrt{w_i} - w_i^2)\Delta_i x = \int_0^1 (\sqrt{x} - x^2)dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

$m_y = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} w_i(\sqrt{w_i} - w_i^2)\Delta_i x = \int_0^1 (x^{3/2} - x^3)dx = \left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right] = \frac{3}{20}$

Thus $\bar{x} = \frac{1}{A} \cdot m_y = 3 \cdot \frac{3}{20} = \frac{9}{20}$. The centroid is at $\left(\frac{9}{20}, \frac{9}{20}\right)$ because the region is symmetric with respect to $y = x$.

14. By symmetry, $\bar{x} = 0$. $A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{4}{9}(9 - w_i^2)\Delta_i x = 2 \cdot \frac{4}{9}\int_0^3 (9 - x^2)dx = \frac{8}{9}\left[9x - \frac{1}{3}x^3\right]_0^3 = \frac{8}{9} \cdot 18 = 16.$

$m_x = 2\int_0^3 \frac{1}{2} \cdot \frac{4}{9}(9 - x)^2 \cdot \frac{4}{9}(9 - x^2)dx = \frac{16}{81}\int_0^3 (81 - 18x^2 + x^4)dx = \frac{16}{81}\left[81x - 6x^3 + \frac{1}{5}x^5\right]_0^3 = \frac{16}{81} \cdot \frac{648}{5} = \frac{128}{5}.$

$\bar{y} = \frac{1}{16} \cdot \frac{128}{5} = \frac{8}{5}$. The centroid is at $(0, \frac{8}{5})$.

15. Half a sphere of radius 4 is obtained by revolving about the x axis the region in the first quadrant bounded by the x axis, the y axis, and the semi-circle $x^2 + y^2 = 16$.

$m_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2 \cdot \frac{1}{2}[\sqrt{16 - w_i^2}]^2 \Delta_i x = \int_0^4 (16 - x^2)dx = \left[16x - \frac{1}{3}x^3\right]_0^4 = 64 - \frac{64}{3} = \frac{128}{3}$

By the theorem of Pappus $V = 2\pi\bar{y}A = 2\pi\left(\frac{m_x}{A}\right)A = 2\pi m_x = 2\pi \cdot \frac{128}{3} = \frac{256}{3}\pi.$

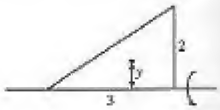
16. Use the Theorem of Pappus to find the volume of a right-circular cone with base radius 2 m and height 3 m.

▹ The cone is the solid generated by revolving the right triangle shown in the figure about the 3 m base. Because in Exercise 6.3.34 it was shown that the distance $\bar{y}$ meters from the centroid G of a triangle to any side of the triangle is one-third the length of the altitude to that side, then

$$\bar{y} = \frac{1}{3}(2) = \frac{2}{3}$$

If the the area of the triangle is A square meters, then

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2} \cdot 3 \cdot 2 = 3$$



If V cubic meters is the volume of the cone, then by the theorem of Pappus

$$V = 2\pi\bar{y}A = 2\pi(\tfrac{2}{3})3 = 4\pi$$

• The volume of the cone is 4π cubic meters.

In Exercises 17–20, use your graphics calculator to find to four significant digits the centroid $(\bar{x}, \bar{y})$ of the region.

17. (4 Rev 99) Bounded by $y = \sqrt[3]{x^2 - 7}$ and the x axis. By symmetry, $\bar{x} = 0$.

$A = 2\int_0^{\sqrt{7}} \sqrt[3]{x^2 - 7}\,dx \approx 8.5160$. $M_x = -2 \cdot \frac{1}{2}\int_0^{\sqrt{7}} (7 - x^2)^{2/3}dx = -7.1564$. $\bar{y} = \frac{-7.1564}{8.5160} = -0.8404$

18. (4 Rev 105) Bounded by $y = x^3 - 6x^2 + 9x - 1$ and $y = x^2 - 2x + 2$ and not intersecting $y = 4$.

▹ $[a, b] = [0.34456, 1.78924]$, $y_1 = x^3 - 6x^2 + 9x - 1$, $y_2 = x^2 - 2x + 2$. $A = \int_a^b (y_1 - y_2)dx \approx 1.9093$.

$M_y = \int_a^b x(y_1 - y_2)dx \approx 1.9846$. $\bar{x} = \frac{1.9846}{1.9093} = 1.039$. $M_x = \frac{1}{2}\int_a^b (y_1^2 - y_2^2)dx \approx 3.6319$. $\bar{y} = \frac{3.6319}{1.9093} = 1.902$

19. (4 Rev 103) Bounded by $y = \cos x^2$ and $y = x^3$ and the y axis. $b = 0.889281$. $A = \int_0^b (\cos x^2 - x^3)dx \approx 0.6789$.

$M_y = \int_0^b x(\cos x^2 - x^3)dx \approx 0.24435$. $\bar{x} = \frac{0.24435}{0.6789} = 0.3597$.

$M_x = \frac{1}{2}\int_0^b [(\cos x^2)^2 - (x^3)^2]dx \approx 0.36369$. $\bar{y} = \frac{0.36369}{0.6789} = 0.5357$

20. The region bounded by the graphs of $y = \cos\sqrt{x}$ and $y = x^2$ and the y axis.

▹ In Exercise 4 Rev 104 we sketched the graph and found the intersection to be at $b = 0.79308$.

$$A = \int_0^b (\cos\sqrt{x} - x^2)dx \approx 0.47635$$

$$M_y = \int_0^b x(\cos\sqrt{x} - x^2)dx \approx 0.13648 \qquad \bar{x} = \frac{M_y}{A} = \frac{0.13648}{0.47635} = 0.2856$$

$$M_x = \frac{1}{2}\int_0^b [(\cos\sqrt{x})^2 - (x^2)^2]dx \approx 0.23353 \qquad \bar{y} = \frac{M_x}{A} = \frac{0.23353}{0.47635} = 0.4903$$

• The centroid is at $(0.2856, 0.4903)$

In Exercises 21–24, find the length L of arc to 4 significant digits by using NINT to compute the definite integral.

21. $y = 4x^{-2}$; $y' = -8x^{-3}$. $L = \int_1^2 \sqrt{1 + (8x^{-3})^2}dx \approx 3.21488 = 3.215$

22. $y = 8x^{-3}$; $y' = -24x^{-4}$. $L = \int_1^2 \sqrt{1 + (24x^{-4})^2}dx \approx 7.12412 = 7.124$

23. $y = \tan x$; $y' = \sec^2 x$. $L = \int_0^1 \sqrt{1 + \sec^4 x}\,dx \approx 1.876$

24. The arc of the sine curve from the point where $x = 1$ to the point where $x = 2$.

▷ An equation of the curve is $y = f(x) = \sin x$.

$$f'(x) = \cos x$$
$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + \cos^2 x}$$

Therefore, by Theorem 6.1.2, we have $\quad L = \displaystyle\int_1^2 \sqrt{1 + \cos^2 x}\,dx \approx 1.04025$

using NINT. The arc is approximately 1.040 units long.

25. $y = \cosh x$, $y' = \sinh x$. $L = \displaystyle\int_{\ln 2}^{\ln 3} \sqrt{1 + \sinh^2 x}\,dx = \int_{\ln 2}^{\ln 3} \sqrt{\cosh^2 x}\,dx = \int_{\ln 2}^{\ln 3} \cosh x\,dx = \sinh x\Big]_{\ln 2}^{\ln 3}$
$= \frac{1}{2}(e^x - e^{-x})\Big]_{\ln 2}^{\ln 3} = \frac{1}{2}(3 - \frac{1}{3} - 2 + \frac{1}{2}) = \frac{7}{12}$

26. A force of 500 lb is required to compress a spring whose natural length is 10 in. to a length of 9 in. Find the work done to compress the spring to a length of 8 in.

▷ Let $f(x)$ pounds be the force needed to compress the spring x inches. By Hooke's law we have $f(x) = kx$. Because a force of 500 lb compresses the spring 1 in., then $500 = k(1)$ and so $f(x) = 500x$. When the spring is compressed to a length of 8 in., then $x = 2$ and so we have $0 \le x \le 2$. If W inch-pounds is the total work done, then $W = \displaystyle\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x = \int_0^2 500x\,dx = 250x^2\Big]_0^2 = 250(4) = 1000$

• The work done is 1000 inch-pounds.

27. Place the spring along the x axis with the origin at the point where the stretching starts. By Hooke's law, $f(x) = kx$. Then $f(5) = 600$; $5k = 600$; $k = 120$; so $f(x) = 120x$. W ergs is the work done in stretching the spring from 30 cm to 40 cm; $x \in [0, 10]$. $W = \displaystyle\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 120w_i\Delta_i x = 120 \int_0^{10} x\,dx = 60x^2\Big|_0^{10} = 6000$

28. The work necessary to stretch a spring from 9 in. to 10 in. is $\frac{3}{2}$ times the work necessary to stretch it from 8 in. to 9 in. What is the natural length of the spring?

▷ Let $f(x)$ pounds be the force needed to compress the spring x inches and let a inches be the natural length of the spring. By Hooke's law, $f(x) = kx$. Let W_1 inch-pounds be the work to stretch the spring from 9 in. to 10 in.; then x is in $[9 - a, 10 - a]$. Let W_2 inch-pounds be the work to stretch the spring from 8 in. to 9 in.; then x is in $[8 - a, 9 - a]$. Therefore,

$$W_1 = \tfrac{3}{2}W_2; \quad \int_{9-a}^{10-a} kx\,dx = \tfrac{3}{2}\int_{8-a}^{9-a} kx\,dx; \quad \tfrac{1}{2}kx^2\Big]_{9-a}^{10-a} = \tfrac{3}{4}kx^2\Big]_{8-a}^{9-a}$$

$$2[(10-a)^2 - (9-a)^2] = 3[(9-a)^2 - (8-a)^2]; \qquad 2(19 - 2a) = 3(17 - 2a); \qquad 2a = 13; \qquad a = \tfrac{13}{2}$$

• The length of the spring is $6\frac{1}{2}$ inches.

29. Choose the positive x axis downward with the origin at the top of the pole, $x \in [0, 20]$.
If W ft-lb is the work done in raising the entire cable to the top of the pole, then
$W = \displaystyle\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2w_i\Delta_i x = \int_0^{20} 2x\,dx = x^2\Big|_0^{20} = 400$

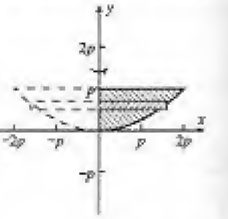
30. Choose the positive x axis downward with the origin at the top of the tank, $x \in [0, 1]$. W ft-lb is the work done in pumping the water to the top of the tank. An element of volume is a rectangular slab of thickness $\Delta_i x$, length 6 ft, width $2\sqrt{1 - w_i^2}$ ft, and is raised w_i ft. Let $u = 1 - x^2$, $du = -2x\,dx$.
$W = 62.4(6) \displaystyle\int_{x=0}^{1} \sqrt{1 - x^2}(2x\,dx) = 374.4 \int_{u=1}^{0} u^{1/2}(-du) = \tfrac{2}{3}(374.4)u^{3/2}\Big]_0^1 = 249.6$

31. Choose the positive x axis downward with the origin at the top of the tank, $x \in [0, 4]$. W joules is the work done in pumping the water in the tank up to a level $\frac{1}{2}$ m above the top of the tank. An element of volume is a rectangular slab of thickness $\Delta_i x$, length 15 m, width 30 m, and is raised $(w_i + \frac{1}{2})$m.
$W = \displaystyle\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (1000)(9.81)(15)(30)(w_i + \tfrac{1}{2})\Delta_i x = 4{,}414{,}500 \int_0^4 (x + \tfrac{1}{2})dx = 4{,}414{,}500\Big[\tfrac{1}{2}x^2 + \tfrac{1}{2}x\Big]_0^4$
$= 4{,}414{,}500(8 + 2) = 44{,}145{,}000$

32. A container has the same shape and dimensions as a solid of revolution formed by revolving about the y axis the region in the first quadrant bounded by the parabola $x^2 = 4py$, the y axis, and the line $y = p$, $p > 0$. If the container is full of water, find the work done in pumping all the water up to a point $3p$ ft above the top of the container.

▷ The figure shows the container. An element of volume is a circular disk centered on the y axis, $y \in [0, p]$, of radius x_i, height $\Delta_i y$, volume measure

$$\pi x_i^2 \Delta_i y = \pi(4py_i)\Delta_i y$$



and weight $4\pi p\rho y_i \Delta_i y$ pounds and is raised a mean distance of $4p - y_i$ feet. Thus,

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4p - y_i)(4\pi p\rho y_i \Delta_i y) = 4\pi p\rho \int_0^p (4p - y)y \, dy$$
$$= 4\pi p\rho \int_0^p (4py - y^2) \, dy = 4\pi p\rho \Big[2py^2 - \tfrac{1}{3}y^3\Big]_0^p = 4\pi p\rho[2p(p^2) - \tfrac{1}{3}p^3]$$
$$= \tfrac{20}{3}\pi\rho p^4 = \tfrac{20}{3}\pi(62.4)p^4 = 428\pi p^4$$

■ The total work is $428\pi p^4$ foot-pounds.

33. Choose the positive x axis downward with the origin at the center of the hemisphere. Let W_1 ft-lb, W_2 ft-lb, and W ft-lb be the work done by pumping the water in the cylinder, the hemisphere, and the entire tank, to the top of the tank. An element of volume for W_1 is a circular disk of radius 4 ft and is raised $(w_i + 8)$ft, $x \in [-8, 0]$.

$$W_1 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} w\pi(4)^2(w_i + 8)\Delta_i x = 16w\pi \int_{-8}^{0} (x + 8)dx = 8w(x + 8)^2 \Big|_{-8}^{0} = 512w\pi$$

An element of volume for W_2 is a circular disk of radius $\sqrt{4^2 - w_i^2}$ ft and is raised $(w_i + 8)$ft, $w_i \in [0, 4]$.

$$W_2 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} w\pi(16 - w_i^2)(w_i + 8)\Delta_i x = w\pi \int_0^4 (-x^3 - 8x^2 + 16x + 128)dx = w\pi\Big[-\tfrac{1}{4}x^4 - \tfrac{8}{3}x^3 + 8x^2 + 128x\Big]_0^4$$
$$= w\pi(-\tfrac{1}{4} \cdot 256 - \tfrac{8}{3} \cdot 64 + 8 \cdot 16 + 128 \cdot 4) = \tfrac{1216}{3} w\pi$$

Therefore $W = W_1 + W_2 = 512w\pi + \frac{1216}{3} w\pi = \frac{2752}{3}(62.4)\pi \approx 57{,}262$

34. Choose the positive x axis downward with the origin at the center of the hemisphere. Let W joules be the work done by pumping the water to the top of the tank. An element of volume is a circular disk of radius $\sqrt{25 - w_i^2}$ ft and is raised w_i ft, $x \in [2, 5]$.

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho\pi(25 - w_i^2)w_i\Delta_i x = \rho\pi \int_2^5 (25x - x^3)dx = \rho\pi\Big[\tfrac{25}{2}x^2 - \tfrac{1}{4}x^4\Big]_2^5 = 9810\pi \cdot \tfrac{441}{4} = 3{,}397{,}797$$

35. The region is bounded by $x^2 = 6y$ and $2y = 3$. The pressure on a rectangular element of area $2\sqrt{6w_i}\Delta_i y$ ft^2 is $\rho(\frac{3}{2} - w_i)$lb/ft^2. If F lb is the force on one side of the plate,

$$F = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho(\tfrac{3}{2} - w_i)(2\sqrt{6}w_i^{1/2})\Delta_i y = 2\sqrt{6}\rho \int_0^{3/2} (\tfrac{3}{2}y^{1/2} - y^{3/2})dy = 2\sqrt{6}\rho\Big[y^{3/2} - \tfrac{2}{5}y^{5/2}\Big]_0^{3/2}$$
$$= 2\sqrt{6}\rho(\tfrac{3}{2}\sqrt{\tfrac{3}{2}} - \tfrac{2}{5} \cdot \tfrac{9}{4}\sqrt{\tfrac{3}{2}}) = 2\sqrt{6}\rho(\tfrac{3}{5}\sqrt{\tfrac{3}{2}}) = \tfrac{18}{5}(62.4) = 22.64$$

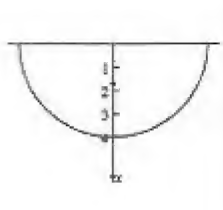
36. A semicircular plate with a radius of 4 ft is submerged vertically in a tank of water, with its diameter lying in the surface. Use Formula 6.5.3 to find the force due to water pressure on one side of the plate.

▷ The figure shows the plate. Formula 6.5.3 states that the force due to liquid pressure on one side of the plate is given by

$$F = \rho\bar{x}A \qquad (1)$$

where A ft^2 is the area of the plate and $\bar{x}$ ft is the depth of its centroid. Now

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(4^2) = 8\pi$$



Because the plate is semicircular of radius $r = 4$, by Exercise 6.5.35 we have

$$\bar{x} = \frac{4r}{3\pi} = \frac{4(4)}{3\pi} = \frac{16}{3\pi}$$

Substituting for ρ, $\bar{x}$ and A into (1), we obtain

$$F = 62.4\Big(\frac{16}{3\pi}\Big)(8\pi) = 2739.2$$

The force on the plate is 2739.2 pounds.

37. Choose the positive x axis downward with the origin at the center of the circle. An equation of the circle is $x^2+y^2=16$. The pressure on a rectangular element of area $2\sqrt{16-w_i^2}\Delta_i x$ ft^2 is $40w_i$ lb/ft^3. If F lb is the force on an end due to fluid pressure,

$$\mathrm{F} = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 40w_i(2\sqrt{16-w_i^2})\Delta_i x = -40\int_0^4 \sqrt{16-x^2}(-2x\,dx) = -\tfrac{80}{3}(16-x^2)^{3/2}\Big]_0^4 = -\tfrac{80}{3}(-64) = \tfrac{5120}{3}$$

38. If F lb is the force on the dam, $\mathrm{F} = \rho\bar{x}\mathrm{A} = 62.4(15\sqrt{2})(100\times 60) = 7{,}942{,}223$

39. The positive y axis is upward with the origin at the bottom of the tank. W ft-lb is the work done in pumping all of the water out over the top. An element of volume is a circular disk of radius $w_i^{1/2}$ ft and is raised $(36-w_i)$ft, $y\in[0,20]$.

$$\mathrm{W} = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} w\pi(w_i^{1/2})^2(36-w_i)\Delta_i y = w\pi\int_0^{20} y(36-y)dy = w\pi\int_0^{20}(36y-y^2)dy = w\pi\Big[18y^2-\tfrac{1}{3}y^3\Big]_0^{20}$$

$$=(62.4)\pi(7200-\tfrac{8000}{3}) = 888{,}694$$

40. If $f(x)=\int_0^x \sqrt{\cos t}\,dt$, find the length of the arc of the graph of f from the point where $x=\frac{1}{3}\pi$ to the point where $x=\frac{1}{2}\pi$. (Hint: Use the first fundamental theorem of the calculus and the identity $\cos^2\frac{1}{2}x=\frac{1}{2}(1+\cos x)$.)

▸ By Theorem the first fundamental theorem of the calculus, we have

$$f'(x)=\frac{d}{dx}\int_0^x \sqrt{\cos t}\,dt = \sqrt{\cos x}$$
$$1+[f'(x)]^2 = 1+\cos x = 2\cos^2\tfrac{1}{2}x$$
$$\sqrt{1+[f'(x)]^2} = \sqrt{2}\cos\tfrac{1}{2}x$$

By Theorem 6.1.2,

$$\mathrm{L}=\int_{\pi/3}^{\pi/2}\sqrt{2}\cos\tfrac{1}{2}x\,dx = 2\sqrt{2}\Big[\sin\tfrac{1}{2}x\Big]_{\pi/3}^{\pi/2} = 2\sqrt{2}(\sin\tfrac{1}{4}\pi-\sin\tfrac{1}{6}\pi) = 2\sqrt{2}(\tfrac{1}{2}\sqrt{2}-\tfrac{1}{2}) = 2-\sqrt{2}$$

▪ The length of the arc is $(2-\sqrt{2})$ units.

SEVEN

TECHNIQUES OF INTEGRATION, INDETERMINATE FORMS, AND IMPROPER INTEGRALS

7.1 INTEGRATION BY PARTS

INTEGRATION FORMULAS You should memorize the following integration formulas.

1. $\int du = u + C$
2. $\int a\, du = au + C$ where a is any constant
3. $\int [a\, f(u) + bg(u)] = a \int f(u)\, du + b \int g(u)\, du$
4. $\int u^n\, du = \frac{u^{n+1}}{n+1} + C \quad n \neq -1$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int a^u\, du = \frac{a^u}{\ln a} + C$ where $a > 0$ and $a \neq 1$
7. $\int e^u\, du = e^u + C$
8. $\int \sin u\, du = -\cos u + C$
9. $\int \cos u\, du = \sin u + C$
10. $\int \sec^2 u\, du = \tan u + C$
11. $\int \csc^2 u\, du = -\cot u + C$
12. $\int \sec u \tan u\, du = \sec u + C$
13. $\int \csc u \cot u\, du = -\csc u + C$
14. $\int \tan u\, du = \ln|\sec u| + C$
15. $\int \cot u\, du = \ln|\sin u| + C$
16. $\int \sec u\, du = \ln|\sec u + \tan u| + C$
17. $\int \csc u\, du = \ln|\csc u - \cot u| + C = \ln\left|\tan \tfrac{1}{2}u\right| + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$ where $a > 0$

21. $\int \sinh u\, du = \cosh u + C$

22. $\int \cosh u\, du = \sinh u + C$

23. $\int \operatorname{sech}^2 u\, du = \tanh u + C$

24. $\int \operatorname{csch}^2 u\, du = -\coth u + C$

25. $\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$

26. $\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$

27. $\int \frac{du}{\sqrt{u^2+a^2}} = \sinh^{-1}\frac{u}{a} + c = \ln(u + \sqrt{u^2+a^2}) + c$ if $a > 0$

28. $\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\frac{u}{a} + c = \ln(u + \sqrt{u^2-a^2}) + c$ if $u > a > 0$

29. $\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\frac{u}{a} + c & \text{if } |u| < a \\ \frac{1}{a}\coth^{-1}\frac{u}{a} + c & \text{if } |u| > a \end{cases} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + c$ if $u \neq a$ and $a \neq 0$

Integration by Parts The formulas for integration by parts for indefinite and definite integrals are

$$\int u\, dv = uv - \int v\, du \quad \text{and} \quad \int_a^b u\, dv = uv\Big]_a^b - \int_a^b v\, du$$

To use the formula, we must factor the given integrand in such a way that we can integrate one of the factors—which we designate dv—and differentiate the other factor—which we designate u. And we must choose u and v in such a way that $\int v\, du$ is easier to integrate. We shall shorten some solutions by framing dv instead of defining it explicitly.

Sometimes it is convenient to choose a nonzero constant of integration; see Exercises 24 and 32. In any integration problem, the first step is to substitute for the argument of a trigonometric or exponential function if it is more complicated than a constant times x; see Exercises 16 and 24. Next, if the integrand is a power of x times

(i) a trigonometric function or an exponential function, let the power of x be u; see Exercises 4, 8, and 12.

(ii) an inverse trigonometric function or a logarithmic function, let the power of x with dx be dv; see Exercises 24 and 32.

Sometimes we may have to integrate by parts more than once; see Exercises 12, 16, and 32. The following formulas are derived in Example 5 and Exercises 49 and 50.

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx \tag{3}$$

$$\int e^{au}\sin u\, du = \frac{e^{au}}{a^2+n^2}(a \sin nu - n\cos nu) + C \tag{4}$$

$$\int e^{au}\cos u\, du = \frac{e^{au}}{a^2+n^2}(a \cos nu + n\sin nu) + C \tag{5}$$

Exercises 7.1

In Exercises 1–24, evaluate the indefinite integral. Check your answer by differentiation or support it graphically.

1. Let $u = x$ and $dv = e^{3x}dx$. Then $du = dx$ and $v = \frac{1}{3}e^{3x}$.

$\int xe^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

2. Let $u = x$ and $dv = \cos 2x$. Then $du = dx$ and $v = \frac{1}{2}\sin 2x$.

$\int x\cos 2x\, dx = \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\, dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$

3. Let $u = x$ and $dv = \sec x\tan x\, dx$. Then $du = dx$ and $v = \sec x$.

$\int x\sec x\tan x\, dx = x\sec x - \int \sec x\, dx = x\sec x - \ln|\sec x + \tan x| + C$

4. $\int x3^x\,dx$

▷ We use integration by parts with

$$u = x \qquad dv = e^x dx$$
$$du = dx \qquad v = \frac{3^x}{\ln 3}$$

Thus,

$$\int x3^x\,dx = \frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\int 3^x\,dx = \frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\frac{3^x}{\ln 3} + C = \frac{3^x(x\ln 3 - 1)}{\ln^2 3} + C$$

5. Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\,dx$ and $v = x$.

$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - x + C$

6. Let $u = \sin^{-1}w$ and $dv = dw$. Then $du = \frac{1}{\sqrt{1-w^2}}$ and $v = w$.

$\int \sin^{-1}w\,dw = w\sin^{-1}w - \int\frac{w}{\sqrt{1-w^2}}dw = w\sin^{-1}w + \sqrt{1-w^2} + C$

7. $\int\frac{(\ln t)^2}{t}dt = \int(\ln t)^2\,d(\ln t) = \frac{1}{3}(\ln t)^3 + C$

8. $\int x\sec^2 x\,dx$

▷ We use integration by parts with

$$u = x \qquad dv = \sec^2 x\,dx$$
$$du = dx \qquad v = \tan x$$

Thus,

$\int x\sec^2 x\,dx = x\tan x - \int\tan x\,dx = x\tan x - \ln|\sec x| + C$

9. Let $u = \tan^{-1}x$ and $dv = x\,dx$. Then $du = \frac{dx}{1+x^2}$ and $v = \frac{1}{2}x^2$.

$\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\frac{x^2dx}{1+x^2} = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{dx}{1+x^2} = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$
$= \frac{1}{2}\tan^{-1}x(x^2+1) - \frac{1}{2}x + C$

Alternatively, let $v = \frac{1}{2}x^2 + \frac{1}{2}$. Then $v\,du = \frac{1}{2}dx$ and

$\int x\tan^{-1}x\,dx = \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}\int dx = \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x + C$

10. Let $u = \ln(x^2+1)$, $dv = dx$. Then $du = \frac{2x}{x^2+1}dx$ and $v = x$. $\int\ln(x^2+1)dx =$

$x\ln(x^2+1) - \int x\cdot\frac{2x}{x^2+1}dx = x\ln(x^2+1) - 2\int\left(1 - \frac{1}{x^2+1}\right)dx = x\ln(x^2+1) - 2x + 2\tan^{-1}x + C$

11. Let $u = xe^x$ and $dv = \frac{dx}{(x+1)^2}$. Then $du = (xe^x + e^x)dx$ and $v = -\frac{1}{x+1}$.

$\int\frac{xe^x}{(x+1)^2}dx = -\frac{xe^x}{x+1} + \int\frac{1}{x+1}[e^x(x+1)]dx = -\frac{xe^x}{x+1} + \int e^x dx = = -\frac{xe^x}{x+1} + e^x + C$

$= \frac{-xe^x + xe^x + e^x}{x+1} + C = \frac{e^x}{x+1} + C$. Note that $\int\frac{xe^{ax}}{(bx+1)^2}dx$ can be integrated only if $a = b$.

12. $\int x^2\sin 3x\,dx$

▷ We let

$$u = x^2 \qquad dv = \sin 3x\,dx$$
$$du = 2x\,dx \qquad v = -\frac{1}{3}\cos 3x$$

Thus,

$$\int x^2\sin 3x\,dx = -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\int x\cos 3x\,dx \tag{1}$$

For the integral on the right side of (1), we let

$$\bar{u} = x \qquad d\bar{v} = \cos 3x\, dx$$
$$d\bar{u} = dx \qquad \bar{v} = \tfrac{1}{3}\sin 3x$$

Thus,

$$\int x \cos 3x\, dx = \frac{1}{3}\, x \sin 3x - \frac{1}{3}\int \sin 3x\, dx = \tfrac{1}{3}\, x \sin 3x + \tfrac{1}{9}\cos 3x + C \qquad (2)$$

Substituting from (2) into (1), we obtain

$$\int x^2 \sin 3x\, dx = -\frac{1}{3}\, x^2 \cos 3x + \frac{2}{9}\, x \sin 3x + \frac{2}{27}\cos 3x + C$$

13. Substitute $x = \ln y$, $y = e^x$, $dy = e^x dx$.

$$\int \sin(\ln y)dy = \int \sin x\, e^x dx \overset{\text{Example 4}}{=} \tfrac{1}{2}e^x(\sin x - \cos x) + C = \tfrac{1}{2}y[\sin(\ln y) - \cos(\ln y)] + C$$

14. First substitute $t = \cos x$, $dt = -\sin x\, dx$. Thus $I = \int \sin x \ln(\cos x)dx = -\int \ln t\, dt$.

Now let $u = \ln t$, $dv = dt$. Then $du = \frac{1}{t}\, dt$ and $v = t$.

$$I = -\left(t \ln t - \int t \cdot \tfrac{1}{t}\, dt\right) = -t \ln t + \int dt = -t \ln t + t + C = -\cos x \ln(\cos x) + \cos x + C$$

15. Let $u = e^x$ and $dv = \cos x\, dx$. Then $du = e^x dx$ and $v = \sin x$.

$$I = \int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$$

Let $\bar{u} = e^x$ and $d\bar{v} = \sin x\, dx$. Then $d\bar{u} = e^x dx$ and $\bar{v} = -\cos x$.

$$I = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\, dx\right) = e^x \sin x + e^x \cos x - I$$

$$2I = e^x(\sin x + \cos x) + 2C;\ I = \tfrac{1}{2}e^x(\sin x + \cos x) + C$$

16. $\int x^5 e^{x^2} dx$

▹ To simplify the exponential, we let $z = x^2$ and $dz = 2x\, dx$. Then

$$\int x^5 e^{x^2} dx = \int (x^2)^2 e^{x^2}(x\, dx) = \int z^2 e^z\, dz$$

Let

$$u = \tfrac{1}{2}z^2 \qquad dv = e^z\, dz$$
$$du = z\, dz \qquad v = e^z$$

Thus

$$\int x^5 e^{x^2} dx = \tfrac{1}{2}z^2 e^z - \int z e^z\, dz \qquad (1)$$

For the integral on the right side of (1), we let

$$\bar{u} = z \qquad d\bar{v} = e^z\, dz$$
$$d\bar{u} = dz \qquad \bar{v} = e^z$$

Thus

$$\int z e^z\, dz = z e^z - \int e^z\, dz = z e^z - e^z + \bar{C} \qquad (2)$$

Substituting from (2) into (1), we obtain

$$\int x^5 e^{x^2} dx = \tfrac{1}{2}z^2 e^z - z e^z + e^z - \bar{C} = (\tfrac{1}{2}z^2 - z + 1)e^z + C$$

where $C = -\bar{C}$. Finally, substituting $z = x^2$, we get

$$\int x^5 e^{x^2} dx = (\tfrac{1}{2}x^4 - x^2 + 1)e^{x^2} + C$$

17. Let $u = x^2$ and $dv = \dfrac{x\, dx}{\sqrt{1 - x^2}}$. Then $du = 2x\, dx$ and $v = -\sqrt{1 - x^2}$.

$$\int \frac{x^3 dx}{\sqrt{1 - x^2}} = -x^2\sqrt{1 - x^2} + \int 2x\sqrt{1 - x^2}dx = -x^2\sqrt{1 - x^2} - \tfrac{2}{3}(1 - x^2)^{3/2} + C$$

18. Let $u = e^{-x}$ and $dv = \sin 2x$. Then $du = -e^{-x}dx$ and $v = -\frac{1}{2}\cos 2x$.

$I = \int e^{-x}\sin 2x\, dx = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\int e^{-x}\cos 2x\, dx$

Let $\bar{u} = e^{-x}$ and $d\bar{v} = \cos 2x$. Then $d\bar{u} = -e^{-x}dx$ and $\bar{v} = \frac{1}{2}\sin 2x$.

$I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}(\frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\int e^{-x}\sin 2x\, dx) = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}I$

$\frac{5}{4}I = -e^{-x}(\frac{1}{2}\cos 2x + \frac{1}{4}\sin 2x)$; $I = -\frac{1}{5}e^{-x}(2\cos 2x + \sin 2x) + C$

19. Let $u = x^2$ and $dv = \sinh x\, dx$. Then $du = 2x\, dx$ and $v = \cosh x$.

$\int x^2\sinh x\, dx = x^2\cosh x - 2\int x\cosh x\, dx$

Let $\bar{u} = x$ and $d\bar{v} = \cosh x\, dx$. Then $d\bar{u} = dx$, and $\bar{v} = \sinh x$.

$\int x^2\sinh x\, dx = x^2\cosh x - 2\left(x\sinh x - \int \sinh x\, dx\right) = x^2\cosh x - 2x\sinh x + 2\cosh x + C$

20. $\int \frac{e^{2x}}{\sqrt{1-e^x}}dx$

▹ Integration by parts is not required. We let $u = 1 - e^x$. Then $du = -e^x\, dx$ and $e^x = 1 - u$. With these substitutions we obtain

$$\int \frac{e^{2x}}{\sqrt{1-e^x}}\, dx = \int \frac{e^x(e^x dx)}{\sqrt{1-e^x}} = \int \frac{(1-u)(-du)}{\sqrt{u}} = \int (u^{1/2} - u^{-1/2})\, du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C$$
$$= \frac{2}{3}u^{1/2}(u-3) + C = \frac{2}{3}\sqrt{1-e^x}(1 - e^x - 3) + C = -\frac{2}{3}\sqrt{1-e^x}(2 + e^x) + C$$

21. First substitute $t = \sqrt{z}$, $dt = \frac{1}{2\sqrt{z}}$ Thus $I = \int \frac{\cot^{-1}\sqrt{z}}{\sqrt{z}}\, dz = 2\int \cot^{-1}t\, dt$.

Now let $u = \cot^{-1}t$ and $dv = dt$. Then $du = \frac{-1}{1+t^2}dt$ and $v = t$.

$I = 2t\cot^{-1}t + \int \frac{2t\, dt}{1+t^2} = 2t\cot^{-1}t + \ln(1+t^2) + C = 2\sqrt{z}\cot^{-1}\sqrt{z} + \ln(1+z) + C$

22. Let $u = \cos^{-1} 2x$ and $dv = dx$. Then $du = -\frac{2\, dx}{\sqrt{1-4x^2}}$ and $v = x$.

$\int \cos^{-1} 2x\, dx = x\cos^{-1} 2x + \int \frac{2x\, dx}{\sqrt{1-4x^2}} = x\cos^{-1} 2x - \frac{1}{2}\sqrt{1-4x^2} + C$

23. First substitute $x = t^2$, $dx = 2t\, dt$. Thus $I = \int \cos\sqrt{x}\, dx = \int 2t\cos t\, dt$.

Now let $u = 2t$, $dv = \cos t\, dt$. Then $du = 2\, dt$ and $v = \sin t$.

$I = 2t\sin t - \int 2\sin t\, dt = 2t\sin t + 2\cos t + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

24. $\int \tan^{-1}\sqrt{x}\, dx$

▹ First, we let $z = \sqrt{x}$. Then $x = z^2$, and $dx = 2z\, dz$. Thus,

$$\int \tan^{-1}\sqrt{x}\, dx = \int 2z\tan^{-1} z\, dz \tag{1}$$

Next, we integrate by parts on the right side of Eq. (1). Let

$$u = \tan^{-1}z \qquad dv = 2z\, dz$$
$$du = \frac{dz}{1+z^2} \qquad v = z^2 + 1$$

Note that we are free to choose the most convenient antiderivative. Thus,

$$\int \tan^{-1}\sqrt{x}\, dx = (z^2+1)\tan^{-1}z - \int \frac{z^2+1}{1+z^2}\, dz = (z^2+1)\tan^{-1}z - \int dz$$
$$= (z^2+1)\tan^{-1}z - z + C = (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$$

In Exercises 25–32, find the exact value of the definite integral. Check by NINT.

25. To compute $\int x^2 3^x dx$, let $u = x^2$ and $dv = 3^x dx$. Then $du = 2x\,dx$ and $v = \frac{3^x}{\ln 3}$.

$\int x^2 3^x dx = \frac{3^x x^2}{\ln 3} - \frac{2}{\ln 3}\int x3^x dx$

Let $\bar{u} = x$ and $d\bar{v} = 3^x dx$. Then $d\bar{u} = dx$ and $\bar{v} = \frac{3^x}{\ln 3}$.

$\int x^2 3^x dx = \frac{3^x x^2}{\ln 3} - \frac{2}{\ln 3}\left(\frac{3^x\,x}{\ln 3} - \frac{1}{\ln 3}\int 3^x dx\right) = \frac{3^x x^2}{\ln 3} - \frac{2x\,3^x}{(\ln 3)^2} + \frac{2\cdot 3^x}{(\ln 3)^3} + C$

Hence $\int_0^2 x^2 3^x dx = \left[\frac{3^x x^2}{\ln 3} - \frac{2x\,3^x}{(\ln 3)^2} + \frac{2\cdot 3^x}{(\ln 3)^3}\right]_0^2 = \left(\frac{36}{\ln 3} - \frac{36}{\ln^2 3} + \frac{18}{\ln^3 3}\right) - \frac{2}{\ln^3 3} = \frac{36}{\ln 3} - \frac{36}{\ln^2 3} + \frac{16}{\ln^3 3} \approx 15.00798$

26. Let $u = \ln(x+2)$ and $dv = dx$. Then $du = \frac{dx}{x+2}$, and $v = x+2$.

$\int_{-1}^{2} \ln(x+2)dx = (x+2)\ln(x+2)\Big]_{-1}^{2} - \int_{-1}^{2} dx = 4\ln 4 - x\Big]_{-1}^{2} = 8\ln 2 - 3 \approx 2.545177$

27. To compute $\int \sin 3x \cos x\,dx$, let $u = \sin 3x$, $dv = \cos x\,dx$. Then $du = 3\cos 3x\,dx$, $v = \sin x$.

$\int \sin 3x \cos x\,dx = \sin 3x \sin x - 3\int \cos 3x \sin x\,dx$

Let $\bar{u} = \cos 3x$ and $d\bar{v} = \sin x\,dx$. Then $d\bar{u} = -\sin 3x\,dx$ and $\bar{v} = -\cos x$.

$\int \sin 3x \cos x\,dx = \sin 3x \sin x - 3\left(-\cos 3x \cos x - 3\int \sin 3x \cos x\,dx\right)$

$-8\int \sin 3x \cos x\,dx = \sin 3x \sin x + 3\cos 3x \cos x - 8C$

$\int \sin 3x \cos x\,dx = -\frac{1}{8}\sin 3x \sin x - \frac{3}{8}\cos 3x \cos x + C$

(A shorter method is given in Example 7.2.4.)

$\int_0^{\pi/3} \sin 3x \cos x\,dx = -\frac{1}{8}\sin 3x \sin x - \left[\frac{3}{8}\cos 3x \cos x\right]_0^{\pi/3}$

$= \left(-\frac{1}{8}\sin\pi \sin\frac{\pi}{3} - \frac{3}{8}\cos\pi\cos\frac{\pi}{3}\right) - \left(-\frac{1}{8}\sin 0 \sin 0 - \frac{3}{8}\cos 0 \cos 0\right) = -\frac{3}{8}(-1)\left(\frac{1}{2}\right) + \frac{3}{8} = \frac{3}{16} + \frac{3}{8} = \frac{9}{16}$

28. $\int_{-\pi}^{\pi} x^2 \cos 2x\,dx$

▹ Because x^2 and $\cos 2x$ are even functions, the integral is $I = \int_0^{\pi} x^2\cos 2x(2\,dx)$. Let

$u = x^2 \qquad dv = \cos 2x(2\,dx)$

$du = 2x\,dx \qquad v = \sin 2x$

$I = x^2\sin 2x\Big]_0^{\pi} - \int_0^{\pi} 2x \sin 2x\,dx$

Let

$\bar{u} = x \qquad d\bar{v} = \sin 2x(2\,dx)$

$du = dx \qquad \bar{v} = -\cos 2x$

$I = 0 - \left(-x\cos 2x\Big]_0^{\pi} + \int_0^{\pi}\cos 2x\,dx\right) = \pi - \left[\frac{1}{2}\sin 2x\right]_0^{\pi} = \pi \approx 3.141593$

29. Let $u = \ln x$, $dv = x^{1/2}\,dx$. Then $du = x^{-1}dx$ and $v = \frac{2}{3}x^{3/2}$ and $v\,du = \frac{2}{3}x^{1/2}dx$.

$\int_1^4 x^{1/2}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x\Big]_1^4 - \int_1^4 \frac{2}{3}x^{1/2}dx = \frac{2}{3}(8)\ln 4 - \frac{4}{9}x^{3/2}\Big]_1^4 = \frac{32}{3}\ln 2 - \frac{28}{9} \approx 4.282459$

30. Let $u = x$, $dv = \cot x \csc x\,dx$. Then $du = dx$ and $v = -\csc x$.

$\int_{\pi/4}^{3\pi/4} x\cot x\csc x\,dx = -x\csc x\Big]_{\pi/4}^{3\pi/4} + \int_{\pi/4}^{3\pi/4}\csc x\,dx = -\frac{3}{4}\pi\sqrt{2} + \frac{1}{4}\pi\sqrt{2} - \ln|\csc x + \cot x|\Big]_{\pi/4}^{3\pi/4}$

$= -\frac{1}{2}\sqrt{2}\pi - \ln(\sqrt{2}-1) + \ln(\sqrt{2}+1) \approx -0.46869$

31. First substitute $t = x^2$, $dt = 2x\,dx$. Thus $I = \int_2^4 \sec^{-1}\sqrt{t}\,dt = \int_{\sqrt{2}}^{2} 2x\sec^{-1}x\,dx$.

Now let $u = \sec^{-1}x$ and $dv = 2x\,dx$. Then $du = \frac{dx}{x\sqrt{x^2-1}}$ and $v = x^2$; so $v\,du = \frac{x\,dx}{\sqrt{x^2-1}}$.

$I = x^2\sec^{-1}x\Big]_{\sqrt{2}}^{2} - \int_{\sqrt{2}}^{2}\frac{x\,dx}{\sqrt{x^2-1}} = 4\left(\frac{\pi}{3}\right) - 2\left(\frac{\pi}{4}\right) - \left[\sqrt{x^2-1}\right]_{\sqrt{2}}^{2} = \frac{5}{6}\pi - \sqrt{3} + 1$

32. $\int_0^1 x \sin^{-1} x \, dx$

▸ Method 1. We use integration by parts with

$$u = \sin^{-1} x \qquad dv = x$$

$$du = \frac{dx}{\sqrt{1 - x^2}} \qquad v = \tfrac{1}{2}x^2 - \tfrac{1}{2}$$

where the constant in v was chosen to simplify $v\,du$. Then

$$\int_0^1 x \sin^{-1} x \, dx = \tfrac{1}{2}(x^2 - 1)\sin^{-1} x \Big]_0^1 - \int_0^1 \frac{(x^2 - 1)dx}{2\sqrt{1 - x^2}} = 0 + \tfrac{1}{2}\int_0^1 \sqrt{1 - x^2}\, dx$$

The integral on the right side represents the area of a quarter-circle of radius 1 and so has value $\frac{1}{4}\pi$. (The antiderivative can be found using the substitution $x = \sin\theta$.) Thus

$$\int_0^1 x \sin^{-1} x \, dx = 0 + \tfrac{1}{2}\cdot\tfrac{1}{4}\pi = \tfrac{1}{8}\pi$$

Method 2. Let $y = \sin^{-1} x$. Then $x = \sin y$ and $dx = \cos y \, dy$; when $x = 0, y = 0$; when $x = 1$, $y = \frac{1}{2}\pi$. Hence

$$\int_0^1 x \sin^{-1} x \, dx = \int_0^{\pi/2} (\sin y) y (\cos y \, dy) = \int_0^{\pi/2} y(\tfrac{1}{2}\sin 2y)\, dy$$

Now we integrate by parts with

$$u = y \qquad dv = \tfrac{1}{2}\sin 2y \, dy$$

$$du = dy \qquad v = -\tfrac{1}{4}\cos 2y$$

Hence

$$\int_0^1 x \sin^{-1} x \, dx = -\tfrac{1}{4} y \cos 2y \Big]_0^{\pi/2} + \tfrac{1}{4}\int_0^{\pi/2} \cos 2y \, dy = -\tfrac{1}{4}\cdot\tfrac{\pi}{2}(-1) + \tfrac{1}{4}\cdot\tfrac{1}{2}\sin 2y\Big]_0^{\pi/2} = \tfrac{1}{8}\pi + 0 = \tfrac{1}{8}\pi$$

In Exercises 33 and 34, use equation (3) to evaluate the integral. Check by NINT.

33. $\int x^3 e^x dx = x^3 e^x - \int 3x^2 e^x dx = x^3 e^x - 3x^2 e^x + \int 6x e^x dx = x^3 e^x - 3x^2 e^x + 6x e^x - \int 6e^x dx$
$= (x^3 - 3x^2 + 6x - 6)e^x$. $\int_1^3 x^3 e^x dx = (27 - 27 + 18 - 6)e^3 - (1 - 3 + 6 - 6)e = 12e^3 + 2e \approx 246.463$

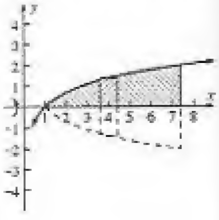
34. Let $u = \frac{1}{2}x$, $du = \frac{1}{2}dx$.
$\int_{x=0}^4 x^2 e^{x/2} dx = \int_{u=0}^2 (2u)^2 e^u (2du) = 8\int_0^2 u^2 e^u du = 8\Big[u^2 e^u - 2ue^u + 2e^u\Big]_0^2 = 8(2e^2 - 2) = 16e^2 - 16 \approx 102.225$

35. A square units is the area of the region bounded by $y = \ln x$, the x axis, and $x = e^2$.
$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \ln w_i \Delta_i x = \int_0^{e^2} \ln x \, dx$
Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\, dx$ and $v = x$.
$A = x \ln x\Big]_1^{e^2} - \int_1^{e^2} dx = x \ln x - x\Big]_1^{e^2} = (2e^2 - e^2) - (-1) = e^2 + 1$

36. Find the volume of the solid generated by revolving about the x axis the region bounded by the curve $y = \ln x$, the x axis, and the line $x = e^2$.

▸ The figure shows the region ($e^2 \approx 7.39$) and a plane section of the solid of revolution. The element of volume is a circular disk with thickness $\Delta_i x$ units and radius $\ln w_i$ units. If V cubic units is the volume of the solid of revolution, then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\ln w_i)^2 \, \Delta_i x = \int_1^{e^2} \pi \ln^2 x \, dx \qquad (1)$$



To find the indefinite integral we use integration by parts with

$$u = \ln^2 x \qquad dv = dx$$

$$du = \frac{2\ln x}{x}\, dx \qquad v = x$$

Hence,

$$\int \ln^2 x \, dx = x \ln^2 x - 2\int \ln x \, dx \qquad (2)$$

Now we integrate by parts again with

$$\bar{u} = \ln x \qquad d\bar{v} = dx$$

$$d\bar{u} = \frac{1}{x}\, dx \qquad \bar{v} = x$$

Thus,

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x \qquad (3)$$

We have taken $C = 0$ in the indefinite integral because we intend to use the result to evaluate a definite integral. Substituting from (3) into (2) and then into (1), we obtain

$$V = \pi\Big[x\ln^2 x - 2x\ln x + 2x\Big]_1^{e^2} = \pi[(e^2\ln^2 e^2 - 2e^2\ln e^2 + 2e^2) - (\ln^2 1 - 2\ln 1 + 2)]$$
$$= \pi[4e^2 - 4e^2 + 2e^2 - 2] = 2\pi(e^2 - 1)$$

- The volume of the solid of revolution is $2\pi(e^2 - 1)$ cubic units.

37. V cubic units is the volume when the region of Ex. 35 is revolved about the y axis. An element of volume is a cylindrical shell centered on the y axis, of mean radius m_i and altitude $\ln m_i$.

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i \ln m_i \Delta_i x = 2\pi\int_1^{e^2} x\ln x\,dx$$

Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{1}{2}x^2$.

$$V = 2\pi\left(\frac{1}{2}x^2\ln x\Big]_1^{e^2} - \frac{1}{2}\int_1^{e^2} x\,dx\right) = 2\pi\Big[\frac{1}{2}x^2\ln x - \frac{1}{2}x^2\Big]_1^{e^2} = 2\pi\Big[\Big(\frac{1}{2}e^4\cdot 2 - \frac{1}{4}e^4\Big) - \Big(-\frac{1}{4}\Big)\Big] = \frac{\pi}{2}(3e^4 + 1)$$

38. A square units is the area of the region bounded by $y = x\csc^2 x$, the x axis, and $x = \frac{1}{6}\pi$ and $x = \frac{1}{4}\pi$.

$A = \int_{\pi/6}^{\pi/4} x\csc^2 x\,dx$. Let $u = x$, $dv = \csc^2 x$. Then $du = dx$ and $v = -\cot x$. $A = -x\cot x\Big]_{\pi/6}^{\pi/4} + \int_{\pi/6}^{\pi/4}\cot x\,dx$

$= \frac{1}{6}\pi\sqrt{3} - \frac{1}{4}\pi\cdot 1 + \ln|\sin x|\Big]_{\pi/6}^{\pi/4} = \pi(\frac{1}{6}\sqrt{3} - \frac{1}{4}) + \ln(\frac{1}{2}\sqrt{2}) - \ln\frac{1}{2} = \pi(\frac{1}{6}\sqrt{3} - \frac{1}{4}) + \frac{1}{2}\ln 2$

39. A square units is the area of the region bounded by $y = 2xe^{-x/2}$, the x axis, and $x = 4$.

$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2w_i e^{-w_i/2}\Delta_i x = 2\int_0^4 xe^{-x/2}dx$$

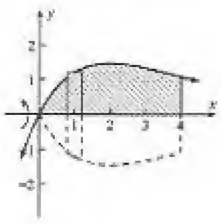
Let $u = x$ and $dv = e^{-x/2}dx$. Then $du = dx$ and $v = -2e^{x/2}$. Therefore

$$A = 2\left(-2xe^{-x/2}\Big]_0^4 + 2\int_0^4 e^{-x/2}dx\right) = 2\Big[-2xe^{-x/2} - 4e^{-x/2}\Big]_0^4 = 2[(-8e^{-2} - 4e^{-2}) - (-4)] = 8 - 24e^{-2}$$

40. Find the volume of the solid of revolution generated by revolving about the x axis the region bounded by the curve $y = 2xe^{-x/2}$, the x axis, and the line $x = 4$.

- Let $f(x) = 2xe^{-x/2}$. Because $f'(x) = e^{-x/2}(2 - x)$, then f has a relative maximum value at $x = 2$. The figure shows the region and a plane section of the solid of revolution. The element of volume is a circular disk with thickness $\Delta_i x$ units and radius $f(w_i)$ units. If V cubic units is the volume of the solid of revolution, then

$$V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\pi[f(w_i)]^2\,\Delta_i x = \int_0^4 \pi(2xe^{-x/2})^2\,dx = 4\pi\int_0^4 x^2e^{-x}\,dx \qquad (1)$$



We use integration by parts. Let

$$u = x^2 \qquad dv = e^{-x}\,dx$$
$$du = 2x\,dx \qquad v = -e^{-x}$$

Then

$$\int x^2e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx \qquad (2)$$

We integrate by parts again with

$$\bar{u} = x \qquad d\bar{v} = e^{-x}$$
$$d\bar{u} = dx \qquad \bar{v} = -e^{-x}$$

Thus,

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} \qquad (3)$$

Substituting from (3) into (2) and then into (1), we obtain

$$V = 4\pi\Big[-x^2e^{-x} + 2(-xe^{-x} - e^{-x})\Big]_0^4 = 4\pi[(-16e^{-4} - 8e^{-4} - 2e^{-4}) - (-2e^0)] = 4\pi(2 - 26e^{-4})$$
$$= 8\pi(1 - 13e^{-4})$$

- The volume of the solid of revolution is $8\pi(1 - 13e^{-4})$ cubic units.

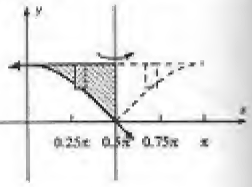
41. If M kg is the mass of the rod, then $M = \int_0^6 2e^{-x}dx = -2e^{-x}\Big]_0^6 = -2e^{-6} + 2$. $M_{\ } = \int_0^6 2xe^{-x}dx$.
Let $u = 2x$ and $dv = e^{-x}dx$. Then $du = 2\,dx$ and $v = -e^{-x}$.
$M_{\ } = -2xe^{-x}\Big]_0^6 + \int_0^6 2e^{-x}dx = -12e^{-6} + (2 - 2e^{-6}) = 2 - 14e^{-6}$. $\bar{x} = \dfrac{M_{\bar{x}}}{M} = \dfrac{2 - 14e^{-6}}{2 - 2e^{-6}} = \dfrac{e^6 - 7}{e^6 - 1}$.
The mass is $(2 - 2e^{-6})$kg and the centroid is $\dfrac{e^6 - 7}{e^6 - 1}$ m from one end.

42. The region is bounded by $y = e^x$, the axes and $x = 3$.
$A = \int_0^3 e^x dx = e^x\Big]_0^3 = e^3 - 1$. $M_y = \int_0^3 x\boxed{e^x dx} = xe^x\Big]_0^3 - \int_0^3 e^x dx = xe^x - e^x\Big]_0^3 = 2e^3 + 1$. $\bar{x} = \dfrac{2e^3 + 1}{e^3 - 1}$
$M_x = \frac{1}{2}\int_0^3 e^{2x}dx = \frac{1}{4}e^{2x}\Big]_0^3 = \frac{1}{4}(e^6 - 1)$. $\bar{y} = \dfrac{\frac{1}{4}(e^6 - 1)}{e^3 - 1} = \dfrac{e^3 + 1}{4}$

43. The region is in the first quadrant bounded by $y = \sin x$, $y = \cos x$, and the y axis.
$A = \int_0^{\pi/4}(\cos x - \sin x)dx = \sin x + \cos x\Big]_0^{\pi/4} = \sqrt{2} - 1$
$M_x = \frac{1}{2}\int_0^{\pi/4}(\cos^2 x - \sin^2 x)dx = \frac{1}{2}\int_0^{\pi/4}\cos 2x\, dx = \frac{1}{4}\sin 2x\Big]_0^{\pi/4} = \frac{1}{4}$. $M_y = \int_0^{\pi/4} x(\cos x - \sin x)dx$.
Let $u = x$ and $dv = (\cos x - \sin x)dx$. Then $du = dx$ and $v = \sin x + \cos x$.
$M_y = x(\sin x + \cos x)\Big]_0^{\pi/4} - \int_0^{\pi/4}(\sin x + \cos x)dx = \frac{\pi}{4}\sqrt{2} - \Big[-\cos x + \sin x\Big]_0^{\pi} = \frac{\pi}{4}\sqrt{2} - 1$
$\bar{x} = \dfrac{M_y}{A} = \dfrac{\frac{1}{4}\pi\sqrt{2} - 1}{\sqrt{2} - 1} \approx 0.267$ and $\bar{y} = \dfrac{M_x}{A} = \dfrac{\frac{1}{4}}{\sqrt{2} - 1} \approx 0.604$. The centroid is at $(0.267, 0.604)$.

44. The region in the first quadrant bounded by the curve $y = \cos x$, the lines $y = 1$ and $x = \frac{1}{2}\pi$, is revolved about the line $x = \frac{1}{2}\pi$. Find the volume of the solid generated.

▸ The figure shows the region and a plane section of the solid of revolution. The element of area is rectangle of width $\Delta_i x$ units, and element of volume is a cylindrical shell of thickness $\Delta_i r = \Delta_i x$ units, mean radius $r_i = \frac{1}{2}\pi - m_i$ units, and altitude $h_i = 1 - \cos m_i$ units. Thus,



$$V = \lim_{\|\Delta\| \to 0}\sum_{i=1}^{n} 2\pi r_i h_i \Delta_i r = \lim_{\|\Delta\| \to 0}\sum_{i=1}^{n} 2\pi(\tfrac{1}{2}\pi - m_i)(1 - \cos m_i)\Delta_i x$$
$$= 2\pi\int_0^{\pi/2}(\tfrac{1}{2}\pi - x)(1 - \cos x)\,dx$$

We use integration by parts with

$$u = \tfrac{1}{2}\pi - x \qquad dv = 1 - \cos x$$
$$du = -1 \qquad v = x - \sin x$$

Thus
$$V = 2\pi\Big\{(\tfrac{1}{2}\pi - x)(x - \sin x)\Big]_0^{\pi/2} + \int_0^{\pi/2}(x - \sin x)\,dx\Big\} = 2\pi\Big\{0 + \tfrac{1}{2}x^2 + \cos x\Big]_0^{\pi/2}\Big\} = 2\pi(\tfrac{1}{8}\pi^2 - 1)$$

■ The volume of the solid of revolution is $2\pi(\frac{1}{8}\pi^2 - 1)$ cubic units.

45. Choose the origin at the top of the tank with the positive x axis downward. Let W ft-lb be the work done in pumping all the water to the top of the tank. An element of volume is a circular disk of radius e^{-w_i}, $w_i \in [0, 4]$. Let $u = x$, $dv = e^{-2x}$; $du = dx$, $v = -\frac{1}{2}e^{-2x}$.
$W = \lim_{\|\Delta\| \to 0}\sum_{i=1}^{n} \mathrm{w}w_i\left(e^{-w_i}\right)^2\Delta_i x = \mathrm{w}\pi\int_0^4 xe^{-2x}dx = \mathrm{w}\pi\Big[-\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx\Big]_0^4 = \mathrm{w}\pi\Big[-\frac{1}{2}xe^{-2x} - \frac{1}{2}e^{-2x}\Big]_0^4$
$= \mathrm{w}\pi\Big(-2e^{-8} - \frac{1}{4}e^{-8} + \frac{1}{4}\Big) = \frac{1}{4}(1 - 9e^{-8})\mathrm{w}\pi$

46. $s\dfrac{ds}{dt} = t\sin t$; $\int s\,ds = \int t\boxed{\sin t\,dt}$; $\frac{1}{2}s^2 = -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + C$. With $s = 0$ and $t = 0$ we have $0 = C$. $s = \pm\sqrt{2(\sin t - t\cos t)}$. $s(\frac{1}{2}\pi) = \pm\sqrt{2}$

47. $C'(x) = \ln x$; so $C(x) = \int \ln x\,dx$. Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}dx$ and $v = x$.
$C(x) = x\ln x - \int dx = x\ln x - x + C$. Because $C(1) = 5$, $5 = 0 - 1 + C$; $C = 6$. $C(x) = x\ln x - x + 6$.

48. A manufacturer has discovered that if $100x$ units of a particular commodity are produced per week, the marginal cost is determined by $x2^{x/2}$ and the marginal revenue is determined by $8 \cdot 2^{-x/2}$, where both the production cost and the revenue are in thousands of dollars. If the weekly fixed costs amount to \$2000, find the maximum weekly profit that can be obtained.

▸ Let $C(x)$ thousand dollars be the cost of producing $100x$ units of the commodity. Because $C'(x)$ thousand dollars is the marginal cost of producing $100x$ units and $C(0)$ thousand dollars is the fixed cost, we are given $C'(x) = x2^{x/2}$ and $C(0) = 2$

Thus,

$$C(x) = \int x2^{x/2}\,dx$$

We use integration by parts with

$$u = x \qquad dv = 2^{x/2}\,dx$$
$$du = dx \qquad v = \frac{(2)2^{x/2}}{\ln 2}$$

Thus,

$$C(x) = \frac{(2x)2^{x/2}}{\ln 2} - \frac{2}{\ln 2}\int 2^{x/2}\,dx = \frac{(2x)2^{x/2}}{\ln 2} - \frac{2}{\ln 2}\cdot\frac{2}{\ln 2}\cdot 2^{x/2} + A$$

Because

$$C(0) = \frac{-4}{(\ln 2)^2} + A$$

and we are given that $C(0) = 2$, then $A = 4/(\ln 2)^2 + 2$, and

$$C(x) = \frac{(2x)2^{x/2}}{\ln 2} - \frac{(4)2^{x/2}}{(\ln 2)^2} + \frac{4}{(\ln 2)^2} + 2 \qquad (1)$$

Let $R(x)$ thousand dollars be the revenue when $100x$ units are sold. Because $R'(x)$ thousand dollars is the marginal revenue, and the revenue is zero when $x = 0$, we are given that

$R'(x) = (8)2^{-x/2}$ and $R(0) = 0$

Thus,

$$R(x) = \int (8)2^{-x/2}\,dx = \frac{(-16)2^{-x/2}}{\ln 2} + B$$

Because

$$R(0) = \frac{-16}{\ln 2} + B$$

and we are given that $R(0) = 0$, then $B = 16/\ln 2$, and

$$R(x) = \frac{(-16)2^{-x/2}}{\ln 2} + \frac{16}{\ln 2} \qquad (2)$$

If $P(x)$ thousand dollars is the weekly profit when $100x$ units are produced and sold per week, then

$$P(x) = R(x) - C(x)$$
$$P'(x) = R'(x) - C'(x) = (8)2^{-x/2} - x2^{x/2}$$

By trial, we find $P'(2) = 0$. Furthermore, $P'(x) > 0$ if $x < 2$ and $P'(x) < 0$ if $x > 2$. Thus the absolute maximum occurs when $x = 2$. We find the profit when $x = 2$. From Eq. (2) we get

$$R(2) = \frac{(-16)2^{-1}}{\ln 2} + \frac{16}{\ln 2} = \frac{8}{\ln 2}$$

and from Eq. (1) we obtain

$$C(2) = \frac{8}{\ln 2} - \frac{8}{(\ln 2)^2} + \frac{4}{(\ln 2)^2} + 2 = \frac{8}{\ln 2} - \frac{4}{(\ln 2)^2} + 2$$

Thus,

$$P(2) = R(2) - C(2) = \frac{8}{\ln 2} - \left[\frac{8}{\ln 2} - \frac{4}{(\ln 2)^2} + 2\right] = \frac{4}{(\ln 2)^2} - 2 = 6.325$$

▪ Because $1000P(2) = 6325$, we conclude that the maximum weekly profit is \$6325.

49. $I = \int \sin nu \boxed{e^{au}\,du} = \frac{1}{a}e^{au}\sin nu - \frac{n}{a}\int \cos nu \boxed{e^{au}\,du} = \frac{1}{a}e^{au}\sin nu - \frac{n}{a}\left[\frac{1}{a}\cos nu - \frac{n}{a}\int e^{au}(\sin nu)du\right]$

$= \frac{1}{a}e^{au} - \frac{n}{a}\left[\frac{1}{a}e^{au}\cos nu + \frac{n}{a}\,I\right] = \frac{1}{a}e^{au}\sin nu - \frac{n}{a^2}e^{au}\cos nu - \frac{n^2}{a^2}\,I$. Therefore

$\frac{a^2+n^2}{a^2}I = \frac{ae^{au}\sin nu - ne^{au}\cos nu}{a^2}$ and so $I = \frac{e^{au}}{a^2+n^2}(a\sin nu - n\cos nu) + C$

50. $I = \int \cos nu \boxed{e^{au}du} = \frac{1}{a}e^{au}\cos nu + \frac{n}{a}\int \sin nu \boxed{e^{au}du} = \frac{1}{a}e^{au}\cos nu + \frac{n}{a}\left[\frac{1}{a}\sin nu - \frac{n}{a}\int e^{au}\cos nu\,du\right]$

$= \frac{1}{a}e^{au}\cos nu + \frac{n}{a}\left[\frac{1}{a}\sin nu - \frac{n}{a}I\right] = \frac{1}{a}e^{au}\cos nu + \frac{n}{a^2}e^{au}\sin nu - \frac{n^2}{a^2}I$. Therefore

$\frac{a^2+n^2}{a^2}I = \frac{ae^{au}\cos nu + ne^{au}\sin nu}{a^2}$ and so $I = \frac{e^{au}}{a^2+n^2}(a\cos nu + n\sin nu) + C$

In Exercises 51 and 52, use formula (4) or (5) to evaluate the definite integral to 6 digits. Check by NINT.

51. $\int_{\pi/6}^{\pi/3} e^{4x}\sin 3x\,dx = \frac{e^{4x}}{16+9}(4\sin 3x - 3\cos 3x)\Big]_{\pi/6}^{\pi/3} = \frac{1}{25}[e^{4\pi/3}(0+3) - e^{2\pi/3}(4-0)] = \frac{3}{25}e^{4\pi/3} - \frac{4}{25}e^{2\pi/3}$
≈ 6.61387

52. $\int_{\pi/8}^{\pi/4} e^{3x}\cos 4x\,dx$

▸ Using formula (5) with $a = 3$ and $n = 4$, we have

$$\int_{\pi/8}^{\pi/4} e^{3x}\cos 4x\,dx = \frac{e^{3x}}{9+16}(3\cos 4x + 4\sin 4x)\Big]_{\pi/8}^{\pi/4} = \frac{1}{25}[e^{3\pi/4}(3(-1)+4(0)) - e^{3\pi/8}(3(0)+4(1))]$$
$$= -\frac{3}{25}e^{3\pi/4} - \frac{4}{25}e^{3\pi/8} \approx -1.78580$$

53. (a) If $r = -1$, then $\int x^r \ln x\,dx = \int \ln x\,\frac{dx}{x} = \frac{1}{2}(\ln x)^2 + C$.

If $r \neq -1$, let $u = \ln x$ and $dv = x^r\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{x^{r+1}}{r+1}$.

$\int x^r \ln x\,dx = \frac{x^{r+1}}{r+1}\ln x - \int \frac{x^r}{r+1}\,dx = \frac{x^{r+1}}{r+1}\ln x - \frac{x^{r+1}}{(r+1)^2} + C$

(b) With $r = 3$ we have $\int_1^3 x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16}\Big]_1^3 = \frac{81}{4}\ln 3 - 5 \approx 17.2469$

54. (a) If $r \neq -1$, $\int (\ln x)^q \boxed{x^r\,dx} = (\ln x)^q\frac{x^{r+1}}{r+1} - \int \frac{x^{r+1}}{r+1}\cdot q(\ln x)^{q-1}\cdot\frac{1}{x}dx = \frac{x^{r+1}(\ln x)^q}{r+1} - \frac{q}{r+1}\int x^r(\ln x)^{q-1}dx$.

(b) If $r = -1$ and $q \neq -1$, $\int (\ln x)^q\frac{dx}{x} = \int (\ln x)^q\,d(\ln x) = \frac{(\ln x)^{q+1}}{q+1} + C$

(c) If $r = -1$ and $q = -1$, $\int \frac{1}{\ln x}\cdot\frac{dx}{x} = \int \frac{1}{\ln x}d(\ln x) = \ln|\ln x| + C$

55. $i(t) = \int_0^t x\boxed{e^{-x}dx} = x(-e^{-x})\Big]_0^t + \int_0^t e^{-x}dx = -te^{-t} - e^{-t}\Big]_0^t = 1 - te^{-t} - e^{-t}$

$E(1) = 50\int_0^1 (1 - te^{-t} - e^{-t})^2dt = 50(6e^{-1} - \frac{7}{4} - \frac{13}{4}e^{-2}) \approx 0.8718$

7.2 TRIGONOMETRIC INTEGRALS

Case 1: $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$, where n is a positive odd integer and so $n-1$ is even.

(i) $\sin^n x\,dx = (\sin^{n-1}x)(\sin x\,dx) = (\sin^2 x)^{(n-1)/2}(\sin x\,dx) = (1-\cos^2 x)^{(n-1)/2}(-d\cos x)$

(ii) $\cos^n x\,dx = (\cos^{n-1}x)(\cos x\,dx) = (\cos^2 x)^{(n-1)/2}(\cos x\,dx) = (1-\sin^2 x)^{(n-1)/2}(d\sin x)$

Case 2: $\int \sin^n x\cos^m x\,dx$ where either m (i) or n (ii) is a positive odd integer.

(i) $\sin^n x\cos^m x\,dx = \sin^{n-1}x\cos^m x(\sin x\,dx) = (\sin^2 x)^{(n-1)/2}\cos^m(\sin x\,dx) = (1-\cos^2 x)^{(n-1)/2}\cos^m x(-d\cos x)$

(ii) $\sin^n x\cos^m x\,dx = \sin^n x\cos^{m-1}x(\cos x\,dx) = \sin^n x(\cos^2 x)^{(n-1)/2}(\cos x\,dx) = \sin^n x(1-\sin^2 x)^{(m-1)/2}(d\sin x)$

You may also use reduction formulas. See Exercises 67 and 68.

$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du$ (Formula 73)

$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1}u\sin u + \frac{n-1}{n}\int \cos^{n-2}u\,du$ (Formula 74)

Case 3: (i) $\int \sin^n x\,dx$, (ii) $\int \cos^m dx$, or (iii) $\int \sin^n x \cos^m dx$ where m and n are positive even integers.
Use the identities $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, $\sin x \cos x = \frac{1}{2}\sin 2x$ to reduce to Case 2.

(iv) $\int \sin ax\, bx\,dx$, (v) $\int \cos ax \cos bx\,dx$, (vi) $\int \sin ax \cos bx\,dx$. Use the identities

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$, $\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$, $\sin a \cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$

Case 4: (i) $\int \tan^n x\,dx$ or (ii) $\int \cot^n x\,dx$, where n is a positive integer

(i) $\tan^n x = \tan^{n-2}x \tan^2 x\,dx = \tan^{n-2}x(\sec^2 x - 1)dx$ (ii) $\cot^n x\,dx = \cot^{n-2}x \cot^2 x\,dx = \cot^{n-2}x(\csc^2 x - 1)dx$

Case 5: (i) $\int \sec^n x\,dx$ or (ii) $\int \csc^n x\,dx$ where n is a positive even integer and so $n-2$ is even.

(i) $\sec^n x\,dx = \sec^{n-2}x(\sec^2 x\,dx) = (\sec^2 x)^{(n-2)/2}(\sec^2 x\,dx) = (\tan^2 x + 1)^{(n-2)/2}(d \tan x)$

(ii) $\csc^n x\,dx = \csc^{n-2}x(\csc^2 x\,dx) = (\csc^2 x)^{(n-2)/2}(\csc^2 x\,dx) = (\cot^2 x + 1)^{(n-2)/2}(-d \cot x)$

Case 6: (i) $\int \tan^n x \sec^m x\,dx$ or (ii) $\int \cot^n x \csc^m x\,dx$ where m is a positive even integer and so $m-2$ is even.

(i) $\tan^n x \sec^m x\,dx = \tan^n x \sec^{m-2}x(\sec^2 x\,dx) = \tan^n x(\sec^2 x)^{(m-2)/2}(\sec^2 x\,dx) = \tan^n x(\tan^2 x + 1)^{(m-2)/2}(d \tan x)$

(ii) $\cot^n x \csc^m x\,dx = \cot^n x \csc^{m-2}x(\csc^2 x\,dx) = \cot^n x(\csc^2 x)^{(m-2)/2}(\csc^2 x\,dx) = \cot^n x(\cot^2 x + 1)^{(m-2)/2}(-d \cot x)$

Case 7: (i) $\int \tan^n x \sec^m dx$ or (ii) $\int \cot^n x \csc^m x\,dx$ where n is a positive odd integer so $k = \frac{1}{2}(n-1)$ is an integer.

(i) $\tan^n x \sec^m dx = \tan^{n-1}x \sec^{m-1}x(\sec x \tan x\,dx) = (\tan^2 x)^k \sec^{m-1}x(d \sec x) = (\sec^2 x - 1)^k \sec^{m-1}x(d \sec x)$

(ii) $\cot^n x \csc^m x\,dx = \cot^{n-1}x \csc^{m-1}x(\csc x \cot x\,dx) = (\cot^2 x)^k \csc^{m-1}x(-d \csc x) = (\csc^2 x - 1)^k \csc^{m-1}x(-d \csc x)$

Case 8: Use the reduction formulas

(i) $\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x \tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$ (Formula 77)

(ii) $\int \csc^n x\,dx = -\frac{1}{n-1}\csc^{n-2}x \cot x + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx$ (Formula 78)

Case 9: (i) $\int \tan^n x \sec^m x\,dx$ or (ii) $\int \cot^n x \csc^m x\,dx$ where n is even and m is odd . Apply Case 8 to:

(i) $\tan^n x \sec^m x = (\tan^2 x)^{n/2}\sec^m x = (\sec^2 x - 1)^{n/2}\sec^m x$ (ii) $\cot^n x \csc^m x = (\cot^2 x)^{n/2}\csc^m = (\csc^2 x - 1)^{n/2}\csc^m x$

Wallis' Formula: $\lim\limits_{n\to\infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot \ldots \cdot (2n)(2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)(2n-1)} = \frac{\pi}{2}$. See Exercise 67.

Exercises 7.2

In Exercises 1–34, evaluate the indefinite integral. Check your answer by differentiation or graphically.

1. (a) $\int \sin^4 x(\cos x\,dx) = \int \sin^4 x\,d(\sin x) = \frac{1}{5}\sin^5 x + C$

(b) $\int \cos^3 4x \sin 4x\,dx = -\frac{1}{4}\int \cos^3 4x(-4 \sin 4x\,dx) = -\frac{1}{16}\cos^4 4x + C$

2. (a) $\int \sin^5 x \cos x\,dx = \int \sin^5 x\,d(\sin x) = \frac{1}{6}\sin^6 x + C$

(b) $\int \cos^6 \frac{1}{2}x \sin \frac{1}{2}x\,dx = -2\int \cos^6 \frac{1}{2}x\,d(\cos \frac{1}{2}x) = -\frac{2}{7}\cos^7 \frac{1}{2}x + C$

3. (a) $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int (1 - \cos^2 x)\sin x\,dx = \int \sin x\,dx + \int \cos^2 x(-\sin x\,dx)$
$= -\cos x + \frac{1}{3}\cos^3 x + C$

(b) $\int \cos^2 \frac{1}{2}x\,dx = \int \frac{1}{2}(1 + \cos x)dx = \frac{1}{2}(x + \sin x) + C$

4. (a) $\int \cos^5 x\,dx$; (b) $\int \sin^2 3x\,dx$

▸ (a) Because we have an odd power of cosine, we use the method of case 1. Let $u = \sin x$, $du = \cos x$.

$$\int \cos^5 x\,dx = \int \cos^4 x(\cos x\,dx) = \int (1 - \sin^2 x)^2(\cos x\,dx) = \int (1-u^2)^2 du = \int (1 - 2u^2 + u^4)du$$
$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

(b) Because we have an even power of sine, we use the method of case 3.

$$\int \sin^2 3x\, dx = \int \tfrac{1}{2}(1-\cos 6x)dx = \tfrac{1}{2}x - \tfrac{1}{12}\sin 6x + C$$

5. (a) $\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx = \int \sin^2 x(1-\sin^2 x)\cos x\, dx = \int (\sin^2 x - \sin^4 x)\cos x\, dx$

$$= \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C$$

(b) $\int \sqrt{\cos x}\,\sin^3 x\, dx = \int \cos^{1/2} x \sin^2 x(\sin x\, dx) = \int \cos^{1/2} x(1-\cos^2 x)(-d\cos x)$

$$= \int (\cos^{5/2} x - \cos^{1/2} x)d(\cos x) = \tfrac{2}{7}\cos^{7/2} x - \tfrac{2}{3}\cos^{3/2} x + C$$

6. (a) $\int \sin^3 x \cos^3 x\, dx = \int \sin^3 x(1-\sin^2 x)(\cos x\, dx) = \int (\sin^3 x - \sin^5 x)(d\sin x) = \tfrac{1}{4}\sin^4 x - \tfrac{1}{6}\sin^6 x + C$

(b) $\int \dfrac{\cos^3 3x}{\sqrt[3]{\sin 3x}}\, dx = \int \dfrac{\cos^2 3x \cos 3x}{\sqrt[3]{\sin 3x}}\, dx = \int (1-\sin^2 3x)(\sin 3x)^{-1/3}\cos 3x\, dx$

$$= \frac{1}{3}\int [(\sin 3x)^{-1/3} - (\sin 3x)^{5/3}](3\cos 3x\, dx) = \tfrac{1}{2}(\sin 3x)^{2/3} - \tfrac{1}{8}(\sin 3x)^{8/3} + C$$

7. $\int \cos 4x \cos 3x\, dx = \tfrac{1}{2}\int (\cos x + \cos 7x)dx = \tfrac{1}{2}\sin x + \tfrac{1}{14}\sin 7x + C$

8. $\int \sin 2x \cos 4x\, dx$

▹ We use part (vi) of Case 3.

$$\int \sin 2x \cos 4x = \int \tfrac{1}{2}[\sin(-2x) + \sin 6x]dx = \int \tfrac{1}{2}(-\sin 2x + \sin 6x)dx = \tfrac{1}{2}(\tfrac{1}{2}\cos 2x - \tfrac{1}{6}\cos 6x) + C$$

$$= \tfrac{1}{4}\cos 2x - \tfrac{1}{12}\cos 6x + C$$

9. $\int \sin 3y \cos 5y\, dy = \tfrac{1}{2}\int (-\sin 2y + \sin 8y)dy = \dfrac{1}{4}\cos 2y - \dfrac{1}{16}\cos 8y + C$

10. $\int \cos 3t \cos t\, dt = \int \tfrac{1}{2}(\cos 2t + \cos 4t)dt = \tfrac{1}{2}(\tfrac{1}{2}\sin 2t + \tfrac{1}{4}\sin 4t) + C = \tfrac{1}{4}\sin 2t + \tfrac{1}{8}\sin 4t + C$

11. $\int \tan^2 5x\, dx = \int (\sec^2 5x - 1)dx = \tfrac{1}{5}\tan 5x - x + C$

12. $\int e^x \tan^2(e^x)dx$

▹ First we let $u = e^x$; $du = e^x\, dx$. Then we apply Case 4.

$$\int \tan^2(e^x)(e^x dx) = \int \tan^2 u\, du = \int (\sec^2 u - 1)du = \tan u - u + C = \tan e^x - e^x + C$$

13. Let $u = 2x^2$, $du = 4x\, dx$.

$$\int x\cot^2 2x^2 dx = \frac{1}{4}\int \cot^2 u\, du = \frac{1}{4}\int (\csc^2 u - 1)du = -\frac{1}{4}\cot u - \frac{1}{4}u + C = -\frac{1}{4}\cot 2x^2 - \frac{1}{2}x^2 + C$$

14. $\int \cot^2 4t\, dt = \int (\csc^2 4t - 1)dt = -\tfrac{1}{4}\cot 4t - t + C$

15. $\int \cot^3 t\, dt = \int (\csc^2 t - 1)\cot t\, dt = \int \csc t(\csc t \cot t\, dt) - \int \cot t\, dt = -\dfrac{1}{2}\csc^2 t - \ln|\sin t| + C$

16. $\int \tan^4 x\, dx$

▹ We apply Case 4.

$$\int \tan^4 x\, dx = \int \tan^2 x(\tan^2 x\, dx) = \int \tan^2 x(\sec^2 x - 1)dx = \int \tan^2 x\, d(\tan x) - \int \tan^2 x\, dx$$

$$= \tfrac{1}{3}\tan^3 x - \int (\sec^2 x - 1)dx = \tfrac{1}{3}\tan^3 x - \tan x + x + C$$

17. $\int \tan^6 3x\, dx = \int \tan^4 3x(\sec^2 3x - 1)dx = \dfrac{1}{3}\int \tan^4 3x(3\sec^2 3x\, dx) - \int \tan^4 3x\, dx$

$$= \frac{1}{15}\tan^5 3x - \int \tan^2 3x(\sec^2 3x - 1)dx = \frac{1}{15}\tan^5 3x - \frac{1}{2}\int \tan^2 3x(3\sec^2 3x\, dx) + \int \tan^2 3x\, dx$$

$$= \frac{1}{15}\tan^5 3x - \frac{1}{9}\tan^3 3x + \int (\sec^2 3x - 1)dx = \frac{1}{15}\tan^5 3x - \frac{1}{9}\tan^3 3x + \frac{1}{3}\tan 3x - x + C$$

18. $\int \csc^3 x\, dx = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\int \csc x\, dx = -\frac{1}{2}\csc x \cot x - \frac{1}{2}\ln(\csc x + \cot x) + C$

19. $\int \sec^4 x\, dx = \int (\tan^2 x + 1)\sec^2 x\, dx = \frac{1}{3}\tan^3 x + \tan x + C$

20. $\int \cot^5 2x\, dx$

▹ We apply Case 4.

$$\int \cot^5 2x\, dx = \int \cot^3 2x(\cot^2 2x\, dx) = \int \cot^3 2x(\csc^2 2x - 1)dx = \int \cot^3 2x(-\tfrac{1}{2}\, d\cot 2x) - \int \cot^3 2x\, dx$$
$$= -\tfrac{1}{8}\cot^4 2x - \int \cot 2x(\cot^2 2x\, dx) = -\tfrac{1}{8}\cot^4 2x - \int \cot 2x(\csc^2 2x - 1)$$
$$= -\tfrac{1}{8}\cot^4 2x - \int \cot 2x(-\tfrac{1}{2}\, d\cot 2x) + \int \cot 2x\, dx = -\tfrac{1}{8}\cot^4 2x + \tfrac{1}{4}\cot^2 2x + \tfrac{1}{2}\ln|\sin 2x| + C$$

21. Let $u = e^x$, $du = e^x dx$. $\int e^x \tan^4(e^x)dx = \int \tan^4 u\, du = \int \tan^2 u(\sec^2 u - 1)du$

$= \int \tan^2 u \sec^2 u\, du - \int (\sec^2 u - 1)du = \frac{1}{3}\tan^3 u - \tan u + u + C = \frac{1}{3}\tan^3(e^x) - \tan(e^x) + e^x + C$

22. Let $u = \ln x$, $du = \frac{dx}{x}$. $\int \frac{\sec^4(\ln x)}{x}dx = \int \sec^4 u\, du = \int \sec^2 u \sec^2 u\, du = \int (\tan^2 u + 1)\, d(\tan u)$

$= \frac{1}{3}\tan^3 u + \tan u + C = \frac{1}{3}\tan^3(\ln x) + \tan(\ln x) + C$

23. $\int \tan^6 x \sec^4 x\, dx = \int \tan^6 x(\tan^2 x + 1)\sec^2 x\, dx = \int (\tan^8 x + \tan^6 x)\sec^2 x\, dx = \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C$

24. $\int \tan^5 x \sec^3 x\, dx$

▹ Because we have an odd power of tangent, we use Case 7.

$$\int \tan^5 x \sec^3 x\, dx = \int \tan^4 x \sec^2 x(\sec x \tan x\, dx) = \int (\sec^2 x - 1)^2 \sec^2 x(\sec x \tan x\, dx)$$

Let $u = \sec x$, $du = \sec x \tan x\, dx$. Then

$$\int \tan^5 x \sec^3 x\, dx = \int (u^2 - 1)^2 u^2 du = \int (u^6 - 2u^4 + u^2)du = \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C$$
$$= \tfrac{1}{7}\sec^7 x - \tfrac{2}{5}\sec^5 x + \tfrac{1}{3}\sec^3 x + C$$

25. $\int \cot^2 3x \csc^4 3x\, dx = \int \cot^2 3x(\cot^2 3x + 1)\csc^2 3x\, dx = \frac{1}{3}\int (\cot^4 3x + \cot^2 3x)(3\csc^2 3x\, dx)$

$= -\frac{1}{15}\cot^5 3x - \frac{1}{9}\cot^3 3x + C$

26. $\int (\sec 5x + \csc 5x)^2 dx = \int (\sec^2 5x + 2\sec 5x \csc 5x + \csc^2 5x)dx = \int (\sec^2 5x + 4\csc 10x + \csc^2 5x)dx$

$= \frac{1}{5}\tan 5x + \frac{4}{10}\ln|\tan 5x| - \frac{1}{5}\cot 5x + C$

27. $\int (\tan 2x + \cot 2x)^2 dx = \int (\tan^2 2x + 2 + \cot^2 2x)dx = \int (\sec^2 2x - 1 + 2 + \csc^2 2x - 1)dx$

$= \frac{1}{2}\tan 2x - \frac{1}{2}\cot 2x + C$

28. $\int \frac{dx}{1 + \cos x}$

▹ $\int \frac{dx}{1 + \cos x} = \int \frac{dx}{2\cos^2 \frac{1}{2}x} = \int \sec^2 \frac{1}{2}x(\frac{1}{2}\, dx) = \tan \frac{1}{2}x + C$

29. $\int \frac{2\sin w - 1}{\cos^2 w}\, dw = \int (2\tan w \sec w - \sec^2 w)dw = 2\sec w - \tan w + C$

30. Let $u = \sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$. $\int \frac{\tan^3\sqrt{x}}{\sqrt{x}}dx = 2\int \tan^3 u\, du = 2\int \tan u \tan^2 u\, du = 2\int \tan u(\sec^2 u - 1)du$

$= 2\int \tan u(\sec^2 u\, du) - 2\int \tan u\, du = \tan^2 u - 2\ln|\sec u| + C = \tan^2\sqrt{x} - 2\ln\left|\sec\sqrt{x}\right| + C$

31. $\int \frac{\csc^4 x}{\cot^2 x}dx = \int \frac{1}{\sin^4 x}\cdot\frac{\sin^2 x}{\cos^2 x}dx = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{4\, dx}{\sin^2 2x} = \int 4\csc^2 2x\, dx = -2\cot 2x + C$

32. $\displaystyle\int \frac{\tan^4 y}{\sec^5 y}\,dy$

▹ We switch to sines and cosines.

$$\int \frac{\tan^4 y}{\sec^5 y}\,dy = \int \frac{\sin^4 y}{\cos^4 y}\cdot\cos^5 y\,dy = \int \sin^4 y(\cos y\,dy) = \tfrac{1}{5}\sin^5 y + C$$

33. $\displaystyle\int \frac{\sec^3 x}{\tan^4 x}\,dx = \int \frac{1}{\cos^3 x}\cdot\frac{\cos^4 x}{\sin^4 x}\,dx = \int \frac{\cos x\,dx}{\sin^4 x} = -\frac{1}{3}\sin^{-3} x + C = -\tfrac{1}{3}\csc^3 x + C$

34. $\displaystyle\int \frac{\sin^2 \pi x}{\cos^6 \pi x}dx = \int \frac{\sin^2 \pi x}{\cos^2 \pi x}\cdot\frac{1}{\cos^4 \pi x}dx = \int \tan^2 \pi x\sec^4 \pi x\,dx = \int \tan^2 \pi x(\tan^2 \pi x + 1)(\sec^2 \pi x\,dx)$

$\displaystyle = \int (\tan^4 \pi x + \tan^2 \pi x)(\tfrac{1}{\pi}\,d\tan \pi x) = \tfrac{1}{\pi}(\tfrac{1}{5}\tan^5 \pi x + \tfrac{1}{3}\tan^3 \pi x) + C$

In Exercises 35–48, find the exact value of the definite integral. Check by NINT.

35. $\displaystyle\int_0^{\pi/2} \cos^3 x\,dx = \int_0^{\pi/2} \cos^2 x\cos x\,dx = \int_0^{\pi/2}(1-\sin^2 x)(\cos x\,dx) = \sin x - \tfrac{1}{3}\sin^3 x\Big]_0^{\pi/2} = 1 - \tfrac{1}{3} = \tfrac{2}{3}$

36. $\displaystyle\int_0^{\pi/2} \sin^3 t\cos^2 t\,dt$

▹ Because there is an odd power of the sine function, we let $u = \cos t$, $du = -\sin t\,dt$.

$$\int_0^{\pi/2} \sin^3 t\cos^2 t\,dt = \int_{t=0}^{\pi/2}(1-\cos^2 t)\cos^2 t(\sin t\,dt) = \int_{u=1}^{0}(1-u^2)u^2(-du) = \int_0^1 (u^2 - u^4)du$$

$$= \tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\Big]_0^1 = \tfrac{2}{15}$$

37. $\displaystyle\int_0^1 \sin^4 \tfrac{1}{2}\pi x\,dx = \int_0^1 \left(\frac{1-\cos \pi x}{2}\right)^2 dx = \frac{1}{4}\int_0^1 (1 - 2\cos \pi x + \cos^2 \pi x)dx$

$\displaystyle = \frac{1}{4}x - \frac{1}{2\pi}\sin \pi x\Big]_0^1 + \frac{1}{4}\int_0^1 \left(\frac{1+\cos 2\pi x}{2}\right)dx = \frac{1}{4} + \left[\frac{1}{8}x + \frac{1}{16\pi}\sin 2\pi x\right]_0^1 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

38. $\displaystyle\int_0^1 \sin^3 \tfrac{1}{2}\pi t\,dt = \int_0^1 \sin^2 \tfrac{1}{2}\pi t(\sin \tfrac{1}{2}\pi t\,dt) = \int_0^1 (1 - \cos^2 \tfrac{1}{2}\pi t)(-\tfrac{2}{\pi}\,d\cos \tfrac{1}{2}\pi t) = -\tfrac{2}{\pi}\Big[\cos \tfrac{1}{2}\pi t - \tfrac{1}{3}\cos^3 \tfrac{1}{2}\pi t\Big]_0^1$

$= -\tfrac{2}{\pi}(-\tfrac{2}{3}) = \tfrac{4}{3\pi}$

39. $\displaystyle\int_0^1 \sin^2 \pi t\cos^2 \pi t\,dt = \int_0^1 (\sin \pi t\cos \pi t)^2 dt = \int_0^1 \left(\frac{\sin 2\pi t}{2}\right)^2 dt = \frac{1}{4}\int_0^1 \sin^2 2\pi t\,dt = \frac{1}{4}\int_0^1 \left(\frac{1-\cos 4\pi t}{2}\right)dt$

$= \tfrac{1}{8}\Big[t - \tfrac{1}{4\pi}\sin 4\pi t\Big]_0^1 = \tfrac{1}{8}$

40. $\displaystyle\int_0^{\pi/2} \sin^2 \tfrac{1}{2}x\cos^2 \tfrac{1}{2}x\,dx$

▹ We use the third formula of Case 3.

$$\int_0^{\pi/2}(\sin \tfrac{1}{2}x\cos \tfrac{1}{2}x)^2 dx = \int_0^{\pi/2} \tfrac{1}{4}\sin^2 x\,dx = \int_0^{\pi/2} \tfrac{1}{4}\cdot\tfrac{1}{2}(1-\cos 2x)dx = \tfrac{1}{8}x - \tfrac{1}{16}\sin 2x\Big]_0^{\pi/2} = \tfrac{1}{16}\pi$$

41. $\displaystyle\int_0^{\pi/8} \sin 3x\cos 5x\,dx = \tfrac{1}{2}\int_0^{\pi/8}(-\sin 2x + \sin 8x)dx = \tfrac{1}{4}\cos 2x - \tfrac{1}{16}\cos 8x\Big]_0^{\pi/8} = (\tfrac{1}{8}\sqrt{2} + \tfrac{1}{16}) - (\tfrac{1}{4} - \tfrac{1}{16})$

$= \tfrac{1}{8}(\sqrt{2} - 1)$

42. $\displaystyle\int_0^{\pi/6} \sin 2x\cos 4x\,dx = \tfrac{1}{2}\int_0^{\pi/6}(-\sin 2x + \sin 6x)dx = \tfrac{1}{4}\cos 2x - \tfrac{1}{12}\cos 6x\Big]_0^{\pi/6} = \tfrac{1}{4}(\tfrac{1}{2} - 1) - \tfrac{1}{12}(-1-1) = \tfrac{1}{24}$

43. $\displaystyle\int_{\pi/16}^{\pi/12} \tan^3 4x\,dx = \int_{\pi/16}^{\pi/12} (\sec^2 4x - 1)\tan 4x\,dx = \int_{\pi/16}^{\pi/12} \tan 4x(\sec^2 4x\,dx) - \int_{\pi/16}^{\pi/12} \tan 4x\,dx$

$\displaystyle = \tfrac{1}{8}\tan^2 4x\Big]_{\pi/16}^{\pi/12} - \tfrac{1}{4}\ln|\sec 4x|\Big]_{\pi/16}^{\pi/12} = \tfrac{1}{8}(3-1) - \tfrac{1}{4}(\ln 2 - \ln\sqrt{2}) = \tfrac{1}{4} - \tfrac{1}{8}\ln 2$

44. $\displaystyle\int_{\pi/8}^{\pi/6} 3\sec^4 2t\,dt$

▹ We apply the method of Case 5. Let $v = \tan 2t$ and $dv = 2\sec^2 2t\,dt$.

$$\int_{\pi/8}^{\pi/6} 3\sec^4 2t\,dt = \int_{\pi/8}^{\pi/6} 3\sec^2 2t(\sec^2 2t\,dt) = \int_{t=\pi/8}^{\pi/6} 3(\tan^2 2t + 1)(\sec^2 2t\,dt) = \int_{v=1}^{\sqrt{3}} 3(v^2+1)(\tfrac{1}{2}\,dv)$$

$$= \tfrac{3}{2}\Big[\tfrac{1}{3}v^3 + v\Big]_1^{\sqrt{3}} = \tfrac{3}{2}[\tfrac{1}{3}(3\sqrt{3} - 1) + (\sqrt{3} - 1)] = 3\sqrt{3} - 2$$

45. $\int_{-\pi/4}^{\pi/4} \sec^6 x\, dx = \int_{-\pi/4}^{\pi/4} (1+\tan^2 x)^2 \sec^2 x\, dx = \int_{-\pi/4}^{\pi/4} (1+2\tan^2 x + \tan^4 x)\sec^2 x\, dx$

$= \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x\Big]_{-\pi/4}^{\pi/4} = \left(1+\frac{2}{3}+\frac{1}{5}\right) - \left(-1-\frac{2}{3}-\frac{1}{5}\right) = \frac{56}{15}$

46. Let $u = \cos x$, $du = -\sin x$. $\int_0^{\pi/3} \frac{\tan^3 x}{\sec x} dx = \int_0^{\pi/3} \frac{\sin^3 x}{\cos^3 x} \cdot \cos x\, dx = \int_{x=0}^{\pi/3} \frac{\sin^2 x}{\cos^2 x} \sin x\, dx = \int_{u=1}^{1/2} \frac{1-u^2}{u^2}(-du)$

$= \int_{1/2}^{1} (u^{-2} - 1) du = -\frac{1}{u} - u\Big]_{1/2}^{1} = \frac{1}{2}$

47. $\int_{\pi/4}^{\pi/2} \frac{\cos^4 t}{\sin^6 t} dt = \int_{\pi/4}^{\pi/2} \cot^4 t \csc^2 t\, dt = -\frac{1}{5}\cot^5 t\Big]_{\pi/4}^{\pi/2} = \frac{1}{5}$

48. $\int_{\pi/6}^{\pi/4} \cot^3 w\, dw$

▷ We use the method of Case 4.

$$\int_{\pi/6}^{\pi/4} \cot^3 w\, dw = \int_{\pi/6}^{\pi/4} (\csc^2 w - 1)\cot w\, dw = \int_{\pi/6}^{\pi/4} \cot w(\csc^2 w\, dw) - \int_{\pi/6}^{\pi/4} \cot w\, dw$$

$$= -\tfrac{1}{2}\cot^2 w\Big]_{\pi/6}^{\pi/4} - \ln|\sin w|\Big]_{\pi/6}^{\pi/4} = -\tfrac{1}{2}(1-3) - [\ln(\tfrac{1}{2}\sqrt{2}) - \ln\tfrac{1}{2}] = 1 - \ln\sqrt{2} = 1 - \tfrac{1}{2}\ln 2$$

49. The region is bounded by $y = \sin^2 x$ and the x axis from $x = 0$ to $x = \pi$.

$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sin w_i \Delta_i x = \int_0^{\pi} \sin^2 x\, dx = \frac{1}{2}\int_0^{\pi} (1-\cos 2x) dx = \frac{1}{2}\Big[x - \frac{1}{2}\sin 2x\Big]_0^{\pi} = \frac{1}{2}\pi$

50. An element of volume is a circular disk centered on the x axis, $x \in [0,\pi]$, of radius $\sin w_i$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi \sin^2 w_i\, \Delta_i x = \pi \int_0^{\pi} \sin^2 x\, dx = \frac{1}{2}\pi \int_0^{\pi} (1-\cos 2x) dx = \frac{1}{2}\pi\Big[x - \frac{1}{2}\sin 2x\Big]_0^{\pi} = \frac{1}{2}\pi^2$

51. An element of volume is a circular disk centered on the x axis, $x \in [0,\pi]$, of radius $\sin^2 w_i$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\sin^2 w_i)^2 \Delta_i x = \pi \int_0^{\pi} (\sin^2 x)^2 dx = \pi \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{\pi}{4}\int_0^{\pi} (1 - 2\cos 2x + \cos^2 2x) dx$

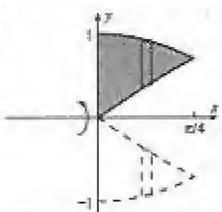
$= \frac{\pi}{4}\Big[x - \sin 2x\Big]_0^{\pi} + \frac{\pi}{4}\int_0^{\pi} \left(\frac{1+\cos 4x}{2}\right) dx = \frac{\pi}{4}\cdot\pi + \frac{\pi}{8}\Big[x + \frac{1}{4}\sin 4x\Big]_0^{\pi} = \frac{\pi^2}{4} + \frac{\pi^2}{8} = \frac{3}{8}\pi^2$

52. The region bounded by the y axis and $y = \sin x$ and $y = \cos x$ for $0 \le x \le \frac{1}{4}\pi$ is revolved about the x axis. Find the volume of the solid of revolution generated.

▷ The figure shows the region. The element of volume is a circular ring of thickness $\Delta_i x$ units, outer radius $\cos w_i$ units and inner radius $\sin w_i$ units. Thus

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\cos^2 w_i - \sin^2 w_i)\Delta_i x = \int_0^{\pi/4} \pi(\cos^2 x - \sin^2 x) dx$$

$$= \pi \int_0^{\pi/4} \cos 2x\, dx = \tfrac{1}{2}\pi \sin 2x\Big]_0^{\pi/4} = \tfrac{1}{2}\pi$$



• The volume is $\frac{1}{2}\pi$ cubic units.

53. An element of volume is a circular ring centered on line $y = 1$, $x \in [0,\pi]$, of radii 1 and $1 - \sin^2 w_i = \cos^2 w_i$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi[1 - (\cos^2 w_i)^2]\Delta_i x = \pi \int_0^{\pi} \left[1 - \left(\frac{1+\cos 2x}{2}\right)^2\right] dx = \pi x\Big]_0^{\pi} - \frac{\pi}{4}\int (1 + 2\cos 2x + \cos^2 2x) dx$

$= \pi^2 - \frac{\pi}{4}\Big[x + \sin 2x\Big]_0^{\pi} - \frac{\pi}{4}\int_0^{\pi} \left(\frac{1+\cos 4x}{2}\right) dx = \pi^2 - \frac{\pi}{4}\cdot\pi - \frac{\pi}{8}\Big[x + \frac{1}{4}\sin 4x\Big]_0^{\pi} = \pi^2 - \frac{\pi^2}{4} + \frac{\pi^2}{8} = \frac{5}{8}\pi^2$

54. An element of volume is a circular ring centered on the x axis, $x \in [0, \frac{1}{2}\pi]$, of radii 1 and $\cos x$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(1 - \cos^2 w_i)\Delta_i x = \pi \int_0^{\pi/2} \sin^2 x\, dx = \frac{1}{2}\pi \int_0^{\pi/2} (1 - \cos 2x) dx = \frac{1}{2}\pi\Big[x - \frac{1}{2}\sin 2x\Big]_0^{\pi/2} = \frac{1}{2}\pi \cdot \frac{1}{2}\pi = \frac{1}{4}\pi^2$

55. The region is bounded by $y = \cos x$, the x axis, $x = 1$ and $x = \frac{1}{2}\pi$.

$A = \int_1^{\pi/2} \cos x\, dx = \sin x\Big]_1^{\pi/2} = 1 - \sin 1$

$M_y = \int_1^{\pi/2} x \cos x\, dx$. Let $u = x$ and $dv = \cos x\, dx$. Then $du = dx$ and $v = \sin x$. Hence

$M_y = x \sin x\Big]_1^{\pi/2} - \int_1^{\pi/2} \sin x\, dx = \frac{1}{2}\pi - \sin 1 + \cos x\Big]_1^{\pi/2} = \frac{1}{2}\pi - \cos 1 - \sin 1$. $\bar{x} = \dfrac{M_y}{A} = \dfrac{\frac{1}{2}\pi - \cos 1 - \sin 1}{1 - \sin 1}$

$M_x = \frac{1}{2}\int_0^{\pi/2} \cos^2 x\, dx = \frac{1}{2}\int_1^{\pi/2} \frac{1 + \cos 2x}{2} dx = \frac{1}{4}\Big[x + \frac{1}{2}\sin 2x\Big]_1^{\pi/2} = \frac{1}{8}\pi - \frac{1}{8}\sin 2 - \frac{1}{4}$. $\bar{y} = \dfrac{M_x}{A} = \dfrac{\frac{1}{8}\pi - \frac{1}{8}\sin 2 - \frac{1}{4}}{1 - \sin 1}$

- The centroid is at the point $\left(\dfrac{\frac{1}{2}\pi - \cos 1 - \sin 1}{1 - \sin 1}, \dfrac{\frac{1}{8}\pi - \frac{1}{8}\sin 2 - \frac{1}{4}}{1 - \sin 1}\right)$

56. Find the centroid of the region of Exercise 52.

▸ The region is bounded by the y axis and $y = \sin x$ and $y = \cos x$ for $0 \le x \le \frac{1}{4}\pi$.

$$A = \int_0^{\pi/4} (\cos x - \sin x)dx = \sin x + \cos x\Big]_0^{\pi/4} = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2} - 1 = \sqrt{2} - 1$$

$$M_y = \int_0^{\pi/4} x\boxed{\cos x - \sin x\, dx} = x(\sin x + \cos x)\Big]_0^{\pi/4} - \int_0^{\pi/4} (\sin x + \cos x)dx$$

$$= \tfrac{1}{4}\pi\sqrt{2} - \Big[-\cos x + \sin x\Big]_0^{\pi/4} = \tfrac{1}{4}\pi\sqrt{2} - 1$$

$$M_x = \tfrac{1}{2}\int_0^{\pi/4} (\cos^2 x - \sin^2 x)dx = \tfrac{1}{2}\int_0^{\pi/4} \cos 2x\, dx = \tfrac{1}{4}\sin 2x\Big]_0^{\pi/4} = \tfrac{1}{4}$$

$$\bar{x} = \frac{M_y}{A} = \frac{\frac{1}{4}\pi\sqrt{2} - 1}{\sqrt{2} - 1} = \tfrac{1}{4}\pi(\sqrt{2} + 2) - (\sqrt{2} + 1) \qquad \bar{y} = \frac{M_x}{A} = \frac{\frac{1}{4}}{\sqrt{2} - 1} = \tfrac{1}{4}(\sqrt{2} + 1)$$

- The centroid is at $(\frac{1}{4}\pi(\sqrt{2} + 2) - (\sqrt{2} + 1), \frac{1}{4}(\sqrt{2} + 1))$.

57. The region is bounded by $y = \tan^2 x$, the x axis, and $x = \frac{1}{4}\pi$.

$A = \int_0^{\pi/4} \tan^2 x\, dx = \int_0^{\pi/4} (\sec^2 x - 1)dx = \tan x - x\Big]_0^{\pi/4} = 1 - \frac{1}{4}\pi$

58. The region is bounded by $y = 3\csc^3 x$, the x axis, $x = \frac{1}{6}\pi$, and $x = \frac{1}{2}\pi$. An element of volume is a circular disk centered on the x axis, $x \in [\frac{1}{6}\pi, \frac{1}{2}\pi]$, of radius $3\csc^3 w_i$. Let $u = \cot x$, $du = -\csc^2 x\, dx$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(3\csc^3 w_i)^2 \Delta_i x = 9\pi \int_{x=\pi/6}^{\pi/2} (\cot^2 x + 1)^2(\csc^2 x\, dx) = 9\pi \int_{u=\sqrt{3}}^{0} (u^2 + 1)^2(-du)$

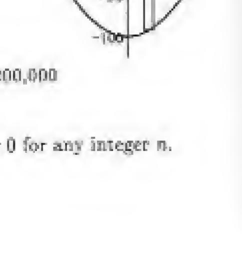
$= 9\pi \int_0^{\sqrt{3}} (u^4 + 2u^2 + 1)du = 9\pi\Big[\frac{1}{5}u^5 + \frac{2}{3}u^3 + u\Big]_0^{\sqrt{3}} = 9\pi\Big[\frac{1}{15}u(3u^4 + 10u^2 + 15)\Big]_0^{\sqrt{3}} = 9\pi \cdot \frac{1}{15}\sqrt{3} \cdot 72 = \frac{216}{5}\sqrt{3}\pi$

59. An element of volume is a circular disk centered on the x axis, $x \in [0, \frac{1}{4}\pi]$, of radius $\sec^2 w_i$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(\sec^2 w_i)^2 \Delta_i x = \pi \int_0^{\pi/4} \sec^2 x \sec^2 x\, dx = \pi \int_0^{\pi/4} (\tan^2 x + 1)\sec^2 x\, dx$

$= \pi\Big[\frac{1}{3}\tan^3 x + \tan x\Big]_0^{\pi/4} = \pi\Big(\frac{1}{3} + 1\Big) = \frac{4}{3}\pi$

60. The face of a dam is in the shape of one arch of $y = -100\cos\frac{1}{200}\pi x$, $x \in [-100, 100]$, and the surface of the water is at the top of the dam. Find the force due to water pressure on the face of the dam if distance is measured in feet.



▸ See the figure. Let $f(x) = -y$. The force on a vertical element, of area $f(w_i)\Delta_i x$ and mean depth $\frac{1}{2}f(w_i)$, is $\frac{1}{2}\rho f(w_i)^2 \Delta x_i$. If F lb is the force on the dam, because of symmetry,

$F = 2\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}\rho f(w_i)^2 \Delta_i x = \rho \int_0^{100} \Big(100\cos\frac{\pi}{200}x\Big)^2 dx$

$= 5000\int_0^{100} \Big(1 + \cos\frac{\pi}{100}\Big)dx = 5000\rho\Big[x + \frac{100}{\pi}\sin\frac{\pi}{100}x\Big]_0^{100} = 5000(62.4)100 = 31{,}200{,}000$

- The force on the dam is 31,200,000 lb = 15,600 tons.

61. $\int_0^{\pi} \sin^2 nx\, dx = \frac{1}{2}\int_0^{\pi} (1 - \cos 2nx)dx = \frac{1}{2}\Big[x - \frac{1}{2n}\sin 2nx\Big]_0^{\pi} = \frac{1}{2}\pi$ because $\sin 2n\pi = 0$ for any integer n.

62. $\int_0^{\pi} \cos^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx + \int_{\pi/2}^{\pi} \cos^n x\, dx$. In the second integral let $x = \pi - u$, $dx = -du$; so
$I = \int_0^{\pi} \cos^n x\, dx = \int_0^{\pi/2} (\cos x)^n dx + \int_{\pi/2}^{0} [\cos(\pi - u)]^n dx = \int_0^{\pi/2} (\cos x)^n dx + \int_0^{\pi/2} (-\cos u)^n dx$
If n is a positive odd integer, the sum of the last two integrals is zero so $I = 0$.

63. Let $x = -u$, $dx = -du$. Then
$I = \int_{-1}^{1} \cos n\pi x \sin m\pi x\, dx = \int_{1}^{-1} \cos(-n\pi u)\sin(-n\pi u)(-du) = \int_{-1}^{1} \cos n\pi u(-\sin n\pi u)du = -1$.
Hence $I = 0$ for any numbers m and n. If f is <u>any</u> odd function then $\int_{-a}^{a} f(x)dx = 0$.

64. If m and n are any positive integers, show that the formula is true.
$\int_{-1}^{1} \cos n\pi x \cos m\pi x\, dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$

► We apply the identity $\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$ with $a = n\pi x$ and $b = m\pi x$. Thus
$$\int_{-1}^{1} \cos n\pi x \cos m\pi x\, dx = \tfrac{1}{2}\int_{-1}^{1} \cos[(n+m)\pi x]dx + \int_{-1}^{1} \cos[(n-m)\pi x]dx \qquad (1)$$
Because $\sin k\pi = 0$ if k is any integer, and by hypothesis $(n+m)$ is a positive integer, then
$$\int_{-1}^{1} \cos(n+m)\pi x\, dx = \frac{1}{(n+m)\pi}\sin(n+m)\pi x\Big]_{-1}^{1} = 0 \qquad (2)$$
If $m \neq n$, then $n - m \neq 0$ and $(n - m)$ is an integer. Thus,
$$\int_{-1}^{1} \cos(n-m)\pi x\, dx = \frac{1}{(n-m)\pi}\sin(n-m)\pi x\Big]_{-1}^{1} = 0 \qquad (3)$$
By substituting from Eqs. (2) and (3) into Eq. (1), we obtain
$$\int_{-1}^{1} \cos n\pi x \cos m\pi x\, dx = 0 \quad \text{if } m \neq n$$
Furthermore, if $m = n$, then $n - m = 0$, and because $\cos 0 = 1$, then
$$\int_{-1}^{1} \cos(n-m)\pi x\, dx = \int_{-1}^{1} dx = 2 \qquad (4)$$
By substituting from Eqs. (2) and (4) into Eq. (1), we obtain $\int_{-1}^{1} \cos n\pi x \cos m\pi x\, dx = 1 \quad \text{if } m = n$

65. (a) $I = \int \sec^n x\, dx = \int \sec^{n-2}x \boxed{\sec^2 x\, dx} = \sec^{n-2}x \tan x - \int \tan x[(n-2)\sec^{n-3}x \sec x \tan x\, dx]$
$= \sec^{n-2}x \tan x - (n-2)\int (\sec^n x - \sec^{n-2}x)dx = \sec^{n-2}x \tan x - (n-2)I + (n-2)\int \sec^{n-2}x\, dx$
Thus $(n-1)I = \sec^{n-2}x \tan x + (n-2)\int \sec^{n-2}x\, dx$ and so $I = \frac{1}{n-1}\sec^{n-2}x \tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\, dx$.

(b) $\int \sec^5 x\, dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x\, dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}(\frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\, dx)$
$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C$

66. (a) $\int \sec^6 x\, dx = \frac{1}{5}\sec^4 x \tan x + \frac{4}{5}\int \sec^4 x\, dx = \frac{1}{5}\sec^4 x \tan x + \frac{4}{5}(\frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\int \sec^2 x\, dx)$
$= \frac{1}{5}\sec^4 x \tan x + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x + C$

(b) $\int \sec^7 x\, dx = \frac{1}{6}\sec^5 x \tan x + \frac{5}{6}\int \sec^5 x\, dx$
$= \frac{1}{6}\sec^5 x \tan x + \frac{5}{6}(\frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x|) + C$
$= \frac{1}{6}\sec^5 x \tan x + \frac{5}{24}\sec^3 x \tan x + \frac{5}{16}\sec x \tan x + \frac{5}{16}\ln|\sec x + \tan x| + C$

67. Prove Formula 73.
$D_u \sin^{n-1}u \cos u = (n-1)\sin^{n-2}u \cos^2 u - \sin^n u = (n-1)\sin^{n-2}u(1 - \sin^2 u) - \sin^n u = (n-1)\sin^{n-2}u - n \sin^n u$
Hence $\sin^{n-1}u \cos u = (n-1)\int \sin^{n-2}u\, du - n\int \sin^n u\, du$; $\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1}u \cos u + \frac{n-1}{n}\int \sin^{n-2}u\, du$

Prove Wallis' formula. From Formula 73 we have $\int_0^{\pi/2} \sin^n u\, du = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}u\, du$. If $n = 2m$,
$\int_0^{\pi/2} \sin^{2m}u\, du = \frac{2m-1}{2m}\cdot \ldots \cdot \frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} \sin^0 u\, du = \frac{1\cdot 3\cdot \ldots \cdot(2m-1)}{2\cdot 4\cdot \ldots \cdot (2n)}\cdot\frac{\pi}{2}$. If $n = 2m - 1$, then

$\int_0^{\pi/2} \sin^{2m-1} u\, du = \frac{2m-2}{2m-1} \cdot \ldots \cdot \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \sin u\, du = \frac{2 \cdot 4 \cdot \ldots \cdot (2m-1)}{3 \cdot 5 \cdot \ldots \cdot (2m-1)}$

If $0 < u < \frac{1}{2}\pi$ then $\sin^{2m-1} u < \sin^{2m} u < \sin^{2m+1} u$ so $\int_0^{\pi/2} \sin^{2m+1} u\, du < \int_0^{\pi/2} \sin^{2m} u\, du < \int_0^{\pi/2} \sin^{2m-1} u\, du$

Thus $\frac{2 \cdot 4 \cdot \ldots \cdot (2m)}{3 \cdot 5 \ldots \cdot (2m+1)} < \frac{1 \cdot 3 \cdot \ldots (2m-1)}{2 \cdot 4 \cdot \ldots \cdot (2m)} \cdot \frac{\pi}{2} < \frac{2 \cdot 4 \cdot \ldots (2m-2)}{3 \cdot 5 \cdot \ldots (2m-1)}$ so $\frac{2m}{2m+1} \cdot \frac{\pi}{2} < \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot \ldots (2m)(2m)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot \ldots (2m-1)(2m+1)} < \frac{\pi}{2}$

Because $\lim_{m \to \infty} \frac{2m}{2m+1} = 1$, Wallis' formula follows by the squeeze principle.

(b) $\int \sin^5 x\, dx = -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5} \int \sin^3 x\, dx = -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5}\left(-\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x\, dx\right) + C$

$= -\frac{1}{5} \sin^4 x \cos x - \frac{4}{15} \sin^2 x \cos x - \frac{8}{15} \cos x + C$ (c) $= -\frac{1}{15} \cos x(3 \sin^4 x + 4 \sin^2 x + 8)$

$= -\frac{1}{15} \cos x[3(1 - \cos^2 x)^2 + 4(1 - \cos^2 x) + 8] + C = -\frac{1}{15} \cos x(3 - 6 \cos^2 x + 3 \cos^4 x + 4 - 4 \cos^2 x + 8) + C$

$= -\frac{1}{15} \cos x(3 \cos^4 x - 10 \cos^2 x + 15) + C = -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$, as in Example 1.

68. (a) Derive the reduction formula where n is an integer greater than 1.

$$\int \cos^n u\, du = \frac{1}{n} \cos^{n-1} u \sin u + \frac{n-1}{n} \int \cos^{n-2} u\, du \qquad \text{(Formula 74)}$$

(b) Apply the reduction formula to evaluate $\int \cos^4 x\, dx$.

(c) Show that your answer is equivalent to the answer of Example 2.

▸ (a) Because $-\sin^2 u = \cos^2 u - 1$, then

$$D_u(\cos^{n-1} u \sin u) = (n-1)\cos^{n-2} u(-\sin u)\sin u + \cos^{n-1} u \cos u = (n-1)\cos^{n-2} u(\cos^2 u - 1) + \cos^n u$$
$$= n \cos^n u - (n-1)\cos^{n-2} u$$

Therefore,

$$\cos^{n-1} u \sin u = n \int \cos^n u\, du - (n-1) \int \cos^{n-2} u\, du$$

which is equivalent to (Formula 74)

(b) With $n = 4$, and then with $n = 2$, we have

$$\int \cos^4 x\, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x\, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2} \cos x \sin x + \frac{1}{2} \int dx\right)$$
$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + C \qquad (2)$$

(c) Applying the double angle formulas $\sin 2x = 2 \sin x \cos x$ and $\cos 2x = 2 \cos^2 x - 1$ to the answer of Example 2, we get $\frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$

$$= \frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{16} \sin 2x \cos 2x + C = \frac{3}{8}x + \frac{1}{16} \sin 2x(4 + \cos 2x)$$
$$= \frac{3}{8}x + \frac{1}{8} \sin x \cos x(3 + 2 \cos^2 x) + C$$

which is equivalent to (2).

69. (a) $\int \cot x \csc^n x\, dx = \int \csc^{n-1} x(\csc x \cot x\, dx) = -\frac{1}{n} \csc^n x + C$

(b) $\int \tan x \sec^n x\, dx = \int \sec^{n-1} x(\sec x \tan x\, dx) = \frac{1}{n} \sec^n x + C$

70. $\int \sin^3 x \cos^3 x\, dx$,

(a) $= \int \sin^3 x \cos^2 x(\cos x\, dx) = \int \sin^3 x(1 - \sin^2 x)(d \sin x) = \int (\sin^3 x - \sin^5 x)(d \sin x) = \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C_1$

(b) $= \int \sin^2 x \cos^3 x(\sin x\, dx) = \int (1 - \cos^2 x)\cos^3 x(-d \cos x) = \int (\cos^5 x - \cos^3 x)(d \cos x) = \frac{1}{6} \cos^6 x - \frac{1}{4} \cos^4 x + C_2$

(c) $= \frac{1}{8} \int (2 \sin x \cos x)^3 dx = \frac{1}{8} \int \sin^3 2x\, dx = \frac{1}{8} \int (1 - \cos^2 2x)(\sin 2x\, dx) = \frac{1}{16} \int (\cos^2 2x - 1)(d \cos 2x)$

$= \frac{1}{16}\left(\frac{1}{3} \cos^3 2x - \cos 2x\right) + C_3$.

(d) In (c), use $\cos 2x = 1 - 2 \sin^2 x$ to get (a) and $\cos 2x = 2 \cos^2 x - 1$ to get (b).

7.3 INTEGRATION OF ALGEBRAIC FUNCTIONS BY TRIGONOMETRIC SUBSTITUTION

Sometimes we can eliminate the square root radical in an expression that we wish to integrate by making a trigonometric substitution. If $a > 0$, the integrand may contain an expression of the form:

Case 1: $\sqrt{a^2 - u^2}$
We let $\theta = \sin^{-1} \frac{u}{a}$.
Then, $u = a \sin \theta$,
$\sqrt{a^2 - u^2} = a \cos \theta$, and
$du = a \cos \theta \, d\theta$

Case 2: $\sqrt{a^2 + u^2}$
We let $\theta = \tan^{-1} \frac{u}{a}$.
Then, $u = a \tan \theta$
$\sqrt{a^2 + u^2} = a \sec \theta$, and
$du = a \sec^2\theta \, d\theta$

Case 3: $\sqrt{u^2 - a^2}$
We let $\theta = \sec^{-1} \frac{u}{a}$.
Then, $u = a \sec \theta$,
$\sqrt{u^2 - a^2} = a \tan \theta$, and
$du = a \sec \theta \tan \theta \, d\theta$

You should memorize the choice for θ in each of the three cases, but do not attempt to memorize the replacement for the expression containing the radical or the replacement for du. Rather, use the trigonometric identities and the differentiation formulas for the trigonometric functions to derive these replacements. If the radical contains a quadratic trinomial, we first complete the square. The identities needed are the following.

$$1 - \sin^2\theta = \cos^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$$

Bear in mind that if the integrand contains an odd power of u, then an algebraic substitution may be easier.
An ellipse of semiaxes a and b has area πab. Thus a circle of radius r has area πr^2. See Exercise 42.
A *fractrix* is a curve such that the length of the segment of every tangent line from the point of tangency is a positive constant a. An equation is (see Ex. 46) $x = a \ln\left(\frac{a + \sqrt{a^2 - y^2}}{y}\right) - \sqrt{a^2 - y^2} = a \cosh^{-1} \frac{a}{y} - \sqrt{a^2 - y^2}$

Exercises 7.3

In Exercises 1–12, evaluate the indefinite integral.

1. Let $x = 2 \sin \theta$ where $0 < \theta \le \frac{1}{2}\pi$ if $x > 0$ and $-\frac{1}{2}\pi \le \theta < 0$ if $x < 0$. Then $dx = 2 \cos \theta \, d\theta$ and
$\sqrt{4 - x^2} = \sqrt{4 - 4 \sin^2\theta} = 2\sqrt{\cos^2\theta} = 2 \cos \theta$.
$\int \frac{dx}{x^2\sqrt{4 - x^2}} = \int \frac{2 \cos \theta \, d\theta}{4 \sin^2\theta(2 \cos \theta)} = \frac{1}{4}\int \csc^2\theta \, d\theta = -\frac{1}{4} \cot \theta + C = -\frac{\sqrt{4 - x^2}}{4x} + C$

2. Let $x = 2 \sin \theta$ where $0 < \theta \le \frac{1}{2}\pi$ if $x > 0$ and $-\frac{1}{2}\pi \le \theta < 0$ if $x < 0$. Then $dx = 2 \cos \theta \, d\theta$ and
$\sqrt{4 - x^2} = \sqrt{4 - 4 \sin^2\theta} = 2\sqrt{\cos^2\theta} = 2 \cos \theta$.
$\int \frac{\sqrt{4 - x^2}}{x^2} dx = \int \frac{2 \cos \theta}{4 \sin^2\theta} 2 \cos \theta \, d\theta = \int \cot^2\theta \, d\theta = \int (\csc^2\theta - 1) d\theta = -\cot \theta - \theta + C = -\frac{\sqrt{4 - x^2}}{x} - \sin^{-1} \frac{x}{2} + C$

3. Let $x = 2 \tan \theta$ where $0 < \theta < \frac{1}{2}\pi$ if $x > 0$ and $-\frac{1}{2}\pi < \theta < 0$ if $x < 0$. Then $dx = 2 \sec^2\theta \, d\theta$ and
$\sqrt{x^2 + 4} = \sqrt{4 \tan^2\theta + 4} = 2\sqrt{\sec^2\theta} = 2 \sec \theta$.
$\int \frac{dx}{x\sqrt{x^2 + 4}} = \int \frac{2 \sec^2\theta \, d\theta}{2 \tan \theta(2 \sec \theta)} = \frac{1}{2}\int \csc \theta \, d\theta = -\ln|\csc \theta - \cot \theta| + C = \frac{1}{2} \ln\left|\frac{\sqrt{x^2 + 4} - 2}{x}\right| + C$

4. $\int \frac{x^2 \, dx}{\sqrt{x^2 + 6}}$

▸ We use the method of Case 2 with $u = x$ and $a = \sqrt{6}$. Thus, we let $\theta = \tan^{-1} \frac{x}{\sqrt{6}}$. Hence,

$$\tan \theta = \frac{x}{\sqrt{6}}, \qquad x = \sqrt{6} \tan \theta, \qquad dx = \sqrt{6} \sec^2\theta \, d\theta \quad \text{and}$$

$$\sqrt{x^2 + 6} = \sqrt{6 \tan^2\theta + 6} = \sqrt{6(\tan^2\theta + 1)} = \sqrt{6 \sec^2\theta} = \sqrt{6} \sec \theta$$

We have

$$\begin{aligned}
\int \frac{x^2 \, dx}{\sqrt{x^2 + 6}} &= \int \frac{(6 \tan^2\theta)\sqrt{6} \sec^2\theta \, d\theta}{\sqrt{6} \sec \theta} = 6 \int \tan^2\theta \sec \theta \, d\theta = 6 \int (\sec^2\theta - 1) \sec \theta \, d\theta \\
&= 6\left[\int \sec^3\theta \, d\theta - \int \sec \theta \, d\theta\right] = 6\left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2}\int \sec \theta \, d\theta - \int \sec \theta \, d\theta\right] \\
&= 6\left[\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln|\sec \theta + \tan \theta|\right] + C = 3 \sec \theta \tan \theta - 3 \ln|\sec \theta + \tan \theta| + C \\
&= 3 \frac{\sqrt{x^2 + 6}}{\sqrt{6}} \cdot \frac{x}{\sqrt{6}} - 3 \ln\left|\frac{\sqrt{x^2 + 6}}{\sqrt{6}} + \frac{x}{\sqrt{6}}\right| + C = \frac{1}{2}x\sqrt{x^2 + 6} - 3 \ln(\sqrt{x^2 + 6} + x) + \bar{C}
\end{aligned}$$

where $\bar{C} = C - 3\ln\sqrt{6}$

5. Let $u^2 = x^2 - 25$, $2u\,du = 2x\,dx$. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-25}} = \int \frac{u\,du}{u} = \int du = u + C = \sqrt{x^2-25} + C$

6. Let $\theta = \tan^{-1}\dfrac{x}{\sqrt{2}}$, $x = \sqrt{2}\tan\theta$, $dx = \sqrt{2}\sec^2\theta\,d\theta$ and $2 + x^2 = 2 + 2\tan^2\theta = 2(1+\tan^2\theta) = 2\sec^2\theta$.

$$\int \frac{dx}{(2+x^2)^{3/2}} = \int \frac{\sqrt{2}\sec^2\theta\,d\theta}{(2\sec^2\theta)^{3/2}} = \frac{1}{2}\int \cos\theta\,d\theta = \frac{1}{2}\sin\theta + C = \frac{x}{2\sqrt{2+x^2}} + C$$

7. Let $x = \frac{3}{2}\sec\theta$ where $0 < \theta < \frac{1}{2}\pi$ if $x > \frac{3}{2}$ and $\pi < \theta < \frac{3}{2}\pi$ if $x < -\frac{3}{2}$. Then $dx = \frac{3}{2}\sec\theta\tan\theta\,d\theta$ and $(4x^2-9)^{3/2} = (9\sec^2 - 9)^{3/2} = 27(\tan^2\theta)^{3/2} = 27\tan^3\theta$.

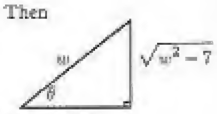
$$\int \frac{dx}{(4x^2-9)^{3/2}} = \int \frac{\frac{3}{2}\sec\theta\tan\theta\,d\theta}{27\tan^3\theta} = \frac{1}{18}\int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{18}\csc\theta + C = -\frac{1}{18}\cdot\frac{2x}{\sqrt{4x^2-9}} + C = -\frac{x}{9\sqrt{4x^2-9}} + C$$

8. $\displaystyle\int \frac{dw}{w^2\sqrt{w^2-7}}$

▶ We use the method of Case 3 with $u = w$ and $a = \sqrt{7}$. Thus we let $\theta = \sec^{-1}\dfrac{w}{\sqrt{7}}$. Then

$$\sec\theta = \frac{w}{\sqrt{7}}, \qquad w = \sqrt{7}\sec\theta, \qquad dw = \sqrt{7}\sec\theta\tan\theta\,d\theta \text{ and}$$

$$\sqrt{w^2-7} = \sqrt{7\sec^2\theta - 7} = \sqrt{7(\sec^2\theta - 1)} = \sqrt{7\tan^2\theta} = \sqrt{7}\tan\theta$$



The figure shows a right triangle satisfying these equations used when substituting back. $\sqrt{7}$

$$\int \frac{dw}{w^2\sqrt{w^2-7}} = \int \frac{\sqrt{7}\sec\theta\tan\theta\,d\theta}{(7\sec^2\theta)(\sqrt{7}\tan\theta)} = \frac{1}{7}\int \cos\theta\,d\theta = \frac{1}{7}\sin\theta + C = \frac{1}{7}\cdot\frac{\sqrt{w^2-7}}{w} + C$$

9. Let $\tan x = 2\sin\theta$ where $0 \le \theta \le \frac{1}{2}\pi$ if $\tan x \ge 0$ and $-\frac{1}{2}\pi \le \theta < 0$ if $\tan x < 0$. Then $\sec^2 x\,dx = 2\cos\theta\,d\theta$ and $(4 - \tan^2 x)^{3/2} = (4 - 4\sin^2\theta)^{3/2} = 8(\cos^2\theta)^{3/2} = 8\cos^3\theta$.

$$\int \frac{\sec^2 x\,dx}{(4-\tan^2 x)^{3/2}} = \int \frac{2\cos\theta\,d\theta}{8\cos^3\theta} = \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta + C = \frac{\tan x}{4\sqrt{4-\tan^2 x}} + C$$

10. $\displaystyle I = \int \frac{dz}{(z^2-6z+18)^{3/2}} = \int \frac{dz}{[(z^2-6z+9)+9]^{3/2}} = \int \frac{dz}{[(z-3)^2+9]^{3/2}}$

Let $\theta = \tan^{-1}\dfrac{z-3}{3}$, $z - 3 = 3\tan\theta$, $dz = 3\sec^2\theta\,d\theta$ and $(z-3)^2 + 9 = 9\tan^2\theta + 9 = 9\sec^2\theta$.

$$I = \int \frac{3\sec^2\theta\,d\theta}{27\sec^3\theta} = \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C = \frac{1}{9}\cdot\frac{z-3}{\sqrt{z^2-6z+18}} + C$$

11. Let $u = \ln w$, $du = \dfrac{dw}{w}$. Then $\displaystyle I = \int \frac{\ln^3 w\,dw}{w\sqrt{\ln^2 w - 4}} = \int \frac{u^3 du}{\sqrt{u^2-4}}$

Trig: Let $u = 2\sec\theta$ where $0 < \theta < \frac{1}{2}\pi$ if $u > 2$ and $\pi < \theta < \frac{3}{2}\pi$ if $u < -2$. Then $du = 2\sec\theta\tan\theta\,d\theta$ and $\sqrt{u^2-4} = \sqrt{4\sec^2\theta - 4} = 2\sqrt{\tan^2\theta} = 2\tan\theta$.

$$I = \int \frac{8\sec^3\theta(2\sec\theta\tan\theta\,d\theta)}{2\tan\theta} = 8\int \sec^4\theta\,d\theta = 8\int (\tan^2\theta + 1)\sec^2\theta\,d\theta = 8\int \tan^2\theta(\sec^2\theta\,d\theta) + 8\int \sec^2\theta\,d\theta$$

$$= \frac{8}{3}\tan^3\theta + 8\tan\theta + C = \frac{8}{3}\cdot\frac{(u^2-4)^{3/2}}{8} + 8\cdot\frac{\sqrt{u^2-4}}{2} + C = \frac{1}{3}\sqrt{u^2-4}(u^2 - 4 + 12) + C$$

$$= \frac{1}{3}\sqrt{\ln^2 w - 4}(8 + \ln^2 w) + C$$

Algebra: Let $u^2 - 4 = t^2$. Then $u^2 = t^2 + 4$ and $u\,du = t\,dt$.

$$I = \int \frac{u^2(u\,du)}{\sqrt{u^2-4}} = \int \frac{(t^2+4)(t\,dt)}{t} = \int (t^2+4)dt = \frac{1}{3}t^3 + 4t + C = \frac{1}{3}t(t^2+12) + C = \frac{1}{3}\sqrt{u^2-4}(u^2+8) + C$$

$$= \frac{1}{3}\sqrt{\ln^2 w - 4}(8 + \ln^2 w) + C$$

12. $\displaystyle\int \frac{e^{-x}\,dx}{(9e^{-2x}+1)^{3/2}}$

▷ $\displaystyle\int \frac{e^{-x}\,dx}{(9e^{-2x}+1)^{3/2}} = \int \frac{e^{-x}dx}{(9e^{-2x}+1)^{3/2}} = \int \frac{e^{2x}dx}{(9+e^{2x})^{3/2}}$. Let $u = 9 + e^{2x}$, $du = 2e^{2x}dx$. Then

$$\int \frac{e^{-x}\,dx}{(9e^{-2x}+1)^{3/2}} = \int \frac{\frac{1}{2}du}{u^{3/2}} = \tfrac{1}{2}\int u^{-3/2}du = -u^{-1/2} + C = -\frac{1}{\sqrt{1+9e^{-2x}}} + C$$

In Exercises 13–30, find the exact value of the definite integral. Check using NINT.

13. Let $u^2 = 25 - x^2$, $2\,du = -2x\,dx$. $\displaystyle\int_1^4 \frac{dx}{x\sqrt{25-x^2}} = \int_{x=1}^4 \frac{x\,dx}{x^2\sqrt{25-x^2}} = \int_{u=2\sqrt{6}}^3 \frac{-u\,du}{(25-u^2)u} = \int_3^{2\sqrt{6}} \frac{du}{25-u^2}$

$$= \frac{1}{10}\ln\left|\frac{5+u}{5-u}\right|\Big]_3^{2\sqrt{6}} = \frac{1}{10}\left(\ln\frac{5+2\sqrt{6}}{5-2\sqrt{6}} - \ln 4\right) = -\tfrac{1}{5}\ln(5-2\sqrt{6}) \approx 0.3199$$

14. $\displaystyle\int_0^1 \sqrt{1-u^2}\,du = \tfrac{1}{4}\pi$, the area of a quarter of the unit circle.

15. Method 1. Let $t^2 = u$, $2t\,dt = du$. Then $\displaystyle I = \int_2^3 \frac{2\,dt}{t\sqrt{t^4+25}} = \int_{t=2}^3 \frac{2t\,dt}{t^2\sqrt{t^4+25}} = \int_{u=4}^9 \frac{du}{u\sqrt{u^2+25}}$.

Now let $u = 5\tan\theta$ where $0 < \theta < \frac{1}{2}\pi$ because $u > 0$, $a = \tan^{-1}\frac{4}{5}$, $b = \tan^{-1}\frac{9}{5}$. Then $du = 5\sec^2\theta\,d\theta$ and $\sqrt{u^2+25} = \sqrt{25\tan^2\theta+25} = 5\sqrt{\sec^2\theta} = 5\sec\theta$. Therefore

$$I = \int_{\theta=a}^b \frac{5\sec^2\theta\,d\theta}{5\tan\theta(5\sec\theta)} = \frac{1}{5}\int_a^b \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{5}\int_a^b \csc\theta\,d\theta = \frac{1}{5}\ln|\csc\theta - \cot\theta|\Big]_a^b$$

$$= \frac{1}{5}\left[\ln\left(\frac{\sqrt{106}}{9} - \frac{5}{9}\right) - \ln\left(\frac{\sqrt{41}}{4} - \frac{5}{4}\right)\right] = \frac{1}{5}\ln\frac{4(\sqrt{106}-5)}{9(\sqrt{41}-5)} \approx 0.10345$$

Method 2. Let $u^2 = t^4 + 25$, $2u\,du = 4t^3dt$. $\displaystyle\int_2^3 \frac{2\,dt}{t\sqrt{t^4+25}} = \int_{t=2}^3 \frac{2t^3dt}{t^4\sqrt{t^4+25}} = \int_{u=\sqrt{41}}^{\sqrt{106}} \frac{u\,du}{(u^2-25)u}$

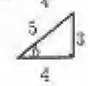
$$= \int_{\sqrt{41}}^{\sqrt{106}} \frac{du}{u^2-25} = \frac{1}{10}\ln\left|\frac{u-5}{u+5}\right|\Big]_{\sqrt{41}}^{\sqrt{106}} = \frac{1}{10}\left[\ln\left(\frac{\sqrt{106}-5}{\sqrt{106}+5}\right) - \ln\left(\frac{\sqrt{41}-5}{\sqrt{41}+5}\right)\right]$$

16. $\displaystyle\int_1^3 \frac{dx}{x^4\sqrt{16+x^2}}$

▷ We use the method of Case 2 with $u = x$ and $a = \sqrt{16} = 4$. We let $\theta = \tan^{-1}\frac{x}{4}$, $a = \tan^{-1}\frac{1}{4}$, $b = \tan^{-1}\frac{3}{4}$.

$$\tan\theta = \frac{x}{4} \qquad x = 4\tan\theta \qquad dx = 4\sec^2\theta\,d\theta$$

$$\sqrt{16+x^2} = \sqrt{16+16\tan^2\theta} = \sqrt{16(1+\tan^2\theta)} = \sqrt{16\sec^2\theta} = 4\sec\theta$$



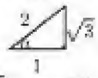
$$\int_{x=1}^3 \frac{dx}{x^4\sqrt{16+x^2}} = \int_{\theta=a}^b \frac{4\sec^2\theta\,d\theta}{(256\tan^4\theta)(4\sec\theta)} = \frac{1}{256}\int_a^b \frac{\sec\theta\,d\theta}{\tan^4\theta} = \frac{1}{256}\int_a^b \frac{\cos^3\theta\,d\theta}{\sin^4\theta}$$

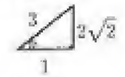
$$= \frac{1}{256}\int_a^b \frac{(1-\sin^2\theta)(\cos\theta\,d\theta)}{\sin^4\theta} = \frac{1}{256}\int_a^b (\sin^{-4}\theta - \sin^{-2}\theta)(d\sin\theta) = \frac{1}{256}\left[-\frac{1}{3\sin^3\theta} + \frac{1}{\sin\theta}\right]_a^b$$

$$= \frac{1}{256}\left(-\frac{17\sqrt{17}}{3} + \sqrt{17} + \frac{125}{3\cdot 27} - \frac{5}{3}\right) = \frac{5+189\sqrt{17}}{10368} \approx 0.075643$$

17. $\displaystyle I = \int_2^4 \frac{dx}{\sqrt{4x+x^2}} = \int_2^4 \frac{dx}{\sqrt{(x^2+4x+4)-4}} = \int_{x=2}^4 \frac{dx}{\sqrt{(x+2)^2-4}}$. Let $x + 2 = 2\sec\theta$, $a = \sec^{-1} 2$, $b = \sec^{-1} 3$.

Then $dx = 2\sec\theta\tan\theta\,d\theta$ and $\sqrt{(x+2)^2-4} = \sqrt{4\sec^2\theta-4} = 2\sqrt{\tan^2\theta} = 2\tan\theta$.





$$I = \int_{\theta=a}^b \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta} = \int_a^b \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big]_a^b = \ln(3+2\sqrt{2}) - \ln(2+\sqrt{3}) \approx 0.4458$$

18. Let $\theta = \tan^{-1}\frac{x}{4}$, $x = 4\tan\theta$, $dx = 4\sec^2\theta\,d\theta$, $16 + x^2 = 16 + 16\tan^2\theta = 16\sec^2\theta$.

$$\int_{x=0}^4 \frac{dx}{(16+x^2)^{3/2}} = \int_{\theta=0}^{\pi/4} \frac{4\sec^2 d\theta}{64\sec^3\theta} = \frac{1}{16}\int_0^{\pi/4}\cos\theta\,d\theta = \frac{1}{16}\sin\theta\Big]_0^{\pi/4} = \tfrac{1}{32}\sqrt{2} \approx 0.04419$$

19. Trig: Let $x = 4\sin\theta$ where $\theta \in [0, \frac{1}{6}\pi]$ because $x \in [0, 2]$. Then

$dx = 4\cos\theta\, d\theta$ and $\sqrt{16 - x^2} = \sqrt{16 - 16\sin^2\theta} = 4\sqrt{\cos^2\theta} = 4\cos\theta$.

$$\int_0^2 \frac{x^3 dx}{\sqrt{16 - x^2}} = \int_0^{\pi/6} \frac{64\sin^3\theta(4\cos\theta\, d\theta)}{4\cos\theta} = 64\int_0^{\pi/6} \sin^3\theta\, d\theta = -64\int_0^{\pi/6} (1 - \cos^2\theta)(-\sin\theta\, d\theta)$$

$$= -64\cos\theta + \frac{64}{3}\cos^3\theta\Big]_0^{\pi/6} = \left(-64\cdot\tfrac{1}{2}\sqrt{3} + \frac{64}{3}\cdot\frac{3}{8}\sqrt{3}\right) - \left(-64 + \frac{64}{3}\right) = -32\sqrt{3} + 8\sqrt{3} + 64 - \frac{64}{3} = \frac{128}{3} - 24\sqrt{3}$$

Algebra: Let $16 - x^2 = u^2$. Then $x^2 = 16 - u^2$ and $x\, dx = -u\, du$.

$$\int_0^2 \frac{x^3 dx}{\sqrt{16 - x^2}} = \int_{x=0}^2 \frac{x^2\cdot x\, dx}{\sqrt{16 - x^2}} = \int_{u=4}^{2\sqrt{3}} \frac{(16 - u^2)(-u\, du)}{u} = \int_{2\sqrt{3}}^4 (16 - u^2)du = 16u - \tfrac{1}{3}u^3\Big]_{2\sqrt{3}}^4$$

$$= \left(64 - \frac{64}{3}\right) - (32\sqrt{3} - 8\sqrt{3}) = \frac{128}{3} - 24\sqrt{3} \approx 1.097$$

20. $\displaystyle\int_1^3 \frac{dx}{\sqrt{4x - x^2}}$

▹ We complete the square. $4x - x^2 = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2$. We let $u = x - 2$. Then $du = dx$.

$$\int_1^3 \frac{dx}{\sqrt{4x - x^2}} = \int_{x=1}^3 \frac{dx}{\sqrt{4 - (x - 2)^2}} = \int_{u=-1}^1 \frac{du}{\sqrt{4 - u^2}} = \sin^{-1}\frac{u}{2}\Big]_{-1}^1 = \tfrac{1}{6}\pi - (-\tfrac{1}{6}\pi) = \tfrac{1}{3}\pi$$

21. $\displaystyle I = \int_{-2}^0 \frac{dx}{(5 - 4x - x^2)^{3/2}} = \int_{-2}^0 \frac{dx}{[9 - (x + 2)^2]^{3/2}}$. Let $\theta = \sin^{-1}\frac{x + 2}{3}$, $b = \sin^{-1}\frac{2}{3}$, $x + 2 = 3\sin\theta$.

$dx = 3\cos\theta\, d\theta$ and $[9 - (x + 2)^2]^{3/2} = [9 - 9\sin^2\theta]^{3/2} = 27(\cos^2\theta)^{3/2} = 27\cos^3\theta$.

$$I = \int_0^b \frac{3\cos\theta\, d\theta}{27\cos^3\theta} = \frac{1}{9}\int_0^b \sec^2\theta\, d\theta = \frac{1}{9}\tan\theta\Big]_0^b = \frac{1}{9}\cdot\frac{2}{\sqrt{5}} = \tfrac{2}{45}\sqrt{5} \approx 0.09938$$

22. $\displaystyle I = \int_2^3 \frac{dx}{x\sqrt{x^4 - 4}} = \int_{x=2}^3 \frac{x^3 dx}{x^4\sqrt{x^4 - 4}}$. Let $u^2 = x^4 - 4$, $2u\, du = 4x^3 dx$. $\displaystyle I = \int_{u=\sqrt{12}}^{\sqrt{77}} \frac{\frac{1}{2}u\, du}{(u^2 + 4)u} = \frac{1}{2}\int_{2\sqrt{3}}^{\sqrt{77}} \frac{du}{u^2 + 4}$

$$= \tfrac{1}{4}\tan^{-1}\frac{u}{2}\Big]_{2\sqrt{3}}^{\sqrt{77}} = \tfrac{1}{4}\tan^{-1}(\tfrac{1}{2}\sqrt{77}) - \tfrac{1}{12}\pi \approx 0.07488$$

23. Trig: Let $x = 3\tan\theta$ where $\theta \in [\frac{1}{6}\pi, \frac{1}{3}\pi]$ because $x \in [\sqrt{3}, 3\sqrt{3}]$. Then

$dx = 3\sec^2\theta\, d\theta$ and $\sqrt{x^2 + 9} = \sqrt{9\tan^2\theta + 9} = 3\sqrt{\sec^2\theta} = 3\sec\theta$.

$$\int_{\sqrt{3}}^{3\sqrt{3}} \frac{dx}{x^2\sqrt{x^2 + 9}} = \int_{\pi/6}^{\pi/3} \frac{3\sec^2\theta\, d\theta}{9\tan^2\theta(3\sec\theta)} = \frac{1}{9}\int_{\pi/6}^{\pi/3} \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{9}\int_{\pi/6}^{\pi/3} \csc\theta\cot\theta\, d\theta = -\frac{1}{9}\csc\theta\Big]_{\pi/6}^{\pi/3}$$

$$= -\frac{1}{9}\left(\frac{2}{3}\sqrt{3} - 2\right) = \frac{1}{27}(6 - 2\sqrt{3}) \approx 0.09392$$

Algebra: Let $u = 1 + 9x^{-2}$, $du = -18x^{-3}dx$.

$$\int_{\sqrt{3}}^{3\sqrt{3}} \frac{dx}{x^2\sqrt{x^2 + 9}} = \int_{x=\sqrt{3}}^{3\sqrt{3}} \frac{x^{-3}dx}{\sqrt{1 + 9x^{-2}}} = \int_{u=4}^{4/3} \frac{-\frac{1}{18}du}{u^{1/2}} = \frac{1}{18}\int_{4/3}^4 u^{-1/2}du = \frac{1}{9}\sqrt{u}\Big]_{4/3}^4 = \frac{1}{9}\left(2 - \frac{2}{3}\sqrt{3}\right)$$

24. $\displaystyle\int_0^1 \frac{x^2 dx}{\sqrt{4 - x^2}}$

▹ We use the method of Case 1 with $u = x$ and $a = 2$. Let $\theta = \sin^{-1}\frac{x}{2}$. Then

$$x = 2\sin\theta, \qquad dx = 2\cos\theta\, d\theta \quad \text{and}$$

$$\sqrt{4 - x^2} = \sqrt{4 - 4\sin^2\theta} = 2\sqrt{1 - \sin^2\theta} = 2\sqrt{\cos^2\theta} = 2\cos\theta$$

Thus

$$\int_{x=0}^1 \frac{x^2 dx}{\sqrt{4 - x^2}} = \int_{\theta=0}^{\pi/6} \frac{(4\sin^2\theta)(2\cos\theta\, d\theta)}{2\cos\theta} = 4\int_0^{\pi/6} \sin^2\theta\, d\theta = 2\int_0^{\pi/6} (1 - \cos 2\theta)d\theta = 2\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/6}$$

$$= 2[(\tfrac{1}{6}\pi - \tfrac{1}{2}\sin\tfrac{1}{3}\pi) - (0 - \tfrac{1}{2}\sin 0)] = \tfrac{1}{3}\pi - \tfrac{1}{2}\sqrt{3} \approx 0.18117$$

25. $\displaystyle\int_4^6 \frac{dx}{x\sqrt{x^2 - 4}} = \frac{1}{2}\sec^{-1}\frac{x}{2}\Big]_4^6 = \frac{1}{2}\sec^{-1}3 - \frac{1}{2}\sec^{-1}2 = \frac{1}{2}\cos^{-1}\frac{1}{6} - \frac{\pi}{6} \approx 0.17808$

26. Let $\theta = \tan^{-1}\frac{x}{\sqrt{3}}$, $x = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\, d\theta$, $\sqrt{x^2+3} = \sqrt{3\tan^2\theta + 3} = \sqrt{3}\sec\theta$

$$\int_{x=1}^{3} \frac{dx}{x^4\sqrt{x^2+3}} = \int_{\theta=\pi/6}^{\pi/3} \frac{\sqrt{3}\sec^2\theta\, d\theta}{9\tan^4\theta\cdot\sqrt{3}\sec\theta} = \frac{1}{9}\int_{\pi/6}^{\pi/3}\frac{\sec\theta\, d\theta}{\tan^4\theta} = \frac{1}{9}\int_{\pi/6}^{\pi/3}\frac{\cos^3\theta\, d\theta}{\sin^4\theta}$$

$$= \frac{1}{9}\int_{\pi/6}^{\pi/3}\frac{(1-\sin^2\theta)(\cos\theta\, d\theta)}{\sin^4\theta} = \frac{1}{9}\int_{\pi/6}^{\pi/3}(\sin^{-4}\theta - \sin^{-2}\theta)(d\sin\theta) = \frac{1}{9}\left[-\frac{1}{3\sin^3\theta} + \frac{1}{\sin\theta}\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{9}\left(-\frac{8}{9\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{8}{3} - 2\right) = \frac{2}{243}(5\sqrt{3}+9) \approx 0.14535$$

27. Let $x = 5\sin\theta$ where $\theta \in [0, \frac{1}{2}\pi]$ because $x \in [0,5]$. Then

$dx = 5\cos\theta\, d\theta$ and $\sqrt{25-x^2} = \sqrt{25 - 25\sin^2\theta} = 5\sqrt{\cos^2\theta} = 5\cos\theta$.

$$\int_0^5 x^2\sqrt{25-x^2}\, dx = \int_0^{\pi/2} 25\sin^2\theta(5\cos\theta)(5\cos\theta\, d\theta) = 625\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta = \frac{625}{4}\int_0^{\pi/2}\sin^2 2\theta\, d\theta$$

$$= \frac{625}{8}\int_0^{\pi/2}(1-\cos 4\theta)d\theta = \frac{625}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{625}{8}\cdot\frac{\pi}{2} = \frac{625}{16}\pi \approx 122.72$$

28. $\displaystyle\int_4^8 \frac{dw}{(w^2-4)^{3/2}}$

▹ Trig: We use the method of Case 3. Let $\theta = \sec^{-1}\frac{w}{2}$, $b = \sec^{-1} 4$. Thus.

$$w = 2\sec\theta, \qquad dw = 2\sec\theta\tan\theta \text{ and}$$

$$(w^2-4)^{3/2} = (4\sec^2\theta - 4)^{3/2} = [4(\sec^2\theta - 1)]^{3/2} = (4\tan^2\theta)^{3/2} = 8\tan^3\theta$$

$$\int_{w=4}^{8}\frac{dw}{(w^2-4)^{3/2}} = \int_{\theta=\pi/3}^{b}\frac{2\sec\theta\tan\theta\, d\theta}{8\tan^3\theta} = \frac{1}{4}\int_{\pi/3}^{b}\frac{\cos\theta\, d\theta}{\sin^2\theta} = \frac{1}{4}\int_{\pi/3}^{b}\csc\theta\cot\theta\, d\theta = -\frac{1}{4}\csc\theta\Big]_{\pi/3}^{b}$$

$$= -\frac{1}{4}\left[\frac{4}{\sqrt{15}} - \frac{2}{\sqrt{3}}\right] = \frac{1}{6}\sqrt{3} - \frac{1}{15}\sqrt{15} \approx 0.03048$$

Algebra: $\displaystyle\int_4^8\frac{dw}{(w^2-4)^{3/2}} = \int_{w=4}^{8}\frac{w^{-3}dw}{(1-4w^{-2})^{3/2}}$

Let $u = 1 - 4w^{-2}$. Then $du = 8w^{-3}dw$. Substituting into the right side of the above, we have

$$\int_4^8\frac{dw}{(w^2-4)^{3/2}} = \int_{u=3/4}^{15/16}\frac{\frac{1}{8}du}{u^{3/2}} = \frac{1}{8}\int_{3/4}^{15/16}u^{-3/2}du = -\frac{1}{4}\left[\frac{1}{\sqrt{u}}\right]_{3/4}^{15/16} = \frac{1}{4}\left[\frac{2}{\sqrt{3}} - \frac{4}{\sqrt{15}}\right] = \frac{1}{6}\sqrt{3} - \frac{1}{15}\sqrt{15}$$

$$\approx 0.03048$$

29. Let $u = e^t$, $du = e^t dt$. $\displaystyle \mathrm{I} = \int_{t=0}^{\ln 2}\frac{e^t dt}{(e^{2t}+8e^t+7)^{3/2}} = \int_{u=1}^{2}\frac{du}{(u^2+8u+7)^{3/2}} = \int_1^2\frac{du}{[(u+4)^2-9]^{3/2}}$

Trig: Let $\theta = \sec^{-1}\frac{u+4}{3}$, $a = \sec^{-1}\frac{5}{3}$, $u + 4 = 3\sec\theta$, $du = 3\sec\theta\tan\theta\, d\theta$ and

$[(u+4)^2-9]^{3/2} = (9\sec^2\theta - 9)^{3/2} = 27(\tan^2\theta)^{3/2} = 27\tan^3\theta$.

$$\mathrm{I} = \int_a^{\pi/3}\frac{3\sec\theta\tan\theta\, d\theta}{27\tan^3\theta} = \frac{1}{9}\int_a^{\pi/3}\frac{\sec\theta\, d\theta}{\tan^2\theta} = \frac{1}{9}\int_a^{\pi/3}\frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{9}\csc\theta\Big]_a^{\pi/3} = \frac{1}{9}\left[\frac{5}{4} - \frac{2}{\sqrt{3}}\right] = \frac{5}{36} - \frac{2}{27}\sqrt{3} \approx .010588$$

Algebra: Let $z = 1 - 9(u+4)^{-2}$, $dz = 18(u+4)^{-3}du$. Thus,

$$\mathrm{I} = \int_{u=1}^{2}\frac{(u+4)^{-3}du}{[1-9(u+4)^{-2}]^{3/2}} = \int_{z=16/25}^{3/4}\frac{\frac{1}{18}dz}{z^{3/2}} = \frac{1}{18}\int_{16/25}^{3/4}z^{-3/2}dz = -\frac{1}{9}\left[\frac{1}{\sqrt{z}}\right]_{16/25}^{3/4} = \frac{1}{9}\left[\frac{5}{4} - \frac{2}{\sqrt{3}}\right] = \frac{5}{36} - \frac{2}{27}\sqrt{3}$$

30. Let $u = e^x$, $du = e^x dx$. $\displaystyle\int_{x=0}^{1}\frac{\sqrt{16-e^{2x}}}{e^x}dx = \int_{u=1}^{e}\frac{\sqrt{16-u^2}}{u}\cdot\frac{du}{u}$. Now let $\theta = \sin^{-1}\frac{u}{4}$, $a = \sin^{-1}\frac{1}{4}$, $b = \sin^{-1}\frac{e}{4}$

$u = 4\sin\theta$, $du = 4\cos\theta\, d\theta$, $\sqrt{16-u^2} = \sqrt{16 - 16\sin^2\theta} = 16\cos\theta$

$$\mathrm{I} = \int_{\theta=a}^{b}\frac{(4\cos\theta)(4\cos\theta\, d\theta)}{(4\sin\theta)^2} = \int_a^b\cot^2\theta\, d\theta = \int_a^b(\csc^2\theta - 1)d\theta = -\cot\theta - \theta\Big]_a^b$$

$$= \sqrt{15} + \sin^{-1}\frac{1}{4} - \frac{\sqrt{16-e^2}}{e} - \sin^{-1}\frac{e}{4} \approx 2.2990$$

In Exercises 31–33, use algebraic substitutions to evaluate the integrals.

31. $\displaystyle\int \frac{3\,dx}{x\sqrt{4x^2-9}} = 3\int \frac{2\,dx}{(2x)\sqrt{(2x)^2-9}} = 3\cdot\tfrac{1}{3}\sec^{-2}\tfrac{2x}{3} + C = \sec^{-1}\tfrac{2}{3}x + C$

32. $\displaystyle\int \frac{5x\,dx}{\sqrt{3-2x^2}}$

▹ Let $3-2x^2 = u^2$, $-4x\,dx = 2u\,du$. Then

$$\int \frac{5x\,dx}{\sqrt{3-2x^2}} = -\frac{5}{4}\int \frac{-4x\,dx}{\sqrt{3-2x^2}} = -\frac{5}{4}\int \frac{2u\,du}{u} = -\frac{5}{2}\int du = -\frac{5}{2}u + C = -\frac{5}{2}\sqrt{3-2x^2} + C$$

33. Let $u^2 = 4-x^2$, $2u\,du = -2x\,dx$. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}dx = \int \frac{\sqrt{4-x^2}}{x^2}x\,dx = \int \frac{u}{4-u^2}(-u\,du) = \int\left(1-\frac{4}{4-u^2}\right)du$

$$= u - \ln\left|\frac{2+u}{2-u}\right| + C = \sqrt{4-x^2} - \ln\left|\frac{2+\sqrt{4-x^2}}{2-\sqrt{4-x^2}}\right| + C$$

34. Let $\theta = \sec^{-1}\frac{x}{3}$, $b = \sec^{-1}\frac{5}{3}$, $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = \sqrt{9\sec^2\theta-9} = 3\tan\theta$

$$A = \int_{x=3}^{b} \frac{\sqrt{x^2-9}}{x^2}dx = \int_{\theta=0}^{b} \frac{3\tan\theta}{9\sec^2\theta}(3\sec\theta\tan\theta\,d\theta) = \int_0^b \frac{\sec^2\theta-1}{\sec\theta}d\theta = \int_0^b(\sec\theta-\cos\theta)d\theta$$

$$= \ln|\sec\theta+\tan\theta| - \sin\theta\Big]_0^b = \ln(\tfrac{5}{3}+\tfrac{4}{3}) - \tfrac{4}{5} = \ln 3 - \tfrac{4}{5}$$

35. $y = \ln x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$. If L units is the length of arc from $x=1$ to $x=3$, then $x>0$ and

$$L = \int_1^3 \sqrt{1+\frac{1}{x^2}}\,dx = \int_1^3 \sqrt{\frac{x^2+1}{x^2}}\,dx = \int_{x=1}^3 \frac{\sqrt{x^2+1}}{x}\,dx$$

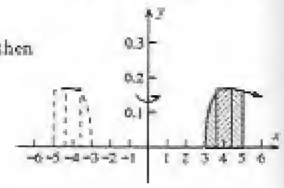
Let $\theta = \tan^{-1}x$, $b = \tan^{-1}3$. Then $x = \tan\theta$, $dx = \sec^2\theta$ and $\sqrt{x^2+1}/x = \csc\theta$.

$$L = \int_{\pi/4}^{b} \csc\theta(\sec^2 d\theta) = \int_{\pi/4}^{b} \csc\theta(\tan^2\theta+1)d\theta = \int_{\pi/4}^{b}(\sec\theta\tan\theta+\csc\theta)d\theta$$

$$= \Big[\sec\theta + \ln|\csc\theta-\cot\theta|\ \Big]_{\pi/4}^{b} = \sqrt{10} + \ln\frac{\sqrt{10}-1}{3} - \sqrt{2} - \ln(\sqrt{2}-1) \approx 2.30199$$

36. Find the volume of the solid generated by revolving the region bounded by the curve $y = \sqrt{x^2-9}/x^2$, the x axis, and the line $x=5$ about the y axis.

▹ See the figure. An element of volume is a cylindrical shell of thickness $\Delta_i x$, mean radius w_i and altitude $y(w_i)$, $x \in [3,5]$. If V cubic units is the volume, then



$$V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\pi w_i y(w_i)\Delta_i x = 2\pi\int_3^5 xy(x)dx$$

Trig: $\displaystyle V = 2\pi\int_{x=3}^{5}\frac{\sqrt{x^2-9}}{x}\,dx$

We use the method of Case 3 with $a=3$. Let $\theta = \sec^{-1}\frac{x}{3}$, $b = \sec^{-1}\frac{5}{3}$. Thus

$$x = 3\sec\theta, \qquad dx = 3\sec\theta\tan\theta\,d\theta \quad \text{and}$$

$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta-9} = 3\tan\theta$$

Thus

$$V = 2\pi\int_{\theta=0}^{b}\frac{(3\tan\theta)(3\sec\theta\tan\theta\,d\theta)}{3\sec\theta} = 6\pi\int_0^b(\sec^2\theta-1)d\theta = 6\pi\Big[\tan\theta-\theta\Big]_0^b = 6\pi(\tfrac{4}{3}-\cos^{-1}\tfrac{3}{5})$$

Algebra: $\displaystyle V = 2\pi\int_{x=3}^{5}\frac{\sqrt{x^2-9}}{x^2}(x\,dx)$

We let $u^2 = x^2-9$. Then $x^2 = u^2+9$ and $x\,dx = u\,du$. Thus

$$V = 2\pi\int_{u=0}^{4}\frac{u}{u^2+9}(u\,du) = 2\pi\int_0^4\left(1-\frac{9}{u^2+9}\right)du = 2\pi\Big[u-3\tan^{-1}\frac{u}{3}\Big]_0^4 = 2\pi(4-3\tan^{-1}\tfrac{4}{3})$$

▪ The volume is $2\pi(4-3\cos^{-1}\frac{3}{5}) = 2\pi(4-3\tan^{-1}\frac{4}{3})$ cubic units.

37. An element of volume is a disk centered on the x axis, $x \in [0, 3]$, of radius $w_i\sqrt[4]{9 - w_i^2}$.

$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\left(w_i\sqrt[4]{9 - w_i^2}\right)^2 \Delta_i x = \pi \int_0^3 x^2\sqrt{9 - x^2}\, dx$

Let $x = 3 \sin \theta$ where $\theta \in [0, \frac{1}{2}\pi]$ because $x \in [0, 3]$. Then $dx = 3 \cos \theta$ and

$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = 3\sqrt{\cos^2\theta} = 3 \cos \theta$.

$V = \pi \int_0^{\pi/2} 9 \sin^2\theta(3 \cos \theta)(3 \cos \theta\, d\theta) = 81\pi \int_0^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta = 81\pi \int_0^{\pi/2} \frac{\sin^2 2\theta}{4}\, d\theta$

$= \frac{81}{4}\pi \int_0^{\pi/2} \frac{1 - \cos 4\theta}{2}\, d\theta = \frac{81}{4}\pi\left[\frac{\theta}{2} - \frac{\sin 4\theta}{8}\right]_0^{\pi/2} = \frac{81}{4}\pi \cdot \frac{\pi}{4} = \frac{81}{16}\pi^2$

38. $y = x^2$, $y' = 2x$. If L units is the length of arc from $x = 0$ to $x = 1$, then $L = \int_{x=0}^{1} \sqrt{1 + (2x)^2}\,dx$.

Let $\theta = \tan^{-1} 2x$, $b = \tan^{-1} 2$. $x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\, d\theta$, $\sqrt{1 + (2x)^2} = \sqrt{1 + \tan^2\theta} = \sec\theta$. Thus,

$L = \int_{\theta=0}^{b} \sec\theta(\frac{1}{2}\sec^2\theta)d\theta = \frac{1}{2}\int_0^b \sec^3\theta\, d\theta = \frac{1}{4}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)\Big]_0^b = \frac{1}{4}(2\sqrt{5} + \ln(\sqrt{5} + 2))$

39. $M_y = \int_0^8 x\sqrt{x^2 + 36} = \frac{1}{2}\int_0^8 (x^2 + 36)^{1/2}(2x\, dx) = \frac{1}{3}(x^2 + 36)^{3/2}\Big]_0^8 = \frac{1}{3}(1000 - 216) = \frac{784}{3}$

$M = \int_{x=0}^{8} \sqrt{x^2 + 36}\, dx$. Let $\theta = \tan^{-1}\frac{x}{6}$, $b = \tan^{-1}\frac{4}{3}$. Then $x = 6\tan\theta$, $dx = 6\sec^2\theta\, d\theta$

and $\sqrt{x^2 + 36} = \sqrt{36\tan^2\theta + 36} = 6\sqrt{\sec^2\theta} = 6\sec\theta$. Therefore

$M = \int_{\theta=0}^{b} 36\sec^3\theta\, d\theta = 18[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_0^b = 18[\frac{5}{3}\cdot\frac{4}{3} + \ln(\frac{5}{3} + \frac{4}{3})] = 40 + 18\ln 3$.

$\bar{x} = M_y \cdot \frac{1}{M} = \frac{784}{3}\cdot\frac{1}{40 + 18\ln 3} = \frac{392}{60 + 27\ln 3}$. The center of mass is $\frac{392}{60 + 27\ln 3}$ cm from the left end.

40. The linear density of a rod at a point x meters from one end is $\sqrt{9 + x^2}$ kg/m. Find the mass and center of mass of the rod if it is 3 m long.

▷ We are given that $\rho(x) = \sqrt{9 + x^2}$. If M kg is the total mass of the rod, then $M = \int_{x=0}^{3} \sqrt{9 + x^2}dx$. Let $\theta = \tan^{-1}(x/3)$. Then $x = 3\tan\theta$, so $dx = 3\sec^2\theta\, d\theta$ and $\sqrt{9 + x^2} = 3\sec\theta$. Thus

$$M = 9\int_{\theta=0}^{\pi/4} \sec^3\theta\, d\theta = \frac{9}{2}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_0^{\pi/4} = \frac{9}{2}[\sqrt{2} + \ln(\sqrt{2} + 1)] \approx 10.3$$

$$M_y = \int_0^3 x\sqrt{9 + x^2}dx = \frac{1}{2}\cdot\frac{2}{3}(9 + x^2)^{2/3}\Big]_0^3 = \frac{1}{3}[18\sqrt{18} - 27] = 9(2\sqrt{2} - 1)$$

$$\bar{x} = \frac{M_y}{M} = \frac{9(2\sqrt{2} - 1)}{\frac{9}{2}[\sqrt{2} + \ln(\sqrt{2} + 1)]} = \frac{2(2\sqrt{2} - 1)}{\sqrt{2} + \ln(1 + \sqrt{2})} \approx 1.59$$

▪ The mass is about 10.3 kg and the center of mass is about 1.59 m from the end.

41. The region is bounded by $y = \frac{\sqrt{x^2 - 9}}{x^2}$, the x axis, and $x = 5$. $A = \int_3^5 \frac{\sqrt{x^2 - 9}}{x^2}\, dx$.

Let $x = 3\sec\theta$ and $\alpha = \sec^{-1}\frac{5}{3}$, where $\theta \in [0, \alpha]$ because $x \in [3, 5]$. Then $dx = 3\sec\theta\tan\theta\, d\theta$

and $\sqrt{x^2 - 9} = \sqrt{9\sec^2\theta - 9} = 3\sqrt{\tan^2\theta} = 3\tan\theta$. Therefore

$A = \int_0^\alpha \frac{3\tan\theta}{9\sec^2\theta}\cdot 3\sec\theta\tan\theta\, d\theta = \int_0^\alpha \frac{\tan^2\theta}{\sec\theta}\, d\theta = \int_0^\alpha \frac{\sec^2\theta - 1}{\sec\theta}\, d\theta = \int_0^\alpha (\sec\theta - \cos\theta)d\theta$

$= \Big[\ln|\sec\theta + \tan\theta| - \sin\theta\Big]_0^\alpha = \left(\ln\left|\frac{5}{3} + \frac{4}{3}\right| - \frac{4}{5}\right) - 0 = \ln 3 - \frac{4}{5} = \frac{5\ln 3 - 4}{5}$. Using the same substitution,

$M_y = \int_3^5 x\cdot\frac{\sqrt{x^2 - 9}}{x^2}\, dx = \int_0^5 \frac{\sqrt{x^2 - 9}}{x}\, dx = \int_0^\alpha \frac{3\tan\theta}{3\sec\theta}\cdot 3\sec\theta\tan\theta\, d\theta = 3\int_0^\alpha \tan^2\theta\, d\theta = 3\int_0^\alpha (\sec^2\theta - 1)d\theta$

$= 3\tan\theta - 3\theta\Big]_0^\alpha = 3\cdot\frac{4}{3} - 3\sec^{-1}\frac{5}{3} = 4 - \cos^{-1}\frac{3}{5}$

$M_x = \frac{1}{2}\int_3^5 \left(\frac{\sqrt{x^2 - 9}}{x^2}\right)^2 dx = \frac{1}{2}\int_3^5 \frac{x^2 - 9}{x^4}\, dx = \frac{1}{2}\int_3^5 \left(\frac{1}{x^2} - \frac{9}{x^4}\right)dx = \frac{1}{2}\left[-11x + \frac{3}{x^3}\right]_3^5 = \frac{1}{2}\left(-\frac{1}{5} + \frac{3}{125} + \frac{1}{3} - \frac{1}{9}\right) = \frac{26}{2}$

The centroid of the given region is at the point $(\bar{x}, \bar{y})$ where

$\bar{x} = \frac{1}{A}\cdot M_y = \frac{5}{5\ln 3 - 4}\cdot\left(4 - 3\cos^{-1}\frac{3}{5}\right) = \frac{20 - 15\cos^{-1}\frac{3}{5}}{5\ln 3 - 4}$, $\bar{y} = \frac{1}{A}\cdot M = \frac{5}{5\ln 3 - 4}\cdot\frac{26}{1125} = \frac{26}{225(5\ln 3 - 4)}$

42. An equation of ellipse $x^2/a^2 + y^2/b^2 = 1$ in the first quadrant is $y = b\sqrt{1 - x^2/a^2}$. Let $\theta = \sin^{-1}(x/a)$, $x = a \sin\theta$, $dx = a \cos\theta\, d\theta$, $\sqrt{1 - x^2/a^2} = \sqrt{1 - \sin^2\theta} = \cos\theta$. Area $= 4b\int_{x=0}^{a}\sqrt{1 - x^2/a^2}\,dx$
$= 4b\int_{\theta=0}^{\pi/2}\cos\theta(a\cos\theta\, d\theta) = 2ab\int_0^{\pi/2}(1 + \cos 2\theta)d\theta = 2ab\Big[\theta - \frac{1}{2}\sin 2\theta\Big]_0^{\pi/2} = 2ab\cdot\frac{1}{2}\pi = \pi ab$

43. Choose the x axis downward and take the y axis at the water level. The face of the gate is the region enclosed by the circle $(x-1)^2 + y^2 = 4$. An element of area is a horizontal strip of width $2\sqrt{4 - (w_i - 1)^2}$ at a mean depth of w_i. If F lb is the force on the gate due to water pressure, then

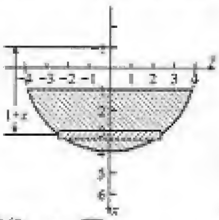
$$F = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\rho w_i\sqrt{4 - (w_i - 1)^2}\,\Delta_i x = 2\rho\int_0^3 x\sqrt{4 - (x-1)^2}\,dx$$

Let $x - 1 = 2\sin\theta$ where $\theta \in [-\frac{1}{6}\pi, \frac{1}{2}\pi]$ because $x \in [0, 3]$. Then $dx = 2\cos\theta\, d\theta$ and $\sqrt{4 - (x-1)^2} = \sqrt{4 - 4\sin^2\theta} = 2\sqrt{\cos^2\theta} = 2\cos\theta$. Therefore

$$F = 2\rho\int_{-\pi/6}^{\pi/2}(1 + 2\sin\theta)(2\cos\theta)(2\cos\theta\, d\theta) = 8\rho\int_{-\pi/6}^{\pi/2}\cos^2\theta\, d\theta + 16\rho\int_{-\pi/6}^{\pi/2}\cos^2\theta\sin\theta\, d\theta$$
$$= 4\rho\int_{-\pi/6}^{\pi/2}(1 + \cos 2\theta)d\theta - \frac{16}{3}\rho\cos^3\theta\Big]_{-\pi/6}^{\pi/2} = 4\rho\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_{-\pi/6}^{\pi/2} + \frac{16}{3}\rho\cdot\Big(\frac{1}{2}\sqrt{3}\Big)^3$$
$$= 4\rho\Big(\frac{1}{2}\pi + \frac{1}{6}\pi + \frac{1}{2}\cdot\frac{1}{2}\sqrt{3}\Big) + 2\sqrt{3}\rho = \Big(\frac{8}{3}\pi + 3\sqrt{3}\Big)(62.4) \approx 847$$

44. A gate in an irrigation ditch is in the shape of a segment of a circle of radius 4 ft. The top of the gate is horizontal and 3 ft above the lowest point of the gate. If the water level is 2 ft above the top of the gate, find the force on the gate due to water pressure.

▷ See the figure. Choose the x axis downward with the origin at the center of the circle. An element of area has a width $2\sqrt{16 - w_i^2}$, height $\Delta_i x$ and is $(w_i + 1)$ft below the water level. If F pounds is the force on the gate,



$$F = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\rho(1 + w_i)\sqrt{16 - w_i^2}\,\Delta_i x = 2\rho\int_1^4(1 + x)\sqrt{16 - x^2}\,dx$$
$$= 2\rho\Big[\int_1^4\sqrt{16 - x^2}\,dx + \int_1^4 x\sqrt{16 - x^2}\,dx\Big] \tag{1}$$

For the second integral of (1) we have

$$\int_1^4 x\sqrt{16 - x^2}\,dx = -\frac{1}{2}\int_1^4(16 - x^2)^{1/2}(-2x\, dx) = -\frac{1}{2}\cdot\frac{2}{3}(16 - x^2)^{3/2}\Big]_1^4 = -\frac{1}{3}[0 - 15^{3/2}] = 5\sqrt{15} \tag{2}$$

For the first integral of (1) we use a trigonometric substitution. We let $\theta = \sin^{-1}\frac{1}{4}x$, $\alpha = \sin^{-1}\frac{1}{4}$. Then

$$x = 4\sin\theta, \qquad dx = 4\cos\theta\, d\theta \quad \text{and}$$
$$\sqrt{16 - x^2} = \sqrt{16 - 16\sin^2\theta} = 4\cos\theta$$

Hence,

$$\int_{x=1}^{4}\sqrt{16 - x^2}\,dx = \int_{\theta=\alpha}^{\pi/2}(4\cos\theta)(4\cos\theta\, d\theta) = 8\int_\alpha^{\pi/2}(1 + \cos 2\theta)d\theta = 8\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_\alpha^{\pi/2}$$
$$= 8(\theta + \sin\theta\cos\theta)\Big]_\alpha^{\pi/2} = 8\Big[\frac{1}{2}\pi - (\sin^{-1}\frac{1}{4} + \frac{1}{4}\sqrt{15})\Big] \tag{3}$$

Substituting from (2) and (3) into (1) we get

$$F = 2\rho(4\pi - 8\sin^{-1}\frac{1}{4} - \frac{1}{2}\sqrt{15} + 5\sqrt{15}) = (8\pi - 16\sin^{-1}\frac{1}{4} + 9\sqrt{15})(62.4) \approx 3491$$

▪ The force on the gate due to water pressure is about 3491 lb.

45. Choose the x axis downward and take the y axis at the liquid level. The end of the tank is the region enclosed by the circle $(x-4)^2 + y^2 = 64$. An element of area is a horizontal strip of width $2\sqrt{64 - (w_i - 4)^2}$ at a mean depth of w_i. If F lb is the force on the end due to liquid pressure, then

$$F = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 2\rho w_i\sqrt{64 - (w_i - 4)^2}\,\Delta_i x = 2\rho\int_0^{12} x\sqrt{64 - (x-4)^2}\,dx$$

Let $x - 4 = 8\sin\theta$ where $\theta \in [-\frac{1}{6}\pi, \frac{1}{2}\pi]$ because $x \in [0, 12]$. Then $dx = 8\cos\theta\, d\theta$ and $\sqrt{64 - (x-4)^2} = \sqrt{64 - 64\sin^2\theta} = 8\sqrt{\cos^2\theta} = 8\cos\theta$. Therefore

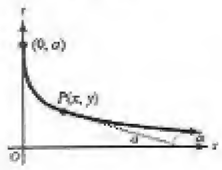
$$F = 2\rho\int_{-\pi/6}^{\pi/2}(4 + 8\sin\theta)(8\cos\theta)(8\cos\theta\, d\theta) = 512\rho\int_{-\pi/6}^{\pi/2}\cos^2\theta\, d\theta + 1024\rho\int_{-\pi/6}^{\pi/2}\cos^2\theta\sin\theta\, d\theta$$

$= 256\rho\int_{-\pi/6}^{\pi/2}(1+\cos 2\theta)d\theta - \frac{1024}{3}\rho\cos^3\theta\Big]_{-\pi/6}^{\pi/2} = 256\rho\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_{-\pi/6}^{\pi/2} + \frac{1024}{3}\rho\cdot\left(\frac{1}{2}\sqrt{3}\right)^3$

$= 256\rho\left(\frac{1}{2}\pi + \frac{1}{6}\pi + \frac{1}{2}\cdot\frac{1}{2}\sqrt{3}\right) + 128\sqrt{3}\rho = \left(\frac{512}{3}\pi + 192\sqrt{3}\right)(0.39) \approx 338.8$ oz

46. Derive the equation of the tractrix. See the figure. $\tan\alpha = \frac{dy}{dx} = \frac{-y}{\sqrt{a^2-y^2}}$ so $x = -\int\frac{\sqrt{a^2-y^2}}{y}dy$.

Substitute $y = a\sin\alpha$, $dy = a\cos\alpha\, d\alpha$, $\sqrt{a^2-y^2} = \sqrt{a^2-a^2\sin^2\alpha} = a\cos\alpha$.



$x = -\int\frac{a\cos\alpha}{a\sin\alpha}(a\cos\alpha\, d\alpha) = -a\int\frac{1-\sin^2\alpha}{\sin\alpha}d\alpha = a\int(\sin\alpha - \csc\alpha)d\alpha$

$= a(-\cos\alpha + \ln|\csc\alpha + \cot\alpha|) = a\left[-\frac{\sqrt{a^2-y^2}}{a} + \ln\left(\frac{a}{y} + \frac{\sqrt{a^2-y^2}}{y}\right)\right]$

The constant is 0 because $y = a$ when $x = 0$.

47. The distance walked is $w = x + \sqrt{a^2-y^2} = a\ln\left(\frac{a+\sqrt{a^2-y^2}}{y}\right) = a\cosh^{-1}\frac{a}{y}$ and so $\frac{a}{y} = \cosh\frac{w}{a}$,

$y = \frac{a}{\cosh(w/a)}$ (a) When $a = 15$ and $y = 12$, $w = 15\ln\left(\frac{15+9}{12}\right) = 15\ln 2 \approx 10.40$

(b) When $a = 15$ and $w = 20$, $y = \frac{15}{\cosh\frac{4}{3}} = \frac{30}{e^{4/3}+e^{-4/3}} \approx 7.394$

48. Suppose a hound moving along the pursuit curve, a tractrix, chasing a rabbit moving along the x axis is always going in the direction pointing at the rabbit. Explain why the hound is always the same distance from the rabbit so that it never catches the rabbit.

▹ Because the hound is always pointing at the rabbit, its distance from the rabbit is along a tangent line to the tractrix. By definition, this tangent line segment always has the same length.

49. (a) $\theta = \sec^{-1}\frac{x}{2}$, $a = \sec^{-1}\frac{3}{2}$, $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$, $4 - x^2 = 4 - 4\sec^2\theta = -4\tan^2\theta$.

$\int_{x=3}^{4}\frac{dx}{4-x^2} = \int_{\theta=a}^{\pi/3}\frac{2\sec\theta\tan\theta\, d\theta}{-4\tan^2\theta} = -\frac{1}{2}\int_a^{\pi/3}\csc\theta\, d\theta = \frac{1}{2}\ln|\csc\theta + \cot\theta|\Big]_a^{\pi/3}$

$= \frac{1}{2}\left[\ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) - \ln\left(\frac{3}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right)\right] = \frac{1}{2}(\ln\sqrt{3} - \ln\sqrt{5}) = \frac{1}{4}\ln\frac{3}{5}$

(b) We cannot use $x = 2\sin\theta$ because $|2\sin\theta| \le 2$.

A.11 PARTIAL FRACTIONS

Partial Fraction for P/D

1. If $\deg(P) \ge \deg(D)$, divide to get quotient $Q(x)$ and remainder $R(x)$. Focus on R/D.
2. Factor D into powers of distinct linear and irreducible quadratic factors.
3. For each $(ax+b)^m$ we get the terms $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$
4. For each $(ax^2+bx+c)^n$ we get $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$

Plan For distinct factors, use the method of undetermined coefficients: multiply by D to get a polynomial identity and substitute the roots. Compare coefficients of the highest power to get another equation if needed.

For repeated factors, just divide repeatedly: the quotient is over a lower power, the remainder is over the original power. When powers are negative, arrange in increasing order. We indicate the result of a long division by $\overset{L}{=}$.

Exercises A.11

Decompose each fraction f into partial fractions or the sum of a polynomial and partial fractions.

1. $\dfrac{12}{x^2-4} = \dfrac{12}{(x-2)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$

$$12 = A(x+2) + B(x-2)$$

$x = 2$: $\quad 12 = 4A,\ A = 3$

$x = -2$: $\quad 12 = -4B,\ B = -3$

$f = \dfrac{3}{x-2} - \dfrac{3}{x+2}$

2. $\dfrac{1}{2x^2-x} = \dfrac{1}{x(2x-1)} = \dfrac{A}{x} + \dfrac{B}{2x-1}$

$$1 = A(2x-1) + Bx$$

$x = 0$: $\quad 1 = -A,\ A = -1$

$x = \frac{1}{2}$: $\quad 1 = \frac{1}{2}B,\ B = 2$

$f = \dfrac{2}{2x-1} - \dfrac{1}{x}$

3. $\dfrac{x-1}{x^2+x} = \dfrac{x-1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1}$

$$x - 1 = A(x+1) + Bx$$

$x = 0$: $\quad -1 = A$

$x = -1$: $\quad -2 = -B,\ B = 2$

$f = \dfrac{2}{x+1} - \dfrac{1}{x}$

4. $\dfrac{x+15}{x^2-9} = \dfrac{x+15}{(x-3)(x+3)} = \dfrac{A}{x-3} + \dfrac{B}{x+3}$

$$x + 15 = A(x+3) + B(x-3)$$

$x = 3$: $\quad 18 = 6A,\ A = 3$

$x = -3$: $\quad 12 = -6B,\ B = -2$

$f = \dfrac{3}{x-3} - \dfrac{2}{x+3}$

5. $\dfrac{x+5}{x^2-4x+3} = \dfrac{x+5}{(x-3)(x-1)} = \dfrac{A}{x-3} + \dfrac{B}{x-1}$

$$x + 5 = A(x-1) + B(x-3)$$

$x = 3$: $\quad 8 = 2A,\ A = 4$

$x = 1$: $\quad 6 = -2B,\ B = -3$

$f = \dfrac{4}{x-3} - \dfrac{3}{x-1}$

6. $\dfrac{3x}{x^2+x-2} = \dfrac{3x}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$

$$3x = A(x+2) + B(x-1)$$

$x = 1$: $\quad 3 = 3A,\ A = 1$

$x = -2$: $\quad -6 = -3B,\ B = 2$

$f = \dfrac{1}{x-1} + \dfrac{2}{x+2}$

7. $\dfrac{x+12}{3x^2-5x-2} = \dfrac{x+12}{(x-2)(3x+1)} = \dfrac{A}{x-2} + \dfrac{B}{3x+1}$

$$x + 12 = A(3x+1) + B(x-2)$$

$x = 2$: $\quad 14 = 7A,\ A = 2$

$x = -\frac{1}{3}$: $\quad \frac{35}{3} = -\frac{7}{3}B,\ B = -5$

$f = \dfrac{2}{x-2} - \dfrac{5}{3x+1}$

8. $\dfrac{3x-7}{4x^2+3x-1} = \dfrac{3x-7}{(4x-1)(x+1)} = \dfrac{A}{4x-1} + \dfrac{B}{x+1}$

$$3x - 7 = A(x+1) + B(4x-1)$$

$x = \frac{1}{4}$: $\quad -\frac{25}{4} = \frac{5}{4}A,\ A = -5$

$x = -1$: $\quad -10 = -5B,\ B = 2$

$f = \dfrac{2}{x+1} - \dfrac{5}{4x-1}$

9. $\dfrac{3x^2+3x-12}{6x^3+5x^2-6x}$

▹ $\dfrac{3x^2+3x-12}{x(3x-2)(2x+3)} = \dfrac{A}{x} + \dfrac{B}{3x-2} + \dfrac{C}{2x+3}$

$3x^2 + 3x - 12 =$
$\quad A(3x-2)(2x+3) + Bx(2x+3) + Cx(3x-2)$

$x = 0$: $\quad -12 = -6A,\ a = 2$

$x = \frac{2}{3}$: $\quad -\frac{26}{3} = \frac{26}{9}B,\ B = -3$

$x = -\frac{3}{2}$: $\quad -\frac{39}{4} = \frac{39}{4}C,\ C = -1$

$f = \dfrac{2}{x} - \dfrac{3}{3x-2} - \dfrac{1}{2x+3}$

10. $\dfrac{2x^2-11x-9}{x^3-2x^2-3x}$

▹ $\dfrac{2x^2-11x-9}{x(x-3)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x-3} + \dfrac{C}{x+1}$

$2x^2 - 11x - 9$
$\quad = A(x-3)(x+1) + Bx(x+1) + Cx(x-3)$

$x = 0$: $\quad -9 = -3A,\ A = 3$

$x = 3$: $\quad -24 = 12B,\ B = -2$

$x = -1$: $\quad 4 = 4C,\ C = 1$

$f = \dfrac{3}{x} - \dfrac{2}{x-3} + \dfrac{1}{x+1}$

11. $\dfrac{2x^3+4}{x^2-4} = 2x + \dfrac{8x+4}{x^2-4}$

$\dfrac{8x+4}{(x-2)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$

$$8x + 4 = A(x+2) + B(x-2)$$

$x = 2$: $\quad 20 = 4A,\ A = 5$

$x = -2$: $\quad -12 = -4B,\ B = 3$

$f = 2x + \dfrac{5}{x-2} + \dfrac{3}{x+2}$

12. $\dfrac{x^3+5}{x^2-1} = x + \dfrac{x+5}{x^2-1}$

$\dfrac{x+5}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

$$x + 5 = A(x+1) + B(x-1)$$

$x = 1$: $\quad 6 = 2A,\ A = 3$

$x = -1$: $\quad 4 = -2B,\ B = -2$

$f = x + \dfrac{3}{x-1} - \dfrac{2}{x+1}$

13. $\dfrac{4x^3-8x^2-10x+30}{2x^2+x-6} = 2x-5+\dfrac{7x}{2x^2+x-6}$

$\dfrac{7x}{(2x-3)(x+2)} = \dfrac{A}{2x-3}+\dfrac{B}{x+2}$

$$\begin{array}{ll} & 7x = A(x+2)+B(2x-3) \\ x=\frac{3}{2}: & \frac{21}{2}=\frac{7}{2}A,\ a=3 \\ x=-2: & -14=-7B,\ B=2 \end{array}$$

$f = 2x-5+\dfrac{3}{2x-3}+\dfrac{2}{x+2}$

14. $\dfrac{6x^3+x^2-5x-7}{3x^2-x-2} = 2x+1+\dfrac{-5}{3x^2-x-2}$

$\dfrac{-5}{(3x+2)(x-1)} = \dfrac{A}{3x+2}+\dfrac{B}{x-1}$

$$\begin{array}{ll} & -5 = A(x-1)+B(3x+2) \\ x=-\frac{2}{3}: & -5=-\frac{5}{3}A,\ A=3 \\ x=1: & -5=5B,\ B=-1 \end{array}$$

$f = 2x+1+\dfrac{3}{3x+2}-\dfrac{1}{x-1}$

15. $\dfrac{3x^2+13x-10}{x^3-2x^2}$

▷ $\dfrac{3x^2+13x-10}{x^2(x-2)} = \dfrac{3+13x^{-1}-10x^{-2}}{x-2}$

$$\begin{array}{r|l} & 5x^{-2}-4x^{-1} \\ \hline -2+x & -10x^{-2}+13x^{-1}+3 \\ & \underline{-10x^{-2}+\ 5x^{-1}} \\ & 8x^{-1}+3 \\ & \underline{8x^{-1}-4} \\ & 7 \end{array}$$

$f = \dfrac{5}{x^2}-\dfrac{4}{x}+\dfrac{7}{x-2}$

16. $\dfrac{x^2+x+1}{x^4-x^3}$

▷ $\dfrac{x^2+x+1}{x^3(x-1)} = \dfrac{x^{-1}+x^{-2}+x^{-3}}{x-1}$

$$\begin{array}{r|l} & -x^{-3}-2x^{-2}-3x^{-1} \\ \hline -1+x & x^{-3}+x^{-2}+x^{-1} \\ & \underline{x^{-3}-x^{-2}} \\ & 2x^{-2}+x^{-1} \\ & \underline{2x^{-2}-2x^{-1}} \\ & 3x^{-1} \\ & \underline{3x^{-1}-3} \\ & 3 \end{array}$$

$f = \dfrac{3}{x-1}-\dfrac{3}{x}-\dfrac{2}{x^2}-\dfrac{1}{x^3}$

17. $\dfrac{x^2-11x+6}{(x+2)(x^2-4x+4)}$

▷ $\dfrac{x^2-11x+6}{(x+2)(x-2)^2} = \dfrac{A}{x+2}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$

$x^2-11x+6$

$\quad = A(x-2)^2+B(x+2)(x-2)+C(x+2)$

$$\begin{array}{ll} x=-2: & 32=16A,\ A=2 \\ x=2: & -12=4C,\ C=-3 \\ \text{coef of } x^2: & 1=A+B=2+B,\ B=-1 \end{array}$$

$f = \dfrac{2}{x+2}-\dfrac{1}{x-2}-\dfrac{3}{(x-2)^2}$

18. $\dfrac{x^2+11}{(x-5)(x^2+2x+1)}$

▷ $\dfrac{x^2+11}{(x-5)(x+1)^2} = \dfrac{A}{x-5}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$

$x^2+11 = A(x+1)^2+B(x-5)(x+1)+C(x-5)$

$$\begin{array}{ll} x=5: & 36=36A,\ A=1 \\ x=-1: & 12=-6C,\ C-=2 \\ \text{coef of } x^2: & 1=A+B=1+B,\ B=0 \end{array}$$

$f = \dfrac{1}{x-5}-\dfrac{2}{(x+1)^2}$

19. $\dfrac{3x+15}{(2x^2-x-1)^2} = \dfrac{3x+15}{(2x+1)^2(x-1)^2}$

Let $x=u+1$. $f = \dfrac{3u+18}{u^2(2u+3)}$

$= \dfrac{1}{3+2u}\cdot\dfrac{18u^{-2}+3u^{-1}}{3+2u}$

$\stackrel{L}{=} \dfrac{1}{3+2u}\left[6u^{-1}-3u^{-1}+\dfrac{6}{3+2u}\right]$

$\stackrel{L}{=} 2u^{-2}-\dfrac{7}{3}u^{-1}+\dfrac{\frac{14}{3}}{3+2u}+\dfrac{6}{3+2u}$

$= \dfrac{2}{(x-1)^2}-\dfrac{\frac{7}{3}}{x-1}+\dfrac{\frac{14}{3}}{2x+1}+\dfrac{6}{(2x+1)^2}$

20. $\dfrac{9x^3-8x^2-4x+48}{(x^2-4)^2} = \dfrac{9x^3-8x^2-4x+48}{(x-2)^2(x+2)^2}$

Let $x=u+2$. $f = \dfrac{9u^3+46u^2+72u+80}{u^2(u+4)^2}$

$= \dfrac{1}{4+u}\cdot\dfrac{80u^{-2}+72u^{-1}+46+9u}{4+u}$

$\stackrel{L}{=} \dfrac{1}{4+u}\left[20u^{-2}+13u^{-1}+9-\dfrac{3}{4+u}\right]$

$\stackrel{L}{=} 5u^{-2}+2u^{-1}+\dfrac{7}{4+u}-\dfrac{3}{(4+u)^2}$

$= \dfrac{5}{(x-2)^2}+\dfrac{2}{x-2}+\dfrac{7}{x+2}-\dfrac{3}{(x+2)^2}$

21. $\dfrac{x^3+6x-4}{(x-2)^3} = \dfrac{1}{(x-2)^2}\cdot\dfrac{x^3+6x-4}{x-2}$

$\stackrel{L}{=} \dfrac{1}{(x-2)^2}\left[x^2+2x+10+\dfrac{16}{x-2}\right]$

$\stackrel{L}{=} \dfrac{1}{x-2}\left[x+4+\dfrac{18}{x-2}+\dfrac{16}{(x-2)^2}\right]$

$\stackrel{L}{=} 1+\dfrac{6}{x-2}+\dfrac{18}{(x-2)^2}+\dfrac{16}{(x-2)^3}$

22. $\dfrac{x^2+2}{(x-3)^3}$. Let $x=u+3$.

$f = \dfrac{u^2+6u+11}{u^3} = \dfrac{1}{u}+\dfrac{6}{u^2}+\dfrac{11}{u^3}$

$= \dfrac{1}{x-3}+\dfrac{6}{(x-3)^2}+\dfrac{11}{(x-3)^3}$

23. $\dfrac{3x^2-x+4}{x^3+x^2+x} = \dfrac{3x^2-x+4}{x(x^2+x+1)}$

$= \dfrac{4x^{-1}-1+3x}{1+x+x^2} \stackrel{L}{=} 4x^{-1}+\dfrac{-5-x}{1+x+x^2}$

$= \frac{4}{x} - \frac{x+5}{x^2+x+1}$

24. $\frac{3x+8}{x^3+4x} = \frac{3x+8}{x(x^2+4)} = \frac{8x^{-1}+3}{4+x^2}$

$\stackrel{L}{=} 2x^{-1} + \frac{3-2x}{4+x^2} = \frac{2}{x} - \frac{2x-3}{x^2+4}$

25. $\frac{3x^2+2x-4}{x^3-8}$

▷ $\frac{3x^2+2x-4}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$

$3x^2+2x-4 = A(x^2+2x+4) + (x-2)(Bx+C)$

$x = 2$: $\quad 12 = 12A$, $A = 1$

$3x^2+2x-4 = x^2+2x+4+(x-2)(Bx+C)$

$2x^2-8 = (x-2)(Bx+C)$

Divide by $x-2$:

$2x+4 = Bx+C$

$f = \frac{1}{x-2} + \frac{2x+4}{x^2+2x+4}$

26. $\frac{x^2-6x+2}{x^3+1}$

▷ $\frac{x^2-6x+2}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$x^2-6x+2 = A(x^2-x+1) + (x+1)(Bx+C)$

$x = -1$: $\quad 9 = 3A$, $A = 3$

$x^2-6x+2 = 3x^2-3x+3+(x+1)(Bx+C)$

$-2x^2-3x-1 = (x+1)(Bx+C)$

Divide by $x+1$:

$-2x-1 = Bx+C$

$f = \frac{3}{x+1} - \frac{2x+1}{x^2-x+1}$

27. $\frac{2x^2-7x+1}{x^3-x^2+x-1}$

▷ $\frac{2x^2-7x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$2x^2-7x+1 = A(x^2+1) + (x-1)(Bx+C)$

$x = 1$: $\quad -4 = 2A$, $A = -2$

$2x^2-7x+1 = -2x^2-2+(x-1)(Bx+C)$

$4x^2-7x+3 = (x-1)(Bx+C)$

Divide by $x-1$:

$4x-3 = Bx+C$

$f = -\frac{2}{x-1} + \frac{4x-3}{x^2+1}$

28. $\frac{3x^2-9x+8}{x^3+x^2+3x+3} = \frac{3x^2-9x+8}{x^2(x+1)+3(x+1)}$

$= \frac{3x^2-9x+8}{(x+1)(x^2+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+3}$

$3x^2-9x+8 = A(x^2+3) + (x+1)(Bx+C)$

$x = -1$: $\quad 20 = 4A$, $A = 5$

$3x^2-9x+8 = 5x^2+15+(x+1)(Bx+C)$

$-2x^2-9x-7 = (x+1)(Bx+C)$

Divide by $x+1$:

$-2x-7 = Bx+C$

$f = \frac{5}{x+1} - \frac{2x+7}{x^2+3}$

29. $\frac{11x^2+11x+8}{2x^3+8x^2+3x+12} = \frac{11x^2+11x+8}{2x^2(x+4)+3(x+4)}$

$= \frac{11x^2+11x+8}{(x+4)(2x^2+3)} = \frac{A}{x+4} + \frac{Bx+C}{2x^2+3}$

$11x^2+11x+8 = A(2x^2+3) + (x+4)(Bx+C)$

$x = -4$: $\quad 140 = 35A$, $A = 4$

$11x^2+11x+8 = 8x^2+12+(x+4)(Bx+C)$

$3x^2+11x-4 = (x+4)$

Divide by $x+4$:

$3x-1 = Bx+C$

$f = \frac{4}{x+4} + \frac{3x-1}{2x^2+3}$

30. $\frac{3x^2+2x+3}{x^4+x^3+x^2+x} = \frac{3x^2+2x+3}{x[x^2(x+1)+(x+1)]}$

$= \frac{3x^2+2x+3}{x(x+1)(x^2+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

$3x^2+2x+3 =$

$A(x+1)(x^2+1) + Bx(x^2+1) + (Cx+D)x(x+1)$

$x = 0$: $\quad 3 = A$

$x = -1$: $\quad 4 = -2B$, $B = -2$

$3x^2+2x+3 =$

$3x^3+3x^2+3x+3-2x^3-2x+(Cx+D)(x^2+x)$

$-x^3+x = (Cx+D)(x^2+x)$

Divide by x^2+x:

$-x+1 = Cx+D$

$f = \frac{3}{x} - \frac{2}{x+1} - \frac{x-1}{x^2+1}$

31. $\frac{3x^2-4x}{(x^2+1)(x^2-x-1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-x-1}$

$3x^2-4x$

$= (Ax+B)(x^2-x-1) + (Cx+D)(x^2+1)$

$= (A+C)x^3 + (-A+B+D)x^2$

$+ (-A-B+C)x + (-B+D)$

$E(x^3)$: $\quad A+C = 0$

$F(x^2)$: $\quad -A+B+D = 0$

$G(x)$: $\quad -A-B+C = -4$

$H(1)$: $\quad -B+D = 0$

$I = E-G$: $\quad 2A+B = 4$

$J = H-F$: $\quad A-2B = 3$

$I-2J$: $\quad 5B = 10$, $B = 2$, $D = 2$

$2I-J$: $\quad 5A = 5$, $A = 1$, $C = -1$

$f = \frac{x+2}{x^2+1} - \frac{x-2}{x^2-x-1}$

32. $\frac{4x-3}{x^4+2x^3+3x^2} = \frac{4x-3}{x^2(x^2+2x+3)}$

$= \frac{x^{-2}+4x^{-1}}{3+2x+x^2} \stackrel{L}{=} -x^{-2}+2x^{-1} + \frac{-3-2x}{3+2x+x^2}$

$= \frac{2}{x} - \frac{1}{x^2} - \frac{2x+3}{x^2+2x+3}$

3. $\dfrac{x+6}{x^4+2x^3+3x^2}=\dfrac{x+6}{x^2(x^2+2x+3)}$

$=\dfrac{6x^{-2}+x^{-1}}{3+2x+x^2}\overset{L}{=}2x^{-2}-x^{-1}+\dfrac{x}{3+2x+x^2}$

$=\dfrac{2}{x^2}-\dfrac{1}{x}+\dfrac{x}{x^2+2x+3}$

4. $\dfrac{3x^4+4}{x^4+4x^2+4}=\dfrac{3x^4+4}{(x^2+2)^2}=\dfrac{1}{x^2+2}\cdot\dfrac{3x^4+4}{x^2+2}$

$\overset{L}{=}\dfrac{1}{x^2+2}\left[3x^2-6+\dfrac{16}{x^2+2}\right]$

$\overset{L}{=}3-\dfrac{12}{x^2+2}+\dfrac{16}{(x^2+2)^2}$

5. $\dfrac{x^3-x^2}{x^4+2x^2+1}=\dfrac{x^3-x^2}{(x^2+1)^2}=\dfrac{1}{x^2+1}\cdot\dfrac{x^3-x^2}{x^2+1}$

$\overset{L}{=}\dfrac{1}{x^2+1}\left[x-1+\dfrac{-x+1}{x^2+1}\right]=\dfrac{x-1}{x^2+1}-\dfrac{x-1}{(x^2+1)^2}$

6. $\dfrac{x^4+2x^2-2x-4}{(x^2+3)^3}=\dfrac{1}{(x^2+3)^2}\cdot\dfrac{x^4+2x^2-2x-4}{x^2+3}$

$\overset{L}{=}\dfrac{1}{x^2+3}\cdot\dfrac{1}{x^2+3}\left[x^2-1-\dfrac{2x+1}{x^2+3}\right]$

$\overset{L}{=}\dfrac{1}{x^2+3}\left[1-\dfrac{4}{x^2+3}-\dfrac{2x+1}{(x^2+3)^2}\right]$

$=\dfrac{1}{x^2+3}-\dfrac{4}{(x^2+3)^2}-\dfrac{2x+1}{(x^2+3)^3}$

7. $\dfrac{x^4+x^3-5x^2-14x-1}{x^5-x^4+4x^3-4x^2+4x-4}$

$=\dfrac{x^4+x^3-5x^2-14x-1}{(x-1)(x^2+2)^2}$

$=\dfrac{A}{x-1}+\dfrac{Q}{x^2+2}+\dfrac{R}{(x^2+2)^2}$

$x^4+x^3-5x^2-4x-1$
$\quad=A(x^2+2)^2+Q(x-1)(x^2+2)+R(x-1)$

$x=1$: $\quad -18=9A,\ A=-2$

$x^4+x^3-5x^2-4x-1$
$=-2x^4-8x^2-8+Q(x-1)(x^2+2)+R(x-1)$

$3x^4+x^3+3x^2-4x+7$
$\quad=Q(x-1)(x^2+2)+R(x-1)$

Divide by $x-1$:
$3x^3+3x^2+7x-7=Q(x^2+2)+R$
Divide by x^2+2: $Q=3x+4$, $R=x-15$

$f=-\dfrac{2}{x-1}+\dfrac{3x+4}{x^2+2}+\dfrac{x-15}{(x^2+2)^2}$

38. $\dfrac{11x-28}{x^5+2x^4+2x^3+4x^2+x+2}$

▸ $\dfrac{11x-28}{x^4(x+2)+2x^2(x+2)+(x+2)}$

$=\dfrac{11x-28}{(x+2)(x^2+1)^2}$. Let $x=u-2$.

$f=\dfrac{11u-50}{u(u^2-4u+5)}=\dfrac{1}{5-4u+u^2}\cdot\dfrac{50u^{-1}}{5-4u+u^2}$

$\overset{L}{=}\dfrac{1}{5-4u+u^2}\left[-10u^{-1}+\dfrac{-29+10}{5-4u+u^2}\right]$

$\overset{L}{=}-2u^{-1}+\dfrac{-8+2u}{5-4u+u^2}+\dfrac{-29+10u}{(5-4u+u^2)^2}$

$=-\dfrac{2}{x+2}+\dfrac{2x-4}{x^2+1}+\dfrac{10x-9}{(x^2+1)^2}$

'.4 INTEGRATION OF RATIONAL FUNCTIONS AND LOGISTIC GROWTH

Partial Fractions The solutions assume that Appendix A.11 above has been studied well.

Logistic Growth If $\dfrac{dy}{dt}=ky(A-y)$ then $y=\dfrac{A}{1+(A/y_0-1)e^{-Akt}}$

Proof Let $u=\dfrac{A}{y}-1$. Then $\dfrac{du}{dt}=-\dfrac{A}{y^2}\dfrac{dy}{dt}=-Ak\left(\dfrac{A}{y}-1\right)=-Aku$ and so

$u=u_0e^{-Akt}$, $\dfrac{A}{y}-1=\left(\dfrac{A}{y_0}-1\right)e^{-Akt}$, $\dfrac{A}{y}=1+\left(\dfrac{A}{y_0}-1\right)e^{-Akt}$, $y=\dfrac{A}{1+(A/y_0-1)e^{-Akt}}$

Mass Action for Second Order Reactions: $\dfrac{dx}{dt}=k(a-x)(b-x)$

Exercises 7.4

n **Exercises 1–10**, evaluate the indefinite integral.

1. $\displaystyle\int\frac{dx}{x^2-4}=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C$

2. $\dfrac{5x-1}{x^2-1}=\dfrac{5x-1}{(x-1)(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}$; $5x-1\equiv A(x+1)+B(x-1)$.
Let $x=1$: $4=2A$, $A=2$. Let $x=-1$: $-6=-2B$, $B=3$.
$\displaystyle\int\frac{5x-1}{x^2-1}dx=2\int\frac{1}{x-1}dx+3\int\frac{1}{x+1}dx=2\ln|x-1|+3\ln|x+1|+\ln|C|=\ln\left|C(x-1)^2(x+1)^3\right|$

3. $\dfrac{4w-11}{2w^2+7w-4} = \dfrac{4w-11}{(2w-1)(w+4)} \equiv \dfrac{A}{2w-1} + \dfrac{B}{w+4}$; $4w - 11 \equiv A(w+4) + B(2w-1)$

Let $w = -4$: $-27 = -9B \Leftrightarrow B = 3$. Let $w = \frac{1}{2}$: $-9 = \frac{9}{2} A \Leftrightarrow A = -2$.

$$\int \frac{4w-11}{(2w-1)(w+4)}\,dw = -\int \frac{2\,dw}{2w-1} + 3\int \frac{dw}{w+4} = -\ln|2w-1| + 3\ln|w+4| + \ln|C| = \ln\left|\frac{C(w+4)^3}{2w-1}\right|$$

4. $\displaystyle\int \frac{4x-2}{x^3-x^2-2x}\,dx$

▹ First we factor the denominator $x^3 - x^2 - 2x = x(x+1)(x-2)$. None of the factors is repeated. We seek A, B, C such that

$$\frac{4x-2}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

Multiplying by $x(x+1)(x-2)$ we obtain

$$4x - 2 \equiv A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

Let $x = 0$: $\quad -2 = -2A$, $A = 1$

Let $x = -1$: $\quad -6 = 3B$, $B = -2$

Let $x = 2$: $\quad 6 = 6C$, $C = 1$

Therefore

$$\int \frac{4x-2}{x^3-x^2-2x}\,dx = \int \frac{dx}{x} - 2\int \frac{dx}{x+1} + \int \frac{dx}{x-2} = \ln|x| - 2\ln|x+1| + \ln|x-2| + \ln|C| = \ln\left|\frac{Cx(x-2)}{(x+1)^2}\right|$$

In Exercise 5 and 6, divide numerator by denominator.

5. $\dfrac{x^2}{x^2+x-6} = 1 - \dfrac{x-6}{(x-2)(x+3)}$. $\dfrac{x-6}{(x-2)(x+3)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x+3}$; $x - 6 \equiv A(x+3) + B(x-2)$

Let $x = 2$: $-4 = 5A$, $A = -\frac{4}{5}$. Let $x = -3$: $-9 = -5B$, $B = \frac{9}{5}$.

$$\int \frac{x^2}{x^2+x-6}\,dx = \int\left(1 + \frac{4}{5}\frac{1}{x-2} - \frac{9}{5}\frac{1}{x+3}\right)dx = x + \frac{4}{5}\ln|x-2| - \frac{9}{5}\ln|x+3| + C$$

6. $\dfrac{9t^2-26t-5}{3t^2-5t-2} = 3 - \dfrac{11t-1}{(t-2)(3t+1)}$. $\dfrac{11t-1}{(t-2)(3t+1)} \equiv \dfrac{A}{t-2} + \dfrac{B}{3t+1}$; $11t - 1 \equiv A(3t+1) + B(t-2)$

$t = 2$: $21 = 7A$, $A = 3$. $t = -\frac{1}{3}$: $-\frac{14}{3} = -\frac{7}{3}B$, $B = 2$

$$\int \frac{9t^2-26t-5}{3t^2-5t-2}\,dt = \int\left(3 - 3\frac{1}{t-2} - 2\frac{1}{3t+1}\right)dt = 3t - 3\ln|t-2| - \frac{2}{3}\ln|3t+1| + C$$

7.
$$\begin{array}{r l}
 & \overline{-(t+2)^{-2} - (t+2)^{-1}} \\
-1 + (t+2) \big) & (t+2)^{-2} \\
 & \underline{(t+2)^{-2} - (t+2)^{-1}} \\
 & \qquad (t+2)^{-1} \\
 & \qquad \underline{(t+2)^{-1} - 1} \\
 & \qquad\qquad 1
\end{array}$$

$$\int \frac{dt}{(t+2)^2(t+1)} = \int \frac{(t+2)^{-2}}{-1+(t+2)}\,dt = \int\left(-(t+2)^{-2} - \frac{1}{t+2} + \frac{1}{t+1}\right)dt$$

$$= \frac{1}{t+2} - \ln|t+2| + \ln|t+1| + C$$

8. $\displaystyle\int \frac{3x^2-x+1}{x^3-x^2}\,dx$

▹ We factor the denominator $x^3 - x^2 = x^2(x-1)$. We divide by x^2 and then by $x - 1$ as shown below.

$$\frac{3x^2-x+1}{x^2(x-1)} \stackrel{L}{=} \frac{3 - x^{-1} + x^{-2}}{x-1} = \frac{3}{x-1} - x^{-2}$$

$$\begin{array}{r l}
 & \overline{-x^{-2}\qquad\quad} \\
-1 + x \big) & x^{-2} - x^{-1}
\end{array}$$

$$\int \frac{3x^2-x+1}{x^3-x^2}\,dx = \int\left(\frac{3}{x-1} - x^{-2}\right)dx = 3\ln|x-1| + \frac{1}{x} + C$$

9. $\int \frac{dx}{x^3+3x^2} = \int \frac{dx}{x^2(x+3)} = \int \frac{x^{-2}}{3+x}dx \overset{L}{=} \int\left(\frac{1}{3}x^{-2} - \frac{1}{9}x^{-1} + \frac{1}{9}\frac{1}{x+3}\right)dx = \frac{1}{3x} - \frac{1}{9}\ln|x| + \frac{1}{9}\ln|x+3| + C$

$$\begin{array}{r|l} & \frac{1}{3}x^{-2} - \frac{1}{9}x^{-1} \\ \hline 3+x & x^{-2} \\ & \underline{x^{-2} + \frac{1}{3}x^{-1}} \\ & -\frac{1}{3}x^{-1} \\ & \underline{-\frac{1}{3}x^{-1} - \frac{1}{9}} \\ & \frac{1}{9} \end{array}$$

10. $\frac{w^2+4w-1}{w^3-w} = \frac{w^2+4w-1}{w(w-1)(w+1)} = \frac{A}{w} + \frac{B}{w-1} + \frac{C}{w+1}$; $w^2+4w-1 \equiv A(w-1)(w+1) + Bw(w+1) + Cw(w-1)$

Let $w=0$: $-1=-A$, $A=1$. Let $w=1$: $4=2B$, $B=2$. Let $w=-1$: $-4=2C$, $C=-2$

$\int \frac{w^2+4w-1}{w^3-w}dw = \int\left(\frac{1}{w} + \frac{2}{w-1} - \frac{2}{w+1}\right)dw = \ln|w| + 2\ln|w-1| - 2\ln|w+1| + \ln C = \ln\left|\frac{Cw(w-1)^2}{(w+1)^2}\right|$

11. $\frac{6x^2-2x-1}{4x^3-x} = \frac{6x^2-2x-1}{x(2x-1)(2x+1)} \equiv \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{2x+1}$

$6x^2-2x-1 \equiv A(2x-1)(2x+1) + Bx(2x+1) + Cx(2x-1)$

Let $x=\frac{1}{2}$: $-\frac{1}{2}=B$. Let $x=-\frac{1}{2}$: $\frac{3}{2}=C$. Let $x=0$: $-1=-A \Leftrightarrow A=1$.

$\int \frac{6x^2-2x-1}{x(2x-1)(2x+1)}dx = \int\frac{dx}{x} - \frac{1}{2}\int\frac{dx}{2x-1} + \frac{3}{4}\int\frac{dx}{2x+1} = \ln|x| - \frac{1}{4}\ln|2x-1| + \frac{1}{4}\ln|C| = \frac{1}{4}\ln\left|\frac{Cx^4(2x+1)^3}{2x-1}\right|$

12. $\int\frac{dx}{2x^3+x}$

▷ Method 1. Factor the denominator into linear and quadratic factors. Clear fractions and equate powers of x.

$\frac{1}{2x^3+x} = \frac{1}{x(2x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{2x^2+1}$; $1 \equiv A(2x^2+1) + (Bx+C)$; $1 \equiv (2A+B)x^2 + Cx + A$

Therefore $2A+B=0$; $C=0$; $A=1$. Hence $B=-2$.

$\int\frac{dx}{2x^3+x} = \int\frac{dx}{x} - 2\int\frac{x\,dx}{2x^2+1} = \ln|x| - \frac{1}{2}\ln|2x^2+1| + \frac{1}{2}\ln|C| = \frac{1}{2}\ln\left|\frac{Cx^2}{2x^2+1}\right|$

Method 2. $\int\frac{dx}{2x^3+x} = \int\frac{x^{-3}dx}{2+x^{-2}} = -\frac{1}{2}\ln|2+x^{-2}| + \frac{1}{2}\ln|C| = \frac{1}{2}\ln\left|\frac{Cx^2}{2x^2+1}\right|$

13. We factor the denominator $x^3+4x = x(x^2+4)$. We divide by x and then by $4+x^2$ as shown below.

$\int\frac{x+4}{x^3+4x}dx = \int\frac{x+4}{x(x^2+4)}dx = \int\frac{1+4x^{-1}}{x^2+4}dx \overset{L}{=} \int\left(x^{-1} - \frac{x}{x^2+4} + \frac{1}{x^2+4}\right)dx = \ln|x| - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{1}{2}x + C$

$$\begin{array}{r|l} & x^{-1} \\ \hline 4+x^2 & 4x^{-1}+1 \\ & \underline{4x^{-1}+x} \\ & -x+1 \end{array}$$

14. Let $u=t^2$, $du=2t\,dt$. $\int\frac{3t\,dt}{2t^4+5t^2+2} = \frac{3}{2}\int\frac{2t\,dt}{(2t^2+1)(t^2+2)} = \frac{3}{2}\int\frac{du}{(2u+1)(u+2)}$. $\frac{1}{(2u+1)(u+2)}$

$\equiv \frac{A}{2u+1} + \frac{B}{u+2}$. $1 \equiv A(u+2) + B(2u+1)$. Let $u=-\frac{1}{2}$: $1=\frac{3}{2}A$, $A=\frac{2}{3}$. Let $u=-2$: $1=-3B$, $B=-\frac{1}{3}$.

$I = \frac{3}{2}\int\left(\frac{2}{3}\cdot\frac{1}{2u+1} - \frac{1}{3}\cdot\frac{1}{u+2}\right)du = \frac{1}{2}\ln|2u+1| - \frac{1}{2}\ln|u+2| + \frac{1}{2}\ln C = \frac{1}{2}\ln\left|\frac{C(2u+1)}{u+2}\right| = \frac{1}{2}\ln\frac{C(2t^2+1)}{t^2+2}$

15. $\dfrac{1}{16x^4-1} = \dfrac{1}{(4x^2-1)(4x^2+1)} \equiv \dfrac{A}{2x-1} + \dfrac{B}{2x+1} + \dfrac{Cx+d}{4x^2+1}$

$1 \equiv A(2x+1)(4x^2+1) + B(2x-1)(4x^2+1) + (Cx+D)(4x^2-1)$. Let $x = \frac{1}{2}$: $1 = 4A$; $A = \frac{1}{4}$.

Let $x = -\frac{1}{2}$: $1 = -4B$; $B = -\frac{1}{4}$. Let $x = 0$: $1 = A - B - D = \frac{1}{4} + \frac{1}{4} - D$; $D = -\frac{1}{2}$.

Compare coefficients of x^3: $0 = 8A + 8B + 4C = 2 - 2 + 4C$; $C = 0$. $\displaystyle\int \frac{dx}{16x^4-1} =$

$$\frac{1}{4}\int\frac{dx}{2x-1} - \frac{1}{4}\int\frac{dx}{2x+1} - \frac{1}{2}\int\frac{dx}{4x^2+1} = \frac{1}{8}\ln|2x-1| - \frac{1}{8}\ln|2x+1| - \frac{1}{4}\tan^{-1}2x + C = \frac{1}{8}\ln\left|\frac{2x-1}{2x+1}\right| - \frac{1}{2}\tan^{-1}2x + C$$

16. $\displaystyle\int \frac{dx}{9x^4+x^2}$

▹ We factor the denominator $9x^4 + x^2 = x^2(9x^2+1)$. We divide by x^2 and then by $9x^2+1$ as shown below.

$$\int\frac{dx}{9x^4+x^2} = \int\frac{dx}{x^2(9x^2+1)} = \int\frac{x^{-2}}{1+9x^2}dx \stackrel{L}{=} \int\left(x^{-2} - \frac{9}{1+9x^2}\right)dx = \int\left(x^{-2} - \frac{1}{x^2+\frac{1}{9}}\right)dx = \frac{1}{x} - 3\tan^{-1}3x + C$$

$$\begin{array}{r|l} & x^{-2} \\ \hline 1+9x^2 & x^{-2} \\ & \underline{x^{-2}+9} \\ & \quad -9 \end{array}$$

17. $\dfrac{x^2+x}{x^3-x^2+x-1} = \dfrac{x^2+x}{(x-1)(x^2+1)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$x^2 + x \equiv A(x^2+1) + (Bx+C)(x-1)$. Let $x = 1$: $2 = 2A \Leftrightarrow A = 1$.

$x^2 + x \equiv (A+B)x^2 + (C-B)x + (A-C)$. $A + B = 1$ (1); $C - B = 1$ (2); $A - C = 0$ (3)

From (1) we get $B = 1 - A = 0$. From (3) we get $C = A = 1$. Therefore

$$\int\frac{(x^2+x)dx}{x^3-x^2+x-1} = \int\frac{dx}{x-1} + \int\frac{dx}{x^2+1} = \ln|x-1| + \tan^{-1}x + C$$

18. $\dfrac{2x^2+3x+2}{x^3+4x^2+6x+4} = \dfrac{2x^2+3x+2}{(x+2)(x^2+2x+2)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+2x+2}$.

$2x^2 + 3x + 2 \equiv A(x^2+2x+2) + (x+2)(Bx+C)$. $x = -2$: $4 = 2A$, $A = 2$. Substituting for A,

$2x^2 + 3x + 2 \equiv 2x^2 + 4x + 4 + (x+2)(Bx+C)$, $-x - 2 \equiv (x+2)(Bx+C)$, $Bx + C \equiv -1$.

$$\int\frac{2x^2+3x+2}{x^3+4x^2+6x+4}dx = \int\left[\frac{2}{x+2} - \frac{1}{(x+1)^2+1}\right]dx = 2\ln|x+2| - \tan^{-1}(x+1) + C$$

19. Let $u = \tan x$, $du = \sec^2 x\,dx$. Then $I = \displaystyle\int\frac{(\sec^2 x+1)\sec^2 x\,dx}{1+\tan^3 x} = \int\frac{(u^2+2)du}{1+u^3} = \int\frac{(u^2+2)du}{(u+1)(u^2-u+1)}$

$\dfrac{u^2+2}{(u+1)(u^2-u+1)} \equiv \dfrac{A}{u+1} + \dfrac{Bu+C}{u^2-u+1}$

$u^2 + 2 \equiv A(u^2-u+1) + (Bu+C)(u+1)$. Let $u = -1$: $3 = 3A \Leftrightarrow A = 1$. Substituting for A,

$u^2 + 2 \equiv u^2 - u + 1 + (Bu+C)(u+1)$, $u + 1 \equiv (Bu+C)(u+1)$, $Bu + C \equiv 1$

$$I = \int\frac{du}{u+1} + \int\frac{du}{(u-\frac{1}{2})^2+\frac{3}{4}} = \ln|u+1| + \frac{2}{\sqrt{3}}\tan^{-1}\frac{u-\frac{1}{2}}{\frac{1}{2}\sqrt{3}} + C = \ln|\tan x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right) + C$$

20. $\displaystyle\int\frac{e^{5x}\,dx}{(e^{2x}+1)^2}$

▹ Method 1. Let $u = e^x$. Then $du = e^x dx$. We then divide by u^2+1 and again by u^2+1

$$\int\frac{e^{5x}\,dx}{(e^{2x}+1)^2} = \int\frac{u^4du}{(u^2+1)^2} = \int\left[\frac{u^2-1}{u^2+1} + \frac{1}{(u^2+1)^2}\right]du = \int du - \int\frac{2\,du}{u^2+1} + \int\frac{du}{(u^2+1)^2}$$

$$= u - 2\tan^{-1}u + \int\frac{du}{(u^2+1)^2}$$

In the last integral, let $\theta = \tan^{-1}u$. Then

$$u = \tan\theta, \quad (u^2+1)^2 = \sec^4\theta, \quad du = \sec^2\theta\, d\theta$$

Thus,

$$\int \frac{du}{(u^2+1} = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \cos^2\theta\, d\theta = \frac{1}{2}\int (1+\cos 2\theta)d\theta = \tfrac{1}{2}(\theta + \tan\sin\theta) + C = \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \tfrac{1}{2}\left(\tan^{-1}u + \frac{u}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{u^2+1}}\right) + C = \tfrac{1}{2}\left(\tan^{-1} + \frac{u}{u^2+1}\right) + C$$

Substituting for the integral, and then substituting for u, we obtain

$$\int \frac{e^{5x}dx}{(e^{2x}+1)^2} = u - 2\tan^{-1}u + \frac{1}{2}\tan^{-1}u + \frac{u}{2(u^2+1)} + C = e^x - \frac{3}{2}\tan^{-1}e^u + \frac{e^x}{2(e^{2x}+1)} + C$$

Method 2. We divide by $e^{2x}+1$ and again by $e^{2x}+1$.

$$\int \frac{e^{5x}dx}{(e^{2x}+1)^2} = \int\left[\frac{e^{3x}-e^x}{e^{2x}+1} + \frac{e^x}{(e^{2x}+1)^2}\right]dx = \int\left[e^x - \frac{2e^x}{e^{2x}+1} + \frac{e^{-x}}{(e^x+e^{-x})^2}\right]dx$$

$$= \int\left[e^x - \frac{e^{2x}}{e^{2x}+1} + \frac{1}{2}\cdot\frac{(e^x+e^{-x})-(e^x-e^{-x})}{(e^x+e^{-x})^2}\right]dx$$

$$= \int\left[e^x - \frac{2e^x}{e^{2x}+1} + \frac{1}{2}\cdot\frac{1}{e^x+e^{-x}} - \frac{1}{2}\cdot\frac{e^x-e^{-x}}{(e^x+e^{-x})^2}\right]dx = \int e^x dx - \frac{3}{2}\int\frac{e^x dx}{e^{2x}+1} - \frac{1}{2}\int\frac{d(e^x+e^{-x})}{(e^x+e^{-x})^2}$$

$$= e^x - \frac{3}{2}\tan^{-1}e^x + \frac{1}{2}\cdot\frac{1}{e^x+e^{-x}} + C$$

In Exercises 21–32, find the exact value of the definite integral and check using NINT to 6 significant digits.

21. $\dfrac{x-3}{x^3+x^2} = \dfrac{x-3}{x^2(x^2+1)} = \frac{A}{2} + \frac{B}{2} + \frac{C}{2}$; $x - 3 \equiv A(x+1) + Bx(x+1) + Cx^2$;

$x - 3 \equiv (B+C)x^2 + (A+B)x + A$. Hence $\quad B + C = 0$ (1) $\quad A + B = 1$ (2) $\quad A = -3$ (3)

Solving in the order (3), (2), (1) we find $A = -3$, $B = 4$, $C = -4$.

Alternatively, by long division, $\dfrac{x-3}{x^2(x+1)} = \dfrac{-\frac{3}{x^2}+\frac{1}{x}}{1+x} = -\dfrac{3}{x^2} + \frac{4}{x} - \dfrac{4}{1+x}$

$$\int_1^2 \frac{x-3}{x^3+x^2}dx = \int_1^2\left(-\frac{3}{x^2} + \frac{4}{x} - \frac{4}{1+x}\right)dx = \frac{3}{x} + 4\ln|x| - 4\ln|x+1|\Big|_1^2 = \left(\tfrac{3}{2} + 4\ln 2 - 4\ln 3\right) - (3 - 4\ln 2)$$

$= 4\ln\frac{4}{3} - \frac{3}{2} \approx -0.349272$

22. $\dfrac{x+4}{2x^2+5x+2} = \dfrac{x+4}{(2x+1)(x+2)} = \dfrac{A}{2x+1} + \dfrac{B}{x+2}$; $x + 4 \equiv A(x+2) + B(2x+1)$. Let $x = -\frac{1}{2}$: $\frac{7}{2} = \frac{3}{2}A$, $A = \frac{7}{3}$.

Let $x = -2$: $2 = -3B$, $B = -\frac{2}{3}$. $\displaystyle\int_0^4 \frac{x+4}{2x^2+5x+2}dx = \frac{7}{3}\int_0^4\frac{dx}{2x+1} - \frac{2}{3}\int_0^4\frac{dx}{x+2} = \frac{7}{6}\ln(2x+1)\Big]_0^4 - \frac{2}{3}\ln(x+2)\Big]_0^4$

$= \frac{7}{6}\ln 9 - \frac{2}{3}\ln 3 = \frac{7}{3}\ln 3 - \frac{2}{3}\ln 3 = \frac{5}{3}\ln 3 \approx 1.831020$

23. $\dfrac{x^2-4x+3}{x(x+1)^2} \equiv \dfrac{A}{x} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+1}$

$x^2 - 4x + 3 \equiv A(x+1)^2 + Bx + Cx(x+1) \equiv A(x^2+2x+1) + Bx + C(x^2+x)$

$x^2 - 4x + 3 \equiv (A+C)x^2 + (2A+B+C)x + A$. Thus $A + C = 1$ (1) $\quad 2A + B + C = -4$ (2) $\quad A = 3$ (3).

Solving in the order (3), (1), (2) we find $A = 3$, $C = -2$, $B = -8$.

Alternatively, by division, $\dfrac{x^2-4x+3}{x(x+1)^2} = \dfrac{\frac{3}{x}-4+x}{(1+x)^2} = \dfrac{\frac{3}{x}+1-\frac{8}{1+x}}{1+x} = \dfrac{3}{x} - \dfrac{2}{1+x} - \dfrac{8}{(1+x)^2}$.

$$\int_1^3 \frac{x^2-4x+3}{x(x+1)^2}dx = \int_1^3\left[\frac{3}{x} - \frac{8}{(x+1)^2} - \frac{2}{x+1}\right]dx = 3\ln|x| + \frac{8}{x+1} - 2\ln|x+1|\Big|_1^3$$

$= (3\ln 3 + 2 - 2\ln 4) - (4 - 2\ln 2) = \ln 27 - \ln 4 - 2 = \ln\frac{27}{4} - 2 \approx -0.090575$

24. $\int_1^4 \frac{2x^2+13x+18}{x^3+6x^2+9x}dx$

▷ $x^3+6x^2+9x = x(x+3)^2$. Thus we find constants A, B, and C such that

$$\frac{2x^2+13x+18}{x^3+6x^2+9x} = \frac{A}{x} + \frac{B}{(x+3)^2} + \frac{C}{x+3}$$

Let $x=0$: $18 = 9A$, $A=2$

Let $x=-3$: $-3=-3B$, $B=1$

Let $x=-2$: $0 = A-2B-2C = 2-2-2C$, $C=0$

$$\int_1^4 \frac{2x^2+13x+18}{x^3+6x^2+9x}dx = \int_1^4 \frac{2}{x}dx + \int_1^4 \frac{dx}{(x+3)^2} = 2\ln x\Big]_1^4 - \frac{1}{x+3}\Big]_1^4 = 2(\ln 4 - \ln 1) - (\tfrac{1}{7}-\tfrac{1}{4})$$
$$= 4\ln 2 + \tfrac{3}{28} \approx 2.87973$$

25. $\frac{4+5x^2}{x^3+4x} = \frac{4+5x^2}{x(x^2+4)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+4}$; $4+5x^2 \equiv A(x^2+4)+(Bx+C)x$; $4+5x^2 \equiv (A+B)x^2+Cx+4A$

Therefore $A+B=5$; $C=0$; $4A=4$. Hence $A=1$, $B=4$, and $C=0$. Thus

$$\int_1^4 \frac{(4+5x^2)dx}{x^3+4x} = \int_1^4 \left(\frac{1}{x}+\frac{4}{x^2+4}\right)dx = \ln|x| + 2\ln|x^2+4|\Big|_1^4 = (\ln 4 + 2\ln 20) - 2\ln 5 = 6\ln 2 \approx 4.15888$$

26. $x^3+2x^2+x+2 = x^2(x+2)+1(x+2) = (x+2)(x^2+1)$. $\frac{x}{(x+2)(x^2+1)} \equiv \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$

$x \equiv A(x^2+1)+(Bx+C)(x+2)$. Let $x=-2$: $-2=5A$, $A=-\frac{2}{5}$.

$x \equiv -\frac{2}{5}(x^2+1)+(Bx+C)$, $\frac{1}{5}(2x^2+5x+2) \equiv (Bx+C)(x+2)$, $Bx+C \equiv \frac{1}{5}(2x+1)$

$$\int_0^1 \frac{x\,dx}{x^3+2x^2+x+2} = -\frac{2}{5}\int_0^1 \frac{dx}{x+2} + \frac{1}{5}\int_0^1 \frac{2x\,dx}{x^2+1} + \frac{1}{5}\int_0^1 \frac{dx}{x^2+1} = -\tfrac{2}{5}\ln(x+2)\Big]_0^1 + \tfrac{1}{5}\ln(x^2+1)\Big]_0^1 + \tfrac{1}{5}\tan^{-1}x\Big]_0^1$$
$$= -\tfrac{2}{5}\ln\tfrac{3}{2} + \tfrac{1}{5}\ln 2 + \tfrac{1}{5}\cdot\tfrac{1}{4}\pi = -\tfrac{2}{5}\ln 3 + \tfrac{3}{5}\ln 2 + \tfrac{1}{20}\pi \approx 0.133523$$

27. $\frac{5x^2-3x+18}{9x-x^3} = \frac{5x^2-3x+18}{x(3-x)(3+x)} \equiv \frac{A}{x} + \frac{B}{3-x} + \frac{C}{3+x}$;

$5x^2-3x+18 \equiv A(3-x)(3+x)+Bx(3+x)+Cx(3-x)$

Let $x=0$: $18=9A \Leftrightarrow A=2$. Let $x=3$: $54=18B \Leftrightarrow B=3$. Let $x=-3$: $72=-18C \Leftrightarrow C=-4$.

$$\int_1^2 \frac{5x^2-3x+18}{9x-x^3}dx = \int_1^2 \left(\frac{2}{x}+\frac{3}{3-x}-\frac{4}{3+x}\right)dx = 2\ln|x| - 3\ln|3-x| - 4\ln|3+x|\Big|_1^2$$
$$= (2\ln 2 - 4\ln 5) - (-3\ln 2 - 4\ln 4) = 13\ln 2 - 4\ln 5 \approx 2.57316$$

28. $\int_3^4 \frac{5x^3-4x}{x^4-16}dx$

▷ Method 1. $\frac{5x^3-4x}{x^4-16} = \frac{5x^3-4x}{(x-2)(x+2)(x^2+4)} \equiv \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$

$5x^3-4x \equiv A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x^2-4)$

Let $A=2$: $32=32A \Leftrightarrow A=1$. Let $x=-2$: $-32=-32B \Leftrightarrow B=1$.

$5x^3-4x \equiv A(x^3+2x^2+4x+8)+B(x^3-2x^2+4x-8)+Cx^3+Dx^2-4Cx-4D$

$5x^3-4x \equiv (A+B+C)x^3+(2A-2B+D)x^2+(4A+4B-4C)x+(8A-8B-4D)$

Therefore $A+B+C=5$; $2A-2B+D=0$; $4A+4B-4C=-4$; $8A-8B-4D=0$.

Solving the first two of these we get $C=3$ and $D=0$. Hence

$$\int_3^4 \frac{(5x^3-4x)dx}{x^4-16} = \int_3^4 \frac{dx}{x-2} + \int_3^4 \frac{dx}{x+2} + 3\int_0^4 \frac{x\,dx}{x^2+4} = \ln|x-2| + \ln|x+2| + \frac{3}{2}\ln|x^4+4|\Big|_3^4$$
$$= \left(\ln 2 + \ln 6 + \frac{3}{2}\ln 20\right) - \left(\ln 5 + \frac{3}{2}\ln 13\right) = \ln\frac{2\cdot 6}{5} + \frac{3}{2}\ln\frac{20}{13} = \ln\frac{12}{5} + \frac{3}{2}\ln\frac{20}{13} \approx 0.423031$$

Method 2. $\int_3^4 \frac{(5x^3-4x)dx}{x^4-16} = \frac{5}{4}\int_3^4 \frac{4x^3\,dx}{x^4-16} - 2\int_3^4 \frac{2x\,dx}{(x^2)^2-16} = \frac{5}{4}\ln|x^4-16| - \frac{1}{4}\ln\frac{x^2-4}{x^2+4}\Big]_3^4$

$= \left(\frac{5}{4}\ln 48 - \frac{1}{4}\ln\frac{3}{4}\right) - \left(\frac{5}{4}\ln 65 - \frac{1}{4}\ln\frac{5}{13}\right) = \ln\frac{12}{5} + \frac{3}{2}\ln\frac{20}{13}$

29. $\frac{x^2+3x+3}{x^3+x^2+x+1} = \frac{x^2+3x+3}{(x+1)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$x^2+3x+3 \equiv A(x^2+1) + (Bx+C)(x+1)$. Let $x = -1$: $1 = 2A \Leftrightarrow A = \frac{1}{2}$.

$x^2+3x+3 \equiv \frac{1}{2}(x^2+1) + (Bx+C)(x+1)$; $\frac{1}{2}(x^2+6x+5) \equiv (Bx+C)(x+1)$; $Bx+C \equiv \frac{1}{2}(x+5)$

$\int_0^1 \frac{(x^2+3x+3)dx}{x^3+x^2+x+1} = \frac{1}{2}\int_0^1 \frac{dx}{x+1} + \frac{1}{2}\int_0^1 \frac{x\,dx}{x^2+1} + \frac{5}{2}\int_0^1 \frac{dx}{x^2+1} = \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| + \frac{5}{2}\tan^{-1}x\Big]_0^1$

$= \left(\frac{1}{2}\ln 2 + \frac{1}{4}\ln 2 + \frac{5}{2}\tan^{-1}\right) - 0 = \frac{3}{4}\ln 2 + \frac{5}{2}\cdot\frac{\pi}{4} = \frac{3}{4}\ln 2 + \frac{5}{8}\pi \approx 2.483356$

30. $\frac{1+t}{1-t^3} = \frac{1+t}{(1-t)(1+t+t^2)} = \frac{A}{1-t} + \frac{Bx+C}{t^2+t+1}$. $1+t \equiv A(t^2+t+1) + (Bt+C)(1-t)$. $t = 1$: $2 = 3A$, $A = \frac{2}{3}$

$1+t \equiv \frac{2}{3}(t^2+t+1)$, $-\frac{1}{3}(2t^2-t-1) \equiv (Bt+C)(1-t)$, $Bt+C \equiv \frac{1}{3}(2t+1)$. $\int_0^{.5} \frac{1+t}{1-t^3}dt =$

$\frac{2}{3}\int_0^{.5} \frac{dt}{1-t} + \frac{1}{3}\int_0^{.5} \frac{(2t+1)dt}{t^2+t+1} = -\frac{2}{3}\ln(1-t)\Big]_0^{1/2} + \frac{1}{3}\ln(t^2+t+1)\Big]_0^{1/2} = -\frac{2}{3}\ln\frac{1}{2} + \frac{1}{3}\ln\frac{7}{4} = \frac{1}{3}\ln 7 \approx 0.648637$

31. $\int_{\ln 2}^{\ln 3} \frac{12\,dt}{e^{2t}+16} = \frac{12}{32}\int_{\ln 2}^{\ln 3} \frac{32e^{-2t}\,dt}{1+16e^{-2t}} = -\frac{3}{8}\ln(1+16e^{-2t})\Big]_{\ln 2}^{\ln 3} = -\frac{3}{8}\left[\ln\left(1+\frac{16}{9}\right) - \ln(1+4)\right] = \frac{3}{8}\ln\frac{9}{5} \approx 0.22042$

32. $\int_{\pi/6}^{\pi/2} \frac{\cos x\,dx}{\sin x + \sin^3 x}$

- Let $u = \sin x$. Then $du = \cos x\,dx$. Using long division, we obtain

$\int_{x=\pi/6}^{\pi/2} \frac{\cos x\,dx}{\sin x + \sin^3 x} = \int_{u=1/2}^{1} \frac{du}{u(u^2+1)} = \int_{1/2}^{1} \frac{u^{-1}}{1+u^2}du = \int_{1/2}^{1}\left(u^{-1} - \frac{u}{1+u^2}\right)du = \ln u\Big]_{1/2}^{1} - \frac{1}{2}\ln(1+u^2)\Big]_{1/2}^{1}$

$= \ln 2 - \frac{1}{2}(\ln 2 - \ln\frac{5}{4}) = \frac{1}{2}\ln\frac{5}{2} \approx 0.458145$

33 and 37. The region is bounded by $y = \frac{x-1}{x^2-5x+6} = \frac{x-1}{(x-3)(x-2)}$, the x axis, $x = 4$, and $x = 6$.

We shall use the identities $\frac{1}{x-3} - \frac{1}{x-2} \equiv \frac{1}{(x-3)(x-2)}$ and $\frac{3}{x-3} - \frac{2}{x-2} \equiv \frac{x}{(x-3)(x-2)}$.

$A = \int_4^6 \frac{x-1}{x^2-5x+6}\,dx = \int_4^6 \left[\frac{x}{(x-3)(x-2)} - \frac{1}{(x-3)(x-2)}\right]dx = \int_4^6 \left[\left(\frac{3}{x-3} - \frac{2}{x-2}\right) - \left(\frac{1}{x-3} - \frac{1}{x-2}\right)\right]dx$

$= \int_4^6 \left(\frac{2}{x-3} - \frac{1}{x-2}\right)dx = 2\ln|x-3| - \ln|x-2|\Big|_4^6 = (2\ln 3 - \ln 4) - (-\ln 2) = 2\ln 3 - \ln 2$

$M_x = \frac{1}{2}\int_4^6 \left(\frac{x-1}{x^2-5x+6}\right)^2 dx = \frac{1}{2}\int_4^6 \left(\frac{2}{x-3} - \frac{1}{x-2}\right)^2 = \frac{1}{2}\int_4^6 \left[\frac{4}{(x-3)^2} - \frac{4}{(x-3)(x-2)} + \frac{1}{(x-2)^2}\right]dx$

$= \frac{1}{2}\int_4^6 \left[\frac{4}{(x-3)^2} - \frac{4}{x-3} + \frac{4}{x-2} + \frac{1}{(x-2)^2}\right]dx = \left[\frac{-2}{x-3} - 2\ln|x-3| + 2\ln|x-2| - \frac{\frac{1}{2}}{x-2}\right]_b^6$

$= (-\frac{2}{3} - 2\ln 3 + 2\ln 4 - \frac{1}{8}) - (-2 + 2\ln 2 - \frac{1}{4}) = \frac{35}{24} + 2\ln 2 - 2\ln 3$

$M_y = \int_4^6 \frac{x(x-1)}{x^2-5x+6}\,dx = \int_4^6 \left[1 + \frac{4x-6}{(x-3)(x-2)}\right]dx = \int_4^6 \left[1 + 4\left(\frac{3}{x-3} - \frac{2}{x-2}\right) - 6\left(\frac{1}{x-3} - \frac{1}{x-2}\right)\right]dx$

$= \int_4^6 \left(1 + \frac{6}{x-3} - \frac{2}{x-2}\right)dx = x + 6\ln|x-3| - 2\ln|x-2|\Big|_4^6 = (6 + 6\ln 3 - 2\ln 4) - (4 - 2\ln 2)$

$= 2 - 2\ln 2 + 6\ln 3$

- The centroid is at the point $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{M_y}{A} = \frac{2 - 2\ln 2 + 6\ln 3}{2\ln 3 - \ln 2}$ and $\bar{y} = \frac{M_x}{A} = \frac{\frac{35}{24} + 2\ln 2 - 2\ln 3}{2\ln 3 - \ln 2}$.

34 and 38. The region is in the first quadrant bounded by $y = \dfrac{4-x}{(x+2)^2}$. Let $u = x+2$, $x = u-2$, $du = dx$

$$A = \int_{x=0}^{4} \frac{4-x}{(x+2)^2}dx = \int_{u=2}^{6} \frac{6-u}{u^2}du = \int_2^6\left(\frac{6}{u^2} - \frac{1}{u}\right)du = -\frac{6}{u} - \ln u = 2 - \ln 3$$

$$M_y = \int_{x=0}^{4} x \cdot \frac{4-x}{(x+2)^2}dx = \int_{u=2}^{6} \frac{(u-2)(6-u)}{u^2} = \int_2^6 \frac{-u^2+8u-12}{u^2}du = \int_2^6\left(-1 + \frac{8}{u} - \frac{12}{u^2}\right)du$$

$$= -u + 8\ln u + \frac{12}{u}\Big]_2^6 = -4 + 8\ln\tfrac{6}{2} - 4 = 8\ln 3 - 8.\ \bar{x} = \frac{8\ln 3 - 8}{2 - \ln 3}$$

$$M_x = \frac{1}{2}\int_{x=0}^{4} \frac{(4-x)^2}{(x+2)^4}dx = \frac{1}{2}\int_{u=2}^{6} \frac{(6-u)^2}{u^4}\,du = \frac{1}{2}\int_2^6 \frac{u^2-12u+36}{u^4}\,du = \frac{1}{2}\int_2^6 (u^{-2} - 12u^{-3} + 36u^{-4})du$$

$$= \frac{1}{2}\left[-\frac{1}{u} + 6\frac{1}{u^2} - 12\frac{1}{u^3}\right]_2^6 = \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{6}\right) + 6\left(\frac{1}{36} - \frac{1}{4}\right) - 12\left(\frac{1}{216} - \frac{1}{8}\right)\right] = \frac{2}{9}.\ \bar{y} = \frac{2}{9(2 - \ln 3)}$$

35. An element of volume is a cylindrical shell, centered on the y axis, of mean radius m_i and altitude $\dfrac{m_i - 1}{m_i^2 - 5m_i + 6}$, $m_i \in [4, 6]$. If V cubic units is the volume of the solid,

$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 2\pi m_i \frac{m_i - 1}{m_i^2 - 5m_i + 6}\Delta_i x = 2\pi\int_4^6 \frac{x^2 - x}{x^2 - 5x + 6}\,dx = 2\pi\int_4^6\left[1 + \frac{4x-6}{(x-3)(x-2)}\right]dx$$

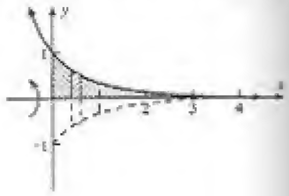
$\dfrac{4x-6}{(x-3)(x-2)} \equiv \dfrac{A}{x-3} + \dfrac{B}{x-2}$; $4x - 6 \equiv A(x-2) + B(x-3)$. Let $x = 2$: $2 = -B \Leftrightarrow B = -2$. Let $x = 3$: $6 = A$.

$$V = 2\pi\int_4^6\left(1 + \frac{6}{x-3} - \frac{2}{x-2}\right)dx = 2\pi\Big[x + 6\ln|x-3| - 2\ln|x-2|\,\Big]_4^6$$

$$= 2\pi[(6 + 6\ln 3 - 2\ln 4) - (4 - 2\ln 2) = 2\pi(2 + 6\ln 3 - 2\ln 2)$$

36. Find the volume of the solid of revolution if the region in the first quadrant bounded by the curve $(x+2)^2 y = 4 - x$ is revolved about the x axis.

▷ Solving the given equation for y, we get $y = f(x) = \dfrac{4-x}{(x+2)^2}$. The figure shows the region and a plane section of the solid of revolution. The element of volume is a circular disk with thickness $\Delta_i x$ units and radius $f(m_i)$ units. If V cubic units is the volume of the solid of revolution, then



$$V = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi[f(m_i)]^2\Delta_i x = \pi\int_{x=0}^{4} \frac{(4-x)^2}{(x+2)^4}dx$$

We let $y = x+2$, $x = y-2$, $4 - x = 6 - y$ and $dx = dy$. Therefore,

$$V = \pi\int_{y=2}^{6} \frac{(6-y)^2}{y^4}\,dy = \pi\int_2^6 \frac{y^2 - 12y + 36}{y^4}\,dy$$

$$= \int_2^6 (y^{-2} - 12y^{-3} + 36y^{-4})\,dy = \pi\left[-\frac{1}{y} + 6\frac{1}{y^2} - 12\frac{1}{y^3}\right]_2^6$$

$$= \pi\left[\left(\frac{1}{2} - \frac{1}{6}\right) + 6\left(\frac{1}{36} - \frac{1}{4}\right) - 12\left(\frac{1}{216} - \frac{1}{8}\right)\right] = \frac{4}{9}\pi$$

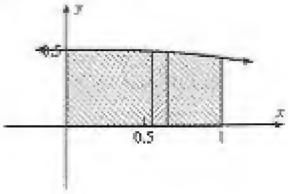
▪ The volume of the solid of revolution is $\frac{4}{9}\pi$ cubic units.

39. The region is bounded by $y = \dfrac{x^3}{(x^2+1)^3}$, the x axis, the y axis, and $x = 1$. Let $x^2 + 1 = u$. Then $x^2 = u - 1$ and $2x\,dx = du$. If A square units is the area of the region, $A = \displaystyle\int_0^1 \frac{x^3dx}{(x^2+1)^3} = \int_0^1 \frac{x^2(x\,dx)}{(x^2+1)^3}$

$$= \frac{1}{2}\int_1^2 \frac{(u-1)du}{u^3} = \frac{1}{2}\int_1^2\left(\frac{1}{u^2} - \frac{1}{u^3}\right)du = \frac{1}{2}\left[-\frac{1}{u} + \frac{1}{2u^2}\right]_1^2 = \frac{1}{2}\left[\left(-\frac{1}{2} + \frac{1}{8}\right) - \left(-1 + \frac{1}{2}\right)\right] = \frac{1}{16}$$

40. Find the area of the region bounded by the curve $y(x^3+8)=4$, the x axis, the y axis, and the line $x=1$.

▸ The figure shows the region. Let f be the function defined by $f(x)=\dfrac{4}{x^3+8}$. The element of area is a rectangle with width $\Delta_i x$ units and altitude $f(m_i)$ units. Thus



$$R=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(m_i)\Delta_i x=\int_0^1 \frac{4\,dx}{x^3+8}$$

We let

$$\frac{4}{x^3+8}=\frac{4}{(x+2)(x^2-2x+4)}\equiv\frac{A}{x+2}+\frac{Bx+C}{x^2-2x+4}$$

$$4\equiv A(x^2-2x+4)+(Bx+C)(x+2)$$

Let $x=-2$: $4=12A$, $A=\frac{1}{3}$. Substitute for A.

$$4\equiv\tfrac{1}{3}(x^2-2x+4)+(Bx+C)(x+2)$$

$$-\tfrac{1}{3}(x^2-2x-8)\equiv(Bx+C)(x+2)$$

$$-\tfrac{1}{3}(x-4)\equiv Bx+C$$

Therefore,

$$R=\frac{1}{3}\int_0^1\left(\frac{1}{x+2}-\frac{x-4}{x^2-2x+4}\right)dx=\frac{1}{3}\int_0^1\left[\frac{1}{x+2}-\frac{x-1}{x^2-2x+4}+\frac{3}{(x-1)^2+3}\right]dx$$

$$=\frac{1}{3}\left[\ln(x+2)-\frac{1}{2}\ln(x^2-2x+4)+\sqrt{3}\tan^{-1}\frac{x-1}{\sqrt{3}}\right]_0^1=\tfrac{1}{3}\left[\ln\tfrac{3}{2}-\tfrac{1}{2}\ln\tfrac{3}{4}+\sqrt{3}(\tfrac{1}{6}\pi)\right]=\tfrac{1}{6}\ln 3+\tfrac{1}{18}\sqrt{3}\pi$$

▪ The area is $\frac{1}{6}\ln 3+\frac{1}{18}\sqrt{3}\pi$ square units.

41 and 42. The region is bounded by $y=\dfrac{4}{x^3+8}$, the x axis, the y axis, and $x=1$. An element of volume is a circular cylinder centered on the y axis, $x\in[0,1]$, of mean radius m_i and altitude $\dfrac{4}{{m_i}^3+8}$. If V cubic units is the volume of the solid of revolution,

$$V=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}2\pi m_i\cdot\frac{4}{{m_i}^3+8}\Delta_i x=8\pi\int_0^1\frac{x\,dx}{x^3+8}\text{ and } M_y=8\int_0^1\frac{x\,dx}{x^3+8}$$

$$\frac{x}{x^3+8}=\frac{x}{(x+2)(x^2-2x+4)}\equiv\frac{A}{x+2}+\frac{Bx+C}{x^2-2x+4}$$

$x\equiv A(x^2-2x+4)+(Bx+C)(x+2)$. Let $x=-2$: $-2=12A\Leftrightarrow A=-\frac{1}{6}$.

$x\equiv-\frac{1}{6}(x^2-2x+4)+(Bx+C)(x+2)$; $\frac{1}{6}(x^2+4x+4)\equiv(Bx+C)(x+2)$; $Bx+C\equiv\frac{1}{6}(x+2)$. V =

$$\frac{8\pi}{6}\left[-\int_0^1\frac{dx}{x+2}+\int_0^1\frac{(x-1)dx}{x^2-2x+4}+3\int_0^1\frac{dx}{(x-1)^2+3}\right]=\frac{4}{3}\pi\left[-\ln|x+2|+\frac{1}{2}\ln|x^2-2x+4|+\sqrt{3}\tan^{-1}\frac{x-1}{\sqrt{3}}\right]_0^1$$

$$=\tfrac{4}{3}\pi(\ln\tfrac{2}{3}+\tfrac{1}{2}\ln\tfrac{3}{4}+\tfrac{1}{6}\pi\sqrt{3})=\tfrac{2}{9}\sqrt{3}\pi^2-\tfrac{2}{3}\pi\ln 3.\ M_y=\tfrac{2}{9}\sqrt{3}\pi-\tfrac{2}{3}\ln 3;\ \bar{x}=\frac{\frac{2}{9}\sqrt{3}\pi-\frac{2}{3}\ln 3}{\frac{1}{6}\ln 3+\frac{1}{18}\sqrt{3}\pi}=\frac{2\pi-2\sqrt{3}\ln 3}{\pi+\sqrt{3}\ln 3}$$

43. (a) $\displaystyle\int\frac{(x^2-4x+6)dx}{x^3-6x^2+18x}=\int\frac{\frac{1}{3}(3x^2-12x+18)dx}{x^3-6x^2+18x}=\ln|x^3-6x^2+18x|+C$

(b) $\displaystyle\int\frac{3x+1}{(x+2)^4}dx=\int\left[\frac{3}{(x+2)^3}-\frac{5}{(x+2)^4}\right]dx=-\frac{3}{2(x+2)^2}+\frac{5}{3(x+2)^3}+C$

44. Show that the graph of the function f defined by $f(t)=\dfrac{A}{1+Be^{-Akt}}$ has a point of inflection at $t=\dfrac{\ln B}{Ak}$.

▸ Method 1. Let $u=Akt$. Because $D_t u=Ak$, then

$$D_t f(u)=(Ak)D_u[A(1+Be^{-u})^{-1}]=A^2k(1+Be^{-u})^{-2}(-Be^{-u})=-A^2Bke^{-u}(1+Be^{-u})^{-2}$$

Applying the product rule,

$$D_t^{\,2}f(u)=-A^3Bk^2[-e^{-u}(1+Be^{-u})^{-2}-2e^{-u}(1+Be^{-u})^{-3}(-Be^{-u})]$$

$$=A^3Bk^2e^{-u}(1+Be^{-u})^{-3}[1+Be^{-u}-2Be^{-u}]=A^3Bk^2e^{-u}(1+Be^{-u})^{-3}(1-Be^{-u})$$

which changes sign when $Be^{-u}=1$, $e^u=B$, $Akt=u=\ln B$, $t=\ln B/(Ak)$.

Method 2. The inflection point can be found from the inverse function. From (6) and partial fractions have

$\frac{dy}{dt} = \frac{1}{A}\left(\frac{1}{y} + \frac{1}{A-y}\right)$ and so $\frac{d^2t}{dy^2} = \frac{1}{A}\left[-\frac{1}{y^2} + \frac{1}{(A-y)^2}\right]$ which changes sign when $y^2 = (A-y)^2$, $y = A - y$, $y = \frac{1}{2}A$, that is, when $Be^{-Akt} = 1$, $e^{Akt} = B$, $Akt = \ln B$, $t = \ln b/(Ak)$

45. After t min, y people have heard the rumor where $y < 5000$. We have a table of boundary conditions.

t	0	10	15	20
y	1	144	y_{15}	y_{20}

$\frac{dy}{dt} = ky(5000-y)$; $\int k\,dt = \int \frac{dy}{y(5000-y)} = \frac{1}{5000}\int\left(\frac{1}{y} + \frac{1}{5000-y}\right)dy$; $kt = \frac{1}{5000}\ln\frac{Cy}{5000-y}$

With $t = 0$ and $y = 1$ we get $0 = \frac{1}{5000}\ln\frac{C}{4999}$; $C = 4999$. Thus $5000kt = \ln\frac{4999y}{5000-y}$.

With $t = 10$ and $y = 144$ we get $(5000k)10 = \ln\frac{4999\cdot 144}{5000-144} = 5$; $5000k = 0.5$.

(a) $0.5t = \ln\frac{4999y}{5000-y}$; $e^{-0.5t} = \frac{5000-y}{4999y}$; $4999e^{-0.5t}y = 5000 - y$, $y = \frac{5000}{1+4999e^{-0.5t}}$

(b) When $t = 15$, $y_{15} = \frac{5000}{1+4999e^{-7.5}} = 1328$. (c) When $t = 20$, $y_{20} = \frac{5000}{1+4999e^{-10}} = 4075$.

(d) Because $\lim_{t\to\infty}\frac{5000}{1+4999e^{-0.5t}} = 5000$, everybody will eventually hear the rumor.

46. From Ex.1.3.26 we have $\frac{dy}{dt} = \frac{1}{16,650,000}z(1,000,000 - z)$ and $y_0 = 500$. Then $B = \frac{A}{y_0} - 1 = \frac{1,000,000}{500} - 1 = 1999$ and $Ak = \frac{1,000,000}{16,650,000} = \frac{20}{333}$. Thus, (a) $y = \frac{1,000,000}{1+1999e^{-20t/333}}$ (b) $y(30) = 3023$ (c) $y(60) = 18,043$

(d) $y(90) = 100,201$ (e) $y(120) = 402,948$ (f) $y(150) = 803,546$ (g) $y(390) = 999,999.9$.

(h) $500,000 = \frac{1,000,000}{1+1999e^{-20t/333}}$, $1 + 1999e^{-20t/333} = 2$, $e^{20t/333} = 1999$, $\frac{20t}{333} = \ln 1999$, $t = \frac{\ln 1999}{\frac{20}{333}} = 126.5$

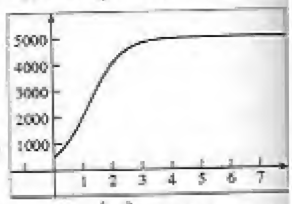
47. From Ex. 1.3.27 we have $\frac{dy}{dt} = \frac{9}{490,000}z(5000 - z)$ and $y_0 = 20$. Then $B = \frac{A}{y_0} - 1 = \frac{5000}{20} - 1 = 249$ and $Ak = \frac{9\cdot 5000}{490,000} = \frac{9}{98}$. Thus, (a) $y = \frac{5000}{1+249e^{-9t/98}}$ (b) $y(10) = 50$ (c) $y(20) = 123$ (d) $y(30) = 297$

(e) $y(60) = 2491$ (f) $y(180) = 4999.9 \approx 5000$ (g) $2500 = \frac{5000}{1+249e^{-9t/98}}$, $1 + 249e^{-9t/98} = 2$; $e^{9t/98} = 249$; $\frac{9t}{98} = \ln 249$; $t = \frac{\ln 249}{9/98} = 60.0$ days

48. At 8AM in a town of 5000 people, 500 residents heard a radio announcement about a local political scandal. The rate of growth of the spread of information about the scandal was jointly proportional to the number of people who had heard it and the number of people who had not heard it. (a) If at 9AM 2000 residents had heard about the scandal, find a mathematical model describing the spread of information. (b) Plot the graph of the mathematical model on your graphics calculator. Estimate from the graph (c) how many residents had heard about the scandal at 10AM, and (d) at what time half the population had heard about it. Confirm your estimates analytically. (e) Show that by 3PM the entire population had heard about the scandal.

▷ (a) If y resident have heard the story t hours after 8AM, then $dy/dt = ky(5000-y)$ and $y_0 = 500$.

$B = \frac{A}{y_0} - 1 = \frac{5000}{500} - 1 = 9$ and so $y = \frac{5000}{1+9e^{-Akt}}$.



At 9AM, $t = 1$ and $y = 2000$. Thus

$2000 = \frac{5000}{1+9e^{-Ak}}$; $\quad 1 + 9e^{-Ak} = \frac{5}{2}$; $9e^{-Ak} = \frac{3}{2}$; $\quad e^{-Ak} = \frac{1}{6}$; $y = \frac{5000}{1+9(\frac{1}{6})^t}$

(b) A plot is shown at the right. (c) At 10AM, $t = 2$ and $y = \frac{5000}{1+9(\frac{1}{6})^2} = 4000$

(d) We solve for t:

$2500 = \frac{5000}{1+9(\frac{1}{6})^t}$; $\quad 1 + 9(\frac{1}{6})^t = 2$; $\quad (\frac{1}{6})^t = \frac{1}{9}$; $\quad 6^t = 9$; $\quad t\ln 6 = \ln 9$; $\quad t = \frac{\ln 9}{\ln 6} \approx 1.226 = 1{:}14$

By 10:14, half the town had heard. (e) At 3PM $t = 7$ and $y(7) = \frac{5000}{1+9(\frac{1}{6})^7} \approx 4999.8$, that is the whole town.

49. Choose $A = 1$. Then $y_0 = .2$ and $B = \frac{1}{.2} - 1 = 4$ and so $y = \frac{1}{1+4e^{-Akt}}$. When $t = 1$, $y = .5$. Thus

$.5 = \frac{1}{1+4e^{-Ak}}$; $1 + 4e^{-Ak} = 2$; $e^{-Ak} = \frac{1}{4}$ and so $y = \frac{1}{1+4(\frac{1}{4})^t}$. Then $.8 = \frac{1}{1+4(\frac{1}{4})^t}$; $1 + 4(\frac{1}{4})^t = \frac{5}{4}$; $4(\frac{1}{4})^t = \frac{1}{4}$; $t = 2$

50. After t weeks y percent of the susceptibles have been infected. We have a table of boundary conditions.

t	0	3	6
y	10	25	y_6

$\frac{dy}{dt} = ky(100-y)$; $\int k\,dt = \int \frac{dy}{y(100-y)} = \frac{1}{100}\int\left(\frac{1}{y} + \frac{1}{100-y}\right)dy$; $100kt = \ln\frac{Cy}{100-y}$

With $t = 0$ and $y = 10$ we get $0 = \ln\frac{10C}{90}$; $C = 9$. Thus $100kt = \ln\frac{9y}{100-y}$.

With $t = 3$ and $y = 25$ we get $(100k)3 = \ln\frac{225}{75} = \ln 3$; $100k = \frac{1}{3}\ln 3$.

Thus $\frac{t}{3}\ln 3 = \ln\frac{9y}{100-y}$; $3^{-t/3} = \frac{100-y}{9y}$; $9y\cdot 3^{-t/3} = 100 - y$; $y = \frac{100}{1 + 9\cdot 3^{-t/3}} = \frac{100}{1+3^{2-t/3}}$.

When $t = 6$, $y = \frac{100}{1+1} = 50$. After 6 weeks 50 of the susceptibles have been infected.

51. If x grams of substance C are present at t min, then $\frac{dx}{dt} = k(5-x)(4-x)$. We have a table of boundary conditions.

t	0	5	10
x	0	1	x_{10}

$\int\frac{dx}{(5-x)(4-x)} = k\int dt$. $\frac{1}{(5-x)(4-x)} \equiv \frac{D}{5-x} + \frac{E}{4-x}$; $1 \equiv D(4-x) + E(5-x)$

Let $x = 5$: $1 = -D \Leftrightarrow D = -1$. Let $x = 4$: $1 = E$. Therefore

$-\int\frac{dx}{5-x} + \int\frac{dx}{4-x} = kt$; $\ln|5-x| - \ln|4-x| + \ln|F| = kt$; $\ln\left|\frac{4-x}{F(5-x)}\right| = -kt$; $\frac{4-x}{5-x} = Fe^{-kt}$

Because $x = 0$ when $t = 0$, $\frac{4}{5} = F$. Thus $\frac{4-x}{5-x} = \frac{4}{5}e^{-kt}$.

Because $x = 1$ when $t = 5$, $\frac{3}{4} = \frac{4}{5}e^{-5k}$; $\frac{15}{16} = e^{-5k}$. Hence $\frac{4-x}{5-x} = \left(\frac{15}{16}\right)^{t/5}$. When $t = 10$ we have

$\frac{4-x_{10}}{5-x_{10}} = \frac{4}{5}\left(\frac{15}{16}\right)^2 = \frac{45}{64}$; $256 - 64x_{10} = 225 - 45x_{10}$; $31 = 19x_{10}$; $x_{10} = \frac{31}{19}$

- Therefore $\frac{31}{19}$ g of substance C will be formed in 10 min.

52. Suppose in the law of mass action that $a = 6$ and $b = 3$ and 1 g of substance C is formed in 4 min. How long will it take 2 g of substance C to be formed?

▹ If x grams of substance C are present at t min, then

$\frac{dx}{dt} = k(6-x)(3-x)$. We have a table of boundary conditions.

t	0	4	T
x	0	1	2

$\int\frac{dx}{(6-x)(3-x)} = k\int dt$. $\frac{1}{(6-x)(3-x)} \equiv \frac{D}{6-x} + \frac{E}{3-x}$; $1 \equiv D(3-x) + E(6-x)$

Let $x = 6$: $1 = -3D \Leftrightarrow D = -\frac{1}{3}$. Let $x = 3$: $1 = 3E \Leftrightarrow E = \frac{1}{3}$. Therefore

$-\int\frac{dx}{6-x} + \int\frac{dx}{3-x} = 3kt$; $\ln|6-x| - \ln|3-x| + \ln|F| = 3kt$; $t = \frac{1}{3k}\ln\left|\frac{6-x}{F(3-x)}\right|$.

Because $x = 0$ when $t = 0$, $\frac{6}{3F} = 1$; F=2. Thus $t = \frac{1}{3k}\ln\frac{6-x}{6-2x}$.

Because $x = 1$ when $t = 4$, $4 = \frac{1}{3k}\ln\frac{5}{4}$; $\frac{1}{3k} = \frac{4}{\ln(5/4)}$. Thus $t = \frac{4}{\ln(5/4)}\ln\frac{6-x}{6-2x}$.

When $x = 2$, $T = \frac{4}{\ln(5/4)}\ln 2 \approx 12.42$. Two grams will be formed in about 12 min 26 sec.

53. t min after the salt begins to dissolve, x lb is the amount of salt present, $(10-x)$ lb of it is dissolved, and $(10-x)/20$ is its concentration. We have a table of boundary conditions.

t	0	10	20
x	10	5	x_{20}

$\frac{dx}{dt} = kx\left(\frac{10-x}{20} - 3\right)$; $20dx = kx(-x-50)dt$; $\int\frac{20\,dx}{x(x+50)} = -k\int dt$. $\frac{20}{x(x+50)} \equiv \frac{A}{x} + \frac{B}{x+50}$; $20 \equiv A(x+50) + Bx$

Let $x = -50$: $20 = -50B \Leftrightarrow B = -\frac{2}{5}$. Let $x = 0$: $20 = 50A \Leftrightarrow A = \frac{2}{5}$. Therefore

$\frac{2}{5}\int\left(\frac{1}{x} - \frac{1}{x+50}\right)dx = -k\int dt$; $\frac{2}{5}(\ln|x| - \ln|x+50| + C) = -kt$; $\ln\left|\frac{Cx}{x+50}\right| = -2.5kt$; $\frac{Cx}{x+50} = e^{-2.5kt}$

Because $x = 10$ when $t = 0$, $\frac{C}{6} = 1$; $C = 6$. Thus $\frac{6x}{x+50} = e^{-2.5kt}$.

Because $x = 5$ when $t = 10$, $\frac{30}{55} = \frac{6}{11} = e^{-25k}$. Thus $\frac{6x}{x+50} = (e^{-25k})^{t/10} = \left(\frac{6}{11}\right)^{t/10}$.

When $t = 20$, $\frac{6x_{20}}{x_{20}+50} = \left(\frac{6}{11}\right)^2 = \frac{36}{121}$; $121x_{20} = 6x_{20} + 300$; $115x_{20} = 300$; $x_{20} = \frac{300}{115} = \frac{60}{23}$.

• Therefore $10 - \frac{60}{23} = \frac{170}{23} \approx 7.4$ lb of salt remains.

54. If y millions of dollars is the company's yearly income t years after the company begins operating, then

$\frac{dy}{dt} = \frac{t^3+3t^2+6t+7}{t^2+3t+2} = t + \frac{4t+7}{(t+2)(t+1)}$; $\frac{4t+7}{(t+2)(t+1)} = \frac{A}{t+2} + \frac{B}{t+1}$; $4t+7 = A(t+1) + B(t+2)$

Let $t = -2$: $-1 = -A \Leftrightarrow A = 1$. Let $t = -1$: $3 = B$. Therefore

$y = \int\left(t + \frac{1}{t+2} + \frac{3}{t+1}\right)dt = \frac{1}{2}t^2 + \ln(t+2) + 3\ln(t+1) + C$

Because $y = 6$ when $t = 4$ we get $6 = 8 + \ln 6 + 3\ln 5 + C$; $C = -2 - \ln 6 - 3\ln 5$.

When $t = 5$, $y_5 = \frac{1}{2}\cdot 25 + \ln 7 + 3\ln 6 - 2 - \ln 6 - 3\ln 5 = 11.2011$.

• The total expected income is \$11,201,100, to the nearest \$100.

55. If v cm/sec is the velocity of the particle at t sec, then $v = \frac{t^2-t+1}{(t+2)^2(t^2+1)}$. Because $v = \frac{ds}{dt}$, we have

$s = \int \frac{t^2-t+1}{(t+2)^2(t^2+1)}\,dt$. Let $\frac{t^2-t+1}{(t+2)^2(t^2+1)} \equiv \frac{A}{(t+2)^2} + \frac{B}{t+2} + \frac{Ct+D}{t^2+1}$

$t^2 - t + 1 \equiv A(t^2+1) + B(t+2)(t^2+1) + (Ct+D)(t+2)^2$

Let $t = -2$: $7 = 5A \Leftrightarrow A = \frac{7}{5}$. We substitute for A, subtract, factor and divide by $(t+2)$:

$t^2 - t + 1 - \frac{7}{5}(t^2+1) = -\frac{1}{5}(2t^2+5t+2) = -\frac{1}{5}(2t+1)(t+2) = B(t+2)(t^2+1) + (Ct+D)(t+2)$

$-\frac{1}{5}(2t+1) \equiv B(t^2+1) + (Ct+D)(t+2)$

Let $t = -2$: $\frac{3}{5} = 5B \Leftrightarrow B = \frac{3}{25}$. We substitute for B, subtract, factor and divide by $(t+2)$:

$-\frac{5}{25}(2t+1) - \frac{3}{5}(t^2+1) = -\frac{1}{25}(3t^2+10t+8) = -\frac{1}{25}(t+2)(3t+4) = (Ct+D)(t+2)$, $-\frac{1}{25}(3t+4) \equiv Ct + D$

$s = \frac{7}{5}\int\frac{dt}{(t+2)^2} + \frac{3}{25}\int\frac{dt}{t+2} - \frac{3}{25}\int\frac{t\,dt}{t^2+1} - \frac{4}{25}\int\frac{dt}{t^2+1} = -\frac{7}{5}\cdot\frac{1}{t+2} + \frac{3}{25}\ln|t+2| - \frac{3}{50}\ln|t^2+1| - \frac{4}{25}\tan^{-1}t + C$

When $t = 0$, $s = 0$; hence $C = \frac{7}{10} - \frac{3}{25}\ln 2$. When $t = t_1$, $s = s_1$. Therefore

$s_1 = -\frac{7}{5}\cdot\frac{1}{t_1+2} + \frac{3}{25}\ln|t_1+2| - \frac{3}{50}\ln|{t_1}^2+1| - \frac{4}{25}\tan^{-1}t + \frac{7}{10} - \frac{3}{25}\ln 2$

$= \frac{3}{50}\ln\frac{(t_1+2)^2}{4({t_1}^2+1)} - \frac{7}{5}\cdot\frac{1}{t_1+2} - \frac{4}{25}\tan^{-1}t_1 + \frac{7}{10}$

56. A particle is moving along a line so that if v ft/sec is the velocity of the particle at t sec, then $v = \frac{t+3}{t^2+3t+2}$.

Find the distance traveled by the particle from the time when $t = 0$ to the time when $t = 2$.

▷ Let s ft be the distance of the particle from a fixed point on the line at t sec. Because $ds/dt = v$, we are given

that $ds = \frac{(t+3)dt}{t^2+3t+2}$. Thus, $s = \int\frac{(t+3)dt}{t^2+3t+2}$ (1)

Let $\frac{t+3}{t^2+3t+2} = \frac{t+3}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$

Then $t + 3 = A(t+2) + B(t+1)$

If $t = -1$, then $2 = A$. If $t = -2$, then $1 = -B$ or $B = -1$. Thus

$$\frac{t+3}{t^2+3t+2} = \frac{2}{t+1} - \frac{1}{t+2}$$

and from (1) we have

$$s = 2\int\frac{dt}{t+1} - \int\frac{dt}{t+2} = 2\ln|t+1| - \ln|t+2| + C = \ln\frac{(t+1)^2}{|t+2|} + C$$

When $t = 2$, we have $s = \ln\frac{9}{4} + C$. When $t = 0$, we have $s = \ln\frac{1}{2} + C$

Thus $s(2) - s(0) = \ln\frac{9}{4} - \ln\frac{1}{2} = \ln\frac{9}{2}$. The distance traveled is $\ln\frac{9}{2}$ feet.

7.5 INTEGRATION BY OTHER SUBSTITUTION TECHNIQUES AND TABLES

Fractional Powers If an integrand involves fractional powers of a variable x, the integrand can be simplified by the substitution $x = z^n$ where n is the least common denominator of the denominators of the exponents.

Sines and Cosines If an integral is a rational function of $\sin x$ and $\cos x$, it can be reduced to a rational function of z by the substitution

$z = \tan \frac{1}{2}x$

It follows that

$\cos x = \dfrac{1 - z^2}{1 + z^2} \qquad \sin x = \dfrac{2z}{1 + z^2} \qquad dx = \dfrac{2\,dz}{1 + z^2}$

The resulting antiderivative is valid only for $x \in (-\pi, \pi)$. To fix the antiderivative we make use of the formula $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x - y}{1 + xy}$. See Exercise 8.

Table of Integrals A short table of integrals is found on the endpapers of the text and at the back of the Outline.

Exercises 7.5

In Exercises 1–10, evaluate the indefinite integral.

1. Let $z = \sqrt{x}$. Then $z^2 = x$ and $2z\,dz = dx$.

$$\int \frac{x\,dx}{3 + \sqrt{x}} = \int \frac{z^2(2z\,dz)}{3 + z} = 2\int \left(z^2 - 3z + 9 - \frac{27}{3 + z}\right)dz = \frac{2}{3}z^3 - 3z^2 + 18z - 54\ln|3 + z| + C$$
$$= \tfrac{2}{3}x^{3/2} - 3x + 18\sqrt{x} - 54\ln\left|3 + \sqrt{x}\right| + C$$

2. Let $z = \sqrt[3]{x}$. Then $z^3 = x$ and $3z^2dz = dx$.

$$\int \frac{dx}{\sqrt[3]{x} - x} = \int \frac{3z^2dz}{z - z^3} = 3\int \frac{z\,dz}{1 - z^2} = -\tfrac{3}{2}\ln\left|1 - z^2\right| + C = -\tfrac{3}{2}\ln\left|1 - x^{2/3}\right| + C$$

3. Let $z = \sqrt{1 + 4x}$. Then $x = \frac{1}{4}(z^2 - 1)$ and $dx = \frac{1}{2}z\,dz$.

$$\int \frac{dx}{x\sqrt{1 + 4x}} = \int \frac{\frac{1}{2}z\,dz}{\frac{1}{4}(z^2 - 1)z} = 2\int \frac{dz}{z^2 - 1} = \ln\left|\frac{z - 1}{z + 1}\right| + C = \ln\left|\frac{\sqrt{1 + 4x} - 1}{\sqrt{1 + 4x} + 1}\right| + C$$

4. $\displaystyle\int x(1 + x)^{2/3}dx$

▸ Method 1: Substitute $u = (1 + x)^{1/3}$. Then $u^3 = 1 + x$, $dx = 3u^2du$, $(1 + x)^{2/3} = u^2$ and $x = u^3 - 1$. Thus

$$\int x(1 + x)^{2/3}dx = \int (u^3 - 1)u^2(3u^2du) = 3\int (u^7 - u^4)du = \tfrac{3}{8}u^8 - \tfrac{3}{5}u^5 + C = \tfrac{3}{8}(1 + x)^{8/3} - \tfrac{3}{5}(1 + x)^{5/3} + C$$
$$= \tfrac{3}{40}(1 + x)^{5/3}[5(1 + x) - 8] + C = \tfrac{3}{40}(1 + x)^{5/3}(5x - 3) + C$$

Method 2: Integrate by parts:

$$\int x\boxed{(1 + x)^{2/3}dx} = x \cdot \tfrac{3}{5}(1 + x)^{5/3} - \tfrac{3}{5}\int (1 + x)^{5/3}dx = \tfrac{3}{5}x(1 + x)^{5/3} - \tfrac{3}{5}\cdot\tfrac{3}{8}(1 + x)^{8/3} + C$$
$$= \tfrac{3}{40}(1 + x)^{5/3}[8x - 3(1 + x)] + C = \tfrac{3}{40}(1 + x)^{5/3}(5x - 3) + C$$

5. Let $z = \sqrt{1 + 2x^3}$. Then $x^3 = \frac{1}{2}(z^2 - 1)$ and $3x^2dx = z\,dz$.

$$\int \frac{(2x^5 + 3x^2)dx}{\sqrt{1 + 2x^3}} = \frac{1}{3}\int \frac{(2x^3 + 3)(3x^2dx)}{\sqrt{1 + 2x^3}} = \frac{1}{3}\int \frac{[(z^2 - 1) + 3]z\,dz}{z} = \frac{1}{3}\int (z^2 + 2)dz = \frac{1}{9}z^3 + \frac{2}{3}z + C$$
$$= \frac{1}{9}z(z^2 + 6) + C = \frac{1}{9}\sqrt{1 + 2x^3}[(1 + 2x^3) + 6] + C = \frac{1}{9}\sqrt{1 + 2x^3}(2x^3 + 7) + C$$

6. $\displaystyle\int \frac{dx}{2\sqrt[3]{x}+\sqrt{x}}$

▷ Because 6 is the least common denominator of the fractional exponents of the given powers of x, which are $x^{1/3}$ and $x^{1/2}$, we let $z = x^{1/6}$. Then $x = z^6$; $dx = 6z^5$; $\sqrt[3]{x} = z^2$; $\sqrt{x} = z^3$. Thus,

$$\int \frac{dx}{2\sqrt[3]{x}+\sqrt{x}} = \int \frac{6z^5\,dz}{2z^2+z^3} = \int \frac{6z^3\,dz}{2+z} = \int (6z^2 - 12z + 24)\,dz - \int \frac{48\,dz}{z+2}$$

$$= 2z^3 - 6z^2 + 24z - 48\ln|z+2| + C = 2x^{1/2} - 6x^{1/3} + 24x^{1/6} - 48\ln(x^{1/6}+2) + C$$

7. Let $z = \tan x$. Then $\cos 2x = \dfrac{1-z^2}{1+z^2}$ and $2\,dx = \dfrac{2\,dz}{1+z^2}$ so $dx = \dfrac{dz}{1+z^2}$. $\displaystyle\int \frac{8\,dx}{3\cos 2x + 1} =$

$$8\int \frac{\dfrac{dz}{1+z^2}}{3\left(\dfrac{1-z^2}{1+z^2}\right)+1} = 8\int \frac{dz}{3-3z^2+1+z^2} = 4\int \frac{dz}{2-z^2} = \frac{4}{2\sqrt{2}}\ln\left|\frac{z+\sqrt{2}}{z-\sqrt{2}}\right| + C = \sqrt{2}\ln\left|\frac{\tan x+\sqrt{2}}{\tan z-\sqrt{2}}\right| + C$$

8. $\displaystyle\int \frac{\cos x\,dx}{3\cos x - 5}$

▷ We let $z = \tan \frac{1}{2}x$. Then

$$\int \frac{\cos x\,dx}{3\cos x - 5} = \int \frac{\dfrac{1-z^2}{1+z^2}\cdot\dfrac{2\,dz}{1+z^2}}{3\cdot\dfrac{1-z^2}{1+z^2}-5} = \int \frac{(z^2-1)\,dz}{(z^2+1)(4z^2+1)} \tag{1}$$

For partial fractions only we let $u = z^2$.

$$\frac{z^2-1}{(z^2+1)(4z^2+1)} = \frac{u-1}{(u+1)(4u+1)} = \frac{A}{u+1} + \frac{B}{4u+1} \tag{2}$$

$$u - 1 = A(4u+1) + B(u+1)$$

Let $u = -1$: $-2 = -3A$, $A = \frac{2}{3}$

Let $u = -\frac{1}{4}$: $-\frac{5}{4} = \frac{3}{4}B$, $B = -\frac{5}{3}$

Substituting for A, B and u in (2), we get from (1)

$$\int \frac{\cos x\,dx}{3\cos x - 5} = \frac{2}{3}\int \frac{dz}{z^2+1} - \frac{5}{3}\int \frac{dz}{4z^2+1} = \tfrac{2}{3}\tan^{-1} z - \tfrac{5}{6}\tan^{-1}(2z) + C$$

$$= \tfrac{2}{3}\tan^{-1}(\tan \tfrac{1}{2}x) - \tfrac{5}{6}\tan^{-1}(2\tan\tfrac{1}{2}x) + C \tag{3}$$

$$= -\tfrac{1}{6}\tan^{-1}(\tan\tfrac{1}{2}x) + \tfrac{5}{6}[\tan^{-1}(\tan\tfrac{1}{2}x) - \tan^{-1}(2\tan\tfrac{1}{2}x)]$$

$$= -\tfrac{1}{12}x + \tfrac{5}{6}\tan^{-1}\frac{\tan\frac{1}{2}x - 2\tan\frac{1}{2}x}{1+2\tan^2\frac{1}{2}x} + C = -\tfrac{1}{12}x + \tfrac{5}{6}\tan^{-1}\frac{-2\cos\frac{1}{2}x\sin\frac{1}{2}x}{2\cos^2\frac{1}{2}x + 4\sin^2\frac{1}{2}x} + C$$

$$= -\tfrac{1}{12}x + \tfrac{5}{6}\tan^{-1}\frac{-\sin x}{1+\cos x + 2(1-\cos x)} + C = -\tfrac{1}{12}x - \tfrac{5}{6}\tan^{-1}\frac{\sin x}{3-\cos x} + C \tag{4}$$

Equation (3) is valid only for $x \in (-\pi, \pi)$ while equation (4) is valid for all x.

9. $$\int \frac{3\,dx}{8+7\cos x} = 3\int \frac{\dfrac{2\,dz}{1+z^2}}{8+7\left(\dfrac{1-z^2}{1+z^2}\right)} = 6\int \frac{dz}{8(1+z^2)+7(1-z^2)} = 6\int \frac{dz}{z^2+15} = \frac{6}{\sqrt{15}}\tan^{-1}\frac{z}{\sqrt{15}} + C$$

$$= \frac{6}{\sqrt{15}}\tan^{-1}\left(\frac{1}{\sqrt{15}}\tan\frac{x}{2}\right) + C \text{ if } x \in (-\pi,\pi) = \frac{6}{\sqrt{15}}\left[\tan^{-1}\left(\frac{1}{\sqrt{15}}\tan\frac{x}{2}\right) - \tan^{-1}\left(\tan\frac{x}{2}\right)\right] + \frac{6}{\sqrt{15}}\cdot\frac{x}{2} + C$$

$$= \frac{6}{\sqrt{15}}\tan^{-1}\frac{\left(\dfrac{1}{\sqrt{15}}-1\right)\tan\dfrac{x}{2}}{1+\dfrac{1}{\sqrt{15}}\tan^2\dfrac{x}{2}} + \frac{3}{\sqrt{15}}x + C = \frac{6}{\sqrt{15}}\tan^{-1}\frac{(1-\sqrt{15})2\sin\frac{x}{2}\cos\frac{x}{2}}{\sqrt{15}\,2\cos^2\frac{x}{2} + 2\sin^2\frac{x}{2}} + \frac{3}{\sqrt{15}}x + C$$

$$= \frac{6}{\sqrt{15}}\tan^{-1}\frac{(1-\sqrt{15})\sin x}{\sqrt{15}(1+\cos x)+(1-\cos x)} + \frac{3}{\sqrt{15}}x + C = -\frac{6}{\sqrt{15}}\tan^{-1}\frac{7\sin x}{7\cos x + \sqrt{15} + 8} + \frac{3}{\sqrt{15}}x + C \text{ for all } x$$

10. $\int \frac{dx}{1+\sin x}$. Method 1. Let $z = \tan \frac{1}{2}x$. $\int \frac{\frac{2\,dz}{1+z^2}}{1+\frac{2z}{1+z^2}} = \int \frac{2\,dz}{1+2z+z^2} = \int \frac{2\,dz}{(1+z)^2} = -\frac{2}{1+z} + C = C - \frac{2}{1+\tan \frac{1}{2}x}$

Method 2. Multiply and divide by $1-\sin x$. $\int \frac{1-\sin x}{\cos^2 x}dx = \int (\sec^2 x - \tan x \sec x)dx = \tan x - \sec x + C$

11. $\int \frac{dx}{4\sin x - 3\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{4\left(\frac{2z}{1+z^2}\right) - 3\left(\frac{1-z^2}{1+z^2}\right)} = 2\int \frac{dz}{8z-3+3z^2} = \frac{2}{3}\int \frac{dz}{z^2+\frac{8}{3}z-1} = \frac{2}{3}\int \frac{dz}{(z^2+\frac{8}{3}z+\frac{16}{9})-\frac{25}{9}}$

$= \frac{2}{3}\int \frac{dz}{(z+\frac{4}{3})^2-(\frac{5}{3})^2} = \frac{2}{3}\cdot\frac{1}{\frac{10}{3}}\ln\left|\frac{(z+\frac{4}{3})-\frac{5}{3}}{(z+\frac{4}{3})+\frac{5}{3}}\right| + C = \frac{1}{5}\ln\left|\frac{z-\frac{1}{3}}{z+3}\right| + C = \frac{1}{5}\ln\left|\frac{\tan\frac{1}{2}x-\frac{1}{3}}{\tan\frac{1}{2}x+3}\right| + C$

12. $\int \frac{dx}{\sin x + \tan x}$

▷ Let $z = \tan \frac{1}{2}x$. Then

$$\int \frac{dx}{\sin x + \tan x} = \int \frac{\frac{2\,dz}{1+z^2}}{\frac{2z}{1+z^2}+\frac{2z}{1-z^2}} = \int \frac{(1-z^2)}{2z}\,dz = \frac{1}{2}\int \frac{dz}{z} - \frac{1}{2}\int z\,dz = \frac{1}{2}\ln|z| - \frac{1}{4}z^2 + C$$

$= \frac{1}{2}\ln\left|\tan\frac{1}{2}x\right| - \frac{1}{4}\tan^2\frac{1}{2}x + C$

In Exercises 13–24, evaluate the definite integral. Check using NINT to six significant digits.

13. Let $z = \sqrt{x}$. Then $x = z^2$ and $dx = 2z\,dz$. When $x = 0$, $z = 0$; when $x = 4$, $z = 2$. Therefore
$\int_0^4 \frac{dx}{1+\sqrt{x}} = \int_0^2 \frac{2z\,dz}{1+z} = 2\int_0^2 \left(1 - \frac{1}{1+z}\right)dz = 2\left[z - \ln|z+1|\ \right]_0^2 = 2[(2-\ln 3) - 0] = 4 - 2\ln 3 \approx 1.80278$

14. Let $x = u^2$, $dx = 2u\,du$.
$\int_{x=0}^1 \frac{x^{3/2}dx}{x+1} = \int_{u=0}^1 \frac{u^3\cdot 2u\,du}{u^2+1} = 2\int_0^1\left(u^2-1+\frac{1}{u^2+1}\right)du = 2\left[\frac{1}{3}u^3 - u + \tan^{-1}u\right]_0^1 = 2\left(\frac{1}{4}\pi - \frac{2}{3}\right) = \frac{1}{2}\pi - \frac{4}{3} \approx .237463$

15. Let $z = \sqrt{2x}$. Then $x = \frac{1}{2}z^2$ and $dx = z\,dz$. When $x = \frac{1}{2}$, $z = 1$; when $x = 2$, $z = 2$. Therefore
$\int_{1/2}^2 \frac{dx}{\sqrt{2x}(\sqrt{2x}+9)} = \int_1^2 \frac{z\,dz}{z(z+9)} = \int_1^2 \frac{dz}{z+9} = \ln|z+9|\,\Big]_1^2 = \ln 11 - \ln 10 = \ln 1.1 \approx 0.0953102$

16. $\int_{16}^{18} \frac{dx}{\sqrt{x}-\sqrt[4]{x^3}}$

▷ Let $u = x^{1/4}$ and $a = 18^{1/4}$. Then $\sqrt{x} = u^2$, $\sqrt[4]{x^3} = u^3$, $x = u^4$, $dx = 4u^3du$. Thus
$\int_{x=16}^{18} \frac{dx}{\sqrt{x}-\sqrt[4]{x^3}} = \int_{u=2}^a \frac{u\,du}{u^2-u^3} = -4\int_2^a \frac{u\,du}{u-1} = -\int_2^a\left(1+\frac{1}{u-1}\right)du = -4\Big[u+\ln(u-1)\Big]_2^a$
$= -4[\sqrt[4]{18} - 2 + \ln(\sqrt[4]{18}-1)] \approx -0.471265$

17. Let $z = \tan\frac{1}{2}x$. Then $\sin x = \frac{2z}{1+z^2}$ and $dx = \frac{2\,dz}{1+z^2}$.

$$\int_0^{\pi/2} \frac{dx}{5\sin x + 3} = \int_0^1 \frac{\frac{2\,dz}{1+z^2}}{5\left(\frac{2z}{1+z^2}\right)+3} = \int_0^1 \frac{2\,dz}{10z+3+3z^2} = \frac{2}{3}\int_0^1 \frac{dz}{z^2+\frac{10}{3}z+1} = \frac{2}{3}\int_0^1 \frac{dz}{(z+\frac{5}{3})^2-\frac{16}{9}}$$

$= \frac{2}{3}\cdot\frac{1}{\frac{8}{3}}\ln\left|\frac{(z+\frac{5}{3})-\frac{4}{3}}{(z+\frac{5}{3})+\frac{4}{3}}\right|\Bigg]_0^1 = \frac{1}{4}\left(\ln\frac{4}{4} - \ln\frac{1}{3}\right) = \frac{1}{4}\ln 3 \approx 0.274653$

18. $\displaystyle\int_0^{\pi/2} \frac{dx}{3+\cos 2x} = \int_0^{\pi/2} \frac{dx}{2+2\cos^2 x} \approx 0.555360.$

Method 1. Let $z = \tan \frac{1}{2}x$. $\displaystyle I = \int_{z=0}^1 \frac{\frac{2\,dz}{1+z^2}}{2\left[1+\left(\frac{1-z^2}{1+z^2}\right)^2\right]} = \int_0^1 \frac{(1+z^2)dz}{(1+z^2)^2+(1-z^2)^2} = \frac{1}{2}\int_0^1 \frac{1+z^2}{1+z^4}dz$

$$= \frac{1}{2}\int_0^1 \frac{1+z^2}{(1+2z^2+z^4)-2z^2} = \frac{1}{2}\int_0^1 \frac{1+z^2}{(1+z^2-\sqrt{2}z)(1+z^2+\sqrt{2}z)}dz = \frac{1}{4}\int_0^1\left(\frac{1}{1+z^2-\sqrt{2}z}+\frac{1}{1+z^2-\sqrt{2}}\right)dz$$

$$= \frac{1}{4}\int_0^1\left[\frac{1}{(z-\frac{1}{2}\sqrt{2})^2+\frac{1}{2}}+\frac{1}{(z+\frac{1}{2}\sqrt{2})^2+\frac{1}{2}}\right]dz = \frac{1}{4}\sqrt{2}\left[\tan^{-1}(\sqrt{2}z-1)+\tan^{-1}(\sqrt{2}z+1)\right]_0^1 = \frac{1}{4}\sqrt{2}\left(\frac{1}{8}\pi+\frac{3}{8}\pi\right) = \frac{1}{8}\sqrt{2}\pi$$

Method 2. $\displaystyle I = \lim_{b\to\pi/2^-} \frac{1}{2}\int_0^b \frac{\sec^2 x\,dx}{\sec^2 x+1} = \lim_{b\to\pi/2^-}\int_0^b \frac{d(\tan x)}{\tan^2 x+2} = \lim_{b\to\pi/2^-} \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)\Big]_0^b = \frac{1}{2\sqrt{2}}\cdot\frac{\pi}{2} = \frac{1}{8}\sqrt{2}\pi$

19. Let $z = \tan x$. Then $\sin 2x = \frac{2z}{1+z^2}$ and $2\,dx = \frac{2\,dz}{1+z^2}$; so $dx - \frac{dz}{1+z^2}$.

$$\int_{\pi/6}^{\pi/2}\frac{3\,dx}{2\sin 2x+1} = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{\frac{3\,dz}{1+z^2}}{\frac{4z}{1+z^2}+1} = \int_{1/\sqrt{3}}^{\sqrt{3}}\frac{dz}{z^2+4z+1} = 3\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{dz}{(z+2)^2-3} = \frac{\sqrt{3}}{2}\ln\left|\frac{z+2-\sqrt{3}}{z+2+\sqrt{3}}\right|_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2}\left[\ln\frac{1}{\sqrt{3}+1}-\ln\frac{\sqrt{3}-1}{\sqrt{3}+2}\right] = \frac{\sqrt{3}}{2}\ln\left(\frac{1}{\sqrt{3}+1}\cdot\frac{\sqrt{3}+2}{\sqrt{3}-1}\right) = \frac{\sqrt{3}}{2}\ln\frac{\sqrt{3}+2}{2} \approx 0.935716$$

20. $\displaystyle\int_0^{\pi/4}\frac{8\,dx}{\tan x+1}$

▷ $$\int_0^{\pi/4}\frac{8\,dx}{\tan x+1} = \int_0^{\pi/4}\frac{8\cos x}{\sin x+\cos x}dx = 4\int_0^{\pi/4}\frac{\cos x+\sin x}{\sin x+\cos x}\,dx + 4\int_0^{\pi/4}\frac{\cos x-\sin x}{\sin x+\cos x}\,dx$$

$$= 4\int_0^{\pi/4}dx + 4\int_0^{\pi/4}\frac{d(\sin x+\cos x)}{\sin x+\cos x} = 4x\Big]_0^{\pi/4} + 4\ln(\sin x+\cos x)\Big]_0^{\pi/4}$$

$$= 4\cdot\frac{\pi}{4}+4(\ln\sqrt{2}-\ln 1) = \pi+2\ln 2 \approx 4.52789$$

21. Let $z = \tan\frac{1}{2}x$. Then $\cos x = \frac{1-z^2}{1+z^2}$ and $dx = \frac{2\,dz}{1+z^2}$.

$$\int_{-\pi/3}^{\pi/2}\frac{3\,dx}{2\cos x+1} = \int_{-1/\sqrt{3}}^1\frac{\frac{6\,dz}{1+z^2}}{2\left(\frac{1-z^2}{1+z^2}\right)+1} = \int_{-1/\sqrt{3}}^1\frac{dz}{2-2z^2+1+z^2} = 6\int_{-1/\sqrt{3}}^1\frac{dz}{3-z^2}$$

$$= 6\cdot\frac{1}{2\sqrt{3}}\ln\left|\frac{z+\sqrt{3}}{z-\sqrt{3}}\right|\Big]_{-1/\sqrt{3}}^1 = \sqrt{3}\left(\ln\left|\frac{1+\sqrt{3}}{1-\sqrt{3}}\right|-\ln\left|\frac{-1/\sqrt{3}+\sqrt{3}}{-1/\sqrt{3}-\sqrt{3}}\right|\right)$$

$$= \sqrt{3}\left(\ln\left|\frac{(1+\sqrt{3})^2}{-2}\right|-\ln\left|\frac{-1+3}{-1-3}\right|\right) = 2\sqrt{3}\ln(1+\sqrt{3}) \approx 3.48160$$

22. $$\int_0^{\pi/2}\frac{\sin 2x\,dx}{2+\cos x} = 2\int_0^{\pi/2}\frac{\cos x}{2+\cos x}\sin x\,dx2\int_0^{\pi/2}\left(1-\frac{2}{2+\cos x}\right)\sin x\,dx = 2\left[\int_0^{\pi/2}\sin x\,dx - \int_0^{\pi/2}\frac{2\sin x\,dx}{2+\cos x}\right]$$

$$= 2(-\cos x+2\ln|2+\cos x|)\Big]_0^{\pi/2} = 2(1+2\ln\tfrac{2}{3}) = 2+4\ln\tfrac{2}{3} \approx 0.378140$$

23. Let $x = z^6$. Then $dx = 6z^5dz$. When $x = 0$, $z = 0$; when $x = 1$, $z = 1$. Therefore

$$\int_0^1\frac{\sqrt{x}\,dx}{1+\sqrt[3]{x}} = \int_0^1\frac{z^3(6z^5dz)}{1+z^2} = 6\int_0^1\frac{z^8}{z^2+1}\,dz = 6\int_0^1\left(z^6-z^4+z^2-1+\frac{1}{z^2+1}\right)dz$$

$$= 6\left[\frac{1}{7}z^7-\frac{1}{5}z^5+\frac{1}{3}z^3-z+\tan^{-1}z\right]_0^1 = 6\left(\frac{1}{7}-\frac{1}{5}+\frac{1}{3}-1+\frac{\pi}{4}\right)-0 = \frac{3}{2}\pi-\frac{152}{35}$$

24. $\displaystyle\int_2^{11}\frac{x^3\,dx}{\sqrt[3]{x^2+4}}$

▷ Let $u = \sqrt[3]{x^2+4}$. Then $u^3 = x^2+4$, so $3u^2du = 2x\,dx$. Hence $x\,dx = \frac{3}{2}u^2du$ and $x^2 = u^3-4$.

$$\int_{x=2}^{11}\frac{x^3\,dx}{\sqrt[3]{x^2+4}} = \int_{u=2}^5\frac{(u^3-4)(\frac{3}{2}u^2du)}{u} = \frac{3}{2}\int_2^5(u^4-4u)du = \frac{3}{2}\left[\frac{1}{5}u^5-2u^2\right]_2^5 = \frac{3}{2}\left[(625-50)-\left(\frac{32}{5}-8\right)\right] = \frac{8{,}649}{10}$$

In Exercises 23–46, use the table of integrals on the endpapers to evaluate the integral.

25. From formula 9 with $a = 6$ and $b = -1$,

$$\int \frac{x^2 dx}{(6-x)^2} = -1\left[6 - x - \frac{36}{6-x} - 12 \ln|6-x|\right] + \bar{C} = x + \frac{36}{6-x} + 12 \ln|6-x| + C$$

26. From formula 10 with $a = 5$ and $b = -2$, $\displaystyle\int \frac{x\,dx}{(5-2x)^3} = \frac{1}{4}\left[\frac{5}{2(5-2x)^2} - \frac{1}{5-2x}\right] + C$

27. From formula 14 with $a = 1$ and $b = 2$

$$\int x\sqrt{1+2x}\,dx = \frac{2}{15\cdot 8}(3\cdot 2x - 2)(1+2x)^{3/2} + C = \frac{1}{30}(3x-1)(1+2x)^{3/2} + C$$

28. The integrand is a form containing $\sqrt{a+bu}$. Use one of the formulas 14–23.

$$\int \frac{dx}{x^2\sqrt{1+2x}}$$

▷ Formula 21 in the table of integrals is

$$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

We apply the formula with $n = 2$, $u = x$, $a = 1$, and $b = 2$. Thus

$$\int \frac{dx}{x^2\sqrt{1+2x}} = -\frac{\sqrt{1+2x}}{x} - \int \frac{dx}{x\sqrt{1+2x}} \tag{1}$$

Formula 21 is an example of a reduction formula. Note that in Eq. (1) we have used the formula to reduce the given integral to one that is simpler. Formula 20 in the table of integrals is

$$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}} \ln\left|\frac{\sqrt{a+bu} - \sqrt{a}}{\sqrt{a+bu} + \sqrt{a}}\right| + C \quad \text{if } a > 0$$

We apply Formula 20 to evaluate the integral that remains on the right side of Eq. (1). Thus,

$$\int \frac{dx}{x\sqrt{1+2x}} = \ln\left|\frac{\sqrt{1+2x} - 1}{\sqrt{1+2x} + 1}\right| + C \tag{2}$$

Substituting from Eq. (2) into Eq. (1), we obtain

$$\int \frac{dx}{x^2\sqrt{1+2x}} = -\frac{\sqrt{1+2x}}{x} - \ln\left|\frac{\sqrt{1+2x} - 1}{\sqrt{1+2x} + 1}\right| + C$$

29. From formula 25 with $a = 2$, $\displaystyle\int \frac{dx}{4-x^2} = \frac{1}{4}\ln\left|\frac{x+2}{x-2}\right| + C.$

30. From formula 26 with $a = 5$, $\displaystyle\int \frac{dx}{x^2-25} = \frac{1}{10}\ln\left|\frac{x-5}{x+5}\right| + C$

31. From formula 27, with $a^2 = 9$, $u = x + 3$ and choosing the minus sign,

$$\int \frac{dx}{\sqrt{x^2+6x}} = \int \frac{dx}{\sqrt{(x+3)^2 - 9}} = \ln\left|x + 3 + \sqrt{(x+3)^2 - 9}\right| + C = \ln\left|x + 3 + \sqrt{x^2+6x}\right| + C$$

32. The integrand is a form containing $\sqrt{u^2+a^2}$. Use one of the formulas 27–38.

$$\int \sqrt{4x^2+1}\,dx$$

▷ Formula 28 in the table of integrals is

$$\int \sqrt{u^2+a^2}\,du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln(u + \sqrt{u^2+a^2}) + C$$

If $u^2 = 4x^2$, then $u = 2x$, and $du = 2\,dx$. We apply the formula with $a = 1$. Thus,

$$\int \sqrt{4x^2+1}dx = \frac{1}{2}\int \sqrt{u^2+1}\,du = \frac{1}{2}x\sqrt{4x^2+1} + \frac{1}{4}\ln(2x + \sqrt{4x^2+1}) + C$$

33. From formula 42 with $a = 3$ and $u = 2x$,

$$\int \frac{\sqrt{9-4x^2}}{x}\,dx = \int \frac{\sqrt{9-(2x)^2}}{2x}(2\,dx) = \sqrt{9-4x^2} - 3\ln\left|\frac{3+\sqrt{9-4x^2}}{2x}\right| + C$$

34. From formula 46 with $a = \frac{5}{3}$, $\displaystyle\int \frac{dx}{x^2\sqrt{25-9x^2}} = \frac{1}{3}\int \frac{dx}{x^2\sqrt{\frac{25}{9}-x^2}} = -\frac{1}{3}\cdot\frac{\sqrt{\frac{25}{9}-x^2}}{\frac{25}{9}x} + C = -\frac{\sqrt{25-9x^2}}{25x} + C$

35. From formula 50 with $2a = 4$, $a = 2$, $\displaystyle\int x\sqrt{4x-x^2}\,dx = \frac{2x^2-2x-3\cdot 2^2}{6}\sqrt{4x-x^2} + 4\cos^{-1}\left(1-\frac{x}{2}\right) + C$

$$= \tfrac{1}{3}(x^2-x-6)\sqrt{4x-x^2} + 4\cos^{-1}\left(\frac{2-x}{2}\right) + C$$

36. The integrand is a form containing $2au - u^2$. Use one of the formulas 49–58.

$$\int \frac{x^2dx}{\sqrt{4x-x^2}}$$

- Formula 55 in the table of integrals is

$$\int \frac{u^2du}{\sqrt{2au-u^2}} = -\frac{(u+3a)}{2}\sqrt{2au-u^2} + \frac{3a^2}{2}\cos^{-1}\left(1-\frac{u}{a}\right) + C$$

We apply the formula with $a = 2$. Thus,

$$\int \frac{x^2dx}{\sqrt{4x-x^2}} = -\frac{u+6}{2}\sqrt{4x-x^2} + 6\cos^{-1}(1-\tfrac{1}{2}x) + C$$

37. From formula 73 with $n = 5$ and $n = 3$,

$$\int \sin^5x\,dx = -\tfrac{1}{5}\sin^4x\cos x + \tfrac{4}{5}\int \sin^3x\,dx = -\tfrac{1}{5}\sin^4x\cos x + \tfrac{4}{5}\left(-\tfrac{1}{3}\sin^2x\cos x + \tfrac{2}{3}\int \sin x\,dx\right)$$

$$= -\tfrac{1}{5}\sin^4x\cos x - \tfrac{4}{15}\sin^2x\cos x - \tfrac{8}{15}\cos x + C$$

38. From formula 74 with $n = 6$, 4 and 2,

$$\int \cos^6x\,dx = \tfrac{1}{6}\cos^5x\sin x + \tfrac{5}{6}\int \cos^4x\,dx = \tfrac{1}{6}\cos^5x\sin x + \tfrac{5}{6}\left(\tfrac{1}{4}\cos^3x\sin x + \tfrac{3}{4}\int \cos^2x\,dx\right)$$

$$= \tfrac{1}{6}\cos^5x\sin x + \tfrac{5}{24}\cos^3x\sin x + \tfrac{5}{8}\left(\tfrac{1}{2}\cos x\sin x + \tfrac{1}{2}\int dx\right)$$

$$= \tfrac{1}{6}\cos^5x\sin x + \tfrac{5}{24}\cos^3x\sin x + \tfrac{5}{16}\cos x\sin x + \tfrac{5}{16}x + C$$

39. From formulas 87 with $n = 4$, 86 with $n = 3$, 87 with $n = 2$, and 82,

$$\int t^4\cos t\,dt = t^4\sin t - 4\int t^3\sin t\,dt = t^4\sin t - 4\left(-t^3\cos t + 3\int t^2\cos t\,dt\right)$$

$$= t^4\sin t + 4t^3\cos t - 12\left(t^2\sin t - 2\int t\sin t\,dt\right) = t^4\sin t + 4t^3\cos t - 12t^2\sin t + 24(\sin t - t\cos t) + C$$

$$= t^4\sin t + 4t^3\cos t - 12t^2\sin t - 24t\cos t + 24\sin t + C$$

40. The integrand is a form containing trigonometric functions. Use one of the formulas 59–88.

$$\int \sin 3w\cos 5w\,dw$$

- Formula 81 in the table of integrals is

$$\int \sin mu\cos nu\,du = -\frac{\cos(m+n)u}{2(m+n)} - \frac{\cos(m-n)u}{2(m-n)} + C$$

We apply the formula with $m = 3$ and $n = 5$. Thus,

$$\int \sin 3w\cos 5w\,dw = -\frac{\cos 8w}{16} - \frac{\cos(-2w)}{-4} + C = -\frac{\cos 8w}{16} + \frac{\cos 2w}{4} + C$$

41. From formula 93 with $u = 3x$, $\displaystyle\int \sec^{-1}3x\,dx = \frac{1}{3}\int \sec^{-1}3x(3\,dx) = \tfrac{1}{3}\left(3x\sec^{-1}3x - \ln\left|3x+\sqrt{9x^2-1}\right|\right) + C$

$$= x\sec^{-1}3x - \tfrac{1}{3}\ln\left|3x+\sqrt{9x^2-1}\right| + C$$

42. From formula 91 with $u = 4t$,

$$\int \tan^{-1}4t\,dt = \frac{1}{4}\int \tan^{-1}4t(4\,dt) = \tfrac{1}{4}(4t\tan^{-1}4t - \ln\sqrt{1+16t^2}) + C = t\tan^{-1}4t - \tfrac{1}{8}\ln(1+16t^2) + C$$

43. From formula 98 with $n = 2$ and formula 97, both with $u = 4x$,

$$\int x^2e^{4x}dx = \frac{1}{64}\int (4x)^2e^{4x}(4\,dx) = \frac{1}{64}\left[(4x)^2e^{4x} - 2\int 4xe^{4x}(4\,dx)\right] = \frac{1}{4}x^2e^{4x} - \frac{1}{32}e^{4x}(4x-1) + C$$

$$= \frac{e^{4x}}{32}(8x^2-4x+1) + C$$

44. The integrand is a form containing an exponential function. Use one of the formulas 95–106.

$$\int x^3 2^x \, dx$$

▹ Formula 99 in the table of integrals is

$$\int u^n a^u \, du = \frac{u^n a^u}{\ln a} - \frac{n}{\ln a}\int u^{n-1} a^u \, du$$

We apply the formula with $u = x$, $n = 3$, and $a = 2$. Thus,

$$\int x^3 2^x \, dx = \frac{x^3 2^x}{\ln 2} - \frac{3}{\ln 2}\int x^2 2^x \, dx$$

Then we reapply the formula with $n = 2$ and $n = 1$. Thus,

$$\int x^3 2^x \, dx = \frac{x^3 2^x}{\ln 2} - \frac{3}{\ln 2}\left[\frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2}\int x 2^x \, dx\right] = \frac{x^3 2^x}{\ln 2} - \frac{3x^2 2^x}{\ln^2 2} + \frac{6}{\ln^2 2}\left[\frac{x 2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x \, dx\right]$$

$$= \frac{x^3 2^x}{\ln 2} - \frac{3x^2 2^x}{\ln^2 2} + \frac{6x 2^x}{\ln^3 2} - \frac{6 \cdot 2^x}{\ln^4 2} + C$$

Note that we used formula 96 in the table of integrals to evaluate the last integral.

45. From formula 103 with $n = 3$ and $u = 3x$,

$$\int x^3 \ln(3x) dx = \frac{1}{81}\int (3x)^3 \ln(3x)(3 \, dx) = \frac{1}{81} \cdot \frac{(3x)^4}{16}(4 \ln 3x - 1) + C = \frac{1}{16} x^4 (4 \ln 3x - 1) + C$$

46. From formula 105 with $a = 2$ and $n = 5$, so that $a^2 + n^2 = 29$, $\int e^{2\theta} \sin 5\theta \, d\theta = \frac{e^{2\theta}}{29}(2 \sin 5\theta - 5 \cos 5\theta) + C$

47. From formula 121 with $u = 5y$,

$$\int 3y \sinh 5y \, dy = \frac{3}{25}\int 5y \sinh 5y(5 \, dy) = \frac{3}{25}\Big(5y \cosh 5y - \sinh 5y\Big) + C = \frac{3}{5} y \cosh 5y - \frac{3}{25} \sinh 5y + C$$

48. The integrand is a form containing a hyperbolic function. Use one of the formulas 107–124.

$$\int e^{3x} \cosh 5x \, dx$$

▹ Formula 124 in the table of integrals is

$$\int e^{au} \cosh nu \, du = \frac{e^{au}}{a^2 - n^2}(a \cosh nu - n \sinh nu) + C$$

We apply the formula with $u = x$, $a = 3$, and $n = 5$. Thus,

$$\int e^{3x} \cosh 5x \, dx = -\frac{e^{3x}}{16}(3 \cosh 5x - 5 \sinh 5x) + C$$

In Exercises 49–64, use the table of integrals on the endpapers to evaluate the definite integral.

49. From formula 13 with $a = 5$ and $b = -1$,

$$\int_1^2 \frac{dx}{x(5 - x)^2} = \frac{1}{5(5 - x)} + \frac{1}{25} \ln\left|\frac{x}{5 - x}\right|\Bigg]_0^2 = \left(\frac{1}{5 \cdot 3} + \frac{1}{25} \ln \frac{2}{3}\right) - \left(\frac{1}{5 \cdot 4} + \frac{1}{25} \ln \frac{1}{4}\right) = \frac{1}{60} + \frac{1}{25} \ln \frac{8}{3}$$

50. From formula 8 with $a = 1$ and $b = 1$,

$$\int_0^3 \frac{x \, dx}{(1 + x)^2} = \frac{1}{1 + x} + \ln(1 + x)\Bigg]_0^3 = -\frac{3}{4} + \ln 4$$

51. From formula 33 with $a^2 = 16$ and choosing the plus sign,

$$\int_0^3 \frac{x^2 dx}{\sqrt{x^2 + 16}} = \frac{x}{2}\sqrt{x^2 + 16} - \frac{16}{2} \ln\left|x + \sqrt{x^2 + 16}\right|\Bigg]_0^3 = \left(\frac{3}{2} \cdot 5 - 8 \ln 8\right) - (-8 \ln 4) = \frac{15}{2} - 8 \ln 2$$

52. $\displaystyle\int_0^2 \frac{dx}{(9 + 4x^2)^{3/2}}$

▹ Formula 38 in the table of integrals is

$$\int \frac{du}{(u^2 + a^2)^{3/2}} = \frac{u}{a^2\sqrt{u^2 + a^2}} + C$$

We apply the formula with $a^2 = \frac{9}{4}$. Thus,

$$\int_0^2 \frac{dx}{(9 + 4x^2)^{3/2}} = \frac{1}{8}\int_0^2 \frac{dx}{(\frac{9}{4} + x^2)^{3/2}} = \frac{1}{8} \cdot \frac{x}{\frac{9}{4}\sqrt{x^2 + \frac{9}{4}}}\Bigg]_0^2 = \frac{x}{9\sqrt{4x^2 + 9}}\Bigg]_0^2 = \frac{2}{45}$$

53. From formula 103 with $n = 4$.

$$\int_1^2 x^4 \ln x\, dx = \frac{x^5}{25}(5 \ln x - 1)\Big]_0^2 = \frac{32}{25}(5 \ln 2 - 1) - \frac{1}{25}(-1) = \frac{32}{5} \ln 2 - \frac{31}{25}$$

54. From formula 98 with $n = 2$ and $u = -x$, followed by formula 97,

$$\int_{x=0}^1 x^2 e^{-x} dx = \int_{u=0}^{-1} u^2 e^u(-du) = \int_{-1}^0 u^2 e^u du = u^2 e^u\Big]_{-1}^0 - 2\int_{-1}^0 ue^u du = -e^{-1} - 2e^u(u-1)\Big]_{-1}^0$$
$$= -e^{-1} + 2 - 4e^{-1} = 2 - 5e^{-1}$$

55. From formula 28 with $a^2 = 16$, $u = x + 1$, and choosing the minus sign,

$$\int_3^4 \sqrt{x^2 + 2x - 15}\, dx = \int_3^4 \sqrt{(x+1)^2 - 16}\, dx = \frac{x+1}{2}\sqrt{(x+1)^2 - 16} - \frac{16}{2} \ln\Big[x + 1 + \sqrt{(x+1)^2 - 16}\Big]_3^4$$
$$= \Big[\tfrac{5}{2}\sqrt{9} - 8 \ln(5 + \sqrt{9})\Big] - 8 \ln 4 = \frac{15}{2} - 8 \ln 8 + 8 \ln 4 = \frac{15}{2} - 8 \ln 2$$

56. $\displaystyle\int_3^5 x^2\sqrt{x^2 - 9}\, dx$

▷ Formula 29 in the table of integrals is

$$\int u^2\sqrt{u^2 - a^2}du = \tfrac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8} \ln\Big|u + \sqrt{u^2 - a^2}\Big| + C$$

We apply the formula with $u = x$, and $a = 3$. Thus,

$$\int_3^5 x^2\sqrt{x^2 - 9}\, dx = \tfrac{x}{8}(2x^2 - 9)\sqrt{x^2 - 9} - \frac{81}{8} \ln\Big|x + \sqrt{x^2 - 9}\Big|\Big]_3^5 = \tfrac{205}{2} - \tfrac{81}{8} \ln 3$$

57. From formula 49 with $2a = 4$, $a = 2$, $u = w$,

$$\int_1^2 \sqrt{4w - w^2}\, dw = \frac{w-2}{2}\sqrt{4w - w^2} + 2\cos^{-1}\Big(1 - \frac{w}{2}\Big)\Big]_1^2 = 2\cos^{-1}0 - [-\tfrac{1}{2}\sqrt{3} + 2\cos^{-1}\tfrac{1}{2}] = \pi + \tfrac{1}{2}\sqrt{3} - \tfrac{2}{3}\pi$$
$$= \tfrac{1}{3}\pi + \tfrac{1}{2}\sqrt{3}$$

58. From formula 77 with $n = 5$ and then $n = 3$,

$$\int_0^{\pi/3} \sec^5 x\, dx = \tfrac{1}{4}\sec^3 x \tan x\Big]_0^{\pi/3} + \tfrac{3}{4}\int_0^{\pi/3} \sec^3 x\, dx = \tfrac{1}{4}\cdot 8\sqrt{3} + \tfrac{3}{4}\Big[\tfrac{1}{2}\sec x \tan x\Big]_0^{\pi/3} + \tfrac{1}{2}\int_0^{\pi/3} \sec x\, dx\Big]$$
$$= 2\sqrt{3} + \tfrac{3}{4}\Big[\tfrac{1}{2}\cdot 2\sqrt{3} + \tfrac{1}{2}\ln|\sec x + \tan x|\Big]_0^{\pi/3}\Big] = 2\sqrt{3} + \tfrac{3}{4}\sqrt{3} + \tfrac{1}{2}\ln(2 + \sqrt{3})\Big] = \tfrac{11}{4}\sqrt{3} + \tfrac{3}{8}\ln(2 + \sqrt{3})$$

59. From formula 79 with $m = 3$ and $n = 5$,

$$\int_{\pi/8}^{\pi/4} \sin 3t \sin 5t\, dt = -\frac{\sin 8t}{2\cdot 8} + \frac{\sin 2t}{2\cdot 2}\Big]_0^{\pi} = \Big(-\frac{\sin 2\pi}{16} + \frac{\sin \frac{1}{2}\pi}{4}\Big) - \Big(-\frac{\sin \pi}{16} + \frac{\sin \frac{1}{4}\pi}{4}\Big) = \frac{1}{4} - \frac{1}{8}\sqrt{2}$$

60. $\displaystyle\int_0^{\pi/4} \tan^6\theta\, d\theta$

▷ Formula 75 in the table of integrals is

$$\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1}u - \int \tan^{n-2}u\, du$$

We apply the formula successively with $n = 6$, $n = 4$, and $n = 2$.

$$\int_0^{\pi/4} \tan^6\theta\, d\theta = \Big[\tfrac{1}{5}\tan^5\theta - \int \tan^4\theta\, d\theta\Big]_0^{\pi/4} = \tfrac{1}{5} - \Big[\tfrac{1}{3}\tan^3 - \int \tan^2\theta\, d\theta\Big]_0^{\pi/4} = \tfrac{1}{5} - \tfrac{1}{3} + \Big[\tan\theta - \int d\theta\Big]_0^{\pi/4}$$
$$= \tfrac{1}{5} - \tfrac{1}{3} + 1 - \tfrac{1}{4}\pi = \tfrac{13}{15} - \tfrac{1}{4}\pi$$

61. From formula 74 with $n = 3$ and $u = 4x$,

$$\int_0^{\pi/2} \sin^3 2x \cos^3 2x\, dx = \frac{1}{32}\int_0^{\pi/2} \sin^3 4x(4\, dx) = \frac{1}{32}\Big[-\tfrac{1}{3}\sin^2 4x \cos 4x\Big]_0^{\pi/2} + \frac{1}{32}\cdot\frac{2}{3}\int_0^{\pi/2} \sin 4x(4\, dx)$$
$$= 0 - \frac{1}{48}\cos 4x\Big]_0^{\pi/2} = 0$$

62. From formula 36 with $a^2 = 16$, $\displaystyle\int_5^6 \frac{dw}{w^2\sqrt{w^2 - 16}} = -\frac{\sqrt{w^2 - 16}}{16w}\Big]_5^6 = \frac{3}{80} - \frac{\sqrt{20}}{96} = \tfrac{3}{80} - \tfrac{1}{48}\sqrt{5}$

63. From formula 98 with $n = 3$ and 2, and formula 97, both with $u = 2x$.

$$\int_0^1 x^3 e^{2x} dx = \frac{1}{16}\int_0^1 (2x)^3 e^{2x}(2\, dx) = \frac{1}{16}\Big[(2x)^3 e^{2x} - 3\int (2x)^2 e^{2x}(2\, dx)\Big]_0^1$$
$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{16}\Big((2x)^2 e^{2x} - 2\int (2x)e^{2x}(2\, dx)\Big)\Big]_0^1 = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{8}e^{2x}(2x - 1)\Big]_0^1 = \Big(\frac{1}{2} - \frac{3}{4} + \frac{3}{8}\Big)e^2 + \frac{3}{8} = \frac{1}{8}(e^2 + 3)$$

64. $\int_0^{\pi/6} e^{2t} \sin 3t \, dt$

▷ Formula 105 in the table of integrals is

$$\int e^{au} \sin nu \, du = \frac{e^{au}}{a^2+n^2}(a \sin nu - n \cos nu) + C$$

We apply the formula with $u = t$, $a = 2$ and $n = 3$. Thus,

$$\int_0^{\pi/6} e^{2t} \sin 3t \, dt = \frac{e^{2t}}{2^2+3^2}(2 \sin 3t - 3 \cos 3t)\Big]_0^{\pi/6} = \tfrac{1}{13}[e^{\pi/3}(2 \sin \tfrac{1}{2}\pi - 3 \cos \tfrac{1}{2}\pi) - e^0(2 \sin 0 - 3 \cos 0)]$$

$$= \tfrac{1}{13}(2e^{\pi/3} + 3)$$

65. (a) Let $x = z^2$. Then $dx = 2z \, dz$. Therefore

$$\int \frac{dx}{x - \sqrt{x}} = \int \frac{2z \, dz}{z^2 - z} = 2\int \frac{dz}{z-1} = 2 \ln|z-1| + C = 2 \ln\left|\sqrt{x} - 1\right| + C$$

(b) Let $u = \sqrt{x} - 1$. Then $du = \dfrac{dx}{2\sqrt{x}}$. Therefore

$$\int \frac{dx}{x - \sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x}-1)} = \int \frac{2\,du}{u} = 2 \ln|u| + C = 2 \ln\left|\sqrt{x} - 1\right| + C$$

66. If $x \in (-\pi, \pi)$, $\displaystyle\int \sin x \, dx = \int \frac{2z}{1+z^2} \cdot \frac{2\,dz}{1+z^2} = \int \frac{2(2z\,dz)}{(1+z^2)^2} = -\frac{2}{1+z^2} + 1 + C = -\frac{1-z^2}{1+z^2} + C = -\cos x + C$

67. $\displaystyle\int \sec x \, dx = \int \frac{dx}{\cos x} = \int \frac{\cos x \, dx}{\cos^2 x} = \int \frac{\cos x \, dx}{1 - \sin^2 x} = \int \frac{1}{(1+\sin x)(1-\sin x)} \cos x \, dx$

$$= \frac{1}{2}\int\left(\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right) d \sin x = \tfrac{1}{2}[\ln(1+\sin x) - \ln(1-\sin x)] + C = \frac{1}{2} \ln\frac{1+\sin x}{1-\sin x} + C$$

Because $|\sin x| \le 1$, then $1 \pm \sin x \ge 0$.

68. Use the substitution $z = \tan \frac{1}{2}x$ to prove that $\displaystyle\int \csc x \, dx = \frac{1}{2} \ln \frac{1-\cos x}{1+\cos x} + C$ (1)

▷ Let $z = \tan \frac{1}{2}x$. Then

$$\int \csc x \, dx = \int \frac{1}{\sin x} dx = \int \frac{1+z^2}{2z} \cdot \frac{2\,dz}{1+z^2} = \int \frac{dz}{z} = \ln|z| + C = \tfrac{1}{2} \ln z^2 + C \qquad (2)$$

Because $\qquad \cos x = \dfrac{1-z^2}{1+z^2}$

it follows that $\qquad z^2 = \dfrac{1-\cos x}{1+\cos x}$ (3)

Substituting from (3) into (2), we obtain (1).

69. From formula (2) we have

$$\int \sec x \, dx = \frac{1}{2} \ln\frac{1+\sin x}{1-\sin x} + C = \frac{1}{2} \ln\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)} + C = \frac{1}{2} \ln\frac{(1+\sin x)^2}{\cos^2 x} + C$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| + C = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C = \ln|\sec x + \tan x| + C$$

70. From the result of Exercise 68, we have

$$\int \csc x \, dx = \frac{1}{2} \ln\frac{1-\cos x}{1+\cos x} + C = \frac{1}{2} \ln\frac{(1-\cos x)^2}{(1+\cos x)(1-\cos x)} + C = \frac{1}{2} \ln\frac{(1-\cos x)^2}{\sin^2 x} + C$$

$$= \ln\left|\frac{1-\cos x}{\sin x}\right| + C = \ln\left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right| + C = \ln|\csc x - \cot x| + C$$

71. (a) $\displaystyle\int \frac{\tan \frac{1}{2}x}{\sin x} dx = \int \frac{z}{\frac{2z}{1+z^2}} \cdot \frac{2\,dz}{1+z^2} = \int dz = z + C = \tan \tfrac{1}{2}x + C$

(b) $\displaystyle\int \frac{\tan \frac{1}{2}x}{\sin x} dx = \int \frac{\frac{\sin \frac{1}{2}x}{\cos \frac{1}{2}x}}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x} dx = \int \sec^2 \tfrac{1}{2}x (\tfrac{1}{2}\,dx) = \tan \tfrac{1}{2}x + C$

7.6 NUMERICAL INTEGRATION

7.6.1 Theorem Trapezoidal Rule If the function f is continuous on the closed interval $[a, b]$ and the numbers $a = x_0, x_1, x_2, \ldots, x_n = b$ form a regular partition of $[a, b]$, and

$$T = \tfrac{1}{2}\Delta x[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where n is any positive integer and $\Delta x = (b - a)/n$, then

$$\lim_{n \to +\infty} T = \int_a^b f(x)\, dx$$

7.6.2 Theorem Let the function f be continuous on the closed interval $[a, b]$, and f' and f'' both exist on (a, b). If

$$\epsilon_T = \int_a^b f(x)\, dx - T$$

where T is the approximate value of $\int_a^b f(x)\, dx$ found by the trapezoidal rule, then there is some number η (eta) in (a, b) such that

$$\epsilon_T = -\tfrac{1}{12}(b - a)f''(\eta)(\Delta x)^2$$

Interval Halving If T′ is the approximate value with half as many intervals each twice as long, then

$$\begin{aligned}\int_a^b f(x)\, dx - (\tfrac{4}{3}T - \tfrac{1}{3}T') &= \tfrac{4}{3}\left[\int_a^b f(x)\, dx - T\right] - \tfrac{1}{3}\left[\int_a^b f(x)\, dx - T'\right] \\ &= -\tfrac{4}{3} \cdot \tfrac{1}{12}(b - a)f''(\eta)(\Delta x)^2 + \tfrac{1}{3} \cdot \tfrac{1}{12}(b - a)f''(\eta')(2\Delta x)^2 \\ &= -\tfrac{1}{9}(b - a)(\Delta x)^2[f''(\eta) - f''(\eta')]\end{aligned}$$

If $\eta = \eta'$, then the error is 0; in any event, we expect the error of $\frac{4}{3}T - \frac{1}{3}T'$ to be considerably smaller than the error in T. Because

$$\tfrac{4}{3}T - \tfrac{1}{3}T' = \tfrac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

we are led to

7.6.4 Theorem Simpson's Rule If the function f is continuous on the closed interval $[a, b]$, n is an even integer, and the numbers $a = x_0, x_1, x_2, \ldots, x_n = b$ form a regular partition of $[a, b]$, and

$$S_n = \tfrac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where $\Delta x = (b - a)/n$, then

$$\lim_{n \to +\infty} S_n = \int_a^b f(x)\, dx$$

Proof

$$\begin{aligned}\lim_{n \to +\infty} S_n &= \tfrac{4}{3}\lim_{n \to +\infty} T - \tfrac{1}{3}\lim_{n \to +\infty} T' \\ &= \tfrac{4}{3}\int_a^b f(x)\, dx - \tfrac{1}{3}\int_a^b f(x)\, dx \\ &= \int_a^b f(x)\, dx\end{aligned}$$

Theorem Simpson's rule is exact for any cubic polynomial $Ax^3 + Bx^2 + Cx + D$.

Proof Let $I(f(x)) = \int_a^b f(x)dx$ and let $S(f(x))$ denote Simpson's approximation.

$$S(1) = \tfrac{1}{6}(b - a)(1 + 4 + 1) = b - a = I(1)$$

$$S(x) = \tfrac{1}{6}(b - a)[b + \tfrac{4}{2}(b + a) + a] = \tfrac{1}{6}(b - a) \cdot 3(b + a) = \tfrac{1}{2}(b^2 - a^2) = I(x)$$

$$S(x^2) = \tfrac{1}{6}(b - a)[b^2 + 4(\tfrac{1}{2}(b + a))^2 + a^2] = \tfrac{1}{6}(b - a) \cdot 2[b^2 + ab + a^2] = \tfrac{1}{3}(b^3 - a^3) = I(x^2)$$

$$S(x^3) = \tfrac{1}{6}(b - a)[b^3 + 4(\tfrac{1}{2}(b + a))^3 + a^3] = \tfrac{1}{6}(b - a) \cdot \tfrac{3}{2}[b^3 + b^2a + ba^2 + a^3] = \tfrac{1}{4}(b^4 - a^4) = I(x^3)$$

$$\begin{aligned}S(Ax^3 + Bx^2 + Cx + D) &= AS(x^3) + BS(x^2) + CS(x) + DS(1) \\ &= AI(x^3) + BI(x^2 + CI(x) + DI(1) = I(Ax^3 + Bx^2 + Cx + D)\end{aligned}$$

7.6.5 Theorem Let the function f be continuous on the closed interval $[a,b]$ and f', f'', f''' and $f^{(4)}$ all exist on (a,b). If

$$\epsilon_S = \int_a^b f(x)dx - S$$

where S is the approximate value of $\int_a^b f(x)$ found by Simpson's rule, then there is some number η in (a,b) such that

$$\epsilon_S = -\tfrac{1}{180}(b-a)f^{(4)}(\eta)(\Delta x)^4$$

A substitution may change the integrand from one whose derivatives are unbounded to to one whose derivatives are bounded. See Exercises 32 and 36.

Exercises 7.6

In Exercises 1–8, (a) compute to three decimal places the approximate value of the definite integral by the trapezoidal rule for the indicated value of n. (b) Compare the result in part (a) with the exact value of the definite integral.

1. $\int_0^2 x^3dx = \frac{1}{4}x^4\Big|_0^2 = 4$. $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$. By the trapezoidal rule
$\int_0^2 x^3dx \approx \frac{1}{2}\cdot\frac{1}{2}[0^3 + 2(\frac{1}{2})^3 + 2(1)^3 + 2(\frac{3}{2})^3 + 2^3] = -[0.000 + 0.250 + 2.000 + 6.750 + 8.000] = 4.250$. $\epsilon_T = -0.250$

2. $\int_0^2 x\sqrt{4-x^2}dx = -\frac{1}{3}(4-x^2)^{3/2}\Big]_0^2 = \frac{8}{3}$. $\Delta x = \frac{b-a}{n} = \frac{2}{8} = .25$. By the trapezoidal rule
$\int_0^2 x\sqrt{4-x^2}dx = \frac{1}{2}\cdot\frac{1}{4}[0 + 2(.25)\sqrt{4-.25^2} + 2(.5)\sqrt{4-.5^2} + 2(.75)\sqrt{4-.75^2} + 2(1)\sqrt{3}$
$+ 2(1.25)\sqrt{4-1.25^2} + 2(1.5)\sqrt{4-1.75} + 0 = 2.679$. $\epsilon_T = \frac{8}{3} - 2.679 = -0.012$

3. $\int_0^\pi \cos x\, dx = -\sin x\Big|_0^\pi = \sin\pi - \sin 0 = 0$. $\Delta x = \frac{b-a}{n} = \frac{1}{2} = -\pi$. By the trapezoidal rule
$\int_0^\pi \cos x\, dx \approx \frac{1}{2}\cdot\frac{\pi}{4}(\cos 0 + 2\cos\frac{1}{4}\pi + 2\cos\frac{1}{2}\pi + 2\cos\frac{3}{4}\pi + \cos\pi) = -(1 + \sqrt{2} + 0 - \sqrt{2} - 1) = 0$. $\epsilon_T = 0$

4. $\int_0^\pi \sin x\, dx$; $n = 6$

▸ (a) $\Delta x = \frac{\pi - 0}{6} = \frac{\pi}{6}$. By the trapezoidal rule

$$\int_0^\pi \sin x\, dx \approx \frac{1}{2}\cdot\frac{\pi}{6}\left[\sin\frac{\pi}{6} + 2\sin\frac{2\pi}{6} + 2\sin\frac{3\pi}{6} + 2\sin\frac{4\pi}{6} + 2\sin\frac{5\pi}{6} + \sin\pi\right]$$
$$= \frac{\pi}{12}\left(0 + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{2}\sqrt{3} + 2\cdot 1 + 2\cdot\frac{1}{2}\sqrt{3} + 2\cdot\frac{1}{2} + 0\right) = \frac{\pi}{12}(4 + 2\sqrt{3}) \approx 1.954$$

(b) $$\int_0^\pi \sin x\, dx = -\cos x\Big]_0^\pi = -\cos\pi + \cos 0 = 2$$

Thus $\epsilon_T = 2 - 1.954 = 0.046$

5. $\int_1^2 \frac{dx}{x} = \ln x\Big]_1^2 = \ln 2 \approx 0.693$. $\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = 0.2$. By the trapezoidal rule
$\int_1^2 \frac{dx}{x} \approx -(0.2)\left[\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right] = 0.696$. $\epsilon_T = 0.693 - 0.696 = -0.003$

6. $\int_2^{10} \frac{dx}{1+x} = \ln(1+x)\Big]_2^{10} = \ln 11 - \ln 3 = \ln\frac{11}{3} \approx 1.299$. $\Delta x = \frac{b-a}{8} = 1$. By the trapezoidal rule
$\int_2^{10} \frac{dx}{1+x} = \frac{1}{2}(1)[\frac{1}{2} + 2(\frac{1}{3}) + 2(\frac{1}{4}) + 2(\frac{1}{5}) + 2(\frac{1}{6}) + 2(\frac{1}{7}) + 2(\frac{1}{8}) + 2(\frac{1}{9}) + \frac{1}{10}] \approx 1.308$. $\epsilon_T = 1.299 - 1.308 = -0.009$

7. $\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}\Big]_0^1 = \sinh^{-1}1 \approx 0.881$. $\Delta x = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$. By the trapezoidal rule
$\int_0^1 \frac{dx}{\sqrt{1+x^2}} \approx \frac{1}{2}(0.2)\left[\frac{1}{1} + \frac{2}{\sqrt{1+.2^2}} + \frac{2}{\sqrt{1+.4^2}} + \frac{2}{\sqrt{1+.6^2}} + \frac{2}{\sqrt{1+.8^2}} + \frac{1}{\sqrt{2}}\right] = 0.880$. $\epsilon_T = .881 - .880 = .001$

8. $\int_2^3 \sqrt{1+x^2}\,dx;\ n=6$

▸ (a) $\Delta x = \frac{3-2}{6} = \frac{1}{6}$. By the trapezoidal rule $\int_2^3 \sqrt{1+x^2}\,dx$

$= \frac{1}{2}\cdot\frac{1}{6}\left[\sqrt{1+(2)^2} + 2\sqrt{1+(\frac{13}{6})^2} + 2\sqrt{1+(\frac{14}{6})^2} + 2\sqrt{1+(\frac{15}{6})^2} + 2\sqrt{1+(\frac{16}{6})^2} + 2\sqrt{1+(\frac{17}{6})^2} + \sqrt{1+(3)^2}\right]$

$= 2.69488$

(b) $\int_2^3 \sqrt{1+x^2}dx = \frac{1}{2}\left[x\sqrt{1+x^2} + \ln(x+\sqrt{1+x^2})\right]_2^3 = \frac{1}{2}[3\sqrt{10} - 2\sqrt{5} + \ln(3+\sqrt{10}) - \ln(2+\sqrt{5})] \approx 2.69475$

Thus $\epsilon_T = 2.69475 - 2.69488 = -0.00013$.

In Exercises 9–12, compute to three decimal places the approximate value of the definite integral by th trapezoidal rule for the indicated value of n.

9. $\Delta x = \frac{1}{n}(b-a) = \frac{1}{6}(\frac{3}{2}\pi - \frac{1}{2}\pi) = \frac{1}{6}\pi$. By the trapezoidal rule, $\int_{\pi/2}^{3\pi/2} \frac{\sin x}{x}\,dx$

$\approx \frac{1}{2}\cdot\frac{\pi}{6}\left[\frac{2}{\pi}\sin\frac{\pi}{2} + 2\cdot\frac{3}{2\pi}\sin\frac{2\pi}{3} + 2\cdot\frac{6}{5\pi}\sin\frac{5\pi}{6} + 2\cdot\frac{1}{\pi}\sin\pi + 2\cdot\frac{6}{7\pi}\sin\frac{7\pi}{6} + 2\cdot\frac{3}{4\pi}\sin\frac{4\pi}{3} + \frac{2}{3\pi}\sin\frac{3\pi}{2}\right]$

$= \frac{\pi}{12}\left[\frac{2}{\pi} + \frac{3\sqrt{3}}{2\pi} + \frac{6}{6\pi} + 0 - \frac{6}{7\pi} - \frac{3\sqrt{3}}{4\pi} - \frac{2}{3\pi}\right] = \frac{\pi}{12}\cdot\frac{2.975}{\pi} = 0.248$ {Exact: 0.238}

10. $\Delta x = \frac{b-a}{n} = \frac{1}{4} = .25$. By the trapezoidal rule.

$\int_0^1 \sqrt{1+x^3}dx \approx \frac{1}{2}\cdot\frac{1}{4}(1 + 2\sqrt{1+.25^3} + 2\sqrt{1+.5^3} + 2\sqrt{1+.75^3} + \sqrt{2}) \approx 1.117$ {Exact: 1.111}

11. $\Delta x = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$. By the trapezoidal rule $\int_0^2 \sqrt{1+x^4}\,dx$

$\approx \frac{1}{2}\cdot\frac{1}{3}\left[1 + 2\sqrt{1+(\frac{1}{3})^4} + 2\sqrt{1+(\frac{2}{3})^4} + 2\sqrt{2} + 2\sqrt{1+(\frac{4}{3})^4} + 2\sqrt{1+(\frac{5}{3})^4} + 2\sqrt{17}\right] = 3.689$ {3.6535}

12. $\int_0^\pi \frac{\sin x}{1+x}\,dx;\ n=6$

▸ $\Delta x = \frac{\pi-0}{6} = \frac{1}{6}\pi$. By the trapezoidal rule,

$$\int_0^\pi \frac{\sin x}{1+x}\,dx \approx \frac{1}{2}\cdot\frac{1}{6}\pi\left[\frac{\sin 0}{1} + 2\cdot\frac{\sin\frac{1}{6}\pi}{1+\frac{1}{6}\pi} + 2\cdot\frac{\sin\frac{2}{6}\pi}{1+\frac{2}{6}\pi} + 2\cdot\frac{\sin\frac{3}{6}\pi}{1+\frac{3}{6}\pi} + 2\cdot\frac{\sin\frac{4}{6}\pi}{1+\frac{4}{6}\pi} + 2\cdot\frac{\sin\frac{5}{6}\pi}{1+\frac{5}{6}\pi} + \frac{\sin\pi}{1+\pi}\right] = 0.816$$

The value of the integral is 0.844 to three decimal places.

In Exercises 13–18, find bounds for the truncation error in the approximation of the indicated exercise.

13. $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$. f'' is increasing on $[0,2]$. Therefore, the absolute minimum value of f'' on $[0,2]$ is $f''(0) = 0$, and the absolute maximum value of f'' on $[0,2]$ is $f''(2) = 12$. If ϵ_T is the error in the resul of Exercise 1, because $\Delta x = \frac{1}{2}$, there is some number η in $[0,2]$ such that

$\epsilon_T = -\frac{1}{12}(2-0)f''(\eta)(\frac{1}{2})^2 = -\frac{1}{24}f''(\eta)$

Thus $-\frac{1}{24}f''(2) \le \epsilon_T \le -\frac{1}{24}f''(0)$; $-\frac{1}{24}(12) \le \epsilon_T \le -\frac{1}{24}(0)$; $-0.5 \le \epsilon_T \le 0$.

14. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$. Therefore, the absolute minimum value of f'' on $[0,\pi]$ is $f''(\frac{1}{2}\pi) = -1$, and the absolute maximum value of f'' on $[0,\pi]$ is $f''(0) = 0$. If ϵ_T is the error in the result o Exercise 4, because $\Delta x = \frac{1}{6}\pi$, there is some number η in $[0,\pi]$ such that

$\epsilon_T = -\frac{1}{12}(\pi-0)f''(\eta)(\frac{1}{6}\pi)^2 = -\frac{1}{432}\pi^3 f''(\eta)$

Thus $-\frac{1}{432}\pi^3 f''(0) \le \epsilon_T \le -\frac{1}{432}\pi^3 f''(\frac{1}{2}\pi)$; $-\frac{1}{432}\pi^3(0) \le \epsilon_T \le -\frac{1}{432}\pi^3(-1)$; $0 \le \epsilon_T \le \frac{1}{432}\pi^3 \approx 0.072$

15. $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$. The absolute maximum value of f'' on $[0,\pi]$ is $f''(\pi) = 1$ and the absolute minimum value of f'' on $[0,\pi]$ is $f''(0) = -1$. If ϵ_T is the error in the result of Exercise 3, becaus $\Delta x = \frac{1}{4}\pi$ there is some number η in $[0,\pi]$ such that

$\epsilon_T = -\frac{1}{12}(\pi-0)f''(\eta)\left(\frac{\pi}{4}\right)^2$. Thus $-\frac{\pi^3}{192}(1) \le \epsilon_T \le -\frac{\pi^3}{192}(-1)$; $-0.161 \le \epsilon_T \le 0.161$ $\{\epsilon_T = 0\}$

16. Exercise 6.

▷ By Theorem 7.6.2, if

$$\epsilon_T = \int_2^{10} \frac{dx}{1+x} - T$$

then $b - a = 8$ and $\Delta x = 1$. Thus, there is some number η in $[2, 10]$ such that

$$\epsilon_T = -\tfrac{1}{12}(8) f''(\eta)(1)^2 = -\tfrac{2}{3} f''(\eta) \tag{1}$$

We find the absolute extrema of f'' on $[2, 10]$. We have

$$f(x) = (1+x)^{-1}, \quad f'(x) = -(1+x)^{-2}, \quad f''(x) = 2(1+x)^{-3}$$

Because f'' is decreasing on $[2, 10]$, the absolute maximum value of f'' is at 2 and the absolute minimum value of f'' is at 10. From Eq. (1), we have

$$-\tfrac{2}{3} f''(2) = -\frac{2}{3}\left(\frac{2}{27}\right) = -0.494 \quad -\frac{2}{3} f''(10) = -\frac{2}{3}\left(\frac{2}{11^3}\right) = -0.001$$

We conclude that

$$-0.494 < \epsilon_T < -0.001$$

17. $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$. f'' is decreasing on $[0, 2]$. Therefore, the absolute maximum value of f'' on $[1, 2]$ is $f''(1) = 2$, and the absolute minimum value of f'' on $[1, 2]$ is $f''(2) = \frac{1}{4}$. If ϵ_T is the error in the result of Exercise 5, because $\Delta x = 0.2$ there is some number η in $[1, 2]$ such that

$$\epsilon_T = -\tfrac{1}{12}(2-1) f''(\eta)(0.2)^2 = -0.0033 f''(\eta)$$

$$-.0033 f''(1) \le \epsilon_T \le -.0033 f''(2); \; -.0033(2) \le \epsilon_T \le -.0033(\tfrac{1}{4}); \; -.007 \le \epsilon_T \le -.001$$

18. $f(x) = \sqrt{1+x^2}$, $f'(x) = \dfrac{x}{\sqrt{1+x^2}}$, $f''(x) = \dfrac{1}{(1+x^2)^{3/2}}$. f'' is decreasing on $[2, 3]$. Thus, the absolute maximum value of f'' on $[2, 3]$ is $f''(2) = 5^{-3/2}$, and the absolute minimum value of f'' on $[2, 3]$ is $f''(3) = 10^{-3/2}$. If ϵ_T is the error in the result of Exercise 8, because $\Delta x = \frac{1}{6}$ there is some number η in $[2, 3]$ such that

$$\epsilon_T = -\tfrac{1}{12}(2-1) f''(\eta)(\tfrac{1}{6})^2 = -\tfrac{1}{432} f''(\eta)$$

$$-\tfrac{1}{432} f''(2) \le \epsilon_T \le -\tfrac{1}{432} f''(3); \; -\tfrac{1}{432}(5^{-3/2}) \le \epsilon_T \le -\tfrac{1}{432}(10^{-3/2}); \; -.00021 \le \epsilon_T \le -.00007$$

19. $\displaystyle\int_0^2 x^3 dx = \tfrac{1}{4}x^4\Big|_0^2 = 4$. $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$. By Simpson's rule

$$\int_0^2 x^3 dx = \tfrac{1}{3} \cdot \tfrac{1}{2}\left[0^3 + 4(\tfrac{1}{2})^3 + 2(1)^3 + 4(\tfrac{3}{2})^3 + 2^3\right] = \tfrac{1}{6}[0 + 0.5 + 2 + 13.5 + 8] = \tfrac{1}{6} \cdot 24 = 4$$

20. Approximate $\int_0^\pi \sin x \, dx$ by Simpson's rule with $n = 6$. Compare the result with those obtained in Exercise 4 and observe that Simpson's rule gives better accuracy than the trapezoidal rule with the same number of subintervals.

▷ $\Delta x = \dfrac{\pi - 0}{6} = \dfrac{\pi}{6}$. By Simpson's rule

$$\int_0^\pi \sin x \, dx \approx \frac{1}{3} \cdot \frac{\pi}{6}\left[\sin 0 + 4 \sin \frac{\pi}{6} + 2 \sin \frac{2\pi}{6} + 4 \sin \frac{3\pi}{6} + 2 \sin \frac{4\pi}{6} + 4 \sin \frac{5\pi}{6} + \sin \pi\right]$$

$$= \frac{\pi}{18}\left[0 + 4 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2}\sqrt{3} + 4 \cdot 1 + 2 \cdot \frac{1}{2}\sqrt{3} + 4 \cdot \frac{1}{2} + 0\right] = \frac{\pi}{9}[4 + \sqrt{3}] = 2.00086$$

In Exercise 4, we found the exact value to be 2 and the trapezoidal approximation to be 1.954. Thus, with $n = 6$, the error in the trapezoidal rule is 0.046 and in Simpson's rule it is -0.00086. Simpson's rule is best here. But for any even integer the trapezoidal approximation to $\int_{-1}^{1} |x| \, dx$ is exact but Simpson's is not.

In Exercises 21–24 compute to four decimal places the approximate value of the definite integral by Simpson's rule for the indicated value of n. (b) Compare the result in part (a) with the exact value of the definite integral.

21. $\displaystyle\int_{-1}^0 \frac{dx}{1-x} = -\ln(1-x)\Big]_{-1}^0 = \ln 2 \approx .6931$. $\Delta x = \frac{b-a}{n} = \frac{0-(-1)}{4} = 0.25$. By Simpson's rule $\displaystyle\int_0^1 \frac{dx}{1-x}$

$$\approx \frac{1}{3} \cdot \frac{1}{4}\left(\frac{1}{2} + \frac{4}{1.75} + \frac{2}{1.5} + \frac{4}{1.25} + 1\right) = \tfrac{1}{12}(0.5 + 2.286 + 1.333 + 3.2 + 1) = \tfrac{1}{12}(8.319) = .6932$$

22. $\int_{-0.5}^{0} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big]_{-0.5}^{0} = \sin^{-1} \frac{1}{2} = \frac{1}{6}\pi \approx .52360$. $\Delta x = \frac{b-a}{n} = \frac{0-(-0.5)}{4} = .125$. By Simpson's rule,

$$\int_{-0.5}^{0} \frac{dx}{\sqrt{1-x^2}} = \frac{1}{3}(.125)\left(\frac{1}{\sqrt{1-(-.5)^2}} + \frac{4}{\sqrt{1-(-.375)^2}} + \frac{2}{\sqrt{1-(-.25)^2}} + \frac{4}{\sqrt{1-(-.125)^2}} + 1\right) = .52362$$

23. $\int_0^1 \frac{dx}{x^2+x+1} = \int_0^1 \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{x+\frac{1}{2}}{\sqrt{3}/2}\Big]_0^1 = \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3\sqrt{3}} \approx .6046.$

$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$. By Simpson's rule

$$\int_0^1 \frac{dx}{x^2+x+1} \approx \frac{1}{3}\cdot\frac{1}{4}\left[1 + 4\cdot\frac{16}{21} + 2\cdot\frac{4}{7} + 4\cdot\frac{16}{37} + \frac{1}{3}\right] = \frac{1}{12}\cdot 7.2535 = 0.6045$$

24. $\int_1^2 \frac{dx}{x+1}$; $n = 8$

▷ (a) $\Delta x = \frac{1-0}{8} = 0.125$. By Simpson's rule

$$\int_1^2 \frac{dx}{x+1} \approx \frac{1}{3}(0.125)\left[\frac{1}{2} + 4\cdot\frac{1}{2.125} + 2\cdot\frac{1}{2.25} + 4\cdot\frac{1}{2.375} + 2\cdot\frac{1}{2.5} + 4\cdot\frac{1}{2.675} + 2\cdot\frac{1}{2.75} + 4\cdot\frac{1}{2.875} + \frac{1}{3}\right]$$
$$= 0.4054655$$

(b) $\int_1^2 \frac{dx}{x+1} = \ln(x+1)\Big]_1^2 = \ln 3 - \ln 2 = \ln 1.5 \approx 0.4054651$

In Exercises 25–28, find bounds for the truncation error in the approximation of the indicated exercise.

25. $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) \equiv 0$, $f^{(4)}(x) \equiv 0$. If ϵ_S is the error in the result of Exercise 19, because $\Delta x = \frac{1}{2}$ there is some number η in $[0,2]$ such that $\epsilon_S = -\frac{1}{180}(2-0)f^{(4)}(\eta)(\frac{1}{2})^4$.

Because $f^{(4)}(x) \equiv 0$, then $\epsilon_S = 0$. Thus the result in Exercise 19 is exact.

26. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) \equiv -\cos x$, $f^{(4)}(x) \equiv \sin x$. The absolute minimum value of $f^{(4)}$ on $[0,\pi]$ is $f^{(4)}(0) = 0$, and the absolute maximum value of $f^{(4)}$ on $[0,\pi]$ is $f^{(4)}(\frac{1}{2}\pi) = 1$. If ϵ_S is the error in the result of Exercise 20, because $\Delta x = \frac{1}{6}\pi$ there is some number η in $[0,\pi]$ such that

$\epsilon_S = -\frac{1}{180}[\pi - 0)]f^{(4)}(\eta)(\frac{1}{6}\pi)^4 = -.0013f^{(4)}(\eta)$. Thus $-.0013\cdot 1 \le \epsilon_S \le -.0013\cdot 0$; $-0.0013 \le \epsilon_S \le 0$

27. $f(x) = \frac{1}{1-x}$, $f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, $f'''(x) = \frac{6}{(1-x)^4}$, $f^{(4)}(x) = \frac{24}{(1-x)^5}$

Because $f^{(4)}$ is increasing on $[-1,0]$, the absolute minimum value of $f^{(4)}$ on $[-1,0]$ is $f^{(4)}(-1) = \frac{3}{4}$, and the absolute maximum value of $f^{(4)}$ on $[-1,0]$ is $f^{(4)}(0) = 24$. If ϵ_S is the error in the result of Exercise 21, because $\Delta x = \frac{1}{4}$ there is some number η in $[-1,0]$ such that

$\epsilon_S = -\frac{1}{180}[0-(-1)]f^{(4)}(\eta)(\frac{1}{4})^4 = -\frac{1}{46080}f^{(4)}(\eta)$

Thus $-\frac{1}{46080}\cdot 24 \le \epsilon_S \le -\frac{1}{46080}\cdot\frac{3}{4}$; $-0.0005 \le \epsilon_S \le 0$ $\{\epsilon_S = -0.0001\}$

28. Find bounds for the truncation error in Exercise 24.

▷ We use Theorem 7.6.5.

$$f(x) = (x+1)^{-1}; \quad f'(x) = -(x+1)^{-2}; \quad f''(x) = 2(x+1)^{-3}; \quad f'''(x) = -6(x+1)^{-4} f^{(4)}(x) = 24(x+1)^{-5}$$

Because $f^{(4)}$ is decreasing on $[1,2]$, the absolute minimum value of $f^{(4)}$ is at 2, and the absolute maximum value of $f^{(4)}$ is at 1.

$$f^{(4)}(1) = \frac{24}{32} = \frac{3}{4} \quad f^{(4)}(2) = \frac{24}{243} = \frac{8}{81}$$

Thus,

$$-\frac{1}{180}(b-a)f^{(4)}(1)(\Delta x)^4 = -\frac{1}{180}(1)(\frac{3}{4})(\frac{1}{8})^4 = -1.02\times 10^{-6}$$

And

$$-\frac{1}{180}(b-a)f^{(4)}(2)(\Delta x)^4 = -\frac{1}{180}(1)(\frac{8}{81})(\frac{1}{8})^4 = -1.34\times 10^{-7}$$

We conclude that

$$-1.02\times 10^{-6} < \epsilon_S < -1.34\times 10^{-7}$$

In Exercises 29–34, compute to four decimal places the approximate value of the definite integral by Simpson's rule for the indicated value of n. The integral cannot be evaluated exactly in terms of elementary functions.

29. $\Delta x = \frac{1}{n}(b-a) = \frac{1}{6}(\frac{3}{2}\pi - \frac{1}{2}\pi) = \frac{1}{6}\pi$. By Simpson's rule $\int_{\pi/2}^{3\pi/2} \frac{\sin x}{x} dx$

$$\approx \frac{1}{3}\cdot\frac{\pi}{6}\left[\frac{\sin\frac{1}{2}\pi}{\frac{1}{2}\pi} + 4\cdot\frac{\sin\frac{2}{3}\pi}{\frac{2}{3}\pi} + 2\cdot\frac{\sin\frac{5}{6}\pi}{\frac{5}{6}\pi} + 4\cdot\frac{\sin\pi}{\pi} + 2\cdot\frac{\sin\frac{7}{6}\pi}{\frac{7}{6}\pi} + 4\cdot\frac{\sin\frac{4}{3}\pi}{\frac{4}{3}\pi} + \frac{\sin\frac{3}{2}\pi}{\frac{3}{2}\pi}\right]$$

$\frac{\pi}{18}[0.6366 + 1.6540 + 0.3820 + 0 - 0.2728 - 0.8270 - 0.2122] = \frac{\pi}{18}\cdot 1.3606 = 0.2375$ {Exact: 0.2376}

30. $\Delta x = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$. By Simpson's rule $\int_0^2 \sqrt{1+x^4}$

$\approx \frac{1}{3}\cdot\frac{1}{3}\left(1 + 4\sqrt{1+(\frac{1}{3})^4} + 2\sqrt{1+(\frac{2}{3})^4} + 4\sqrt{2} + 2\sqrt{1+(\frac{4}{3})^4} + 4\sqrt{1+(\frac{5}{3})^4} + \sqrt{17}\right) = 3.65354$ {Exact: 3.65348}

31. $\Delta x = \frac{b-a}{n} = \frac{1.8-1}{4} = 0.2$. By Simpson's rule $\int_1^{1.8} \sqrt{1+x^3}\, dx$

$\approx \frac{1}{3}(0.2)[\sqrt{2} + 4\sqrt{1+1.2^3} + 2\sqrt{1+1.4^3} + 4\sqrt{1+1.6^3} + \sqrt{1+1.8^3}] = 1.5690$ {1.5689}

32. $\int_0^1 \sqrt[3]{1-x^2}\, dx;\ n = 4$

▸ $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. By Simpson's rule,

$$\int_0^1 \sqrt[3]{1-x^2}\, dx \approx \frac{1}{3}\cdot\frac{1}{4}\left[1 + 4\sqrt[3]{1-(\tfrac{1}{4})^2} + 2\sqrt[3]{1-(\tfrac{2}{4})^2} + 4\sqrt[3]{1-(\tfrac{3}{4})^2} + 0\right] = 0.814$$

This result is suspect because the derivatives of $\sqrt[3]{1-x^2}$ are unbounded on $(0,1)$ and so Theorem 7.6.5 does not give a bound for the error. Applying Simpson's rule with $n = 8$ we get 0.8305 and with $n = 16$ we get 0.8370. Thus we cannot even be certain of the second digit. The substitution $x = 1 - u^3$, $dx = -3u^2\, du$ gives $\int_0^1 3u^3\sqrt[3]{2-u^3}\, du$ whose integrand has bounded derivatives on $(0,1)$. Applying Simpson's rule to this integral, we get 0.8378 for $n = 4$, 0.8410 for $n = 8$, and 0.8413 for $n = 16$, which is the value of the integral to four decimal places.

33. $\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = 0.25$. By Simpson's rule $\int_0^2 \frac{dx}{\sqrt{1+x^3}}$

$$\approx \frac{0.25}{3}\left[1 + \frac{4}{\sqrt{1+.25^3}} + \frac{2}{\sqrt{1+.5^3}} + \frac{4}{\sqrt{1+.75^3}} + \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{1+1.25^3}} + \frac{2}{\sqrt{1+1.5^3}} + \frac{4}{\sqrt{1+1.75^3}} + \frac{1}{3}\right]$$

$= 1.40223$ {Exact: 1.40218}

34. $\Delta x = \frac{b-a}{n} = \frac{\pi/2-0}{6} = \frac{\pi}{12}$. We use 15°. By Simpson's rule $\int_0^{\pi/2} \sqrt{\sin x}\, dx$

$\approx \frac{1}{3}\cdot\frac{\pi}{12}(0 + 4\sqrt{\sin 15^\circ} + 2\sqrt{\sin 30^\circ} + 4\sqrt{\sin 45^\circ} + 2\sqrt{\sin 60^\circ} + 4\sqrt{\sin 75^\circ} + 1) = 1.1873$. The result is suspect because the derivatives of $\sqrt{\sin x}$ are unbounded on $(0, \frac{1}{2}\pi)$ and so Theorem 7.6.5 does not give a bound for the error. Applying Simpson's rule with $n = 12$ we get 1.19430. Thus we cannot even be certain of the second decimal place. The substitution $x = u^2$, $dx = 2u\, du$ gives $\int_0^{\sqrt{\pi/2}} \sqrt{\sin u^2}(2u\, du)$ whose integrand has bounded derivative in $(0, \frac{1}{2}\pi)$. Applying Simpson's rule to this integral, we get 1.1977 with $n = 6$ and 1.1981 with $n = 12$, which is the value of the integral to four decimal places.

35. The value of the integral $\int_0^2 \sqrt{4-x^2}\, dx$ is the measure of the area of the region in the first quadrant enclosed by the quarter circle of radius 2 centered at the origin. Thus

$\int_0^2 \sqrt{4-x^2}\, dx = \frac{1}{4}\cdot\pi(2)^2 = \pi \approx 3.142$. $\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = 0.25$. By the trapezoidal rule $\int_0^2 \sqrt{4-x^2}\, dx$

$\approx \frac{.25}{2}(2 + 2\sqrt{4-0.25^2} + 2\sqrt{4-0.5^2} + 2\sqrt{4-0.75^2} + 2\sqrt{3} + 2\sqrt{4-1.25^2} + 2\sqrt{4-1.5^2} + 2\sqrt{4-1.75^2} + 0)$

$= 3.090$. Accuracy is poor because f' is unbounded on $(0,2)$. The substitution $x = 2 - u^2$, $dx = -2u\, du$ gives $2\int_0^{\sqrt{2}} u^2\sqrt{4-u^2}\, du$ whose integrand has bounded derivatives on $(0, \sqrt{2})$. Applying the trapezoidal rule to this integral, we get 3.152 with $n = 8$ and 3.144 with $n = 16$.

36. Show that the exact value of the integral $\int_0^1 4\sqrt{1-x^2}\,dx$ is π by interpreting it as the measure of the area of a region. Use Simpson's rule with $n = 6$ to get an approximate value of the definite integral. Compare the value so obtained with the exact value.

▷ Because the graph of $y = \sqrt{1-x^2}$ for $0 \le x \le 1$ is the first quadrant arc of the circle $x^2 + y^2 = 1$, the integral $\int_0^1 4\sqrt{1-x^2}\,dx$ measures the area of a circle of radius 1. Since

$$A = \pi r^2 = \pi 1^2 = \pi$$

we conclude that the integral has exact value π. Let $\Delta x = \frac{1-0}{6} = \frac{1}{6}$. By Simpson's rule,

$$4\int_0^1 \sqrt{1-x^2}\,dx \approx 4\cdot\frac{1}{3}\cdot\frac{1}{2}\left[\sqrt{1} + 4\sqrt{1-(\tfrac{1}{6})^2} + 2\sqrt{1-(\tfrac{2}{6})^2} + 4\sqrt{1-(\tfrac{3}{6})^2} + 2\sqrt{1-(\tfrac{4}{6})^2} + 4\sqrt{1-(\tfrac{5}{6})^2} + \sqrt{0}\right]$$
$$= 3.110$$

This result is suspect because the derivatives of $\sqrt{1-x^2}$ are unbounded on $(0,1)$ and so Theorem 7.6.5 does not give a bound for the error. The substitution $x = 1 - u^2$, $dx = -2u\,du$ gives $8\int_0^1 u^2\sqrt{1-u^2}\,du$ whose integrand has bounded derivatives on $(0,1)$. Applying Simpson's rule to this integral, we get $3.1408 \approx 3.141$ which is the value of π to three decimal places.

Exercises 37 and 38 refer to standardized normal probability.

37. $P([0,1]) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^1 e^{-x^2/2}dx$. $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{4} = \dfrac{1}{4}$.

(a) By the trapezoidal rule $\displaystyle\int_0^1 e^{-x^2/2}dx \approx \tfrac{1}{2}\cdot\tfrac{1}{4}[e^0 + 2e^{-1/32} + 2e^{-1/8} + 2e^{-9/32} + e^{-1/2}]$

$$= \tfrac{1}{8}[1 + 2(0.9692) + 2(0.8825) + 2(0.7548) + 0.6065)] = \tfrac{1}{8}\cdot 6.8195 = 0.8524$$

Thus $P([0,1]) \approx \dfrac{1}{\sqrt{2\pi}}(0.8524) = 0.3401$.

(b) By Simpson's rule $\displaystyle\int_0^1 e^{-x^2/2}dx \approx \tfrac{1}{3}\cdot\tfrac{1}{4}[e^0 + 4e^{-1/32} + 2e^{-1/8} + 4e^{-9/32} + e^{-1/2}]$

$$= \tfrac{1}{12}[1 + 4(0.96923) + 2(0.88250) + 4(0.75484) + 0.60653] = \tfrac{1}{12}\cdot 10.26782 = 0.85565$$

$P([0,1]) \approx \dfrac{1}{\sqrt{2\pi}}(0.85565) \approx 0.34136$, while $P([0,1]) = 0.34134$ to five decimal places.

38. (a) Because $n = 6$, $\Delta x = \dfrac{b-a}{n} = \dfrac{3+3}{6} = 1$. By the trapezoidal rule with $f(x) = e^{-x^2/2}$,

$$P([-3,3]) = \frac{1}{\sqrt{2\pi}}\int_{-3}^3 e^{-x^2/2}dx \approx \frac{1}{2\sqrt{2\pi}}[e^{-4.5} + 2e^{-2} + 2e^{-0.5} + 2 + 2e^{-0.5} + 2e^{-2} + e^{-4.5}] = 0.9953$$

The exact value of the probability is 0.9973 to four decimal places.

(b) By Simpson's rule with $n = 6$ and $\Delta x = 1$,

$$P([-3,3]) = \frac{1}{\sqrt{2\pi}}\int_{-3}^3 e^{-x^2/2}dx \approx \frac{1}{3\sqrt{2\pi}}[e^{-4.5} + 4e^{-2} + 2e^{-0.5} + 4 + 2e^{-0.5} + 4e^{-2} + e^{-4.5}] = 1.0015$$

In this case, the error in Simpson's rule is twice as great as the error in the trapezoidal rule. The answer is worthless, because a probability cannot exceed 1. If we apply the same rule to the equivalent integral $2P([0,3])$ we get 0.9972 which is off by only one unit in the fourth decimal place.

39. $y = \sin x$; $D_x y = \cos x$. By Theorem 6.1.2 and Simpson's rule with $n = 8$, $\Delta x = \frac{1}{8}\pi$

$$L = \int_0^\pi \sqrt{1+\cos^2 x}\,dx \approx \tfrac{1}{3}(\tfrac{1}{8}\pi)[\sqrt{1+\cos 0} + 4\sqrt{1+\cos^2 \tfrac{1}{8}\pi} + 2\sqrt{1+\cos^2 \tfrac{1}{4}\pi} + 4\sqrt{1+\cos^2 \tfrac{3}{8}\pi}$$
$$+ 2\sqrt{1+\cos^2 \tfrac{1}{2}\pi} + 4\sqrt{1+\cos^2 \tfrac{5}{8}\pi} + 2\sqrt{1+\cos^2 \tfrac{3}{4}\pi} + 4\sqrt{1+\cos^2 \tfrac{7}{8}\pi} + \sqrt{1+\cos^2 \pi}] = 3.8203$$

40. In Exercise 6.1.28, you used NINT to find to four significant digits the length of the arc of the cosine curve from the point $(0,1)$ to the point $(\frac{1}{3}\pi, \frac{1}{2})$. Now compute this length by Simpson's rule with $n = 8$.

▷ Let $f(x) = \cos x$. Then $f'(x) = -\sin x$ and

$$\sqrt{1 + [f'(x)]^2} = \sqrt{1+\sin^2 x}$$

By Theorem 6.1.2, we have

$$L = \int_0^{\pi/3} \sqrt{1+\sin^2 x}\,dx$$

By Simpson's rule, $\Delta x = \frac{1}{8}\cdot\frac{1}{3}\pi = \frac{1}{24}\pi$ and

$$L \approx \frac{1}{3}\cdot\frac{1}{24}\pi[1 + 4\sqrt{1+\sin^2\tfrac{1}{24}\pi} + 2\sqrt{1+\sin^2\tfrac{2}{24}\pi} + 4\sqrt{1+\sin^2\tfrac{3}{24}\pi} + 2\sqrt{1+\sin^2\tfrac{4}{24}\pi}$$
$$+ 4\sqrt{1+\sin^2\tfrac{5}{24}\pi} + 2\sqrt{1+\sin^2\tfrac{6}{24}\pi} + 4\sqrt{1+\sin^2\tfrac{7}{24}\pi} + \sqrt{1+\sin^2\tfrac{1}{24}\pi}]$$
$$= 1.185950$$

The length of the curve is 1.185952 correct to six decimal places.

41. $\Delta x = 4/8 = 0.5$. (a) By the trapezoidal rule $\int_0^4 f(x)dx$
$\approx \frac{1}{2}(.5)[3.25 + 2(4.17) + 2(4.60) + 2(3.84) + 2(3.59) + 2(4.23) + 2(4.01) + 2(3.96) + 3.75] = 15.95$
(b) By Simpson's rule $\int_0^4 f(x)dx$
$\approx \frac{1}{3}(.5)[3.25 + 4(4.17) + 2(4.60) + 4(3.84) + 2(3.59) + 4(4.23) + 2(4.01) + 4(3.96) + 3.75] = 16.03$

42. $\Delta x = 4/10 = 0.4$ (a) By the trapezoidal rule $\int_0^4 f(x)dx$
$\approx \frac{1}{2}(.4)[8.4 + 2(8.1) + 2(7.9) + 2(7.5) + 2(7.6) + 2(7.2) + 2(6.8) + 2(6.3) + 2(6.5) + 2(6.0) + 5.7] = 28.38$
(b) By Simpson's rule $\int_0^4 f(x)dx$
$\approx \frac{1}{3}(.4)[8.4 + 4(8.1) + 2(7.9) + 4(7.5) + 2(7.6) + 4(7.2) + 2(6.8) + 4(6.3) + 2(6.5) + 4(6.0) + 5.7] = 28.28$

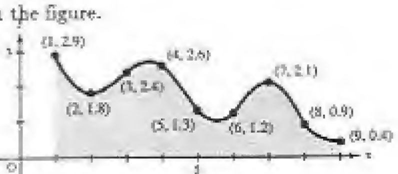
43. $\Delta x = 1$. A square units is the area of the region. By Simpson's rule
$A = \int_0^6 f(x)dx \approx \frac{1}{3}(1)[3.2 + 4(4.5) + 2(5.4) + 4(4.4) + 2(2.7) + 4(4.9) + 5.3] = 26.6$

44. Use Simpson's rule to approximate the area of the shaded region in the figure.

▷ From the figure we see that $\Delta x = 1$ and $n = 8$. By Simpson's rule,

$$A \approx \frac{1}{3}\cdot 1[2.9 + 4\cdot 1.8 + 2\cdot 2.4 + 4\cdot 2.6 + 2\cdot 1.3 + 4\cdot 1.2 + 2\cdot 2.1 + 4\cdot 0.9 + 0.4]$$
$$= 13.9$$

■ The area of the shaded region is approximately 13.9 square units.



45. $\Delta t = \frac{1}{60}$. s miles is the distance traveled. By Simpson's rule $s = \int_0^{1/6} v(t)dt \approx$
$\frac{1}{3}\cdot\frac{1}{60}[0 + 4(30) + 2(33) + 4(41) + 2(38) + 4(32) + 2(42) + 4(45) + 2(41) + 4(37) + 22] = 5.9$

46. A square feet is the area of the lot. $\Delta x = 20$. By Simpson's rule
$A = \int_0^{240} f(x)dx \approx \frac{1}{3}\cdot 20[150 + 4(154) + 2(158) + 4(165) + 2(163) + 4(172) + 175] = 39{,}080$

47. A square units is the area of the loop of $y^2 = 8x^2 - x^5 = x^2(8 - x^3)$, which is symmetric. $y = 0$ when $x = 0, 2$.
With $n = 8$, $\Delta x = \frac{2}{8} = 0.25$. By Simpson's rule $A = 2\int_0^2 x\sqrt{8 - x^3}dx$
$$\approx \frac{1}{3}(0.25)2[0 + 4(0.25)\sqrt{8 - .25^3} + 2(0.5)\sqrt{8 - .5^3} + 4(0.75)\sqrt{8 - .75^3} + 2(1)\sqrt{8 - 1^3}$$
$$+ 4(1.25)\sqrt{8 - 1.25^3} + 2(1.5)\sqrt{8 - 1.5^3} + 4(1.75)\sqrt{8 - 1.75^3} + 0]$$
$= 8.218$. The result is suspect because y' is unbounded on $(0, 2)$. Let $x = 2 - u^2$, $dx = -2\,du$. Then $A = 4\int_0^{\sqrt{2}}(2u^2 - u^4)\sqrt{u^4 - 6u^2 + 12}du$. Applying Simpson's rule to this integral with $n = 8$, we get 8.36275; the exact answer is 8.36280 to five decimal places.

48. The study of light diffraction (dispersion or bending of light around corners) at a rectangular aperture involves the Fresnel integrals $C(t) = \int_0^t \cos\frac{1}{2}\pi x^2 dx$ and $S(t) = \int_0^t \sin\frac{1}{2}\pi x^2 dx$. Complete the following table of values to four decimal places by Simpson's rule.

▷ Simpson's rule with $n = 8$ was used to complete the table.

t	0.2	0.4	0.6	0.8	1.0
$C(t)$	0.1999	0.3975	0.5811	0.7228	0.7799
$S(t)$	0.0042	0.0334	0.1105	0.2493	0.4383

49. By Simpson's rule we find $\int_0^2 e^{-t^2/2}dt = 1.19629$. $v = \sqrt{\frac{2E(2)}{m}} = \sqrt{\frac{2 \times 9.8}{\sqrt{2\pi}}(1.19629)} = 3.06$

50. Apply Simpson's rule to $\int_a^b f(x)\,dx$ where $f(x)$ is a third degree polynomial to prove the *prismoidal formula*.

▷ Simpson's rule is exact for third-degree polynomials. With $n = 2$, $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$ and $\Delta x = \frac{b-a}{2}$, we have $\int_a^b f(x)\,dx = \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + f(x_2)] = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

In Exercises 51–54, evaluate the definite integral by two methods: (a) use the prismoidal formula given in Exercise 50. (b) use the second fundamental theorem of the calculus.

51. $\int_1^3 (4x^3 - 3x^2 + 1)dx = x^4 - x^3 + x\Big|_1^3 = 57 - 1 = 56$. Let $f(x) = 4x^3 - 3x^2 + 1$. By the prismoidal formula, $\int_1^3 f(x)dx = \frac{3-1}{6}[f(1) + 4f(2) + f(3)] = \frac{1}{3}[2 + 4(21) + 82] = \frac{168}{3} = 56$

52. $\int_{-2}^{2} (x^3 + x^2 - 4x - 2)\,dx$

▷ (a) Let $f(x) = x^3 + x^2 - 4x - 2$. By the prismoidal formula, we have

$$a = -2,\ b = 2,\ \frac{b-a}{6} = \frac{2-(-2)}{6} = \frac{2}{3},\ \frac{a+b}{2} = \frac{-2+2}{2} = 0$$

$$\begin{aligned}\int_{-2}^{2} (x^3 + x^2 - 4x - 2)\,dx &= \tfrac{2}{3}[f(-2) + 4f(0) + f(2)] = \tfrac{2}{3}\{[(-2)^3 + (-2)^2 - 4(-2) - 2] + 4(-2) + [2^3 + 2^2 - 4(2) - 2]\} \\ &= \tfrac{2}{3}\{2 - 8 + 2\} = -\tfrac{8}{3}\end{aligned}$$

(b) By the second fundamental theorem of the calculus we have

$$\begin{aligned}\int_{-2}^{2} (x^3 + x^2 - 4x - 2)dx &= \tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - 2x^2 - 2\Big]_{-2}^{2} = \tfrac{1}{4}[2^4 - (-2)^4] + \tfrac{1}{3}[2^3 - (-2)^3] - 2[2^2 - (-2)^2] - 2[2 - (-2)] = -\tfrac{8}{3}\end{aligned}$$

53. $\int_{-1}^{5} (x^3 + 3x^2 - 2x - 6)dx = \left[\frac{1}{4}x^4 + x^3 - x^2 - 6x\right]_{-1}^{5} = 226.25 - 4.25 = 222$. Let $f(x) = x^3 + 3x^2 - 2x - 6$. By the prismoidal formula, $\int_{-1}^{5} f(x)dx = \frac{5-(-1)}{6}[f(-1) + 4f(2) + f(5)] = 1[-2 + 4(10) + 184] = 222$

54. $\int_2^6 (2x^3 - 2x - 3)dx = \left[\frac{1}{2}x^4 - x^2 - 3x\right]_2^6 = 594 - (-2) = 596$. Let $f(x) = 2x^3 - 2x - 3$. By the prismoidal formula $\int_2^6 f(x)dx = \frac{6-2}{6}[f(2) + 4f(4) + f(6)] = \frac{2}{3}[9 + 4(117) + 417] = 596$

55. If T is the approximate value of $\int_a^b f(x)$ by using the trapezoidal rule, then

$$T = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)] \quad (1)$$

If S is the approximate value of $\int_a^b f(x)dx$ by using Simpson's rule, then where n is even

$$S = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \quad (2)$$

From(1): $\frac{2T}{\Delta x} = f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)$ (3)

From(2): $\frac{3S}{\Delta x} = f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)$ (4)

Subtracting members of (3) from corresponding members of (4), we have

$\frac{3S}{\Delta x} - \frac{2T}{\Delta x} = 2f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-1})$; $S = \frac{2}{3}[T + \Delta x(f(x_1) + f(x_3) + \cdots + f(x_{n-1}))]$

7.7 THE INDETERMINATE FORM 0/0 AND CAUCHY'S MEAN-VALUE THEOREM

By Limit Theorem 9, if

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = c \neq 0$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0$$

By Limit Theorem 12, if

$$\lim_{x\to a} f(x) = c \neq 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0 \text{ through positive or negative values}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \pm\infty$$

7.7.1 Definition If f and g are two functions such that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then the function f/g has the *indeterminate form 0/0 at a*.

We have evaluated many such limits in earlier chapters. We now have two forms of L'Hopital's rule that can sometimes be used to find such limits. We use the symbol $\stackrel{0/0}{=}$ to indicate that we have applied one of them. Sometimes the rule is used more than once. Be sure to check the hypotheses each time. It is advisable to simplify after each application. The converse of L'Hopital's rule is not true; if the limit of f'/g' does not exist, no conclusion can be drawn. See Exercise 16, where the one-sided limits exist, and Exercise 41.

7.7.2 Theorem (L'Hopital's rule) Let f and g be functions that are differentiable on an open interval I, except possibly at the number a in I. Suppose that for all $x \neq a$ in I, $g'(x) \neq 0$. If $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$ and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \text{L}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \text{L}$$

The theorem is valid if all the limits are right-hand limits or all the limits are left-hand limits.

7.7.4 Theorem (L'Hopital's rule) Let f and g be functions that are differentiable for all $x > \text{N}$, where N is a positive constant, and suppose that for all $x > \text{N}$, $g'(x) \neq 0$. If $\lim_{x\to+\infty} f(x) = 0$ and $\lim_{x\to+\infty} g(x) = 0$ and

$$\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = \text{L}$$

then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \text{L}$$

The theorem is also valid if "$x\to+\infty$" is replaced by "$x\to-\infty$".

The proof of L'Hopital's rule depends on the following theorem.

7.7.3 Theorem (Cauchy's mean-value theorem) If f and g are two functions such that

(i) f and g are continuous on the closed interval $[a, b]$
(ii) f and g are differentiable on the open interval (a, b)
(iii) for all x in the open interval (a, b), $g'(x) \neq 0$

then there exists a number z in the open interval (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(z)}{g'(z)}$$

Exercises 7.7

In Exercises 1–10, (a) Plot the graph of f and state the apparent limit as x approaches a. (b) Compute $\lim\limits_{x\to a} f(x)$

1. $\lim\limits_{x\to 0}\dfrac{x}{\tan x}\overset{0/0}{=}\lim\limits_{x\to 0}\dfrac{1}{\sec^2 x}=1$

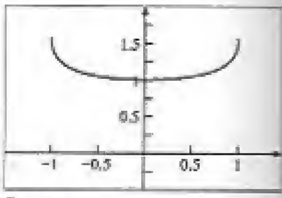
3. $\lim\limits_{x\to 2}\dfrac{\sin \pi x}{2-x}\overset{0/0}{=}\lim\limits_{x\to 2}\dfrac{\pi\cos \pi x}{-1}=-\pi\approx -3.142$

2. $\lim\limits_{x\to 0}\dfrac{\tan x - x}{x-\sin x}\overset{0/0}{=}\lim\limits_{x\to 0}\dfrac{\sec^2 x-1}{1-\cos x}\overset{0/0}{=}\lim\limits_{x\to 0}\dfrac{2\sec^2 x\tan x}{\sin x}=\lim\limits_{x\to 0}\dfrac{2}{\cos^3 x}=2$

4. $f(x)=\dfrac{\sin^{-1}x}{x}$; $a=0$

▷ (a) From the plot at the right, the limit appears to be 1.
(b) Because $\lim\limits_{x\to 0}\sin^{-1}x=0$ and $\lim\limits_{x\to 0}x=0$, L'Hôpital's rule applies.

$$\lim_{x\to 0}\frac{\sin^{-1}x}{x}\overset{0/0}{=}\lim_{x\to 0}\frac{1/\sqrt{1-x^2}}{1}=1$$



5. $\lim\limits_{x\to 0}\dfrac{2^x-3^x}{x}\overset{0/0}{=}\lim\limits_{x\to 0}\dfrac{2^x\ln 2-3^x\ln 3}{1}=\ln 2-\ln 3=\ln\frac{2}{3}\approx -0.405$

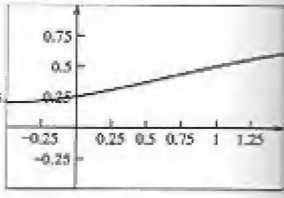
6. $\lim\limits_{x\to 0}\dfrac{\tanh 2x}{\tanh x}\overset{0/0}{=}\dfrac{2\,\text{sech}^2 2x}{\text{sech}^2 x}=2$

7. $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{\sin x^2}=\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^2\dfrac{x^2}{\sin x^2}=1^2\cdot 1=1$

8. $f(x)=\dfrac{x^3-1}{x^3+3x-4}$; $a=1$

▷ (a) From the plot of $f(x)$ at the right, the limit appears to be 0.5.
(b) Because $\lim\limits_{x\to 1}(x^3-1)=0$ and $\lim\limits_{x\to 1}(x^3+3x-4)=0$, L'Hôpital's rule applies.

$$\lim_{x\to 1}\frac{x^3-1}{x^3+3x-4}\overset{0/0}{=}\lim_{x\to 1}\frac{3x^2}{3x^2+3}=\frac{1}{2}$$



9. $\lim\limits_{x\to -2}\dfrac{x^3+8}{x^3+x^2+4}\overset{0/0}{=}\lim\limits_{x\to -2}\dfrac{3x^2}{3x^2+2x}=\dfrac{12}{8}=\dfrac{3}{2}$

10. $\lim\limits_{x\to \pi/2}\dfrac{3\cos x}{2x-\pi}\overset{0/0}{=}\lim\limits_{x\to \pi/2}\dfrac{-3\sin x}{2}=-\dfrac{3}{2}$

In Exercises 11–16, find the limit if it exists, and support your answer graphically.

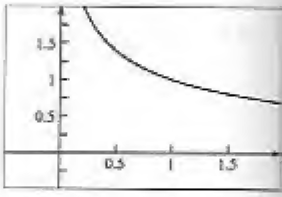
11. $\lim\limits_{x\to 0}\dfrac{\tan 3x}{\tan 2x}\overset{0/0}{=}\lim\limits_{x\to 0}\dfrac{3\sec^2 3x}{2\sec^2 2x}=\dfrac{3}{2}$

12. $\lim\limits_{x\to 1}\dfrac{\ln x}{x-1}$

▷ Because $\lim\limits_{x\to 1}\ln x=0$ and $\lim\limits_{x\to 1}(x-1)=0$, L'Hôpital's rule applies.

$$\lim_{x\to 1}\frac{\ln x}{x-1}\overset{0/0}{=}\lim_{x\to 1}\frac{1/x}{1}=1$$

The plot at the right of $\ln x/(x-1)$ supports this limit.



13. $\lim\limits_{x\to +\infty}\sin\frac{2}{x}=0$ and $\lim\limits_{x\to +\infty}\frac{1}{x}=0$. By L'Hôpital's rule

$$\lim_{x\to +\infty}\frac{\sin\frac{2}{x}}{\frac{1}{x}}=\lim_{x\to +\infty}\frac{\cos\frac{2}{x}\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}}=2\lim_{x\to +\infty}\cos\frac{2}{x}=2$$

14. $\lim\limits_{\theta\to 0}(\theta-\sin\theta)=0$ and $\lim\limits_{\theta\to 0}\tan^3\theta=0$. By L'Hôpital's rule

$$\lim_{\theta\to 0}\frac{\theta-\sin\theta}{\tan^3\theta}\overset{0/0}{=}\lim_{\theta\to 0}\frac{1-\cos\theta}{3\tan^2\theta\sec^2\theta}=\lim_{\theta\to 0}\frac{1}{\sec^2\theta}\cdot\lim_{\theta\to 0}\frac{1-\cos\theta}{3\tan^2\theta}\overset{0/0}{=}1\cdot\lim_{\theta\to 0}\frac{\sin\theta}{6\tan\sec^2\theta}=\lim_{\theta\to 0}\frac{\cos^3\theta}{6}=\frac{1}{6}$$

Because $\lim\limits_{\theta\to 0}(1-\cos\theta)=0$ and $\lim\limits_{\theta\to 0}(3\tan^2\theta)=0$, we applied L'Hôpital's rule again.

15. $\lim\limits_{x\to \pi/2}\ln(\sin x)=0$ and $\lim\limits_{x\to \pi/2}(\pi-2x)^2=0$. By L'Hôpital's rule

$$\lim_{x\to \pi/2}\frac{\ln(\sin x)}{(\pi-2x)^2}\overset{0/0}{=}\lim_{x\to \pi/2}\frac{\frac{\cos x}{\sin x}}{-4(\pi-2x)}=-\frac{1}{4}\lim_{x\to \pi/2}\frac{\cot x}{\pi-2x}\overset{0/0}{=}-\frac{1}{4}\lim_{x\to \pi/2}\frac{-\csc^2 x}{-2}=-\frac{1}{8}$$

Because $\lim\limits_{x\to \pi/2}\cot x=0$ and $\lim\limits_{x\to \pi/2}(\pi-2x)=0$, we applied L'Hôpital's rule again.

16. $\lim_{x\to 0} \dfrac{e^x - \cos x}{x \sin x}$

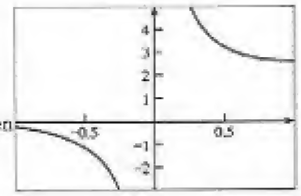
▷ Because $\lim_{x\to 0}(e^x - \cos x) = 0$ and $\lim_{x\to 0} x \sin x = 0$, we apply L'Hôpital's rule. Thus,

$$\lim_{x\to 0} \frac{e^x - \cos x}{x \sin x} \overset{0/0}{=} \lim_{x\to 0} \frac{e^x + \sin x}{x \cos x + \sin x}$$

We have $\lim_{x\to 0}(e^x + \sin x) = 1$ and $\lim_{x\to 0}(x \cos x + \sin x) = 0$. Because $x \cos x + \sin x$ approaches zero through positive values as $x \to 0^+$, we have

$$\lim_{x\to 0^+} \frac{e^x - \cos x}{x \sin x} = +\infty$$

Because $x \cos x + \sin x$ approaches zero through negative values as $x \to 0^-$, then



$$\lim_{x\to 0^-} \frac{e^x - \cos x}{x \sin x} = -\infty$$

Therefore the given limit does not exist. This is supported by the plot.

In Exercises 17–28, evaluate the limit if it exists.

17. $\lim_{x\to 1^+} \dfrac{x - 1}{x - 1 - 2\sqrt{x-1}} = \lim_{x\to 1^+} \dfrac{\sqrt{x-1}}{\sqrt{x-1} - 2} = \dfrac{0}{-2} = 0$

18. $\lim_{x\to 0} \dfrac{e^{2x^2} - 1}{\sin^2 x} \overset{0/0}{=} \lim_{x\to 0} \dfrac{4xe^{2x^2}}{2 \sin x \cos x} = \lim_{x\to 0} 2\dfrac{x}{\sin x} \cdot \lim_{x\to 0} \dfrac{e^{2x^2}}{\cos x} = 2 \cdot 1 = 2$

19. $\lim_{x\to +\infty} (1 - e^{1/x}) = 0$ and $\lim_{x\to +\infty} \dfrac{-3}{x} = 0$. By L'Hôpital's rule

$$L = \lim_{x\to +\infty} \frac{1 - e^{1/x}}{-\frac{3}{x}} \overset{0/0}{=} \lim_{x\to +\infty} \frac{-e^{1/x}\left(-\frac{1}{x^2}\right)}{\frac{3}{x^2}} = \frac{1}{3} \lim_{x\to 0} e^{1/x} = \frac{1}{3}$$

Alternatively, let $x = \frac{1}{x}$. Then $L = \lim_{x\to 0^+} \dfrac{1 - e^x}{-3x} \overset{0/0}{=} \lim_{x\to 0^+} \dfrac{-e^x}{-3} = \dfrac{1}{3}$.

20. $\lim_{y\to 0} \dfrac{y^2}{1 - \cosh y}$

▷ The hypothesis of L'Hôpital's rule is satisfied. Thus, two applications of the rule yield

$$\lim_{y\to 0} \frac{y^2}{1 - \cosh y} \overset{0/0}{=} \lim_{y\to 0} \frac{2y}{-\sinh y} \overset{0/0}{=} \lim_{y\to 0} \frac{2}{-\cosh y} = -2$$

21. $\lim_{t\to 0} \sin t = 0$ and $\lim_{t\to 0} \ln(2e^t - 1) = 0$. By L'Hôpital's rule $\lim_{t\to 0} \dfrac{\sin t}{\ln(2e^t - 1)} \overset{0/0}{=} \lim_{t\to 0} \dfrac{\cos t}{\frac{2e^t}{2e^t - 1}} = \dfrac{1}{2}$.

22. $\lim_{x\to 0} \dfrac{5x}{5^x - e^x} \overset{0/0}{=} \lim_{x\to 0} \dfrac{5}{5^x \ln 5 - e^x} = \dfrac{5}{\ln 5 - 1}$

23. $\lim_{x\to 0}[(1 + x)^{1/5} - (1 - x)^{1/5}] = 0$ and $\lim_{x\to 0} [(1 + x)^{1/3} - (1 - x)^{1/3}] = 0$. By L'Hôpital's rule

$$\lim_{x\to 0} \frac{(1 + x)^{1/5} - (1 - x)^{1/5}}{(1 + x)^{1/3} - (1 - x)^{1/3}} = \lim_{x\to 0} \frac{\frac{1}{5}(1 + x)^{-4/5} + \frac{1}{5}(1 - x)^{-4/5}}{\frac{1}{3}(1 + x)^{-2/3} + \frac{1}{3}(1 - x)^{-2/3}} = \frac{\frac{1}{5} + \frac{1}{5}}{\frac{1}{3} + \frac{1}{3}} = \frac{\frac{2}{5}}{\frac{2}{3}} = \frac{3}{5}$$

24. $\lim_{x\to +\infty} \dfrac{\frac{1}{x^2} - 2 \tan^{-1} \frac{1}{x}}{\frac{1}{x}}$

▷ We simplify by substituting $u = \frac{1}{x}$ before applying L'Hopital's rule. Thus,

$$\lim_{x\to +\infty} \frac{\frac{1}{x^2} - 2 \tan^{-1} \frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{u^2 - 2 \tan^{-1} u}{u} \overset{0/0}{=} \lim_{x\to 0^+} \frac{2u - \frac{2}{1 + u^2}}{1} = -2$$

25. $\lim_{x\to \pi} \dfrac{1 + \cos 2x}{1 - \sin x} = \dfrac{1 + \cos 2\pi}{1 - \sin \pi} = \dfrac{1 + 1}{1 - 0} = 2$

26. $\lim_{x\to 0} \dfrac{e^x - 10^x}{x} \overset{0/0}{=} \lim_{x\to 0} \dfrac{e^x - 10^x \ln 10}{1} = 1 - \ln 10$

27. $\lim_{x\to 0}(\cos x - \cosh x) = 0$ and $\lim_{x\to 0} x^2 = 0$. By L'Hôpital's rule

$$\lim_{x\to 0} \frac{\cos x - \cosh x}{x^2} = \lim_{x\to 0} \frac{-\sin x - \sinh x}{2x} = \lim_{x\to 0} \frac{-\cos x - \cosh x}{2} = -1$$

Because $\lim_{x\to 0}(-\sin x - \sinh x) = 0$ and $\lim_{x\to 0} 2x = 0$, we applied L'Hôpital's rule again.

28. $\lim_{x\to 0} \dfrac{\sinh x - \sin x}{\sin^3 x}$

▹ Before applying L'Hôpital's rule the second and third time, we remove $\cos x$, which has a nonzero limit.

$$\lim_{x\to 0} \frac{\sinh x - \sin x}{\sin^3 x} \overset{0/0}{=} \lim_{x\to 0} \frac{\cosh x - \cos x}{3\sin^2 x \cos x} = \lim_{x\to 0} \frac{1}{\cos x} \cdot \lim_{x\to 0} \frac{\cosh x - \cos x}{3\sin^2 x} \overset{0/0}{=} 1 \cdot \lim_{x\to 0} \frac{\sinh x + \sin x}{6 \sin x \cos x}$$
$$\overset{0/0}{=} 1 \cdot \lim_{x\to 0} \frac{\cosh x + \cos x}{6\cos x} = \frac{2}{6} = \frac{1}{3}$$

In Exercises 29–36, find all values of z in the interval satisfying the conclusion of Cauchy's mean-value theorem.

29. $f(x) = x^3$ and $g(x) = x^2$. Then $f'(x) = 3x^2$ and $g'(x) = 2x$. We wish to find all values of z in $(0,2)$ such that

$$\frac{f(2)-f(0)}{g(2)-g(0)} = \frac{f'(z)}{g'(z)};\ \frac{8-0}{4-0} = \frac{3z^2}{2z};\ \ z = \frac{4}{3}$$

30. $f(x) = \dfrac{2x}{1+x^2}$, $g(x) = \dfrac{1-x^2}{1+x^2}$. Then $f'(x) = \dfrac{2-2x^2}{(1+x^2)^2}$, $g'(x) = \dfrac{-4x}{(1+x^2)^2}$. We seek values of z in $(0,2)$ such that

$$\frac{f(2)-f(0)}{g(2)-g(0)} = \frac{f'(z)}{g'(z)};\ \frac{\frac{4}{5}-0}{-\frac{3}{5}-1} = \frac{2-2z^2}{-4z};\ z^2+z-1=0;\ z = \tfrac{1}{2}(-1\pm\sqrt{5}).\ z = \tfrac{1}{2}(-1+\sqrt{5}) \in (0,2)$$

31. $f(x) = \sin x$, $g(x) = \cos x$. Then $f'(x) = \cos x$, $g'(x) = -\sin x$. We seek all values of z in $(0,\pi)$ such that

$$\frac{f(\pi)-f(0)}{g(\pi)-g(0)} = \frac{f'(z)}{g'(z)};\ \ \frac{0-0}{-1-1} = \frac{\cos z}{-\sin z};\ \ -\cot z = 0;\ z = \tfrac{1}{2}\pi$$

32. $f(x) = \cos 2x$, $g(x) = \sin x$, $(a,b) = (0, \frac{1}{2}\pi)$

▹ $$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{\cos\pi - \cos 0}{\sin\frac{1}{2}\pi - \sin 0} = -2; \qquad \frac{f'(x)}{g'(x)} = \frac{-2\sin 2x}{\cos x}$$

Thus, we want to find all z in $(0,\frac{1}{2}\pi)$ such that

$$\frac{-2\sin 2z}{\cos z} = -2; \qquad \frac{2\sin z\cos z}{\cos z} = 1; \qquad \sin z = \tfrac{1}{2}$$

Hence, $z = \frac{1}{6}\pi$.

33. $f(x) = \ln x$, $g(x) = x^2$. Then $f'(x) = \frac{1}{x}$, $g'(x) = 2x$. We seek all values of z in $(1,3)$ such that

$$\frac{f(3)-f(1)}{g(3)-g(1)} = \frac{f'(z)}{g'(z)};\ \frac{\ln 3 - \ln 1}{3^2-1^2} = \frac{1/z}{2z};\ \frac{\ln 3}{8} = \frac{1}{2z^2};\ z^2 = \frac{4}{\ln 3};\ z = \frac{2}{\sqrt{\ln 3}}$$

34. $f(x) = \sqrt{x+5}$, $g(x) = x+3$. Then $f'(x) = \dfrac{1}{2\sqrt{x+5}}$, $g'(x) = 1$. We seek all values of z in $(-4,-1)$ such that

$$\frac{f(-1)-f(-4)}{g(-1)-g(-4)} = \frac{f'(z)}{g'(z)};\ \ \frac{2-1}{2-(-1)} = \frac{1}{2\sqrt{z+5}};\ \ \sqrt{z+5} = \frac{3}{2};\ z+5 = \frac{9}{4};\ \ z = -\frac{11}{4}$$

35. $f(x) = e^{2x}$, $g(x) = e^x$. Then $f'(x) = 2e^{2x}$, $g'(x) = e^x$. We seek all values of z in $(0,2)$ such that

$$\frac{f(2)-f(0)}{g(2)-g(0)} = \frac{f'(z)}{g'(z)};\ \frac{e^4-e^0}{e^2-e^0} = \frac{2e^{2z}}{e^z};\ \frac{e^4-1}{e^2-1} = e^2+1 = 2e^z;\ z = \ln[\tfrac{1}{2}(e^2+1)]$$

36. $f(x) = \ln(x+1)$, $g(x) = \ln x$; $(a,b) = (1,2)$

▹ $f'(x) = \dfrac{1}{x+1}$ and $g'(x) = \dfrac{1}{x}$. We seek all values of z in $(1,2)$ such that

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(z)}{g'(z)}; \qquad \frac{\frac{1}{3}-\frac{1}{2}}{\frac{1}{2}-1} = \frac{\frac{1}{z+1}}{\frac{1}{z}}; \qquad \frac{1}{3} = \frac{z}{z+1}; \qquad z = \frac{1}{2}$$

37. $\lim_{R\to 0^+} \mathrm{E}(1-e^{-Rt/L}) = 0$ and $\lim_{R\to 0^+} \mathrm{R} = 0$, where t, E and L are constants. By L'Hôpital's rule

$$\lim_{R\to 0^+} \mathrm{I} = \lim_{R\to 0^+} \frac{\mathrm{E}(1-e^{-Rt/L})}{\mathrm{R}} = \lim_{R\to 0^+} \frac{\mathrm{E}\left(\frac{t}{L}e^{-Rt/L}\right)}{1} = \frac{\mathrm{Et}}{\mathrm{L}}$$

38. Because the hypothesis of L'Hôpital's rule is satisfied, we have $\lim\limits_{r\to 1} S = \lim\limits_{r\to 1} \dfrac{a(r^n-1)}{r-1} = \lim\limits_{r\to 1} \dfrac{anr^{n-1}}{1} = an$.

If $r = 1$, then each term is equal to the first term a. Hence the sum of the first n terms is $a + a + \cdots + a = na$. Thus, $\lim_{r\to 1} S$ is consistent with the sum of the first n terms if $r = 1$.

39. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x} & \text{if } x \neq 0 \\ \lim\limits_{x\to 0} f(x) & \text{if } x = 0 \end{cases}$

$\lim\limits_{x\to 0}(\cos x - 1) = 0$ and $\lim\limits_{x\to 0} x = 0$. By L'Hôpital's rule $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = \lim\limits_{x\to 0} \dfrac{-\sin x}{1} = 0$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{(\cos x - 1)/x - 0}{x} = \lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}$$

Because $\lim\limits_{x\to 0}(\cos x - 1) = 0$, $\lim\limits_{x\to 0} x^2 = 0$; and $\lim\limits_{x\to 0}(-\sin x) = 0$, $\lim\limits_{x\to 0} 2x = 0$; we used L'Hôpital's rule twice.

40. (a) Prove that if $a > 0$, $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \ln a$. (b) From the result of part (a) show that if $r > 0$ and $s > 0$, then $\lim\limits_{x\to 0} \dfrac{(rs)^x - 1}{x} = \lim\limits_{x\to 0} \dfrac{r^x - 1}{x} + \lim\limits_{x\to 0} \dfrac{s^x - 1}{x}$.

▹ (a) $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} \overset{0/0}{=} \lim\limits_{x\to 0} \dfrac{a^x \ln a}{1} = \ln a$

(b) Apply the result of (a), with a successively rs, r, then s. Thus

$$\lim_{x\to 0} \frac{(rs)^x - 1}{x} = \ln rs = \ln r + \ln s = \lim_{x\to 0} \frac{r^x - 1}{x} + \lim_{x\to 0} \frac{s^x - 1}{x}$$

41. $f(x) = x^2 \sin \frac{1}{x}$, $g(x) = x$. Because $|\sin x| \leq 1$, then $-x^n \leq x^n \sin \frac{1}{x} \leq x^n$, $n > 0$. It follows by the squeeze theorem with $n = 2$ that $\lim\limits_{x\to 0} f(x) = 0$ and with $n = 1$ that $\lim\limits_{x\to 0} f(x)/g(x) = \lim\limits_{x\to 0}(x \sin \frac{1}{x}) = 0$. However, $\dfrac{f'(x)}{g'(x)} = \dfrac{2x \sin(1/x) - \cos(1/x)}{1}$ oscillates between -1 and 1 as $x \to 0$ and does not approach any limit.

42. $\lim\limits_{x\to +\infty} f(x) = \mathrm{M} \Leftrightarrow |f(x) - \mathrm{M}| < \epsilon$ when $x > k \Leftrightarrow |f(1/t) - \mathrm{M}| < \epsilon$ when $0 < t < 1/k = \delta \Leftrightarrow \lim\limits_{t\to 0^+} \mathrm{F}(t) = \mathrm{M}$

43. We wish to find values of a and b such that $\lim\limits_{x\to 0} \dfrac{\sin 3x + ax + \mathrm{b}x^3}{x^3} = 0$. $\lim\limits_{x\to 0}(\sin 3x + ax + \mathrm{b}x^3) = 0$ and $\lim\limits_{x\to 0} x^3 = 0$. By L'Hôpital's rule $\lim\limits_{x\to 0} \dfrac{\sin 3x + ax + \mathrm{b}x^3}{x^3} = \lim\limits_{x\to 0} \dfrac{3\cos 3x + \mathrm{a} + 3\mathrm{b}x^2}{3x^2}$.

$\lim\limits_{x\to 0} 3x^2 = 0$. Because $\lim\limits_{x\to 0}(3\cos 3x + \mathrm{a} + 3b^2) = 3 + \mathrm{a}$, we can apply L'Hôpital's rule if $a = -3$.

If $\mathrm{a} \neq -3$ the given limit is not 0. Let $a = -3$ and we have

$$\lim_{x\to 0} \frac{3\cos 3x - 3 + 3\mathrm{b}x^2}{3x^2} = \lim_{x\to 0} \frac{-9\sin 3x + 6\mathrm{b}x}{6x} = \lim_{x\to 0} \frac{-27\cos 3x + 6\mathrm{b}}{6} = -27 + 6\mathrm{b}$$

Because $\lim\limits_{x\to 0}(-9\sin 3x + 6\mathrm{b}x) = 0$ and $\lim\limits_{x\to 0} 6x = 0$, we applied L'Hôpital's rule again. The limit will be 0 if $-27 + 6b = 0$; $b = \frac{9}{2}$. Thus the given limit is 0 if $a = -3$ and $b = \frac{9}{2}$.

44. Prove Theorem 7.7.2 (ii).

▹ We wish to prove that if $\lim\limits_{x\to a^-} f(x) = 0$, $\lim\limits_{x\to a^-} g(x) = 0$, and $\lim\limits_{x\to a^-} \dfrac{f'(x)}{g'(x)} = \mathrm{L}$, then $\lim\limits_{x\to a^-} \dfrac{f(x)}{g(x)} = \mathrm{L}$. If we reflect the graphs in the y axis, then an approach from the right becomes an approach from the left and we can use case (i). Let $\mathrm{F}(x) = f(-x)$, $\mathrm{G}(x) = g(-x)$, and $t = -x$. Then

$\lim\limits_{x\to -a^+} \mathrm{F}(x) = \lim\limits_{t\to a^-} f(t) = 0$, $\lim\limits_{x\to -a^+} \mathrm{G}(x) = \lim\limits_{t\to a^-} g(t) = 0$, and $\lim\limits_{x\to -a^+} \dfrac{\mathrm{F}'(x)}{\mathrm{G}'(x)} = \lim\limits_{t\to a^-} \dfrac{-f'(t)}{-g'(t)} = \mathrm{L}$

Thus the hypotheses of case(ii) are satisfied for $\mathrm{F}(x)$ and $\mathrm{G}(x)$ at $-a$ and so

$\lim\limits_{x\to -a^+} \dfrac{\mathrm{F}(x)}{\mathrm{G}(x)} = \mathrm{L}$. Therefore $\lim\limits_{x\to a^-} \dfrac{f(x)}{g(x)} = \mathrm{L}$.

45. Prove Theorem 7.7.4 for $x \to -\infty$.

▹ In the proof of 7.7.(ii) above, replace a with ∞.

46. (a) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\mathrm{L}_1}{\mathrm{L}_2}$ (b) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{0}{\mathrm{L}_2} = 0$ (c) Nothing (d) We may apply L'Hopital's rule again.

7.8 OTHER INDETERMINATE FORMS

We may also use L'Hôpital's rule when both the numerator and denominator of a fraction have limits $\pm\infty$.

7.8.1 2 Theorem (L'Hôpital's rule) Let f and g be functions that are differentiable on an open interval I, except possibly at the number a in I, and suppose that for all $x \neq a$ in I, $g'(x) \neq 0$. If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, $\lim_{x\to a} g(x) = +\infty$ or $-\infty$, and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \mathrm{L}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \mathrm{L}$$

The theorem is valid if a is $+\infty$ or $-\infty$ and all the limits are right-hand limits or if all the limits are left-hand limits.

Proof Because $\lim_{x\to a} \dfrac{f'(x)}{g'(x)} = \mathrm{L}$, then for any $\epsilon > 0$, there is an open subinterval I′ of I containing a such that for any $z \in \mathrm{I}'$

$$\mathrm{L} - \epsilon < \frac{f'(z)}{g'(z)} < \mathrm{L} + \epsilon \tag{1}$$

By the Cauchy mean-value theorem, for any x and y in I′, and for some z between x and y,

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(z)}{g'(z)} \quad \text{or, equivalently,} \quad \frac{\dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)}}{1 - \dfrac{g(x)}{g(y)}} = \frac{f'(z)}{g'(z)} \tag{2}$$

Substituting from (2) into (1), we have

$$\mathrm{L} - \epsilon < \frac{\dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)}}{1 - \dfrac{g(x)}{g(y)}} < \mathrm{L} + \epsilon \tag{3}$$

We shall now let y approach a, keeping x fixed. Because $|g(y)|$ approaches $+\infty$, we may assume that

$$1 - \frac{g(x)}{g(y)} > 0 \tag{4}$$

It follows from (3) and (4) that

$$(\mathrm{L} - \epsilon)\left[1 - \frac{g(x)}{g(y)}\right] < \frac{f(y)}{g(y)} - \frac{f(x)}{g(y)} < (\mathrm{L} + \epsilon)\left[1 - \frac{g(x)}{g(y)}\right]$$

$$(\mathrm{L} - \epsilon)\left[1 - \frac{g(x)}{g(y)}\right] + \frac{f(x)}{g(y)} < \frac{f(y)}{g(y)} < (\mathrm{L} + \epsilon)\left[1 - \frac{g(x)}{g(y)}\right] + \frac{f(x)}{g(y)} \tag{5}$$

As y approaches a, the left and right members of (5) approach $\mathrm{L} - \epsilon$ and $\mathrm{L} + \epsilon$, respectively. Thus for y sufficiently close to a, we have

$$\mathrm{L} - 2\epsilon < \frac{f(y)}{g(y)} < \mathrm{L} + 2\epsilon$$

Because ϵ was arbitrary, this implies that

$$\lim_{y\to a} \frac{f(y)}{g(y)} = \mathrm{L}$$

which is equivalent to what we set out to prove. Notice that we did not require the hypothesis $\lim_{x\to a} f(x) = \pm\infty$. We use $\overset{\infty/\infty}{=}$ to indicate that the hypotheses of this theorem are satisfied and the we have applied the rule.

Other Forms For the indeterminate forms $0 \cdot \pm\infty$ and $+\infty - (+\infty)$, we must first write the given expression in the indeterminate form $0/0$ or $\pm\infty/\pm\infty$. For the indeterminate forms 0^0, $(\pm\infty)^0$, or $1^{\pm\infty}$, we introduce the ln function in order to obtain the indeterminate form $0/0$ or $\pm\infty/\pm\infty$. We may also us the following known results:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \lim_{x\to 0^+} x\ln x = 0 \quad \lim_{x\to 0}(1+x)^{1/x} = e \quad \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$$

Rate of Growth For any $p > 0$, e^x increases faster than x^p and x^p increases faster than $\ln x$. See Ex. 37–38.

Exercises 7.8

In Exercises 1–16, find the limit and support your answer graphically.

1. $\lim_{x\to+\infty} x^2 = +\infty$ and $\lim_{x\to+\infty} e^x = +\infty$. By L'Hôpital's rule $\lim_{x\to+\infty}\frac{x^2}{e^x} = \lim_{x\to+\infty}\frac{2x}{e^x} = \lim_{x\to+\infty}\frac{2}{e^x} = 0$.
Because $\lim_{x\to+\infty} 2x = +\infty$ and $\lim_{x\to+\infty} e^x = +\infty$, we applied L'Hôpital's rule again.

2. $\lim_{x\to+\infty}\frac{(\ln x)^3}{x} = \left(\lim_{x\to+\infty}\frac{\ln x}{x^{1/3}}\right)^3 \overset{\infty/\infty}{=} \left(\lim_{x\to+\infty}\frac{1/x}{\frac{1}{3}x^{-2/3}}\right)^3 = \left(\lim_{x\to+\infty}\frac{3}{x^{1/3}}\right)^3 = 0^3 = 0.$

Using L'Hopital's rule directly would require many applications if the exponent 3 were replaced with a larger integer, and would not work at all if the exponent was not an integer.

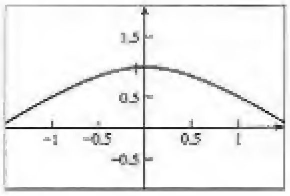
3. $\lim_{x\to 0^+}\tan x = 0$ so that $\lim_{x\to 0^+}\cot x = +\infty$, and $\lim_{x\to 0^+}\ln x = -\infty$. By L'Hôpital's rule

$$\lim_{x\to 0^+}\tan x(\ln x) = \lim_{x\to 0^+}\frac{\ln x}{\cot x} = \lim_{x\to 0^+}\frac{1/x}{-\csc^2 x} = -\lim_{x\to 0^+}\sin x \cdot \lim_{x\to 0^+}\frac{\sin x}{x} = 0\cdot 1 = 0$$

4. $\lim_{x\to 0^+}\tan^{-1}x\cot x$

▸ From the plot at the right it appears that the limit is 1. We express the limit as a quotient. Thus,

$$\lim_{x\to 0^+}\tan^{-1}x\cot x = \lim_{x\to 0^+}\frac{\tan^{-1}x}{\tan x}$$



Because $\lim_{x\to 0^+}\tan^{-1}x = 0$ and $\lim_{x\to 0^+}\tan x = 0$, we apply the case $0/0$ of L'Hôpital's rule as follows.

$$\lim_{x\to 0^+}\tan^{-1}x\cot x = \lim_{x\to 0^+}\frac{\frac{1}{1+x^2}}{\sec^2 x} = 1$$

5. $\lim_{x\to 1}(x - 1 - \ln x) = 0$ and $\lim_{x\to 1}(x-1)\ln x = 0$. By L'Hôpital's rule

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1}\frac{x - 1 - \ln x}{(x-1)\ln x} \overset{0/0}{=} \lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x + 1 - \frac{1}{x}} = \lim_{x\to 1}\frac{x-1}{\ln x + x - 1} \overset{0/0}{=} \lim_{x\to 1}\frac{1}{\ln x + 2} = \frac{1}{2}$$

Because $\lim_{x\to 1}(x-1) = 0$ and $\lim_{x\to 1}(x\ln x + x - 1) = 0$, we applied L'Hôpital's rule again.

6. $\lim_{x\to 0^+}(1+x)^{\ln x} = \lim_{x\to 0^+}[(1+x)^{1/x}]^{x\ln x} = e^0 = 1$

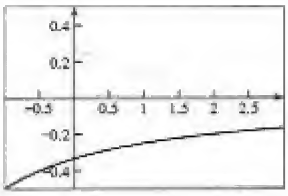
7. $\lim_{x\to 0^+}\ln x = -\infty$ and $\lim_{x\to 0^+}\csc x = +\infty$. By L'Hôpital's rule

$$\lim_{x\to 0^+}\ln x^{\sin x} = \lim_{x\to 0^+}\frac{\ln x}{\csc x} \overset{\infty/\infty}{=} \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc x\cot x} = \lim_{x\to 0^+} -\tan x \cdot \lim_{x\to 0^+}\frac{\sin x}{x} = 0\cdot 1 = 0$$

Therefore $\ln \lim_{x\to 0^+} x^{\sin x} = 0$, $\lim_{x\to 0^+} x^{\sin x} = 1$.

8. $\lim_{x\to 2}\left(\frac{5}{x^2+x-6} - \frac{1}{x-2}\right)$

▸ From the plot at the right, the limit seems to be -0.2.
We first combine into a single quotient.



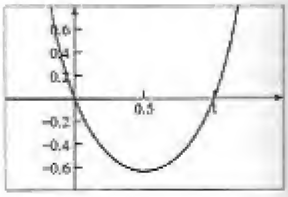
$$\begin{aligned}\lim_{x\to 2}\left(\frac{5}{x^2+x-6} - \frac{1}{x-2}\right) &= \lim_{x\to 2}\left(\frac{5}{(x-2)(x+3)} - \frac{1}{x-2}\cdot\frac{x+3}{x+3}\right)\\ &= \lim_{x\to 2}\frac{-x+2}{(x-2)(x+3)} = \lim_{x\to 2}\frac{-1}{x+3} = -\frac{1}{5}\end{aligned}$$

9. $\displaystyle\lim_{x\to+\infty}\frac{\ln x}{x}\overset{\infty/\infty}{=}\lim_{x\to+\infty}\frac{1/x}{1}=0$

10. $\displaystyle\lim_{x\to+\infty}\frac{\ln(x+100)}{\ln x}\overset{\infty/\infty}{=}\lim_{x\to+\infty}\frac{1/(x+100)}{1/x}=\lim_{x\to+\infty}\frac{x}{x+100}=\lim_{x\to+\infty}\frac{1}{1+100/x}=1$

11. $\displaystyle\lim_{x\to0}x\csc x=\lim_{x\to0}\frac{x}{\sin x}=1$

12. $\displaystyle\lim_{x\to1/2}(2x-1)\tan\pi x$



▷ From the plot at the right, the limit appears to be -0.6.
We first express the limit as a quotient.

$\displaystyle\lim_{x\to1/2}(2x-1)\tan\pi x=\lim_{x\to1/2}\frac{2x-1}{\cot\pi x}$

Because $\displaystyle\lim_{x\to1/2}(2x-1)=0$ and $\displaystyle\lim_{x\to1/2}\cot\pi x=0$,
we apply the case 0/0 of L'Hôpital's rule.

$\displaystyle\lim_{x\to1/2}(2x-1)\tan\pi x=\lim_{x\to1/2}\frac{2}{-\pi\csc^2\pi x}=-\frac{2}{\pi}\approx0.6366$

13. $\displaystyle\lim_{x\to+\infty}(x^2-\sqrt{x^4-x^2+2})=\lim_{x\to+\infty}(x^2-\sqrt{x^4-x^2+2})\cdot\frac{x^2+\sqrt{x^4-x^2+2}}{x^2+\sqrt{x^4-x^2+2}}$

$\displaystyle=\lim_{x\to+\infty}\frac{x^2-2}{x^2+\sqrt{x^4-x^2+2}}=\lim_{x\to+\infty}\frac{1-x^{-2}}{1+\sqrt{1-x^{-2}+2x^{-4}}}=\frac{1}{1+1}=\frac{1}{2}$

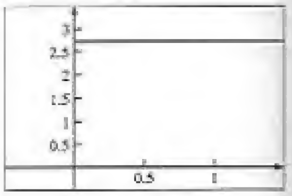
14. $\displaystyle\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to0}\frac{x-\sin x}{x\sin x}\overset{0/0}{=}\lim_{x\to0}\frac{1-\cos x}{\sin x+x\cos x}\overset{0/0}{=}\lim_{x\to0}\frac{\sin x}{2\cos x-x\sin x}=0$

15. $\displaystyle\lim_{x\to0}\ln(1+3x)=0$ and $\displaystyle\lim_{x\to0}x=0$. By L'Hôpital's rule

$\displaystyle\lim_{x\to0}\ln(1+3x)^{1/x}=\lim_{x\to0}\frac{\ln(1+3x)}{x}=\lim_{x\to0}\frac{3}{1+3x}=3$

Therefore $\displaystyle\ln\lim_{x\to0}(1+3x)^{1/x}=3$; $\displaystyle\lim_{x\to0}(1+3x)^{1/x}=e^3$.

16. $\displaystyle\lim_{x\to0^+}x^{1/\ln x}$



▷ From the plot at the right, the function appears to be a constant 2.7.
Because $x=e^{\ln x}$ if $x>0$, then

$\displaystyle\lim_{x\to0^+}x^{1/\ln x}=\lim_{x\to0^+}(e^{\ln x})^{1/\ln x}=\lim_{x\to0^+}e=e\approx2.718$

In Exercises 17–34, find the limit if it exists.

17. $\displaystyle\lim_{x\to1/2^-}\ln(1-2x)=-\infty$ and $\displaystyle\lim_{x\to1/2^-}\tan\pi x=+\infty$. By L'Hôpital's rule

$\displaystyle\lim_{x\to1/2^-}\frac{\ln(1-2x)}{\tan\pi x}\overset{0/0}{=}\lim_{x\to1/2^-}\frac{\frac{-2}{1-2x}}{\pi\sec^2\pi x}=\lim_{x\to1/2^-}\frac{-2\cos^2\pi x}{(1-2x)\pi}\overset{0/0}{=}\lim_{x\to1/2^-}\frac{4\pi\cos\pi x\sin\pi x}{-2\pi}=0$

Because $\displaystyle\lim_{x\to1/2^-}(-2\cos^2\pi x)=0$ and $\displaystyle\lim_{x\to1/2^-}(1-2x)\pi=0$, we applied L'Hôpital's rule again.

18. $\displaystyle\lim_{x\to\pi/2^-}\frac{\ln(\cos x)}{\ln(\tan x)}\overset{\infty/\infty}{=}\lim_{x\to\pi/2^-}\frac{-\sin x/\cos x}{\sec^2x/\tan x}=\lim_{x\to\pi/2^-}(-\sin^2x)=-1$

19. $\displaystyle\lim_{x\to+\infty}2\ln(e^x+x)=+\infty$ and $\displaystyle\lim_{x\to+\infty}x=+\infty$. By L'Hôpital's rule

$\displaystyle\lim_{x\to+\infty}\ln(e^x+x)^{2/x}=\lim_{x\to+\infty}\frac{2\ln(e^x+x)}{x}\overset{\infty/\infty}{=}\lim_{x\to+\infty}\frac{2(e^x+1)}{e^x+x}\overset{\infty/\infty}{=}\lim_{x\to+\infty}\frac{2e^x}{e^x+1}=\lim_{x\to+\infty}\frac{2}{1+e^{-x}}=2$

Because $\displaystyle\lim_{x\to+\infty}2(e^x+1)=+\infty$ and $\displaystyle\lim_{x\to+\infty}(e^x+x)=+\infty$, we applied L'Hôpital's rule again.

Therefore $\displaystyle\ln\lim_{x\to+\infty}(e^x+x)^{2/x}=2$; $\displaystyle\lim_{x\to+\infty}(e^x+x)^{2/x}=e^2$.

20. $\lim_{x\to 0^+}(\sinh x)^{\tan x}$

▷ Because $\lim_{x\to 0^+}\sinh x = 0$ and $\lim_{x\to 0^+}\tan x = 0$, we have the indeterminate form 0^0. We note that if $x > 0$, then $\sinh x > 0$, so $\ln(\sinh x)$ is defined, and we may let

$$y = (\sinh x)^{\tan x}$$

and

$$\ln y = \ln(\sinh x)^{\tan x} = \tan x \ln(\sinh x) = \frac{\ln(\sinh x)}{\cot x}$$

Because $\lim_{x\to 0^+}\ln(\sinh x) = -\infty$ and $\lim_{x\to 0^+}\cot x = +\infty$, we may now apply L'Hôpital's rule as follows.

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(\sinh x)}{\cot x} = \lim_{x\to 0^+}\frac{\frac{\cosh x}{\sinh x}}{-\csc^2 x} = \lim_{x\to 0^+}\frac{-\sin^2 x}{\tanh x} = \lim_{x\to 0^+}\frac{-2\sin x\cos x}{\operatorname{sech}^2 x} = 0$$

Because $\lim_{x\to 0^+}\sin^2 x = 0$ and $\lim_{x\to 0^+}\tanh x = 0$, we applied L'Hôpital's rule again. Thus,

$$\ln\left(\lim_{x\to 0^+} y\right) = 0$$
$$\lim_{x\to 0^+} y = e^0 = 1$$

Therefore,

$$\lim_{x\to 0^+}(\sinh x)^{\tan x} = 1$$

21. $\lim_{x\to 0^+}\ln\sin x = -\infty$ and $\lim_{x\to 0^+}x^{-2} = +\infty$. By L'Hôpital's rule

$$\lim_{x\to 0^+}\ln(\sin x)^x = \lim_{x\to 0^+}\frac{\ln\sin x}{x^{-1}} = \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-x^{-2}} = \lim_{x\to 0^+}\frac{x\cos x}{-1}\cdot\lim_{x\to 0^+}\frac{x}{\sin x} = 0\cdot 1 = 0$$

Therefore $\ln\lim_{x\to 0^+}(\sin x)^x = 0$; $\lim_{x\to 0^+}(\sin x)^x = 1$.

22. $\lim_{x\to 0}\ln(x+e^{2x})^{1/x} = \lim_{x\to 0}\dfrac{\ln(x+e^{2x})}{x} \overset{0/0}{=} \lim_{x\to 0}\dfrac{1+2e^{2x}}{x+e^{2x}} = 3$. Therefore, $\lim_{x\to 0}(x+e^{2x})^{1/x} = e^3$.

23. $\lim_{x\to+\infty}\dfrac{x^2+2x}{e^{3x}-1} \overset{\infty/\infty}{=} \lim_{x\to+\infty}\dfrac{2x+2}{3e^{3x}} \overset{\infty/\infty}{=} \lim_{x\to+\infty}\dfrac{2}{9e^{3x}} = 0$

24. $\lim_{x\to+\infty}\left(1+\dfrac{1}{2x}\right)^{x^2}$

▷ Method 1. To simplify the base, we let $t = 1/x$. Thus

$$\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^{x^2} = \lim_{t\to 0^+}(1+\tfrac{1}{2}t)^{1/t^2}$$

Because we have the indeterminate form $1^{+\infty}$, we let

$$y = (1+\tfrac{1}{2}t)^{1/t^2}$$

Then

$$\ln y = \ln(1+\tfrac{1}{2}t)^{1/t^2} = \frac{\ln(1+\frac{1}{2}t)}{t^2} \tag{1}$$

Because

$$\lim_{t\to 0^+}\ln(1+\tfrac{1}{2}t) = 0 \text{ and } \lim_{t\to 0^+}t^2 = 0$$

we use L'Hôpital's rule on the right side of (1). Thus,

$$\lim_{t\to 0^+}\ln y = \lim_{t\to 0^+}\frac{\ln(1+\frac{1}{2}t)}{t^2} = \lim_{t\to 0^+}\frac{\frac{1}{2}}{1+\frac{1}{2}t}\cdot\frac{1}{2t} = +\infty$$

Thus,

$$\lim_{t\to 0^+} y = +\infty$$

or, equivalently,

$$\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^{x^2} = +\infty$$

Method 2. Because $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$, then

$$\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^{x^2} = \lim_{x\to+\infty}\left[\left(1+\frac{1}{2x}\right)^{2x}\right]^{x/2} = \lim_{x\to+\infty} e^{x/2} = +\infty$$

25. $\lim\limits_{x\to0} 2\ln(1+\sinh x) = 0$ and $\lim\limits_{x\to0} x = 0$. By L'Hôpital's rule

$$\lim_{x\to0}\ln(1+\sinh x)^{2/x} = \lim_{x\to0}\frac{2\ln(1+\sinh x)}{x} = \lim_{x\to0}\frac{2\cosh x}{1+\sinh x} = 2$$

Therefore $\ln\lim\limits_{x\to0}(1+\sinh x)^{2/x} = 2$; $\lim\limits_{x\to0}(1+\sinh x)^{2/x} = e^2$.

26. $\lim\limits_{x\to2}(x-2)\tan\frac{1}{4}\pi x = \lim\limits_{x\to2}\frac{x-2}{\cot\frac{1}{4}\pi x} \overset{0/0}{=} \lim\limits_{x\to2}\frac{1}{-\frac{1}{4}\pi\csc^2\frac{1}{4}\pi} = -\frac{4}{\pi}$

27. $\lim\limits_{x\to0} 4(\ln\cos x + \frac{1}{2}x^2) = 0$ and $\lim\limits_{x\to0} x^4 = 0$. By L'Hôpital's rule

$$\lim_{x\to0}\ln[(\cos x)e^{x^2/2}]^{4/x^4} = \lim_{x\to0}\frac{4(\ln\cos x + \frac{1}{2}x^2)}{x^4} = \lim_{x\to0}\frac{4\left(\frac{-\sin x}{\cos x}\right)}{4x^3} = \lim_{x\to0}\frac{x-\tan x}{x^3}$$

$\lim\limits_{x\to0}(x-\tan)x = 0$ and $\lim\limits_{x\to0} x^3 = 0$; so we apply L'Hôpital's rule again and get

$$\lim_{x\to0}\frac{x-\tan x}{x^3} = \lim_{x\to0}\frac{1-\sec^2 x}{3x^2} = \lim_{x\to0}\frac{-\tan^2 x}{3x^2} = \lim_{x\to0}\frac{-1}{3\cos^2 x}\cdot\left(\lim_{x\to0}\frac{\sin x}{x}\right)^2 = -\frac{1}{3}\cdot1^2 = -\frac{1}{3}$$

Therefore $\ln\lim\limits_{x\to0}[(\cos x)e^{x^2/2}]^{4/x^4} = -\frac{1}{3}$; $\lim\limits_{x\to0}[(\cos x)e^{x^2/2}]^{4/x^4} = e^{-1/3}$.

28. $\lim\limits_{x\to0}(\cos x)^{1/x^2}$

▸ Because $\lim\limits_{x\to0}\cos x = 1$ and $\lim\limits_{x\to0}\frac{1}{x^2} = +\infty$

we have the indeterminate form $1^{+\infty}$. Thus we let

$$y = (\cos x)^{1/x^2}$$

$$\ln y = \ln(\cos x)^{1/x^2} = \frac{\ln(\cos x)}{x^2} \qquad (1)$$

Because

$\lim\limits_{x\to0}\ln(\cos x) = 0$ and $\lim\limits_{x\to0} x^2 = 0$

we use L'Hôpital's rule on the right-hand side of (1). Thus,

$$\lim_{x\to0}\ln y = \lim_{x\to0}\frac{\ln(\cos x)}{x^2} = \lim_{x\to0}\frac{-\sin x}{2x\cos x} = \lim_{x\to0}\frac{\sin x}{x}\cdot\lim_{x\to0}\frac{-1}{2\cos x} = 1(-\tfrac{1}{2}) = -\tfrac{1}{2}$$

Thus,

$$\ln(\lim_{x\to0} y) = -\tfrac{1}{2}$$

$$\lim_{x\to0} y = e^{-1/2}$$

or, equivalently,

$$\lim_{x\to0}(\cos x)^{1/x^2} = e^{-1/2}$$

29. $\lim\limits_{x\to+\infty}[(x^6+3x^5+4)^{1/6} - x] = \lim\limits_{x\to+\infty}\dfrac{\left(1+\frac{3}{x}+\frac{4}{x^6}\right)^{1/6} - 1}{\frac{1}{x}} \overset{0/0}{=} \lim\limits_{x\to+\infty}\dfrac{(1+\frac{3}{x}+\frac{4}{x^6})^{1/6} - 1}{\frac{1}{x}}$

$$= \lim_{x\to+\infty}\frac{\frac{1}{6}\left(1+\frac{3}{x}+\frac{4}{x^6}\right)^{-5/6}\left(-\frac{3}{x^2}-\frac{24}{x^7}\right)}{-\frac{1}{x^2}} = \lim_{x\to+\infty}\left(1+\frac{3}{x}+\frac{4}{x^6}\right)^{-5/6}\left(\frac{1}{2}+\frac{4}{x^5}\right) = \frac{1}{2}$$

30. $\lim\limits_{x\to+\infty}\dfrac{\ln(x+e^x)}{3x} \overset{\infty/\infty}{=} \lim\limits_{x\to+\infty}\dfrac{1+e^x}{3(x+e^x)} = \lim\limits_{x\to+\infty}\dfrac{1+e^{-x}}{3(1+xe^{-x})} = \dfrac{1}{3}$ because $\lim\limits_{x\to+\infty}\frac{x}{e^x} = 0$.

31. Let $y = \frac{1}{x}$. Then $\lim\limits_{x\to0^+}\dfrac{e^{-1/x}}{x} = \lim\limits_{x\to0^+}\dfrac{\frac{1}{x}}{e^{1/x}} = \lim\limits_{y\to+\infty}\dfrac{y}{e^y}$. $\lim\limits_{y\to+\infty} y = +\infty$ and $\lim\limits_{y\to+\infty} e^y = +\infty$.

By L'Hôpital's rule $\lim\limits_{y\to+\infty}\dfrac{y}{e^y} = \lim\limits_{y\to+\infty}\dfrac{1}{e^y} = 0$.

32. $\lim_{x\to0^+} x^{x^x}$

▶ First we find $\lim_{x\to0^+} x^x$. Let

$$y = x^x$$
$$\ln y = x\ln x$$
$$\lim_{x\to0^+}\ln y = \lim_{x\to0^+} x\ln x$$
$$= \lim_{x\to0^+}\frac{\ln x}{1/x} \tag{1}$$

Because $\lim_{x\to0^+}\ln x = -\infty$ and $\lim_{x\to0^+}\frac{1}{x} = +\infty$ we apply L'Hôpital's rule on the right-hand side of (1). Thus,

$$\lim_{x\to0^+}\ln y = \lim_{x\to0^+}\frac{x^{-1}}{-x^{-2}} = \lim_{x\to0^+} -x = 0$$

Thus,

$$\ln \lim_{x\to0^+} y = 0; \quad \lim_{x\to0^+} y = e^0; \ \lim_{x\to0^+} x^x = 1$$

Therefore

$$\lim_{x\to0^+} x^{x^x} = 0^1 = 0$$

33. $\lim_{x\to+\infty}\frac{x}{\sqrt{1+x^2}} = \lim_{x\to+\infty}\frac{1}{\sqrt{x^{-2}+1}} = \frac{1}{\sqrt{0+1}} = 1$

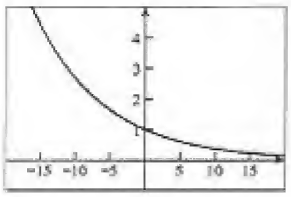
34. $\lim_{x\to+\infty}(x-\sqrt{x^2+x}) = \lim_{x\to+\infty}(x-\sqrt{x^2+x})\cdot\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}} = \lim_{x\to+\infty}\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}} = \lim_{x\to+\infty}\frac{-x}{x+\sqrt{x^2+x}}$

$$= \lim_{x\to+\infty}\frac{-1}{1+\sqrt{1+1/x}} = -\frac{1}{2}$$

In Exercises 35 and 36, plot the graph of $f(x)$ and from the graph predict the behavior of $f(x)$ as (a) $x\to-\infty$, and (b) $x\to+\infty$. Confirm your answers in parts (a) and (b) by determining (c) $\lim_{x\to-\infty} f(x)$ and (d) $\lim_{x\to+\infty} f(x)$.

35. $\lim_{x\to-\infty}\frac{2^x}{e^x} = \lim_{x\to-\infty}\left(\frac{2}{e}\right)^x = +\infty$ and $\lim_{x\to+\infty}\left(\frac{2}{e}\right)^x = 0$ because $\frac{2}{e} < 1$.

36. $f(x) = \frac{e^x}{3^x}$



▶ From the plot at the right we predict that $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} f(x) = 0$. These are confirmed because $\frac{e}{3} < 1$ and so $f(x) = \left(\frac{e}{3}\right)^x$ is a decreasing exponential function.

37. $\lim_{x\to+\infty}\frac{e^x}{x^p} = \left(\lim_{x\to+\infty}\frac{e^{x/p}}{x}\right)^p \overset{\infty/\infty}{=} \left(\lim_{x\to+\infty} pe^{x/p}\right)^p = +\infty$

38. $\lim_{x\to+\infty}\frac{\ln x}{x^p} \overset{\infty/\infty}{=} \lim_{x\to+\infty}\frac{1/x}{px^{p-1}} = \lim_{x\to+\infty}\frac{1}{px^p} = 0$

39. $f(x) = \begin{cases}(1-e^{4x})^x & \text{if } x<0\\ k & \text{if } 0\le x\end{cases}$

Because $f(0) = k$ and $\lim_{x\to0^+} f(x) = k$, then f will be continuous at 0 if $\lim_{x\to0^-} f(x) = k$.

$\lim_{x\to0^-}\ln(1-e^{4x}) = -\infty$ and $\lim_{x\to0^-}\frac{1}{x} = -\infty$. By L'Hôpital's rule

$$\lim_{x\to0^-}\ln f(x) = \lim_{x\to0^-} x\ln(1-e^{4x}) = \lim_{x\to0^-}\frac{\ln(1-e^{4x})}{\frac{1}{x}} = \lim_{x\to0^-}\frac{\frac{-4e^{4x}}{1-e^{4x}}}{-\frac{1}{x^2}} = \lim_{x\to0^-}\frac{4x^2}{e^{-4x}-1} = \lim_{x\to0^-}\frac{8x}{-4e^{-4x}} = 0$$

Because $\lim_{x\to0^-} 4x^2 = 0$ and $\lim_{x\to0^-}(e^{-4x}-1) = 0$, we applied L'Hôpital's rule again. Therefore

$\ln \lim_{x\to0^-} f(x) = 0$; $\lim_{x\to0^-} f(x) = 1$. Hence f will be continuous at 0 if $k = 1$.

40. If $f(x) = \begin{cases}(x+1)^{(\ln k)/x} & \text{if } x \neq 0 \\ 5 & \text{if } x = 0\end{cases}$, find k so that f is continuous at $x = 0$.

▹ We must find k so that $\lim_{x\to 0} f(x) = f(0) = 5$. Because

$(x+1)^{(\ln k)/x} = e^{\ln(x+1)(\ln k)/x} = k^{\ln(x+1)/x}$

we apply L'Hôpital's rule to the exponent.

$\lim_{x\to 0} \frac{\ln(x+1)}{x} = \lim_{x\to 0} \frac{1}{x+1} = 1$

Thus,

$\lim_{x\to 0} f(x) = k^1 = k$

Therefore, f will be continuous at 0 if $k = 5$.

41. $\lim_{x\to+\infty} \ln\left(\frac{nx+1}{nx-1}\right) = \ln \lim_{x\to+\infty} \frac{n+\frac{1}{x}}{n-\frac{1}{x}} = \ln 1 = 0$ and $\lim_{x\to+\infty} \frac{1}{x} = 0$. By L'Hôpital's rule

$$\lim_{x\to+\infty} \ln\left(\frac{nx+1}{nx-1}\right)^x = \lim_{x\to+\infty} \frac{\ln\frac{nx+1}{nx-1}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{\frac{n}{nx+1} - \frac{n}{nx-1}}{-\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{2nx^2}{n^2x^2-1} = \lim_{x\to+\infty} \frac{2n}{n^2 - x^{-2}} = \frac{2}{n}$$

Therefore $\ln 9 = \ln \lim_{x\to+\infty} \left(\frac{nx+1}{nx-1}\right)^x = \frac{2}{n}$; $n = \frac{2}{\ln 9} = \frac{2}{\ln 3^2} = \frac{1}{\ln 3}$.

42. Let $t = \frac{1}{2}\pi - x$. $\lim_{x\to\pi/2} \frac{\sec x}{\sec 3x} = \lim_{x\to\pi/2} \frac{\cos 3x}{\cos x} = \lim_{t\to 0} \frac{\cos(\frac{3}{2}\pi - 3t)}{\cos(\frac{1}{2}\pi - t)} = \lim_{t\to 0} \frac{-\sin 3t}{3t} \cdot \frac{t}{\sin t} \cdot 3 = -3$

43. (a) Use one-sided limits. Let $t = \frac{1}{x^2}$. $\lim_{t\to+\infty} t = +\infty$ and $\lim_{t\to+\infty} e^{2t/n} = +\infty$. By L'Hôpital's rule

$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = \lim_{t\to+\infty} \frac{t^{n/2}}{e^t} = \left(\lim_{t\to+\infty} \frac{t}{e^{2t/n}}\right)^{n/2} = \left(\lim_{t\to+\infty} \frac{1}{(2/n)e^{2t/n}}\right)^{n/2} = 0^{n/2} = 0$ because n is positive.

Because n is a positive integer $\lim_{x\to 0^-} \frac{e^{-1/x^2}}{x^n} = \pm \lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = 0$. Therefore $\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} = 0$.

(b) $\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-1/x^2} = 0$. $f'(x) = \frac{2e^{-1/x^2}}{x^3}$ and every derivative of f is a sum of terms of the form $\frac{ke^{-1/x^2}}{x^n}$. Hence the limits at 0 of all the derivatives of f are 0.

44. Suppose $f(x) = \int_0^x e^{3t}\sqrt{9t^4+1}\,dt$ and $g(x) = x^n e^{3x}$. If $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = 1$, find n.

▹ By the first fundamental theorem of the calculus (4.8.1), $f'(x) = e^{3x}\sqrt{9x^4+1}$

Differentiating $g(x)$ we obtain

$$g'(x) = nx^{n-1}e^{3x} + x^n(3e^{3x}) = e^{3x}x^n(3 + nx^{-1})$$

Therefore,

$$\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to+\infty} \frac{e^{3x}\sqrt{9x^4+1}}{e^{3x}x^n(3+nx^{-1})} = \lim_{x\to+\infty} \frac{\sqrt{9x^4+1}}{x^n(3+nx^{-1})} \tag{1}$$

If $n > 0$, the hypothesis of L'Hôpital's rule is satisfied for the limit on the right-hand side of (1). However, the rule is difficult to apply. To find the limit we divide the numerator and denominator by x^2. Thus

$$\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to+\infty} \frac{\sqrt{9+x^{-4}}}{x^{n-2}(3+nx^{-1})} = \lim_{x\to+\infty} \frac{\sqrt{9+x^{-4}}}{3+nx^{-1}} \cdot \lim_{x\to+\infty} \frac{1}{x^{n-2}} = \frac{\sqrt{9}}{3} \cdot \lim_{x\to+\infty} \frac{1}{x^{n-2}} = \lim_{x\to+\infty} \frac{1}{x^{n-2}} \tag{2}$$

If $n > 2$, then $\lim_{x\to+\infty} \frac{1}{x^{n-2}} = 0$. If $n < 2$, then $\lim_{x\to+\infty} \frac{1}{x^{n-2}} = +\infty$

We are given that $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = 1 \qquad (3)$

By comparing Eqs. (2) and (3), we conclude that $n = 2$.

45. $y = x^p$, $y' = px^{p-1}$. Slope of normal line at $(u, u^p) = -\frac{1}{y'(u)} = \frac{-1}{pu^{p-1}} = \frac{u^p - 0}{u - a}$. Thus,

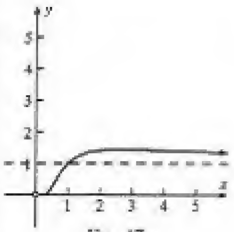
$$\lim_{u \to +\infty} (a - u) = \lim_{u \to +\infty} pu^{2p-1} = \begin{cases} 0 & \text{if } 0 < p < \frac{1}{2} \\ \frac{1}{2} & \text{if } p = \frac{1}{2} \\ +\infty & \text{if } p > \frac{1}{2} \end{cases}$$

46. $y = \ln x$, $y' = \frac{1}{x}$. Slope of normal line at $(u, \ln u) = -\frac{1}{y'(u)} = -u = \frac{\ln u - 0}{u - a}$. Thus,

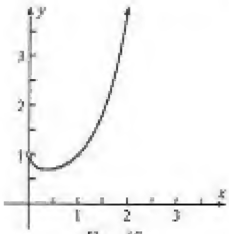
$$\lim_{u \to +\infty} (a - u) = \lim_{u \to +\infty} \frac{\ln u}{u} = 0$$

In Exercises 47 and 48, sketch the graph of f by first finding the extrema of f and any horizontal asymptotes.

47. $f(x) = x^{1/x} = e^{(\ln x)/x}$, $x > 0$. See the figure, below left. $f'(x) = e^{(\ln x/x)}\left(\frac{1 - \ln x}{x^2}\right) = x^{1/x}\left(\frac{1 - \ln x}{x^2}\right)$. Because $f'(x) > 0$ if $0 < x < e$ and $f'(x) < 0$ if $x > e$, the graph of $y = f(x)$ has an absolute maximum value at the point $(e, e^{1/e}) \approx (2.78, 1.44)$. $f''(x) = x^{1/x}(2x \ln x + \ln^2 x - 2 \ln x + 1)/x^4$. Using trace and zoom we find inflection points when x is 0.5819 and 4.3678. By L'Hôpital's rule $\lim_{x \to +\infty} \ln f(x) = \lim_{x \to +\infty} \frac{\ln x}{x} = \lim_{x \to +\infty} \frac{1}{x} = 0$. Therefore $\lim_{x \to 0} f(x) = e^0 = 1$ and $y = 1$ is a horizontal asymptote. Furthermore, $\lim_{x \to 0^+} f(x) = 0$.



Ex. 47



Ex. 48

48. $f(x) = x^x$, $x > 0$

▹ Because $f(x) = e^{x \ln x}$, then

$$f'(x) = e^{x \ln x}\, D_x(x \ln x) = x^x(\ln x + 1)$$

Because $f'(x) < 0$ if $x < e^{-1}$ and $f'(x) > 0$ if $x > e^{-1}$, then f has an absolute minimum value at $(1/e, 1/e^{1/e}) \approx (0.368, 0.692)$. Because $\lim_{x \to 0^+} (x \ln x) = 0$, then $\lim_{x \to 0^+} f(x) = e^0 = 1$.
Finally, $\lim_{x \to +\infty} f(x)/x = \lim_{x \to +\infty} x^{x-1} = +\infty$ so there are no asymptotes. See the figure above right.

49. $\lim_{x \to +\infty} \log_x(x + 10) = \lim_{x \to +\infty} \frac{\ln(x + 10)}{\ln x} \overset{\infty/\infty}{=} \lim_{x \to +\infty} \frac{1/(x + 10)}{1/x} = \lim_{x \to +\infty} \frac{1}{1 + 10/x} = 1$

50. $\lim_{x \to +\infty} \frac{x - \cos x}{x + \sin x} = \lim_{x \to +\infty} \frac{1 - \cos x/x}{1 + \sin x/x} = \frac{1}{1} = 1$
L'Hopital's rule cannot be used because $\lim_{x \to +\infty} \frac{1 + \sin x}{1 + \cos x}$ does not exist.

51. (a) $\lim_{x \to 0^+} \sin x = 0^+$ and $\lim_{x \to 0^+} \csc x = +\infty$. Hence $\lim_{x \to 0^+} (\sin x)^{\csc x} = 0$.

(b) $\lim_{x \to +\infty} \sin \frac{1}{x} = 0^+$. Hence $\lim_{x \to +\infty} \left(\sin \frac{1}{x}\right)^x = 0$. $0^{+\infty}$ is not an indeterminate form.

7.9 IMPROPER INTEGRALS WITH INFINITE LIMITS OF INTEGRATION

7.9.1 Definition Infinite Upper Limit If f is continuous for all $x \geq a$, then
$$\int_a^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx$$
if this limit exists.

7.9.2 Definition Infinite Lower Limit If f is continuous for all $x \leq b$, then
$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx$$
if this limit exists.

7.9.3 Definition Both Limits Infinite If f is continuous for all values of x, and c is any real number, then
$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{a \to -\infty} \int_a^c f(x)dx + \lim_{b \to +\infty} \int_c^b f(x)dx$$
if these limits exist.

Principal Value of $\int_{-\infty}^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_{-b}^{b} f(x)dx$ if the limit exists.

Convergent, Divergent The integral is *convergent* if the limits exist; otherwise it is *divergent*. If $f(x) \geq c > 0$ for $x \geq a$ then
$\int_a^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx \geq \lim_{b \to +\infty} \int_a^b c\, dx = \lim_{b \to +\infty} c(b-a) = +\infty$. Hence the given integral is divergent. Certainly if $\lim_{x \to +\infty} f(x) = +\infty$, the integral is divergent.

p Test $\int_1^{+\infty} \frac{dx}{x^p}$ is convergent if and only if $p > 1$. See Exercise 22.

Probability of Interval If f is a probability density function (pdf) for a particular event occurring, then the probability that the event will occur over the interval $[a, b]$ is
$$P([a,b]) = \int_a^b f(x)dx$$

Mean of PDF $\mu = \int_{-\infty}^{+\infty} xf(x)dx$ if the integral exists.

Exponential Density Function: $f(x) = \begin{cases} ke^{-kx} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$. Its mean is $\frac{1}{k}$. See Exercises 32.

Uniform Density Function: $f(x) = \begin{cases} 1/(d-c) & \text{if } c \leq x \leq d \\ 0 & \text{otherwise} \end{cases}$, where $c < d$.

Present Value of a flow of income of $f(t)$ dollars per year after t years when the annual rate rate of interest is $100i$ percent compounded continuously is $V = \int_0^{+\infty} f(t)e^{-it}dt$.

The following limit theorems are helpful when applying these definitions.

$$\lim_{x \to +\infty} \ln x = +\infty, \quad \lim_{x \to +\infty} a^x = +\infty \quad \text{if } a > 1, \quad \lim_{x \to -\infty} a^x = 0 \quad \text{if } a > 1$$

$$\lim_{x \to +\infty} \tan^{-1} x = \tfrac{1}{2}\pi, \quad \lim_{x \to -\infty} \tan^{-1} x = -\tfrac{1}{2}\pi, \quad \lim_{x \to +\infty} \frac{x^n}{a^x} = 0 \text{ if } a > 1 \text{ and } n \text{ is any real number}$$

Exercises 7.9

In Exercises 1–18 determine if the improper integral is convergent or divergent. If it is convergent, evaluate it.

1. $\int_0^{+\infty} e^{-x/3}dx = \lim_{b \to +\infty} \int_0^b e^{-x/3}dx = \lim_{b \to +\infty} \left[-3e^{-x/3}\right]_0^b = \lim_{b \to +\infty} (-3e^{-b/3} + 3e^0) = 3$

2. $\int_{-\infty}^{1} e^x dx = \lim_{a \to -\infty} \int_a^1 e^x dx = \lim_{a \to -\infty} e^x \Big]_a^1 = \lim_{a \to -\infty} (e - e^a) = e$

3. $\int_{-\infty}^{0} x5^{-x^2}\, dx = \lim_{a \to -\infty} \int_a^0 5^{-x^2}\,(x\,dx) = -\frac{1}{2} \lim_{a \to -\infty} \left[\frac{5^{-x^2}}{\ln 5}\right]_a^0 = -\frac{1}{2} \lim_{a \to -\infty} \left(\frac{1 - 5^{-a^2}}{\ln 5}\right) = -\frac{1}{2 \ln 5}$

4. $\int_1^{+\infty} 2^{-x}dx$

▷ We use Definition 7.9.1. Thus,
$$\int_1^{+\infty} 2^{-x}dx = \lim_{b \to +\infty} \int_1^b 2^{-x}dx = \lim_{b \to +\infty} \left[\frac{-2^{-x}}{\ln 2}\right]_1^b = \lim_{b \to +\infty} \left[\frac{-2^{-b}}{\ln 2} + \frac{1}{2 \ln 2}\right] = \frac{1}{2 \ln 2}$$

■ Thus the integral is convergent with value [illegible].

5. Let $u = x$ and $dv = 3^{-x}dx$. Then $du = dx$ and $v = -\frac{3^{-x}}{\ln 3}$. Therefore

$$\int_{0}^{+\infty} x3^{-x}dx = \lim_{b\to+\infty}\int_0^b x3^{-x}dx = \lim_{b\to+\infty}\left[\cdots\right]_0^b = \lim_{b\to+\infty}\left[\cdots\right]$$

$$= \lim_{b\to+\infty}\left[\cdots\right] = \frac{1}{(\ln 3)^2}$$

6. $\int_{4}^{+\infty}\frac{dx}{\cdots} = \int_{\cdots}^{+\infty}\frac{du}{u^{\cdots}}$ is same as the p-test with $p = \frac{1}{2} \le 1$.

7. Because $\lim_{x\to+\infty} x\cosh x \ne 0$, $\int_{-\infty}^{+\infty} x\cosh x\,dx$ is divergent. The integrand is odd so the principal value is 0.

8. $\int_{-\infty}^{0} x^2e^x dx$

□ We use Definition 7.8.2. Then

$$\int_{-\infty}^{0} x^2e^xdx = \lim_{a\to-\infty}\int_a^0 x^2e^xdx \tag{1}$$

We use integration by parts. Let

$$u = x^2 \qquad dv = e^xdx$$
$$du = 2x\,dx \qquad v = e^x$$

Thus

$$\int x^2e^xdx = x^2e^x - \int 2xe^xdx \tag{2}$$

We integrate by parts again. Let

$$\bar{u} = 2x \qquad d\bar{v} = e^xdx$$
$$d\bar{u} = 2\,dx \qquad \bar{v} = e^x$$

Then from (2) we obtain

$$\int x^2e^xdx = x^2e^x - 2xe^x + \int 2e^xdx = x^2e^x - 2xe^x + 2e^x \tag{3}$$

Substituting from (3) into (1), we obtain

$$\int_{-\infty}^{0} x^2e^xdx = \lim_{a\to-\infty}\left[e^x(x^2-2x+2)\right]_a^0 = \lim_{a\to-\infty}\left[2 - e^a(a^2-2a+2)\right] = 2 - \lim_{a\to-\infty}\left(\frac{a^2}{e^{-a}} - 2\frac{a}{e^{-a}} + \frac{2}{e^{-a}}\right) = 2$$

■ Thus the integral is convergent and has value 2.

9. Because $\lim_{x\to+\infty}\frac{x}{\cdots} = \lim_{x\to+\infty}\frac{\cdots}{\cdots} = \cdots$, $\int^{+\infty}\frac{x\,dx}{\cdots}$ is divergent.

10. $$\int_{-\infty}^{+\infty}\frac{2x\,dx}{(3x^2+2)^3} = \lim_{a\to-\infty}\int_a^0\frac{\frac{1}{3}d(3x^2+2)}{(3x^2+2)^3} + \lim_{b\to+\infty}\int_0^b\frac{\frac{1}{3}d(3x^2+2)}{(3x^2+2)^3}$$

$$= \lim_{a\to-\infty}\left[-\frac{1}{6(3x^2+2)^2}\right]_a^0 + \lim_{b\to+\infty}\left[-\frac{1}{6(3x^2+2)^2}\right]_0^b = \lim_{a\to-\infty}\left(\cdots\right) + \lim_{b\to+\infty}\left(\cdots\right)$$

$$= -\frac{1}{24} + \frac{1}{24} = 0$$

11. $$\int_{\sqrt{3}}^{+\infty}\frac{2\,dx}{x^2+9} = 2\lim_{b\to+\infty}\int_{\sqrt{3}}^b\frac{dx}{x^2+9} = \cdots\lim_{b\to+\infty}\tan^{-1}\frac{x}{3}\Big]_{\sqrt{3}}^b = \lim_{b\to+\infty}\left[\cdots\right] = \cdots$$

12. $\int_{2}^{+\infty}\frac{dx}{x\ln x}$

□ By Definition 7.8.1, we have

$$\int_2^{+\infty}\frac{dx}{x\ln x} = \lim_{b\to+\infty}\int_2^b\frac{dx}{x\ln x} = \lim_{b\to+\infty}\ln(\ln x)\Big]_2^b = \lim_{b\to+\infty}[\ln(\ln b) - \ln(\ln 2)] = +\infty$$

■ Thus the integral is divergent.

13. $\int_{-\infty}^{+\infty} e^{-|x|}dx = \lim_{a \to -\infty}\int_a^0 e^x dx + \lim_{b \to +\infty}\int_0^b e^{-x}dx = \lim_{a \to -\infty}\left[e^x\right]_a^0 + \lim_{b \to +\infty}\left[-e^{-x}\right]_0^b$

$= \lim_{a \to -\infty}(1 - e^a) + \lim_{b \to +\infty}(-e^{-b} + 1) = 1 - 0 - 0 + 1 = 2$

14. $\int_{-\infty}^{+\infty} xe^{-x^2}dx = \lim_{a \to -\infty}\int_a^0 e^{-x^2} \cdot -\frac{1}{2}d(-x^2) + \lim_{b \to +\infty}\int_0^b e^{-x^2} \cdot -\frac{1}{2}d(-x^2) = \lim_{a \to -\infty} -\frac{1}{2}e^{-x^2}\Big]_a^0 + \lim_{b \to +\infty} -\frac{1}{2}e^{-x^2}\Big]_0^b$

$\lim_{a \to -\infty} \frac{1}{2}(e^{-a^2} - 1) + \lim_{b \to +\infty} \frac{1}{2}(1 - e^{-b^2}) = -\frac{1}{2} + \frac{1}{2} = 0$

15. $\int_e^{+\infty} \frac{dx}{x(\ln x)^2} = \lim_{b \to +\infty}\int_e^b \frac{1}{(\ln x)^2}\frac{dx}{x} = \lim_{b \to +\infty}\left[-\frac{1}{\ln x}\right]_e^b = \lim_{b \to +\infty}\left(-\frac{1}{\ln b} + \frac{1}{\ln e}\right) = -0 + 1 = 1$

16. $\int_{-\infty}^{+\infty} \frac{dx}{16 + x^2}$

▷ We use Definition 7.9.3 with $c = 0$. Thus

$$\int_{-\infty}^{+\infty} \frac{dx}{16 + x^2} = \lim_{a \to -\infty}\int_a^0 \frac{dx}{16 + x^2} + \lim_{b \to +\infty}\int_0^b \frac{dx}{16 + x^2} = \lim_{a \to -\infty}\left[\frac{1}{4}\tan^{-1}\frac{x}{4}\right]_a^0 + \lim_{b \to +\infty}\left[\frac{1}{4}\tan^{-1}\frac{x}{4}\right]_0^b$$

$$= \lim_{a \to -\infty} -\frac{1}{4}\tan^{-1}\frac{a}{4} + \lim_{b \to +\infty}\frac{1}{4}\tan^{-1}\frac{b}{4} = -\frac{1}{4}(-\frac{1}{2}\pi) + \frac{1}{4}(\frac{1}{2}\pi) = \frac{1}{4}\pi$$

▪ Thus the integral is convergent and has value $\frac{1}{4}\pi$.

17. Because $\lim_{x \to +\infty} \ln x = +\infty$, $\int_1^{+\infty} \ln x\, dx$ is divergent.

18. From formula #106, $\int_0^{+\infty} e^{-x}\cos x\, dx = \lim_{b \to +\infty}\int_0^b e^{-x}\cos x\, dx = \lim_{b \to +\infty} \frac{1}{2}e^{-x}(-\cos x + \sin x)\Big]_0^b$

$= \lim_{b \to +\infty}[\frac{1}{2}e^{-b}(-\cos b + \sin b) - \frac{1}{2}(-1)] = \frac{1}{2}$

19. (a) $\int_{-\infty}^{+\infty} \sin x\, dx = \lim_{a \to -\infty}\int_a^0 \sin x\, dx + \lim_{b \to +\infty}\int_0^b \sin x\, dx = \lim_{a \to -\infty}\left[-\cos x\right]_a^0 + \lim_{b \to +\infty}\left[-\cos x\right]_0^b$

$= \lim_{a \to -\infty}(-1 + \cos a) + \lim_{b \to +\infty}(-\cos b + 1)$

Because $\lim_{a \to -\infty} \cos a$ does not exist or because $\lim_{b \to +\infty} \cos b$ does not exist, $\int_{-\infty}^{+\infty} \sin x\, dx$ diverges.

(b) $\lim_{r \to +\infty}\int_{-r}^{r} \sin x\, dx = \lim_{r \to +\infty}\left[-\cos x\right]_{-r}^{r} = \lim_{r \to +\infty}[-\cos r + \cos(-r)] = \lim_{r \to +\infty}(-\cos r + \cos r) = \lim_{r \to +\infty} 0 = 0$

20. Prove that if $\int_{-\infty}^{b} f(x)dx$ is convergent, then $\int_{-b}^{+\infty} f(-x)dx$ is also convergent and has the same value.

▷ We are given that $\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty}\int_a^b f(x)dx$ (1)

exists. Let $z = -x$, $dz = -dx$. Then

$$\int_{-b}^{+\infty} f(-x)dx = \lim_{c \to +\infty}\int_{x=-b}^{c} f(-x)dx = \lim_{c \to +\infty}\int_{z=b}^{-c} f(z)(-dz) = \lim_{c \to +\infty}\int_{-c}^{b} f(z)dz = \lim_{c \to +\infty}\int_{c}^{b} f(x)dx \quad (2)$$

Comparing (1) and (2), we conclude that $\int_{-b}^{+\infty} f(-x)dx$ is convergent and has the same value as $\int_{-\infty}^{b} f(x)dx$.

21. (a) $\int_{-\infty}^{+\infty} \frac{x\, dx}{(1 + x^2)^2} = \lim_{a \to -\infty}\int_a^0 \frac{x\, dx}{(1 + x^2)^2} + \lim_{b \to +\infty}\int_0^b \frac{x\, dx}{(1 + x^2)^2} = \lim_{a \to -\infty}\left[-\frac{1}{2(1 + x^2)}\right]_a^0 + \lim_{b \to +\infty}\left[-\frac{1}{2(1 + x^2)}\right]_0^b$

$= \lim_{a \to -\infty}\left(-\frac{1}{2} + \frac{1}{2(1 + a^2)}\right) + \lim_{b \to +\infty}\left(-\frac{1}{2} + \frac{1}{2(1 + b^2)}\right) = -\frac{1}{2} + \frac{1}{2} = 0$

(b) $\int_{-\infty}^{+\infty} \frac{x\, dx}{1 + x^2} = \lim_{a \to -\infty}\int_a^0 \frac{x\, dx}{1 + x^2} + \lim_{b \to +\infty}\int_0^b \frac{x\, dx}{1 + x^2} = \lim_{a \to -\infty}\left[\frac{1}{2}\ln(1 + x^2)\right]_a^0 + \lim_{b \to +\infty}\left[\frac{1}{2}\ln(1 + x^2)\right]_0^b$

$= \lim_{a \to -\infty}\left[-\frac{1}{2}\ln(1 + a^2)\right] + \lim_{b \to +\infty}\left[\frac{1}{2}\ln(1 + b^2)\right]$

Neither limit exists. Hence the given integral is divergent.

22. Prove the p test for convergence.

▷ If $p = 1$, $\int_1^{+\infty} \frac{dx}{x} = \lim_{b \to +\infty} \ln x\Big]_1^b = \lim_{b \to +\infty}(\ln b - \ln 1) = +\infty$. If $p \neq 1$,

$\int_1^{+\infty} x^{-p}dx = \lim_{b \to +\infty}\frac{x^{1-p}}{1 - p}\Big]_1^b = \lim_{b \to +\infty} \frac{1}{1 - p}(b^{1-p} - 1)$ which converges if and only if $1 - p < 0 \Leftrightarrow p > 1$.

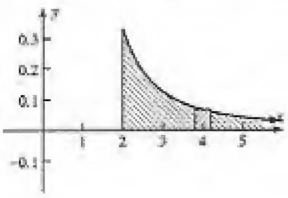
23. Let A square units be the area of the region bounded by $y = 1/(e^x + e^{-x})$, the x axis, the line $x = a$ $(a < 0)$ and the line $x = b$ $(b > 0)$. Then

$$A = \int_a^b \frac{dx}{e^x + e^{-x}} = \int_a^b \frac{e^x dx}{e^{2x} + 1} = \tan^{-1} e^x \Big]_a^b = \tan^{-1} e^b - \tan^{-1} e^a$$

Because $\lim\limits_{b \to +\infty} \tan^{-1} e^b = \frac{1}{2}\pi$ and $\lim\limits_{a \to -\infty} \tan^{-1} e^a = -\frac{1}{2}\pi$, we assign π as the measure of the area.

24. Determine whether it is possible to assign a finite number to represent the measure of the area of the region bounded by the x axis, the line $x = 2$, and the curve whose equation is $y = 1/(x^2 - 1)$. If a finite number can be assigned, find it.

▸ Let f be the function defined by $f(x) = \dfrac{1}{x^2 - 1}$.



We find the area of the region R, bounded above by the curve $y = f(x)$, bounded below by the x axis, bounded on the left by the line $x = 2$, and bounded on the right by the line $x = b$, where $b > 2$ as illustrated in the figure. The number of square units in the area of R is given by

$$A = \int_2^b \frac{dx}{x^2 - 1} = \frac{1}{2}\int_2^b \frac{dx}{x - 1} - \frac{1}{2}\int_2^b \frac{dx}{x + 1} = \frac{1}{2} \ln \frac{x-1}{x+1}\Big]_2^b = \frac{1}{2}\left(\ln \frac{b-1}{b+1} - \ln \frac{1}{3}\right)$$

where partial fractions were used for the integral. Then,

$$\lim_{b \to +\infty} A = \lim_{b \to +\infty} \frac{1}{2}\left(\ln \frac{b-1}{b+1} + 3\right) = \frac{1}{2}(\ln 1 + \ln 3) = \frac{1}{2} \ln 3$$

Because the above limit exists, the area is $\frac{1}{2} \ln 3$ square units.

25. Consider the region bounded by $y = x^{-3/2}$, the x axis, $x = 1$ and $x = b$ $(b > 1)$. An element of volume is a circular disk, centered on the x axis, of radius $w_i^{-3/2}$. Then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(w_i^{-3/2})^2 \Delta_i x = \pi \int_1^b x^{-3} dx = -\frac{\pi}{2x^2}\Big]_1^b = \pi\left(-\frac{1}{2b^2} + \frac{1}{2}\right)$$

Because $\lim\limits_{b \to +\infty} V = \lim\limits_{b \to +\infty} \pi\left(-\dfrac{1}{2b^2} + \dfrac{1}{2}\right) = \dfrac{1}{2}\pi$, we assign $\frac{1}{2}\pi$ as the measure of the volume.

26. Consider the region bounded by $y = e^{-2x}$, the x axis, $x = 0$ and $x = b$ $(b > 1)$. An element of volume is a circular disk, centered on the x axis, of radius e^{-2w_i}. Then

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(e^{-2w_i})^2 \Delta_i x = \pi \int_0^b e^{-4x} dx = -\frac{1}{4}\pi e^{-4x}\Big]_0^b = \frac{1}{4}\pi(1 - e^{-4b})$$

Because $\lim\limits_{b \to +\infty} V = \lim\limits_{b \to +\infty} \frac{1}{4}\pi(1 - e^{-4b}) = \frac{1}{4}\pi$, we assign $\frac{1}{4}\pi$ as the measure of the volume.

27. $f(x) = \begin{cases} \frac{1}{60} e^{-x/60} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

(a) The probability that the life of a battery is not more than 50 hours is

$$P((-\infty, 50]) = \int_{-\infty}^{50} f(x)dx = \int_{-\infty}^{0} 0\, dx + \int_0^{50} \frac{1}{60} e^{-x/60} dx = 0 + \left[-e^{-x/60}\right]_0^{50} = -e^{-5/6} + 1 \approx 0.565$$

(b) The probability that the life of a battery is at least 75 hours is $P([75, +\infty))$

$$= \int_{75}^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_{75}^{b} \frac{1}{60} e^{-x/60} dx = \lim_{b \to +\infty} \left[-e^{-x/60}\right]_{75}^{b} = \lim_{b \to +\infty} (-e^{-b/60} + e^{-5/4}) = e^{-5/4} \approx 0.287$$

28. For a certain type of light bulb, the probability density function that x hours will be the life of a bulb selected at random is given by $f(x) = \begin{cases} \frac{1}{40} e^{-x/40} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$. Find the probability that the life of a bulb selected at random will be (a) between 40 and 60 hours, and (b) at least 60 hours.

▸ (a) The probability that the life of a bulb selected at random will be between 40 and 60 hours is

$$P([40, 60]) = \int_{40}^{60} \frac{1}{40} e^{-x/40} dx = -e^{-x/40}\Big]_{40}^{60} = -e^{-3/2} + e^{-1} = 0.145$$

(b) The probability that the life of a bulb selected at random will be at least 60 hours is

$$P([60, +\infty)) = \int_{60}^{+\infty} \frac{1}{40} e^{-x/40} dx = \lim_{b \to +\infty} \int_{60}^{b} \frac{1}{40} e^{-x/40} dx = \lim_{b \to +\infty} -e^{-40/x}\Big]_{60}^{b} = \lim_{b \to +\infty} [-e^{-b/40} + e^{-3/2}$$

$$= 0 + e^{-3/2} = 0.223$$

29. $f(x) = \begin{cases} \frac{1}{3}e^{-x/3} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

(a) $P([1,2]) = \int_1^2 f(x)dx = \int_1^2 \frac{1}{3}e^{-x/3}dx = \left[-e^{-x/3}\right]_1^2 = -e^{-2/3} + e^{-1/3} \approx 0.203$

(b) $P([5,+\infty)) = \int_5^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_5^b \frac{1}{3}e^{-x/3}dx = \lim_{b\to+\infty} \left[-e^{-x/3}\right]_5^b = \lim_{b\to+\infty} (-e^{-b/3} + e^{-5/3}) = e^{-5/3}$

30. $f(x) = \begin{cases} 0.02e^{-0.02x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

$P([12,+\infty)) = \int_{12}^{+\infty} 0.02e^{-0.02x}dx = \lim_{b\to+\infty} -e^{-0.02x}\Big]_{12}^b = \lim_{b\to+\infty} (e^{-0.24} - e^{-0.02b}) = e^{-0.24} \approx 0.7866$

31. $v^2 = 2gR^2 \int_R^{+\infty} x^{-2}dx = 2gR^2 \lim_{b\to+\infty} -x^{-1}\Big]_R^b = 2gR^2 \lim_{b\to+\infty} (R^{-1} - b^{-1}) = 2gR^2 \cdot R^{-1} = 2gR$

$v = \sqrt{2gR} = \sqrt{2(0.00609)(3960)} = 6.9498 = 6.95$

32. Find the mean of the probabilities obtained from the exponential density function.

▷ $f(x) = \begin{cases} ke^{-kx} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

$\mu = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^0 xf(x)dx + \int_0^{+\infty} xf(x)dx = 0 + \lim_{b\to+\infty} \int_0^b x(ke^{-kx})dx = \lim_{b\to+\infty} \left[-xe^{-kx} - \frac{1}{k}e^{-kx}\right]_0^b$

$= \lim_{b\to+\infty} \left(-\frac{b}{e^{kb}} - \frac{1}{ke^{kb}} + \frac{1}{k}\right) = \frac{1}{k}$ after integrating by parts. Hence, the mean of the probabilities is $\frac{1}{k}$.

33. Let $f(t) = 1000 \cdot 2^{-t}$ and $i = 0.08$. Because $2e^{-t} = e^{-t\ln 2}$, the present value is

$V = \int_0^{+\infty} f(t)e^{-it}dt = \int_0^{+\infty} 1000 \cdot 2^{-t}e^{-0.08t}dt = \lim_{b\to+\infty} \int 1000e^{(-\ln 2 - 0.08)t}dt$

$= \lim_{b\to+\infty} \left[\frac{1000}{-\ln 2 - 0.08}e^{(-\ln 2 - 0.08)t}\right]_0^b = -\frac{1000}{\ln 2 + 0.08} \lim_{b\to+\infty} [e^{(-\ln 2 - 0.08)b} - 1]$

$= -\frac{1000}{\ln 2 + 0.08}[0 - 1] = 1293.41$. Thus, the present value is \$1293.41.

34. $V = \int_0^{+\infty} 12000e^{-0.1t}dt = \lim_{b\to+\infty} -120{,}000e^{-0.1t}\Big]_0^b = \lim_{b\to+\infty} (120{,}000 - 120{,}000e^{-0.1b}) = 120{,}000$

35. Let S dollars be the fair selling price. Then

$S = \int_0^{+\infty} Re^{-it}dt = \lim_{b\to+\infty} \int_0^b Re^{-it}dt = \lim_{b\to+\infty} \left[\frac{-Re^{-it}}{i}\right] = \lim_{b\to+\infty} \left(\frac{-Re^{-ib}}{i} + \frac{R}{i}\right) = \left(0 + \frac{R}{i}\right) = \frac{R}{i}$

36. The continuous flow of profit for a company is increasing with time, and at t years the number of dollars in the profit per year is proportional to t. show that the present value of the company is inversely proportional to i^2, where $100i$ percent is the interest rate compounded continuously.

▷ Let $f(t)$ dollars be the profit per year after t years. We are given that $f(t) = kt$, with $k > 0$, because the profit per year is proportional to t. This, if V dollars is the present value of the company, then

$$V = \int_0^{+\infty} f(t)e^{-it}dt = \int_0^{+\infty} kte^{-it}dt = k \lim_{b\to+\infty} \int_0^b te^{-it}dt \tag{1}$$

We integrate by parts with

$u = t \qquad dv = e^{-it}$

$du = dt \qquad v = -\frac{1}{i}e^{-it}$

Thus

$$\int te^{-it}dt = -\frac{t}{i}e^{-it} + \frac{1}{i}\int e^{-it}dt = -\frac{t}{i}e^{-it} - \frac{1}{i^2}e^{-it} = -\frac{it+1}{i^2e^{it}} \tag{2}$$

Substituting the value of the indefinite integral given in (2) into (1), we obtain

$$V = k \lim_{b\to+\infty}\left[-\frac{it+1}{i^2e^{it}}\right]_0^b = -\frac{k}{i^2}\left[\lim_{b\to+\infty} \frac{ib+1}{e^{ib}} - 1\right] = -\frac{k}{i^2}[0-1] = \frac{k}{i^2}$$

Thus, the present value of the company is inversely proportional to i^2.

37. The substitution $x = b \tan \theta$, $dx = b \sec^2\theta \, d\theta$ gives a proper integral.

$$F = 2kb\int_{x=0}^{+\infty} \frac{dx}{(b^2+x^2)^{3/2}} = 2kb\int_{\theta=0}^{\pi/2} \frac{b \sec^2\theta \, d\theta}{b^3 \sec^3\theta} = \frac{2k}{b}\int_0^{\pi/2} \cos\theta \, d\theta = \frac{2k}{b}\sin\theta\Big]_0^{\pi/2} = \frac{2k}{b}$$

38. The substitution $x = b \tan \theta$, $dx = b \sec^2\theta \, d\theta$ gives a proper integral.

$$F = 2kb\int_{x=-\infty}^{+\infty} \frac{dx}{x^2+b^2} = 2kb\int_{\theta=-\pi/2}^{\pi/2} \frac{b \sec^2\theta \, d\theta}{b^2 \sec^2\theta} = 2k\int_{-\pi/2}^{\pi/2} d\theta = 2k\pi$$

39. If $n \le 1$, $\displaystyle\int_e^{+\infty} \frac{dx}{x(\ln x)^n} \ge \lim_{b\to+\infty}\int_e^b \frac{1}{\ln x}\cdot\frac{dx}{x} = \lim_{b\to+\infty} \ln(\ln x)\Big]_e^b = \lim_{b\to+\infty} \ln(\ln b) = +\infty$

Thus the integral diverges. If $n > 1$, then

$$\int_e^{+\infty} \frac{1}{(\ln x)^n}\cdot\frac{dx}{x} = \lim_{b\to+\infty} \frac{(\ln x)^{-n+1}}{-n+1}\Big]_e^b = \lim_{b\to+\infty}\left[\frac{(\ln b)^{1-n}}{1-n} + \frac{1}{n-1}\right] = \frac{1}{n-1}$$

Hence the integral converges if and only if $n > 1$.

40. Determine a value of n for which the improper integral $\displaystyle\int_1^{+\infty}\left(\frac{n}{x+1} - \frac{3x}{2x^2+n}\right)dx$ is convergent, and evaluate the integral for this value of n.

▸ $$\int_1^{+\infty}\left(\frac{n}{x+1} - \frac{3x}{2x^2+n}\right)dx = \lim_{b\to+\infty}\int_1^b\left(\frac{n}{x+1} - \frac{3x}{2x^2+n}\right)dx = \lim_{b\to+\infty}\left[n\ln(x+1) - \tfrac{3}{4}(2x^2+n)\right]_1^b$$
$$= \lim_{b\to+\infty}\left[n\ln(b+1) - \tfrac{3}{4}\ln(2b^2+n) - n\ln 2 + \tfrac{3}{4}\ln(2+n)\right]$$
$$= \lim_{b\to+\infty}\left\{n\left[\ln b\left(1+\tfrac{1}{b}\right)\right] - \frac{3}{4}\ln\left[b^2\left(2+\tfrac{n}{b^2}\right)\right] - n\ln 2 + \frac{3}{4}\ln(2+n)\right\}$$
$$= \lim_{b\to+\infty}\left\{n\left[\ln b + \ln\left(1+\tfrac{1}{b}\right)\right] - \frac{3}{4}\left[\ln b^2 + \ln\left(2+\tfrac{n}{b^2}\right)\right] - n\ln 2 + \frac{3}{4}\ln(2+n)\right\}$$
$$= \lim_{b\to+\infty}\left[\left(n-\tfrac{3}{2}\right)\ln b + n\ln\left(1+\tfrac{1}{b}\right) - \tfrac{3}{4}\ln\left(2+\tfrac{n}{b^2}\right) - n\ln 2 + \tfrac{3}{4}\ln(2+n)\right] \qquad (1)$$

The limit in the second and third terms exist for any value of n and the limit in the first exists only if $n = \frac{3}{2}$. Substituting this value of n in (1), the limit is

$0 + \frac{3}{2}\ln 1 - \frac{3}{4}\ln 2 - \frac{3}{2}\ln 2 + \frac{3}{4}\ln\frac{7}{2} = 0 + 0 - \frac{3}{4}\ln 2 - \frac{3}{2}\ln 2 + \frac{3}{4}\ln 7 - \frac{3}{4}\ln 2 = \frac{3}{4}\ln 7 - 3\ln 2$

41. $$I = \int_1^{+\infty}\left(\frac{nx^2}{x^3+1} - \frac{1}{3x+1}\right)dx = \lim_{b\to+\infty}\int_1^b\left(\frac{n}{3}\cdot\frac{3x^2}{x^3+1} - \frac{1}{3}\cdot\frac{3}{3x+1}\right)dx = \frac{1}{3}\lim_{b\to+\infty}\Big[n\ln|x^3+1| - \ln|3x+1|\Big]_1^b$$
$$= \frac{1}{3}\lim_{b\to+\infty}\left[n\ln(b^3+1) - \ln(3b+1) - n\ln 2 + \ln 4\right]$$

If $n \le 0$, both variable terms are negative, the limit is $-\infty$ and the integral is divergent. If $n > 0$, then

$$I = \frac{1}{3}\lim_{b\to+\infty}\left[\ln\frac{(b^3+1)^n}{3b+1} + (2-n)\ln 2\right] = \frac{1}{3}\lim_{b\to+\infty}\left[\ln\frac{(b^{3-1/n}+b^{-1/n})^n}{3+b^{-1}} + (2-n)\ln 2\right]$$

If $0 < n < \frac{1}{3}$ the numerator approaches 0 and its logarithm approaches $-\infty$ while if $n > \frac{1}{3}$ the numerator approaches $+\infty$ and the limit is $+\infty$; in either case the integral does not exist. If $n = \frac{1}{3}$

$$I = \frac{1}{3}\lim_{b\to+\infty}\left[\ln\frac{(1+b^{-3})^{1/3}}{3+b^{-1}} + \frac{5}{3}\ln 2\right] = \frac{1}{3}\left[\ln\frac{1}{3} + \frac{5}{3}\ln 2\right] = \frac{1}{9}\left[3\ln\frac{1}{3} + 5\ln 2\right] = \frac{1}{9}\ln\frac{32}{27}$$

42. $$\lim_{a\to-\infty}\int_a^d f(x)dx + \lim_{b\to+\infty}\int_d^b f(x)dx = \lim_{a\to-\infty}\left(\int_a^c f(x)dx + \int_c^d f(x)dx\right) + \lim_{b\to+\infty}\left(\int_d^c f(x)dx + \int_c^b f(x)dx\right) = L + M$$

because $\displaystyle\int_c^d f(x)dx + \int_d^c f(x)dx = 0$.

43. Show that the uniform density function qualifies as a probability density function.

▸ Because $c < d$, then $d - c > 0$ and so $f(x) \ge 0$. Furthermore

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^d f(x)dx + \int_d^{+\infty} f(x)dx = \int_{-\infty}^c 0\,dx + \int_c^d \frac{1}{d-c}dx + \int_d^{-\infty} 0\,dx$$
$$= 0 + (d-c)\frac{1}{d-c} + 0 = 1$$

7.10 OTHER IMPROPER INTEGRALS

The definite integral $\int_a^b f(x)dx$ is improper if f is unbounded on the closed interval $[a,b]$. We consider the three cases covered by the following definitions.

7.10.1 Definition If f is continuous at all x in the interval half open on the left $(a,b]$, and if $\lim_{x \to a^+} |f(x)| = +\infty$, then $\int_a^b f(x)dx = \lim_{s \to a^+} \int_s^b f(x)dx$ if this limit exists.

7.10.2 Definition If f is continuous at all x in the interval half open on the right $[a,b)$ and if $\lim_{x \to b^-} |f(x)| = +\infty$, then $\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$ if this limit exists.

7.10.3 Definition If f is continuous at all x in the interval $[a,b]$ except c, where $a < c < b$ and if $\lim_{x \to c} |f(x)| = +\infty$, then $\int_a^b f(x)dx = \lim_{t \to c^-} \int_a^t f(x)dx + \lim_{s \to c^+} \int_s^b f(x)dx$ if these limits exist.

If $\int_a^b f(x)dx$ is an improper integral, it is *convergent* if the corresponding limit exists; otherwise it is *divergent*. If there is more than one number in $[a,b]$ at which the function f is unbounded, we must express the given integral as a sum of two or more integrals of the types covered by 7.10.1 and 7.10.2. The following theorem lets us sometimes avoid limits. See Exercise 12.

***p* Test** $\int_0^b \frac{dx}{x^p}$ is convergent if and only if $p < 1$. See Exercise 32.

Theorem A If in the open interval (a,b) f is continuous and has an antiderivative $F(x)$, and $F(x)$ is continuous in the closed interval $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof Let $c \in (a,b)$. Then by the second fundamental theorem of the calculus,

$$\int_a^b f(x)dx = \lim_{s \to a^+} \int_s^c f(x)dx + \lim_{t \to b^-} \int_c^t f(x)dx = \lim_{s \to a^+} [F(c) - F(a)] + \lim_{t \to b^-} [F(t) - F(c)]$$
$$= F(b) - F(a)$$

Exercises 7.10

In Exercises 1–20 determine if the improper integral is convergent or divergent. If it is convergent, evaluate it.

1. $\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{t \to 1^-} \int_0^t \frac{dx}{\sqrt{1-x}} = \lim_{t \to 1^-} \left[-2\sqrt{1-x}\,\right]_0^t = \lim_{t \to 1^-} (-2\sqrt{1-t} + 2) = 2$

2. $\int_0^{16} \frac{dx}{x^{3/4}} = \lim_{s \to 0^+} \int_s^{16} x^{-3/4}dx = \lim_{s \to 0^+} 4x^{1/4}\Big]_s^{16} = \lim_{s \to 0^+} 4(16^{1/4} - s^{1/4}) = 4 \cdot 2 = 8$

3. $\int_{-5}^{-3} \frac{x\,dx}{\sqrt{x^2-9}} = \frac{1}{2} \lim_{t \to -3^-} \int_{-5}^t 2x(x^2-9)^{-1/2}dx = \frac{1}{2} \lim_{t \to -3^-} \left[2\sqrt{x^2-9}\,\right]_{-5}^t = \lim_{t \to -3^-} [\sqrt{t^2-9} - \sqrt{(-5)^2-9}] = -4$

4. $\int_0^4 \frac{x\,dx}{\sqrt{16-x^2}}$

▸ Because there is an infinite discontinuity at 4, we use Definition 7.10.2. Thus,

$$\int_0^4 \frac{x\,dx}{\sqrt{16-x^2}} = \lim_{t \to 4^-} \int_0^t \frac{x\,dx}{\sqrt{16-x^2}} = \lim_{t \to 4^-} \int_0^t -\tfrac{1}{2}(16-x^2)^{-1/2}(-2x\,dx) = \lim_{t \to 4^-} -\sqrt{16-x^2}\,\Big]_0^t$$
$$= \lim_{t \to 4^-} [-\sqrt{16-t^2} + 4] = 4$$

5. $\int_2^4 \frac{dt}{\sqrt{16-t^2}} = \lim_{t \to 4^-} \int_2^t \frac{dt}{\sqrt{16-t^2}} = \lim_{t \to 4^-} \left[\sin^{-1} \frac{t}{4}\right]_2^t = \lim_{t \to 4^-} \left[\sin^{-1}\frac{t}{4} - \sin^{-1}\frac{1}{2}\right] = \frac{1}{2}\pi - \frac{1}{6}\pi = \frac{1}{3}\pi$

6. $\int_{-4}^3 \frac{dz}{(z+3)^3} = \lim_{t \to -3^-} \int_{-4}^t \frac{dz}{(z+3)^3} + \lim_{s \to -3^+} \int_s^3 \frac{dz}{(z+3)^3} = \lim_{t \to -3^-} \frac{-1}{2(z+3)^2}\Big]_{-4}^t + \lim_{s \to -3^+} \frac{-1}{2(z+3)^2}\Big]_s^3$

$= \lim_{t \to -3^-}\left[\frac{-1}{2(t+3)^2} + \frac{1}{2}\right] + \lim_{s \to -3^+}\left[-\frac{1}{2} + \frac{1}{2(s+3)^2}\right]$. Because neither limit exits, the integral is divergent.

7. $\int_{\pi/4}^{\pi/2} \sec\theta\, d\theta = \lim_{t\to\pi/2^-} \int_{\pi/4}^{t} \sec\theta\, d\theta = \lim_{t\to\pi/2^-} \left[\ln|\sec\theta + \tan\theta|\;\right]_{\pi/4}^{t}$

$= \lim_{t\to\pi/2^-} [\ln|\sec t + \tan t| - |\sqrt{2}+1|] = +\infty$. Therefore, the given integral is divergent.

8. $\int_{-2}^{0} \frac{dx}{\sqrt{4-x^2}}$

▶ Because there is an infinite discontinuity at -2, we use Definition 7.10.1. Thus,

$$\int_{-2}^{0} \frac{dx}{\sqrt{4-x^2}} = \lim_{s\to-2^+} \int_{s}^{0} \frac{dx}{\sqrt{4-x^2}} = \lim_{s\to-2^+} \sin^{-1}\tfrac{1}{2}x\Big]_{s}^{0} = \lim_{s\to-2^+} [\sin^{-1}0 - \sin^{-1}\tfrac{1}{2}s] = -\sin^{-1}(-1) = \tfrac{1}{2}\pi$$

9. $\int_{0}^{+\infty}\frac{dx}{x^3} = \lim_{t\to0^+}\int_{t}^{1}\frac{dx}{x^3} + \lim_{b\to+\infty}\int_{1}^{b}\frac{dx}{x^3} = \lim_{t\to0^+}\left[-\frac{1}{2x^2}\right]_{t}^{1} + \lim_{b\to+\infty}\left[-\frac{1}{2x^2}\right]_{1}^{b} = \lim_{t\to0^+}\left(-\frac{1}{2}+\frac{1}{2t^3}\right) + \lim_{b\to+\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right)$

Because $\lim_{t\to0^+}\left(-\frac{1}{2}+\frac{1}{2t^2}\right) = +\infty$, the integral diverges. Alternatively, $\int_{0}^{1}\frac{dx}{x^3}$ diverges by the p test with $p=3$.

10. $\int_{0}^{\pi/2}\tan\theta\, d\theta = \lim_{t\to\pi/2^-}\int_{0}^{t}\tan\theta\, d\theta = \lim_{t\to\pi/2^-} \ln\sec\theta\Big]_{0}^{t} = \lim_{t\to\pi/2^-}[\ln\sec t - \ln 1] = +\infty$. Integral diverges.

11. $\int_{0}^{\pi/2}\frac{dy}{1-\sin y} = \lim_{t\to\pi/2^-}\int_{0}^{t}\frac{1+\sin y}{1+\sin y}\cdot\frac{dy}{1-\sin y} = \lim_{t\to\pi/2^-}\int_{0}^{t}\frac{1+\sin y}{\cos^2 y}dy = \lim_{t\to\pi/2^-}\int_{0}^{t}(\sec^2 y + \sec y\tan y)dy$

$= \lim_{t\to\pi/2^-} |\tan t + \sec t - 1| = +\infty$. Therefore, the given integral is divergent.

12. $\int_{0}^{2}\frac{dx}{(x-1)^{2/3}}$

▶ Because there is an infinite discontinuity at 1, we use Definition 7.10.3. Thus,

$$\int_{0}^{2}\frac{dx}{(x-1)^{2/3}} = \lim_{t\to1^-}\int_{0}^{t}(x-1)^{-2/3}dx + \lim_{s\to1^+}\int_{s}^{2}(x-1)^{-2/3}dx = \lim_{t\to1^-} 3(x-1)^{1/3}\Big]_{0}^{t} + \lim_{s\to1^+} 3(x-1)^{1/3}\Big]_{s}^{2}$$

$$= \lim_{t\to1^-}[3(t-1)^{1/3}+3] + \lim_{s\to1^+}[3-3(s-1)^{1/3}] = 3+3 = 6$$

Alternatively, because $3(x-1)^{1/3}$ is continuous on the closed interval $[0,2]$, we may apply Theorem A. Thus

$$\int_{0}^{2}\frac{dx}{(x-1)^{2/3}} = 3(x-1)^{1/3}\Big]_{0}^{2} = 3-(-3) = 6$$

13. $\int_{2}^{\infty}\frac{dx}{x\sqrt{x^2-4}} = \lim_{s\to2^+}\int_{s}^{6}\frac{dx}{x\sqrt{x^2-4}} + \lim_{t\to+\infty}\int_{6}^{t}\frac{dx}{x\sqrt{x^2-4}} = \lim_{s\to2^+}\tfrac{1}{2}\sec^{-1}\tfrac{1}{2}x\Big]_{s}^{6} + \lim_{t\to+\infty}\tfrac{1}{2}\sec^{-1}\tfrac{1}{2}x\Big]_{6}^{t}$

$= \tfrac{1}{2}\sec^{-1}3 - 0 + \tfrac{1}{2}\cdot\tfrac{1}{2}\pi - \tfrac{1}{2}\sec^{-1}3 = \tfrac{1}{4}\pi$

14. $\int_{0}^{4}\frac{dx}{x^2-2x-3} = \int_{0}^{4}\frac{dx}{(x-3)(x+1)} = \lim_{t\to3^-}\int_{0}^{t}\frac{dx}{(x-3)(x+1)} + \lim_{t\to3^+}\int_{t}^{4}\frac{dx}{(x-3)(x+1)} = I + J$

But $I = \lim_{t\to3^-}\frac{1}{4}\int_{0}^{t}\left(\frac{1}{x-3}-\frac{1}{x+1}\right)dx = \lim_{t\to3^-}\left[\frac{1}{4}\ln\left|\frac{x-3}{x+1}\right|\right]_{0}^{t} = -\infty$ (J also does not converge.)

Therefore, the given integral is divergent.

15. Because $\lim_{x\to+\infty}\ln x = +\infty$, $\int_{0}^{+\infty}\ln x\, dx$ is divergent.

16. $\int_{0}^{2}\frac{dx}{\sqrt{2x-x^2}}$

▶ There are infinite discontinuities at both 0 and 2. Thus, we express the given integral as a sum of two integrals each with an infinite discontinuity at one endpoint.

$$\int_{0}^{2}\frac{dx}{\sqrt{2x-x^2}} = \int_{0}^{1}\frac{dx}{\sqrt{2x-x^2}} + \int_{1}^{2}\frac{dx}{\sqrt{2x-x^2}} = \lim_{s\to0^+}\int_{s}^{1}\frac{dx}{\sqrt{1-(x-1)^2}} + \lim_{t\to2^-}\int_{1}^{t}\frac{dx}{\sqrt{1-(x-1)^2}}$$

$$= \lim_{s\to0^+}\sin^{-1}(x-1)\Big]_{s}^{1} + \lim_{t\to2^-}\sin^{-1}(x-1)\Big]_{1}^{t} = \lim_{s\to0^+}[-\sin^{-1}(s-1)] + \lim_{t\to2^-}\sin^{-1}(t-1)$$

$$= \tfrac{1}{2}\pi + \tfrac{1}{2}\pi = \pi$$

17. $\displaystyle\int_{-2}^{0}\frac{dw}{(w+1)^{1/3}}=\lim_{t\to-1^-}\int_{-2}^{t}\frac{dw}{(w+1)^{1/3}}+\lim_{t\to-1^+}\int_{t}^{0}\frac{dw}{(w+1)^{1/3}}$

$\displaystyle=\lim_{t\to-1^-}\Big[\tfrac{3}{2}(w+1)^{2/3}\Big]_{-2}^{t}+\lim_{t\to-1^+}\tfrac{3}{2}(w+1)^{2/3}\Big]_{t}^{0}=\lim_{t\to-1^-}\Big[\tfrac{3}{2}(t+1)-\tfrac{3}{2}\Big]+\lim_{t\to-1^+}\Big[\tfrac{3}{2}-\tfrac{3}{2}(t+1)\Big]=-\tfrac{3}{2}+\tfrac{3}{2}=0$

18. $\displaystyle\int_{-1}^{1}\frac{dx}{x^2}=\lim_{t\to0^-}\int_{-1}^{t}\frac{dx}{x^2}+\lim_{s\to0^+}\int_{s}^{1}\frac{dx}{x^2}=\lim_{t\to0^-}-\frac{1}{x}\Big]_{-1}^{t}+\lim_{s\to0^+}-\frac{1}{x}\Big]_{s}^{1}=\lim_{t\to0^-}\Big[1-\frac{1}{t}\Big]+\lim_{s\to0^+}\Big[\frac{1}{s}-1\Big]$, diverges

19. $\displaystyle\int_{-2}^{2}\frac{dx}{x^3}=\lim_{t\to0^-}\int_{-2}^{t}\frac{dx}{x^3}+\lim_{t\to0^+}\int_{t}^{2}\frac{dx}{x^3}=\lim_{t\to0^-}\Big[-\frac{1}{2x^2}\Big]_{-2}^{t}+\lim_{t\to0^+}\Big[-\frac{1}{2x^2}\Big]_{t}^{2}=\lim_{t\to0^-}\Big[-\frac{1}{2t^2}+\frac{1}{8}\Big]+\lim_{t\to0^+}\Big[-\frac{1}{8}+\frac{1}{2t^2}\Big]$

Because neither limit exists, the integral is divergent. Alternatively, $\displaystyle\int_0^2\frac{dx}{x^3}$ diverges by the p test with $p=3$.

20. $\displaystyle\int_0^{+\infty}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$

▸ There is an infinite discontinuity at 0 and an infinite upper limit. Thus, we express the given integral as a sum of two improper integrals.

$$\int_0^{+\infty}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx=\int_0^{1}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx+\int_1^{+\infty}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx=\lim_{s\to0^+}\int_s^{1}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx+\lim_{t\to+\infty}\int_1^{t}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx \qquad (1)$$

To find the integral, we let $u=-\sqrt{x}$. Then $du=-\frac{1}{2}x^{-1/2}dx$. Thus

$$\int\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx=\int e^u(-2\,du)=-2e^u=-2e^{-\sqrt{x}} \qquad (2)$$

Substituting from (2) into (1), we obtain

$$\int_0^{+\infty}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx=\lim_{s\to0^+}-2e^{-\sqrt{x}}\Big]_s^1+\lim_{t\to+\infty}-2e^{-\sqrt{x}}\Big]_1^t=\lim_{s\to0^+}[-2e^{-1}+2e^{-\sqrt{a}}+\lim_{t\to+\infty}[-2e^{-\sqrt{b}}+2e^{-1}]$$

$$=[-2e^{-1}+2]+2e^{-1}=2$$

21. $\displaystyle\int_{1/2}^{2}\frac{dx}{x(\ln x)^{1/5}}=\lim_{t\to1^-}\int_{1/2}^{t}(\ln x)^{-1/5}\frac{dx}{x}+\lim_{t\to1^+}\int_{t}^{2}(\ln x)^{-1/5}\frac{dx}{x}=\lim_{t\to1^-}\Big[\tfrac{5}{4}(\ln x)^{4/5}\Big]_{1/2}^{t}+\lim_{t\to1^+}\Big[\tfrac{5}{4}(\ln x)^{4/5}\Big]_{t}^{2}$

$\displaystyle=\tfrac{5}{4}\Big[\lim_{t\to1^-}(\ln t)^{4/5}-\Big(\ln\tfrac{1}{2}\Big)^{4}+(\ln 2)^{4/5}-\lim_{t\to1^+}(\ln t)^{4/5}\Big]=\tfrac{5}{4}[0-(-\ln 2)^{4/5}+(\ln 2)^{4/5}-0]=0$

22. $\displaystyle\int_0^2\frac{x\,dx}{1-x}=\lim_{t\to1^-}\int_0^t\Big(\frac{1}{1-x}-1\Big)dx+\lim_{s\to1^+}\int_s^2\Big(\frac{1}{1-x}-1\Big)dx=\lim_{t\to1^-}\Big[-\ln|1-x|-x\Big]_0^t+\lim_{s\to1^+}\Big[-\ln|1-x|-x\Big]_s^2$

$\displaystyle=\lim_{t\to1^-}[-\ln(1-t)-t]+\lim_{s\to1^+}[\ln(s-1)+s-2]$. Because neither limit exists, the integral is divergent.

23. $\displaystyle\int_1^2\frac{dx}{x\sqrt{x^2-1}}=\lim_{t\to1^+}\int_t^2\frac{dx}{x\sqrt{x^2-1}}=\lim_{t\to1^+}\Big[\sec^{-1}x\Big]_t^2=\lim_{t\to1^+}(\sec^{-1}2-\sec^{-1}t)=\tfrac{1}{3}\pi-0=\tfrac{1}{3}\pi$

24. $\displaystyle\int_0^1\frac{dx}{x\sqrt{4-x^2}}$

▸ There is an infinite discontinuity at 0. Thus

$$\int_0^1\frac{dx}{x\sqrt{4-x^2}}=\lim_{s\to0^+}\int_s^1\frac{dx}{x\sqrt{4-x^2}}>\lim_{s\to0^+}\int_s^1\frac{dx}{2x}=\lim_{s\to0^+}\tfrac{1}{2}\ln x\Big]_s^1=\lim_{s\to0^+}-\tfrac{1}{2}\ln a=+\infty$$

Therefore, the integral is divergent.

25. $\displaystyle\int_1^{+\infty}\frac{dx}{x^2-1}=\lim_{s\to1^+}\int_s^2\frac{1}{2}\Big(\frac{1}{x-1}-\frac{1}{x+1}\Big)dx+\lim_{t\to+\infty}\int_2^t\frac{1}{2}\Big(\frac{1}{x-1}-\frac{1}{x+1}\Big)dx$

$\displaystyle=\lim_{s\to1^+}\tfrac{1}{2}\ln\Big|\frac{x-1}{x+1}\Big|\Big]_s^2+\lim_{t\to+\infty}\tfrac{1}{2}\ln\Big|\frac{x-1}{x+1}\Big|\Big]_2^t=\lim_{s\to0}\tfrac{1}{2}\Big(\ln\tfrac{1}{3}-\ln\frac{s-1}{s+1}\Big)+\lim_{t\to+\infty}\tfrac{1}{2}\Big(\ln\frac{t-1}{t+1}-\ln\tfrac{1}{3}\Big)$

Although the second limit exists, the first does not and so the integral is divergent.

26. $\displaystyle\int_1^3\frac{dy}{\sqrt[3]{y-2}}=\lim_{t\to2^-}\int_1^t(y-2)^{-1/3}dy+\lim_{t\to2^+}\int_t^3(y-2)^{-1/3}dy=\lim_{t\to2^-}\Big[\tfrac{3}{2}(y-2)^{2/3}\Big]_1^t+\lim_{t\to2^+}\Big[\tfrac{3}{2}(y-2)^{2/3}\Big]_t^3$

$\displaystyle=\lim_{t\to2^-}\Big[\tfrac{3}{2}(t-2)-\tfrac{3}{2}\Big]+\lim_{t\to2^+}\Big[\tfrac{3}{2}-\tfrac{3}{2}(t-2)\Big]=-\tfrac{3}{2}+\tfrac{3}{2}=0$

In Exercises 27–29, find the values of n for which the improper integral converges and evaluate the integral for these values of n.

27. If $n \le -1$, $\int_0^1 x^n dx \ge \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} \left[\ln|x|\right]_t^1 = \lim_{t\to 0^+} [0 - \ln t] = +\infty$. If $n > -1$, then

$$\int_0^1 x^n dx = \lim_{t\to 0^+} \int_t^1 x^n dx = \lim_{t\to 0^+} \left[\frac{x^{n+1}}{n+1}\right]_t^1 = \lim_{t\to 0^+} \left(\frac{1}{n+1} - \frac{t^{n+1}}{n+1}\right) = \frac{1}{n+1}$$

28. $\int_0^1 x^n \ln x\, dx$

▹ First, we find an indefinite integral. If $n \ne -1$, we may integrate by parts. Let

$$u = \ln x \qquad dv = x^n dx$$
$$du = \frac{dx}{x} \qquad v = \frac{x^{n+1}}{n+1}$$

Thus, if $n \ne -1$

$$\int x^n \ln x\, dx = \frac{x^{n+1}\ln x}{n+1} - \int \frac{x^n}{n+1} = \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2}$$

If $n = -1$, then

$$\int x^n \ln x\, dx = \int (\ln x)x^{-1} dx = \tfrac{1}{2}(\ln x)^2 \tag{2}$$

If $n > -1$, we use the antiderivative in (1), and we have

$$\int_0^1 x^n \ln x\, dx = \lim_{s\to 0^+} \int_s^1 x^n \ln x\, dx = \lim_{s\to 0^+} \left[\frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2}\right]_s^1$$

$$= \lim_{s\to 0^+} \left[\frac{\ln 1}{n+1} - \frac{1}{(n+1)^2} - \frac{s^{n+1}\ln s}{n+1} + \frac{s^{n+1}}{(n+1)^2}\right] = -\frac{1}{(n+1)^2} - \frac{1}{n+1} \lim_{s\to 0^+} s^{n+1}\ln s \tag{3}$$

Because $n+1 > 0$, $\lim_{s\to 0^+} s^{-(n+1)} = +\infty$. Applying L'Hôpital's rule,

$$\lim_{s\to 0^+} s^{n+1}\ln s = \lim_{s\to 0^+} \frac{\ln s}{s^{-(n+1)}} = \lim_{s\to 0^+} \frac{s^{-1}}{-(n+1)s^{-(n+2)}} = \lim_{s\to 0^+} \frac{-s^{n+1}}{n+1} = 0 \tag{4}$$

Substituting from (4) into (3), we have

$$\int_0^1 x^n \ln x\, dx = -\frac{1}{(n+1)^2} \quad \text{if } n > -1$$

If $n \le -1$, Because $0 \le x \le 1$, then $x^n \ge x^{-1}$. Using the antiderivative in (2), we have

$$\int_0^1 x^n \ln x\, dx = \lim_{s\to 0^+} \int_s^1 x^{-1} \ln x\, dx = \lim_{s\to 0^+} \left[\tfrac{1}{2}(\ln x)^2\right]_s^1 = \lim_{s\to 0^+} [-\tfrac{1}{2}(\ln s)^2] = -\infty$$

Thus the integral is divergent if $n \le -1$.

29. If $n \le -1$, $\int_0^1 x^n \ln^2 x\, dx \ge \lim_{t\to 0^+} \int_t^1 \ln^2 x \frac{dx}{x} = \lim_{t\to 0^+} \frac{1}{3}\ln^3 x\Big]_t^1 = \lim_{t\to 0^+} -\frac{1}{3}\ln^3 t = +\infty$.

If $n > -1$ we use integration by parts twice and get

$$\int_0^1 x^n \ln^2 x\, dx = \lim_{t\to 0^+} \int_t^1 x^n \ln^2 x\, dx = \lim_{t\to 0^+} \left[\frac{x^{n+1}\ln^2 x}{n+1} - \frac{2x^{n+1}\ln x}{(n+1)^2} + \frac{2x^{n+1}}{(n+1)^3}\right]_t^1$$

$$= \lim_{t\to 0^+} \left[\frac{2}{(n+1)^3} - \frac{t^{n+1}\ln^2 t}{n+1} + \frac{2t^{n+1}\ln t}{(n+1)^2} - \frac{2t^{n+1}}{(n+1)^3}\right] = \frac{2}{(n+1)^3}$$

30. The area is represented by $\int_0^1 \frac{dx}{\sqrt{x}} = 2x^{1/2}\Big]_0^1 = 2$, by Theorem A, while the volume of revolution is represented by $\pi \int_0^1 \frac{dx}{x} = \pi \lim_{s\to 0^+} \int_s^1 \frac{dx}{x} = \pi \lim_{s\to 0^+} \ln x\Big]_s^1 = -\pi \lim_{s\to 0^+} \ln s = -\infty$

31. Consider the region bounded by $y = x^{-1/3}$, $x = t$ $(t > 0)$, $x = 8$, and the x axis. An element of volume is a circular disk centered on the x axis, $x \in [t,8]$, of radius $w_i^{-1/3}$. If V cubic units represents the volume of the given solid, then

$$V = \lim_{t\to 0^+} \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \pi(w_i^{-1/3})^2 \Delta_i x = \lim_{t\to 0^+} \pi \int_t^8 x^{-2/3} dx = \pi \lim_{t\to 0^+} \left[3x^{1/3}\right]_t^8 = \pi \lim_{t\to 0^+} [6 - 3t^{1/3}] = 6\pi$$

32. Determine if the improper integral $\int_a^b \frac{dx}{(x-a)^n}$, $b > a$ is convergent or divergent in each case: (a) $0 < n < 1$; (b) $n = 1$; (c) $n > 1$. If the integral is convergent, evaluate it.

▷ In each case there is an infinite discontinuity at $x = a$.

(a) If $0 < n < 1$, then $1 - n > 0$. Thus,

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{s\to a^+}\int_s^b (x-a)^{-n}dx = \lim_{s\to a^+} \frac{(x-a)^{-n+1}}{-n+1}\bigg]_s^b = \frac{1}{1-n}\lim_{s\to a^+}[(b-a)^{1-n} - (s-a)^{1-n}] = \frac{(b-a)^{1-n}}{1-n}$$

Thus, the improper integral is convergent.

(b) If $n = 1$, then

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{s\to a^+}\int_s^b \frac{dx}{x-a} = \lim_{s\to a^+} \ln(x-a)\bigg]_s^b = \lim_{s\to a^+}[\ln(b-a) - \ln(s-a)] = +\infty$$

Thus, the improper integral is divergent.

(c) If $n > 1$, then $n - 1 > 0$. Hence,

$$\int_a^b \frac{dx}{(x-a)^n}dx = \lim_{s\to a^+}\int_s^b (x-a)^{-n}dx = \lim_{s\to a^+} \frac{(x-a)^{-n+1}}{-n+1}\bigg]_s^b = \frac{-1}{n-1}\lim_{s\to a^+}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{(s-a)^{n-1}}\right] = +\infty$$

Thus, the improper integral is divergent.

33. An equation of the quarter circle in the first quadrant is $y = \sqrt{a^2 - x^2}$. Then $y' = -x/\sqrt{a^2-x^2}$. If L units is the circumference of the circle,

$$\text{L} = 4\int_0^a \sqrt{1 + \frac{x^2}{a^2 - x^2}}\,dx = 4\text{a} \lim_{t\to 0^+}\int_t^a \frac{dx}{\sqrt{a^2-x^2}} = 4\text{a} \lim_{t\to 0^+}\left[\sin^{-1}\frac{x}{a}\right]_t^a = 4\text{a} \lim_{t\to 0^+}\left(\sin^{-1}1 - \sin^{-1}\frac{t}{a}\right)$$

$$= 4\text{a}\cdot\tfrac{1}{2}\pi = 2\pi\text{a}$$

Miscellaneous Exercises for Chapter 7

In Exercises 1–54, evaluate the indefinite integral.

1. $\int \tan^2 4x \cos^4 4x\, dx = \int \sin^2 4x \cos^2 4x\, dx = \frac{1}{4}\int \sin^2 8x\, dx = \frac{1}{8}\int (1 - \cos 16x)dx = \frac{x}{8} - \frac{1}{128}\sin 16x + \text{C}$

2. Method 1: $\frac{5x^2-3}{x^3-x} = \frac{5x^2-x}{x(x-1)(x+1)} = \frac{\text{A}}{x} + \frac{\text{B}}{x-1} + \frac{\text{C}}{x+1}$, $5x^2 - 3 = \text{A}(x-1)(x+1) + \text{B}x(x+1) + \text{C}x(x-1)$

$x = 0$: $-3 = -\text{A}$, $\text{A} = 3$; $x = 1$: $2 = 2\text{B}$, $\text{B} = 1$; $x = -1$: $2 = 2\text{C}$, $\text{C} = 1$

$\int \frac{5x^2-3}{x^3-x}dx = \int\left(\frac{3}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right)dx = 3\ln|x| + \ln|x-1| + \ln|x+1| + \text{C}$

Method 2: $\int \frac{5x^2-3}{x^3-x}dx = \int \frac{5x^4-3x^2}{x^5-x^3}dx = \int \frac{d(x^5-x^3)}{x^5-x^3} = \ln|x^5 - x^3| + \text{C}$

3. Let $u = 4 - e^x$, $du = -e^x dx$. Then $\int \frac{e^x dx}{\sqrt{4-e^x}} = \int \frac{-du}{\sqrt{u}} = -2\sqrt{\text{u}} + \text{C} = -2\sqrt{4-e^x} + \text{C}$.

4. $\int \frac{dx}{x^2\sqrt{a^2+x^2}}$

▷ Let $\theta = \tan^{-1}(x/a)$. Then $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$, $\sqrt{a^2+x^2} = a\sec\theta$

Thus,

$$\int \frac{dx}{x^2\sqrt{a^2+x^2}} = \int \frac{a\sec^2\theta\, d\theta}{(a^2\tan^2\theta)(a\sec\theta)} = \frac{1}{a^2}\int \cot^2\theta\sec\theta\, d\theta = \frac{1}{a^2}\int \cot\theta\csc\theta\, d\theta = -\frac{\csc\theta}{a^2} + \text{C} \tag{1}$$

From the figure we have

$$\csc\theta = \frac{\sqrt{a^2+x^2}}{x} \tag{2}$$

And by substitution from (2) into (1) we get

$$\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x} + \text{C}$$

5. Let $x = t^2$, $dx = 2t\, dt$. Then $\text{I} = \int \tan^{-1}\sqrt{x}\, dx = \int 2t\tan^{-1}t\, dt$.

Now let $u = \tan^{-1}t$ and $dv = 2$t. Then $du = \frac{dt}{1+t^2}$ and $v = t^2 + 1$ so that $v\, du = dt$. Hence

$\text{I} = (t^2+1)\tan^{-1} - \int dt = (t^2+1)\tan^{-1} - t + \text{C} = (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + \text{C}$

6. $\dfrac{1}{2t^2+5t+3} = \dfrac{1}{(2t+3)(t+1)} = \dfrac{A}{2t+3} + \dfrac{B}{t+1}$; $1 = A(t+1) + B(2t+3)$. $t = -\frac{3}{2}$: $1 = -\frac{1}{2}A$; $A = -2$.
$t = -1$: $1 = B$. $\displaystyle\int \frac{dt}{2t^2+5t+3} = \int\left(\frac{-2}{2t+3} + \frac{1}{t+1}\right)dt = -\ln|2t+3| + \ln|t+1| + C = \ln\left|\frac{t+1}{2t+3}\right| + C$

7. $\displaystyle\int \cos^2 \tfrac{1}{3}x\, dx = \frac{1}{2}\int\left(1 + \cos\frac{2}{3}x\right)dx = \frac{1}{2}x + \frac{3}{4}\sin\frac{2}{3}x + C$

8. $\displaystyle\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx$

▸ Let $u = \sqrt{x+1}$. Then $u^2 = x+1$, and $dx = 2u\, du$. Hence,

$$\begin{aligned}\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx &= \int \frac{(u+1)(2u\,du)}{u-1} = 2\int (u+2)du + 4\int \frac{dx}{u-1} = u^2 + 4u + 4\ln|u-1| + \bar{C}\\ &= (x+1) + 4\sqrt{x+1} + 4\ln\left|\sqrt{x+1}-1\right| + \bar{C} = x + 4\sqrt{x+1} + 4\ln\left|\sqrt{x+1}-1\right| + C\end{aligned}$$

where $C = 1 + \bar{C}$.

9. Let $u = -1$, $du = dx$. Then
$$\int \frac{x^2+1}{(x-1)^3}dx = \int \frac{u^2+2u+2}{u^3}du = \int\left(\frac{1}{u} + 2u^{-2} + 2u^{-3}\right)du = \ln|u| - 2u^{-1} - u^{-2} + C$$

10. Let $y = u^2$, $dy = 2u\, du$. Then
$$\int \frac{dy}{\sqrt{y}+1} = \int \frac{2u\,du}{u+1} = \int\left(2 - \frac{2}{u+1}\right)du = 2u - 2\ln|u+1| + C = 2\sqrt{y} - 2\ln\left|\sqrt{y}+1\right| + C$$

11. $\displaystyle\int \sin x \sin 3x\, dx = \int \tfrac{1}{2}(-\cos 4x + \cos 2x)dx = -\tfrac{1}{8}\sin 4x + \tfrac{1}{4}\sin 2x + C$

12. $\displaystyle\int \cos\theta \cos 2\theta\, d\theta$

▸ We use the identity $\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$. Thus,
$$\int \cos\theta\cos 2\theta\, d\theta = \frac{1}{2}\int \cos 3\theta\, d\theta + \frac{1}{2}\int \cos\theta\, d\theta = \tfrac{1}{6}\sin 3\theta + \tfrac{1}{2}\sin\theta + C$$

13. Let $u = x^{1/3}$; then $x = u^3$ and $dx = 3u^2du$. Hence
$$\int \frac{dx}{x+x^{4/3}} = \int \frac{3u^2du}{u^3+u^4} = 3\int \frac{du}{u(1+u)} = 3\int\left(\frac{1}{u} - \frac{1}{1+u}\right)du = 3(\ln|u| - \ln|1+u|) + C = 3\ln\left|\frac{x^{1/3}}{1+x^{1/3}}\right| + C$$
Alternatively, $\displaystyle\int \frac{dx}{x+x^{4/3}} = \int \frac{x^{-4/3}dx}{x^{-1/3}+1} = -3\ln\left|x^{-1/3}+1\right| + C = 3\ln\left|\frac{x^{1/3}}{1+x^{1/3}}\right| + C$

14. Let $\theta = \sin^{-1}(t-1)$, $t-1 = \sin\theta$, $dt = \cos\theta\, d\theta$. Then $\displaystyle\int t\sqrt{2t-t^2}dt = \int t\sqrt{1-(t-1)^2}dt$
$$\begin{aligned}&= \int(\sin\theta+1)\cos\theta(\cos\theta\, d\theta) = \int\cos^2\theta(\sin\theta\, d\theta) + \int\tfrac{1}{2}(1+\cos 2\theta)d\theta = -\tfrac{1}{3}\cos^3\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C\\ &= \tfrac{1}{2}\theta - \tfrac{1}{3}\cos^3\theta + \tfrac{1}{2}\sin\theta\cos\theta + C = \tfrac{1}{2}\sin^{-1}(t-1) - \tfrac{1}{3}(2t-t^2)^{3/2} + \tfrac{1}{2}(t-1)\sqrt{2t-t^2} + C\end{aligned}$$

15. $\displaystyle\int(\sec 3x + \csc 3x)^2dx = \int(\sec^2 3x + 2\sec 3x\csc 3x + \csc^2 3x)dx = \int(\sec^2 3x + 4\csc 6x + \csc^2 3x)dx$
$= \frac{1}{3}\tan 3x + \frac{2}{3}\ln|\tan 3x| - \frac{1}{3}\cot 3x + C$

16. $\displaystyle\int \frac{dx}{\sqrt{e^x-1}}$

▸ Let $u = \sqrt{e^x-1}$. Then $\quad u^2 = e^x - 1;\quad x = \ln(u^2+1);\quad dx = \dfrac{2u\,du}{u^2+1}$

Thus,
$$\int \frac{dx}{\sqrt{e^x-1}} = \int \frac{2u\,du}{(u^2+1)u} = 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{e^x-1} + C$$

17. $\dfrac{2t^3+11t+8}{t^3+4t^2+4t}=2+\dfrac{-8t^2+3t+8}{t(t+2)^2}$. $\dfrac{-8t^2+3t+8}{t(t+2)^2}\equiv\dfrac{A}{t}+\dfrac{B}{t+2}+\dfrac{C}{(t+2)^2}$;

$-8t^2+3t+8\equiv A(t+2)^2+Bt(t+2)+Ct.$

Let $t=0$: $8=4A \Leftrightarrow A=2$. Let $t=-2$: $-30=-2C\Leftrightarrow C=15$.

Let $t=1$: $3=9A+3B+C$; $3=18+3B+15\Leftrightarrow B=-10$. Thus

$$\int\frac{2t^3+11t+8}{t^3+4t^2+4t}\,dt=\int\left[2+\frac{2}{t}-\frac{10}{t+2}+\frac{15}{(t+2)^2}\right]dt=2t+2\ln|t|-10\ln|t+2|-\frac{15}{t+2}+C$$
$$=2t+\ln\frac{t^2}{(t+2)^{10}}-\frac{15}{t+2}+C$$

18. $\displaystyle\int x^3\boxed{e^{3x}dx}=x^3\cdot\tfrac{1}{3}e^{3x}-\int x^2\boxed{e^{3x}dx}=\tfrac{1}{3}x^3e^{3x}-x^2\cdot\tfrac{1}{3}e^{3x}+\int\tfrac{2}{3}x\boxed{e^{3x}dx}=\tfrac{1}{3}x^3e^{3x}-\tfrac{1}{3}x^2e^{3x}+\tfrac{2}{9}xe^{3x}-\int\tfrac{2}{9}e^{3x}dx$

$=\tfrac{1}{3}x^3e^{3x}-\tfrac{1}{3}x^2e^{3x}+\tfrac{2}{9}xe^{3x}-\tfrac{2}{27}e^{3x}+C=\tfrac{1}{27}e^{3x}(9x^3-9x^2+6x-2)+C$

19. $\displaystyle\int\frac{x^4+1}{x^4-1}\,dx=\int\left[1+\frac{2}{(x^2-1)(x^2+1)}\right]dx=\int\left[1+\frac{1}{x^2-1}-\frac{1}{x^2+1}\right]dx=x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|-\tan^{-1}x+C$

20. $\displaystyle\int\frac{\sqrt{x^2-4}}{x^2}dx$

▹ Let $\theta=\sec^{-1}\frac{1}{2}x$. Then $x=2\sec\theta$, $dx=2\sec\theta\tan\theta\,d\theta$, and $\sqrt{x^2-4}=2\tan\theta$. Thus,

$$\int\frac{\sqrt{x^2-4}}{x^2}dx=\int\frac{(2\tan\theta)(2\sec\theta\tan\theta\,d\theta)}{4\sec^2\theta}=\int\frac{\tan^2\theta\,d\theta}{\sec\theta}=\int\frac{(\sec^2\theta-1)d\theta}{\sec\theta}=\int(\sec\theta-\cos\theta)d\theta$$
$$=\ln|\sec\theta+\tan\theta|-\sin\theta+C=\ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|-\frac{\sqrt{x^2-4}}{x}+C$$

$=\ln\left|x+\sqrt{x^2-4}\right|-\dfrac{\sqrt{x^2-4}}{x}+C$ where $C=C-\ln 2$.

21. $\displaystyle\int\sin^4 3x\cos^2 3x\,dx=\int(\sin 3x\cos 3x)^2\sin^2 3x\,dx=\int\frac{\sin^2 6x}{4}\cdot\frac{1-\cos 6x}{2}\,dx$

$\displaystyle=\frac{1}{8}\int\frac{1-\cos 12x}{2}\,dx-\frac{1}{8}\int\sin^2 6x\cos 6x\,dx=\frac{1}{16}x-\frac{1}{192}\sin 12x-\frac{1}{144}\sin^3 6x+C$

22. $\displaystyle\int t\sin^2 2t\,dt=\int t\cdot\tfrac{1}{2}(1-\cos 4t)dt=\int\tfrac{1}{2}t\,dt-\int\tfrac{1}{2}t\boxed{\cos 4t\,dt}=\tfrac{1}{4}t^2-\tfrac{1}{8}t\sin 4t+\int\tfrac{1}{8}\sin 4t\,dt$

$=\tfrac{1}{4}t^2-\tfrac{1}{8}t\sin 4t-\tfrac{1}{32}\cos 4t+C$

23. $\displaystyle\int\frac{dr}{\sqrt{3-4r-r^2}}=\int\frac{dr}{\sqrt{7-(r+2)^2}}=\sin^{-1}\left(\frac{r+2}{\sqrt{7}}\right)+C$

24. $\displaystyle\int\frac{4x^2+x-2}{x^3-5x^2+8x-4}dx$

▹ We factor the denominator. Thus, $x^3-5x^2+8x-4=(x-1)(x-2)^2$. let

$$\frac{4x^2+x-2}{(x-1)(x-2)^2}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$$

Then

$$4x^2+x-2=A(x-2)^2+B(x-1)(x-2)+C(x-1)$$

If $x=1$, we get $3=A$. If $x=2$, we have $16=C$. If $x=0$, we get $-2=4A+2B-C$ and because $A=3$ and $C=16$, then $-2=12+2B-16$, so $B=1$. Therefore

$$\int\frac{4x^2+x-2}{x^3-5x^2+8x-4}dx=\int\frac{3\,dx}{x-1}+\int\frac{dx}{x-2}+\int\frac{16\,dx}{(x-2)^2}=3\ln|x-1|+\ln|x-2|-16(x-2)^{-1}+C$$
$$=\ln\left|(x-1)^3(x-2)\right|-\frac{16}{x-2}+C$$

25. Let $t=x^2$, $dt=2x\,dx$. Then $I=\displaystyle\int x^3\cos x^2dx=\int x^2\cos x^2(x\,dx)=\int\frac{1}{2}t\cos t\,dt.$

Now let $u=\frac{1}{2}t$ and $dv=\cos dt$. Then $du=\frac{1}{2}dt$ and $v=\sin t$. Thus

$I=\dfrac{1}{2}t\sin t-\displaystyle\int\frac{1}{2}\sin t\,dt=\frac{1}{2}t\sin t+\frac{1}{2}\cos t+C=\frac{1}{2}x^2\sin x^2+\frac{1}{2}\cos x^2+C$

26. Let $u = 4y^2$; $du = 8y\,dy$. $\displaystyle\int \frac{y\,dy}{9+16y^2} = \int \frac{\frac{1}{8}du}{9+u^2} = \frac{1}{8}\cdot\frac{1}{3}\tan^{-1}\frac{u}{3} + C = \frac{1}{24}\tan^{-1}\frac{4}{3}y^2 + C$

27. Let $u = e^{t/2}$ and $dv = \cos 2t\,dt$. Then $du = \frac{1}{2}e^{t/2}dt$ and $v = \frac{1}{2}\sin 2t$. Therefore

$$\int e^{t/2}\cos 2t\,dt = \frac{1}{2}e^{t/2}\sin 2t - \frac{1}{4}\int e^{t/2}\sin 2t\,dt$$

Let $\bar{u} = e^{t/2}$ and $d\bar{v} = \sin 2t\,dt$. Then $d\bar{u} = \frac{1}{2}e^{t/2}$ and $\bar{v} = -\frac{1}{2}\cos 2t$. Hence

$$\int e^{t/2}\cos 2t\,dt = \frac{1}{2}e^{t/2}\sin 2t - \frac{1}{4}\left[-\frac{1}{2}\cos 2t + \frac{1}{4}\int e^{t/2}\cos 2t\,dt\right]$$

Adding $\frac{1}{16}\int e^{t/2}\cos 2t\,dt$ to both members, we obtain

$$\frac{17}{16}\int e^{t/2}\cos 2t\,dt = \frac{1}{2}e^{t/2}\sin 2t + \frac{1}{8}e^{t/2}\cos 2t + \frac{17}{16}C$$

$$\int e^{t/2}\cos 2t\,dt = \frac{2}{17}e^{t/2}(4\sin 2t + \cos 2t) + C$$

28. $\displaystyle\int \frac{du}{u^{5/8} - u^{1/8}}$

▸ Let $z = u^{1/8}$. Then $u = z^8$; $du = 8z^7dz$. Thus

$$\int \frac{du}{u^{5/8}-u^{1/8}} = \int \frac{8z^7dz}{z^5 - z} = \int \frac{8z^6dz}{z^4-1} = \int 8z^2dz + \int \frac{8z^2}{z^4-1}dz = \int 8z^2dz + \int\left(\frac{4}{z^2+1} + \frac{4}{z^2-1}\right)dz$$

$$= \int 8z^2dz + \int\left(\frac{4}{z^2+1} + \frac{2}{z-1} - \frac{2}{z+1}\right)dz = \frac{8}{3}z^3 + 4\tan^{-1}z + 2\ln|z-1| - 2\ln|z+1| + C$$

$$= \frac{8}{3}u^{3/8} + 4\tan^{-1}u^{1/8} + 2\ln\left|\frac{u^{1/8}-1}{u^{1/8}+1}\right| + C$$

29. Let $u = \sin^2 x$, $du = 2\sin x\cos x\,dx$. Therefore

$$\int \frac{\sin x\cos x}{4+\sin^4 x}dx = \frac{1}{2}\int\frac{du}{4+u^2} = \frac{1}{4}\tan^{-1}\frac{u}{2} + C = \frac{1}{4}\tan^{-1}\left(\tfrac{1}{2}\sin^2 x\right) + C$$

30. Method 1. Let $x + \frac{1}{2} = \frac{1}{2}\sqrt{3}\tan\theta$, $dx = \frac{1}{2}\sqrt{3}\sec^2\theta\,d\theta$. $\displaystyle\int\frac{dx}{x\sqrt{x^2+x+1}} = \int\frac{dx}{x\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$

$$= \int\frac{\frac{1}{2}\sqrt{3}\sec^2\theta\,d\theta}{(\frac{1}{2}\sqrt{3}\tan\theta - \frac{1}{2})\cdot\frac{1}{2}\sqrt{3}\sec\theta} = \int\frac{\sec\theta\,d\theta}{\frac{1}{2}\sqrt{3}\tan\theta - \frac{1}{2}} = \int\frac{d\theta}{\frac{1}{2}\sqrt{3}\sin\theta - \frac{1}{2}\cos\theta} = \int\frac{d\theta}{\sin(\theta - \frac{1}{6}\pi)} = \int\csc(\theta - \tfrac{1}{6}\pi)d\theta$$

$$= -\ln\left|\csc(\theta - \tfrac{1}{6}\pi) + \cot(\theta - \tfrac{1}{6}\pi)\right| + \bar{C} = \ln\left|\frac{\sin(\theta - \frac{1}{6}\pi)}{1 + \cos(\theta - \frac{1}{6}\pi)}\right| + \bar{C} = \ln\left|\frac{\frac{1}{2}\sqrt{3}\sin\theta - \frac{1}{2}\cos\theta}{1 + \frac{1}{2}\sqrt{3}\cos\theta + \frac{1}{2}\sin\theta}\right| + \bar{C}$$

$$= \ln\left|\frac{\frac{1}{2}\sqrt{3}(x+\frac{1}{2})/\sqrt{x^2+x+1} - \frac{1}{2}\cdot\frac{1}{2}\sqrt{3}/\sqrt{x^2+x+1}}{1 + \frac{1}{2}\sqrt{3}\cdot\frac{1}{2}\sqrt{3}/\sqrt{x^2+x+1} + \frac{1}{2}(x+\frac{1}{2})/\sqrt{x^2+x+1}}\right| + \bar{C} = \ln\left|\frac{\frac{1}{2}\sqrt{3}x}{\sqrt{x^2+x+1} + \frac{1}{2}x + 1}\right| + \bar{C}$$

$$= \ln\left|\frac{x}{\sqrt{x^2+x+1} + \frac{1}{2}x + 1}\right| + C \text{ where } C = \ln(\tfrac{1}{2}\sqrt{3}) + \bar{C}.$$

Method 2. Let $x = \frac{1}{u}$, $dx = -\frac{1}{u^2}du$. Then let $u + \frac{1}{2} = \frac{1}{2}\sqrt{3}\tan\theta$, $du = \frac{1}{2}\sqrt{3}\sec^2\theta\,d\theta$. $\displaystyle\int\frac{dx}{x\sqrt{x^2+x+1}}$

$$= \int\frac{-\frac{1}{u^2}du}{\frac{1}{u}\sqrt{\frac{1}{u^2}+\frac{1}{u}+1}} = -\int\frac{du}{\sqrt{1+u+u^2}} = -\int\frac{du}{\sqrt{(u+\frac{1}{2})^2+\frac{3}{4}}} = -\int\frac{\frac{1}{2}\sqrt{3}\sec^2\theta\,d\theta}{\frac{1}{2}\sqrt{3}\sec\theta} = -\int\sec\theta\,d\theta$$

$$= -\ln|\sec\theta + \tan\theta| + \bar{C} = -\ln\left|\frac{\sqrt{u^2+u+1}}{\frac{1}{2}\sqrt{3}} + \frac{u+\frac{1}{2}}{\frac{1}{2}\sqrt{3}}\right| + \bar{C} = -\ln\left|\sqrt{\frac{1}{x^2}+\frac{1}{x}+1} + \frac{1}{x} + \frac{1}{2}\right| + C$$

$$= -\ln\left|\frac{\sqrt{1+x+x^2} + 1 + \frac{1}{2}x}{x}\right| + C = \ln\left|\frac{x}{\sqrt{x^2+x+1} + \frac{1}{2}x + 1}\right| + C$$

31. Let $u = \cos x$, $du = -\sin x\,dx$. Then $\displaystyle\int\frac{\sin x\,dx}{1+\cos^2 x} = -\int\frac{du}{1+u^2} = -\tan^{-1}u + C = -\tan^{-1}(\cos x) + C$

32. $\displaystyle\int \frac{dx}{x \ln x(\ln x - 1)}$

▸ Let $u = \ln x$, and $du = dx/x$. Then

$$\int \frac{dx}{x \ln x(\ln x - 1)} = \int \frac{du}{u(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u} = \ln|u-1| + \ln|u| + C = \ln\left|\frac{u-1}{u}\right| + C = \ln\left|\frac{\ln x - 1}{\ln x}\right| + C$$

33. $I = \displaystyle\int \sqrt{4t - t^2}\, dt = \int \sqrt{4 - (t-2)^2}\, dt$. Let $t - 2 = 2 \sin\theta$ where $0 \le \theta \le \frac{1}{2}\pi$ if $t \ge 2$ and $-\frac{1}{2}\pi \le \theta < 0$ if $t < 2$. Then $dt = 2\cos\theta\, d\theta$ and $\sqrt{4-(t-2)^2} = \sqrt{4 - 4\sin^2\theta} = 2\sqrt{\cos^2\theta} = 2\cos\theta$. Therefore

$$I = \int 2\cos\theta(2\cos\theta\, d\theta) = 2\int (1 + \cos 2\theta)d\theta = 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{t-2}{2}\right) + \tfrac{1}{2}(t-2)\sqrt{4t-t^2} + C$$

34.
$$\int \frac{dx}{\sqrt{1 - x + 3x^2}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{x^2 - \frac{1}{3}x + \frac{1}{3}}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{(x - \frac{1}{6})^2 + \frac{11}{36}}} \tag{1}$$

Let $\theta = \tan^{-1}\left(\dfrac{x - \frac{1}{6}}{\frac{1}{6}\sqrt{11}}\right)$. Then
$$x - \tfrac{1}{6} = \tfrac{1}{6}\sqrt{11}\tan\theta \tag{2}$$

$$dx = \tfrac{1}{6}\sqrt{11}\sec^2\theta$$

$$\sqrt{(x-\tfrac{1}{6})^2 + \tfrac{11}{36}} = \sqrt{\tfrac{11}{36}\tan^2\theta + \tfrac{11}{36}} = \tfrac{1}{6}\sqrt{11}\sec\theta \tag{3}$$

Substituting into (1), we obtain

$$\int \frac{dx}{\sqrt{1-x+3x^2}} = \frac{1}{\sqrt{3}}\int \frac{\frac{1}{6}\sqrt{11}\sec^2\theta\, d\theta}{\frac{1}{6}\sqrt{11}\sec\theta} = \frac{1}{\sqrt{3}}\ln|\sec\theta + \tan\theta| + \bar{C}$$

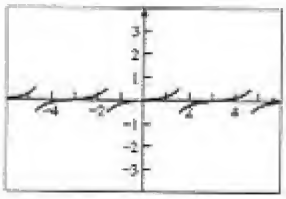
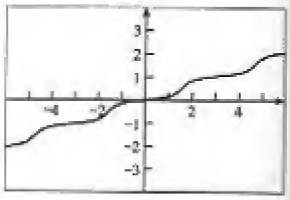
Solving (2) for $\tan\theta$, solving (3) for $\sec\theta$, and substituting into (4) we obtain

$$\int \frac{dx}{\sqrt{1-x+3x^2}} = \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{(x-\frac{1}{6})^2 + \frac{11}{36}}}{\frac{1}{6}\sqrt{11}} + \frac{x - \frac{1}{6}}{\frac{1}{6}\sqrt{11}}\right| + \bar{C} = \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{(6x-1)^2 + 11} + 6x - 1}{\sqrt{11}}\right| + C$$

$$= \tfrac{1}{3}\sqrt{3}\ln(2\sqrt{3}\sqrt{3x^2 - x + 1} + 6x - 1) + C$$

35. $\displaystyle\int \frac{dx}{x^3 - x} = \frac{1}{2}\int \frac{2x^{-3}dx}{1 - x^{-2}} = \frac{1}{2}\ln|1 - x^{-2}| + C = \frac{1}{2}\ln\left|\frac{x^2-1}{x^2}\right| + C$

36. $\displaystyle\int \frac{dx}{5 + 4\cos 2x}$

▸ First, let $u = 2x$ and $du = 2\, dx$.

$$\int \frac{dx}{5 + 4\cos 2x} = \frac{1}{2}\int \frac{du}{5 + 4\cos u}$$

Now let $z = \tan\frac{1}{2}u$. Then

$$du = \frac{2\, dz}{1 + z^2} \quad \text{and} \quad \cos u = \frac{1 - z^2}{1 + z^2}$$

Hence,

$$\int \frac{dx}{5 + 4\cos 2x} = \frac{1}{2}\int \frac{\frac{2\,dz}{1+z^2}}{5 + 4\frac{1-z^2}{1+z^2}} = \int \frac{dz}{9 + z^2} = \frac{1}{3}\tan^{-1}\frac{z}{3} + C = \frac{1}{3}\tan^{-1}(\tfrac{1}{3}\tan\tfrac{1}{2}u) + C = \frac{1}{3}\tan^{-1}(\tfrac{1}{3}\tan x) + C$$

The first plot shows this to be discontinuous; the second is of the required antiderivative.

To eliminate the discontinuity at $\pm\frac{1}{2}\pi$, $\pm\frac{3}{2}\pi$ we use $\tan^{-1}a - \tan^{-1}b = \tan^{-1}\dfrac{a-b}{1+ab}$. Thus,

$$\int \frac{dx}{5 + 4\cos 2x} = \tfrac{1}{3}x + \tfrac{1}{3}[\tan^{-1}(\tfrac{1}{3}\tan x) - \tan^{-1}(\tan x)] + C = \tfrac{1}{3}x + \frac{1}{3}\tan^{-1}\frac{-\frac{2}{3}\tan x}{1 + \frac{1}{3}\tan^2 x} + C$$

$$= \frac{1}{3}x - \frac{1}{3}\tan^{-1}\frac{2\sin x\cos x}{3\cos^2 x + \sin^2 x} + C = \frac{1}{3}x - \frac{1}{3}\tan^{-1}\frac{\sin 2x}{\frac{3}{2}(1 + \cos 2x) + \frac{1}{2}(1 - \cos 2x)} + C$$

$$= \frac{1}{3}x - \frac{1}{3}\tan^{-1}\frac{\sin 2x}{2 + \cos 2x} + C$$

37. Let $u = 3e^x$, $du = 3e^x dx$. Then $\displaystyle\int \frac{e^x dx}{\sqrt{4-9e^{2x}}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3}\sin^{-1}\frac{u}{2} + C = \frac{1}{3}\sin^{-1}\left(\frac{3}{2}e^x\right) + C.$

38. Let $u = \sqrt{t}$, $t = u^2$, $dt = 2u\, du$. Then,

$$\int \frac{\sqrt{t}-1}{\sqrt{t}+1}dt = \int \frac{u-1}{u+1}(2u\, du) = \int \left(2u - 4 + \frac{4}{u+1}\right)du = u^2 - 4u + 4\ln|u+1| + C = t - 4\sqrt{t} + 4\ln(\sqrt{t}+1) + C$$

39. $\displaystyle\int \cot^2 3x \csc^4 3x\, dx = \int \csc^2 3x(\cot^2 3x + 1)\csc^2 3x\, dx = -\frac{1}{3}\int (\cot^4 3x + \cot^2 3x)(-3\csc^2 3x\, dx)$

$$= -\frac{1}{15}\cot^5 3x - \frac{1}{9}\cot^3 3x + C$$

40. $\displaystyle\int \frac{dx}{x\sqrt{5x-6-x^2}}$

▹ We let $x = z^{-1}$ and $dx = -z^{-2}dz$. Note that $2 < x < 3$ and so $z > \frac{1}{3}$.

$$\int \frac{dx}{x\sqrt{5x-6-x^2}} = \int \frac{-z^{-2}dz}{z^{-1}\sqrt{5z^{-1}-6-z^{-2}}} = -\int \frac{dz}{z\sqrt{\frac{5z-6z^2-1}{z^2}}} = -\int \frac{dz}{5z-6z^2-1} = -\frac{1}{\sqrt{6}}\int \frac{dz}{\sqrt{\frac{5}{6}z - z^2 - \frac{1}{6}}}$$

$$= -\frac{1}{\sqrt{6}}\int \frac{dz}{\sqrt{\frac{1}{144} - (z - \frac{5}{12})^2}} = -\frac{1}{\sqrt{6}}\sin^{-1}\left(\frac{z - \frac{5}{12}}{\frac{1}{12}}\right) + C = -\frac{1}{\sqrt{6}}\sin^{-1}(12z - 5) + C$$

$$= -\frac{1}{\sqrt{6}}\sin^{-1}\left(\frac{12}{x} - 5\right) + C = -\frac{1}{\sqrt{6}}\sin^{-1}\left(\frac{5x-12}{x}\right) + C$$

41. Let $u = \sin^{-1}x$ and $dv = x^2 dx$. Then $du = \dfrac{dx}{\sqrt{1-x^2}}$ and $v = \frac{1}{3}x^3$. Therefore

$$I = \int x^2 \sin^{-1}x\, dx = \frac{1}{3}x^3 \sin^{-1}x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}dx$$

For this integral let $t^2 = 1 - x^2$; then $x^2 = 1 - t^2$ and $x\, dx = -t\, dt$. Thus

$$\int \frac{x^2(x\, dx)}{\sqrt{1-x^2}} = \int \frac{(1-t^2)(-t\, dt)}{t} = \int (t^2 - 1)dt = \frac{1}{3}t^3 - t + \bar{C} = \frac{1}{3}t(t^2 - 3) = -\frac{1}{3}\sqrt{1-x^2}(x^2+2) + \bar{C}$$

Therefore $I = \frac{1}{3}x^3 \sin^{-1}x + \frac{1}{9}\sqrt{1-x^2}(x^2+2) + C$.

41. $\displaystyle\int \frac{\cot x\, dx}{3 + 2\sin x} = \int \frac{\cos x\, dx}{\sin x(3 + 2\sin x)} = \frac{1}{3}\int \left(\frac{1}{\sin x} - \frac{2}{3 + 2\sin x}\right)d(\sin x) = \frac{1}{3}(\ln|\sin x| - \ln|3 + 2\sin x|) + C$

$$= \frac{1}{3}\ln\left|\frac{\sin x}{3 + 2\sin x}\right| + C$$

43. Let $u = \cos x$; then $du = -\sin x\, dx$. Therefore

$$\int \frac{dx}{\sin x - 2\csc x} = \int \frac{\sin x\, dx}{\sin^2 x - 2} = \int \frac{-\sin x\, dx}{1 + \cos^2 x} = \int \frac{du}{1+u^2} = \tan^{-1}u + C = \tan^{-1}(\cos x) + C$$

44. $\displaystyle\int \cos x \ln(\sin x)dx$

▹ We let $u = \sin x$, $du = \cos x\, dx$ and then integrate by parts.

$$\int \cos x \ln(\sin x)dx = \int \ln u \boxed{du} = u\ln u - \int u \cdot \frac{1}{u}du = u\ln u - u + C = \sin x \ln(\sin x) - \sin x + C$$

45. Let $u = \sin 3t$; then $du = 3\cos 3t\, dt$. Thus

$$\int \frac{\cos 3t\, dt}{\sin 3t\sqrt{\sin^2 3t - \frac{1}{4}}} = \frac{2}{3}\int \frac{2\, du}{2u\sqrt{(2u)^2 - 1}} = \frac{2}{3}\sec^{-1}2u + C = \frac{2}{3}\sec^{-1}(2\sin 3t) + C$$

46. $\displaystyle\int \tan x \sin x\, dx = \int \frac{\sin^2 x}{\cos x}dx = \int \frac{1 - \cos^2 x}{\cos x}dx = \int (\sec x - \cos x)dx = \ln|\sec x + \tan x| - \sin x + C$

47. Let $x = \sqrt{2t}$. Then $2t = x^2$ and $dt = x\,dx$. Hence $\mathrm{I} = \int \frac{\sin^{-1}\sqrt{2t}}{\sqrt{1-2t}}\,dt = \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}(x\,dx)$.

Now let $z = \sin^{-1}x$. Then $dz = \frac{dx}{\sqrt{1-x^2}}$ and $x = \sin z$. Thus $\mathrm{I} = \int z \sin z\,dz$.

Finally, let $u = z$, $dv = \sin z$; so $du = dz$ and $v = -\cos z$. Therefore

$\mathrm{I} = -z\cos z + \int \cos z\,dz = -z\cos z + \sin z + \mathrm{C} = -\sin^{-1}x\sqrt{1-x^2} + x + \mathrm{C} = -\sqrt{1-2t}\,\sin^{-1}\sqrt{2t} + \sqrt{2t} + \mathrm{C}$

48. $\int \ln(x^2+1)dx$

▷ We use integration by parts. Let

$$u = \ln(x^2+1) \qquad dv = dx$$
$$du = \frac{2x\,dx}{x^2+1} \qquad v = x$$

Then,

$$\int \ln(x^2+1)dx = x\ln(x^2+1) - \int \frac{2x^2dx}{x^2+1} = x\ln(x^2+1) - 2\int dx + 2\int \frac{dx}{x^2+1} = x\ln(x^2+1) - 2x + 2\tan^{-1}x + \mathrm{C}$$

49. $\mathrm{I} = \int \frac{dx}{5+4\sec x} = \int \frac{\cos x\,dx}{5\cos x + 4} = \frac{1}{5}\int dx - \frac{4}{5}\int \frac{dx}{5\cos x + 4} = \frac{1}{5}x - \frac{4}{5}\mathrm{J}$. Let $z = \tan\frac{1}{2}x$. Then $dx = \frac{2\,dz}{1+z^2}$ and

$\cos x = \frac{1-z^2}{1+z^2}$. $\mathrm{J} = \int \frac{2\,dz/(1+z^2)}{5(1-z^2)/(1+z^2)+4} = \int \frac{2\,dz}{9-z^2} = \frac{1}{3}\int\left(\frac{1}{3+z} - \frac{1}{3-z}\right)dz = \frac{1}{3}\ln\left|\frac{3+z}{3-z}\right| + \mathrm{C}$. Thus

$$\mathrm{I} = \frac{1}{5}x - \frac{4}{15}\ln\left|\frac{3+\tan\frac{1}{2}x}{3-\tan\frac{1}{2}x}\right| + \mathrm{C}$$

50. Let $z = \tan\frac{1}{2}x$. Then $dx = \frac{2\,dz}{1+z^2}$ and $\cos x = \frac{1-z^2}{1+z^2}$. $\int \frac{dx}{2+2\sin x + \cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{2 + 2\cdot\frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}}$

$$= \int \frac{2\,dz}{z^2+4z+3} = \int\left(\frac{1}{z+1} - \frac{1}{z+3}\right)dz = \ln\left|\frac{z+1}{z+3}\right| + \mathrm{C} = \ln\left|\frac{\tan\frac{1}{2}x+1}{\tan\frac{1}{2}x+3}\right| + \mathrm{C}$$

51. If $n = 0$, $\int \sin^5 nx\,dx = \int 0\,dx = \mathrm{C}$. If $n \neq 0$, $\int \sin^5 nx\,dx = \int (1-\cos^2 nx)^2 \sin nx\,dx$

$$= \int (1 - 2\cos^2 nx + \cos^4 nx)\sin nx\,dx = \frac{1}{n}\left(-\cos nx + \frac{2}{3}\cos^3 nx - \frac{1}{5}\cos^5 nx\right) + \mathrm{C}$$

52. $\int \tan^n x \sec^4 x\,dx$, $n > 0$

▷ Let $u = \tan x$, $du = \sec^2 x\,dx$. Then

$$\int \tan^n x\sec^4 x\,dx = \int \tan^n x\sec^2 x(\sec^2 x\,dx) = \int u^n(u^2+1)du = \int (u^{n+2}+u^n)du = \frac{u^{n+3}}{n+3} + \frac{u^{n+1}}{n+1} + \mathrm{C}$$
$$= \frac{\tan^{n+3}x}{n+3} + \frac{\tan^{n+1}x}{n+1} + \mathrm{C}$$

53. If $n = -1$: $\int x^n \ln x\,dx = \int \ln x\frac{dx}{x} = \frac{1}{2}\ln^2 x + \mathrm{C}$. If $n \neq -1$, let $u = \ln x$ and $dv = x^n dx$. Then $du = \frac{dx}{x}$ and

$v = \frac{x^{n+1}}{n+1}$. $\int x^n \ln x\,dx = \frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^n dx = \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + \mathrm{C}$

54. Let $u = \sqrt{\tan x}$. Then $u^2 = \tan x$ and $2u\,du = \sec^2 x\,dx$; so that $dx = \frac{2u\,du}{1+\tan^2 x} = \frac{2u\,du}{1+u^4}$.

Method 1. $\mathrm{I} = \int \sqrt{\tan x}\,dx = 2\int \frac{u^2}{1+u^4}\,du = 2\int \frac{u^2du}{(u^4+2u^2+1)-2u^2} = \int \frac{2u^2du}{(u^2+1)^2-2u^2}$

$$\frac{2u^2}{(u^2+1)^2-2u^2} \equiv \frac{\mathrm{A}(2u+\sqrt{2})+\mathrm{B}}{u^2+\sqrt{2}u+1} + \frac{\mathrm{C}(2u-\sqrt{2})+\mathrm{D}}{u^2-\sqrt{2}u+1}$$

$$2u^2 \equiv [\mathrm{A}(2u+\sqrt{2}) + \mathrm{B}(u^2-\sqrt{2}+1) + [\mathrm{C}(2u-\sqrt{2})+\mathrm{D}](u^2+\sqrt{2}u+1)$$

$$2u^2 \equiv (2\mathrm{A}+2\mathrm{C})u^3 + (-\sqrt{2}\mathrm{A}+\mathrm{B}+\sqrt{2}\mathrm{C}+\mathrm{D})u^2 + (-\sqrt{2}\mathrm{B}+\sqrt{2}\mathrm{D})u + (\sqrt{2}\mathrm{A}+\mathrm{B}-\sqrt{2}\mathrm{C}+\mathrm{D})$$

$2A + 2C = 0$ (1); $-\sqrt{2}A + B + \sqrt{2}C + D = 2$ (2); $-\sqrt{2}B + \sqrt{2}D = 0$ (3); $\sqrt{2}A + B - \sqrt{2}C + D = 0$ (4)

(2) + (4) gives $2B + 2D = 2$ and from (3) we get $B = D = \frac{1}{2}$. Then (2) yields $-\sqrt{2}A + \sqrt{2}C = 1$ and from (1) we get $A = -\frac{1}{4}\sqrt{2}$, $C = \frac{1}{4}\sqrt{2}$. Therefore

$$I = -\frac{1}{4}\sqrt{2}\int\frac{(2u+\sqrt{2})du}{u^2+\sqrt{2}+1} + \frac{1}{2}\int\frac{du}{u^2\sqrt{2}+1} + \frac{1}{4}\sqrt{2}\int\frac{(2u-\sqrt{2})du}{u^2-\sqrt{2}+1} + \frac{1}{2}\int\frac{du}{u^2-\sqrt{2}+1}$$

$$= -\frac{1}{4}\sqrt{2}\ln(u^2+\sqrt{2}u+1) + \frac{1}{2}\int\frac{du}{(u+\frac{1}{2}\sqrt{2})^2+\frac{1}{2}} + \frac{1}{4}\sqrt{2}\ln(u^2-\sqrt{2}u+1) + \frac{1}{2}\int\frac{du}{(u-\frac{1}{2}\sqrt{2})^2+\frac{1}{2}}$$

$$= \frac{1}{4}\sqrt{2}\ln\left(\frac{u^2-\sqrt{2}u+1}{u^2+\sqrt{2}u+1}\right) + \frac{1}{2}\sqrt{2}\tan^{-1}\left(\frac{u+\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}}\right) + \frac{1}{2}\sqrt{2}\tan^{-1}\left(\frac{u-\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}}\right) + C$$

$$= \frac{1}{4}\sqrt{2}\ln\left|\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right| + \frac{1}{2}\sqrt{2}\tan^{-1}(\sqrt{2\tan x}+1) + \frac{1}{2}\sqrt{2}\tan^{-1}(\sqrt{2\tan x}-1) + C$$

Method 2. $\int\sqrt{\tan x}\,dx = \int\frac{2u^2}{u^4+1}du = \int\frac{u^2-1}{u^4+1}du + \int\frac{u^2+1}{u^4+1}du = \int\frac{1-u^{-2}}{u^2+u^{-2}}du + \int\frac{1+u^{-2}}{u^2+u^{-2}}du$

$$= \int\frac{d(u+u^{-1})}{(u+u^{-1})^2-2} + \int\frac{d(u-u^{-1})}{(u-u^{-1})^2+2} = \frac{1}{2\sqrt{2}}\ln\left|\frac{u+u^{-1}-\sqrt{2}}{u+u^{-1}+\sqrt{2}}\right| + \frac{1}{\sqrt{2}}\tan^{-1}\frac{u-u^{-1}}{\sqrt{2}} + C$$

$$= \frac{1}{4}\sqrt{2}\ln\left|\frac{u^2-\sqrt{2}u+1}{u^2+\sqrt{2}u+1}\right| + \frac{1}{2}\sqrt{2}\tan^{-1}\left(\frac{u^2-1}{\sqrt{2}u}\right) + C$$

$$= \frac{1}{4}\sqrt{2}\ln\left|\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right| + \frac{1}{2}\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right) + C$$

In Exercises 55–84, find the exact value of the definite interval.

55. $\int_0^\pi\sqrt{2+2\cos x}\,dx = 2\int_0^\pi\sqrt{\frac{1+\cos x}{2}}\,dx = 2\int_0^\pi\cos\frac{x}{2}\,dx = 4\sin\frac{x}{2}\Big]_0^\pi = 4$

56. $\int_{1/2}^1\sqrt{\frac{1-x}{x}}dx$

▷ Let $\theta = \sin^{-1}\sqrt{x}$. Then $x = \sin^2\theta$ and $dx = 2\sin\theta\cos\theta\,d\theta$ and

$$\sqrt{\frac{1-x}{x}} = \sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}} = \sqrt{\frac{\cos^2\theta}{\sin^2\theta}} = \frac{\cos\theta}{\sin\theta}$$

Thus,

$$\int_{x=1/2}^1\sqrt{\frac{1-x}{x}}dx = \int_{\theta=\pi/4}^{\pi/2}\frac{\cos\theta}{\sin\theta}(2\sin\theta\cos\theta\,d\theta) = \int_{\pi/4}^{\pi/2}2\cos^2\theta\,d\theta = \int_{\pi/4}^{\pi/2}(1+\cos 2\theta)d\theta$$

$$= \theta + \frac{1}{2}\sin 2\theta\Big]_{\pi/4}^{\pi/2} = (\frac{1}{2}\pi + \frac{1}{2}\sin\pi) - (\frac{1}{4}\pi + \frac{1}{2}\sin\frac{1}{2}\pi) = \frac{1}{4}\pi - \frac{1}{2}$$

57. $\int_1^2\frac{2x^2+x+4}{x^3+4x^2}dx = \int_1^2\left(\frac{2x^2}{x^2(x+4)} + \frac{x+4}{x^2(x+4)}\right)dx = \int_1^2\left(\frac{2}{x+4}+\frac{1}{x^2}\right)dx = 2\ln|x+4| - \frac{1}{x}\Big]_1^2 = \frac{1}{2} + 2\ln\frac{6}{5}$

58. Let $u = e^x$, $du = e^x dx$. $\int_0^1\frac{dx}{e^x+e^{-x}} = \int_{x=0}^1\frac{e^x dx}{e^{2x}+1} = \int_{u=1}^e\frac{du}{u^2+1} = \tan^{-1}u\Big]_1^e = \tan^{-1}e - \frac{1}{4}\pi$

59. Method 1. Let $t = 2\tan\theta$ where $0 \le \theta \le \frac{1}{4}\pi$. Then $dt = 2\sec^2\theta\,d\theta$ and

$$\sqrt{4+t^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{\sec^2\theta} = 2\sec\theta$$

$$\int_0^2\frac{t^3dt}{\sqrt{4+t^2}} = \int_0^{\pi/4}\frac{8\tan^3\theta(2\sec^2\theta\,d\theta)}{2\sec\theta} = 8\int_0^{\pi/4}(\sec^2\theta-1)(\tan\theta\sec\theta\,d\theta) = \frac{8}{3}\sec^3\theta - 8\sec\theta\Big]_0^{\pi/4} = \frac{16}{3} - \frac{8}{3}\sqrt{2}$$

Method 2. Let $u = \sqrt{4+t^2}$. Then $u^2 = 4+t^2$ and $u\,du = t\,dt$. Thus

$$\int_0^2\frac{t^3dt}{\sqrt{4+t^2}} = \int_0^2\frac{t^2(t\,dt)}{\sqrt{4+t^2}} = \int_2^{2\sqrt{2}}\frac{(u^2-4)(u\,du)}{u} = \int_2^{2\sqrt{2}}(u^2-4)du = \frac{u^3}{3} - 4u\Big]_2^{2\sqrt{2}} = \frac{16}{3} - \frac{8}{3}\sqrt{2}$$

Method 3. Let $u = t^2$ and $dv = \frac{t\,dt}{\sqrt{4+t^2}}$. Then $du = 2t\,dt$ and $v = \sqrt{4+t^2}$. Thus

$$\int_0^2\frac{t^3dt}{\sqrt{4+t^2}} = \int_0^2 t^2\cdot\frac{t\,dt}{\sqrt{4+t^2}} = t^2\sqrt{4+t^2}\Big]_0^2 - \int_0^2 2t\sqrt{4+t^2}\,dt = 8\sqrt{2} - \frac{2}{3}(4+t^2)^{3/2}\Big]_0^2 = \frac{16}{3} - \frac{8}{3}\sqrt{2}$$

60. $\int_0^{\pi/2} \sin^3 t \cos^3 t \, dt$

▸ Let $u = \sin t$ and $du = \cos t \, dt$. Then

$$\int_0^{\pi/2} \sin^3 t \cos^3 t \, dt = \int_{t=0}^{\pi/2} \sin^3 t(1 - \sin^2 t)\cos t \, dt = \int_0^1 u^3(1 - u^2)du = \tfrac{1}{4}u^4 - \tfrac{1}{6}u^6\Big]_0^1 = \tfrac{1}{12}$$

61. Let $u = x$ and $dv = \dfrac{x\,dx}{(16 - x^2)^{3/2}}$. Then $du = dx$ and $v = \dfrac{1}{(16 - x^2)^{1/2}}$. Therefore

$$\int_{-2}^{2\sqrt{3}} \frac{x^2 dx}{(16 - x^2)^{3/2}} = \frac{x}{\sqrt{16 - x^2}}\Big]_{-2}^{2\sqrt{3}} - \int_{-2}^{2\sqrt{3}} \frac{dx}{\sqrt{16 - x^2}} = \sqrt{3} + \tfrac{1}{3}\sqrt{3} - \Big[\sin^{-1}\frac{x}{4}\Big]_{-2}^{2\sqrt{3}}$$

$$= \tfrac{4}{3}\sqrt{3} - \Big[\sin^{-1}(\tfrac{1}{2}\sqrt{3}) - \sin^{-1}(-\tfrac{1}{2})\Big] = \tfrac{4}{3}\sqrt{3} - \tfrac{1}{2}\pi$$

62. Let $u = x + 1$, $du = dx$. $\displaystyle\int_{x=0}^{1} \frac{xe^x}{(1+x)^2} = \int_{u=1}^{2} \frac{(u-1)e^{u-1}}{u^2}du = \int_1^2 u^{-1}\boxed{e^{u-1}du} - \int_1^2 u^{-2}e^{u-1}du$

$$= u^{-1}e^{u-1}\Big]_1^2 + \int_1^2 u^{-2}e^{u-1}du - \int_1^2 u^{-2}e^{u-1}du = \tfrac{1}{2}e - 1$$

63. $\displaystyle\int_0^{\pi/4} \sec^4 x \, dx = \int_0^{\pi/4}(\tan^2 x + 1)\sec^2 x \, dx = \frac{1}{3}\tan^3 x + \tan x\Big]_0^{\pi/4} = \frac{4}{3}$

64. $\displaystyle\int_0^2 \frac{(1 - x)dx}{x^2 + 3x + 2}$

▸ Because $x^2 + 3x + 2 = (x + 1)(x + 2)$, we let

$$\frac{1 - x}{x^2 + 3x + 2} = \frac{A}{x + 1} + \frac{B}{x + 2}$$

$$1 - x = A(x + 2) + B(x + 1)$$

If $x = -1$, then $2 = A$; if $x = -2$, then $3 = -B$, so $B = -3$. Thus

$$\int_0^2 \frac{(1 - x)dx}{x^2 + 3x + 2} = \int_0^2\Big(\frac{2}{x + 1} - \frac{3}{x + 2}\Big)dx = 2\ln|x + 1| - 3\ln|x + 2|\Big]_0^2 = (2\ln 3 - 3\ln 4) - (2\ln 1 - 3\ln 2)$$

$$= 2\ln 3 - 3\ln 4 + 3\ln 2 = \ln\frac{(3^2)(2^3)}{4^3} = \ln\frac{9}{8}$$

65. $\displaystyle\int_{\pi/12}^{\pi/6} \cot^3 2y \, dy = \int_{\pi/12}^{\pi/6} \cot 2y(\csc^2 2y - 1)dy = -\frac{1}{4}\cot^2 2y - \frac{1}{2}\ln|\sin 2y|\Big]_{\pi/12}^{\pi/6}$

$$= \Big(-\frac{1}{4} - \frac{1}{2}\ln\frac{1}{\sqrt{2}}\Big) - \Big(-\frac{3}{4} - \frac{1}{2}\ln\frac{1}{2}\Big) = -\frac{1}{4} + \frac{1}{4}\ln 2 + \frac{3}{4} - \frac{1}{2}\ln 2 = \frac{1}{2} - \frac{1}{4}\ln 2$$

66. $\displaystyle\int_0^2 (2^x + x^2)dx = \frac{2^x}{\ln 2} + \frac{x^3}{3}\Big]_0^2 = \frac{3}{\ln 2} + \frac{8}{3}$

67. Let $y + 1 = \sec\theta$ where $\theta \in [0, \pi/3]$. Then $dy = \sec\theta\tan\theta\,d\theta$ and

$$\sqrt{2y + y^2} = \sqrt{(y + 1)^2 - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$$

$$\int_0^1 \sqrt{2y + y^2}\,dy = \int_0^{\pi/3} \tan\theta(\sec\theta\tan\theta\,d\theta) = \int_0^{\pi/3} \sec\theta(\sec^2\theta - 1)d\theta = \int_0^{\pi/3}(\sec^3\theta - \sec\theta)d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/3} = \sqrt{3} - \frac{1}{2}\ln(2 + \sqrt{3})$$

68. $\displaystyle\int_1^2 (\ln x)^2 dx$

▸ We use integration by parts.

$$u = (\ln x)^2 \qquad dv = dx$$

$$du = \frac{2\ln x\,dx}{x} \qquad v = x$$

Thus,

$$\int_1^2 (\ln x)^2 dx = x(\ln x)^2\Big]_1^2 - 2\int_1^2 \ln x\,dx \qquad (1)$$

For the integral in the right-hand side of (1), we integrate by parts.

$$\bar{u} = \ln x \qquad d\bar{v} = dx$$

$$d\bar{u} = \frac{dx}{x} \qquad \bar{v} = x$$

Thus, from (1) we obtain

$$\int_1^2 (\ln x)^2 dx = 2(\ln 2)^2 - 2\left[x \ln x - \int dx\right]_1^2 = 2(\ln 2)^2 - 2\left[x \ln x - x\right]_1^2 = 2(\ln 2)^2 - 2[(2 \ln 2 - 2) - (0 - 1)]$$
$$= 2(\ln 2)^2 - 4 \ln 2 + 2 = 2(\ln 2 - 1)^2$$

69. $\dfrac{2x^2 - 2x + 1}{x^3 + x} \equiv \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1}$; $2x^2 - 2x + 1 = A(x^2 + 1) + (Bx + C)x = (A + B)x^2 + Cx + A$

Therefore $A + B = 2$, $C = -2$, $A = 1$ so that $B = 1$. Thus

$$\int_{\sqrt{3}/3}^1 \frac{2x^2 - 2x + 1}{x^3 + x} dx = \int_{\sqrt{3}/3}^1 \left(\frac{1}{x} + \frac{x - 2}{x^2 + 1}\right) dx = \ln|x| + \tfrac{1}{2}\ln(x^2 + 1) - 2\tan^{-1}x\Big]_{\sqrt{3}/3}^1$$
$$= \left(\tfrac{1}{2}\ln 2 - \tfrac{1}{2}\pi\right) - \left(\ln \tfrac{1}{3}\sqrt{3} + \tfrac{1}{2}\ln\tfrac{4}{3} - \tfrac{1}{3}\pi\right) = \tfrac{1}{2}\ln\tfrac{9}{2} - \tfrac{1}{6}\pi$$

70. Let $u^2 = 2 - x^2$, $2u\,du = -2x\,dx$. $\displaystyle\int_{\sqrt{2}/2}^1 \frac{x^3 dx}{\sqrt{2 - x^2}} = \int_{x=\sqrt{2}/2}^1 \frac{x^2(x\,dx)}{\sqrt{2 - x^2}} = \int_{u=\sqrt{6}/2}^1 \frac{(2 - u^2)(-u\,du)}{u}$

$$= \int_1^{\sqrt{6}/2} (2 - u^2) du = \left[2u - \tfrac{1}{3}u^3\right]_1^{\sqrt{6}/2} = \tfrac{3}{4}\sqrt{6} - \tfrac{5}{3}$$

71. $\displaystyle\int_1^{10} \log_{10}\sqrt{ex}\,dx = \frac{1}{2\ln 10}\int_1^{10} \ln ex\,dx = \frac{1}{2\ln 10}\int_1^{10} (1 + \ln x) dx = \frac{1}{\ln 10}\Big[x + x\ln x - x\Big]_1^{10}$

$$= \frac{1}{2\ln 10}\Big[x \ln x\Big]_1^{10} = 5$$

72. $\displaystyle\int_0^{2\pi} |\sin x - \cos x|\, dx$

▷ $$\int_0^{2\pi} |\sin x - \cos x|\, dx = \sqrt{2}\int_0^{2\pi} \left|\tfrac{1}{2}\sqrt{2}\sin x - \tfrac{1}{2}\sqrt{2}\cos x\right| dx = \sqrt{2}\int_0^{2\pi} \left|\sin(x - \tfrac{1}{4}\pi)\right| dx$$
$$= \sqrt{2}\left[\int_0^{\pi/4} -\sin(x - \tfrac{1}{4}\pi) dx + \int_{\pi/4}^{5\pi/4} \sin(x - \tfrac{1}{4}\pi) dx + \int_{5\pi/4}^{2\pi} -\sin(x - \tfrac{1}{4}) dx\right]$$
$$= \sqrt{2}\left\{\cos(x - \tfrac{1}{4}\pi)\Big]_0^{\pi/4} - \cos(x - \tfrac{1}{4}\pi)\Big]_{\pi/4}^{5\pi/4} + \cos(x - \tfrac{1}{4}\pi)\Big]_{5\pi/4}^{2\pi}\right\}$$
$$= \sqrt{2}\{(1 - \tfrac{1}{2}\sqrt{2}) - (-1 - 1) + (\tfrac{1}{2}\sqrt{2} + 1)\} = 4\sqrt{2}$$

73. Let $u = x + 1$; then $x = u - 1$ and $dx = du$. Therefore

$$\int_1^2 \frac{x + 2}{(x + 1)^2} dx = \int_2^3 \frac{u + 1}{u^2} du = \int_2^3 \left(\frac{1}{u} + \frac{1}{u^2}\right) du = \ln|u| - \frac{1}{u}\Big]_2^3 = \left(\ln 3 - \tfrac{1}{3}\right) - \left(\ln 2 - \tfrac{1}{2}\right) = \ln\frac{3}{2} + \frac{1}{6}$$

74. Let $u = x^2$, $du = 2x\,dx$ and use Eq. 7.1.5.

$$\int_{x=0}^{\sqrt{\pi/2}} x e^{x^2}\cos x^2 dx = \tfrac{1}{2}\int_{u=0}^{\pi/2} e^u \cos u\,du = \tfrac{1}{4}e^u(\cos u + \sin u)\Big]_0^{\pi/2} = \tfrac{1}{4}(e^{\pi/2} - 1)$$

75. By symmetry, $\displaystyle\int_0^{\pi} |\cos^3 x|\, dx = 2\int_0^{\pi/2} \cos^3 x\,dx = 2\int_0^{\pi/2} (1 - \sin^2 x)\cos x\,dx = 2\left[\sin x - \tfrac{1}{3}\sin^3 x\right]_0^{\pi/2} = \tfrac{4}{3}$

76. $\displaystyle\int_{-\pi/4}^{\pi/4} |\tan^5 x|\, dx$

▷ Because $|\tan^5(-x)| = |\tan^5 x|$, then

$$\int_{-\pi/4}^{\pi/4} |\tan^5 x|\, dx = 2\int_0^{\pi/4} |\tan^5 x|\, dx = 2\int_0^{\pi/4} \tan^5 x\,dx = 2\int_0^{\pi/4} \tan^3 x(\sec^2 x - 1) dx$$
$$= 2\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx - 2\int_0^{\pi/4} \tan x(\sec^2 x - 1) dx$$
$$= \tfrac{1}{2}\tan^4 x\Big]_0^{\pi/4} - 2\int_0^{\pi/4} \tan x \sec^2 x\,dx + 2\int_0^{\pi/4} \tan x\,dx$$
$$= \tfrac{1}{2} - \tan^2 x\Big]_0^{\pi/4} + 2\ln|\sec x|\Big]_0^{\pi/4} = \tfrac{1}{2} - 1 + 2\ln\sqrt{2} = -\tfrac{1}{2} + \ln 2$$

77. $\dfrac{2x}{x^3 - x^2 - x + 1} \equiv \dfrac{A}{x + 1} + \dfrac{B}{x - 1} + \dfrac{C}{(x - 1)^2}$; $2x \equiv A(x - 1)^2 + B(x^2 - 1) + C(x + 1)$

Let $x = 1$: $2 = 2C \Leftrightarrow C = 1$. Let $x = -1$: $-2 = 4A \Leftrightarrow A = -\tfrac{1}{2}$.

Let $x = 0$: $0 = A - B + C$; $0 = -\tfrac{1}{2} - B + 1 \Leftrightarrow B = \tfrac{1}{2}$. Therefore

$$\int_0^{1/2} \frac{2x\,dx}{x^3 - x^2 - x + 1} = \int_0^{1/2} \left(-\frac{1}{2}\cdot\frac{1}{x + 1} + \frac{1}{2}\cdot\frac{1}{x - 1} + \frac{1}{(x - 1)^2}\right) dx = \frac{1}{2}\ln\left|\frac{x - 1}{x + 1}\right| - \frac{1}{x - 1}\Big]_0^{1/2} = 1 - \frac{1}{2}\ln 3$$

78. Let $u = \sqrt{x^2+1}$. Then $u^2 = x^2 + 1$, $u\,du = x\,dx$. $\int_0^1 x^3\sqrt{1+x^2}dx = \int_{x=0}^1 x^2\sqrt{1+x^2}(x\,dx)$

$= \int_{u=0}^{\sqrt{2}}(u^2-1)u(u\,du) = \int_0^{\sqrt{2}}(u^4-u^2)du = \frac{1}{5}u^5 - \frac{1}{3}u^3\Big]_1^{\sqrt{2}} = \frac{1}{5}(4\sqrt{2}) - \frac{1}{3}(2\sqrt{2}) - (\frac{1}{5}-\frac{1}{3}) = \frac{2}{15}\sqrt{2} + \frac{2}{15}$

79. $\int_0^{1/2}\frac{x\,dx}{\sqrt{1-4x^4}} = \frac{1}{4}\int_0^{1/2}\frac{4x\,dx}{\sqrt{1-(2x^2)^2}} = \frac{1}{4}\sin^{-1}(2x^2)\Big]_0^{1/2} = \frac{1}{4}\sin^{-1}\frac{1}{2} = \frac{1}{24}\pi$

80. $\int_0^{\pi/12}\frac{dx}{\cos^4 3x}$

▷ $\int_0^{\pi/12}\frac{dx}{\cos^4 3x} = \int_0^{\pi/12}\sec^4 3x\,dx = \int_0^{\pi/12}(1+\tan^2 3x)x\,dx = \Big[\frac{1}{3}\tan 3x + \frac{1}{9}\tan^3 3x\Big]_0^{\pi/12} = \frac{1}{3}+\frac{1}{9} = \frac{4}{9}$

81. Let $z = \tan\frac{1}{2}t$. Then $\cos t = \frac{1-z^2}{1+z^2}$ and $dt = \frac{2\,dz}{1+z^2}$. Therefore

$$\int_0^{\pi/2}\frac{dt}{12+13\cos t} = \int_0^1\frac{\frac{2\,dz}{1+z^2}}{12+13\left(\frac{1-z^2}{1+z^2}\right)} = \int_0^1\frac{2\,dz}{25-z^2} = \frac{1}{5}\ln\left|\frac{z+5}{z-5}\right|\Big]_0^1 = \frac{1}{5}\ln\frac{3}{2}$$

82. $\int_{2\pi/3}^{\pi}\frac{\sin\frac{1}{2}t}{1+\cos\frac{1}{2}t}dt = \int_{2\pi/3}^{\pi}\tan\frac{1}{4}t\,dt = -4\ln\left|\cos\frac{1}{4}t\right|\Big]_{2\pi/3}^{\pi} = -4[\ln(\cos\frac{1}{4}\pi) - \ln(\cos\frac{1}{6}\pi)] = -4\ln\frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{3}} = \ln\frac{9}{4}$

83. Let $u = \sqrt{x}$. then $x = u^2$ and $dx = 2u\,du$. Thus $I = \int_0^{16}\sqrt{4-\sqrt{x}}\,dx = 2\int_0^4\sqrt{4-u}\,u\,du$.

Now let $v = \sqrt{4-u}$. Then $u = 4 - v^2$ and $du = -2v\,dv$. Therefore

$I = 2\int_2^0 v(4-v^2)(-2v\,dv) = 4\int_2^0(v^4-4v^2)dv = 4\Big[\frac{1}{5}v^5 - \frac{4}{3}v^3\Big]_2^0 = \frac{256}{15}$

84. $\int_0^3\frac{dr}{(r+2)\sqrt{r+1}}$

▷ Let $x = \sqrt{r+1}$. Then $x^2 = r+1$, so $dr = 2x\,dx$ and $r+2 = x^2+1$.

$$\int_{r=0}^3\frac{dr}{(r+2)\sqrt{r+1}} = \int_{x=1}^2\frac{2x\,dx}{(x^2+1)x} = 2\int_1^2\frac{dx}{1+x^2} = 2\tan^{-1}x\Big]_1^2 = 2(\tan^{-1}2 - \frac{1}{4}\pi) = 2\tan^{-1}2 - \frac{1}{2}\pi$$

In Exercises 85–89, approximate the integral using the given rule with $n = 4$; express the result to 3 decimal places.

85. $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$. By the trapezoidal rule $\int_0^2\sqrt{1+x^2}\,dx$

$\approx \frac{0.5}{2}[1 + 2\sqrt{1+.5^2} + 2\sqrt{2} + 2\sqrt{1+1.5^2} + \sqrt{5}] = 2.977$ {Exact: 2.9579}

86. $\Delta x = \frac{b-a}{n} = \frac{1.8-1}{4} = 0.2$. By the trapezoidal rule $\int_1^{9/5}\sqrt{1+x^3}dx$

$\approx \frac{0.2}{2}[\sqrt{2} + 2\sqrt{1+1.2^3} + 2\sqrt{1+1.4^3} + 2\sqrt{1+1.6^3} + \sqrt{1+1.8^3}] = 1.5716 = 1.572$ {Exact: 1.5689}

87. $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$. By Simpson's rule $\int_0^2\sqrt{1+x^2}\,dx$

$\approx \frac{0.5}{3}[1 + 4\sqrt{1+.5^2} + 2\sqrt{2} + 4\sqrt{1+1.5^2} + \sqrt{5}] = 2.95796 = 2.958$ {Exact: 2.95789}

88. $\int_1^{9/5}\sqrt{1+x^3}\,dx$; Simpson's rule

▷ $\Delta x = \frac{\frac{9}{5}-1}{4} = 0.2$. By Simpson's rule,

$$\int_1^{9/5}\sqrt{1+x^3}\,dx \approx \tfrac{1}{3}(0.2)\Big[\sqrt{1+2^3} + 4\sqrt{1+(1.2)^3} + 2\sqrt{1+(1.4)^3} + 4\sqrt{1+(1.6)^3} + \sqrt{1+(1.8)^3}\Big]$$
$$= 1.568953 = 1.569$$

The value of the integral is 1.568949 correct to six decimal places.

89. $\Delta x = \frac{1}{n}(b-a) = \frac{1}{4}(\frac{1}{2}-\frac{1}{10}) = 0.1$. $I = \int_{1/10}^{1/2}\frac{\cos x}{x}\,dx$. Put your calculator in radian mode.

(a) By the trapezoidal rule $I \approx \frac{0.1}{2}\left(\frac{\cos .1}{.1} + 2\cdot\frac{\cos .2}{.2} + 2\cdot\frac{\cos .3}{.3} + 2\cdot\frac{\cos .4}{.4} + \frac{\cos .5}{.5}\right) = 0.05[32.48] = 1.624$

(b) By Simpson's rule $I \approx \frac{0.1}{3}\left(\frac{\cos .1}{.1} + 4\cdot\frac{\cos .2}{.2} + 2\cdot\frac{\cos .3}{.3} + 4\cdot\frac{\cos .4}{.4} + \frac{\cos .5}{.5}\right) = 1.563$

The exact value is 1.55008. Accuracy is poor near 0 because the integrand is unbounded. We integrate by parts with $u = \cos x$, $dv = dx/x$ to get a bounded integrand and then apply Simpson's rule with $n = 4$:
$I = \ln x \cos x\Big]_{.1}^{.5} + \int_{.1}^{.5} \sin x \ln x\, dx = 1.68278 - 0.13267 = 1.55012$ which is correct to 4 decimal places.

90. $\Delta x = 0.2$. By (a) the trapezoidal rule, (b) Simpson's rule

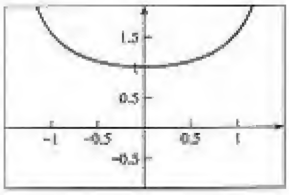
(a) $I \approx \frac{0.2}{2}(5.2 + 2\times5.7 + 2\times5.8 + 2\times6.3 + 2\times6.1 + 2\times6.0 + 2\times6.5 + 2\times6.8 + 2\times6.7 + 6.4) = 12.42$

(b) $I \approx \frac{0.2}{3}(5.2 + 4\times5.7 + 2\times5.8 + 4\times6.3 + 2\times6.1 + 4\times6.0 + 2\times6.5 + 4\times6.8 + 2\times6.7 + 6.4) = 12.49$

In Exercises 91–106, (a) from a plot state what $f(x)$ appears to approach as x approaches a; (b) compute $\lim_{x\to a} f(x)$.

91. $\lim_{x\to0}\frac{\tan^{-1}x}{x}\overset{0/0}{=}\lim_{x\to0}\frac{1/(1+x^2)}{1} = 1$

92. $f(x) = \frac{\tan x}{x}$; $a = 0$



▹ (a) From the plot at the right, the limit appears to be 1.

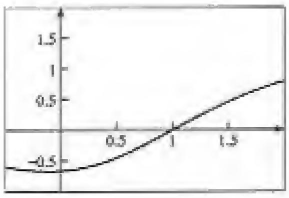
(b) By L'Hôpital's rule, $\lim_{x\to0}\frac{\tan x}{x}\overset{0/0}{=}\lim_{x\to0}\frac{\sec^2 x}{1} = 1$

93. $\lim_{x\to2}\frac{x^3-8}{x^3-2x^2+4x-8}\overset{0/0}{=}\lim_{x\to2}\frac{3x^2}{3x^2-4x+4} = \frac{3}{2}$

94. $\lim_{x\to1/2}\frac{\cos \pi x}{2x-1}\overset{0/0}{=}\lim_{x\to1/2}\frac{-\pi\sin\pi x}{2} = 0$

95. $\lim_{x\to0}\frac{\sin 2x}{x-\sin 5x}\overset{0/0}{=}\lim_{x\to0}\frac{2\cos 2x}{1-5\cos 5x} = \frac{2}{-4} = -\frac{1}{2}$

96. $f(x) = \frac{x^3-3x+2}{2x^3-3x^2+4x-3}$; $a = 1$



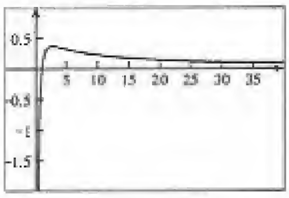
▹ (a) From the plot at the right, the limit appears to be 0.

(b) By L'Hôpital's rule, $\lim_{x\to1}\frac{x^3-3x+2}{2x^3-3x^2+4x-3}\overset{0/0}{=}\lim_{x\to1}\frac{3x^2-3}{6x^2-6x+4} = 0$

97. $\lim_{x\to3}\frac{\ln(x-2)}{x-3}\overset{0/0}{=}\lim_{x\to3}\frac{1/(x-2)}{1} = 1$ **98.** $\lim_{x\to0}\frac{x^2}{e^x-1}\overset{0/0}{=}\lim_{x\to0}\frac{2x}{e^x} = 0$

99. $\lim_{x\to0^+}\frac{\ln(\sin x)}{\ln(\cot x)}\overset{\infty/\infty}{=}\lim_{x\to0^+}\frac{\cos x/\sin x}{-\csc^2 x/\cot x} = \lim_{x\to0^+}(-\cos^2 x) = 1$

100. $\lim_{x\to+\infty}\frac{\ln x}{x}$



▹ (a) From the plot at the right, the limit appears to be 0.

(b) By L'Hôpital's rule, $\lim_{x\to+\infty}\frac{\ln x}{x}\overset{\infty/\infty}{=}\lim_{x\to+\infty}\frac{1/x}{1} = 0$

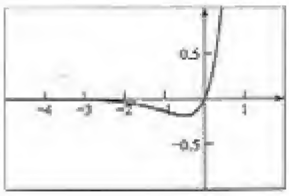
101. $\lim_{x\to+\infty}\frac{x}{e^x}\overset{\infty/\infty}{=}\lim_{x\to+\infty}\frac{1}{e^x} = 0$

102. Using known limits, $\lim_{x\to0^+}x^{\tan x} = \lim_{x\to0^+}(x^x)^{\tan x/x} = 1^1 = 1$

103. $\ln L = \lim_{x\to\pi/2}\ln(\sin^2 x)^{\tan x} = \lim_{x\to\pi/2}(\tan x\ln\sin^2 x)$

$= \lim_{x\to\pi/2} 2\sin x\cdot\frac{\ln\sin x}{\cos x}\overset{0/0}{=}2\cdot\lim_{x\to\pi/2}\frac{\cos x/\sin x}{-\sin x} = 0$. $L = 1$

104. $\lim_{x\to-\infty} xe^{2x}$



▹ (a) From the plot at the right, the limit appears to be 0.

(b) By L'Hôpital's rule, $\lim_{x\to-\infty}\frac{x}{e^{-2x}}\overset{\infty/\infty}{=}\lim_{x\to-\infty}\frac{1}{-2e^{-2x}} = 0$

105. $\lim_{x\to0^+}\frac{\tan 2x}{\sin^2 x}\overset{0/0}{=}\lim_{x\to0^+}\frac{2\sec^2 x}{2\sin x\cos x} = +\infty$

106. $\lim_{x\to\pi/2}\left(\frac{1}{1-\sin x}-\frac{1}{\cos^2 x}\right) = \lim_{x\to\pi/2}\left(\frac{1+\sin x}{1-\sin^2 x}-\frac{1}{\cos^2 x}\right) = \lim_{x\to\pi/2}\frac{\sin x}{\cos^2 x} = +\infty$

In Exercises 107–118, find the limit if it exists, and support your answer graphically.

107. We use $\lim_{x\to 0}\frac{x}{\sin x} = 1$ to simplify the derivatives of the denominator.

$$L = \lim_{x\to 0}\left(\csc^2 x - \frac{1}{x^2}\right) = \lim_{x\to 0}\frac{x^2 - \sin^2 x}{x^2\sin^2 x} = \left(\lim_{x\to 0}\frac{x}{\sin x}\right)^2 \cdot \lim_{x\to 0}\frac{x^2 - \sin^2 x}{x^4} = \lim_{x\to 0}\frac{x^2 - \sin^2 x}{x^4}$$

$\lim_{x\to 0}(x^2 - \sin^2 x) = 0$ and $\lim_{x\to 0} x^4 = 0$. By L'Hôpital's rule

$$L = \lim_{x\to 0}\frac{2x - 2\sin x\cos x}{4x^3} = \lim_{x\to 0}\frac{2x - \sin 2x}{4x^3} \overset{0/0}{=} \lim_{x\to 0}\frac{2 - 2\cos 2x}{12x^2} \overset{0/0}{=} \lim_{x\to 0}\frac{4\sin 2x}{24x} \overset{0/0}{=} \lim_{x\to 0}\frac{8\cos 2x}{24} = \frac{1}{3}$$

108. $\lim_{x\to 0}\frac{e - (1+x)^{1/x}}{x}$

▹ From Section 5.4 we have the limit

$$\lim_{x\to 0}(1+x)^{1/x} = e \qquad (1)$$

Therefore, $\lim_{x\to 0}[e - (1+x)^{1/x}] = 0$ and $\lim_{x\to 0} x = 0$. Thus the hypothesis of L'Hôpital's rule is satisfied for the given function. Because $D_x e = 0$, and $D_x x = 1$, we have

$$\lim_{x\to 0}\frac{e - (1+x)^{1/x}}{x} = \lim_{x\to 0}[-D_x(1+x)^{1/x}] \qquad (2)$$

Let

$$y = (1+x)^{1/x}$$

Then

$$\ln y = \frac{\ln(1+x)}{x}$$

$$\frac{1}{y}\cdot D_x y = \frac{\frac{x}{1+x} - \ln(1+x)}{x^2}$$

$$D_x y = y\cdot\frac{\frac{x}{1+x} - \ln(1+x)}{x^2}$$

$$D_x(1+x)^{1/x} = (1+x)^{1/x}\frac{\frac{x}{1+x} - \ln(1+x)}{x^2} \qquad (3)$$

Substituting from (3) into (2) and applying L'Hôpital's rule, we obtain

$$\lim_{x\to 0}\frac{e - (1+x)^{1/x}}{x} = \lim_{x\to 0}(1+x)^{1/x}\cdot\lim_{x\to 0}\frac{\ln(1+x) - \frac{x}{1+x}}{x^2} \overset{0/0}{=} e\cdot\lim_{x\to 0}\frac{\frac{1}{1+x} - \frac{1}{(1+x)^2}}{2x} = \lim_{x\to 0}\frac{1}{2(1+x)^2} = \frac{1}{2}e$$

109. $$\lim_{t\to +\infty}\frac{\ln\left(1 + \frac{e^{2t}}{t}\right)}{t^{1/2}} = \lim_{t\to +\infty}\frac{\ln\left[e^t\left(e^{-t} + \frac{e^t}{t}\right)\right]}{t^{1/2}} = \lim_{t\to +\infty}\frac{t + \ln\left(e^{-t} + \frac{e^t}{t}\right)}{t^{1/2}} = \lim_{t\to +\infty}\left[t^{1/2} + \frac{\ln\left(e^{-t} + \frac{e^t}{t}\right)}{t^{1/2}}\right] = +\infty$$

The second term is positive because $e^t > t$.

110. $$\lim_{x\to +\infty} x\ln\frac{x+1}{x-1} = \lim_{x\to +\infty}\frac{\ln(x+1) - \ln(x-1)}{x^{-1}} \overset{0/0}{=} \lim_{x\to +\infty}\frac{\frac{1}{x+1} - \frac{1}{x-1}}{-x^{-2}} = \lim_{x\to +\infty}\frac{2}{1 - x^{-2}} = 2$$

111. $\lim_{t\to 0}(1+4t)^{3/t} = [\lim_{t\to 0}(1+4t)^{1/4t}]^{12} = e^{12}$

112. $\lim_{y\to +\infty}(1 + e^{2y})^{-2/y}$

▹ $$\lim_{y\to 0}(1 + e^{2y})^{-2/y} = \lim_{y\to +\infty}[(1 + e^{-2y})\cdot e^{2y}]^{-2/y} = \lim_{y\to +\infty}(1 + e^{-2y})^{-2/y}\cdot e^{-4} = 1^0\cdot e^{-4} = e^{-4}$$

113. Because $\lim_{x\to +\infty}\frac{\ln x}{x} = \lim_{x\to +\infty}\frac{1}{x} = 0$, then $\lim_{x\to 0}(x - \ln x) = \lim_{x\to 0} x\left(1 - \frac{\ln x}{2}\right) = +\infty$. Also, $\lim_{x\to 0}\ln(\ln x) = +\infty$.

$$\lim_{x\to +\infty}\frac{\ln(\ln x)}{\ln(x - \ln x)} \overset{\infty/\infty}{=} \lim_{x\to +\infty}\frac{\frac{1}{x\ln x}}{\frac{x-1}{x - \ln x}} = \lim_{x\to +\infty}\frac{x - \ln x}{x(x-1)\ln x} \overset{\infty/\infty}{=} \lim_{x\to +\infty}\frac{1 - \frac{\ln x}{x}}{(x-1)\ln x} = 0$$

114. $\displaystyle \lim_{x\to 0}\frac{x-\tan^{-1}x}{4x^3}\overset{0/0}{=}\lim_{x\to 0}\frac{1-\dfrac{1}{1+x^2}}{12x^2}=\lim_{x\to 0}\frac{1}{12(1+x^2)}=\frac{1}{12}$

115. $\displaystyle \lim_{\theta\to\pi/2}\frac{\tan\theta+3}{\sec\theta-1}=\lim_{\theta\to\pi/2}\frac{\sin\theta+3\cos\theta}{1-\cos\theta}=1$

116. $\displaystyle \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x}$

▷ Because $\displaystyle \lim_{x\to 0}\frac{\sin x}{x}=1$

we have the indeterminate form 1^∞. We let

$$y=\left(\frac{\sin x}{x}\right)^{1/x},\qquad \ln y=\frac{\ln(\sin x/x)}{x}$$

By L'Hôpital's rule

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(\sin x/x)}{x}=\lim_{x\to 0}\frac{\ln(\sin x)-\ln x}{x}\overset{0/0}{=}\lim_{x\to 0}\frac{\dfrac{\cos x}{\sin x}-\dfrac{1}{x}}{1}=\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin x}$$

$$\overset{0/0}{=}\lim_{x\to 0}\frac{-x\sin x}{-x\cos x+\sin x}\overset{0/0}{=}\lim_{x\to 0}\frac{-x\cos x-\sin x}{-x\sin x+2\cos x}=0$$

Therefore $\displaystyle \lim_{x\to 0}y=e^0$ or, equivalently, $\displaystyle \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x}=1$. Note: $\displaystyle \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}=e^{-1/6}$

117. Because $\displaystyle \lim_{x\to+\infty}\frac{e^x}{x}=\lim_{x\to+\infty}e^x=+\infty$ then $\displaystyle \lim_{x\to+\infty}\ln(e^x-x)=\lim_{x\to+\infty}x\left(\frac{e^x}{x}-1\right)=+\infty$. By L'Hôpital's rule

$$\lim_{x\to+\infty}\ln(e^x-x)^{1/x}=\lim_{x\to+\infty}\frac{\ln(e^x-x)}{x}=\lim_{x\to+\infty}\frac{e^x-1}{e^x-x}=\lim_{x\to 0}\frac{1-e^{-x}}{1-x/e^x}=1$$

Thus $\displaystyle \ln\lim_{x\to+\infty}(e^x-x)^{1/x}=1$; $\displaystyle \lim_{x\to+\infty}(e^x-x)^{1/x}=e$.

118. $\displaystyle \lim_{x\to\pi/2}2\ln\sin x=0$ and $\displaystyle \lim_{x\to\pi/2}\cot x=0$. By L'Hôpital's rule

$$\lim_{x\to\pi/2}\ln(\sin^2x)^{\tan x}=\lim_{x\to\pi/2}\tan x\ln\sin^2x=\lim_{x\to\pi/2}\frac{2\ln\sin x}{\cot x}=\lim_{x\to\pi/2}\frac{\dfrac{2\cos x}{\sin x}}{-\csc^2x}=\lim_{x\to\pi/2}(-2\sin x\cos x)=0$$

Thus $\displaystyle \ln\lim_{x\to\pi/2}(\sin^2x)^{\tan x}=0$; $\displaystyle \lim_{x\to\pi/2}(\sin^2x)^{\tan x}=1$.

In Exercises 119–132 determine if the improper integral is convergent or divergent. If it is convergent, evaluate it.

119. $\displaystyle \int_{-2}^{0}\frac{dx}{2x-3}=\lim_{t\to-3/2^-}\int_{-2}^{t}\frac{dx}{2x+3}+\lim_{t\to-3/2^+}\int_{t}^{0}\frac{dx}{2x+3}=\lim_{t\to-3/2^-}\left[\tfrac12\ln|2x+3|\right]_{-2}^{t}+\lim_{t\to-3/2^+}\left[\tfrac12\ln|2x+3|\right]_{t}^{0}$

$\displaystyle =\lim_{t\to-3/2^-}\tfrac12\ln|2t+3|+\lim_{t\to-3/2^+}\left[\tfrac12\ln 3-\tfrac12\ln|2t+3|\right]$

Because neither of these limits exists, the given integral is divergent.

120. $\displaystyle \int_{0}^{+\infty}\frac{dx}{\sqrt{e^x}}$

▷ $\displaystyle \int_{0}^{+\infty}\frac{dx}{\sqrt{e^x}}=\lim_{b\to+\infty}\int_{0}^{b}e^{-x/2}dx=\lim_{b\to+\infty}\left[-2e^{-x/2}\right]_0^b=\lim_{b\to+\infty}\left[-2e^{-b/2}+2e^0\right]=2$

121. $\displaystyle \int_{-\infty}^{0}\frac{dx}{(x-2)^2}=\lim_{a\to-\infty}\int_{a}^{0}\frac{dx}{(x-2)^2}=\lim_{a\to-\infty}\left[-\frac{1}{x-2}\right]_a^0=\lim_{a\to-\infty}\left(\frac12+\frac{1}{a-2}\right)=\frac12$

122. Let $u=\sqrt{x-2}$, $x-2=u^2$, $dx=2u\,du$ to get a proper integral.

$\displaystyle \int_{x=2}^{4}\frac{x\,dx}{\sqrt{x-2}}=\int_{u=0}^{\sqrt2}\frac{(u^2+2)(2u\,du)}{u}=\int_0^{\sqrt2}(2u^2+4)du=\tfrac23u^3+4u\Big]_0^{\sqrt2}=\tfrac43\sqrt2+4\sqrt2=\tfrac{16}{3}\sqrt2$

123. $\displaystyle \int_0^{\pi/4}\cot^2\theta\,d\theta=\lim_{t\to 0^+}\int_t^{\pi/4}(\csc^2\theta-1)d\theta=\lim_{t\to 0^+}\Big[-\cot\theta-\theta\Big]_t^{\pi/4}=\lim_{t\to 0^+}\left(-1-\frac{\pi}{4}+\cot t+t\right)=+\infty$

Therefore, the given integral is divergent.

124. $\displaystyle \int_1^{+\infty}\frac{dt}{t^4+t^2}$

▷ $\displaystyle \int_1^{+\infty}\frac{dt}{t^4+t^2}=\lim_{b\to+\infty}\int_1^b\frac{dt}{t^2(t^2+1)}=\lim_{b\to+\infty}\int_1^b\left(\frac{1}{t^2}-\frac{1}{t^2+1}\right)dt=\lim_{b\to+\infty}\left[-\frac1t-\tan^{-1}t\right]_1^b$

$\displaystyle =\lim_{b\to+\infty}\left[-\frac1b-\tan^{-1}b+1+\tfrac14\pi\right]=-\tfrac12\pi+1+\tfrac14\pi=1-\tfrac14\pi$

125. $\int_{-\infty}^{3} 4^x dx = \lim_{a\to-\infty} \int_a^3 4^x dx = \lim_{a\to-\infty} \left[\frac{4^x}{\ln 4}\right]_a^3 = \lim_{a\to-\infty} \left(\frac{4^3}{\ln 4} - \frac{4^a}{\ln 4}\right) = \frac{64}{2\ln 2} = \frac{32}{\ln 2}$

126. $\int_{-\infty}^{0} xe^x dx = \lim_{a\to-\infty} \int_a^0 x\boxed{e^x dx} = \lim_{a\to-\infty} \left[xe^x - \int e^x dx\right]_a^0 = \lim_{a\to-\infty} \left[xe^x - e^x\right]_a^0 = \lim_{a\to-\infty} [-1 - ae^a + e^a] = -1$

127. $\int_0^1 \frac{(\ln x)^2}{x} dx = \lim_{t\to0^+} \int_t^1 (\ln x)^2 \frac{dx}{x} = \lim_{t\to0^+} \left[\frac{(\ln x)^3}{3}\right]_t^1 = \lim_{t\to0^+} \left[-\frac{(\ln t)^3}{3}\right] = +\infty$. The integral is divergent.

128. $\int_0^{+\infty} \frac{3^{-\sqrt{x}}}{\sqrt{x}} dx$

▷ We let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u\, du$. Therefore

$$\int_{x=0}^{+\infty} \frac{3^{-\sqrt{x}}}{\sqrt{x}} dx = \int_{u=0}^{+\infty} \frac{3^{-u}}{u}(2u\, du) = \lim_{b\to+\infty} 2\int_0^b 3^{-u} du = \lim_{b\to+\infty} -\frac{2}{\ln 3} 3^{-u}\Big]_0^b = \lim_{b\to+\infty} -\frac{2}{\ln 3}3^{-b} + \frac{2}{\ln 3} = \frac{2}{\ln 3}$$

129. $\int_{-\infty}^{+\infty} \frac{dx}{4x^2+4x+5} = \lim_{a\to-\infty} \int_a^{-1/2} \frac{dx}{(2x+1)^2+4} + \lim_{b\to+\infty} \int_{-1/2}^{b} \frac{dx}{(2x+1)^2+4}$

$= \lim_{a\to-\infty} \frac{1}{4}\tan^{-1}\left(x+\frac{1}{2}\right)\Big]_a^{-1/2} + \lim_{b\to+\infty} \frac{1}{4}\tan^{-1}\left(x+\frac{1}{2}\right)\Big]_{-1/2}^{b}$

$= -\frac{1}{4} \lim_{a\to-\infty} \tan^{-1}\left(a+\frac{1}{2}\right) + \frac{1}{4} \lim_{b\to+\infty} \tan^{-1}\left(b+\frac{1}{2}\right) = -\frac{1}{4}\left(-\frac{1}{2}\pi\right) + \frac{1}{4}\left(\frac{1}{2}\pi\right) = \frac{1}{4}\pi$

130. $\int_0^1 \frac{\ln x}{x} = \lim_{a\to0^+} \int_a^1 \ln x\, d(\ln x) = \lim_{a\to0^+} \frac{1}{2}(\ln x)^2\Big]_a^1 = \lim_{a\to0^+} -\frac{1}{2}(\ln a)^2 = -\infty$. The integral is divergent.

131. Because $x \in [0,1]$, $1 + x^2 \le 2$. Therefore by Theorem 4.6.2

$\int_0^1 \frac{dx}{x+x^3} = \lim_{t\to0^+} \int_t^1 \frac{dx}{x(1+x^2)} \ge \lim_{t\to0^+} \int_t^1 \frac{dx}{2x} = \lim_{t\to0^+} \frac{1}{2}\ln|x|\Big]_t^1 = -\lim_{t\to0^+} \frac{1}{2}\ln t = +\infty$

Hence, the given integral is divergent.

132. $\int_{-3}^{0} \frac{dx}{\sqrt{3-2x-x^2}}$

▷ Because $3 - 2x - x^2 = (3+x)(1-x)$, there is an infinite discontinuity at -3. Thus

$$\int_{-3}^{0} \frac{dx}{\sqrt{3-2x-x^2}} = \lim_{a\to-3^+} \int_a^0 \frac{dx}{\sqrt{4-(x+1)^2}} = \lim_{a\to-3^+} \left[\sin^{-1}\frac{x+1}{2}\right]_a^0 = \lim_{a\to-3^+} [\sin^{-1}\tfrac{1}{2} - \sin^{-1}\tfrac{1}{2}(a+1)]$$

$$= \tfrac{1}{6}\pi - (-\tfrac{1}{2}\pi) = \tfrac{2}{3}\pi$$

133. If $n \le 1$, then $\int_1^{+\infty} \frac{\ln x}{x^n} > \int_1^{+\infty} \frac{1}{x} dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x} dx = \lim_{b\to+\infty} \ln x\Big]_1^b = \lim_{b\to+\infty} \ln b = +\infty$. Integral diverges.

If $n > 1$, then let $u = \ln x$, $dx = e^u du$. $\int_{x=1}^{+\infty} \frac{\ln x}{x^n} = \int_{u=0}^{+\infty} \frac{u}{e^{nu}}(e^u du) = \lim_{b\to+\infty} \int_0^b u\boxed{e^{(1-n)u} du} =$

$= \lim_{b\to+\infty} \left[\frac{ue^{(1-n)u}}{1-n} - \int \frac{e^{(1-n)u}}{1-n} du\right]_0^b = \lim_{b\to+\infty} \left[\frac{ue^{(1-n)u}}{1-n} - \frac{e^{(1-n)u}}{(1-n)^2}\right]_0^b$

$= \lim_{b\to+\infty} \left[\frac{b}{e^{(n-1)b}(1-n)} - \frac{1}{e^{(n-1)u}(1-n)^2} + \frac{1}{(1-n)^2}\right] = \frac{1}{(n-1)^2}$

134. (a) Because $\lim_{x\to0} \sinh x\, dx = +\infty$, $\int_0^{+\infty} \sinh x\, dx$, and hence $\int_0^{+\infty} \sinh x\, dx$, is divergent.

(b) $\lim_{r\to+\infty} \int_{-r}^{r} \sinh x\, dx = \lim_{r\to+\infty} \left[\cosh x\right]_{-r}^{r} = \lim_{r\to+\infty} [\cosh r - \cosh(-r)] = \lim_{r\to+\infty} [\cosh r - \cosh r] = \lim_{r\to+\infty} 0 = 0$

135. $M = \int_0^3 ke^{-3x} dx = -\frac{k}{3}\int_0^3 e^{-3x}(-3\, dx) = -\frac{k}{3}\left[e^{-3x}\right]_0^3 = \frac{k}{3}(1-e^{-9})$

Let $u = kx$ and $dv = e^{-3x}\, dx$. Then $du = k\, dx$ and $v = -\frac{1}{3}e^{-3x}$. Therefore

$M = \int_0^3 kxe^{-3x} dx = -\frac{k}{3} xe^{-3x}\Big]_0^3 + \int_0^3 e^{-3x} dx = -ke^{-9} - \frac{k}{9}\left[e^{-3x}\right]_0^3 = -ke^{-9} + \frac{k}{9}(1-e^{-9}) = \frac{k}{9}(1-10e^{-9})$

$\bar{x} = \frac{M_y}{M} = \frac{k(1-10e^{-9})/9}{k(1-e^{-9})/3} = \frac{e^9-10}{3(e^9-1)}$. The centroid is $\frac{e^9-10}{3(e^9-1)} \approx 0.3330$ m from one end.

136. Find the center of mass of a rod 4 m long if the linear density x meters from the left end is $\sqrt{9+x^2}$ kg/m.

▸ If M kg is the total mass of the rod, then

$$\mathrm{M} = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{9+w_i{}^2}\,\Delta_i x = \int_{x=0}^{4} \sqrt{9+x^2}\,dx$$

Let $\theta = \tan^{-1}(x/3)$. Then $x = 3\tan\theta$, so $dx = 3\sec^2\theta\,d\theta$ and $\sqrt{9+x^2} = 3\sec\theta$. Let $\tan b = \frac{4}{3}$ so $\sec b = \frac{5}{3}$. Then, with the help of Example 7.2.10,

$$\mathrm{M} = 9\int_{\theta=0}^{b} \sec^3\theta\,d\theta = 9\Big[\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta+\tan\theta|\Big]_0^b = 9\Big(\tfrac{1}{2}\cdot\tfrac{5}{3}\cdot\tfrac{4}{3} + \tfrac{1}{2}\ln\Big|\tfrac{5}{3}+\tfrac{4}{3}\Big|\Big) = 10 + \tfrac{9}{2}\ln 3$$

Furthermore, if the center of mass of the rod is $\bar{x}$ meters from the left end, then

$$\mathrm{M}\bar{x} = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{9+w_i{}^2}\,w_i\Delta_i x = \int_0^4 \sqrt{9+x^2}(x\,dx) = \tfrac{1}{2}\cdot\tfrac{2}{3}(9+x^2)^{3/2}\Big]_0^4 = \tfrac{1}{3}(25^{3/2} - 9^{3/2}) = \tfrac{98}{3}.$$

Therefore,

$$\bar{x} = \frac{98/3}{\mathrm{M}} = \frac{98}{3(10+\frac{9}{2}\ln 3)} \approx 2.19$$

• The center of mass of the rod is about 2.19 meters from the left end of the rod.

137. $x = \frac{1}{6}y^2$. Hence $\frac{dx}{dy} = \frac{1}{3}y$. Let L units be the length of arc from $(6,6)$ to $(12, 6\sqrt{2})$.

Let $y = 3\tan\theta$ where $0 < \theta < \frac{1}{2}\pi$ and let $a = \tan^{-1}2$ and $b = \tan^{-1}2\sqrt{2}$ so that $\sec a = \sqrt{5}$ and $\sec b = 3$.

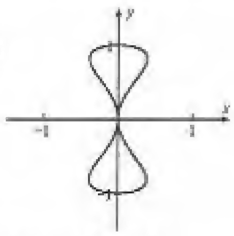
Then $dy = 3\sec^2\theta\,d\theta$ and $\sqrt{1+\frac{1}{9}y^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$. Therefore

$$\mathrm{L} = \int_6^{6\sqrt{2}} \sqrt{1+\tfrac{1}{9}y^2}\,dy = \int_a^b \sec\theta(3\sec^2\theta\,d\theta) = 3\int_a^b \sec^3\theta\,d\theta = \frac{3}{2}\sec\theta\tan\theta + \frac{3}{2}\ln|\sec\theta+\tan\theta|\Big]_a^b$$

$$= \tfrac{3}{2}\cdot 3\cdot 2\sqrt{2} + \tfrac{3}{2}\ln(3+2\sqrt{2}) - \tfrac{3}{2}\cdot\sqrt{5}\cdot 2 - \tfrac{3}{2}\ln(\sqrt{5}+2) = 9\sqrt{2} - 3\sqrt{5} + \tfrac{3}{2}\ln\Big(\frac{3+2\sqrt{2}}{\sqrt{5}+2}\Big)$$

138. Use horizontal elements. $\mathrm{A} = \int_0^{\pi/3} (\frac{1}{4}\sqrt{3} - \frac{1}{2}\sin y)dy = \Big[\frac{1}{4}\sqrt{3}y + \frac{1}{2}\cos y\Big]_0^{\pi/3} = \frac{1}{4}\sqrt{3}\cdot\frac{1}{3}\pi + \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2} = \frac{1}{12}\sqrt{3}\pi - \frac{1}{4}$

139. The figure shows the graph of $x^2 = y^4(1-y^2)$, $y \in [0,1]$. Parametric equations for plotting are $x = \sin^2 t\cos t$, $y = \sin t$. The area enclosed by one loop is twice the area of the region in the first quadrant. If A square units is the area enclosed by one loop,



$$\mathrm{A} = 2\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} w_i{}^2\sqrt{1-w_i{}^2}\,\Delta_i y = 2\int_0^1 y^2\sqrt{1-y^2}\,dy$$

Let $y = \sin\theta$ where $0 \le \theta \le \frac{1}{2}\pi$. Then $dy = \cos\theta\,d\theta$ and

$$\sqrt{1-y^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$$

$$\mathrm{A} = 2\int_0^{\pi/2} \sin^2\theta\cos\theta(\cos\theta\,d\theta) = 2\int_0^{\pi/2} \frac{\sin^2 2\theta}{42}\,d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2}\,d\theta = \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta\Big]_0^{\pi/2} = \frac{1}{8}\pi$$

140. Find the length of the arc of the curve $y = \ln x$ from $x = 1$ to $x = e$.

▸ We apply Theorem 6.1.2. Let $f(x) = \ln x$. Then

$$f'(x) = \tfrac{1}{x}, \quad \sqrt{1+[f'(x)]^2} = \sqrt{1+\frac{1}{x^2}} = \frac{\sqrt{x^2+1}}{x}$$

If L units is the length of the arc, then

$$\mathrm{L} = \int_1^e \frac{\sqrt{x^2+1}}{x}dx = \int_{x=1}^{e} \frac{\sqrt{x^2+1}}{x^2}(x\,dx)$$

Let $u = \sqrt{x^2+1}$. Then $u^2 = x^2+1$, so $u\,du = x\,dx$ and $x^2 = u^2-1$. Thus,

$$\mathrm{L} = \int_{u=\sqrt{2}}^{\sqrt{e^2+1}} \frac{u(u\,du)}{u^2-1} = \int_{\sqrt{2}}^{\sqrt{e^2+1}} \Big(1+\frac{1}{u^2-1}\Big)du = \Big[u + \frac{1}{2}\ln\Big|\frac{u-1}{u+1}\Big|\Big]_{\sqrt{2}}^{\sqrt{e^2+1}}$$

$$= \sqrt{e^2+1} - \sqrt{2} + \tfrac{1}{2}\ln\Big(\frac{\sqrt{e^2+1}-1}{\sqrt{e^2+1}+1}\Big) - \tfrac{1}{2}\ln\Big(\frac{\sqrt{2}-1}{\sqrt{2}+1}\Big)$$

$$= \sqrt{e^2+1} - \sqrt{2} + \frac{1}{2}\ln\left[\frac{(\sqrt{e^2+1}-1)^2}{e^2}\right] - \frac{1}{2}\ln(\sqrt{2}-1)^2$$

$$= \sqrt{e^2+1} - \sqrt{2} + \ln(\sqrt{e^2+1}-1) - 1 - \ln(\sqrt{2}-1) \approx 2.003$$

■ The length of arc is about 2.00 units.

141. The region is bounded by $y = \ln 2x$, the x axis, and $x = e$. An element of volume is a cylindrical shell centered on the y axis of mean radius m_i and altitude $\ln 2m_i$, $m_i \in [\frac{1}{2}, e]$. If V cubic units is the volume of the solid of revolution, then

$V = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} 2\pi m_i(\ln 2m_i)\Delta_i x = 2\pi \int_{1/2}^{e} x(\ln 2x)dx$. Let $u = \ln 2x$ and $dv = x\,dx$. Then $du = \frac{1}{x}$ and $v = \frac{1}{2}x^2$

and $V = 2\pi \int_{1/2}^{e} x(\ln 2x)dx = 2\pi\left[\frac{1}{2}x^2\ln 2x - \frac{1}{2}\int x\,dx\right]_{1/2}^{e} = \pi\left[x^2\ln 2x - \frac{1}{2}x^2\right]_{1/2}^{e}$

$= \pi\left(e^2\ln 2e - \frac{1}{2}e^2 + \frac{1}{8}\right) = \pi\left(e^2\ln 2 + \frac{1}{2}e^2 + \frac{1}{8}\right)$

142. Let $u = x + 1$, $du = dx$.

$$V = \int_{x=0}^{5} \pi\left[\frac{5-x}{(x+1)^2}\right]^2 dx = \int_{u=1}^{6} \pi\left(\frac{6-u}{u^2}\right)^2 du = \int_1^6 \pi\left(\frac{36}{u^4} - \frac{12}{u^3} + \frac{1}{u^2}\right)du = \pi\left[-\frac{12}{u^3} + \frac{6}{u^2} - \frac{1}{u}\right]_1^6 = \frac{125}{18}\pi$$

143. Let x pounds of chemical C be present at t hours.
We have a table of boundary conditions.

t	0	1	3
x	0	15	x_3

$\frac{dx}{dt} = k(60 - \frac{3}{5}x)(60 - \frac{2}{5}x)$; $\int \frac{dx}{(60-\frac{3}{5}x)(60-\frac{2}{5}x)} = \int k\,dt$; $\frac{1}{12}\int\left(\frac{\frac{3}{5}}{60-\frac{3}{5}x} - \frac{\frac{2}{5}}{60-\frac{2}{5}}\right)dx = k\int dt$

$\frac{1}{12}\ln\left|\frac{60-\frac{3}{5}x}{60-\frac{2}{5}x}\right| = kt + C_1$; $\frac{300-2x}{300-3x} = Ce^{12kt}$

When $t = 0$, $x = 0$. Therefore $C = \frac{300}{300} = 1$. Thus $\frac{300-2x}{300-3x} = e^{12kt}$.

When $t = 1$, $x = 15$. Hence $e^{12k} = \frac{270}{255} = \frac{18}{17}$, so that $\frac{300-2x}{300-3x} = \left(\frac{18}{17}\right)^t$.

(a) $300(17)^t - 2(17)^t x = 300(18)^t - 3(18)^t x$; $(3 \cdot 18^t - 2 \cdot 17^t)x = 300(18^t - 17^t)$; $x = 300\left(\frac{18^t - 17^t}{3 \cdot 18^t - 2 \cdot 17^t}\right)$

(b) When $t = 3$ we have $x_3 = 300\left(\frac{18^3 - 17^3}{3 \cdot 18^3 - 2 \cdot 17^3}\right) = 35.94$.

144. A tank is in the shape of the solid formed by rotating about the x axis the region bounded by the curve $y = \ln x$, the x axis, and the line $x = e$ and $x = e^2$. If the tank is full of water, find the work done in pumping all the water to the top of the tank. Distance is measured in feet. Take the positive x axis downward.

▷ The figure shows a plane section of the tank. The element of volume is a circular disk of thickness $\Delta_i x$ and radius $\ln w_i$. Thus the number of cubic feet in an element of volume is

$\Delta_i V = \pi(\ln w_i)^2\Delta_i x$

If w is the density of water, then the number of pounds in the force on the element is

$\Delta_i F = w\Delta_i V = \pi w(\ln w_i)^2\Delta_i x$

Because the top of the tank is at the point where $x = e$, the number of feet in the element of displacement is

$D_i = w_i - e$

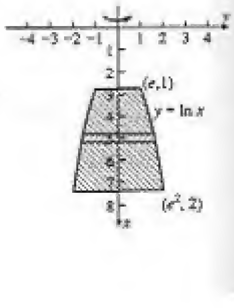
Because the tank is bounded by the lines $x = e$ and $x = e^2$, then

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} D_i\Delta_i F = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (w_i - e)\pi w(\ln w_i)^2\Delta_i x = \pi w\int_e^{e^2} (x-e)(\ln x)^2 dx$$

We use integration by parts.

$u = (\ln x)^2 \qquad dv = (x - e)dx$

$du = \frac{2\ln x\,dx}{x} \qquad v = \frac{1}{2}x^2 - ex$



Thus,

$$W = \pi w\left[(\tfrac{1}{2}x^2 - ex)(\ln x)^2 - \int (x - 2e)\ln x\,dx\right]_e^{e^2}$$

We integrate by parts again.

$\bar{u} = \ln x \qquad dv = (x - 2e)dx$

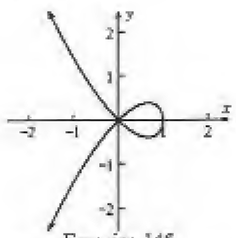
$d\bar{u} = \frac{dx}{x} \qquad v = \frac{1}{2}x^2 - 2ex$

$W = \pi w\Big[(\frac{1}{2}x^2 - ex)(\ln x)^2 - (\frac{1}{2}x^2 - 2ex)\ln x + \frac{1}{4}x^2 - 2ex\Big]_e^{e^2}$

$= \pi w\{[(\frac{1}{2}e^4 - e^3)4 - (\frac{1}{2}e^4 - 2e^3)2 + \frac{1}{4}e^4 - 2e^3] - [(\frac{1}{2}e^2 - e^2) - (\frac{1}{2}e^2 - 2e^2) + \frac{1}{4}e^2 - 2e^2)]\}$

$= 62.4\pi(\frac{5}{4}e^4 - 2e^3 + \frac{3}{4}e^2)$

- The work is about 6590.4 ft-lb.



Exercise 146

145. From Exercise 139, $A = \pi/8$. $M = 2\int_0^1 y\sqrt{y^4(1-y^2)}dy = 2\int_0^1 y^2\sqrt{1-y^2}\, y\, dy$

Let $u = \sqrt{1-y^2}$. Then $u^2 = 1 - y^2$, $u\,du = -y\,dy$ and $y^2 = 1 - u^2$. Then

$M_x = 2\int_1^0 (1-u^2)u(-u\,du) = 2\int_0^1 (u^2 - u^4)du = 2\Big[\frac{1}{3}u^3 - \frac{1}{5}u^5\Big]_0^1 = \frac{4}{15}$

The centroid is at $(0, \bar{y})$ where $y = \frac{1}{A}M_x = \frac{8}{\pi}\cdot\frac{4}{15} = \frac{32}{15\pi}$.

146. The region is enclosed by the loop of $y^2 = x^2 - x^3 = x^2(1-x)$. Parametric equations for plotting are $x = 1 - t^2$, $y = t(1-t^2)$. See the figure above. By symmetry $\bar{y} = 0$. Let $u = \sqrt{1-x}$, $x = 1 - u^2$, $dx = -2u\,du$.

$A = \int_{x=0}^1 x\sqrt{1-x}dx = \int_{u=1}^0 (1-u^2)u(-2u\,du) = \int_0^1 (2u^2 - u^4)du = \Big[\frac{2}{3}u^3 - \frac{2}{5}u^5\Big]_0^1 = \frac{4}{15}$

$M_y = \int_{x=0}^1 x\cdot x\sqrt{1-x}\,dx = \int_{u=1}^0 (1-u^2)^2u(-2u\,du) = \int_0^1 (2u^2 - 4u^4 - 2u^6)du = \Big[\frac{2}{3}u^3 - \frac{4}{5}u^5 + \frac{2}{7}u^7\Big]_0^1 = \frac{16}{105}$.

$\bar{x} = M_y \cdot \frac{1}{A} = \frac{16}{105}\cdot\frac{15}{4} = \frac{4}{7}$

147. The region is bounded by the y axis, $y = \sin x - \cos x$ and $y = \sin x + \cos x$.

$A = \int_0^{\pi/2} [(\sin x + \cos x) - (\sin x - \cos x)]dx = 2\int_0^{\pi/2} \cos x\,dx = 2\sin x\Big]_0^{\pi/2} = 2$

$M_y = \int_0^{\pi/2} x[(\sin x + \cos x) - (\sin x - \cos x)]dx = 2\int_0^{\pi/2} x\boxed{\cos x\,dx} = 2\Big[x\sin x + \cos x\Big]_0^{\pi/2} = 2\Big(\frac{\pi}{2} - 1\Big) = \pi - 2$

$M = \frac{1}{2}\int_0^{\pi/2}[(\sin x + \cos x) + (\sin x - \cos x)][(\sin x + \cos x) - (\sin x - \cos x)]dx$

$= \frac{1}{2}\int_0^{\pi/2}(2\sin x)(2\cos x)dx = \int_0^{\pi/2} \sin 2x\,dx = -\frac{1}{2}\cos 2x\Big]_0^{\pi/2} = -\frac{1}{2}(-1-1) = 1$

$\bar{x} = \frac{M_y}{A} = \frac{\pi - 2}{2} = \frac{1}{2}\pi - 1$ and $\bar{y} = \frac{M_x}{A} = \frac{1}{2}$. The centroid is at the point $\Big(\frac{1}{2}\pi - 1, \frac{1}{2}\Big)$.

148. Find the centroid of the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$.

▹ If A square units is the area of the region, then

$A = \int_0^{\pi/2}\cos x\,dx = \sin x\Big]_0^{\pi/2} = 1$

$M_y = \int_0^{\pi/2} x\,\boxed{\cos x\,dx} = x\sin x\Big]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx = x\sin x + \cos x\Big]_0^{\pi/2} = \frac{1}{2}\pi - 1$

$M_x = \int_0^{\pi/2}\frac{1}{2}\cos^2 x\,dx = \frac{1}{4}\int_0^{\pi/2}(1 + \cos 2x)dx = \frac{1}{4}\Big[x + \frac{1}{2}\sin 2x\Big]_0^{\pi/2} = \frac{1}{8}\pi$

If $(\bar{x}, \bar{y})$ is the centroid of the region, then $\bar{x} = M_y \cdot \frac{1}{A} = \frac{1}{2}\pi - 1$ and $\bar{y} = M_x\cdot\frac{1}{A} = \frac{1}{8}\pi$

149. Let the x axis be at the top of the tank and take the positive y axis downward. Then the region is bounded by $y = 2\sin\frac{1}{3}\pi x$, $x \in [0,3]$, and the x axis. An element of area is a vertical strip of length $2\sin\frac{1}{3}\pi w_i$ and mean depth $\sin\frac{1}{3}\pi w_i$. F lb is the force on the end.

$F = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\Big(\rho\sin\frac{\pi}{3}w_i\Big)\Big(2\sin\frac{\pi}{3}w_i\Big)\Delta_i x = \rho\int_0^3 2\sin^2\frac{\pi}{3}x\,dx = \rho\int_0^3\Big(1 - \cos\frac{2\pi}{3}x\Big)dx = \rho\Big[x - \frac{3}{2\pi}\sin\frac{2\pi}{3}\Big]_0^3 = 62.4\cdot 3$

150. Let the curve be $y = \sin x$, $x \in [0,\pi]$. $y = 2$ is the water level. F lb is the force on one side of the board. An element of area is a vertical strip of altitude $\sin w_i$ ft and mean depth $2 - \frac{1}{2}\sin w_i$ ft.

$F = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\rho(2 - \frac{1}{2}\sin w_i)\sin w_i\Delta_i x = \rho\int_0^{\pi}(2\sin x - \frac{1}{2}\sin^2 x)dx = \rho\int_0^{\pi}[2\sin x - \frac{1}{4}(1 - \cos 2x)]dx$

$= \rho\Big[-2\cos x - \frac{1}{4}x + \frac{1}{8}\sin 2x\Big]_0^{\pi} = \rho[2 - \frac{1}{4}\pi - (-2)] = 62.4(4 - \frac{1}{4}\pi) \approx 200.6$

E I G H T

POLYNOMIAL APPROXIMATIONS, SEQUENCES, AND INFINITE SERIES

8.1 POLYNOMIAL APPROXIMATIONS BY TAYLOR'S FORMULA

8.1.1 Theorem Let f be a function such that f and its first n derivatives are continuous on the closed interval $[a, b]$. Furthermore, let $f^{(n+1)}(x)$ exist in the open interval (a, b). Then there is a number z in the open interval (a, b) such that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(b-a)^{(n+1)} \quad (1)$$

If in (1) b is replaced by x, *Taylor's formula* is obtained. It is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{(n+1)} \quad (2)$$

where z is between a and x.

Taylor Polynomial If $f(a)$ exists and if the first n derivatives of f exist at the number a, we define the nth degree *Taylor polynomial* of the function f at the number a as follows.

$$\mathrm{P}_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \quad (3)$$

Maclaurin Polynomial If $a = 0$ in Eq. (3) we have

$$\mathrm{P}_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \quad (3)$$

which is called the nth degree *Maclaurin polynomial* for the function f.

Maclaurin for $f(x^m)$ If m is a positive integer and $\mathrm{P}_n(x)$ is the nth degree Maclaurin polynomial for $f(x)$, then $x^m\mathrm{P}_n(x)$ is the $(m+n)$th degree Maclaurin polynomial for $x^m f(x)$ and $\mathrm{P}_n(x^m)$ is the (mn)th degree Maclaurin polynomial for $f(x^m)$. See Exercise 8.

If the Taylor polynomial P_n is used to approximate the function f, then the error, $f(x) - \mathrm{P}_n(x)$, is denoted by $\mathrm{R}_n(x)$. We have two forms of the remainder.

Lagrange Form $\mathrm{R}_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$ where z is between a and x, if $f^{(n+1)}$ exists

Theorem 8.1.2 Integral Form $\mathrm{R}_n(x) = \dfrac{1}{n!}\displaystyle\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$ if $f^{(n+1)}$ is continuous

Taylor's formula (2) can be written as

$$f(x) = \mathrm{P}_n(x) + \mathrm{R}_n(x)$$

Sometimes $|\mathrm{R}_n(x)|$ is small, in which case the Taylor polynomial $\mathrm{P}_n(x)$ is a good approximation of $f(x)$. However, sometimes $|\mathrm{R}_n(x)|$ is large, in which case $\mathrm{P}_n(x)$ is of no use for approximating $f(x)$. Whether $|\mathrm{R}_n(x)|$ is small or large depends on the function f, the value of $(x-a)$, and the value of n. We need $|\mathrm{R}_n(x)| < 5 \times 10^{-(n+1)}$ for n decimal place accuracy.

Exercises 8.1

In Exercises 1–10, find the Maclaurin polynomial P_n of the stated degree for the function f with the Lagrange form of the remainder. Plot the graphs of f and P_n and observe how the graph of P_n approximates the graph of f near the point where $x = 0$.

1. $f(x) = (x-2)^{-1}$; $f'(x) = -(x-2)^{-2}$; $f''(x) = 2(x-2)^{-3}$; $f'''(x) = -3!(x-2)^{-4}$; $f^{(4)}(x) = 4!(x-2)^{-5}$

   $f(0) = -\frac{1}{2}$; $f'(0) = -\frac{1}{4}$; $f''(0)/2! = -\frac{2}{8}/2! = -\frac{1}{8}$; $f'''(0)/3! = -\frac{3!}{16}/3! = -\frac{1}{16}$; $f^{(4)}(0)/4! = -\frac{4!}{32}/4! = -\frac{1}{32}$

   $P_4(x) = -\frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{1}{32}x^4$. $f^{(5)}(x) = -5!(x-2)^{-6}$. $R_4(x) = \dfrac{x^5}{(z-2)^6}$, z between 0 and x

2. $f(x) = (x+3)^{-1}$; $f'(x) = -(x+3)^{-2}$; $f''(x) = 2(x+3)^{-3}$; $f'''(x) = -3!(x+3)^{-4}$; $f^{(4)}(x) = 4!(x+3)^{-5}$;

   $f^{(5)}(x) = -5!(x+3)^{-6}$. $f(0) = \frac{1}{3}$; $f'(0) = -\frac{1}{9}$; $f''(0)/2! = \frac{2}{27}/2! = \frac{1}{27}$; $f'''(0)/3! = -\frac{3!}{81}/3! = -\frac{1}{81}$;

   $f^{(4)}(0)/4! = \frac{4!}{243}/4! = \frac{1}{243}$; $f^{(5)}(0)/5! = -\frac{5!}{729}/5! = -\frac{1}{729}$. $P_5(x) = \frac{1}{3} - \frac{1}{9}x + \frac{1}{27}x^2 - \frac{1}{81}x^3 + \frac{1}{243}x^4 - \frac{1}{729}x^5$

   $f^{(6)}(x) = 6!(x+3)^{-7}$. $R_5(x) = \dfrac{x^6}{(z+3)^7}$, z between 0 and x

3. $f(x) = e^{-x}$; $f'(x) = -e^{-x}$; $f''(x) = e^{-x}$; $f'''(x) = -e^{-x}$; $f^{(4)}(x) = e^{-x}$; $f^{(5)}(x) = -e^{-x}$

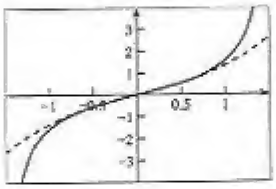
   $f(0) = 1$; $f'(0) = -1$; $f''(0)/2! = 1/2!$; $f'''(0)/3! = -1/3!$; $f^{(4)}(0)/4! = 1/4!$; $f^{(5)}(0)/5! = -1/5!$

   $P_5(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$. $f^{(6)}(x) = e^{-x}$. $R_5(x) = \frac{e^{-z}}{6!}x^6$, z between 0 and x

4. $f(x) = \tan x$; degree 3

   ▶ The third-degree Maclaurin polynomial for the function f is defined by

$$P_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \qquad (1)$$



and the Lagrange form of the remainder is given by

$$R_3(x) = \frac{f^{(4)}(z)}{4!}x^4 \quad \text{where } z \text{ is between 0 and } x. \qquad (2)$$

Thus, we find the first four derivatives of f.

$$f'(x) = \sec^2 x$$

$$f''(x) = 2\sec^2 x \tan x = 2\tan^3 x + 2\tan x$$

$$f'''(x) = 6\tan^2 x \sec^2 x + 2\sec^2 x = 6\tan^4 x + 8\tan^2 x + 2$$

$$f^{(4)}(x) = 24\tan^3 x \sec^2 x + 16\tan x \sec^2 x = 24\tan^5 x + 40\tan^3 x + 16\tan x$$

from which we obtain

$$f(0) = \tan 0 = 0; \quad f'(0) = \sec^2 0 = 1; \quad f''(0) = 2\tan^3 0 + 2\tan 0 = 0; \quad f'''(0) = 6\tan^4 0 + 8\tan^2 0 + 2 = 2$$

Substituting in (1), we obtain $\qquad P_3(x) = 0 + 1 \cdot x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$

Substituting in (2), we obtain

$$R_3(x) = \frac{24\tan^5 z + 40\tan^3 z + 16\tan z}{4!}x^4 = \tfrac{1}{3}(3\tan^5 z + 5\tan^3 z + 2\tan z)x^4 \text{ where } z \text{ is between 0 and } x$$

The figure shows a plot of $f(x)$ and $P_3(x)$ (dashed).

5. $f(x) = \cos x$; $f'(x) = -\sin x$; $f''(x) = -\cos x$; $f'''(x) = \sin x$; $f^{(4)}(x) = \cos x$; $f^{(5)}(x) = -\sin x$; $f^{(6)}(x) = -\cos x$

   $f(0) = 1$; $f'(0) = 0$; $f''(0)/2! = -1/2!$; $f'''(0)/3! = 0$; $f^{(4)}(0)/4! = 1/4!$; $f^{(5)}(0)/5! = 0$; $f^{(6)}(0)/6! = 1/6!$

   $P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$. $f^{(7)}(x) = \sin x$. $R_7(x) = \frac{\sin z}{7!}x^7$, z between 0 and x

6. $f(x) = \cosh x$; $f'(x) = \sinh x$; $f''(x) = \cosh x$; $f'''(x) = \sinh x$; $f^{(4)}(x) = \cosh x$

   $f(0) = 1$; $f'(0) = 0$; $f''(0)/2! = 1/2!$; $f'''(0)/3! = 0$; $f^{(4)}(0)/4! = 1/4!$; $f^{(5)}(z)/5! = \sinh z/5!$

   $P_4(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$. $R_4(x) = \frac{\sinh z}{5!}x^5$, z between 0 and x

7. $f(x) = \sinh x$; $f'(x) = \cosh x$, $f''(x) = \sinh x$, $f'''(x) = \cosh x$, $f^{(4)}(x) = \sinh x$, $f^{(5)}(x) = \cosh x$

$f(0) = 0$, $f'(0) = 1$, $\frac{f''(0)}{2!} = 0$, $\frac{f'''(0)}{3!} = \frac{1}{6}$, $\frac{f^{(4)}(0)}{4!} = 0$, $\frac{f^{(5)}(z)}{5!} = \frac{\cosh z}{120}$

$P_4(x) = 0 + x + 0 + \frac{1}{6}x^3 + 0 = x + \frac{1}{6}x^3$; $R_4(x) = \frac{\cosh z}{120}x^5$ where z is between 0 and x.

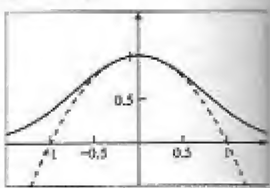
8. $f(x) = e^{-x^2}$; degree 3

▸ We must find $\quad P_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}$

and $\quad R_3(x) = \frac{f^{(4)}(z)}{4!}x^4$ where z is between 0 and x

We have

$$\begin{aligned}
f(x) &= e^{-x^2} \\
f'(x) &= -2xe^{-x^2} \\
f''(x) &= (-2x)(-2x)e^{-x^2} + (-2)e^{-x^2} = 2e^{-x^2}(2x - 1) \\
f'''(x) &= 2e^{-x^2}(4x) + (2x^2 - 1)(-4xe^{-x^2}) = -4e^{-x^2}(2x^3 - 3x) \\
f^{(4)}(x) &= -4e^{-x^2}(6x^2 - 3) + (2x^3 - 3x)(8xe^{-x^2}) = 4e^{-x^2}[-(6x^2 - 3) + 2x(2x^3 - 3x)] \\
&= 4e^{-x^2}(4x^4 - 12x^2 + 3) \qquad (3)
\end{aligned}$$



Thus,

$$f(0) = 1; \quad f'(0) = 0; \quad f''(0) = -2; \quad f'''(0) = 0$$

Substituting into Eq. (1), we obtain

$$P_3(x) = 1 - x^2$$

which is both a second and third degree Maclaurin polynomial. Substituting from (3) into (2), we obtain

$$R_3(x) = \frac{4e^{-z^2}(4z^4 - 12z^2 + 3)x^4}{4!} = \frac{(4z^4 - 12z^2 + 3)x^4}{6e^{z^2}} \text{ where } z \text{ is between 0 and } x$$

The figure shows a plot of $f(x)$ and $P_3(x)$ (dashed).

Alternatively, let $f(x) = e^{-x}$ and so $f'(x) = -e^{-x}$; $f''(x) = e^{-x}$. Then $f(0) = 1$, $f'(0) = -1$. Thus,

$P_1(x) = 1 - x$ and $\quad R_1(x) = \frac{e^{-z}}{2!}x^2$ where z is between 0 and x

are the first degree Maclaurin polynomial and error for $f(x)$. Therefore,

$P_1(x^2) = 1 - x^2$

is the second and third degree Maclaurin polynomial for $f(x^2) = e^{-x^2}$ and the error is

$R_1(x^2) = \frac{e^{-z}}{2}x^4$ where z is between 0 and x^2

or, equivalently

$R_1(x^2) = \frac{e^{-z^2}x^4}{2!}$ where z is between 0 and x

9. $f(x) = (1+x)^{3/2}$; $f'(x) = \frac{3}{2}(1+x)^{1/2}$, $f''(x) = \frac{3}{4}(1+x)^{-1/2}$, $f'''(x) = -\frac{3}{8}(1+x)^{-3/2}$, $f^{(4)}(x) = \frac{9}{16}(1+x)^{-5/2}$

$f(0) = 1$, $f'(0) = \frac{3}{2}$, $\frac{f''(0)}{2!} = \frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}$, $\frac{f'''(0)}{3!} = \frac{1}{6}\left(-\frac{3}{8}\right) = -\frac{1}{16}$, $\frac{f^{(4)}(z)}{4!} = \frac{1}{24}\left[\frac{9}{16}(1+z)^{-5/2}\right]$

$P_3(x) = 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3$; $R_3(x) = \frac{3}{128}(1+z)^{-5/2}x^4$ where z is between 0 and x.

10. $f(x) = (1-x)^{-1/2}$; $f'(x) = \frac{1}{2}(1-x)^{-3/2}$; $f''(x) = \frac{3}{4}(1-x)^{-5/2}$; $f'''(x) = \frac{15}{8}(1-x)^{-7/2}$; $f^{(4)}(x) = \frac{105}{16}(1-x)^{-9/2}$

$f(0) = 1$; $f'(0) = \frac{1}{2}$; $\frac{f''(0)}{2!} = \frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}$; $\frac{f'''(0)}{3!} = \frac{1}{6}\cdot\frac{15}{8} = \frac{5}{16}$; $\frac{f^{(4)}(0)}{4!} = \frac{1}{24}\cdot\frac{105}{16} = \frac{35}{128}$; $\frac{f^{(5)}(z)}{5!} = \frac{1}{120}\cdot\frac{945}{32}(1-z)^{-11/2}$

$P_4(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4$; $R_4(x) = \frac{63}{256}(1-z)^{-11/2}x^5$

In Exercises 11–18, find the Taylor polynomial P_n of the stated degree at the number a for the function f with the Lagrange form of the remainder. Plot the graphs of f and P_n and observe how the graph of P_n approximates the graph of f near the point where $x = a$.

11. $a = 4$. $f(x) = x^{3/2}$; $f'(x) = \frac{3}{2}x^{1/2}$, $f''(x) = \frac{3}{4}x^{-1/2}$, $f'''(x) = -\frac{3}{8}x^{-3/2}$, $f^{(4)}(x) = \frac{9}{16}x^{-5/2}$

$f(4) = 8$, $f'(4) = 3$, $\dfrac{f''(4)}{2!} = \frac{1}{2}\cdot\frac{3}{8} = \frac{3}{16}$, $\dfrac{f'''(4)}{3!} = \frac{1}{6}\left(-\frac{3}{64}\right) = -\frac{1}{128}$, $\dfrac{f^{(4)}(z)}{4!} = \frac{1}{24}\left(\frac{9}{16}z^{-5/2}\right)$

$P_3(x) = 8 + 3(x-4) + \frac{3}{16}(x-4)^2 - \frac{1}{128}(x-4)^3$; $R_3(x) = \dfrac{3(x-4)^4}{128z^{5/2}}$ where z is between 4 and x.

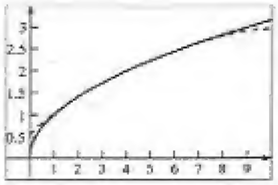
12. $f(x) = \sqrt{x}$; $a = 4$; degree 4

▹ The fourth-degree Taylor polynomial at 4 for the function f is defined by

$$P_4(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3 + \frac{f^{(4)}(4)}{4!}x^4 \qquad (1)$$

and the Lagrange form of the remainder is given by

$$R_4(x) = \frac{f^{(5)}(z)}{5!}(x-4)^5 \quad \text{where } z \text{ is between 4 and } x \qquad (2)$$



We have $f(4) = 2$, and

$f'(x) = \frac{1}{2}x^{-1/2}$ thus, $f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ thus, $f''(4) = -\frac{1}{32}$

$f'''(x) = \frac{3}{8}x^{-5/2}$ thus, $f'''(4) = \frac{3}{256}$

$f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$ thus, $f^{(4)}(4) = -\frac{15}{248}$

Substituting in (1), we obtain

$$P_4(x) = 2 + \tfrac{1}{4}(x-4) - \frac{\frac{1}{32}}{2!}(x-4)^2 + \frac{\frac{3}{256}}{3!}(x-4)^3 - \frac{\frac{15}{2048}}{4!}(x-4)^4$$

$$= 2 + \tfrac{1}{4}(x-4) - \tfrac{1}{64}(x-4)^2 + \tfrac{1}{512}(x-4)^3 - \tfrac{5}{16384}(x-4)^4$$

Because $\quad f^{(5)}(x) = \frac{105}{32}x^{-9/2}$

from (2) we obtain

$$R_4(x) = \frac{\frac{105}{32}z^{-9/2}}{5!}(x-5)^5 = \tfrac{7}{256}z^{-9/2}(x-4)^5, \text{ where } z \text{ is between 4 and } x$$

The figure shows a plot of $f(x)$ and $P_4(x)$ (dashed).

13. $a = \frac{1}{6}\pi$; $f(x) = \sin x$; $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$

$f(\frac{1}{6}\pi) = \frac{1}{2}$, $f'(\frac{1}{6}\pi) = \frac{1}{2}\sqrt{3}$, $\dfrac{f''(\frac{1}{6}\pi)}{2!} = \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{1}{4}$, $\dfrac{f'''(\frac{1}{6}\pi)}{3!} = \frac{1}{6}\left(-\frac{1}{2}\sqrt{3}\right) = -\frac{1}{12}\sqrt{3}$, $\dfrac{f^{(4)}(z)}{4!} = \dfrac{\sin z}{24}$

$P_3(x) = \frac{1}{2} + \frac{1}{2}\sqrt{3}(x - \frac{1}{6}\pi) - \frac{1}{4}(x - \frac{1}{6}\pi)^2 - \frac{1}{12}\sqrt{3}(x - \frac{1}{6}\pi)^3$; $R_3(x) = \frac{\sin z}{24}(x - \frac{1}{6}\pi)^4$ where z is between $\frac{1}{6}\pi$ and x.

14. $a = \frac{1}{3}\pi$; $f(x) = \cos x$; $f'(x) = -\sin x$; $f''(x) = -\cos x$; $f'''(x) = \sin x$; $f^{(4)}(x) = \cos x$; $f^{(5)}(x) = -\sin x$

$f(\frac{1}{3}\pi) = \frac{1}{2}$, $f'(\frac{1}{3}\pi) = -\frac{1}{2}\sqrt{3}$, $\dfrac{f''(\frac{1}{3}\pi)}{2!} = \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{1}{4}$, $\dfrac{f'''(\frac{1}{3}\pi)}{3!} = \frac{1}{6}\left(-\frac{1}{2}\sqrt{3}\right) = -\frac{1}{12}\sqrt{3}$, $\dfrac{f^{(4)}(\frac{1}{3}\pi)}{4!} = \frac{1}{24}\cdot\frac{1}{2} = \frac{1}{48}$

$P_4(x) = \frac{1}{2} - \frac{1}{2}\sqrt{3}(x - \frac{1}{3}\pi) - \frac{1}{4}(x - \frac{1}{3}\pi)^2 - \frac{1}{12}\sqrt{3}(x - \frac{1}{3}\pi)^3 + \frac{1}{48}(x - \frac{1}{3}\pi)^4$; $R_4(x) = -\frac{\cos z}{120}(x - \frac{1}{3}\pi)^5$ $z \in (\frac{1}{3}\pi, x)$

15. $a = 1$; $f(x) = \ln x$; $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -3!x^{-4}$; $f^{(5)}(x) = 4!x^{-5}$

$f(1) = 0$, $f'(1) = 1$, $\dfrac{f''(1)}{2!} = \dfrac{-1}{2}$, $\dfrac{f'''(1)}{3!} = \frac{2}{6} = \frac{1}{3}$; $\dfrac{f^{(4)}(1)}{4!} = \dfrac{-3!}{4!} = -\frac{1}{4}$; $\dfrac{f^{(5)}(1)}{5!} = \dfrac{4!}{5!} = \frac{1}{5}$; $\dfrac{f^{(6)}(z)}{6!} = -\frac{1}{6}z^{-6}$

$P_5(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$; $R_5(x) = -\frac{1}{6}z^{-6}(x-1)^6$, z is between 1 and x.

16. $f(x) = \ln(x+2)$; $a = -1$, degree 3

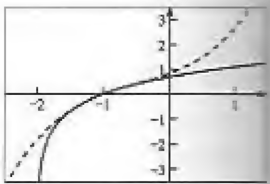
▸ Because $x - a = x + 1$, we must find

$$P_3(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 \qquad (1)$$

and

$$R_3(x) = \frac{f^{(4)}(z)}{4!}(x+1)^4, \text{ where } z \text{ is between } -1 \text{ and } x \qquad (2)$$

We have



$$\begin{aligned} f(x) &= \ln(x+2) & f(-1) &= \ln 1 = 0 \\ f'(x) &= (x+2)^{-1} & f'(-1) &= 1 \\ f''(x) &= -(x+2)^{-2} & f''(-1) &= -1 \\ f'''(x) &= 2(x+2)^{-3} & f''(-1) &= 2 \\ f^{(4)}(x) &= -6(x+2)^{-4} \end{aligned}$$

Substituting into (1) and (2) we obtain

$$P_3(x) = (x+1) - \tfrac{1}{2}(x+1)^2 + \tfrac{1}{3}(x+1)^3 \text{ and } R_3(x) = -\tfrac{1}{4}(z+2)^{-4}(x+1)^4 \text{ where } z \text{ is between } -1 \text{ and } x$$

The figure shows a plot of $f(x)$ and $P_3(x)$ (dashed).

17. $f(x) = \ln \cos x$; $a = \frac{1}{3}\pi$; $n = 3$

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$, $f''(x) = -\sec^2 x$, $f'''(x) = -2\sec^2 x \tan x$, $f^{(4)}(x) = -4\sec^2 x \tan^2 x - 2\sec^4 x$

$f(\frac{1}{3}\pi) = \ln\frac{1}{2} = -\ln 2$, $f'(\frac{1}{3}\pi) = -\sqrt{3}$, $\frac{f''(\frac{1}{3}\pi)}{2!} = \frac{-4}{2} = -2$, $\frac{f'''(\frac{1}{3}\pi)}{3!} = \frac{-8\sqrt{3}}{6} = -\frac{4}{3}\sqrt{3}$

$P_3(x) = -\ln 2 - \sqrt{3}(x - \frac{1}{3}\pi) - 2(x - \frac{1}{3}\pi)^2 - \frac{4}{3}\sqrt{3}(x - \frac{1}{3}\pi)^3$

$R_3(x) = \frac{1}{24}(-4\sec^2 z \tan^2 z - 2\sec^4 z)(x - \frac{1}{3}\pi)^4 = -\frac{1}{12}(2\sec^2 z\tan^2 z + \sec^4 z)(x - \frac{1}{3}\pi)^4$, z is between $\frac{1}{3}\pi$ and x

18. $f(x) = x\sin x$, $f(\frac{1}{6}\pi) = \frac{1}{12}\pi$; $f'(x) = x\cos x + \sin x$, $f'(\frac{1}{6}\pi) = \frac{1}{12}\sqrt{3}\pi + \frac{1}{2}$; $f''(x) = -x\sin x + 2\cos x$,
$f''(\frac{1}{6}\pi) = -\frac{1}{12}\pi + \sqrt{3}$; $f'''(x) = -x\cos x - 3\sin x$, $f'''(\frac{1}{6}\pi) = -\frac{1}{12}\sqrt{3}\pi - \frac{3}{2}$; $f^{(4)}(x) = x\sin x - 4\cos x$,
$f^{(4)}(\frac{1}{6}\pi) = \frac{1}{12}\pi - 2\sqrt{3}$; $f^{(5)}(x) = x\cos x + 5\sin x$, $f^{(5)}(\frac{1}{6}\pi) = \frac{1}{12}\sqrt{3}\pi + \frac{5}{2}$; $f^{(6)}(x) = -x\sin x + 6\cos x$. $P_5(x) =$
$\frac{\pi}{12}\pi + (\frac{\pi}{12}\sqrt{3} + \frac{1}{2})(x - \frac{\pi}{6}) + (-\frac{\pi}{24} + \frac{1}{2}\sqrt{3})(x - \frac{\pi}{6})^2 - (\frac{\pi}{72} + \frac{1}{4})(x - \frac{\pi}{6})^3 + (\frac{\pi}{288} - \frac{1}{12}\sqrt{3})(x - \frac{\pi}{6})^4 + (\frac{\pi}{1440}\sqrt{3} + \frac{1}{48})(x - \frac{\pi}{6})^5$
$R_5(x) = \frac{1}{720}(-z\sin z + 6\cos z)(x - \frac{\pi}{6})^6$ where z is between $\frac{\pi}{6}$ and x.

In Exercises 19–22, compute the function value accurate to five decimal places by using a Taylor polynomial and prove that your answer has the required accuracy. Assume that e is less than 4.

19. $f(x) = e^x$; $a = 1$; $|\text{error}| < 0.000\,005$. For all n, $f^{(n)}(x) = e^x$, $\frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$. $P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$.

$e = e^1 \approx P_9(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320} + \frac{1}{362{,}880} = 2.718282$

with $|\text{error}| = \frac{e^z}{10!}1^{10} < \frac{e^1}{10!} < \frac{4}{10!} = 0.000001$ because e^x is increasing and z is between 0 and 1.

20. $e^{-1/2}$; five decimal places

▸ We require $f(-\frac{1}{2})$ where $f(x) = e^x$ with $|R_n| < 5 \times 10^{-6}$. From equation (8) we have

$$R_n(x) = \frac{e^z}{(n+1)!}x^{n+1} \text{ where } z \text{ is between 0 and } x.$$

If $x = -\frac{1}{2}$, then $-\frac{1}{2} < z < 0$ and $e^z < e^0 = 1$. Thus,

$$\left|R_n(-\tfrac{1}{2})\right| = \frac{e^z}{(n+1)!}(\tfrac{1}{2})^{n+1} < \frac{1}{2^{n+1}(n+1)!}$$

If $n = 6$,

$$\left|R_6(-\tfrac{1}{2})\right| < \frac{1}{2^7(7!)} = 0.00000155$$

From (7) we have

$$P_6(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

and so

$$P_6(-\tfrac{1}{2}) = 1 - \tfrac{1}{2} + \frac{1}{2 \cdot 2^2} - \frac{1}{3! \cdot 2^3} + \frac{1}{4! \cdot 2^4} - \frac{1}{5! \cdot 2^5} + \frac{1}{6! \cdot 2^6} = 0.60653$$

Thus $e^{-1/2} = 0.60653$ with the required accuracy.

21. Using $P_2(x)$ from Exercise 13, with $|\text{error}| < \frac{1}{6} \cdot \left(\frac{\pi}{180}\right)^3 = 8.9 \times 10^{-7}$, we have $\sin 31^\circ$

$= \sin\left(\frac{\pi}{2} + \frac{\pi}{180}\right) \approx P_3\left(\frac{\pi}{2} + \frac{\pi}{180}\right) \approx 0.5000 + \frac{1}{2}\sqrt{3}\left(\frac{\pi}{180}\right) - \frac{1}{4}\left(\frac{\pi}{180}\right)^2 = 0.51504$

22. Using $P_2(x)$ from Exercise 14, with $|\text{error}| < \frac{1}{6}\left(\frac{\pi}{180}\right)^3 = 8.9 \times 10^{-7}$, we have

$\cos 59^\circ = \cos\left(\frac{\pi}{3} - \frac{\pi}{180}\right) = \frac{1}{2} - \frac{1}{2}\sqrt{3}\left(-\frac{\pi}{180}\right) - \frac{1}{4}\left(-\frac{\pi}{180}\right)^2 \approx 0.51504$

23. $f(x) = \cos x$; $a = 0$; $n = 3$; $|x| < 0.1$

$f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$

$f(0) = 1$, $f'(0) = 0$, $\dfrac{f''(0)}{2!} = \dfrac{1}{2}$, $\dfrac{f'''(0)}{3!} = 0$, $\dfrac{f^{(4)}(z)}{4!} = \dfrac{\cos z}{24}$

$P_3(x) = 1 + 0 - \frac{1}{2}x^2 + 0 = 1 - \frac{1}{2}x^2$ and $|\text{error}| = \left|\frac{\cos z}{24}x^4\right| \le \frac{1}{24}(0.1)^4 < 0.000\,005$.

24. Estimate the error that results when $\sin x$ is replaced by $x - \frac{1}{6}x^3$ if $|x| < 0.05$.

▶ We have $a = 0$, $n = 4$; $|x| < 0.05$

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x \end{aligned}$$

Thus,

$$P_4(x) = 0 + 1x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 = x - \tfrac{1}{6}x^3$$

and

$$R_4(x) = \frac{\cos z}{5!}x^5$$

Because $|\cos z| \le 1$ and $|x| \le 0.05$, then

$$|\text{error}| < \frac{1}{120}(0.05)^5 = 2.6 \times 10^{-9}$$

25. $a = 0$, $0 < x < 0.01$. $f(x) = (1 + x)^{1/2}$, $f(0) = 1$; $f'(x) = \frac{1}{2}(1 + x)^{-1/2}$, $f'(0) = \frac{1}{2}$; $f''(x) = -\frac{1}{4}(1 + x)^{-3/2}$

$P_1(x) = 1 + \frac{1}{2}x$. $|\text{error}| = \left|\frac{1}{2!} \cdot \frac{x^2}{4\sqrt{1 + z}}\right| < \frac{1}{2} \cdot \frac{.01^2}{4} = 0.0000125$

26. $a = 1$, $0.99 < x < 1.01$. $f(x) = x^{-1/2}$, $f(1) = 1$; $f'(x) = -\frac{1}{2}x^{-3/2}$, $f'(1) = -\frac{1}{2}$; $f''(x) = \frac{3}{4}x^{-5/2}$

$P_1(x) = 1 - \frac{1}{2}(x - 1) = \frac{3}{2} - \frac{1}{2}x$. $|\text{error}| = \left|\frac{1}{2!} \cdot \frac{3(x - 1)^2}{4z^{5/2}}\right| < \frac{1}{2} \cdot \frac{3(0.01)^2}{4(.99)^{5/2}} = 0.0000385$

27. $a = 0$, $|x| < 0.01$. $f(x) = e^x$. $P_2(x) = 1 + x + \frac{1}{2}x^2$. $|\text{error}| = \left|\frac{e^z}{3!}x^3\right| < \frac{e^{0.01}}{6}(0.01)^3 = 1.68 \times 10^{-7}$

28. Use the Maclaurin polynomial for the function defined by $f(x) = \ln(1 + x)$ to compute the value of ln 1.2, accurate to four decimal places.

▷
$$
\begin{aligned}
f(0) &= \ln 1 = 0 & & \\
f'(x) &= (1+x)^{-1} & f'(0) &= 1 \\
f''(x) &= -(1+x)^{-2} & f''(0) &= -1 \\
f'''(x) &= 2(1+x)^{-3} & f'''(0) &= 2 \\
f^{(4)}(x) &= -3!(1+x)^{4} & f^{(4)}(0) &= -3! \\
f^{(5)}(x) &= 4!(1+x)^{5} & f^{(5)}(0) &= 4!
\end{aligned}
$$

Thus, the fifth-degree Maclaurin polynomial is

$$P_5(x) = 0 + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{3!}{4!}x^4 + \frac{4!}{5!}x^5 = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 \tag{1}$$

Because $f(0.2) = \ln(1 + 0.2) = \ln 1.2$, we replace x by 0.2 in (1). Thus,

$$
\begin{aligned}
P_5(0.2) &= 0.2 - \tfrac{1}{2}(0.2)^2 + \tfrac{1}{3}(0.2)^3 - \tfrac{1}{4}(0.2)^4 + \tfrac{1}{5}(0.2)^5 \\
&= 0.20000 - 0.02000 + 0.002667 - 0.00040 + 0.00006 = 0.18233 \qquad (2)
\end{aligned}
$$

We find $\left|R_5(0.2)\right|$. Because $f^{(6)}(x) = -5!(1+x)^{-6}$, then

$$R_5(x) = -\frac{5!(1+z)^{-6}}{6!}x^6 = -\frac{1}{6(1+z)^6} \text{ where } z \text{ is between 0 and } x$$

Thus,

$$\left|R/0.2)\right| = \frac{(0.2)^6}{6(1+z)^6} \text{ where } 0 < z < 0.2$$

Because $0 < z < 0.2$, then

$$\frac{1}{(1+z)^6} < 1$$

Thus,

$$\left|R_5(0.2)\right| < \frac{(0.2)^6}{6} = 0.000011 = 1.1 \times 10^{-5}$$

Because the error is less than 5×10^{-5} when $P_5(0.2)$ is used to approximate $f(0.2)$, we conclude that (2) gives an approximation for $f(0.2)$ that is accurate to four decimal places. Rounding off the result in (2) we have $\ln 2 = 0.1823$

29. $f(x) = \ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$; $a = 0$; $x = \frac{1}{11}$; $|\text{error}| < 0.00005$

$f'(x) = (1+x)^{-1} + (1-x)^{-1}$, $f''(x) = -(1+x)^{-2} + (1-x)^{-2}$, $f'''(x) = 2(1+x)^{-3} + 2(1-x)^{-3}$,

$f^{(4)}(x) = -6(1+x)^{-4} + 6(1-x)^{-4}$, $f^{(5)}(x) = 24(1+x)^{-5} + 24(1-x)^{-5}$

$f(0) = 0$, $f'(0) = 2$, $\dfrac{f''(0)}{2!} = 0$, $\dfrac{f'''(0)}{3!} = \frac{4}{6} = \frac{2}{3}$, $\dfrac{f^{(4)}(0)}{4!} = 0$, $\dfrac{f^{(5)}(z)}{5!} = \frac{1}{5}[(1+z)^{-5} + (1-z)^{-5}]$

$P_4(x) = 2x + \frac{2}{3}x^3$. $\ln 1.2 = f\left(\frac{1}{11}\right) \approx P_4\left(\frac{1}{11}\right) = 2 \cdot \frac{1}{11} + \frac{2}{3}\left(\frac{1}{11}\right)^3 = 0.182319 \approx 0.1823$

with $|\text{error}| = \frac{1}{5}[(1+z)^{-5} + (1-z)^{-5}]\left(\frac{1}{11}\right)^5 < \left[1 + \dfrac{1}{(10/11)^5}\right]\dfrac{1}{11^5} = \dfrac{1}{11^5} + \dfrac{1}{10^5} = 0.000016$ because $z \in (0, \frac{1}{11})$.

30. From Exercise 24, $\left|R_4(x)\right| = \left|\frac{\cos z}{5!}x^5\right| < \frac{1}{120}\left(\frac{1}{2}\right)^5 = \frac{1}{3840}$

31. From Exercise 30 if $0 \le u \le \frac{1}{2}$, $\left|\sin u - \left(u - \frac{1}{6}u^3\right)\right| < \frac{1}{3840}$. Let $u = x^2$. Therefore

if $0 \le x \le \dfrac{1}{\sqrt{2}}$ then $\left|\sin x^2 - \left(x^2 - \frac{1}{6}x^6\right)\right| < \frac{1}{3840}$ so that

$$\int_0^{1/\sqrt{2}} \sin x^2\,dx \approx \int_0^{1/\sqrt{2}} \left(x^2 - \tfrac{1}{6}x^6\right)dx = \tfrac{1}{3}x^3 - \tfrac{1}{42}x^7\Big]_0^{1/\sqrt{2}} = \frac{1}{3}\cdot\frac{1}{2^{3/2}} - \frac{1}{42}\cdot\frac{1}{2^{7/2}} \approx 0.115747$$

with $|\text{error}| < \displaystyle\int_0^{1/\sqrt{2}} \frac{1}{3840}\,dx = \frac{1}{7680}\sqrt{2} = 0.000184.$

In fact the solution of Exercise 30 shows $\left|\sin u - \left(u - \frac{1}{6}u^3\right)\right| = \frac{|\sin z|}{120}u^5 < \frac{u^5}{120}$ so that

$\left|\sin x^2 - \left(x^2 - \frac{1}{6}x^6\right)\right| < \frac{x^{10}}{120}$ and $|\text{error}| < \int_0^{1/\sqrt{2}} \frac{x^{10}}{120}\,dx = \frac{x^{11}}{1320}\Big]_0^{1/\sqrt{2}} = \frac{1}{1320}\cdot\frac{\sqrt{2}}{64} = 0.000017.$

32. Show that the formula $(1-x)^{3/2} \approx 1 + \frac{3}{2}x$ is accurate to three decimal places if $-0.03 \le x \le 0$.

▶ Let f be the function defined by

$$f(x) = (1+x)^{3/2}$$

We show that $1 + \frac{3}{2}x$ is the first-degree Maclaurin polynomial for the function f. We have $f(0) = 1$.

Because $f'(x) = \frac{3}{2}(1+x)^{1/2}$, then $f'(0) = \frac{3}{2}$. Hence,

$$P_1(x) = f(0) + f'(0)x = 1 + \tfrac{3}{2}x$$

The Lagrange form of the remainder for $P_1(x)$ is given by

$$R_1(x) = \frac{f''(z)}{2!}x^2 \text{ where } z \text{ is between 0 and } x$$

Because $f''(x) = \frac{3}{4}(1+x)^{-1/2}$, then we have

$$R_1(x) = \frac{\frac{3}{4}(1+z)^{-1/2}}{2!}x^2 = \frac{3x^2}{8(1+z)^{1/2}} \text{ where } z \text{ is between 0 and } x \tag{1}$$

Because $-0.03 \le x \le 0$, and $x < z < 0$, then

$$R_1(x) < \frac{3(0.03)^2}{8(0.97)^{1/2}} = 0.00034 = 3.4 \times 10^{-4} \tag{2}$$

By (1) and (2) we have

$$R_1(x) < 5 \times 10^{-4} \text{ if } -0.03 \le x \le 0$$

Thus, the first-degree Maclaurin polynomial $1 + \frac{3}{2}x$ is accurate to three decimal places if used to approximate $(1+x)^{3/2}$ when $-0.03 \le x \le 0$.

33. $f(x) = (1+x)^{-1/2}, f(0) = 1; f'(x) = -\frac{1}{2}(1+x)^{-3/2}, f'(0) = -\frac{1}{2}; f''(x) = \frac{3}{4}(1+x)^{-5/2}, \frac{f''(z)}{2!} = \frac{1}{2}\cdot\frac{3}{4}(1+z)^{-5/2}$

$P_1(x) = 1 - \frac{1}{2}x$ and $R_1(x) = \frac{3}{8}(1+z)^{-5/2}x^2$ where z is between 0 and x.

Because $-0.1 \le x \le 0$, then $|R_1(x)| \le \frac{3}{8}(0.9)^{-5/2}(0.1)^2 = 0.00488 < 0.005$.

Hence the formula $(1+x)^{-1/2} \approx 1 - \frac{1}{2}x$ is accurate to two decimal places if $-0.1 \le x \le 0$.

34. $f(x) = \sin x - mx$, $f(\pi) = -m\pi$; $f'(x) = \cos x - m$, $f'(\pi) = -1 - m$. $P_1(x) = -m\pi - (1+m)(x-\pi) = 0$ when $x - \pi = -\frac{m\pi}{1+m}$, $x = \frac{\pi}{1+m}$

35. Drawing sketches of $y = \cot x$ and $y = mx$ on the same set of axes we note that if m is positive and close to 0 the graphs intersect near the point $(\frac{1}{2}\pi, 0)$.

Let $f(x) = \cot x - mx$; $a = \frac{1}{2}\pi$; $n = 2$. Then $f'(x) = -\csc^2 x - m$ and $f''(x) = 2\csc^2 x \cot x$.

$f(\frac{1}{2}\pi) = -\frac{1}{2}\pi m$, $f'(\frac{1}{2}\pi) = -1 - m$, $f''(\frac{1}{2}\pi)/2! = 0$. $P_2(x) = -\frac{1}{2}\pi - (1+m)(x - \frac{1}{2}\pi)$

To solve $\cot x = mx$ approximately we solve $P_2(x) = 0$. Then

$-\frac{1}{2}\pi m - (1+m)x + \frac{1}{2}\pi + \frac{1}{2}\pi m = 0$; $\frac{1}{2}\pi = (1+m)x$; $x = \frac{\frac{1}{2}\pi}{1+m}$

Using Newton's method with $x_1 = \frac{1}{2}\pi$ we find $x_2 = \frac{\frac{1}{2}\pi}{1+m}$ again.

36. (a) Use the first-degree Maclaurin polynomial to approximate e^k if $0 < k < 0.01$.
(b) Estimate the error in terms of k.

▹ (a) Let $f(x) = e^x$. Because $f'(x) = e^x$, we have $f(0) = 1$ and $f'(0) = 1$. Thus, the first-degree Maclaurin polynomial for the function f is given by

$$P_1(x) = 1 + x$$

Because $f(k) = e^k$ and $f(k) \approx P_1(k)$, then

$$e^k \approx 1 + k$$

(b) The Lagrange form of the remainder for $P_1(x)$ is given by

$$R_1(x) = \frac{f''(z)x^2}{2!} \quad \text{where } z \text{ is between 0 and } x$$

Because $f''(x) = e^x$, then

$$R_1(k) = \frac{e^z k^2}{2} \quad \text{where } z \text{ is between 0 and } k$$

If $z < k$, then $e^z < e^k$, so

$$R_1(k) < \frac{e^k k^2}{2} \tag{1}$$

which estimates the error in terms of k. We are given that $0 < k < 0.01$. Thus,

$$e^k < e^{0.01} = 1.01 \tag{2}$$

and

$$k^2 < 10^{-4} \tag{3}$$

From (1), (2), and (3), we conclude that

$$R_1(k) < \frac{1.01 \times 10^{-4}}{2} = 5.05 \times 10^{-5}$$

Therefore, for all k where $0 < k < 0.01$, the error is less than 5.05×10^{-5} when the Maclaurin polynomial $1 + k$ is used to approximate e^k.

37. $f(x) = P(x) = x^4 - x^3 + 2x^2 - 3x + 1$; $a = 1$; $n = 4$

$f'(x) = 4x^3 - 3x^2 + 4x - 3$, $f''(x) = 12x^2 - 6x + 4$, $f'''(x) = 24x - 6$, $f^{(4)}(x) = 24$, $f^{(5)}(x) = 0$

$f(1) = 0$, $f'(1) = 2$, $\frac{f''(1)}{2!} = \frac{10}{2} = 5$, $\frac{f'''(1)}{3!} = \frac{18}{6} = 3$, $\frac{f^{(4)}(1)}{4!} = \frac{24}{24} = 1$, $\frac{f^{(5)}(z)}{5!} = 0$

$P_4(x) = 2(x-1) + 5(x-1)^2 + 3(x-1)^3 + (x-1)^4 = P(x)$ because $|\text{error}| = 0$.

38. $f(x) = e^x$. $P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$. (a) $D_x P_n(x) = 1 + x + \cdots + \frac{x^{n-1}}{(n-1)!}$

(b) $e^0 + \int_0^x P_n(t)dt = 1 + \left[t + \frac{t^2}{2!} + \cdots + \frac{t^{n+1}}{n+1}\right]_0^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n+1}}{(n+1)^2}$

39. $f(x) = \sin x$. $P_n(x) = x - \frac{x^3}{3!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!}$, $n-1 \le 2k+1 \le n$. (a) $D_x P_n(x) = 1 - \frac{x^2}{2!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!}$

(b) $-\cos 0 + \int_0^x P_n(t)dt = -1 + \left[\frac{t^2}{2!} - \frac{t^4}{4!} + \cdots + (-1)^k \frac{t^{2k+2}}{(2k+2)!}\right]_0^x = -\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^{k+1} \frac{x^{2k+2}}{(2k+2)!}\right)$

40. Prove Theorem 8.1.2 (Integral Form of Remainder)

▹ Let $P(n)$ be the proposition that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$$

$P(0)$ is:

$$f(x) = f(a) + \int_a^x f'(t)dt$$

$P(0)$ is true because it is equivalent to the second fundamental theorem of calculus. Suppose that $P(k)$ is true for some integer $k \ge 0$. Then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)dt \tag{1}$$

Bearing in mind that t is the variable and x is a constant, we can integrate by parts with

$$u = f^{(k+1)}(t) \qquad dv = \frac{(x-t)^k}{k!}$$

$$du = f^{(k+2)}(t) \qquad v = -\frac{(x-t)^{k+1}}{(k+1)!}$$

$$\frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)dt = -f^{(k+1)}(t)\frac{(x-t)^{k+1}}{(k+1)!}\Big]_{t=a}^{x} + \frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)dt$$

$$= \frac{f^{(k+1)}(a)}{(k+1)}(x-a)^{k+1} \tag{2}$$

Substituting from (2) into (1) we find that we have established $P(k+1)$. Hence, by the principal of mathematical induction, $P(n)$ is true for any integer n.

41. In Exercise 2.8.50 we established that differentiation reverses parity. Furthermore, if $f(x)$ is odd, then $f(0) = 0$. (a) Let $f(x)$ be odd. Then so are $f''(x)$, $f^{(4)}(x)$, ... Thus $0 = f(0) = f''(0) = f^{(4)}(0) = \cdots$ and so the even powers of the Maclaurin polynomial all vanish. (b) Let $f(x)$ be even. Then $f'(x)$, $f'''(x)$, ... are odd. Thus $0 = f'(0) = f'''(0) = \cdots$ and so the odd powers all vanish.

8.2 SEQUENCES

8.2.1 Definition A *sequence* $\{a_n\}$ is a function whose domain is the set $\{1, 2, 3, \ldots, n, \ldots\}$ of all positive integers. The numbers a_n are the *elements* of the sequence.

8.2.2 Definition A sequence $\{a_n\}$ has the limit L if for any $\epsilon > 0$ there exists a number N such that if n is an integer and

if $n > N$ then $|a_n - L| < \epsilon$

and we write

$$\lim_{n\to+\infty} a_n = L$$

8.2.3 Theorem If $\lim_{n\to+\infty} f(x) = L$ and f is defined for every positive integer, then also $\lim_{n\to+\infty} f(n) = L$ when n is any positive integer.

Warning Theorem 8.2.3 does NOT say that if $\lim_{x\to+\infty} f(x)$ does not exist then $\lim_{n\to+\infty} f(n)$ does not exist. To see this, consider $a_n = f(n) = \sin n\pi$ and $f(x) = \sin \pi x$.

By Theorem 8.2.3 we may use the limit theorems of Chapter 1 and L'Hôpital's rule to find the limit of a sequence function.

Convergent Divergent If a sequence $\{a_n\}$ has a limit, the sequence is said to be *convergent*, and a_n *converges* to that limit. If the sequence is not convergent, it is *divergent*. Thus, if $\lim_{n\to+\infty} a_n = \pm\infty$ or if $\lim_{n\to+\infty} a_n$ does not exist, the sequence $\{a_n\}$ is divergent.

8.2.4 Theorem If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

(i) the constant sequence $\{c\}$ has c as its limit;

(ii) $\lim_{n\to+\infty} ca_n = c \lim_{n\to+\infty} a_n$;

(iii) $\lim_{n\to+\infty} (a_n \pm b_n) = \lim_{n\to+\infty} a_n \pm \lim_{n\to+\infty} b_n$;

(iv) $\lim_{n\to+\infty} a_n b_n = (\lim_{n\to+\infty} a_n)(\lim_{n\to+\infty} b_n)$

(v) $\lim_{n\to+\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n}$ if $\lim_{n\to+\infty} b_n \neq 0$.

The result of Example 3 is often stated as the following theorem.

Theorem If $|r| < 1$, the sequence $\{r^n\}$ is convergent and $\lim_{n\to+\infty} r^n = 0$.

Sometimes we want to know whether or not a sequence is convergent without actually finding the limit. A monotonic sequence is convergent if and only if it is bounded. Formal statements of the definitions and theorems follow.

8.2.5 Definition A sequence $\{a_n\}$ is said to be

(i) *increasing* if $a_{n+1} \geq a_n$ for all n
(ii) *decreasing* if $a_{n+1} \leq a_n$ for all n

If a sequence is increasing or if it is decreasing, it is called *monotonic*.

8.2.6, 8 Definition The number C is called a *lower bound* of the sequence $\{a_n\}$ if $C \leq a_n$ for all positive integers n, and the number D is called an *upper bound* of the sequence $\{a_n\}$ if $a_n \leq D$ for all positive integers n. A sequence $\{a_n\}$ is said to be *bounded* if and only if it has an upper bound and a lower bound.

8.2.7 Definition If A is a lower bound of a sequence $\{a_n\}$ and if A has the property that for every lower bound C of $\{a_n\}$, $C \leq A$, then A is called the *greatest lower bound* of the sequence. Similarly, if B is an upper bound of a sequence $\{a_n\}$ and if B has the property that for every upper bound D of $\{a_n\}$, $B \leq D$, then B is called the *least upper bound* of the sequence.

8.2.9 Axiom of Completeness Every nonempty set of real numbers that has a lower bound has a greatest lower bound. Also, every nonempty set of real numbers that has an upper bound has a least upper bound.

8.2.10 Theorem A bounded monotonic sequence is convergent.
In fact, an increasing sequence that is bounded above converges (to its least upper bound) and a decreasing sequence that is bounded below converges (to its greatest lower bound).

We may use either Definition 8.2.5 or the following theorem to show that the sequence $\{a_n\}$ is monotonic.

Theorem If f is a function such that $f(n) = a_n$ for each positive integer n, and

(i) if $f'(x) > 0$ for all $x > 0$, then $\{a_n\}$ is decreasing
(ii) if $f'(x) < 0$ for all $x > 0$, then $\{a_n\}$ is decreasing.

If the sequence $\{a_n\}$ is increasing, then $a_n \geq a_1$ for all n. Thus, a_1 is a lower bound for $\{a_n\}$, and $\{a_n\}$ is convergent if and only if $\{a_n\}$ has an upper bound. Similarly, if $\{a_n\}$ is decreasing, then a_1 is an upper bound and $\{a_n\}$ is convergent if and only if $\{a_n\}$ has a lower bound. For arbitrary sequences we have the following theorem.

8.2.13 Theorem A convergent sequence is bounded.

Proof To prove that $\{a_n\}$ is bounded, it must be shown that it has a lower bound and an upper bound. Because $\{a_n\}$ is convergent, the sequence has a limit; call this limit L. Therefore $\lim_{n\to+\infty} a_n = L$, and so by Definition 8.2.2, for $\epsilon = 1$ there exists a number $N > 0$ such that if n is an integer and

$$\begin{aligned} & \text{if } n > N \text{ then } |a_n - L| < 1 \\ \Leftrightarrow\quad & \text{if } n > N \text{ then } -1 < a_n - L < 1 \\ \Leftrightarrow\quad & \text{if } n > N \text{ then } L - 1 < a_n < L + 1 \end{aligned}$$

It follows from this statement that the smallest of the numbers

$$a_1, a_2, \ldots, a_N, L - 1$$

is a lower bound for the sequence, and an upper bound is the largest of the numbers

$$a_1, a_2, \ldots, a_N, L + 1$$

Exercises 8.2

In Exercises 1–20, write the first 4 elements of the sequence and determine if it is convergent. If so, find its limit.

1. $a_n = \dfrac{n+1}{2n-1}$, $a_1 = \dfrac{1+1}{2-1} = 2$, $a_2 = \dfrac{2+1}{4-1} = 1$, $a_3 = \dfrac{3+1}{6-1} = \frac{4}{5}$, $a_4 = \dfrac{4+1}{8-1} = \frac{5}{7}$.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \dfrac{n+1}{2n-1} = \lim\limits_{n\to+\infty} \dfrac{1+\frac{1}{n}}{2-\frac{1}{n}} = \frac{1}{2}$. Therefore $\{a_n\}$ is convergent and its limit is $\frac{1}{2}$.

2. $a_n = \dfrac{2n^2+1}{3n^2-n}$, $a_1 = \dfrac{2+1}{3-1} = \frac{3}{2}$, $a_2 = \dfrac{8+1}{12-2} = \frac{9}{10}$, $a_3 = \dfrac{18+1}{27-3} = \frac{19}{24}$, $a_4 = \dfrac{32+1}{48-4} = \frac{33}{44}$. $\lim\limits_{n\to+\infty} a_n = \dfrac{2+\frac{1}{n^2}}{3-\frac{1}{n}} = \frac{2}{3}$

3. $a_n = \frac{n^2+1}{n}$. $a_1 = \frac{1+1}{1} = 2$, $a_2 = \frac{4+1}{2} = \frac{5}{2}$, $a_3 = \frac{9+1}{3} = \frac{10}{3}$, $a_4 = \frac{16+1}{4} = \frac{17}{4}$.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \frac{n^2+1}{n} = \lim\limits_{n\to+\infty} \left(n + \frac{1}{n}\right) = +\infty$. Therefore $\{a_n\}$ is divergent.

4. $\left\{\frac{3n^3+1}{2n^2+n}\right\}$

▷ Let $a_n = \frac{3n^3+1}{2n^2+n}$

The first four elements of the sequence are found by replacing n by 1, 2, 3, and 4. Thus,

$$a_1 = \frac{3(1^3)+1}{2(1^2)+1} = \frac{4}{3}, \qquad a_2 = \frac{3(2^3)+1}{2(2^2)+2} = \frac{5}{2}, \qquad a_3 = \frac{3(3^3)+1}{2(3^2)+3} = \frac{82}{21}, \qquad a_4 = \frac{3(4^3)+1}{2(4^2)+4} = \frac{193}{36}$$

Using techniques developed for limits of functions, we have

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{3n^3+1}{2n^2+n} = \lim_{n\to+\infty} \frac{3n + \frac{1}{n^2}}{2 + \frac{1}{n}} = +\infty$$

Therefore the sequence $\{a_n\}$ is divergent.

5. $a_n = \frac{3-2n^2}{n^2-1}$. a_1 is not defined, $a_2 = \frac{3-8}{4-1} = -\frac{5}{3}$, $a_3 = \frac{3-18}{9-1} = -\frac{15}{8}$, $a_4 = \frac{3-32}{16-1} = -\frac{29}{15}$.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \frac{3-2n^2}{n^2-1} = \lim\limits_{n\to+\infty} \frac{\frac{3}{n^2}-2}{1-\frac{1}{n^2}} = -2$. Therefore $\{a_n\}$ is convergent and its limit is -2.

6. $a_n = \frac{e^n}{n}$. $a_1 = e \approx 2.7$, $a_2 = \frac{e^2}{2} = 3.7$, $a_3 = \frac{e^3}{3} \approx 6.7$, $a_4 = \frac{e^4}{4} \approx 13.6$. $\lim\limits_{n\to+\infty} \frac{e^n}{n} = \lim\limits_{x\to+\infty} \frac{e^x}{x} = +\infty$. divergent

7. $a_n = \frac{\ln n}{n^2}$. $a_1 = \frac{\ln 1}{1} = 0$, $a_2 = \frac{\ln 2}{4}$, $a_3 = \frac{\ln 3}{9}$, $a_4 = \frac{\ln 4}{16}$. Let $f(x) = \frac{\ln x}{x^2}$. By L'Hôpital's rule

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{\ln x}{x^2} = \lim\limits_{x\to+\infty} \frac{1/x}{2x} = \lim\limits_{x\to+\infty} \frac{1}{2x^2} = 0$

By Theorem 8.2.3 $\lim\limits_{n\to+\infty} f(n) = 0$ when n is a positive integer. Therefore $\{a_n\}$ is convergent and its limit is 0.

8. $\left\{\frac{\log_b n}{n}\right\}$, $b > 1$

▷ Let $a_n = \frac{\log_b n}{n} = \frac{\ln n}{n \ln b}$

The first four elements of the sequence are

$$a_1 = \frac{\ln 1}{1 \ln b} = 0, \qquad a_2 = \frac{\ln 2}{2 \ln b}, \qquad a_3 = \frac{\ln 3}{3 \ln b}, \qquad a_4 = \frac{\ln 4}{4 \ln b}$$

We apply Theorem 8.2.3 to determine whether the sequence is convergent or divergent. Let

$$f(x) = \frac{\ln x}{x \ln b}$$

Then $f(n) = a_n$ for every positive integer n. Furthermore, by L'Hôpital's rule

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln x}{x \ln b} = \frac{1}{\ln b} \lim_{x\to+\infty} \frac{1}{x} = 0$$

Therefore, the sequence $\{a_n\}$ is convergent and has limit 0.

9. $a_n = \tanh n$. $a_1 = \tanh 1 \approx 0.762$, $a_2 = \tanh 2 \approx 0.964$, $a_3 = \tanh 3 \approx 0.995$, $a_4 = \tanh 4 \approx 0.999$.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \tanh n = \lim\limits_{n\to+\infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim\limits_{n\to+\infty} \frac{1-e^{-2n}}{1+e^{-2n}} = 1$. Thus $\{a_n\}$ is convergent and its limit is 1.

10. $a_n = \sinh n$. $a_1 = \sinh 1 \approx 1.2$, $a_2 = \sinh 2 \approx 3.6$, $a_3 = \sinh 3 \approx 10$, $a_4 = \sinh 4 \approx 27$.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \frac{1}{2}(e^n - e^{-n}) = +\infty$. divergent

11. $a_n = \frac{n}{n+1} \sin \frac{n\pi}{2}$. $a_1 = \frac{1}{1+1} \sin \frac{1}{2}\pi = \frac{1}{2}$, $a_2 = \frac{2}{2+1} \sin \pi = 0$, $a_3 = \frac{3}{4} \sin \frac{3}{2}\pi = -\frac{3}{4}$, $a_4 = \frac{4}{5} \sin 2\pi = 0$.

Because $|a^n| \geq \frac{1}{2}$ if n is odd and $a_n = 0$ if n is even, $\lim\limits_{n\to+\infty} a_n$ does not exist. Therefore $\{a_n\}$ is divergent.

12. $\left\{\frac{\sinh n}{\sin n}\right\}$

▷ If $a_n = \frac{\sinh n}{\sin n}$, then the first four elements are

$$a_1 = \frac{\sinh 1}{\sin 1} \approx 1.4, \qquad a_2 = \frac{\sinh 2}{\sin 2} \approx 4.0, \qquad a_3 = \frac{\sinh 3}{\sin 3} \approx 71.5, \qquad a_4 = \frac{\sinh 4}{\sin 4} \approx -36.1$$

Because $\lim\limits_{n\to+\infty} \sin n$ does not exist, and $\lim\limits_{n\to+\infty} \sinh n = +\infty$, we conclude that $\lim\limits_{n\to+\infty} a_n$ does not exist. Hence, the sequence $\{a_n\}$ is divergent.

13. $a_n = \dfrac{1}{\sqrt{n^2+1}-n}$, $a_1 = \dfrac{1}{\sqrt{2}-1}$, $a_2 = \dfrac{1}{\sqrt{5}-2}$, $a_3 = \dfrac{1}{\sqrt{10}-3}$, $a_4 = \dfrac{1}{\sqrt{17}-4}$.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \dfrac{\sqrt{n^2+1}+n}{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)} = \lim\limits_{n\to+\infty} \dfrac{\sqrt{n^2+1}+n}{1} = +\infty$. Therefore $\{a_n\}$ is divergent.

14. $a_n = \sqrt{n+1}-\sqrt{n}$. $a_1 = \sqrt{2}-1 \approx 0.41$, $a_2 = \sqrt{3}-\sqrt{2} \approx 0.32$, $a_3 = 2-\sqrt{3} \approx 0.27$, $a_4 = \sqrt{5}-2 \approx 0.24$

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \dfrac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to+\infty} \dfrac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

15. $a_n = \left(1+\frac{1}{3n}\right)^n$. $a_1 = 1+\frac{1}{3} = \frac{4}{3} \approx 1.333$, $a_2 = \left(1+\frac{1}{6}\right)^2 = \frac{49}{36} \approx 1.361$, $a_3 = \left(1+\frac{1}{9}\right)^3 = \frac{1000}{729} \approx 1.372$,

$a_4 = \left(1+\frac{1}{12}\right)^4 = \frac{28,561}{20,736} \approx 1.377$. $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \left[\left(1+\frac{1}{3n}\right)^{3n}\right]^{1/3} = e^{1/3}$. $\{a_n\}$ is converges to $e^{1/3} \approx 1.396$.

16. $\left\{\left(1+\frac{2}{n}\right)^n\right\}$

▷ Let $a_n = \left(1+\frac{2}{n}\right)^n$. Then the first four elements are

$$a_1 = 3, \qquad a_2 = 2^2 = 4, \qquad a_3 = \left(\tfrac{5}{3}\right)^3 = \tfrac{125}{27} \approx 4.63, \qquad a_4 = \left(\tfrac{3}{2}\right)^4 = \tfrac{81}{16} \approx 5.06$$

If $f(x) = \left(1+\frac{2}{x}\right)^x$ then

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^x = \left[\lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^{x/2}\right]^2 = e^2 \approx 7.39$$

Thus the sequence is convergent and has limit e^2.

17. $a_n = 2^{1/n}$. $a_1 = 2$, $a_2 = 2^{1/2} \approx 1.41$, $a_3 = 2^{1/3} \approx 1.26$, $a_4 = 2^{1/4} \approx 1.19$. $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} e^{\ln 2/n} = e^0 = 1$

18. $a_n = \left(\frac{1}{2}\right)^{1/n}$. $a_1 = \frac{1}{2} = 0.50$, $a_2 = \left(\frac{1}{2}\right)^{1/2} \approx 0.71$, $a_3 = \left(\frac{1}{2}\right)^{1/3} \approx 0.79$, $a_4 = \left(\frac{1}{2}\right)^{1/4} \approx 0.84$

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} e^{-\ln 2/n} = e^0 = 1$

19. $a_n = \frac{n}{2^n}$. $a_1 = \frac{1}{2} = 0.50$, $a_2 = \frac{2}{4} = 0.50$, $a_3 = \frac{3}{8} \approx 0.38$, $a_4 = \frac{4}{16} = 0.25$. Let $f(x) = \frac{x}{2^x}$. By L'Hôpital's rule

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{x}{2^x} = \lim\limits_{x\to+\infty} \dfrac{1}{2^x \ln 2} = 0$. By Th. 8.2.3 $\lim\limits_{n\to+\infty} a_n = 0$. Thus $\{a_n\}$ converges to 0.

20. $\{\cos n\pi\}$

▷ Let $a_n = \cos n\pi$. Then the first four elements are

$$a_1 = \cos\pi = -1, \qquad a_2 = \cos 2\pi = 1, \qquad a_3 = \cos 3\pi = -1, \qquad a_4 = \cos 4\pi = 1$$

Because $\cos n\pi = -1$ if n is an odd integer and $\cos n\pi = 1$ if n is an even integer, $\lim\limits_{n\to+\infty} a_n$ does not exist. Thus the sequence $\{a_n\}$ is divergent.

In Exercises 21–24, estimate graphically the limit of the convergent sequence. Confirm your estimate analytically.

21. (a) $\lim\limits_{n\to+\infty} \dfrac{3}{n+1} = 0$ (b) $\lim\limits_{n\to+\infty} \dfrac{8n}{2n+3} = \lim\limits_{n\to+\infty} \dfrac{8}{2+(3/n)} = \dfrac{8}{2} = 4$

22. (a) $\lim\limits_{n\to+\infty} \dfrac{4}{2n-1} = 0$ (b) $\lim\limits_{n\to+\infty} \dfrac{1-7n}{2n+5} = \lim\limits_{n\to+\infty} \dfrac{(1/n)-7}{2+(5/n)} = -\dfrac{7}{2}$

23. (a) $\lim\limits_{n\to+\infty} \dfrac{3n^2}{6n^2+1} = \lim\limits_{n\to+\infty} \dfrac{3}{6+(1/n^2)} = \dfrac{3}{6} = \dfrac{1}{2}$ (b) $\lim\limits_{n\to+\infty} \dfrac{n^2-1}{2+n^2} = \lim\limits_{n\to+\infty} \dfrac{1-(1/n^2)}{(2/n^2)+1} = \dfrac{1}{1} = 1$

24. (a) $\left\{\frac{9-2n}{3+n}\right\}$; (b) $\left\{\frac{6n}{n^2+4}\right\}$

▸ (a) The limit appears to be -2. Analytically, $\lim\limits_{n\to+\infty}\frac{9-2n}{3+n}=\lim\limits_{n\to+\infty}\frac{\frac{9}{n}-2}{\frac{3}{n}+1}=\frac{-2}{1}=-2$

(b) The limit appears to be 0. Analytically, $\lim\limits_{n\to+\infty}\frac{6n}{n^2+4}=\lim\limits_{n\to+\infty}\frac{\frac{6}{n}}{1+\frac{4}{n^2}}=\frac{0}{1}=0$

25. $\{a_n\}=\left\{\frac{n^2}{n-3}\right\}$ and $\{b_n\}=\left\{\frac{n^2}{n+4}\right\}$ are both divergent but $\{c_n\}=\left\{\frac{n^2}{n-3}-\frac{n^2}{n+4}\right\}$ is convergent (with limit 7) because $\lim\limits_{n\to+\infty}a_n=\lim\limits_{n\to+\infty}\frac{n}{1-(3/n)}=+\infty$, $\lim\limits_{n\to+\infty}b_n=\lim\limits_{n\to+\infty}\frac{n}{1+(4/n)}=+\infty$, but

$$\lim_{n\to+\infty}c_n=\lim_{n\to+\infty}\frac{n^2(n+4)-n^2(n-3)}{(n-3)(n-4)}=\lim_{n\to+\infty}\frac{7n^2}{(n-3)(n-4)}=\lim_{n\to+\infty}\frac{7}{(1-3/n)(1-4/n)}=7$$

26. $a_n=\frac{n}{c^n}$. If $|c|>1$, let $f(x)=\frac{x}{|c|^x}$. By L'Hôpital's rule $\lim\limits_{x\to+\infty}f(x)=\lim\limits_{x\to+\infty}\frac{x}{|c|^x}=\lim\limits_{x\to+\infty}\frac{1}{|c|^x\ln|c|}=0$. By Th. 8.2.3 and Exercise 60, $\lim\limits_{n\to+\infty}a_n=0$. Thus $\{a_n\}$ converges to 0. If $0<|c|\le 1$, then $|a_n|\ge n$. Thus $\{a_n\}$ is unbounded and hence divergent.

In Exercises 27–42, determine if the sequence is increasing, decreasing, or not monotonic.

27. $a_n=\frac{3n-1}{4n+5}=\frac{3}{4}-\frac{\frac{11}{4}}{4n+5}$ by long division. Because $\frac{\frac{11}{4}}{4n+5}$ is decreasing, a_n is increasing.

28. $\left\{\frac{2n-1}{4n-1}\right\}$

▸ Let $a_n=\frac{2n-1}{4n-1}$. By long division, we find $a_n=\frac{1}{2}-\frac{1}{2(4n-1)}$. Because $\frac{1}{2(4n-1)}$ is decreasing, then $\{a_n\}$ is increasing.

29. $a_n=\frac{1-2n^2}{n^2}=\frac{1}{n^2}-2$. Because $\frac{1}{n^2}$ is decreasing, a_n is decreasing.

30. $a_n=\sin n\pi$. Because $\sin n\pi=0$ for every integer n, then $\{a_n\}$ is a constant sequence which is both increasing and decreasing, and hence monotonic.

31. $a_n=\cos\frac{1}{3}n\pi$. $a_1=\cos\frac{1}{3}\pi=\frac{1}{2}$, $a_2=\cos\frac{2}{3}\pi=-\frac{1}{2}$, $a_3=\cos\pi=-1$, $a_4=\cos\frac{4}{3}\pi=-\frac{1}{2}$. Thus $a_2>a_3$ and $a_3<a_4$. For some values of n, $a_n>a_{n+1}$, and for some values of n, $a_n<a_{n+1}$. Thus the sequence is not monotonic.

32. $\left\{\frac{n^3-1}{n}\right\}$

▸ $\frac{n^3-1}{n}=n^2-\frac{1}{n}$. Because n^2 is increasing and $\frac{1}{n}$ is decreasing, the given sequence is increasing.

33. $a_n=\frac{1}{n+\sin n^2}$. $a_1=\frac{1}{1+\sin 1}\approx 0.543$, $a_2=\frac{1}{2+\sin 4}\approx 0.804$, $a_3=\frac{1}{3+\sin 9}\approx 0.293$. Thus $a_1<a_2$ and $a_2>a_3$. For some values of n, $a_n>a_{n+1}$, and for some values of n, $a_n<a_{n+1}$. The sequence is not monotonic.

34. $a_n=\frac{2^n}{1+2^n}=\frac{1}{2^{-n}+1}$. Because $2^{-n}+1$ is decreasing, a_n is increasing.

35. $a_n=\frac{5^n}{1+5^{2n}}=\frac{1}{5^n+5^{-n}}$. Let $f(x)=\frac{1}{5^x+5^{-x}}$. Then $f'(x)=-\frac{\ln 5(5^x-5^{-x})}{(5^x+5^{-x})^2}<0$ for $x>0$. Hence $f(x)$ is decreasing for $x>0$ so that a_n is decreasing for positive integers n.

36. $\left\{\frac{(2n)!}{5^n}\right\}$

▸ Let $a_n=\frac{(2n)!}{5^n}$. We have $a_1=\frac{2}{5}$, $a_2=\frac{4!}{5^2}=\frac{24}{25}$, $a_3=\frac{6!}{5^3}=\frac{144}{25}$. Because $a_1<a_2<a_3$, the sequence may be increasing. Because

$$\frac{a_{n+1}}{a_n}=\frac{(2n+2)!}{5^{n+1}}\cdot\frac{5^n}{(2n)!}=\frac{(2n+2)(2n+1)}{5}>1$$

when $n\ge 1$, we conclude that the sequence $\{a_n\}$ is increasing.

37. $a_n = \frac{n!}{3^n}$. $\frac{a_{n+1}}{a^n} = \frac{(n+1)!}{3^{n+1}} / \frac{n!}{3^n} = \frac{(n+1)n!}{3^{n+1}} \cdot \frac{3^n}{n!} = \frac{n+1}{3} > 1$ if $n > 2$.
Hence the sequence is increasing after the first two terms.

38. $a_n = \frac{n}{2^n}$. $\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \le 1$ if $n+1 \le 2n$, $n \ge 1$. Hence a_n is decreasing.

39. $a_n = \frac{n^n}{n!}$. $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} / \frac{n^n}{n!} = \frac{(n+1)^n(n+1)}{(n+1)n!} \cdot \frac{n!}{n^n} = \left(\frac{n+1}{n}\right) > 1$. Hence a_n is increasing.

40. $\{n^2 + (-1)^n n\}$

▸ Let $a_n = n^2 + (-1)^n n$. We have $a_1 = 1 - 1 = 0$, $a_2 = 4 + 2 = 6$, $a_3 = 9 - 3 = 6$, $a_4 = 16 + 4 = 20$, $a_5 = 25 - 5 = 20$. Because $a_1 < a_2 \le a_3 < a_4 \le a_5$, the sequence may be increasing.

$$a_{n+1} - a_n = [(n+1)^2 + (-1)^{n+1}(n+1)] - [n^2 + (-1)^n n] = 2n + 1 + (-1)^{n+1}(2n+1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 4n+2 & \text{if } n \text{ is odd} \end{cases}$$

Hence $a_{n+1} - a_n \ge 0$ and so the sequence $\{a_n\}$ is increasing.

41. $a_n = \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$. $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} / \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{n+1}{2n+1} < 1$
Therefore, the sequence is decreasing.

42. $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n n!}$. $\frac{a_{n+1}}{a_n} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)(2n+1)}{2^{n+1}(n+1)!} \cdot \frac{2^n n!}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} = \frac{2n+1}{2(n+1)} < 1$
Therefore, the sequence is decreasing.

In Exercises 43 and 44, determine if the sequence is bounded.

43. $a_n = \frac{n^2+3}{n+1}$. Because $\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} \frac{n^2+3}{n+1} = \lim_{n \to +\infty} \frac{n + (3/n)}{1 + (1/n)} = +\infty$ the sequence is not bounded.

44. $\{3 - (-1)^{n-1}\}$

▸ Let $a_n = 3 - (-1)^{n-1}$. If n is an odd integer, then $n - 1$ is even, so $(-1)^{n-1} = 1$ and $a_n = 2$. If n is an even integer, then $n - 1$ is odd, so $(-1)^{n-1} = -1$ and $a_n = 4$. Thus, $2 \le a_n \le 4$ for all positive integers n, and thus the sequence $\{a_n\}$ is bounded.

In Exercises 45–54, prove that the sequence is convergent by using Theorem 8.2.10.

45. In Exercise 27 the sequence $\{a_n\} = \left\{\frac{3n-1}{4n+5}\right\}$ was shown to be increasing. Furthermore $0 < a_n < \frac{3}{4}$ for all positive integers n. Thus the sequence is bounded and monotonic, and so by Theorem 8.2.10 it is convergent.

46. $a_n = \frac{n}{3^{n+1}}$. $\frac{a_{n+1}}{a_n} = \frac{n+1}{3^{n+2}} \cdot \frac{3^{n+1}}{n} = \frac{n+1}{3n} < \frac{2}{3}$ for $n \ge 1$. a_n is decreasing and $a_n > 0$. Convergent by Th. 8.2.10.

47. $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}$. $\frac{a_{n+1}}{a_n} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)(2n+2)} / \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} = \frac{2n+1}{2n+2} < 1$.
Hence the sequence is decreasing. Furthermore $0 < a_n < 1$ for all positive integers n.
Thus the sequence is bounded and monotonic, and so by Theorem 8.2.10 it is convergent.

48. The sequence of Exercise 34.

▸ The sequence $\{a_n\}$ is defined by $a_n = \frac{2^n}{1+2^n}$

In Exercise 34 we showed that $\{a_n\}$ is increasing. Thus, $\{a_n\}$ is monotonic. We show that $\{a_n\}$ is bounded. Because $a_n > 0$, 0 is a lower bound for $\{a_n\}$. Because $2^n < 1 + 2^n$, then

$$\frac{2^n}{1+2^n} < \frac{1+2^n}{1+2^n} = 1$$

Thus, $a_n < 1$, and hence, 1 is an upper bound for $\{a_n\}$. Because $\{a_n\}$ is monotonic and bounded, by Theorem 8.2.10 it is convergent.

49. In Exercise 35 the sequence $\{a_n\} = \left\{\frac{5^n}{1+5^{2n}}\right\}$ was shown to be decreasing; and $0 < a_n < 1$.
Thus the sequence is bounded and monotonic, and so by Theorem 8.2.10 it is convergent.

50. $a_n = \frac{n}{2^n}$. In Exercise 38 the sequence $\{a_n\}$ was shown to be decreasing, and $a_n > 0$. It converges by Th 8.2.10.

51. In Exercise 15 $\{a_n\} = \left\{\frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}\right\}$ was shown to be decreasing; and $0 < a_n < 1$.

Thus the sequence is bounded and monotonic, and so by Theorem 8.2.10 it is convergent.

52. The sequence of Exercise 42.

▹ The sequence $\{a_n\}$ is defined by

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n \cdot n!}$$

In Exercise 42 we showed that $\{a_n\}$ is decreasing. Thus, $a_n \le a_1$ for all n, and hence a_1 is an upper bound for $\{a_n\}$. Because $a_n > 0$ for all n, then 0 is a lower bound for $\{a_n\}$. Because $\{a_n\}$ is bounded and monotonic, by Theorem 8.2.10 $\{a_n\}$ is convergent.

53. $a_n = \frac{n^2}{2^n}$. $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 < 1$ when $1 + \frac{1}{n} < \sqrt{2} \approx 1.4$; $n \ge 3$. Therefore, after the first three terms, the sequence is decreasing. Further, $0 < a_n < a_3$. Thus the sequence is bounded and monotonic, and so by Theorem 8.2.10 it is convergent.

54. $k > 1$, $a_n = k^{1/n} = e^{\ln k/n}$. Because $\ln k > 0$, then $\ln k/n$ is decreasing, and so a_n is decreasing. Also, $a_n > 0$.

55. The sequence $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots$ is clearly bounded (by 0 and 1), convergent (to 0) but not monotonic.

56. Prove Theorem 8.2.3.

▹ We are given

$$\lim_{x \to +\infty} f(x) = L$$

which is equivalent to

For every $\epsilon > 0$ there is a number N such that for all real numbers x, if $x > N$ then $|f(x) - L| < \epsilon$

which implies

For every $\epsilon > 0$ there is a number N such that for all positive integers n, if $n > N$ then $|f(n) - L| < \epsilon$

which is equivalent to

$$\lim_{n \to +\infty} f(n) = L$$

57. Let $y = 1 - x^{-1}$. Then $\lim\limits_{n \to +\infty} \frac{1-(1-n^{-1})^a}{1-(1-n^{-1})^b} = \lim\limits_{x \to +\infty} \frac{1-(1-x^{-1})^a}{1-(1-x^{-1})^b} = \lim\limits_{y \to 1^-} \frac{1-y^a}{1-y^b} \overset{0/0}{=} \lim\limits_{y \to 1^-} \frac{-ay^{a-1}}{-by^{b-1}} = \frac{a}{b}$

58. Suppose $\lim\limits_{n \to +\infty} a_n = L$ and $\lim\limits_{n \to +\infty} a_n = M$ and $|M - L| = \epsilon > 0$. For some N, if $n > N$, then $|a_n - L| < \frac{1}{2}\epsilon$ and $|a_n - M| < \frac{1}{2}\epsilon$. By the triangle inequality, $|L - M| = |(a_n - L) - (a_n - M)| \le |a_n - L| + |a_n - M| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ which contradicts the hypothesis.

59. If $r = 0$, $\{nr^n\} = \{0\}$ and $\lim\limits_{n \to +\infty} 0 = 0$. Now suppose $0 < |r| < 1$, and let $f(x) = xr^x = \frac{x}{r^{-x}}$.

Because $\lim\limits_{x \to +\infty} x = +\infty$ and $\lim\limits_{x \to +\infty} r^{-x} = \lim\limits_{x \to +\infty} \left(\frac{1}{r}\right) = +\infty$, we apply L'Hôpital's rule, and we have

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x}{r^{-x}} = \lim_{x \to +\infty} \frac{1}{(-\ln r)r^{-x}} = \lim_{x \to +\infty} \frac{r^x}{-\ln r} = 0$$

$\lim\limits_{n \to +\infty} nr^n = 0$ and so the sequence is convergent and its limit is 0.

60. Prove that if the sequence $\{a_n\}$ is convergent and $\lim\limits_{n \to +\infty} a_n = L$, then the sequence $\{|a_n|\}$ is also convergent and $\lim\limits_{n \to +\infty} |a_n| = |L|$.

▹ By the triangle inequality,

$$|a_n - L| \ge |a_n| - |L|$$

and

$$|a_n - L| = |L - a_n| \ge |L| - |a_n|$$

Therefore

$$|a_n - L| \ge \big||a_n| - |L|\big| \tag{1}$$

Because $\lim\limits_{n \to +\infty} a_n = L$, by Definition 8.2.2 for any $\epsilon > 0$ there is a number N such that if n is an integer and

if $n > N$ then $|a_n - L| < \epsilon$

By (1) we have

$$\text{if } n > N \text{ then } \big||a_n| - |L|\big|$$

Hence, by Definition 8.2.2, $\lim_{n\to+\infty} |a_n| = |L|$

61. To prove $\lim_{n\to+\infty} a_n{}^2 = L^2$ we must show that for any $\epsilon > 0$ there is a number $N > 0$ such that
$|a_n{}^2 - L^2| = |a_n - L|\,|a_n + L| < \epsilon$ for every integer $n > N$ (1)
Because $\lim_{n\to+\infty} a^n = L$, then there is a number N_1 such that for every integer $n > N_1$
$|a_n - L| < 1$; $-1 < a_n - L < 1$; $2L - 1 < a_n + L < 2L + 1$; $|a_n + L| < 2|L| + 1$
There is also a number N_2 such that $|a^n - L| < \dfrac{\epsilon}{2|L|+1}$ for every integer $n > N_2$. Hence
$|a_n{}^2 - L^2| = |a_n - L|\,|a_n + L| < \dfrac{\epsilon}{2|L|+1} \cdot (2|L|+1) = \epsilon$ for every integer $n > N = \max(N_1, N_2)$.

62. Because $0 < k \le 1$, then $a_{n+1} < ka_n \le a_n$. Also $a_n > 0$. Hence a_n is decreasing and bounded below.

8.3 INFINITE SERIES OF CONSTANT TERMS

Let $\{u_n\}$ be a sequence and define $\{s_n\}$, the *sequence of partial sums*, as follows:

$s_1 = u_1$
$s_2 = u_1 + u_2$
$s_3 = u_1 + u_2 + u_3$
and so on, where
$s_n = u_1 + u_2 + \cdots + u_n$

The sequence $\{s_n\}$ is called an *infinite series*. If $\{s_n\}$ converges, then we define

$$\sum_{n=1}^{+\infty} u_n = \lim_{n\to+\infty} s_n$$

and we say that the infinite series is *convergent*. The formal definitions are as follows.

8.3.1 Definition If $\{u_n\}$ is a sequence and

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n$$

then the sequence $\{s_n\}$ is called an *infinite series*. This infinite series is denoted by

$$\sum_{n=1}^{+\infty} u_n = u_1 + u_2 + \cdots + u_n + \cdots$$

The numbers $u_1, u_2, u_3, \ldots, u_n, \ldots$ are called the *terms* of the infinite series. The numbers $s_1, s_2, s_3, \ldots, s_n, \ldots$ are called the *partial sums* of the infinite series.

8.3.2 Definition Let $\sum_{n=1}^{+\infty} u_n$ be a given infinite series, and let $\{s_n\}$ be the sequence of partial sums defining this infinite series. If $\lim_{n\to+\infty} s_n$ exists and is equal to S, we say that the given series is *convergent* and that S is the *sum* of the given infinite series. If $\lim_{n\to+\infty} s_n$ does not exist, the series is said to be *divergent* and the series does not have a sum.

We will not usually be able to use Definition 8.3.2 to tell whether or not an infinite series is convergent. The following theorem allows us to prove that an infinite series is divergent.

8.3.3 Theorem If the infinite series $\sum_{n=1}^{+\infty} u_n$ is convergent, then $\lim_{n\to+\infty} u_n = 0$.

From Theorem 8.3.3 it follows that if $\lim_{n\to+\infty} u_n \neq 0$, then the series $\sum_{n=1}^{+\infty} u_n$ is divergent. The converse of Theorem 8.3.3 is false. That is, if $\lim_{n\to+\infty} u_n = 0$, it does not follow that $\sum_{n=1}^{+\infty} u_n$ is convergent. For example, the

Harmonic Series $\sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ is divergent.

Euler's Constant $\lim_{n\to+\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}-\ln n\right)$ converges. The limit is $\gamma \approx 0.5772157$.

Proof Because $\frac{1}{n} < \int_{n-1}^{n}\frac{1}{x}dx < \frac{1}{n-1}$ (1)

by adding we obtain

$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < \int_{1}^{n}\frac{1}{x}dx = \ln n < 1+\frac{1}{2}+\cdots+\frac{1}{n+1}$

and so $\frac{1}{n} < 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}-\ln n < 1$. Thus the sequence $s_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}-\ln n$ is bounded by 0 and 1. Consider $s_n - s_{n-1} = \frac{1}{n}-\ln n+\ln(n-1)$. From (1) we also get $\frac{1}{n} < \ln n-\ln(n-1) < \frac{1}{n-1}$ and so $\frac{1}{n}-\frac{1}{n-1} < \frac{1}{n}-\ln n+\ln(n-1) < 0$ so that s_n is decreasing. Thus s_n converges.

It follows that $1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} \approx \ln n+\gamma$. A better approximation is $\ln n+\gamma+\frac{1}{2n}$.

Geometric Series If the ratio of successive terms in a series is constant, the series is called a *geometric series*. That is, a geometric series has the form

$\sum_{n=1}^{+\infty} ar^{n-1} = a+ar+ar^2+\cdots+ar^{n-1}+\cdots$

where a and r are any constants. The nth partial sum is given by $s_n = a\frac{1-r^n}{1-r}$. We have the following theorem.

8.3.5 Theorem The geometric series converges to the sum $a/(1-r)$ if $|r|<1$, and diverges if $|r|\geq 1$. Thus

$\sum_{n=1}^{+\infty} ar^{n-1} = \frac{a}{1-r}$ if $|r|<1$

Telescoping Series $(a_0-a_1)+(a_1-a_2)+\cdots+(a_{n-2}-a_{n-1})+(a_{n-1}-a_n) = a_0-a_n$

When testing an infinite series to determine whether or not it is convergent, we may disregard the first N terms, where N is any particular positive integer. furthermore, it does not affect the convergence or divergence of a series if we multiply each term by the same nonzero constant. If we add or subtract the corresponding terms of two convergent series, the resulting series is also convergent. If we add the terms of a convergent series to the corresponding terms of a divergent series, the resulting series is divergent. However, if we add the corresponding terms of two divergent series, the resulting series may be divergent or it may be convergent. These results are stated formally in the following theorems.

8.3.9 Theorem If $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ are two infinite series differing only in their first m terms (i.e. $a_k = b_k$ if $k > m$), then either both series converge or both series diverge.

8.3.6 Theorem Let c be any nonzero constant.

(i) If the series $\sum_{n=1}^{+\infty} u_n$ is convergent and its sum is S, then the series $\sum_{n=1}^{+\infty} cu_n$ is also convergent and its sum is cS.

(ii) If the series $\sum_{n=1}^{+\infty} u_n$ is divergent, then the series $\sum_{n=1}^{+\infty} cu_n$ is also divergent.

8.3.7 Theorem If $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ are convergent infinite series whose sums are R and S respectively, then

(i) $\sum_{n=1}^{+\infty}(a_n+b_n)$ is a convergent series and its sum is S + R.

(ii) $\sum_{n=1}^{+\infty}(a_n-b_n)$ is a convergent series and its sum is S − R.

8.3.8 Theorem If the series $\sum_{n=1}^{+\infty} a_n$ is convergent and the series $\sum_{n=1}^{+\infty} b_n$ is divergent, then the series $\sum_{n=1}^{+\infty}(a_n+b_n)$ is divergent.

As we progress through this chapter, we will develop techniques for determining whether or not a series is convergent. In this section we have the following methods.

To show that a series is *convergent*, either

1. find the partial sum s_n, and show $\lim\limits_{n \to +\infty} s_n$ exists
2. show that the series is a geometric series with $|r| < 1$
3. show that the series is a multiple of a convergent series or the sum of two convergent series

To show that a series is *divergent*, either

1. find the partial sum s_n, and show that $\lim\limits_{n \to +\infty} s_n$ does not exist
2. show that $\lim\limits_{n \to +\infty} u_n \neq 0$
3. show that the series is a geometric series with $|r| \geq 1$
4. show that the series is a multiple of a divergent series or is the sum of a convergent and divergent series.

Exercises 8.3

In Exercises 1–8, find the first four elements of the sequence of partial sums $\{s_n\}$ and find a formula for s_n in terms of n. Also, determine if the infinite series is convergent or divergent, and if it is convergent, find its sum.

1. $\sum\limits_{n=1}^{+\infty} \frac{1}{(2n-1)(2n+1)} = \sum\limits_{n=1}^{+\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$. $s_1 = \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{3}$

$s_2 = \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)\right] = \frac{1}{2}\left(1 - \frac{1}{5}\right) = \frac{2}{5}$; $s_3 = \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right)\right] = \frac{1}{2}\left(1 - \frac{1}{7}\right) = \frac{3}{7}$

$s_4 = \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right)\right] = \frac{1}{2}\left(1 - \frac{1}{9}\right) = \frac{4}{9}$.

$s_n = \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right] = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}$

Because $\lim\limits_{n \to +\infty} s_n = \lim\limits_{n \to +\infty} \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2} \cdot 1 = \frac{1}{2}$, the series is convergent and its sum is $\frac{1}{2}$.

2. $\sum\limits_{n=1}^{+\infty} n$. $s_1 = 1$, $s_2 = 1 + 2 = 3$, $s_3 = 1 + 2 + 3 = 6$, $s_4 = 1 + 2 + 3 + 4 = 10$. $s_n = \frac{1}{2}n(n+1)$.

Because $\lim\limits_{n \to +\infty} s_n = +\infty$, the series is divergent.

3. $\sum\limits_{n=1}^{+\infty} \frac{5}{(3n+1)(3n-2)} = \sum\limits_{n=1}^{+\infty} \frac{5}{3}\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)$. $s_1 = \frac{5}{3}\left(1 - \frac{1}{4}\right) = \frac{5}{4}$.

$s_2 = \frac{5}{3}\left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right)\right] = \frac{5}{3}\left(1 - \frac{1}{7}\right) = \frac{10}{7}$; $s_3 = \frac{5}{3}\left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right)\right] = \frac{5}{3}\left(1 - \frac{1}{10}\right) = \frac{3}{2}$

$s_4 = \frac{5}{3}\left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{13}\right)\right] = \frac{5}{3}\left(1 - \frac{1}{13}\right) = \frac{20}{13}$.

$s_n = \frac{5}{3}\left[\left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{3n-5} - \frac{1}{3n-2}\right) + \left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)\right] = \frac{5}{3}\left(1 - \frac{1}{3n+1}\right) = \frac{5n}{3n+1}$

Because $\lim\limits_{n \to +\infty} s_n = \lim\limits_{n \to +\infty} \frac{5}{3}\left(1 - \frac{1}{3n+1}\right) = \frac{5}{3} \cdot 1 = \frac{5}{3}$, the series is convergent and its sum is $\frac{5}{3}$.

4. $\sum\limits_{n=1}^{+\infty} \frac{2}{(4n-3)(4n+1)}$

▹ We have $u_n = \frac{2}{(4n-3)(4n+1)}$ (1)

Thus, by making natural number replacements for n in (1), we have

$u_1 = \frac{2}{5} \quad u_2 = \frac{2}{45} \quad u_3 = \frac{2}{117} \quad u_4 = \frac{2}{221}$

from which we obtain the first four elements of the sequence of partial sums

$s_1 = u_1 = \frac{2}{5}$ $\qquad$ $s_2 = u_1 + u_2 = \frac{2}{5} + \frac{2}{45} = \frac{4}{9}$

$s_3 = u_1 + u_2 + u_3 = s_2 + u_3 = \frac{4}{9} + \frac{2}{117} = \frac{6}{13}$ $\qquad$ $s_4 = u_1 + u_2 + u_3 + u_4 = s_3 + u_4 = \frac{6}{13} + \frac{2}{221} = \frac{8}{17}$

To find a formula for s_n, we use partial fractions to get a telescoping series. Let

$u_n = \frac{2}{(4n-3)(4n+1)} = \frac{A}{4n-3} + \frac{B}{4n+1}$

We find $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Thus $u_n = \frac{\frac{1}{2}}{4n-3} - \frac{\frac{1}{2}}{4n+1}$

and so

$$s_n = u_1 + u_2 + \cdots + u_n = \left(\frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}}{5}\right) + \left(\frac{\frac{1}{2}}{5} - \frac{\frac{1}{2}}{9}\right) + \cdots + \left(\frac{\frac{1}{2}}{4n-3} - \frac{\frac{1}{2}}{4n+1}\right) = \frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}}{4n+1} = \frac{2n}{4n+1}$$

Because

$$\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} \frac{2n}{4n+1} = \lim_{n \to +\infty} \frac{2}{4 + \frac{1}{n}} = \frac{1}{2}$$

we conclude that the infinite series is convergent and

$$\sum_{n=1}^{+\infty} \frac{2}{(4n-3)(4n+1)} = \frac{1}{2}$$

5. $\displaystyle\sum_{n=1}^{+\infty} \ln \frac{n}{n+1} = \sum_{n=1}^{+\infty} [\ln n - \ln(n+1)]$. $s_1 = \ln 1 - \ln 2 = -\ln 2$;

$s_2 = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) = -\ln 3$; $s_3 = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) = -\ln 4$

$s_4 = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + (\ln 4 - \ln 5) = -\ln 5$

$s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + [\ln(n-1) - \ln n] + [\ln n - \ln(n+1)] = -\ln(n+1)$

Because $\displaystyle\lim_{n \to +\infty} s_n = -\infty$, the series is divergent.

6. $\displaystyle\sum_{n=1}^{+\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{+\infty} \frac{(n+1)^2 - n^2}{n^2(n+1)^2} = \sum_{n=1}^{+\infty}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]$. $s_1 = 1 - \frac{1}{4} = \frac{3}{4}$; $s_2 = (1 - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{9}) = 1 - \frac{1}{9} = \frac{8}{9}$;

$s_3 = (1 - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{9}) + (\frac{1}{9} - \frac{1}{16}) = 1 - \frac{1}{16} = \frac{15}{16}$; $s_4 = (1 - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{9}) + (\frac{1}{9} - \frac{1}{16}) + (\frac{1}{16} - \frac{1}{25}) = 1 - \frac{1}{25} = \frac{24}{25}$

$s_n = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \cdots + \left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] = 1 - \frac{1}{(n+1)^2}$. $\displaystyle\lim_{n \to +\infty} s_n = 1$. The series is convergent with sum 1.

7. $\displaystyle\sum_{n=1}^{+\infty} \frac{2}{5^{n-1}}$. $s_1 = u_1 = 2$; $s_2 = s_1 + u_2 = 2 + \frac{2}{5} = \frac{12}{5}$; $s_3 = s_2 + u_3 = \frac{12}{5} + \frac{2}{25} = \frac{62}{25}$;

$s_4 = s_3 + u_4 = \frac{62}{25} + \frac{2}{125} = \frac{312}{125}$; $s_n = 2\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots + \frac{1}{5^{n-1}}\right) = 2\,\frac{1 - 1/5^n}{1 - 1/5} = \frac{5}{2}\left(1 - \frac{1}{5^n}\right)$

Because $\displaystyle\lim_{n \to +\infty} s_n = \frac{5}{2} \cdot 1 = \frac{5}{2}$, the series is convergent and its sum is $\frac{5}{2}$.

8. $\displaystyle\sum_{n=1}^{+\infty} \frac{2^{n-1}}{3^n}$

▷ We have

$$u_n = \frac{2^{n-1}}{3^n}$$

Thus, $u_1 = \frac{1}{3}$, $u_2 = \frac{2}{9}$, $u_3 = \frac{4}{27}$, and $u_4 = \frac{8}{81}$, from which we obtain the first four elements of the sequence of partial sums.

$s_1 = u_1 = \frac{1}{3}$; $s_2 = u_1 + u_2 = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}$; $s_3 = s_2 + u_3 = \frac{5}{9} + \frac{4}{27} = \frac{19}{27}$; $s_4 = s_3 + u_4 = \frac{19}{27} + \frac{8}{81} = \frac{65}{81}$

Because

$$u_{n+1} \div u_n = \frac{2^n}{3^{n+1}} \cdot \frac{3^n}{2^{n-1}} = \frac{2}{3}$$

the series is a geometric series with $a = 1$ and $r = \frac{2}{3}$, and so we may us Formula (13) in the text to obtain the sum. Thus,

$$s_n = a\frac{1 - r^n}{1 - r} = \frac{1}{3} \cdot \frac{1 - (\frac{2}{3})^n}{1 - \frac{2}{3}} = 1 - (\tfrac{2}{3})^n$$

Because

$$\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} [1 - (\tfrac{2}{3})^n] = 1$$

we conclude that the infinite series is convergent, and

$$\sum_{n=1}^{+\infty} \frac{2^{n-1}}{3^n} = 1$$

We may also use Theorem 8.3.5 to find the sum, as follows:

$$\sum_{n=1}^{+\infty} \frac{2^{n-1}}{3^n} = \sum_{n=1}^{+\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{\frac{1}{3}}{1 - \frac{2}{3}} = 1$$

In Exercises 9-13, find the infinite series that is the given sequence of partial sums. Also determine of the infinite series is convergent or divergent, and if it is convergent, find its sum.

9. $s_n = \frac{2n}{3n+1}$. $u_n = s_n - s_{n-1} = \frac{2n}{3n+1} - \frac{2(n-1)}{3(n-1)+1} = \frac{2}{(3n-2)(3n+1)}$. The series is $\sum_{n=1}^{+\infty} \frac{2}{(3n-2)(3n+1)}$.
Because $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \frac{2n}{3n+1} = \lim_{n\to+\infty} \frac{2}{3+1/n} = \frac{2}{3}$, the series is convergent and its sum is $\frac{2}{3}$.

10. $s_n = \frac{n^2}{n+1}$. $u_n = s_n - s_{n-1} = \frac{n^2}{n+1} - \frac{(n-1)^2}{n} = \frac{n^3-(n^3-n^2-n+1)}{n(n+1)} = \frac{n^2+n-1}{n(n+1)}$. $\lim_{n\to+\infty} s_n = +\infty$, divergent

11. $s_n = \frac{1}{3^n}$. $u_1 = s_1 = \frac{1}{3}$. If $n > 1$, $u_n = s_n - s_{n-1} = \frac{1}{3^n} - \frac{1}{3^{n-1}} = -\frac{2}{3^n}$. Hence the series is $\frac{1}{3^n} - \sum_{n=2}^{+\infty} \frac{2}{3^n}$.
Because $\lim_{n\to+\infty} s_n = 0$, the series is convergent and its sum is 0.

12. $\{s_n\} = \{3^n\}$

► Let the infinite series be represented by
$\sum_{n=1}^{+\infty} u_n$
Because $s_1 = 3$, then $u_1 = 3$. If $n > 1$, then
$$u_n = s_n - s_{n-1} = 3^n - 3^{n-1} = 3^{n-1}(3-1) = 2 \cdot 3^{n-1}$$
Thus, the infinite series is
$3 + \sum_{n=1}^{+\infty} 2 \cdot 3^{n-1}$
Because
$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} 3^n = +\infty$
then the series is divergent.

13. $s_n = \ln(2n+1)$. $u_n = s_n - s_{n-1} = \ln(2n+1) - \ln(2n-1) = \ln\left(\frac{2n+1}{2n-1}\right)$. Hence the series is $\sum_{n=1}^{+\infty} \ln\left(\frac{2n+1}{2n-1}\right)$.
Because $\lim_{n\to+\infty} s_n = +\infty$, the series is divergent.

In Exercises 14-24, write the first four terms of the infinite series and determine whether the series is convergent or divergent. If the series is convergent, find its sum.

15. $\sum_{n=1}^{+\infty} \frac{2n+1}{3n+2} = \frac{3}{5} + \frac{5}{8} + \frac{7}{11} + \frac{9}{14} + \cdots$. Because $\lim_{n\to+\infty} \frac{2n+1}{3n+2} = \lim_{n\to+\infty} \frac{2+\frac{1}{n}}{3+\frac{2}{n}} = \frac{2}{3} \neq 0$, it follows from Theorem 8.3.3 that the series is divergent.

16. $\sum_{n=1}^{+\infty} [1+(-1)^n]$

► The first four terms of the infinite series are

$u_1 = 1+(-1)^1 = 0$; $u_2 = 1+(-1)^2 = 2$; $u_3 = 1+(-1)^3 = 0$; $u_4 = 1+(-1)^4 = 2$

Because $\lim_{n\to+\infty} u_n \neq 0$, then by Theorem 8.3.3 the series is divergent.

17. $\sum_{n=1}^{+\infty} \left(\frac{2}{3}\right) = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$ is a geometric series with $a = \frac{2}{3}$ and $r = \frac{2}{3} < 1$.
Thus it is convergent and its sum is $\frac{\frac{2}{3}}{1-\frac{2}{3}} = 2$.

18. $\sum_{n=1}^{+\infty} \frac{3n^2}{n^2+1} = \frac{3}{2} + \frac{12}{5} + \frac{27}{10} + \frac{48}{17} + \cdots$. $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{3n^2}{n^2+1} = \lim_{n\to+\infty} \frac{3}{1+(1/n^2)} = 3 \neq 0$. The series diverges.

19. $\sum_{n=1}^{+\infty} \ln\frac{1}{n} = 0 + \ln\frac{1}{2} + \ln\frac{1}{3} + \ln\frac{1}{4} + \cdots$. Since $\lim_{n\to+\infty} \ln\frac{1}{n} = \lim_{n\to+\infty} (-\ln n) = -\infty$ the series diverges by Th 8.3.3.

20. $\sum_{n=1}^{+\infty} \frac{2}{3^{n-1}}$

▷ $\sum_{n=1}^{+\infty} 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots$

Because

$$\sum_{n=1}^{+\infty} \frac{2}{3^{n-1}} = \sum_{n=1}^{+\infty} 2\left(\frac{1}{3}\right)^{n-1}$$

the series is a geometric series with $a = 2$ and $r = 1$. By Theorem 8.3.5 we conclude that the series is convergent and

$$\sum_{n=1}^{+\infty} \frac{2}{3^{n-1}} = \frac{2}{1-\frac{1}{3}} = 3$$

21. $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{3}{2^n} = \frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \frac{3}{16} + \cdots$ is a geometric series with $a = \frac{3}{2}$ and $r = -\frac{1}{2}$.

Because $|r| < 1$, the series is convergent. Its sum is $\frac{\frac{3}{2}}{1-(-\frac{1}{2})} = 1$.

22. $\sum_{n=1}^{+\infty} \tan^n \frac{1}{6}\pi = \sum_{n=1}^{+\infty}\left(\frac{1}{\sqrt{3}}\right)^n = \frac{1}{\sqrt{3}} + \frac{1}{3} + \frac{1}{3\sqrt{3}} + \frac{1}{9} + \cdots$ is a geometric series with $a = r = \frac{1}{\sqrt{3}}$.

Because $|r| < 1$, the series is convergent. Its sum is $\frac{1/\sqrt{3}}{1-1/\sqrt{3}} = \frac{1}{\sqrt{3}-1} = \frac{1}{2}(\sqrt{3}+1)$

23. $\sum_{n=1}^{+\infty} e^{-n} = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} + \cdots$ is a geometric series with $a = \frac{1}{e}$ and $r = \frac{1}{e} < 1$.

Thus it is convergent and its sum is $\frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}$.

24. $\sum_{n=1}^{+\infty} \frac{\sinh n}{n}$

▷ $\sum_{n=1}^{+\infty} \frac{\sinh n}{n} = \sinh 1 + \frac{\sinh 2}{2} + \frac{\sinh 3}{3} + \frac{\sinh 4}{4} + \cdots \approx 1.18 + 1.83 + 3.34 + 6.82 + \cdots$

Because $u_1 < u_2 < u_3 < u_3$, the sequence $\{u_n\}$ may not have limit 0. By L'Hopital's rule

$$\lim_{x \to +\infty} \frac{\sinh x}{x} = \lim_{x \to +\infty} \frac{\cosh x}{1} = +\infty$$

Thus, by Theorem 8.3.3 the series is divergent.

In Exercises 25–44, determine whether the series is convergent or divergent. If it is convergent, find its sum.

25. $\sum_{n=1}^{+\infty} \frac{1}{n+2} = \sum_{n=3}^{+\infty} \frac{1}{n}$. By Theorem 8.3.9, the given series diverges since $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges.

26. $\sum_{n=3}^{+\infty} \frac{1}{n-1} = \sum_{n=2}^{+\infty} \frac{1}{n}$. By Theorem 8.3.9, the given series diverges since $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges.

27. $\sum_{n=1}^{+\infty} \frac{3}{2n} = \frac{3}{2}\sum_{n=1}^{+\infty} \frac{1}{n}$. By Theorem 8.3.6, the given series diverges since $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges.

28 $\sum_{n=1}^{+\infty} \frac{2}{3n}$

▷ Because

$$\sum_{n=1}^{+\infty} \frac{2}{3n} = \frac{2}{3}\sum_{n=1}^{+\infty} \frac{1}{n}$$

each term is $\frac{2}{3}$ times the corresponding term in the harmonic series. Because the harmonic series is divergent, by Theorem 8.3.6(ii) we conclude that the given series is divergent.

29. $\sum_{n=1}^{+\infty} \frac{3}{2^n}$ is a geometric series with $a = \frac{3}{2}$ and $r = \frac{1}{2} < 1$. Hence the series is convergent and its sum is $\frac{\frac{3}{2}}{1-\frac{1}{2}} = 3$.

30. $\sum_{n=1}^{+\infty} \frac{2}{3^n}$ is a geometric series with $a = \frac{2}{3}$ and $r = \frac{1}{3} < 1$. Hence the series is convergent and its sum is $\frac{\frac{2}{3}}{1-\frac{1}{3}} = 1$.

31. $\sum_{n=1}^{+\infty} \frac{4}{3}\left(\frac{5}{7}\right)$ is a geometric series with $a = \frac{20}{21}$ and $r = \frac{5}{7} < 1$. The series is convergent with sum $\frac{\frac{20}{21}}{1-\frac{5}{7}} = \frac{7}{2}\cdot\frac{20}{21} = \frac{10}{3}$.

32. $\sum_{n=1}^{+\infty} \frac{7}{5}\left(\frac{3}{4}\right)^n$

▷ Because

$$\sum_{n=1}^{+\infty} \tfrac{7}{5}\left(\tfrac{3}{4}\right)^n = \sum_{n=1}^{+\infty} \tfrac{7}{5}\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)^{n-1} = \sum_{n=1}^{+\infty} \tfrac{21}{20}\left(\tfrac{3}{4}\right)^{n-1}$$

the given series is a geometric series with $a = \frac{21}{20}$ and $r = \frac{3}{4}$. By Theorem 8.3.5 we conclude that the series is convergent and

$$\sum_{n=1}^{+\infty} \frac{7}{5}\left(\frac{3}{4}\right)^n = \frac{a}{1-r} = \frac{\frac{21}{20}}{1-\frac{3}{4}} = \frac{21}{5}$$

33. $[\![\sin \frac{4}{1}\pi]\!] = [\![0]\!] = 0$, $[\![\sin \frac{4}{2}\pi]\!] = [\![0]\!] = 0$, $[\![\sin \frac{4}{3}\pi]\!] = [\![-\frac{1}{2}\sqrt{3}]\!] = -1$, $[\![\sin \frac{4}{4}\pi]\!] = [\![0]\!] = 0$, $[\![\sin \frac{4}{5}\pi]\!] = [\![0.587]\!] = 0$, $[\![\sin \frac{4}{6}\pi]\!] = [\![\frac{1}{2}\sqrt{3}]\!] = 0$, $[\![\sin \frac{4}{7}\pi]\!] = [\![0.975]\!] = 0$, $[\![\sin \frac{4}{8}\pi]\!] = [\![1]\!] = 1$. If $n > 8$, then

$0 < \frac{4}{n}\pi < \frac{1}{2}\pi$ and $0 < \sin \frac{4}{n}\pi < 1$ so $[\![\sin \frac{4}{n}\pi]\!] = 0$. Therefore $\sum_{n=1}^{+\infty} \frac{[\![\sin \frac{4}{n}\pi + 3]\!]}{4^n} = -\frac{1}{4^3} + \frac{1}{4^8} + \sum_{n=1}^{+\infty} \frac{3}{4^n}$.

We have a geometric series with $a = \frac{3}{4}$ and $r = \frac{1}{4} < 1$. Hence the given series is convergent and its sum is

$$\frac{\frac{3}{4}}{1-\frac{1}{4}} - \frac{1}{4^3} + \frac{1}{4^8} = 1 - \frac{1}{2^6} + \frac{1}{2^{16}} = \frac{63 \cdot 2^{10} + 1}{2^{16}}$$

34. $S = \sum_{n=1}^{+\infty} \frac{[\![\cos \pi/n + 1]\!]}{2^n}$. $[\![\cos \pi/1 + 1]\!] = [\![-1+1]\!] = 0$, $[\![\cos \pi/2 + 1]\!] = [\![0+1]\!] = 1$. If $n > 2$, $0 < \pi/n < \pi/2$ and $0 < \cos \pi/n < 1$ so $[\![\cos \pi/n]\!] = 0$. S is a geometric series with $a = \frac{1}{4}$ and $r = \frac{1}{2}$. Thus $S = \sum_{n=2}^{+\infty} \frac{1}{2^n} = \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{1}{2}$

35. $\sum_{n=1}^{+\infty} \left(\frac{1}{2n} + \frac{1}{2^n}\right) = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{1}{n} + \sum_{n=1}^{+\infty} \frac{1}{2^n}$. By Theorem 8.3.8, the given series diverges because $\sum_{n=1}^{+\infty} \frac{1}{n}$ is the divergent harmonic series and $\sum_{n=1}^{+\infty} \frac{1}{2^n}$ is a convergent geometric series.

36. $\sum_{n=1}^{+\infty} \left(\frac{1}{3^n} + \frac{1}{3n}\right)$

▷ Because

$$\sum_{n=1}^{+\infty} \frac{1}{3^n} = \sum_{n=1}^{+\infty} \frac{1}{3}\left(\frac{1}{3}\right)^n \qquad (1)$$

then the series (1) is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{3}$. Because $|r| < 1$, the series is convergent. Furthermore,

$$\sum_{n=1}^{+\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{+\infty} \frac{1}{n} \qquad (2)$$

Because the harmonic series is divergent, then by Theorem 8.3.6(ii), series (2) is divergent. By theorem 8.3.8 we conclude that the series

$$\sum_{n=1}^{+\infty} \left(\frac{1}{3^n} + \frac{1}{3n}\right)$$

is divergent.

37. $\sum_{n=1}^{+\infty} \left(\frac{1}{2^n} + \frac{1}{3^n}\right) = \sum_{n=1}^{+\infty} \frac{1}{2^n} + \sum_{n=1}^{+\infty} \frac{1}{3^n}$. Because these geometric series converge to $\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ and $\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$ respectively, the entire series converges to $1 + \frac{1}{2} = \frac{3}{2}$ by Theorem 8.3.7.

38. $\sum_{n=1}^{+\infty} \left(\frac{1}{3^n} - \frac{1}{4^n}\right) = \sum_{n=1}^{+\infty} \frac{1}{3^n} - \sum_{n=1}^{+\infty} \frac{1}{4^n}$. Because these geometric series converge to $\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$ and $\frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$ respectively, the entire series converges to $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ by Theorem 8.3.7.

39. $\sum_{n=1}^{+\infty} (e^{-n} + e^n)$. $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} (e^{-n} + e^n) = +\infty$. The series diverges by Theorem 8.3.3.

40. $\sum_{n=1}^{+\infty} (2^{-n} + 3^n)$

▷ Because

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} (2^{-n} + 3^n) = +\infty$$

the series diverges by Theorem 8.3.3.

41. $\sum\limits_{n=1}^{+\infty}\left(\frac{1}{2n}-\frac{1}{3n}\right)=\sum\limits_{n=1}^{+\infty}\frac{1}{6n}=\frac{1}{6}\sum\limits_{n=1}^{+\infty}\frac{1}{n}$. The series diverges by Theorem 8.3.6.

42. $\sum\limits_{n=1}^{+\infty}\left(\frac{3}{2n}-\frac{2}{3n}\right)=\sum\limits_{n=1}^{+\infty}\frac{5}{6n}=\frac{5}{6}\sum\limits_{n=1}^{+\infty}\frac{1}{n}$. The series diverges by Theorem 8.3.6.

43. $\sum\limits_{n=1}^{+\infty}\left(\frac{3}{2^n}-\frac{2}{3^n}\right)=\sum\limits_{n=1}^{+\infty}\frac{3}{2^n}-\sum\limits_{n=1}^{+\infty}\frac{2}{3^n}$. These geometric series converge to $\dfrac{\frac{3}{2}}{1-\frac{1}{2}}=3$ and $\dfrac{\frac{2}{3}}{1-\frac{1}{3}}=1$ respectively. Hence by Theorem 8.3.7 the given series converges to $3-1=2$.

44. $\sum\limits_{n=1}^{+\infty}\left(\frac{5}{4^n}+\frac{4}{5^n}\right)$

▹ We apply Theorem 8.3.5 to obtain

$$\sum_{n=1}^{+\infty}\frac{5}{4^n}=\sum_{n=1}^{+\infty}\frac{5}{4}\left(\frac{1}{4}\right)^{n-1}=\frac{\frac{5}{4}}{1-\frac{1}{4}}=\frac{5}{3} \tag{1}$$

and

$$\sum_{n=1}^{+\infty}\frac{4}{5^n}=\sum_{n=1}^{+\infty}\frac{4}{5}\left(\frac{1}{5}\right)^{n-1}=\frac{\frac{4}{5}}{1-\frac{1}{5}}=1 \tag{2}$$

We apply Theorem 8.3.7(i) and Eqs. (1) and (2) to conclude that the given series is convergent and

$$\sum_{n=1}^{+\infty}\left(\frac{5}{4^n}+\frac{4}{5^n}\right)=\frac{5}{3}+1=\frac{8}{3}$$

In Exercises 45–48, express the nonterminating repeating decimal as a common fraction.

45. $0.27\,27\,27\ldots=\frac{27}{100}+\frac{27}{10000}+\frac{27}{1000000}+\cdots$ is a geometric series with $a=\frac{27}{100}$ and $r=\frac{1}{100}<1$.

Therefore it is convergent and its sum is $\dfrac{\frac{27}{100}}{1-\frac{1}{100}}=\frac{27}{99}=\frac{3}{11}$.

46. $x=2.0\,45\,45\,45\ldots$, $100x=204.5\,45\,45\,45\ldots$, $99x=100x-x=202.5$, $x=\frac{202.5}{99}=\frac{405}{198}=\frac{45}{22}$

47. $0.234\,234\,234\ldots=\frac{234}{1000}+\frac{234}{1000000}+\frac{234}{1000000000}+\cdots$ is a geometric series with $a=\frac{234}{1000}$ and

$r=\frac{1}{1000}<1$. Therefore it is convergent and its sum is $\dfrac{\frac{234}{1000}}{1-\frac{1}{1000}}=\frac{234}{999}=\frac{26}{111}$. Hence

$1.234\,234\,234\ldots=1+\frac{26}{111}=\frac{137}{111}$

48. $0.4653\,4653\,4653\ldots$

▹ $0.4653\,4653\,4653\ldots=0.4653+0.0000\,4653+0.0000\,0000\,4653+\cdots$ is a geometric series with $a=0.4653$ and $r=0.0001$ since $0.0000\,4653\div0.4653=0.0001$ and so on. Thus by Theorem 8.3.5 we have

$0.4653+0.0000\,4653+0.0000\,0000\,4653+\cdots=\frac{0.4653}{1-0.0001}=\frac{0.4653}{0.9999}=\frac{4653}{9999}=\frac{47}{101}$

49. Let S cm be the distance traveled by the pendulum bob before it comes to rest. Then

$S=56+56(0.93)+56(0.93)^2+\cdots+56(0.93)^{n-1}+\cdots=\sum\limits_{n=1}^{+\infty}56(0.93)^{n-1}$. This is a geometric series with

$a=56$ and $r=0.93$. Hence $S=\frac{56}{1-0.93}=\frac{56}{0.07}=800$. Thus the distance traveled by the bob is 800 cm = 8 m.

50. Let R be the number of rotations of the wheel before the bicycle stops. Then

$r=200+200\left(\frac{4}{5}\right)+200\left(\frac{4}{5}\right)^2+\cdots=\sum\limits_{n=1}^{+\infty}200\left(\frac{4}{5}\right)$ is a geometric series with $a=200$ and $r=\frac{4}{5}$.

Hence $r=\dfrac{200}{1-\frac{4}{5}}=\dfrac{200}{\frac{1}{5}}=1000$. The wheel rotates 1000 times before the bicycle stops.

51. Let S feet be the distance traveled by the ball before it comes to rest. Then

$S=12+12(\frac{3}{4})+12(\frac{3}{4})+12(\frac{3}{4})^2+12(\frac{3}{4})^2+\cdots=12+2\sum\limits_{n=1}^{+\infty}12(\frac{3}{4})^n$. We have a geometric series with $a=9$ and

$r=\frac{3}{4}$. Hence $S=12+2\cdot\dfrac{9}{1-\frac{3}{4}}=12+\dfrac{18}{\frac{1}{4}}=84$. The ball travels 84 ft before coming to rest.

52. What is the total distance traveled by a tennis ball before coming to rest if it is dropped from a height of 100 ft and if, after each fall, it rebounds eleven-twentieths of the distance from which it fell?

▷ Let S feet be the distance traveled by the tennis ball before it comes to rest. Then

$$S = 100 + 100\left(\frac{11}{20}\right) + 100\left(\frac{11}{20}\right) + 100\left(\frac{11}{20}\right)^2 + \left(\frac{11}{20}\right)^2 + \cdots = 100 + 2\sum_{n=1}^{+\infty} 100\left(\frac{11}{20}\right)^n$$

We have a geometric series with $a = 55$ and $r = \frac{11}{20}$. Hence

$$S = 100 + 2 \cdot \frac{55}{1 - \frac{11}{20}} = 100 + \frac{110}{\frac{9}{20}} = 100 + \frac{2200}{9} = \frac{3100}{9}$$

• The ball travels $\frac{3100}{9}$ ft ≈ 344.44 ft before coming to rest.

53. (a) Because $h = \frac{1}{2}gt^2 = 16t^2$, then $t = \frac{1}{4}\sqrt{h}$ (b) If T sec is the total time in Ex. 51, then

$T = \frac{1}{4}\sqrt{12} + \frac{1}{4}\sqrt{12(\frac{3}{4})} + \frac{1}{4}\sqrt{12(\frac{3}{4})} + \frac{1}{4}\sqrt{12(\frac{3}{4})^2} + \frac{1}{4}\sqrt{12(\frac{3}{4})^2} + \cdots = \frac{1}{2}\sqrt{3} + 2\sum_{n=2}^{+\infty}(\frac{1}{2}\sqrt{3})^n.$

We have a geometric series with $a = \frac{3}{4}$ and $r = \frac{1}{2}\sqrt{3}$. Hence

$$T = \frac{1}{2}\sqrt{3} + 2 \cdot \frac{\frac{3}{4}}{1 - \frac{1}{2}\sqrt{3}} = \frac{1}{2}\sqrt{3} + \frac{3}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{1}{2}\sqrt{3} + 6 + 3\sqrt{3} = 6 + \frac{7}{2}\sqrt{3}.$$ It takes about 12.1 seconds.

54. If T sec is the total time in Ex. 52, $T = \frac{1}{4}\sqrt{100} + \frac{1}{4}\sqrt{100(\frac{11}{20})} + \frac{1}{4}\sqrt{100(\frac{11}{20})} + \frac{1}{4}\sqrt{100(\frac{11}{20})^2} + \frac{1}{4}\sqrt{100(\frac{11}{20})^2} + \cdots$

$$= \frac{5}{2} + 2\sum_{n=1}^{+\infty} \frac{5}{2}(\tfrac{1}{10}\sqrt{55})^n = \frac{5}{2} + 2 \cdot \frac{\frac{5}{2} \cdot \frac{1}{10}\sqrt{55}}{1 - \frac{1}{10}\sqrt{55}} = \frac{5}{2} + \frac{5\sqrt{55}}{10 - \sqrt{55}} \cdot \frac{10 + \sqrt{55}}{10 + \sqrt{55}} = \frac{5}{2} + \frac{50\sqrt{55} + 275}{45} = \frac{155}{18} + \frac{10}{9}\sqrt{55}$$

• It takes about 16.9 seconds for the ball to stop bouncing.

55. Total perimeter $= \sum_{n=1}^{+\infty} 12(\frac{1}{2})^{n-1} = \frac{12}{1 - \frac{1}{2}} = 24$

56. Find an infinite series whose sum is 6 and such that each term is four times the sum of all the terms that follow it.

▷ Let $\sum_{n=1}^{+\infty} u_n$ be the series. Then

$$u_n = 4(u_{n+1} + u_{n+2} + \cdots)$$

and

$$u_{n+1} = 4(u_{n+2} + \cdots)$$

Subtracting, we obtain

$$u_n - u_{n+1} = 4u_{n+1}$$
$$u_{n+1} = \frac{1}{5}u_n$$

Thus, the series is a geometric series with ratio $\frac{1}{5}$. Because the sum of all terms after the first is $\frac{1}{4}$ of the first,

$$u_1 + \frac{1}{4}u_1 = 6$$

and so $u_1 = \frac{24}{5}$. Therefore the series is

$$\sum_{n=1}^{+\infty} \frac{24}{5}(\tfrac{1}{5})^{n-1} = 24\sum_{n=1}^{+\infty} \frac{1}{5^n}$$

In Exercises 57 and 58, we calculate $\sum_{n=1}^{N} \frac{1}{n}$ and compare with $\ln N + 0.5772157 + \frac{1}{2N}$.

57. (a) $\sum_{n=1}^{25} \frac{1}{n} = 3.81596$, $\ln 25 + 0.57722 + \frac{1}{50} = 3.81609$ (b) $\sum_{n=1}^{50} \frac{1}{n} = 4.49921$, $\ln 50 + 0.57722 + \frac{1}{100} = 4.49924$

(c) $\sum_{n=1}^{75} \frac{1}{n} = 4.90136$, $75 + 0.57722 + \frac{1}{150} = 4.90137$ (d) $\sum_{n=1}^{100} \frac{1}{n} = 5.18738$, $\ln 100 + 0.5772 + \frac{1}{200} = 5.18739$

58. (a) $\sum_{n=1}^{250} \frac{1}{n} = 6.100675$, $\ln 250 + 0.577216 + \frac{1}{500} = 6.100677$

(b) $\sum_{n=1}^{500} \frac{1}{n} = 6.792823$, $\ln 500 + 0.577216 + \frac{1}{1000} = 6.792824$

(c) $\sum_{n=1}^{750} \frac{1}{n} = 7.197955$, $\ln 750 + 0.577216 + \frac{1}{1500} = 7.197956$

(d) $\sum_{n=1}^{+\infty} \frac{1}{n} = 7.485471$, $\ln 1000 + 0.577216 + \frac{1}{2000} = 7.485471$

59. $\ln n + 0.5772 = 10$ when $n = e^{9.4228} = 12367$. In fact $\sum_{n=1}^{12366} \frac{1}{n} = 9.99996$ and $\sum_{n=1}^{12367} \frac{1}{n} = 10.00004$

60. Prove Theorem 8.3.7.

▸ Let s_n and r_n be the nth partial sums of the series $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$, respectively. Then

$$s_n = a_1 + a_2 + \cdots + a_n \quad \text{and} \qquad r_n = b_1 + b_2 + \cdots + b_n$$

By Definition 8.3.2 we are given that

$$\lim_{n \to +\infty} s_n = \mathrm{S} \quad \text{and} \quad \lim_{n \to +\infty} r_n = \mathrm{R}$$

If t_n is the nth partial sum of the series $\sum_{n=1}^{+\infty} (a_n \pm b_n)$ then

$$t_n = (a_1 \pm b_1) + (a_2 \pm b_2) + \cdots + (a_n \pm b_n) = (a_1 + a_2 + \cdots + a_n) \pm (b_1 + b_2 + \cdots + b_n) = s_n \pm r_n$$

Thus

$$\lim_{n \to +\infty} t_n = \lim_{n \to +\infty} (s_n \pm r_n) = \lim_{n \to +\infty} s_n \pm \lim_{n \to +\infty} r_n = \mathrm{S} \pm \mathrm{R}$$

By Definition 8.3.2 we conclude that

(i) $\sum_{n=1}^{+\infty} (a_n + b_n) = \mathrm{S} + \mathrm{R}$ and (ii) $\sum_{n=1}^{+\infty} (a_n - b_n) = \mathrm{S} - \mathrm{R}$

8.4 INFINITE SERIES OF POSITIVE TERMS

If every term in a series is positive, we may sometimes determine whether or not the series is convergent by comparing it with a series of positive terms that we know to be convergent or that we know to be divergent.

8.4.1 Theorem An infinite series of positive terms is convergent if and only if its sequence of partial sums has an upper bound.

8.4.2 Theorem Comparison Test Let $\sum_{n=1}^{+\infty} u_n$ be a series of positive terms.

(i) If $\sum_{n=1}^{+\infty} v_n$ is a series of positive terms that is known to be convergent and $u_n \le v_n$ for all positive integers n, then $\sum_{n=1}^{+\infty} u_n$ is convergent.

(ii) If $\sum_{n=1}^{+\infty} v_n$ is a series of positive terms that is known to be divergent and $u_n \ge v_n$ for all positive integers n, then $\sum_{n=1}^{+\infty} u_n$ is divergent.

The comparison test gives no information if a series is less than a divergent series or greater than a convergent series. Sometimes when Theorem 8.4.2 is difficult to apply, we may use the following test to determine whether or not the series is convergent.

8.4.3 Theorem Limit Comparison Test Let $\sum_{n=1}^{+\infty} u_n$ and $\sum_{n=1}^{+\infty} v_n$ be two series of positive terms.

(i) If $\lim_{n \to +\infty} (u_n/v_n) = c > 0$, then the two series either both converge or both diverge.

(ii) If $\lim_{n \to +\infty} (u_n/v_n) = 0$, and if $\sum_{n=1}^{+\infty} v_n$ converges, then $\sum_{n=1}^{+\infty} u_n$ converges.

(iii) If $\lim_{n \to +\infty} (u_n/v_n) = +\infty$, and if $\sum_{n=1}^{+\infty} v_n$ diverges, then $\sum_{n=1}^{+\infty} u_n$ diverges.

For the known series $\sum_{n=1}^{+\infty} v_n$ in the comparison tests, we may use a geometric series, the harmonic series, the convergent series $\sum_{n=1}^{+\infty} \frac{1}{n!}$ of Example 1 of the text, or the p-series.

The p-series Let p be a constant. The p-series $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

If p is an even integer, the sum is known: $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{n=1}^{+\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, $\sum_{n=1}^{+\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$

The extended p-series $\sum_{n=1}^{+\infty} \frac{1}{n(\ln n)^p}$ converges if $p > 1$ and diverges if $p \le 1$; see Ex. 51.

We have the following theorems on rearranging series.

8.4.4-5 Theorem If $\sum_{n=1}^{+\infty} u_n$ is a given convergent series of positive terms, its terms can be grouped in any manner, or rearranged, and the resulting series will converge, and to the same sum.

8.4.6 Theorem (Integral Test) Let f be a function that is continuous, decreasing, and positive valued for all $x \geq 1$. Then the infinite series

$$\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + \cdots + f(n) + \cdots$$

is convergent if the improper integral

$$\int_1^{+\infty} f(x)dx$$

exists, and it is divergent if $\lim_{b\to+\infty} \int_1^b f(x)dx = +\infty$.

We may also use the integral test for an infinite series that does not begin with $n = 1$. Thus, if f is continuous, decreasing, and positive valued for all $x \geq a$, where a is a positive integer, then $\sum_{n=a}^{\infty} f(x)$ converges if and only if $\int_a^{+\infty} f(x)dx$ is convergent.

Exercises 8.4

In Exercises 1–24, determine whether the series is convergent or divergent by applying either the comparison test or the limit comparison test.

1. $\sum_{n=1}^{+\infty} \frac{1}{n2^n}$ is convergent by comparison with a geometric series: $\frac{1}{n2^n} \leq \frac{1}{2^n}$.

2. $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{2n+1}}$ is divergent by a limit comparison with a p-series with $p = \frac{1}{2}$:

$$\lim_{n\to+\infty} \frac{1}{\sqrt{2n+1}} \Big/ \frac{1}{\sqrt{n}} = \lim_{n\to+\infty} \sqrt{\frac{n}{2n+1}} = \lim_{n\to+\infty} \sqrt{\frac{1}{2+1/n}} = \sqrt{\tfrac{1}{2}} > 0$$

3. $\sum_{n=1}^{+\infty} \frac{1}{n^n}$ is convergent by comparison with a geometric series: $\frac{1}{n^n} \leq \frac{1}{2^n}$ when $n \geq 2$.

4. $\sum_{n=1}^{+\infty} \frac{n^2}{4n^3+1}$

▷ For large n, the expression $4n^3 + 1$ is approximately equal to $4n^3$. Thus, for large n,

$$\frac{n^2}{4n^3+1} \approx \frac{n^2}{4n^3} = \frac{1}{4n}$$

We make a limit comparison test with

$$u_n = \frac{n^2}{4n^3+1} \quad \text{and} \quad v_n = \frac{1}{4n}$$

Thus,

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \lim_{n\to+\infty} \frac{4n^3}{4n^3+1} = \lim_{n\to+\infty} \frac{4}{4+\frac{1}{n^3}} = 1$$

Because the harmonic series is divergent, and

$$\sum_{n=1}^{+\infty} v_n = \sum_{n=1}^{+\infty} \frac{1}{4n} = \frac{1}{4}\sum_{n=1}^{+\infty} \frac{1}{n}$$

the series $\sum_{n=1}^{+\infty} v_n$ is divergent. Thus, by Thm 8.4.3(i), $\sum_{n=1}^{+\infty} u_n$ is divergent. That is, the given series is divergent.

5. $\sum_{n=1}^{+\infty} \frac{3n+1}{2n^2+5}$ is divergent by a limit comparison with the harmonic series:

$$\lim_{n\to+\infty} \frac{3n+1}{2n^2+5} \Big/ \frac{1}{n} = \lim_{n\to+\infty} \frac{3+n^{-1}}{2+5n^{-2}} = \frac{3}{2} > 0$$

6. $\sum_{n=1}^{+\infty} \frac{3}{\sqrt{n^3+n}}$ is convergent by a limit comparison with a p-series, $p = \frac{3}{2}$:

$$\lim_{n\to+\infty} \frac{3}{\sqrt{n^3+n}} \Big/ \frac{1}{\sqrt{n^3}} = \lim_{n\to+\infty} 3\sqrt{\frac{n^3}{n^3+n}} = \lim_{n\to+\infty} 3\sqrt{\frac{1}{1+1/n^2}} = 3 > 0$$

7. $\sum_{n=1}^{+\infty} \frac{\cos^2 n}{3^n}$ is convergent by comparison with a geometric series: $\frac{\cos^2 n}{3^n} \leq \frac{1}{3^n}$.

8. $\sum_{n=1}^{+\infty} \frac{1}{\ln(n+1)}$

▷ For all positive integers n we have $0 < \ln(n+1) < n$, and thus

$\frac{1}{\ln(n+1)} > \frac{1}{n}$

We use the comparison test with

$u_n = \frac{1}{\ln(n+1)}$ and $v_n = \frac{1}{n}$

Because $\sum_{n=1}^{+\infty} v_n$ is the divergent harmonic series, and $u_n > v_n$. By Theorem 8.4.2(ii) we conclude that $\sum_{n=1}^{+\infty} u_n$ is divergent. That is, the given series is divergent.

9. $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^2+4n}}$ is divergent by a limit comparison with the harmonic series:

$\lim_{n\to+\infty} \frac{1}{\sqrt{n^2+4n}} / \frac{1}{n} = \lim_{n\to+\infty} \frac{1}{\sqrt{1+4n^{-1}}} = 1 > 0$

10. $\sum_{n=1}^{+\infty} \frac{|\sin n|}{n^2}$ is convergent by comparison with a p-series ($p=2$): $\frac{|\sin n|}{n^2} < \frac{1}{n^2}$

11. $\sum_{n=1}^{+\infty} \frac{n!}{(n+2)!}$ is convergent by comparison with a p-series ($p=2$): $\frac{n!}{(n+2)!} = \frac{1}{(n+2)(n+1)} < \frac{1}{n^2}$

12. $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^3+1}}$

▷ Because

$\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}}$

we let

$u_n = \frac{1}{\sqrt{n^3+1}}$ and $v_n = \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$

Because

$\sum_{n=1}^{+\infty} v_n = \sum_{n=1}^{+\infty} \frac{1}{n^{3/2}}$

is a p-series with $p = \frac{3}{2}$ and $p > 1$, the series converges. Thus, by Theorem 8.4.2(i) the given series $\sum_{n=1}^{+\infty} u_n$ also converges.

13. $\sum_{n=1}^{+\infty} \frac{n}{5n^2+3}$ is divergent by a limit comparison with the harmonic series:

$\lim_{n\to+\infty} \frac{n}{5n^2+3} / \frac{1}{n} = \lim_{n\to+\infty} \frac{n^2}{5n^2+3} = \lim_{n\to+\infty} \frac{1}{5+3n^{-2}} = \frac{1}{5} > 0$

14. $\sum_{n=1}^{+\infty} \frac{(n-1)!}{(n+1)!}$ converges by comparison with a p-series ($p=2$): $\frac{(n-1)!}{(n+1)!} = \frac{1}{n(n+1)} < \frac{1}{n^2}$

15. $\sum_{n=1}^{+\infty} \frac{n!}{(2n)!}$ is convergent by comparison with a geometric series: $\frac{n!}{(2n)!} = \frac{1}{(2n)(2n-1)\cdots(n+1)} < \frac{1}{2^n}$

16. $\sum_{n=1}^{+\infty} \sin \frac{1}{n}$

▷ We use the limit comparison test. Let

$u = \sin \frac{1}{n}$ and $v_n = \frac{1}{n}$

With the help of the substitution $x = 1/n$ we have

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \lim_{n\to+\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to0^+} \frac{\sin x}{x} = 1$$

Furthermore, $\sum_{n=1}^{+\infty} v_n$ is the divergent harmonic series. By Theorem 8.4.3(i), we conclude that the given series is divergent.

17. $\sum_{n=1}^{+\infty} \frac{|\csc n|}{n}$ is divergent by comparison with the harmonic series: $\frac{|\csc n|}{n} > \frac{1}{n}$.

18. $\sum_{n=1}^{+\infty} \frac{1}{n+\sqrt{n}}$ is divergent by comparison with the harmonic series: $\frac{1}{n+\sqrt{n}} > \frac{1}{2n}$

19. $\sum_{n=1}^{+\infty} \frac{1}{n\sqrt{n^2-1}}$ is convergent by a limit comparison with a p-series ($p = 2$):

$$\lim_{n\to+\infty} \frac{1}{n\sqrt{n^2-1}} \Big/ \frac{1}{n^2} = \lim_{n\to+\infty} \sqrt{\frac{n^2}{n^2-1}} = \lim_{n\to+\infty} \sqrt{\frac{1}{1-n^{-2}}} = 1$$

20. $\sum_{n=1}^{+\infty} \frac{2^n}{n!}$

▷ If $n > 2$, then

$$\frac{2^n}{n!} = \frac{2[2\cdot 2\cdot \ldots \cdot 2]2\cdot 2}{1[2\cdot 3\cdot \ldots \cdot (n-2)](n-1)n} \leq \frac{2\cdot 2\cdot 2}{(n-1)n}$$

Furthermore, $n - 1 > \frac{1}{2}n$ and so

$$\frac{2^n}{n!} < \frac{16}{n^2}$$

Thus the given series converges by comparison with a p-series with $p = 2$.

21. $\sum_{n=1}^{+\infty} \frac{3}{2n-\sqrt{n}}$ is divergent by comparison with a harmonic series: $\frac{3}{2n-\sqrt{n}} > \frac{3}{2n}$.

22. $\sum_{n=1}^{+\infty} \frac{\sqrt{n}}{n^2+1}$ is convergent by comparison with a p-series ($p = 2$): $\frac{\sqrt{n}}{n^2+1} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

23. $\sum_{n=1}^{+\infty} \frac{\ln n}{n^2+2}$ is convergent by a limit comparison with a p-series ($p = \frac{3}{2}$):

$$\lim_{n\to+\infty} \frac{\ln n}{n^2+1} \Big/ \frac{1}{n^{3/2}} = \lim_{n\to+\infty} \frac{\ln n}{n^{1/2}+2n^{-3/2}} = \lim_{n\to+\infty} \frac{n^{-1}}{\frac{1}{2}n^{-3/2}-3n^{-5/2}} = \lim_{n\to+\infty} \frac{1}{\frac{1}{2}n^{1/2}-3n^{-3/2}} = 0$$

Because $\lim_{n\to+\infty} \ln n = +\infty$ and $\lim_{n\to+\infty} (n^{1/2} + 2n^{-3/2}) = +\infty$, we used L'Hôpital's rule.

24. $\sum_{n=1}^{+\infty} \frac{1}{3^n - \cos n}$

▷ Because $-1 \leq -\cos n$, then

$$3^n - 1 \leq 3^n - \cos n$$

$$\frac{1}{3^n - \cos n} \leq \frac{1}{3^n - 1} < \frac{2}{3^n}$$

We let

$$u_n = \frac{1}{3^n - \cos n} \quad \text{and} \quad v_n = \frac{2}{3^n}$$

Because $\sum_{n=1}^{+\infty} v_n = \sum_{n=1}^{+\infty} \frac{2}{3^n}$ is a convergent geometric series, then $\sum_{n=1}^{+\infty} u_n$ converges.

In Exercises 25–34, apply the integral test to determine whether the series is convergent or divergent.

25. $\sum_{n=1}^{+\infty} \frac{1}{2n+1}$. If $f(x) = \frac{1}{2x+1}$ then f is continuous, $f(x) > 0$, and f is decreasing for $x \geq 1$.

$$\int_1^{+\infty} \frac{dx}{2x+1} = \lim_{b\to+\infty} \int_1^b \frac{dx}{2x+1} = \lim_{b\to+\infty} \frac{1}{2}\ln|2x+1|\Big]_1^b = \lim_{n\to+\infty} \left(\frac{1}{2}\ln|2b+1| - \frac{1}{2}\ln 3\right) = +\infty$$

Therefore the given series is divergent by the integral test.

26. $\sum_{n=1}^{+\infty} \frac{2}{(3n+5)^2}$. If $f(x) = \frac{2}{(3x+5)^2}$ then f is continuous, $f(x) > 0$, and f is decreasing for $x \geq 1$.

$$\int_1^{+\infty} \frac{2\,dx}{(3x+5)^2} = \lim_{b\to+\infty} \int_1^b \frac{2\,dx}{(3x+5)^2} = \lim_{b\to+\infty} -\frac{1}{3(3x+5)}\Big]_1^b = \lim_{b\to+\infty}\left[\frac{1}{24} - \frac{1}{3(3b+5)}\right] = \frac{1}{24}$$

Therefore the given series is convergent by the integral test.

27. $\sum_{n=1}^{+\infty} \frac{1}{(n+2)^{3/2}}$. If $f(x) = \frac{1}{(x+3)^{3/2}}$ then f is continuous, $f(x) > 0$, and f is decreasing for $x \geq 1$.

$$\int_1^{+\infty} \frac{dx}{(x+2)^{3/2}} = \lim_{b \to +\infty} \int_1^b \frac{dx}{(x+3)^{3/2}} = \lim_{b \to +\infty} \frac{-2}{(x+2)^{1/2}}\Big]_1^b = \lim_{b \to +\infty} \left[-\frac{2}{(b+2)^{1/2}} + \tfrac{2}{3}\sqrt{3} \right] = \tfrac{2}{3}\sqrt{3}$$

Therefore the given series is convergent by the integral test.

28. $\sum_{n=2}^{\infty} \frac{n}{n^2-2}$

▷ Let

$$f(x) = \frac{x}{x^2-2}$$

We show that the hypothesis of the integral test is satisfied. First, we note that f is continuous and positive valued for all $x \geq 2$. We show that f is also decreasing for $x \geq 2$.

$$f'(x) = \frac{1(x^2-2) - x(2x)}{(x^2-2)^2} = -\frac{x^2+2}{(x^2-2)^2}$$

Because $f'(x) < 0$ if $x \geq 2$, then f is decreasing on $[2, +\infty)$. Thus the hypothesis of Theorem 8.4.6 is satisfied.

$$\int_2^{+\infty} f(x)dx = \int_2^{+\infty} \frac{x\,dx}{x^2-2} = \lim_{b \to +\infty} \int_2^b \frac{x\,dx}{x^2-2} = \lim_{b \to +\infty} \tfrac{1}{2} \ln|x^2-2|\Big]_2^b = \tfrac{1}{2} \lim_{b \to +\infty} [\ln(b^2-2) - \ln 2] = +\infty$$

Therefore, the given series is divergent.

29. $\sum_{n=3}^{+\infty} \frac{4}{n^2-4}$. If $f(x) = \frac{4}{x^2-4}$ then f is continuous, $f(x) > 0$, and f is decreasing for $x \geq 3$.

$$\int_3^{+\infty} \frac{4\,dx}{x^2-4} = \lim_{b \to +\infty} \int_3^b \frac{4\,dx}{x^2-4} = \lim_{b \to +\infty} \ln\left|\frac{x-2}{x+2}\right|\Big]_3^b = \lim_{b \to +\infty} \left(\ln\left|\frac{b-2}{b+2}\right| - \ln \tfrac{1}{5} \right) = \lim_{b \to +\infty} \left(\ln\left|\frac{1-2b^{-1}}{1+2b^{-1}}\right| - \ln \tfrac{1}{5} \right)$$

$= \ln 5$. Therefore the given series is convergent by the integral test.

30. $\sum_{n=1}^{+\infty} \frac{2n+3}{(n^2+3n)^2}$. If $f(x) = \frac{2x+3}{(x^2+3x)^2}$ then f is continuous, $f(x) > 0$, and f is decreasing for $x \geq 1$.

$$\int_1^{+\infty} \frac{(2x+3)dx}{(x^2+3x)} = \lim_{b \to +\infty} \int_1^b \frac{d(x^2+3x)}{(x^2+3x)^2} = \lim_{b \to +\infty} -\frac{1}{x^2+3x}\Big]_1^b = \lim_{b \to +\infty} \left[\frac{1}{4} - \frac{1}{b^2+3b} \right] = \frac{1}{4}.$$ Series converges.

31. $\sum_{n=1}^{+\infty} e^{-5n}$. If $f(x) = e^{-5x}$ then f is continuous, $f(x) > 0$ and f is decreasing for $x \geq 1$.

$$\int_1^{+\infty} e^{-5x}dx = \lim_{b \to +\infty} \int_1^b e^{-5x}dx = \lim_{b \to +\infty} \left[-\tfrac{1}{5} e^{-5x} \right]_1^b = \lim_{b \to +\infty} \left(-\tfrac{1}{5} e^{-5b} + \tfrac{1}{5} e^{-5} \right) = \tfrac{1}{5} e^{-5}$$

Therefore the given series is convergent by the integral test.

32. $\sum_{n=1}^{\infty} \frac{2n}{n^4+1}$

▷ Let

$$f(x) = \frac{2x}{x^4+1}$$

Then

$$f'(x) = \frac{2(x^4+1) - 2x(4x^3)}{(x^4+1)^2} = \frac{2-6x^4}{(x^4+1)^2}$$

Because $f'(x) < 0$ if $x \geq 1$, then f is decreasing on $[1, +\infty)$. Furthermore, f is continuous and positive valued for all $x \geq 1$. Thus, the hypothesis of the integral test 8.4.6 is satisfied.

$$\int_1^{+\infty} f(x)dx = \int_1^{+\infty} \frac{2x\,dx}{x^4+1} = \lim_{b \to +\infty} \int_1^b \frac{2x\,dx}{1+(x^2)^2} = \lim_{b \to +\infty} \tan^{-x} x^2\Big]_1^b = \lim_{b \to +\infty} [\tan^{-1} b^2 - \tan^{-1} 1]$$

$$= \tfrac{1}{2}\pi - \tfrac{1}{4}\pi = \tfrac{1}{4}\pi$$

Therefore, the given series is convergent.

In Exercises 33–48, use any method to determine whether the series is convergent or divergent.

33. $\sum_{n=1}^{+\infty} \frac{\ln n}{n}$. If $f(x) = \frac{\ln x}{x}$, then f is continuous and $f(x) > 0$ if $x > 1$. Furthermore, $f(x) = \frac{1 - \ln x}{x^2} < 0$ if $x > e$, and so f is decreasing for all $x \geq e$. $\int_e^{+\infty} \ln x \frac{dx}{x} = \lim_{b \to +\infty} \left[\frac{1}{2} \ln^2 x\right]_e^b = \lim_{b \to +\infty} \left(\frac{1}{2} \ln^2 b - \frac{1}{2}\right) = +\infty$

Therefore, the given series is divergent by the integral test. Alternatively,

$\sum_{n=1}^{+\infty} \frac{\ln n}{n}$ diverges by comparison with the harmonic series: $\frac{\ln n}{n} > \frac{1}{n}$.

34. $\sum_{n=2}^{+\infty} \frac{1}{n \ln n}$. We apply the integral test. If $f(x) = \frac{1}{x \ln x}$, then f is continuous, decreasing and positive $x \geq 2$.

$\int_2^{+\infty} \frac{dx}{x \ln x} = \lim_{b \to +\infty} \ln(\ln x)\Big]_2^b = \lim_{b \to +\infty} [\ln(\ln b) - \ln(\ln 2)] = +\infty$. Thus, the given series diverges.

35. $\sum_{n=1}^{+\infty} \frac{\tan^{-1} n}{n^2 + 1}$. If $f(x) = \frac{\tan^{-1} x}{x^2 + 1}$, then f is continuous and $f(x) > 0$ for $x \geq 1$. Furthermore,

$f'(x) = \frac{1 - 2x \tan^{-1} x}{(x^2 + 1)^2} < 0$ if $x \geq 1$, and so f is decreasing for $x \geq 1$.

$\int_1^{+\infty} \tan^{-1} x \frac{dx}{x^2 + 1} = \lim_{b \to +\infty} \left[\frac{1}{2}(\tan^{-1} x)^2\right]_1^b = \lim_{b \to +\infty} \left[\frac{1}{2}(\tan^{-1} b)^2 - \frac{1}{32}\pi^2\right] = \frac{1}{8}\pi^2 - \frac{1}{32}\pi^2 = \frac{3}{32}\pi^2$

Therefore the given series is convergent by the integral test. Alternatively,

$\sum_{n=1}^{+\infty} \frac{\tan^{-1} n}{n^2 + 1}$ converges by comparison with a p-series ($p = 2$): $\frac{\tan^{-1} n}{n^2 + 1} < \frac{\frac{1}{2}\pi}{n^2}$.

36. $\sum_{n=1}^{+\infty} n e^{-n^2}$

▸ We use the integral test with $f(x) = x e^{-x^2}$. It is clear that f is continuous and positive valued. To show that f is decreasing for $x > 1$, we find the derivative of f. Thus

$f'(x) = e^{-x^2}(1 - 2x^2)$

Because $f'(x) < 0$ if $x \geq 1$, then f is decreasing on $[1, +\infty)$. Thus, the hypothesis of the integral test is satisfied. Furthermore,

$\int_1^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_1^b x e^{-x^2} dx = \lim_{b \to +\infty} \left[-\frac{1}{2} e^{-x^2}\right]_1^b = \lim_{b \to +\infty} \left[-\frac{1}{2} e^{-b^2} + \frac{1}{2} e^{-1}\right] = \frac{1}{2e}$

Thus, the given series is convergent.

37. $\sum_{n=1}^{+\infty} n^2 e^{-n}$ is convergent by a limit comparison with the geometric series $\sum_{n=1}^{+\infty} e^{-n/2}$:

$\lim_{n \to +\infty} \frac{n^2 e^{-n}}{e^{-n/2}} = \lim_{n \to +\infty} \frac{n^2}{e^{n/2}} = \left(\lim_{n \to +\infty} \frac{n}{e^{n/4}}\right)^2 = \left(\lim_{n \to +\infty} \frac{1}{\frac{1}{4} e^{n/4}}\right)^2 = 0$

Because $\lim_{n \to +\infty} e^{n/4} = +\infty$, we applied L'Hôpital's rule.

38.

$$s_n = e^{-1} + 2e^{-2} + 3e^{-3} + \cdots + ne^{-n}$$

$$e^{-1} s_n = \qquad e^{-2} + 2e^{-3} + \cdots + (n-1)e^{-n} + ne^{-n-1}.\ \text{Subtracting,}$$

$$s_n - e^{-1} s_n = e^{-1} + e^{-2} + e^{-3} + \cdots + e^{-n} - ne^{-n-1}$$

$$(1 - e^{-1}) s_n = e^{-1} \frac{1 - e^{-n}}{1 - e^{-1}} - ne^{-n-1},\ s_n = \frac{e^{-1}(1 - e^{-n})}{(1 - e^{-1})^2} - \frac{ne^{-n-1}}{1 - e^{-1}}.\ \lim_{n \to +\infty} s_n = \frac{e^{-1}}{(1 - e^{-1})^2} = \frac{e}{(e-1)^2}$$

Thus the series converges to $e/(e-1)^2$. The series of Exercise 37 sums to $e(e+1)/(e-1)^3$.

39. $\sum_{n=1}^{+\infty} \frac{\ln n}{n^3}$ is convergent by comparison with a p-series ($p = 2$): $\frac{\ln n}{n^3} < \frac{1}{n^2}$ because $\ln n < n$.

40. $\sum\limits_{n=1}^{+\infty} \cot^{-1} n$

▸ Because $\cot^{-1} n = \tan^{-1} \frac{1}{n}$, we use a limit comparison test with

$u_n = \tan^{-1} \frac{1}{n}$ and $v_n = \frac{1}{n}$

We use the substitution $x = \frac{1}{n}$ so that $x \to 0^+$ as $n \to +\infty$, and then apply L'Hôpital's rule. Thus,

$$\lim_{n \to +\infty} \frac{u_n}{v_n} = \lim_{n \to +\infty} \frac{\tan^{-1} \frac{1}{n}}{\frac{1}{n}} = \lim_{x \to 0^+} \frac{\tan^{-1} x}{x} \overset{0/0}{=} \lim_{x \to 0^+} \frac{1}{1+x^2} = 1$$

Because $\sum\limits_{n=1}^{+\infty} v_n$ is the divergent harmonic series, by Theorem 8.4.3(i) we conclude that $\sum\limits_{n=1}^{+\infty} u_n$ is also divergent.

41. $\sum\limits_{n=1}^{+\infty} \operatorname{csch} n = \sum\limits_{n=1}^{+\infty} \frac{2}{e^n - e^{-n}}$ is convergent by comparison with a geometric series: $\frac{2}{e^n - e^{-n}} < \frac{4}{e^n}$.

42. $\sum\limits_{n=1}^{+\infty} \frac{e^{\tan^{-1} n}}{n^2+1}$ is convergent by comparison with a p-series ($p = 2$): $\frac{e^{\tan^{-1} n}}{n^2+1} < \frac{e^{\pi/2}}{n^2}$

43. $\sum\limits_{n=1}^{+\infty} \frac{e^{1/n}}{n^2}$ is convergent by comparison with a p-series ($p = 2$): $\frac{e^{1/n}}{n^2} < \frac{e}{n^2}$.

44. $\sum\limits_{n=1}^{+\infty} \operatorname{sech}^2 n$

▸ Because

$$\operatorname{sech}^2 n = \left(\frac{2}{e^n + e^{-n}}\right)^2 < \left(\frac{2}{e^n}\right)^2 = \frac{4}{e^{2n}}$$

and $\sum\limits_{n=1}^{+\infty} \frac{4}{e^{2n}}$ is a convergent geometric series, the given series converges by comparison.

45. $\sum\limits_{n=1}^{+\infty} \ln \frac{n+3}{n}$ is divergent by a limit comparison with the harmonic series:

$$\lim_{n \to +\infty} \frac{\ln \frac{n+3}{n}}{\frac{1}{n}} = \lim_{n \to +\infty} \frac{\ln(n+3) - \ln n}{\frac{1}{n}} = \lim_{n \to +\infty} \frac{\frac{1}{n+3} - \frac{1}{n}}{-\frac{1}{n^2}} = \lim_{n \to +\infty} \frac{3n^2}{n(n+3)} = \lim_{n \to +\infty} \frac{3}{1 + \frac{3}{n}} = 3 > 0$$

Because $\lim\limits_{n \to +\infty} \ln \frac{n+3}{n} = \ln 1 = 0$ and $\lim\limits_{n \to +\infty} \frac{1}{n} = 0$, we used L'Hôpital's rule.

46. $\sum\limits_{n=2}^{+\infty} \frac{1}{n(\ln n)^3}$. Let $f(x) = \frac{1}{x(\ln x)^3}$. Then f is continuous, decreasing and positive for $x \geq 2$. Furthermore,

$$\int_2^{+\infty} \frac{dx}{x(\ln x)^3} = \lim_{b \to +\infty} -\frac{1}{2(\ln x)^2}\bigg]_2^b = \lim_{b \to +\infty}\left[\frac{1}{2(\ln 2)^2} - \frac{1}{2(\ln b)^2}\right] = \frac{1}{2(\ln 2)^2}.$$ Thus, the given series converges.

47. $\sum\limits_{n=1}^{+\infty} \frac{(n+1)^2}{(n+2)!}$ is convergent by comparison with $\sum\limits_{n=1}^{+\infty} \frac{1}{n!}$: $\frac{(n+1)^2}{(n+2)!} = \frac{(n+1)^2}{n!(n+1)(n+2)} = \frac{n+1}{n+2} \cdot \frac{1}{n!} < \frac{1}{n!}$

48. $\sum\limits_{n=1}^{+\infty} \frac{1}{(n+2)(n+4)}$

▸ Let s_n be the nth partial sum. Then

$$\begin{aligned} s_n &= \frac{1}{3 \cdot 5} + \frac{1}{4 \cdot 6} + \frac{1}{5 \cdot 7} + \frac{1}{6 \cdot 8} + \cdots + \frac{1}{(n-1)(n+1)} + \frac{1}{n(n+2)} + \frac{1}{(n+1)(n+3)} + \frac{1}{(n+2)(n+4)} \\ &= \frac{1}{2}\left[\left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \cdots \left(\frac{1}{n-1} + \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) + \left(\frac{1}{n+1} - \frac{1}{n+3}\right) + \left(\frac{1}{n+2} - \frac{1}{n+4}\right)\right] \\ &= \frac{1}{2}\left[\frac{1}{3} + \frac{1}{4} - \frac{1}{n+1} - \frac{1}{n+3}\right] \end{aligned}$$

Then $\lim\limits_{n \to +\infty} s_n = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right) = \frac{7}{12}$. Thus, the given series converges to $\frac{7}{24}$.

49. Because $\sum\limits_{n=1}^{+\infty} b_n$ is convergent, $\lim\limits_{n \to +\infty} b_n = 0$. Then $\lim\limits_{n \to +\infty} \frac{a_n b_n}{a_n} = \lim\limits_{n \to +\infty} b_n = 0$. Because $\sum\limits_{n=1}^{+\infty} a_n$ is convergent it follows from the limit comparison test that $\sum\limits_{n=1}^{+\infty} a_n b_n$ is convergent.

50. If $p \le 1$, then $\int_1^{+\infty} \frac{dx}{x^p} \le \int_1^{+\infty} \frac{dx}{x} = \lim_{b\to+\infty} \ln x\Big]_1^b = \lim_{b\to+\infty} \ln b = +\infty$ while if $p > 1$, then
$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b\to+\infty} -\frac{1}{(p-1)x^{p-1}}\Big]_1^{+\infty} = \lim_{b\to+\infty} \frac{1}{p-1}\left(1 - \frac{1}{b^{p-1}}\right) = \frac{1}{p-1}$

51. If $p \le 0$, the series $\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^p}$ is divergent by comparison with the harmonic series; assume $p \ge 0$.

Let $f(x) = \frac{1}{x(\ln x)^p}$. f is continuous, $f(x) > 0$, and f is decreasing for $x \ge 2$.

Case 1: $p \ne 1$. $\int_2^{+\infty} \frac{dx}{x(\ln x)^p} = \lim_{b\to+\infty} \int_2^b \frac{1}{(\ln x)^p} \cdot \frac{dx}{x} = \lim_{b\to+\infty} \left[\frac{-1}{(p-1)(\ln x)^{p-1}}\right]_2^b$

$= \lim_{b\to+\infty} \frac{-1}{(p-1)(\ln b)^{p-1}} + \frac{1}{(p-1)(\ln 2)^{p-1}}$

If $p > 1$, $\lim_{b\to+\infty} \frac{-1}{(p-1)(\ln b)^{p-1}} = 0$; if $p < 1$, $\lim_{b\to+\infty} \frac{-1}{(p-1)(\ln b)^{p-1}} = +\infty$. Therefore, by the integral test $\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^p}$ is convergent if $p > 1$ and divergent if $p < 1$.

Case 2: $p = 1$. $\int_2^{+\infty} \frac{dx}{x(\ln x)} = \lim_{b\to+\infty} \int_2^b \frac{1}{\ln x} \cdot \frac{dx}{x} = \lim_{b\to+\infty} \Big[\ln|\ln x|\Big]_2^b = \lim_{b\to+\infty} \ln|\ln b| - \ln(\ln 2) = +\infty$.

Hence the series is divergent if $p = 1$.

52. Prove that the series $\sum_{n=3}^{+\infty} \frac{1}{n(\ln n)[\ln(\ln n)]^p}$ is convergent if and only if $p > 1$.

▶ We apply the integral test. Let

$f(x) = \frac{1}{x(\ln x)[\ln(\ln x)]^p}$

If $x \ge 3$, then $\ln x > 1$, and $\ln(\ln x) > 0$, and thus $f(x) > 0$. Hence f is continuous, decreasing, and positive valued for all $x \ge 3$. Furthermore,

$$\int_3^{+\infty} f(x)dx = \int_3^{+\infty} \frac{dx}{x(\ln x)[\ln(\ln x)]^p} = \lim_{b\to+\infty} \int_3^b \frac{dx}{x(\ln x)[\ln(\ln x)]^p} \tag{1}$$

To evaluate the indefinite integral we let

$u = \ln(\ln x)$ and $du = \frac{dx}{x(\ln x)}$

Thus,

$$\int \frac{dx}{x(\ln x)[\ln(\ln x)]^p} = \int \frac{du}{u^p} = \begin{cases} \ln|u| & \text{if } p = 1 \\ \frac{u^{-p+1}}{-p+1} & \text{if } p \ne 1 \end{cases} \tag{2}$$

We consider two cases.

Case 1: $p \le 1$

Substituting from (2) into (1), we have

$$\int_3^{+\infty} f(x)dx \ge \int_3^{+\infty} \frac{dx}{x(\ln x)\ln(\ln x)} = \lim_{b\to+\infty} \ln|u|\Big]_3^b = \lim_{b\to+\infty} [\ln b - \ln 3] = +\infty$$

Thus, the given series is divergent if $p \le 1$.

Case 2: $p > 1$

Then $p - 1 > 0$, and

$$\int_3^{+\infty} f(x)dx = \lim_{b\to+\infty} \left[\frac{u^{-p+1}}{-p+1}\right]_3^b = \frac{-1}{p-1} \lim_{b\to+\infty} \left[\frac{1}{u^{p-1}}\right]_3^b = \frac{-1}{p-1} \lim_{b\to+\infty} \left[\frac{1}{b^{p-1}} - \frac{1}{3^{p-1}}\right] = \frac{1}{(p-1)3^{p-1}}$$

Thus, the given series is convergent if $p > 1$.

53. If $p \le 1$, $\sum_{n=1}^{+\infty} \frac{\ln n}{n^p}$ is divergent by comparison with the harmonic series: $\frac{\ln n}{n^p} > \frac{1}{n}$. If $p > 1$, let q satisfy $p > q > 1$. Then $\sum_{n=1}^{+\infty} \frac{\ln n}{n^p}$ is convergent by a limit comparison with $\sum_{n=1}^{+\infty} \frac{1}{n^q}$:

$\lim_{n\to+\infty} \frac{\ln n}{n^p} / \frac{1}{n^q} = \lim_{n\to+\infty} \frac{\ln n}{n^{p-q}} = \lim_{n\to+\infty} \frac{1/n}{(p-q)n^{p-q-1}} = \lim_{n\to+\infty} \frac{1}{(p-q)n^{p-q}} = 0$

Because $\lim_{n\to+\infty} \ln n = +\infty$ and $\lim_{n\to+\infty} n^{p-q} = +\infty$, we applied L'Hôpital's rule.

54. See Section 8.3.

55. $\sum_{m=50}^{100} \frac{1}{m} = \sum_{m=50}^{100} \int_m^{m+1} \frac{1}{m}\,dx > \sum_{m=50}^{100} \int_m^{m+1} \frac{1}{x}\,dx = \sum_{m=50}^{100} [\ln(m+1) - \ln m] = \ln 101 - \ln 50 \approx 0.7032$

$\sum_{m=50}^{100} \frac{1}{m} = \sum_{m=50}^{100} \int_{m-1}^{m} \frac{1}{m}\,dx < \sum_{m=50}^{100} \int_{m-1}^{m} \frac{1}{x}\,dx = \sum_{m=50}^{100} [\ln m - \ln(m-1)] = \ln 100 - \ln 49 \approx 0.7134$

By the trapezoidal rule, $\int_{50}^{100} \frac{dx}{x} = \ln x\Big]_{50}^{100} = \ln 100 - \ln 50 \approx \frac{1}{2}\cdot\frac{1}{50} + \frac{1}{51} + \cdots + \frac{1}{99} + \frac{1}{2}\cdot\frac{1}{100}$ so

$\sum_{m=50}^{100} \frac{1}{m} \approx \ln 2 + \frac{1}{2}\cdot\frac{1}{50} + \frac{1}{2}\cdot\frac{1}{100} = 0.70815$. The sum is 0.70817.

56. Suppose f is a function such that $f(n) > 0$ for n any positive integer. Furthermore, suppose that for some number p, $\lim_{n\to+\infty} n^p f(n)$ exists and is positive. Prove that the series $\sum_{n=1}^{+\infty} f(n)$ is convergent if $p > 1$ and divergent if $p \le 1$.

▸ We use a limit comparison with a p-series. Let

$u_n = f(n)$ and $v_n = \frac{1}{n^p}$

By hypothesis, $u_n > 0$ for all positive n. Moreover,

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \lim_{n\to+\infty} n^p f(n) \tag{1}$$

By hypothesis, the limit in (1) exists and is positive. Thus, by Th. 8.4.3(i) the series $\sum_{n=1}^{+\infty} f(n)$ is convergent if $\sum_{n=1}^{+\infty} v_n$ converges, and the series $\sum_{n=1}^{+\infty} f(n)$ is divergent if $\sum_{n=1}^{+\infty} v_n$ diverges. Because $\sum_{n=1}^{+\infty} v_n$ is a p-series, it converges if $p > 1$ and diverges if $p \le 1$. Thus, the given series is convergent if $p > 1$ and divergent if $p \le 1$.

57. Prove Th. 8.4.3(ii): Because $\lim_{n\to+\infty}(u_n/v_n) = 0$, then there exists a number $M > 0$ such that $u_n/v_n < 1$ whenever $n > M$ or, equivalently, $u_n < v_n$. Because $\sum_{n=1}^{+\infty} v_n$ is convergent, $\sum_{n=1}^{+\infty} u_n$ is convergent by comparison with $\sum_{n=1}^{+\infty} v_n$.

58. Prove Th. 8.4.3(iii). Because $\lim_{n\to+\infty}(u_n/v_n) = +\infty$, then there exists a number $M > 0$ such that $u_n/v_n > 1$ whenever $n > M$ or, equivalently, $u_n > v_n$. Because $\sum_{n=1}^{+\infty} v_n$ is divergent, $\sum_{n=1}^{+\infty} u_n$ is divergent by comparison with $\sum_{n=1}^{+\infty} v_n$.

59. The series $\sum_{n=1}^{+\infty} (-1)^{n+1}$ is divergent because $\lim_{n\to+\infty} (-1)^{n+1}$ does not exist. If n is odd then $s_n = 1$; and if n is even $s_n = 0$. Therefore the sequence of partial sums $\{s_n\}$ has an upper bound of 1. Thus Theorem 8.4.1 does not apply to an infinite series of both positive and negative terms.

8.5 INFINITE SERIES OF POSITIVE AND NEGATIVE TERMS

An alternating series is a series whose terms are alternately positive and negative. If the terms in an alternating series are decreasing in absolute value and have limit zero, the alternating series is convergent. Furthermore, if the sum of such an alternating series is approximated by the first k terms, then the absolute value of the error is less than the absolute value of the $(k+1)$st term. We state these facts formally in the following definitions and theorems.

8.5.1 Definition If $a_n > 0$ for all positive integers n, then the series

$$\sum_{n=1}^{+\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1} a_n + \cdots \tag{1}$$

and the series

$$\sum_{n=1}^{+\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n + \cdots \tag{2}$$

are called *alternating series.*

8.5.3 Definition If an infinite series is convergent and its sum is S, then the *remainder* obtained by approximating the the sum of the series by the kth partial sum s_k is denoted by R_k and $R_k = S - s_k$.

8.5.2 Theorem (Alternating Series Test) Suppose we have the alternating series (1) or (2), where $a_n > 0$. If (i) $a_{n+1} \le a_n$ for all positive integers n and (ii) $\lim_{n\to+\infty} a_n = 0$, then the alternating series is convergent.

Note that it is enough if (i) holds for all $n > N$ for some positive integer N. Also, if $a_n = p(n)/q(n)$ is a rational function of n, then (ii) implies (i) because a_n has at most $\deg(p) + \deg(q) - 1$ critical points and is monotonic after its last critical point.

8.5.4 Theorem Furthermore, if R_k is the remainder obtained by approximating the sum of the series by the first k terms, then $|R_k| < a_{k+1}$.

If, in addition, the sequence $\{a_n - a_{n+1}\}$ is decreasing and t_k is s_k with the last term halved, then $|S - t_k| < \frac{1}{2}(a_k - a_{k+1})$.

Proof

$$\begin{aligned} |S - t_k| &= \left|\tfrac{1}{2}(-1)^{k+1}a_k + (-1)^{k+2}a_{k+1} + (-1)^{k+3}a_{k+2} + \cdots\right| \\ &= \left|\tfrac{1}{2}(-1)^{k+1}[(a_k - a_{k+1}) - (a_{k+1} - a_{k+2}) + \cdots]\right| \\ &= \tfrac{1}{2}[(a_k - a_{k+1}) - (a_{k+1} - a_{k+2}) + \cdots] \\ &< \tfrac{1}{2}(a_k - a_{k+1}) \end{aligned}$$

as in the proof of Theorem 8.5.4.

If we replace each term in a convergent series by its absolute value, the resulting series may be either convergent or divergent. If convergent, we say that the original series is *absolutely convergent*; if divergent, we say that the original series is *conditionally convergent*. The formal definitions and theorems follow.

8.5.5 Definition The infinite series $\sum_{n=1}^{+\infty} u_n$ is said to be *absolutely convergent* if the series $\sum_{n=1}^{+\infty} |u_n|$ is convergent.

8.5.6 Definition A series that is convergent, but not absolutely convergent, is said to be *conditionally convergent*.

8.5.7 Theorem If the infinite series $\sum_{n=1}^{+\infty} u_n$ is absolutely convergent, then it is convergent and

$$\left|\sum_{n=1}^{+\infty} u_n\right| \le \sum_{n=1}^{+\infty} |u_n|$$

The tests which follow do not require that we know a comparison series.

8.5.8 Theorem (Ratio Test) Let $\sum_{n=1}^{+\infty} u_n$ be a given infinite series for which every u_n is nonzero. Then

(i) if $\lim_{n\to+\infty} |u_{n+1}/u_n| = L < 1$, the given series is absolutely convergent;

(ii) if $\lim_{n\to+\infty} |u_{n+1}/u_n| = L > 1$ or if $\lim_{n\to+\infty} |u_{n+1}/u_n| = +\infty$, the series is divergent;

(iii) if $\lim_{n\to+\infty} |u_{n+1}/u_n| = 1$ or if $\lim_{n\to+\infty} |u_{n+1}/u_n|$ does not exist, no conclusion regarding convergence may be made from this test.

8.5.9 Theorem (Root Test) Let $\sum_{n=1}^{+\infty} u_n$ be a given infinite series for which every u_n is nonzero. Then

(i) if $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = L < 1$, the given series is absolutely convergent;

(ii) if $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = L > 1$ or if $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = +\infty$, the series is divergent;

(iii) if $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = 1$ or if $\lim_{n\to+\infty} \sqrt[n]{|u_n|}$ does not exist, no conclusion regarding convergence may be made from this test.

If the ratio test fails, as it does with the p series, the following test, which we state without proof, applies.

Theorem (Raabe's Test) Let $\sum_{i=1}^{n} u_n$ be a given infinite series for which $\lim_{n\to+\infty} |u_n/u_{n+1}| = 1$. Then

(i) if $\lim_{n\to+\infty} n(|u_n/u_{n+1}| - 1) = p > 1$ or $\lim_{n\to+\infty} n(|u_n/u_{n+1}| - 1) = +\infty$, the given series is absolutely convergent;

(ii) if $\lim_{n\to+\infty} n(|u_n/u_{n+1}| - 1) = p < 1$ or $\lim_{n\to+\infty} n(|u_n/u_{n+1}| - 1) = -\infty$, the series is divergent;

(iii) if $\lim_{n\to+\infty} n(|u_n/u_{n+1}| - 1) = 1$ or $\lim_{n\to+\infty} n(|u_n/u_{n+1}| - 1)$ does not exist, no conclusion regarding convergence may be made from this test.

Exercises 8.5

In Exercises 1–14, determine whether the alternating series is convergent or divergent.

▷ We use the alternating series test.

1. $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{1}{2n}$ converges because $\frac{1}{2(n+1)} < \frac{1}{2n}$ and $\lim_{n\to+\infty} \frac{1}{2n} = 0$.

2. $\sum_{n=1}^{+\infty} (-1)^{n}\frac{1}{n^2}$ converges because $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ and $\lim_{n\to+\infty} \frac{1}{n^2} = 0$.

3. $\sum_{n=1}^{+\infty} (-1)^{n}\frac{3}{n^2+1}$ converges because $\frac{3}{(n+1)^2+1} < \frac{3}{n^2+1}$ and $\lim_{n\to+\infty} \frac{3}{n^2+1} = 0$.

4. $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{4}{3n-2}$

▷ We apply the alternating series test (8.5.2). Let

$$a_n = \frac{4}{3n-2}$$

Then $a_n > 0$ for all positive integers n. Furthermore, because the denominator of a_n is increasing, then a_n is decreasing. Moreover,

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{4}{3n-2} = \lim_{n\to+\infty} \frac{\frac{4}{n}}{3-\frac{2}{n}} = 0$$

By the alternating series test we conclude that the series $\sum_{n=1}^{+\infty} (-1)^{n+1} a_n$, which is the given series, is convergent.

5. $\sum_{n=1}^{+\infty} (-1)^{n}\frac{1}{\ln n}$ converges because $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$ and $\lim_{n\to+\infty} \frac{1}{\ln n} = 0$.

6. $\sum_{n=1}^{+\infty} (-1)^{n+1}\sin\frac{\pi}{n}$ converges because $\sin\frac{\pi}{n+1} < \sin\frac{\pi}{n}$ if $n \geq 2$ and $\lim_{n\to+\infty} \sin\frac{\pi}{n} = 0$

7. $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{n^2}{n^3+2}$. Because $\frac{d}{dx}\left(\frac{x^2}{x^3+2}\right) = \frac{2x(x^3+2) - x^2(3x^2)}{(x^3+2)^2} = \frac{4x - x^4}{(x^3+2)^2} < 0$ if $x \geq 2$, a_n is decreasing.

Also $\lim_{n\to+\infty} \frac{n^2}{n^3+2} = \lim_{n\to+\infty} \frac{n^{-1}}{1+2n^{-3}} = 0$. Thus the series is convergent.

8. $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{\ln n}{n}$

▷ We apply the alternating series test with

$$a_n = \frac{\ln n}{n}$$

Then $a_1 = 0$ and $a_n > 0$ for all integers n with $n \geq 2$. Let

$$f(x) = \frac{\ln x}{x} \qquad f'(x) = \frac{1-\ln x}{x^2}$$

Because $f'(x) < 0$ if $x > e$, then f is decreasing on $[e, +\infty)$. Thus, $a_{n+1} < a_n$ if $n \geq 3$. Furthermore, by L'Hôpital's rule,

$$\lim_{n\to+\infty} a_n = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{1}{x} = 0$$

Therefore, by Theorem 8.3.9 and the alternating series test, we conclude that the given series is convergent.

9. $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{\ln n}{n^2}$. Because $\frac{d}{dx}\left(\frac{\ln x}{x^2}\right) = \frac{x-(2x)\ln x}{x^4} = \frac{1-2\ln x}{x^3} < 0$ for $x \geq 2$, a_n is decreasing.
Also by L'Hôpital's rule $\lim_{x\to+\infty} \frac{\ln x}{x^2} = \lim_{x\to+\infty} \frac{x^{-1}}{2x} = \lim_{x\to+\infty} \frac{1}{2x^2} = 0$. Thus the series converges.

10. $\sum_{n=1}^{+\infty} (-1)^n \frac{e^n}{n}$ is divergent because $\lim_{n\to+\infty} \frac{e^n}{n} = +\infty$.

11. $\sum_{n=1}^{+\infty} (-1)^n \frac{3^n}{n^2}$ is divergent because $\lim_{x\to+\infty} \frac{3^x}{x^2} = \left(\lim_{x\to+\infty} \frac{3^{x/2}}{x}\right)^2 = \left(\lim_{x\to+\infty} \frac{\frac{1}{2}(\ln 3)3^{x/2}}{1}\right)^2 = +\infty$.

12. $\sum_{n=1}^{+\infty} (-1)^n \frac{\sqrt{n}}{3n-1}$

▹ Let

$$a_n = \frac{\sqrt{n}}{3n-1}$$

Then $a_n > 0$ for all positive integers n. Because

$$a_n = \frac{1}{3\sqrt{n} - \frac{1}{\sqrt{n}}}$$

and the denominator of a_n is increasing, then a_n is decreasing. Furthermore,

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{\sqrt{n}}{3n-1} = \lim_{n\to+\infty} \frac{1/\sqrt{n}}{3-1/n} = 0$$

Therefore, by the alternating series test, the given series is convergent.

13. $\sum_{n=1}^{+\infty} (-1)^n \frac{n}{2^n}$. By L'Hôpital's rule $\lim_{x\to+\infty} \frac{x}{2^x} = \lim_{x\to+\infty} \frac{1}{2^x \ln 2} = 0$. Also $\frac{n+1}{2^{n+1}} < \frac{n}{2^n}$ if and only if $\frac{n+1}{n} < 2$, which is true if $n > 1$. Hence the given series is convergent.

14. $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{3^n}{1+3^{2n}} = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{3^n + 3^{-n}}$. $\frac{d}{dx}(3^x + 3^{-x}) = \ln 3(3^x - 3^{-x}) > 0$ if $x > 0$ so u_n is decreasing and $\lim_{n\to+\infty} \frac{1}{3^n + 3^{-n}} = 0$. Thus, the series converges.

In Exercises 15–22, find an upper bound for the error if the sum of the first four terms is used as an approximation to the sum of the infinite series.

15. $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{1}{n}$. $|R_4| < |u_5| = \left|\frac{1}{5}\right| = \frac{1}{5}$

16. $\sum_{n=1}^{+\infty} (-1)^n \frac{2}{n^2}$

▹ We apply Theorem 8.5.4. We have

$$a_n = \frac{2}{n^2}$$

Because $a_n > 0$ and $a_{n+1} < a_n$ for all positive integers n, and $\lim_{n\to+\infty} a_n = 0$, then the remainder obtained by approximating the sum of the series by the first four terms is R_4 and

$$|R_4| < a_5 = \frac{2}{5^2} = 0.08$$

Thus, the error is less than 0.08 if the sum of the first four terms is used as an approximation to the sum of the given infinite series.

17. $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{(2n-1)^2}$. $|R_4| < |u_5| = \left|-\frac{1}{9^2}\right| = \frac{1}{81}$

18. $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{n}{(n+1)^2}$. $|R_4| < |u_5| = \frac{5}{36}$

19. $\sum_{n=1}^{+\infty} (-1)^n \frac{1}{n^2}$. $|R_4| < |u_5| = \left|-\frac{1}{5^2}\right| = \frac{1}{25}$

20. $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^n}$

▷ Let

$a_n = \frac{1}{n^n}$

Then $a_n > 0$ and $a_{n+1} < a_n$ for all positive integers n. And $\lim_{n\to+\infty} a_n = 0$. By Theorem 8.5.4 the remainder obtained by approximating the sum of the series by the first four terms is R_4 and

$|R_4| < a_5 = \frac{1}{5^5} = 0.00032$

Therefore, the error is less than 0.00032 if the sum of the first four terms is used to approximate the sum of the given infinite series.

21. $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{(n+1)\ln(n+1)}$. $|R_4| < |u_5| = \frac{1}{6\ln 6}$ **22.** $\sum_{n=1}^{+\infty}(-1)^n\frac{1}{n!}$. $|R_4| < |u_5| = \frac{1}{5!} = \frac{1}{120}$

In Exercises 23–28, find the sum of the infinite series, accurate to three decimal places.

▷ We may add terms until we see that the next term will be less than $\frac{1}{2000}$. It is not necessary to determine the number of terms beforehand.

23. $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{2^n} \approx \frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-\frac{1}{64}+\frac{1}{128}-\frac{1}{256}+\frac{1}{512}-\frac{1}{1024} = 0.333$ with $|\text{error}| < \frac{1}{2048}$.

24. $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^4}$

▷ The approximation is accurate to three decimal places if the error is less than 0.0005. By Theorem 8.5.4 if the sum of the first k terms of the series is used to approximate the sum of the series, then the error is less than the absolute value of the next term. We first determine the number of terms that are required. Let

$a_n = \frac{1}{n^4}$

If $a_n < 0.0005$, then $\qquad \frac{1}{n^4} < 0.0005$

Solving for n, we get

$$n^4 > 2000 \qquad n > \sqrt[4]{2000} \approx 6.7$$

Thus $a_7 < 0.0005$, and if we use the sum of the first six terms to approximate the sum of the series, the result will be accurate to three decimal places.

$\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^4} \approx 1-\frac{1}{16}+\frac{1}{81}-\frac{1}{256}+\frac{1}{625}-\frac{1}{1296} \approx 0.94677 \approx 0.947$

Using only half of the last term, we get 0.94715. The sum of the series is $7\pi^4/720 = 0.94703$.

25. $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n!} = 1-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\frac{1}{120}-\frac{1}{720} = 0.632$ with $|\text{error}| < \frac{1}{5040}$.

26. $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{2}{3^n} \approx \frac{2}{3}-\frac{2}{9}+\frac{2}{27}-\frac{2}{81}+\frac{2}{243}-\frac{2}{729}+\frac{2}{2187} = 0.50023 \approx 0.500$ with $|\text{error}| < \frac{2}{6561}$

27. $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{(2n)^3} \approx \frac{1}{8}-\frac{1}{64}+\frac{1}{216}-\frac{1}{512}+\frac{1}{1000}-\frac{1}{1728} = 0.11247 \approx 0.112$ with $|\text{error}| < \frac{1}{2744}$.

Using only half the last term gives 0.11276. The sum is 0.11269.

28. $\sum_{n=1}^{+\infty}(-1)^n\frac{1}{(2n+1)^3}$

▷ We add the terms in the series until the absolute value of the next term is less than 0.0005. Thus

$$\begin{aligned}\sum_{n=1}^{+\infty}(1)^n\frac{1}{(2n+1)^3} &= -\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\frac{1}{13^3}-\cdots \\ &\approx -0.0370+0.0080-0.0029+0.0014-0.0008+0.0005 \approx -0.03088 \approx -0.031\end{aligned}$$

Using only half the last term gives -0.03110. The sum is $\pi^3/32-1 = -0.03105$.

In Exercises 29–48, determine if the series is absolutely convergent, conditionally convergent, or divergent.

29. $\sum_{n=1}^{+\infty}\left(-\frac{2}{3}\right)$ is a geometric series with $|r| = \left|-\frac{2}{3}\right| < 1$ and so is absolutely convergent.

30. $\sum_{n=1}^{+\infty}(-1)^n\frac{2^n}{n^3}$. Because $\lim_{n\to+\infty}\frac{2^n}{n^3} = +\infty$, the series is divergent.

31. $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{2^n}{n!}$ is absolutely convergent by the ratio test:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0 < 1$$

32. $\sum_{n=1}^{+\infty} n(\frac{2}{3})^n$

▶ We use the ratio test, Theorem 8.5.8. Because $u_n = n(\frac{2}{3})^n$, then

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\frac{(n+1)(\frac{2}{3})^{n+1}}{n(\frac{2}{3})^n} = \lim_{n\to+\infty}\frac{2}{3}\cdot\frac{n+1}{n} = \frac{2}{3}$$

Because $\frac{2}{3} < 1$, we conclude by Theorem 8.5.8(i) that the series is absolutely convergent.

33. $\sum_{n=1}^{+\infty}\frac{n^2}{n!}$ is absolutely convergent by the ratio test:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+1)!}\cdot\frac{n!}{n^2}\right| = \lim_{n\to\infty}\frac{n+1}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right) = 0 < 1$$

34. $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{(2n-1)!}$ is absolutely convergent by the ratio test:

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(2n-1)!}{(2n+1)!}\right| = \lim_{n\to+\infty}\frac{1}{(2n+1)(2n)} = 0 < 1$$

35. $\sum_{n=1}^{+\infty}(-1)^n\frac{n!}{2^{n+1}}$ is divergent by the ratio test:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{2^{n+2}}\cdot\frac{2^{n+1}}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{2} = +\infty$$

36. $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n(n+2)}$

▶ Let $\quad u_n = \frac{(-1)^{n+1}}{n(n+2)}$

Then

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\frac{n(n+2)}{(n+1)(n+3)} = \lim_{n\to+\infty}\frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)} = 1$$

Thus the ratio test fails.

Method 1: We use a comparison test with $v_n = \frac{1}{n^2}$. We have

$0 < |u_n| < v_n$

for all positive integers n, and because $\sum_{n=1}^{+\infty} v_n$ is a p-series with $p = 2 > 1$, then $\sum_{n=1}^{+\infty} v_n$ is convergent. Therefore, $\sum_{n=1}^{+\infty}|u_n|$ is convergent, and by Definition 8.5.5, we conclude that the given series is absolutely convergent.

Method 2: Because the ratio test fails, we may use Raabe's test.

$$\lim_{n\to+\infty} n\left(\left|\frac{u_n}{u_{n+1}}\right|-1\right) = \lim_{n\to+\infty} n\left[\frac{(n+1)(n+3)}{n(n+2)}-1\right] = \lim_{n\to+\infty} n\left[\frac{n^2+4n+3}{n^2+2n}-1\right] = \lim_{n\to+\infty}\frac{2n^2+3n}{n^2+2n}$$

$$= \lim_{n\to+\infty}\frac{2+\frac{3}{n}}{1+\frac{2}{n}} = 2$$

Because $p = 2 > 1$, the given series converges absolutely.

37. $\sum_{n=1}^{+\infty}\frac{1-2\sin n}{n^3}$ is absolutely convergent because $\sum_{n=1}^{+\infty}\frac{|1-2\sin n|}{n^3}$ is convergent by comparison with a p-series ($p = 3$): $\frac{|1-2\sin n|}{n^3} \le \frac{3}{n^3}$.

38. $\sum_{n=1}^{+\infty}(-1)^n\frac{1}{(n+1)^3}$ is absolutely convergent because $\sum_{n=1}^{+\infty}|u_n| = \sum_{n=1}^{+\infty}\frac{1}{(n+1)^3}$ is a convergent p-series ($p = 3$).

39. $\sum\limits_{n=1}^{+\infty} (-1)^{n+1}\frac{3^n}{n!}$ is absolutely convergent by the ratio test:

$\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}\right| = \lim\limits_{n\to\infty}\frac{3}{n+1} = 0 < 1$

40. $\sum\limits_{n=1}^{+\infty}(-1)^n\frac{n^2+1}{n^3}$

▶ Let $\qquad u_n = (-1)^n\frac{n^2+1}{n^3}$

Then

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\frac{(n+1)^2+1}{(n+1)^3}\cdot\frac{n^3}{n^2+1} = \lim_{n\to+\infty}\frac{\left(1+\frac{1}{n}\right)+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^3}\cdot\frac{1}{1+\frac{1}{n^2}} = 1$$

Thus the ratio test fails. We apply the alternating series test. Let

$a_n = \frac{n^2+1}{n^3}$

Then $a_n > 0$ for all positive integers n. Because

$a_n = \frac{n^2}{n^3}+\frac{1}{n^3} = \frac{1}{n}+\frac{1}{n^3}$

and each term is decreasing, then a_n is decreasing. Furthermore

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty}\left(\frac{1}{n}+\frac{1}{n^3}\right) = 0$

Therefore by the alternating series test the given series is convergent. We test for absolute convergence.

$$\sum_{n=1}^{+\infty}|u_n| = \sum_{n=1}^{+\infty}\left(\frac{1}{n}+\frac{1}{n^3}\right) \tag{1}$$

Because $\sum\limits_{n=1}^{+\infty}\frac{1}{n}$ is a divergent harmonic series and $\sum\limits_{n=1}^{+\infty}\frac{1}{n^3}$ is a convergent p-series, by Theorem 8.3.8 we conclude that series (1) is divergent. Therefore, by Definition 8.5.6 the given series is conditionally convergent.

41. $\sum\limits_{n=1}^{+\infty} (-1)^{n+1}\frac{1}{n(\ln n)^2}$ is absolutely convergent because $\sum\limits_{n=1}^{+\infty}\frac{1}{n(\ln n)^p}$ is convergent for $p > 1$.

42. $\sum\limits_{n=1}^{+\infty}\frac{\cos n}{n^2}$ is absolutely convergent by comparison with a p-series ($p = 2$): $\left|\frac{\cos n}{n^2}\right| \le \frac{1}{n^2}$

43. $\sum\limits_{n=1}^{+\infty}\frac{\sin \pi n}{n}$ is absolutely convergent because $\sum\limits_{n=1}^{+\infty}\left|\frac{\sin \pi n}{n}\right| = \sum\limits_{n=1}^{+\infty}\frac{0}{n} = 0$.

44. $\sum\limits_{n=2}^{+\infty}(-1)^{n+1}\frac{n}{\ln n}$

▶ Let

$u_n = (-1)^{n+1}\frac{n}{\ln n}$

If f is the function defined by $f(n) = |u_n|$, then

$f(x) = \frac{x}{\ln x}$

and by L'Hôpital's rule

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\frac{x}{\ln x} = \lim\limits_{x\to+\infty}\frac{1}{1/x} = +\infty$

Thus, $\lim\limits_{n\to+\infty} u_n$ does not exist, and by Theorem 8.3.3 we conclude that the given series is divergent.

45. $\sum\limits_{n=2}^{+\infty}\frac{1}{(\ln n)^n}$ is absolutely convergent by the root test: $\lim\limits_{n\to\infty}\sqrt[n]{|u_n|} = \lim\limits_{n\to\infty}\left[\frac{1}{(\ln n)^n}\right]^{1/n} = \lim\limits_{n\to\infty}\frac{1}{\ln n} = 0$ or, because $\ln 3 > 1$, by comparison with the geometric series $\sum\limits_{n=3}^{+\infty}\frac{1}{(\ln 3)^n}$.

46. $\sum\limits_{n=1}^{+\infty}\frac{(1+1/n)^{2n}}{e^n}$ is absolutely convergent by the root test: $\lim\limits_{n\to\infty}\sqrt[n]{|u_n|} = \lim\limits_{n\to\infty}\left[\frac{(1+1/n)^{2n}}{e^n}\right]^{1/n}$

$= \lim\limits_{n\to+\infty}\frac{(1+1/n)^2}{e} = \frac{1}{e} < 1$ or by comparison with the geometric series $\sum\limits_{n=1}^{+\infty}\frac{e^2}{e^n}$.

47. $\sum_{n=1}^{+\infty} \frac{n^n}{n!}$ is divergent by the ratio test:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$$

48. $\sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)}$

▷ We apply the ratio test. Let $u_n = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)}$

Then

$$u_{n+1} = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)(2n+1)}{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)(3n+1)}$$

$$\frac{u_{n+1}}{u_n} = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)(2n+1)}{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)(3n+1)}\cdot\frac{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} = \frac{2n+1}{3n+1}$$

Thus,

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\frac{2n+1}{3n+1} = \lim_{n\to+\infty}\frac{2+1/n}{3+1/n} = \frac{2}{3}$$

By the ratio test, we conclude that the series converges absolutely.

49. $\sum_{n=1}^{+\infty} \frac{1}{2^{n+1+(-1)^n}} = \frac{1}{2^1}+\frac{1}{2^4}+\frac{1}{2^3}+\frac{1}{2^6}+\frac{1}{2^5}+\cdots = \frac{1}{2}+\frac{1}{16}+\frac{1}{8}+\frac{1}{64}+\frac{1}{32}+\cdots$

(a) $\frac{u_{n+1}}{u_n}$ cycles between $\frac{1}{8}$ and 2; so $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}$ does not exist and the ratio test fails.

(b) $\lim_{n\to\infty}\sqrt[n]{|u^n|} = \lim_{n\to\infty}\left(\frac{1}{2^{n+1+(-1)^n}}\right)^{1/n} = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{1}{2^{(1+(-1)^n)/n}} = \frac{1}{2}\cdot\frac{1}{2^0} = \frac{1}{2}$

Hence, by the root test, the series is convergent.

50. If $\lim_{n\to\infty}\sqrt[n]{|u_n|} = L < 1$ and $L < R < 1$, then for some integer N, $\sqrt[n]{|u_n|} < R$ and $|u_n| < R^n$. Hence the series converges by comparison with the geometric series $\sum_{n=1}^{+\infty} R^n$.

51. If $\lim_{n\to\infty}\sqrt[n]{|u_n|} = L > 1$ or if $\lim_{n\to\infty}\sqrt[n]{|u_n|} = +\infty$, then there is a number $R > 1$ and an integer N such that if $n > N$ then $\sqrt{|u_n|} > R$. Hence if $n > N$, $|u_n| > R^n > 1$ and the series diverges by Theorem 8.3.3.

52. Prove part (iii) of the root test by applying it to the two series

$\sum_{n=1}^{+\infty}\frac{1}{n}$ and $\sum_{n=1}^{+\infty}\frac{1}{n^2}$

Hint: Determine $\lim_{n\to+\infty}\sqrt[n]{n}$ by letting $\sqrt[n]{n} = e^{(\ln n)/n}$ and using L'Hôpital's rule to find $\lim_{n\to+\infty}\frac{\ln n}{n}$.

▷ If $u_n = 1/n$, then

$$\lim_{n\to+\infty}\sqrt[n]{|u_n|} = \lim_{n\to+\infty}\sqrt[n]{\frac{1}{n}} = \frac{1}{\lim_{n\to+\infty}\sqrt[n]{n}} \qquad (1)$$

Because $e^{\ln n} = n$, then

$$\sqrt[n]{n} = \sqrt[n]{e^{\ln n}} = (e^{\ln n})^{1/n} = e^{(\ln n)/n} \qquad (2)$$

if $f(n) = (\ln n)/n$, then by L'Hôpital's rule

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{\ln x}{x} = \lim_{x\to+\infty}\frac{1}{x} = 0$$

Thus, from Eq. (2) we have

$$\lim_{n\to+\infty}\sqrt[n]{n} = e^0 = 1 \qquad (3)$$

and from Eq. (1) we conclude that

$$\lim_{n\to+\infty}\sqrt[n]{|u_n|} = \frac{1}{1} = 1$$

Moreover, $\sum_{n=1}^{+\infty} u_n$ is the divergent harmonic series. On the other hand if $v_n = \frac{1}{n^2}$ then

$$\lim_{n\to+\infty}\sqrt[n]{|v_n|} = \lim_{n\to+\infty}\sqrt[n]{\frac{1}{n^2}} = \frac{1}{[\lim_{n\to+\infty}\sqrt[n]{n}]^2} \qquad (4)$$

By substituting from Eq. (3) into (4), we obtain

$$\lim_{n \to +\infty} \sqrt[n]{|v_n|} = \frac{1}{1^2} = 1$$

Furthermore, $\sum_{n=1}^{+\infty} v_n$ is a convergent p series with $p = 2$. Therefore, no conclusion regarding convergence may be made from the root test if $\lim_{n \to +\infty} \sqrt[n]{|v_n|} = 1$. Finally, if $w_n = \begin{cases} e^{-n} & \text{if } n \text{ is even} \\ e^{-2n} & \text{if } n \text{ is odd} \end{cases}$, then $\sqrt[n]{w_n} = \begin{cases} e^{-1} & \text{if } n \text{ is even} \\ e^{-2} & \text{if } n \text{ is odd} \end{cases}$ and $\lim_{n \to +\infty} w_n$ does not exist. However, $w_n \le e^{-n}$ so the series converges by comparison with the geometric series $\sum_{n=1}^{+\infty} e^{-n}$. Hence no conclusion regarding convergence can be made from the root test if $\lim_{n \to +\infty} \sqrt[n]{n}$ does not exist.

53. If $\sum_{n=1}^{+\infty} u_n$ is absolutely convergent then $\sum_{n=1}^{+\infty} |u_n|$ is convergent and $\lim_{n \to +\infty} u_n = 0$. Thus, for some N, $|u_n| < 1$ and $u_n^{\,2} < |u_n|$ if $n > N$. Thus, $\sum_{n=1}^{+\infty} u_n^{\,2}$ converges by comparison with $\sum_{n=1}^{+\infty} |u_n|$.

54. (a) Prove Dirichlet's Test: Suppose we have the series $\sum_{n=1}^{+\infty} a_n b_n$ where (i) $a_{n+1} \le a_n$, (ii) $\lim_{n \to +\infty} a_n = 0$, and (iii) if $t_n = b_1 + b_2 + \cdots + b_n$ then $|t_n| < M$ for some M. The alternating series test is the case $b_n = (-1)^n$.

(b) Show that the series $\sum_{n=1}^{+\infty} \frac{\sin n}{n}$ is convergent.

(a)
$$\begin{aligned} s_n &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = a_1 t_1 + a_2(t_2 - t_1) + \cdots + a_n(t_n - t_{n-1}) \\ &= [t_1(a_1 - a_2) + t_2(a_2 - a_3) + \cdots + t_n(a_n - a_{n-1})] + a_n t_n \end{aligned}$$
The sequence in brackets converges because it converges absolutely. In fact
$$\begin{aligned} &|t_1(a_1 - a_2)| + |t_2(a_2 - a_3)| + \cdots + |t_n(a_n - a_{n-1})| \le M(a_1 - a_2) + M(a_2 - a_3) + \cdots + M(a_n - a_{n-1}) \\ &= M(a_1 - a_n) \le Ma_1 \end{aligned}$$
and a bounded series of positive terms converges. Furthermore, because $0 \le |a_n t_n| \le Ma_n$, $a_n t_n$ converges to 0 by the squeeze theorem. Thus s_n converges because it is the sum of two convergent sequences.

(b) Let $a_n = 1/n$ and $b_n = \sin n$. Because $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$, then

$$\begin{aligned} 2 \sin \tfrac{1}{2}\, t_n &= 2 \sin \tfrac{1}{2}(\sin 1 + \sin 2 + \cdots + \sin n) = 2 \sin \tfrac{1}{2} \sin 1 + 2 \sin \tfrac{1}{2} \sin 2 + \cdots + 2 \sin \tfrac{1}{2} \sin n \\ &= (\cos \tfrac{1}{2} - \cos \tfrac{3}{2}) + (\cos \tfrac{3}{2} - \cos \tfrac{5}{2}) + \cdots + [\cos(n - \tfrac{1}{2}) - \cos(n + \tfrac{1}{2})] = \cos \tfrac{1}{2} - \cos(n + \tfrac{1}{2}) \end{aligned}$$

and so $|t_n| = \left|\cos \tfrac{1}{2} - \cos(n + \tfrac{1}{2})\right| / 2 \sin \tfrac{1}{2} < 1/\sin \tfrac{1}{2}$. Thus, by Dirichlet's test, $\sum_{n=1}^{+\infty} a_n b_n = \sum_{n=1}^{+\infty} \sin n/n$ converges.

8.6 A SUMMARY OF TESTS FOR CONVERGENCE OR DIVERGENCE OF AN INFINITE SERIES

The following steps may often be used to decide if the infinite series $\sum_{n=1}^{+\infty} u_n$ is convergent or divergent.

1. If $u_n = ar^{n-1}$, with $a \ne 0$, then the series is a geometric series.
   a. If $|r| < 1$, the series is convergent and has sum $a/(1 - r)$.
   b. If $|r| \ge 1$, the series is divergent.
2. If $u_n = a_n - a_{n-1}$ with $\lim_{n \to +\infty} a_n = 0$, then the series is a telescoping series with sum a_0.
3. If $u_n = a/n^p$, with $a \ne 0$, the series is a constant multiple of a p-series.
   a. If $p > 1$, the series is convergent.
   b. If $p \le 1$, the series is divergent.
4. Find $\lim_{n \to +\infty} u_n = L$
   a. If $L \ne 0$, the series is divergent.
   b. If $L = 0$ and the series is an alternating series, apply the alternating series test: if $|u_{n+1}| \le |u_n|$ for all positive integers n, the alternating series is convergent. Otherwise, no conclusion can be made.
5. If u_n contains the factor $n!$ or a^n, then apply the ratio test. Find $\lim_{n \to +\infty} |u_{n+1}/u_n| = L$.
   a. If L does not exist, no conclusion can be made.
   b. If $L < 1$, the series is absolutely convergent.
   c. If $L > 1$ or $L = +\infty$, the series is divergent.
   d. If $L = 1$, then apply Raabe's test. Find $\lim_{n \to +\infty} n(|u_n/u_{n+1}| - 1) = p$

(i) if $p > 1$ or $p = +\infty$, the given series is absolutely convergent.
(ii) if $p < 1$ or $p = -\infty$, the series is divergent.
(iii) if $p = 1$ or p does not exist, no conclusion can be made.

6. If u_n contains the factor n^n, then apply the root test. Find $\lim_{n \to +\infty} \sqrt[n]{|u_n|} = L$.
   a. If $L < 1$, the series is absolutely convergent.
   b. If $L > 1$ or $L = +\infty$, the series is divergent.
   c. If $L = 1$ or L does not exist, no conclusion can be made.
7. If $u_n > 0$ for all positive integers n, use a comparison test or a limit comparison test with either a p series or a geometric series.
   a. (Comparison Test) Let the series $\sum_{n=1}^{+\infty} u_n$ be a series of positive terms.
      (i) If $\sum_{n=1}^{+\infty} v_n$ is a series of positive terms that is known to be convergent, and $u_n \le v_n$ for all positive integers n, then $\sum_{n=1}^{+\infty} u_n$ is convergent.
      (ii) If $\sum_{n=1}^{+\infty} v_n$ is a series of positive terms that is known to be divergent, and $u_n \ge v_n$ for all positive integers n, then $\sum_{n=1}^{+\infty} u_n$ is divergent.
   b. (Limit Comparison Test) Let $\sum_{n=1}^{+\infty} u_n$ and $\sum_{n=1}^{+\infty} v_n$ be two series of positive terms.
      (i) If $\lim_{n \to +\infty} (u_n/v_n) = c > 0$, then the two series either both converge or both diverge.
      (ii) If $\lim_{n \to +\infty} (u_n/v_n) = 0$ and if $\sum_{n=1}^{+\infty} v_n$ converges, then $\sum_{n=1}^{+\infty} u_n$ converges.
      (iii) If $\lim_{n \to +\infty} (u_n/v_n) = +\infty$ and if $\sum_{n=1}^{+\infty} v_n$ diverges, then $\sum_{n=1}^{+\infty} u_n$ diverges.
8. If $u_n > 0$ for all positive integers n, use the integral test. Let $f(n) = u_n$. If f is continuous and decreasing for $x \ge a \ge 1$, then $\sum_{n=1}^{+\infty} u_n$ is convergent if and only if $\int_a^{+\infty} f(x)\,dx$ exists.

Exercises 8.6

In Exercises 1 and 2, find the first four elements of the sequence of partial sums $\{s_n\}$, and find a formula for s_n in terms of n. Also determine if the infinite series is convergent or divergent; if it is convergent, find its sum.

1. $\sum_{n=1}^{+\infty} \frac{3}{4^{n+1}}$. $s_1 = u_1 = \frac{3}{16}$, $s_2 = s_1 + u_2 = \frac{3}{16} + \frac{3}{64} = \frac{15}{64}$, $s_3 = s_2 + u_3 = \frac{15}{64} + \frac{3}{256} = \frac{63}{256}$, $s_4 = s_3 + u_4 = \frac{63}{256} + \frac{3}{1024} = \frac{255}{1024}$. $s_n = \frac{3}{4^2} + \frac{3}{4^3} + \frac{3}{4^4} + \cdots + \frac{3}{4^{n+1}} = \frac{3}{4^2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n-1}}\right) = \frac{3}{16} \cdot \frac{1 - (\frac{1}{4})^n}{1 - \frac{1}{4}} = \frac{1}{4}\left(1 - \frac{1}{4^n}\right)$.
   Because $\lim_{n \to \infty} s_n = \frac{1}{4}$, the series is convergent and its sum is $\frac{1}{4}$.
2. $\sum_{n=1}^{+\infty} \ln\left(\frac{2n-1}{2n+1}\right) = \sum_{n=1}^{+\infty} [\ln(2n-1) - \ln(2n+1)]$, $s_1 = \ln 1 - \ln 3 = -\ln 3$, $s_2 = (\ln 1 - \ln 3) + (\ln 3 - \ln 5) = -\ln 5$, $s_3 = s_2 + (\ln 5 - \ln 7) = -\ln 7$, $s_4 = -\ln 7 + (\ln 7 - \ln 9) = -\ln 9$. $s_n = -\ln(2n+1)$. Because $\lim_{n \to +\infty} s_n = -\infty$, the series is divergent.

In Exercises 3–12, determine whether the series is convergent or divergent. If the series is convergent, find its sum.

3. $\sum_{n=1}^{+\infty} \left(\frac{3}{4}\right)$ is a geometric series with $a = \frac{3}{4}$ and $r = \frac{3}{4} < 1$. Thus it is convergent and its sum is $\frac{\frac{3}{4}}{1 - \frac{3}{4}} = 3$.
4. $\sum_{n=1}^{+\infty} e^{-2n}$

▸ $\sum_{n=1}^{+\infty} e^{-2n} = \sum_{n=1}^{+\infty} (e^{-2})^n$
is a geometric series with first term $a = e^{-2}$ and ratio $r = e^{-2}$. Then,
$\sum_{n=1}^{+\infty} e^{-2n} = \frac{a}{1-r} = \frac{e^{-2}}{1 - e^{-2}} = \frac{1}{e^2 - 1}$

5. $\sum_{n=1}^{+\infty} \frac{n-1}{n+1}$ is divergent because $\lim_{n \to \infty} \frac{n-1}{n+1} = \lim_{n \to \infty} \frac{1 - n^{-1}}{1 + n^{n-1}} = 1 \ne 0$.
6. $\sum_{n=1}^{+\infty} [(-1)^n + (-1)^{n+1}] = \sum_{n=1}^{+\infty} (-1)^n[1 + (-1)] = \sum_{n=1}^{+\infty} 0$. The series is convergent with sum 0.

7. $\sum\limits_{n=1}^{+\infty} \sin^n \frac{1}{3}\pi$ is a geometric series with $a = 1$ and $r = \sin \frac{1}{3}\pi = \frac{1}{2}\sqrt{3} < 1$.
Therefore it is convergent and its sum is $\dfrac{1}{1 - \frac{1}{2}\sqrt{3}} = 4 + 2\sqrt{3}$.

8. $\sum\limits_{n=0}^{+\infty} \cos^n \frac{1}{3}\pi$

▷ Because $\cos \frac{1}{3}\pi = \frac{1}{2}$, then
$\sum\limits_{n=0}^{+\infty} \cos^n \frac{1}{3}\pi = \sum\limits_{n=0}^{+\infty} (\frac{1}{2})^n$
is a geometric series with first term $a = (\frac{1}{2})^0 = 1$ and ratio $r = \frac{1}{2}$. Then,
$\sum\limits_{n=0}^{+\infty} \cos^n \frac{1}{3}\pi = \dfrac{a}{1-r} = \dfrac{1}{1-\frac{1}{2}} = 2$

9. $\sum\limits_{n=1}^{+\infty} \dfrac{1}{(3n-1)(3n+2)} = \frac{1}{3}\sum\limits_{n=1}^{+\infty}\left(\dfrac{1}{3n-1} - \dfrac{1}{3n+2}\right)$ by partial fractions.
$s_n = \frac{1}{3}\left[\left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \cdots + \left(\dfrac{1}{3n-4} - \dfrac{1}{3n-1}\right) + \left(\dfrac{1}{3n-1} - \dfrac{1}{3n+2}\right)\right] = \frac{1}{3}\left(\frac{1}{2} - \dfrac{1}{3n+2}\right)$
Because $\lim\limits_{n\to\infty} s_n = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$, the series is convergent and its sum is $\frac{1}{6}$.

10. $\sum\limits_{n=1}^{+\infty} \frac{3}{2}\left(\frac{1}{5}\right)^n$ is a geometric series converging to $\dfrac{a}{1-r} = \dfrac{\frac{3}{10}}{1-\frac{1}{5}} = \frac{3}{10}\cdot\frac{5}{4} = \frac{3}{8}$

11. $[\![\sin \frac{3}{1}\pi]\!] = [\![0]\!] = 0$, $[\![\sin \frac{3}{2}\pi]\!] = [\![-1]\!] = -1$, $[\![\sin \frac{3}{3}\pi]\!] = [\![0]\!] = 0$, $[\![\sin \frac{3}{4}\pi]\!] = [\![\frac{1}{2}\sqrt{2}]\!] = 0$, $[\![\sin \frac{3}{5}\pi]\!] =$

$[\![0.587]\!] = 0$, $[\![\sin \frac{3}{6}\pi]\!] = [\![1]\!] = 1$. If $n > 6$ then $0 < \frac{3}{n}\pi < \frac{1}{2}\pi$ and $0 < \sin \frac{3}{n}\pi < 1$ so $[\![\sin \frac{3}{n}\pi]\!] = 0$. Thus
$\sum\limits_{n=1}^{+\infty} \dfrac{[\![\sin \frac{3}{n}\pi + 2]\!]}{3^n} = \sum\limits_{n=1}^{+\infty} \dfrac{[\![\sin \frac{3}{n}]\!]}{3^n} + \sum\limits_{n=1}^{+\infty} \dfrac{2}{3^n} = -\dfrac{1}{3^2} + \dfrac{1}{3^6} + \sum\limits_{n=1}^{+\infty} \dfrac{2}{3^n}$. We have a geometric series with $a = \frac{2}{3}$ and
$r = \frac{1}{3} < 1$. Hence the given series converges and its sum is $-\frac{1}{9} + \frac{1}{729} + \dfrac{\frac{2}{3}}{1-\frac{1}{3}} = -\frac{1}{9} + \frac{1}{729} + 1 = \frac{649}{729}$.

12. $\sum\limits_{n=1}^{+\infty}\left(\dfrac{1}{4^n} + \dfrac{1}{3^n}\right)$

▷ The given series is the sum of two convergent series.
$\sum\limits_{n=1}^{+\infty}\left(\dfrac{1}{4^n} + \dfrac{1}{3^n}\right) = \sum\limits_{n=1}^{+\infty}\dfrac{1}{4^n} + \sum\limits_{n=1}^{+\infty}\dfrac{1}{3^n} = \dfrac{\frac{1}{4}}{1-\frac{1}{4}} + \dfrac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

In Exercises 13–30, determine whether the series is convergent or divergent.

13. $\sum\limits_{n=1}^{+\infty} \dfrac{2}{n^2+6n}$ is convergent by comparison with a p-series ($p = 2$): $\dfrac{2}{n^2+6n} < \dfrac{2}{n^2}$. By partial fractions we find the sum is $\frac{49}{60}$.

14. $\sum\limits_{n=1}^{+\infty} \dfrac{1}{(2n+1)^3}$ is convergent by comparison with a p-series ($p = 2$): $\dfrac{1}{(2n+1)^3} < \dfrac{1}{8n^3}$.

15. $\sum\limits_{n=1}^{+\infty} \cos\left(\dfrac{\pi}{2n^2-1}\right)$ is divergent because $\lim\limits_{n\to\infty} \cos\left(\dfrac{\pi}{2n^2-1}\right) = \cos 0 = 1 \neq 0$.

16. $\sum\limits_{n=1}^{+\infty} \dfrac{3 + \sin n}{n^2}$

▷ Because $-1 \le \sin n \le 1$, then $\dfrac{2}{n^2} \le \dfrac{3+\sin n}{n^2} \le \dfrac{4}{n^2}$
Because $\sum\limits_{n=1}^{+\infty} \dfrac{4}{n^2}$ is a p-series with $p = 2$, it is convergent. From the comparison test we conclude that the given series is convergent.

17. $\sum\limits_{n=1}^{+\infty} \dfrac{(n!)^2}{(2n)!}$ is convergent by the ratio test:
$\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{[(n+1)!]^2}{[2(n+1)]!}\cdot\dfrac{(2n)!}{(n!)^2}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)^2}{(2n+1)(2n+2)} = \lim\limits_{n\to\infty}\dfrac{n+1}{4(n+\frac{1}{2})} = \frac{1}{4} < 1$

18. $\sum\limits_{n=1}^{+\infty} \dfrac{n}{\sqrt{3n+2}}$ is divergent because $\lim\limits_{n\to+\infty} \dfrac{n}{\sqrt{3n+2}} = \lim\limits_{n\to+\infty} \dfrac{\sqrt{n}}{\sqrt{3+2/n}} = +\infty$

19. $\sum_{n=1}^{+\infty} (-1)^n \ln \frac{1}{n}$ is divergent because $\lim_{n\to\infty} \ln \frac{1}{n} = -\infty$.

20. $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{1+\sqrt{n}}$

▷ We apply the alternating series test. Let

$$a_n = \frac{1}{1+\sqrt{n}}$$

Then $a_n > 0$ and $a_{n+1} < a_n$ for all positive integers n. Furthermore,

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{1}{1+\sqrt{n}} = 0$$

By the alternating series test we conclude that the series is convergent.

21. $\sum_{n=1}^{+\infty} \frac{1}{n(\ln n)^2}$ is convergent by the integral test: Let $f(x) = \frac{1}{x(\ln x)^2}$. For $x \geq 2$, f is continuous, decreasing, $f(x) > 0$ and $\int_2^{+\infty} \frac{1}{(\ln x)^2} \frac{dx}{x} = \lim_{b\to\infty} \left[-\frac{1}{\ln x}\right]_2^b = \lim_{b\to\infty} \left[-\frac{1}{\ln b} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$.

22. $\sum_{n=1}^{+\infty} \frac{\ln n}{n^2}$ is convergent by a limit comparison with $\sum_{n=1}^{+\infty} \frac{1}{n^{3/2}}$: $\lim_{n\to+\infty} \frac{\ln n}{n^2} / \frac{1}{n^{3/2}} = \lim_{n\to+\infty} \frac{\ln n}{n^{1/2}} = 0$

23. $\sum_{n=1}^{+\infty} \left(\frac{2}{5n} - \frac{3}{2n}\right)$ is divergent by comparison with the harmonic series: $\frac{2}{5n} - \frac{3}{2n} = -\frac{11}{10} \cdot \frac{1}{n}$.

24. $\sum_{n=1}^{+\infty} \frac{n!}{10^n}$

▷ We apply the ratio test.

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \frac{n+1}{10}$$

Thus,

$$\lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = +\infty$$

By the ratio test, the series is divergent.

25. $\sum_{n=1}^{+\infty} \frac{1}{1+2\ln n}$ is divergent by comparison with the harmonic series: $\frac{1}{1+2\ln n} > \frac{1}{n+2n} = \frac{1}{3n}$.

26. $\sum_{n=1}^{+\infty} \frac{|\sec n|}{n^{3/4}}$ is divergent by comparison with a p-series ($p = \frac{3}{4}$): $\frac{|\sec n|}{n^{3/4}} \geq \frac{1}{n^{3/4}}$

27. $\sum_{n=1}^{+\infty} \frac{\cos n}{n^3}$ is absolutely convergent, and hence convergent, by comparison with a p-series ($p = 3$): $\left|\frac{\cos n}{n^3}\right| < \frac{1}{n^3}$.

28. $\sum_{n=1}^{+\infty} n(3^{-n^2})$

▷ We apply the root test. Because

$$\lim_{n\to+\infty} n^{1/n} = 1 \quad \text{and} \quad \lim_{n\to+\infty} 3^n = +\infty$$

then

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} (n3^{-n^2})^{1/n} = \lim_{n\to+\infty} \frac{n^{1/n}}{3^n} = 0$$

By the root test, the series is convergent.

29. $\sum_{n=1}^{+\infty} \frac{1}{2^n + \sin n}$ converges by comparison with a geometric series: $\frac{1}{2^n + \sin n} < \frac{1}{2^n - 2^{n-1}} = \frac{1}{2^{n-1}}$.

30. $\sum_{n=1}^{+\infty} \frac{(n+2)^2}{(n+3)!}$ is convergent by the ratio test:

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \frac{(n+3)^2}{(n+4)!} \cdot \frac{(n+3)!}{(n+2)^2} = \lim_{n\to+\infty} \frac{1}{n+4}\left(\frac{n+3}{n+2}\right)^2 = \lim_{n\to+\infty} \frac{1}{n+4}\left(\frac{1+3/n}{1+2/n}\right)^2 = 0$$

In Exercises 31–40, determine if the series is absolutely convergent, conditionally convergent, or divergent.

31. $\sum_{n=1}^{+\infty} (-1)^n \frac{n^2}{3^n}$ is absolutely convergent by the ratio test:

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{3^{n+1}}\cdot\frac{3^n}{n^2}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{3n^2} = \frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = \frac{1}{3} < 1$$

32. $\sum_{n=0}^{+\infty}(-1)^n\frac{5^{2n+1}}{(2n+1)!}$

▷ We apply the ratio test. Let

$$u_n = (-1)^n\frac{5^{2n+1}}{(2n+1)!}$$

Then

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\frac{5^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{5^{2n+1}} = \lim_{n\to+\infty}\frac{5^2(2n+1)!}{(2n+3)(2n+2)(2n+1)!} = \lim_{n\to+\infty}\frac{25}{(2n+3)(2n+2)} = 0$$

Therefore, the series is absolutely convergent.

33. $\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{(n+1)^{3/4}}$ is convergent by the alternating series test but $\sum_{n=1}^{+\infty}\frac{1}{(n+1)^{3/4}}$ is divergent by a limit comparison with a p-series ($p=\frac{3}{4}$). Hence the given series is conditionally convergent.

34. $\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{6^n}{5^{n+1}}$ is a divergent geometric series, $r=-\frac{6}{5}$.

35. $\sum_{n=1}^{+\infty}(-1)^n\frac{n!}{10n}$ is divergent because $\lim_{n\to\infty}\frac{n!}{10n} = \lim_{n\to\infty}\frac{(n-1)!}{10} = +\infty$.

36. $\sum_{n=1}^{+\infty}(-1)^n\frac{\sqrt{2n-1}}{n}$

▷ We have an alternating series with $a_n = \frac{\sqrt{2n-1}}{n}$.

$$\lim_{n\to+\infty}a_n = \lim_{n\to+\infty}\frac{\sqrt{2n-1}}{n} = \lim_{n\to+\infty}\sqrt{\frac{2n-1}{n^2}} = \lim_{n\to+\infty}\sqrt{\frac{2}{n}-\frac{1}{n^2}} = 0$$

Because ${a_n}^2$ is a rational function with limit 0, it is eventually decreasing. By the alternating series test, the given series is convergent. To test for absolute convergence, we make a limit comparison test with

$$u_n = a_n = \frac{\sqrt{2n-1}}{n} \quad\text{and}\quad v_n = \frac{1}{\sqrt{n}}$$

Thus,

$$\lim_{n\to+\infty}\frac{u_n}{v_n} = \lim_{n\to+\infty}\frac{\sqrt{2n-1}}{n}\cdot\frac{\sqrt{n}}{1} = \lim_{n\to+\infty}\sqrt{2-\frac{1}{n}} = \sqrt{2} \tag{1}$$

The series

$$\sum_{n=1}^{+\infty}v_n = \sum_{n=1}^{+\infty}\frac{1}{\sqrt{n}}$$

is divergent because it is a p-series with $p=\frac{1}{2}$. By (1) and Theorem 8.4.3(i), the series $\sum_{n=1}^{+\infty}|u_n|$ is divergent. Thus, the given series is conditionally convergent.

37. $\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{2^{3n}}{n^n}$ converges absolutely by the root test: $\lim_{n\to\infty}\sqrt[n]{|u_n|} = \lim_{n\to\infty}\left(\frac{2^{3n}}{n^n}\right)^{1/n} = \lim_{n\to\infty}\frac{2^3}{n} = 0$.

38. $\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{[\ln(n+2)]^n}$ converges absolutely by comparison with a geometric series: $\frac{1}{[\ln(n+2)]^n} < \frac{1}{(\ln 3)^n}$

39. The given series is $\sum_{n=1}^{+\infty}c_n$ where $c_n = \begin{cases} -\frac{1}{n} & \text{if } n \text{ is a perfect square} \\ \frac{1}{n^2} & \text{if } n \text{ is not a perfect square}\end{cases}$

Therefore $\sum_{n=1}^{+\infty}|c_n|$ consists of terms of the form $\frac{1}{n}$, where n is a perfect square, whose sum is $\sum_{i=1}^{+\infty}\frac{1}{i^2}$, and terms of the form $\frac{1}{n^2}$, where n is not a perfect square, whose sum is less than $\sum_{n=1}^{+\infty}\frac{1}{n^2}$. Because each sum is a p-series ($p=2$), $\sum_{n=1}^{+\infty}|c_n|$ is convergent and so the given series is absolutely convergent.

40. $\sum_{n=1}^{+\infty} c_n$, where $c_n = \begin{cases} -1/n & \text{if } \frac{1}{4}n \text{ is an integer} \\ 1/n^2 & \text{if } \frac{1}{4}n \text{ is not an integer} \end{cases}$

▸ We express the given series as the sum of two series. Let

$$a_n = \begin{cases} 0 & \text{if } \frac{1}{4}n \text{ is an integer} \\ 1/n^2 & \text{if } \frac{1}{4}n \text{ is not an integer} \end{cases} \quad \text{and} \quad b_n = \begin{cases} -1/n & \text{if } \frac{1}{4}n \text{ is an integer} \\ 0 & \text{if } \frac{1}{4}n \text{ is not an integer} \end{cases}$$

Then

$$\sum_{n=1}^{+\infty} c_n = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} b_n$$

By comparison with the p series with $p = 2$, we prove that $\sum_{n=1}^{+\infty} a_n$ is convergent. By comparison with th harmonic series, we prove that $\sum_{n=1}^{+\infty} a_n = \sum_{k=1}^{+\infty} \frac{1}{4k}$ where $\frac{1}{4}n = k$, is divergent. Therefore, by Theorem 8.3.8 the series $\sum_{n=1}^{+\infty} c_n$ is divergent.

41. $1.3242424\ldots = \frac{13}{10} + \left(\frac{24}{1000} + \frac{24}{100{,}000} + \frac{24}{100{,}000{,}000} + \cdots\right)$ is a geometric series with $a = \frac{24}{1000}$ and $r = \frac{1}{100} < 1$.

Thus it is convergent and its sum is $\frac{13}{10} + \dfrac{\frac{24}{1000}}{1 - \frac{1}{100}} = \frac{13}{10} + \frac{24}{990} = \frac{437}{330}$.

42. $18 + 18 \cdot 2[\frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 + \cdots] = 18 + 36 \cdot \dfrac{\frac{2}{3}}{1 - \frac{2}{3}} = 90$ ft.

43. Because $t = \frac{1}{4}\sqrt{h}$, $T = \frac{1}{4}\sqrt{18} + \frac{1}{4}\sqrt{18} \cdot 2[\sqrt{\frac{2}{3}} + (\sqrt{\frac{2}{3}})^2 + (\sqrt{\frac{2}{3}})^3 + \cdots] = \frac{1}{4}\sqrt{18}\left[1 + 2\dfrac{\sqrt{\frac{2}{3}}}{1 - \sqrt{\frac{2}{3}}}\right]$

$$= \frac{3}{4}\sqrt{2}\left[1 + 2\frac{\sqrt{2}}{\sqrt{3} - \sqrt{2}} \cdot \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}}\right] = \frac{3}{4}\sqrt{2}[5 + 2\sqrt{6}] = \frac{15}{4}\sqrt{2} + 3\sqrt{3}$$

44. The path of each swing, after the first, of a pendulum bob is 80% as long as the path of the previous swing from one side to the other side. If the path of the first swing is 18 in. long, how far does the bob travel before it comes to rest?

▸ If S in. is the total distance the bob travels, then

$$S = 18 + 18(\tfrac{4}{5}) + 18(\tfrac{4}{5})^2 + 18(\tfrac{4}{5})^3 + \cdots = \sum_{n=1}^{+\infty} 18(\tfrac{4}{5})^{n-1}$$

is a geometric series with $a = 18$ and $r = \frac{4}{5}$, and sum $S = \dfrac{18}{1 - \frac{4}{5}} = 90$.

Therefore the bob travels 90 in. before coming to rest.

8.7 POWER SERIES

Until now, we have considered infinite series in which each term is a constant. We now consider infinite series that contain terms that are variables. The series

$$\sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

is called a *power series* in x. It is a special case of the power series

$$\sum_{n=0}^{+\infty} c_n (x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n + \cdots$$

Sometimes a power series in x converges for certain replacements of x and diverges for other replacements of x. It depends on the constants $c_0, c_1, c_2, \ldots$. We have the following.

8.7.4 Theorem Let $\sum_{n=0}^{+\infty} c_n (x - a)^n$ be a given power series. Then exactly one of the following conditions holds:

(i) The series converges only when $x = a$.

(ii) The series is absolutely convergent for all values of x.

(iii) There exists a number $R > 0$ such that the series is absolutely convergent for all values of x for which $|x - a| < R$ and is divergent for all values of x for which $|x - a| > R$.

The number R in Theorem 8.7.4(iii) is called the *radius of convergence* of the power series. If condition (i) holds, we take $R = 0$ and if condition (ii) holds we write $R = +\infty$. The set of all x for which the power series converges is called the *interval of convergence*. The following

steps are used to find the interval of convergence of a power series that is represented by $\sum_{n=0}^{+\infty} u_n$ where $u_n = c_n x^n$ or $u_n = c_n(x-a)^n$.

1. Find L, where $L = \lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right|$.
   a. If $L = 0$, the series converges absolutely for all x.
   b. If $L = +\infty$ (except when $x = a$), the series converges only when $x = a$.
   c. If $L < 1$ for $x_1 < x < x_2$, the series converges absolutely for all x in the interval (x_1, x_2).
2. If (x_1, x_2) is the interval found in step 1(c), replace x by x_1 in the given power series and test to determine whether the series converges. Because $L = 1$, the ratio and root tests will fail; Raabe's test is appropriate. Repeat this for $x = x_2$.
3. The interval of convergence is the union of the set of all x found in steps 1 and 2 for which the series converges.

By inverting the ratio test we may calculate the radius of convergence directly:

$R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right|$ if the limit exists.

Geometric Series $1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x}$ if $|x| < 1$ (3)

$1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots = \frac{1}{1+x}$ if $|x| < 1$ (4)

Wallis Inequality $\frac{1}{2\sqrt{n}} < u_n = \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot(2n)} = \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n n!} < \frac{1}{\sqrt{2n}}$

Proof $\frac{2n-1}{2n} = 1 - \frac{1}{2n} = \sqrt{\left(1-\frac{1}{2n}\right)^2} = \sqrt{1-\frac{1}{n}+\frac{1}{4n^2}} > \sqrt{1-\frac{1}{n}} = \sqrt{\frac{n-1}{n}}$ and so

$u_n > \frac{1}{2}\sqrt{\frac{1}{2}}\sqrt{\frac{2}{3}}\cdots\sqrt{\frac{n-1}{n}} = \frac{1}{2\sqrt{n}}$. Furthermore,

$u_n < \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{n} = \frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{1}{2n+1} = \frac{1}{u_n}\cdot\frac{1}{2n+1}$ and so $u_n^2 < \frac{1}{2n+1} < \frac{1}{2n}$.

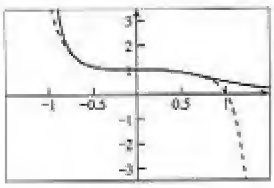
The real Wallis inequality uses the reduction formula for $\int_0^{\pi/2}\sin^n x\,dx$ to show

$\frac{1}{\sqrt{\pi(n+\frac{1}{2})}} < u_n < \frac{1}{\sqrt{\pi n}}$

Exercises 8.7

1. (b) Replacing x by $2x$ in (4) we get $\frac{1}{1+2x} = 1 - 2x + 4x^2 - \cdots = \sum_{n=0}^{+\infty}(-2x)^n$
2. (b) Replacing x by $4x^2$ in (3) we get $\frac{1}{1-4x^2} = 1 + 4x^2 + 16x^4 + \cdots = \sum_{n=0}^{+\infty}(2x)^{2n}$
3. (b) Replacing x by $9x^2$ in (4) we get $\frac{1}{1+9x^2} = 1 - 9x^2 + 81x^4 - \cdots = \sum_{n=0}^{+\infty}(-3x)^{2n}$
4. (a) Use series (4) to find a power-series representation for $f(x) = \frac{1}{1+x^3}$.

   (b) Support graphically that the power series in part (a) converges to $f(x)$, if $|x| < 1$, by plotting f and $P_{10}(x)$ in the same window.



▹ (a) We replace x by x^3 in (4) to obtain

$$\frac{1}{1+x^3} = 1 - x^3 + (x^3)^2 - (x^3)^3 + \cdots + (-1)^n(x^3)^n + \cdots$$
$$= 1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots$$

(b) The plot shows $f(x)$ solid and $P_{10}(x) = 1 - x^3 + x^6 - x^9$ (dashed).

In Exercises 5–32, determine the interval I of convergence of the power series.

5. $\sum_{n=0}^{+\infty} \frac{x^n}{n+1}$. $\lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty} \left|\frac{x^{n+1}}{n+2}\cdot\frac{n+1}{x^n}\right| = \lim_{n\to+\infty} \left|\frac{n+1}{n+2}\right| |x| = |x|$

Therefore the power series is absolutely convergent if $|x| < 1$, that is, if $-1 < x < 1$.

$x = 1$: $\sum_{n=0}^{+\infty} \frac{1}{n+1}$ is a divergent harmonic series. $x = -1$: $\sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1}$ converges by the alternating-series test.

Hence the interval of convergence of the given power series is $[-1,1)$.

6. $\sum_{n=0}^{+\infty} \frac{x^n}{n^2+1}$. $R = \lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{(n+1)^2+1}{n^2+1} = \lim_{n\to+\infty} \frac{(1+1/n)^2+1/n^2}{1+1/n^2} = 1$. If $x = \pm 1$ we get $\sum_{n=0}^{+\infty} \frac{(\pm 1)^n}{n^2+1}$ which converges absolutely by comparison with a p-series, $p = 2$. Thus, I is $[-1, 1]$.

7. $\sum_{n=0}^{+\infty} \frac{x^n}{n^2-3}$. $\lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty} \left|\frac{x^{n+1}}{(n+1)^2-3}\cdot\frac{n^2-3}{x^n}\right| = \lim_{n\to+\infty} \left|\frac{n^2-3}{n^2+2n-2}\right| |x| = |x|$

Therefore the power series is absolutely convergent if $|x| < 1$, that is, if $-1 < x < 1$.

$x = \pm 1$: $\sum_{n=0}^{+\infty} \frac{(\pm 1)^n}{n^2-3}$ is absolutely convergent by a limit comparison with a p-series ($p = 2$):

$\lim_{n\to+\infty} \frac{1}{n^2-3}/\frac{1}{n^2} = \lim_{n\to+\infty} \frac{n^2}{n^2-3} = \lim_{n\to+\infty} \frac{1}{1-3n^{-2}} = 1$. Hence I is $[-1,1]$.

8. $\sum_{n=1}^{+\infty} \frac{n^2x^n}{2^n}$

▷ Let

$$u_n = \frac{n^2x^n}{2^n}$$

Then

$$u_{n+1} = \frac{(n+1)^2x^{n+1}}{2^{n+1}}$$

and

$$\lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty} \left|\frac{(n+1)^2x^{n+1}}{2^{n+1}} \div \frac{n^2x^n}{2^n}\right| = \lim_{n\to+\infty} \left|\frac{(n+1)^2x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n^2x^n}\right| = \lim_{n\to+\infty} \left|\frac{(n+1)^2x}{2n^2}\right|$$

$$= \left|\frac{x}{2}\right| \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^2 = \left|\frac{x}{2}\right|$$

If $\left|\frac{x}{2}\right| < 1$ then $|x| < 2$, $-2 < x < 2$

Thus, by the ratio test we conclude that the given series converges absolutely for all x in the open interval $(-2,2)$. We test the endpoints of the interval for convergence. If $x = \pm 2$, the given power series becomes

$$\sum_{n=0}^{\infty} \frac{n^2(\pm 2)^n}{2^n} = \sum_{n=0}^{+\infty} (\pm 1)^n n^2$$

which is divergent because $\lim_{n\to+\infty} (\pm 1)^n n^2 \neq 0$. Hence, the interval of convergence for the given power series is the open interval $(-2,2)$.

9. $\sum_{n=1}^{+\infty} \frac{2^nx^n}{n^2}$. $\lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty} \left|\frac{2^{n+1}x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^nx^n}\right| = \lim_{n\to+\infty} \left|\frac{2n^2}{n^2+2n+1}\right| |x| = 2|x|$

Therefore, the power series is absolutely convergent if $2|x| < 1$, that is, if $-\frac{1}{2} < x < \frac{1}{2}$.

$x = \pm\frac{1}{2}$: $\sum_{n=1}^{+\infty} |u_n| = \sum_{n=1}^{+\infty} \left|\frac{2^n(\pm\frac{1}{2})^n}{n^2}\right| = \sum_{n=1}^{+\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2$). Hence I is $[-\frac{1}{2}, \frac{1}{2}]$.

10. $\sum_{n=1}^{+\infty} \frac{x^n}{2^n\sqrt{n}}$. $R = \lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{2^{n+1}\sqrt{n+1}}{2^n\sqrt{n}} = \lim_{n\to+\infty} 2\sqrt{1+\frac{1}{n}} = 2$. If $x = -2$, the series is $\sum_{n=1}^{+\infty} \frac{(-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$ which converges by the alternating series test. If $x = 2$, the series is $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}}$, a divergent p-series with $p = \frac{1}{2}$. Hence I is $[-2, 2)$.

11. $\sum_{n=1}^{+\infty} \frac{nx^n}{3^n}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)x^{n+1}}{3^{n+1}}\cdot\frac{3^n}{nx^n}\right| = \lim_{n\to+\infty}\left|\frac{n+1}{3n}\right||x| = \frac{1}{3}|x|$

Therefore, the power series is absolutely convergent if $\frac{1}{3}|x| < 1$, that is, if $-3 < x < 3$. If $x = \pm 3$ the series is $\sum_{n=1}^{+\infty} \frac{n(\pm 3)^n}{3^n} = \sum_{n=1}^{+\infty} (\pm 1)^n n$. Both series diverge because the terms do not approach 0. Hence I is $(-3,3)$.

12. $\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

▷ This is a power series in x^2, and we may proceed as for a power series in x.

$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = x^2 \lim_{n\to+\infty}\frac{(2n)!}{(2n+2)(2n+1)(2n)!} = x^2 \lim_{n\to+\infty}\frac{1}{(2n+2)(2n+1)} = 0$

Because the limit is 0 for all x, by the ratio test we conclude that the series is absolutely convergent for all x. The interval of convergence is $(-\infty,+\infty)$.

13. $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n+1}}{(2n+1)!}\cdot\frac{(2n-1)!}{x^{2n-1}}\right| = \lim_{n\to+\infty}\left|\frac{1}{2n(2n+1)}\right|x^2 = 0 < 1$ for all x.

The given power series converges for all x; its interval of convergence is $(-\infty, +\infty)$.

14. $\sum_{n=1}^{+\infty} \frac{n+1}{n^{2n}} x^n$. $R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{n+1}{n^{2n}}\cdot\frac{(n+1)^{2n+2}}{n+2} = \lim_{n\to+\infty}\frac{n+1}{n+2}\left(1+\frac{1}{n}\right)^{2n}(n+1)^2 = +\infty$. $I = (-\infty,+\infty)$

15. $\sum_{n=0}^{+\infty} \frac{(x+3)^n}{2^n}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(x+3)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x+3)^n}\right| = \frac{1}{2}|x+3|$

The power series is absolutely convergent if $\frac{1}{2}|x+3| < 1$; $|x+3| < 2$; $-2 < x+3 < 2$; $-5 < x < -1$.

$|x+3| = 2$: $\sum_{n=0}^{+\infty} |u_n| = \sum_{n=0}^{+\infty} 1$. Therefore both series diverge. Hence I is $(-5,-1)$.

16. $\sum_{n=0}^{+\infty} \frac{x^n}{(n+1)5^n}$

▷ $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{(n+2)5^{n+1}}\cdot\frac{(n+1)5^n}{x^n}\right| = \left|\frac{x}{5}\right|\lim_{n\to+\infty}\frac{n+1}{n+2} = \left|\frac{x}{5}\right|$

If $|x/5| < 1$ then $-5 < x < 5$. Thus the series converges absolutely for $-5 < x < 5$. We test each endpoint. If $x = -5$, the given power series becomes

$$\sum_{n=0}^{+\infty} \frac{(-5)^n}{(n+1)5^n} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1}$$

which converges by the alternating series test. If $x = 5$, the given power series becomes

$$\sum_{n=0}^{+\infty} \frac{5^n}{(n+1)5^n} = \sum_{n=0}^{+\infty} \frac{1}{n+1}$$

which diverges because it is the harmonic series. Hence, the interval of convergence is $[-5,5)$.

17. $\sum_{n=1}^{+\infty} \frac{(-1)^n x^n}{(2n-1)3^{2n-1}}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{(2n+1)3^{2n+1}}\cdot\frac{(2n-1)3^{2n-1}}{x^n}\right| = \lim_{n\to+\infty}\left|\frac{2n-1}{9(2n+1)}\right||x| = \frac{1}{9}|x|$

Therefore, the power series is absolutely convergent if $\frac{1}{9}|x| < 1$, that is, if $-9 < x < 9$.

$x = -9$: $\sum_{n=1}^{+\infty} \frac{(-1)^n(-1)^n 3^{2n}}{(2n-1)3^{2n-1}} = \sum_{n=1}^{+\infty} \frac{3}{2n-1}$ diverges by comparison with the harmonic series: $\frac{3}{2n-1} > \frac{3}{2}\cdot\frac{1}{n}$.

$x = 9$: $\sum_{n=1}^{+\infty} \frac{(-1)^n 3^{2n}}{(2n-1)3^{2n-1}} = \sum_{n=1}^{+\infty} (-1)^n \frac{3}{2n-1}$ is convergent by the alternating series test. Hence I is $(-9,9]$.

18. $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(n+1)x^n}{n!}$. $R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{(n+1)}{n!}\cdot\frac{(n+1)!}{n+2} = \lim_{n\to+\infty}\frac{n+1}{n+2}(n+1) = +\infty$. I is $(-\infty,+\infty)$.

19. $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(x-1)^{n+1}}{n+1}\cdot\frac{n}{(x-1)^n}\right| = \lim_{n\to+\infty}\left|\frac{n}{n+1}\right||x-1| = |x-1|$

Therefore, the power series is absolutely convergent if $|x-1| < 1$, that is if $0 < x < 2$.

$x = 0$: $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{(-1)^n}{n} = \sum_{n=1}^{+\infty}\left(-\frac{1}{n}\right)$ is a divergent harmonic series.

$x = 2$: $\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{1}{n}$ is convergent by the alternating-series test. Hence I is (0,2].

20. $\sum_{n=1}^{+\infty} \frac{(x+2)^n}{(n+1)2^n}$

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(x+2)^{n+1}}{(n+2)2^{n+1}}\cdot\frac{(n+1)2^n}{(x+2)^n}\right| = \left|\frac{x+2}{2}\right|\lim_{n\to+\infty}\frac{n+1}{n+2} = \left|\frac{x+2}{2}\right|$$

If $\left|\frac{x+2}{2}\right| < 1$ then $-1 < \frac{x+2}{2} < 1$, $-2 < x+2 < 2$, $-4 < x < 0$

Thus the series converges absolutely for all x in $(-4,0)$. If $x = -4$, the given power series is

$$\sum_{n=1}^{+\infty}\frac{(-2)^n}{(n+1)2^n} = \sum_{n=1}^{+\infty}\frac{(-1)^n}{n+1}$$

which converges by the alternating series test. If $x = 0$, the given series is

$$\sum_{n=1}^{+\infty}\frac{2^n}{(n+1)2^n} = \sum_{n=1}^{+\infty}\frac{1}{n+1}$$

which diverges because it is an harmonic series. The interval of convergence of the given power series is $[-4,0)$.

21. $\sum_{n=1}^{+\infty} (\sinh 2n)x^n = \frac{1}{2}\sum_{n=1}^{+\infty}(e^{2n} - e^{-2n})x^n$

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{e^{2(n+1)} - e^{-2(n+1)}}{e^{2n} - e^{-2n}}\right|\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to+\infty}\left|\frac{e^{2n+2}(1 - e^{-4n-4})}{e^{2n}(1 - e^{-4n})}\right||x| = e^2|x|$$

Thus the power series is absolutely convergent if $e^2|x| < 1$, that is, if $-\frac{1}{e^2} < x < \frac{1}{e^2}$.

$x = \pm\frac{1}{e^2}$: $\frac{1}{2}\sum_{n=1}^{+\infty}(\pm 1)^n(1 - e^{-4n})$ is divergent because $\lim_{n\to+\infty}(1 - e^{-4n}) = 1 \neq 0$. Therefore, I is $\left(-\frac{1}{e^2}, \frac{1}{e^2}\right)$.

22. $\sum_{n=1}^{+\infty}\frac{x^n}{\ln(n+1)}$. $R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{\ln(n+2)}{\ln(n+1)} = 1$. If $x = -1$, we have $\sum_{n=1}^{+\infty}\frac{(-1)^n}{\ln(n+1)}$ which converges by the alternating series test. If $x = 1$, we have $\sum_{n=1}^{+\infty}\frac{1}{\ln(n+1)}$ which diverges by comparison with the harmonic series. Thus I is $[-1,1)$.

23. $\sum_{n=2}^{+\infty}\frac{(-1)^{n+1}x^n}{n(\ln n)^2}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{(n+1)[\ln(n+1)]^2}\cdot\frac{n(\ln n)^2}{x^n}\right| = \lim_{n\to+\infty}\frac{n}{n+1}\left|\frac{\ln n}{\ln n + \ln(1+\frac{1}{n})}\right||x|$

$$= \lim_{n\to+\infty}\frac{1}{1+\frac{1}{n}}\cdot\frac{1}{1+[\ln(1+\frac{1}{n})/\ln n]}|x| = |x|$$

Therefore, the power series is absolutely convergent if $|x| < 1$, that is, if $-1 < x < 1$.

$x = \pm 1$: $-\sum_{n=2}^{+\infty}\frac{(\pm 1)^{n+1}}{n(\ln n)^2}$ is absolutely convergent by Exercise 8.4.51 with $p = 2$. Hence I is $[-1,1]$.

24. $\sum_{n=1}^{+\infty}\frac{(x+5)^{n-1}}{n^2}$

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(x+5)^n}{(n+1)^2}\cdot\frac{n^2}{(x+5)^{n-1}}\right| = |x+5|\lim_{n\to+\infty}\frac{n^2}{(n+1)^2} = |x+5|$$

If $|x+5| < 1$, then $-1 < x+5 < 1$, $-6 < x < -4$

Thus the power series converges absolutely for all x in $(-6,-4)$. If $|x+5| = 1$,

$$\sum_{n=1}^{+\infty}|u_n| = \sum_{n=1}^{+\infty}\frac{1}{n^2}$$

which is a convergent p-series with $p = 2$. The interval of convergence of the given power series is $[-6,-4]$.

25. $\sum_{n=1}^{+\infty}\frac{n^2}{5^n}(x-1)^n$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)^2(x-1)^{n+1}}{5^{n+1}}\cdot\frac{5^n}{n^2(x-1)^n}\right| = \lim_{n\to+\infty}\frac{1}{5}\left(\frac{n+1}{n}\right)^2|x-1| = \frac{1}{5}|x-1|$.

Hence, the power series is absolutely convergent if $\frac{1}{5}|x-1| < 1$; $|x-1| < 5$; $-4 < x < 6$.

$|x-1| = 5$: $\sum_{n=1}^{+\infty}\frac{n^2}{5^n}(\pm 5)^n = \sum_{n=1}^{+\infty}(\pm 1)^n n^2$ diverges because $\lim_{n\to+\infty} n^2 = +\infty$. Hence I is $(-4,6)$.

26. $\sum_{n=0}^{+\infty} \frac{4^{n+1}x^{2n}}{n+3}$. $R^2 = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{4^{n+1}}{n+3}\cdot\frac{n+4}{4^{n+2}} = \lim_{n\to+\infty}\frac{1}{4}\cdot\frac{n+4}{n+3} = \frac{1}{4}$; $R = \frac{1}{2}$. If $x = \pm\frac{1}{2}$, we have $\sum_{n=0}^{+\infty}\frac{4^{n+1}(\frac{1}{2})^{2n}}{n+3} = \sum_{n=0}^{+\infty}\frac{4}{n+3}$, a divergent harmonic series. Thus, $I = (-\frac{1}{2},\frac{1}{2})$.

27. $\sum_{n=1}^{+\infty}\frac{\ln n(x-5)^n}{n+1}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{\ln(n+1)(x-5)^{n+1}}{n+2}\cdot\frac{n+1}{\ln n(x-5)^n}\right|$

$= \lim_{n\to+\infty}\left|\frac{n+1}{n+2}\cdot\frac{\ln n+\ln(1+\frac{1}{n})}{\ln n}\right||x-5| = \lim_{n\to+\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\left[1+\frac{\ln(1+\frac{1}{n})}{\ln n}\right]|x-5| = |x-5|$

Thus, the power series is absolutely convergent if $|x-5|<1$, that is, if $4<x<6$.

$x=6$: $\sum_{n=1}^{+\infty}\frac{\ln n}{n+1}$ diverges by comparison with the harmonic series $\sum_{n=1}^{+\infty}\frac{1}{n+1}$.

$x=4$: $\sum_{n=1}^{+\infty}(-1)^n\frac{\ln n}{n+1}$ converges by the alternating-series test; $\lim_{x\to+\infty}\frac{\ln x}{x+1} = \lim_{x\to+\infty}\frac{1/x}{1} = 0$ and $\frac{d}{dx}\frac{\ln x}{x+1} = \frac{1+1/x-\ln x}{(x+1)^2} < 0$ for $x \le 4$; so the terms are decreasing. Hence, I is $[4,6)$.

28. $\sum_{n=1}^{+\infty}\frac{x^n}{n^n}$

▷ We apply the root test.

$\lim_{n\to+\infty}|u_n|^{1/n} = \lim_{n\to+\infty}\frac{|x|}{n} = 0$

Therefore, the given series converges absolutely for all x. The interval of convergence is $(-\infty,+\infty)$.

29. $\sum_{n=1}^{+\infty}(-1)^n\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}x^{2n+1}$.

$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{1\cdot3\cdot5\cdots(2n+1)}{2\cdot4\cdot6\cdots(2n)(2n+2)}\cdot\frac{2\cdot4\cdot6\cdots(2n)}{1\cdot3\cdot5\cdots(2n-1)}\right|\left|\frac{x^{2n+3}}{x^{2n+1}}\right| = \lim_{n\to+\infty}\left|\frac{2n+1}{2n+2}\right|x^2 = x^2$

Therefore, the power series is absolutely convergent if $x^2<1$, that is, if $-1<x<1$. If $x=\pm1$, then $\sum_{n=1}^{+\infty}(-1)^n\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}$ converges by the alternating-series test since by the Wallis inequality, $|u_n| < \frac{1}{\sqrt{2n}}$.

30. $\sum_{n=1}^{+\infty}n^n(x-3)^n$. Apply the root test. $\lim_{n\to+\infty}|u_n|^{1/n} = \lim_{n\to+\infty}n|x-3| = +\infty$. The series converges only if $x=3$.

31. $\sum_{n=1}^{+\infty}\frac{n!x^n}{n^n}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)!x^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!x^n}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)n^n}{(n+1)^{n+1}}\right||x| = \lim_{n\to+\infty}\left(\frac{n}{n+1}\right)^n|x|$

$= \lim_{n\to+\infty}\left[\left(\frac{n+1}{n}\right)^n\right]^{-1}|x| = \left[\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n\right]^{-1}|x| = e^{-1}|x|$

Hence the power series is absolutely convergent if $e^{-1}|x|<1$, that is, if $-e<x<e$.

$x=\pm e$: $\sum_{n=1}^{+\infty}(\pm1)^n\frac{n!e^n}{n^n}$ diverges because $\lim_{n\to+\infty}\frac{n!e^n}{n^n}\neq0$. In fact, because ln is increasing

$\ln n! = \sum_{k=1}^{n}\ln k = \sum_{k=1}^{n}\int_{k-1}^{k}\ln k\,dx > \sum_{i=1}^{n}\int_{k-1}^{k}\ln x\,dx = \int_1^n \ln x\,dx = \Big[x\ln x - x\Big]_1^n = n\ln n - n + 1$

Therefore $n! > \frac{n^n}{e^{n-1}}$ so that $\frac{n!e^n}{n^n} > e$. Hence the interval of convergence of the given power series is $(-e,e)$.

32. $\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots\cdot(2n)}x^n$

▷ $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)(2n+1)x^{n+1}}{2\cdot4\cdot6\cdot\ldots\cdot(2n)(2n+2)}\cdot\frac{2\cdot4\cdot6\cdot\ldots\cdot(2n)}{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)x^n}\right| = |x|\lim_{n\to+\infty}\frac{2n+1}{2n+2} = |x|$

If $|x|<1$ then $-1<x<1$. Thus the series converges absolutely for x in $(-1,1)$. If $x=-1$, the series becomes

$$\sum_{n=1}^{+\infty}\frac{(-1)^{2n+1}1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots\cdot(2n)} = -\sum_{n=1}^{+\infty}\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots\cdot(2n)} \tag{1}$$

By the Wallis inequality

$$\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)} > \frac{1}{2\sqrt{n}}$$

and so the series diverges by comparison with a p-series with $p = \frac{1}{2}$. Next, we replace x by 1 in the given series. The result is

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1} 1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}$$

By the other part of the Wallis inequality

$$\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)} < \frac{1}{\sqrt{2n}}$$

and so the series converges by the alternating series test if $x = 1$. Hence, the interval of convergence of the power series is $(-1, 1]$.

ALTERNATE SOLUTION: We may use Raabe's test to show that the series in (1) diverges.

$$\lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to+\infty} n\left(\frac{2n+2}{2n+1} - 1\right) = \lim_{n\to+\infty} \frac{n}{2n+1} = \frac{1}{2}$$

Because $p = \frac{1}{2} < 1$, by Raabe's test, the given series diverges.

33. $\frac{PV}{RT} = 1 + \frac{B}{V} + \frac{C}{V^2} + \frac{D}{V^3} + \cdots$ (1) $\quad \frac{PV}{RT} = 1 + \bar{B}P + \bar{C}P^2 + \bar{D}P^3 + \cdots$ (2). Solve (1) for P and substitute in (2).

$\frac{PV}{RT} = 1 + \bar{B}\left(\frac{RT}{V} + \frac{BRT}{V^2} + \frac{CRT}{V^3} + \cdots\right) + \bar{C}\left(\frac{RT}{V} + \frac{BRT}{V^2} + \cdots\right) + \bar{D}\left(\frac{RT}{V} + \cdots\right)$. Compare with (1).

coefficient of $\frac{1}{V}$: $B = \bar{B}RT$, $\bar{B} = \frac{B}{RT}$ $\qquad$ coefficient of $\frac{1}{V^2}$: $C = \bar{B}RT + \bar{C}R^2T^2$, $\bar{C} = \frac{C - B\bar{B}RT}{R^2T^2} = \frac{C - B^2}{R^2T^2}$

coefficient of $\frac{1}{V^3}$: $D = \bar{B}CRT + \bar{C}\cdot 2BR^2T^2 + \bar{D}R^3T^3$,

$\bar{D} = \frac{D - C\cdot\bar{B}RT - 2B\cdot\bar{C}R^2T^2}{R^3T^3} = \frac{D - CB - 2B(C - B^2)}{R^3T^3} = \frac{D + 2B^3 - 3BC}{R^3T^3}$. (1) and (2) are the *virial equations*.

34. If $|x| \le 1$, then $|a_n x^n| \le |a_n|$ and so $\sum_{n=1}^{+\infty} |a_n x^n|$ is convergent by comparison with $\sum_{n=1}^{+\infty} |a_n|$.

35. $\sum_{n=1}^{+\infty} \frac{(n+a)!}{n!(n+b)!} x^n$ where a and b are positive integers. $\lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty} \left|\frac{(n+1+a)!x^{n+1}}{(n+1)!(n+1+b)!} \cdot \frac{n!(n+b)!}{(n+a)!x^n}\right|$

$= \lim_{n\to+\infty} \left|\frac{n+1+a}{(n+1)(n+1+b)}\right| |x| = \lim_{n\to+\infty} \frac{1 + a/(n+1)}{n+1+b} |x| = 0 < 1$ for all x.

The power series converges for all x so the radius of convergence is $+\infty$.

36. Prove that if $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = L$ ($L \ne 0$), then the radius of convergence of $\sum_{n=1}^{+\infty} u_n x^n$ is $\frac{1}{L}$.

▶ We apply the root test.

$$\lim_{n\to+\infty} |u_n x^n|^{1/n} = |x| \lim_{n\to+\infty} |u_n|^{1/n} = |x|L$$

The series is absolutely convergent when $|x|L < 1$, that is, when $|x| < 1/L$, so the radius of convergence is $1/L$.

37. Suppose the power series $\sum_{n=0}^{+\infty} c_n x^n$ converges if $-a < x <$. Let the power series converge absolutely if $x = a$. Then the series $\sum_{n=0}^{+\infty} |c^n a^n|$ converges. When $x = -a$ the power series becomes $\sum_{n=0}^{+\infty} (-1)^n c_n a^n$. Because $\sum_{n=0}^{+\infty} |(-1)^n c_n a^n| = \sum_{n=0}^{+\infty} |c_n a^n|$, which is convergent, then the series $\sum_{n=0}^{+\infty} (-1)^n c_n a^n$ is absolutely convergent.

38. This is the contrapositive of Exercise 37.

39. We are given that the radius of convergence of the series $\sum_{n=1}^{+\infty} u_n x^n$ is r. Let $\bar{r}$ be the radius of convergence of the series $\sum_{n=1}^{+\infty} u_n x^{2n}$.

(a) If $|x_1| < \sqrt{r}$, then $|x_1^2| < r$, and so the series $\sum_{n=1}^{+\infty} u_n (x_1^2)^n = \sum_{n=1}^{+\infty} u_n x_1^{2n}$ converges.

(b) If $|x_2| > \sqrt{r}$, then $|x_2^2| > r$, and so the series $\sum_{n=1}^{+\infty} u_n (x_2^2)^n = \sum_{n=1}^{+\infty} u_n x_2^{2n}$ diverges.

From (a) we have $\sqrt{r} \le \bar{r}$ and from (b) we have $\sqrt{r} \ge \bar{r}$. Hence $\bar{r} = \sqrt{r}$.

8.8 DIFFERENTIATION AND INTEGRATION OF POWER SERIES

8.8.3 Theorem Let $\sum_{n=0}^{+\infty} c_n x^n$ be a power series with radius of convergence R > 0. If f is the function defined by

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n \tag{1}$$

then $f'(x)$ exists for every x in the open interval $(-R, R)$, and it is given by

$$f'(x) = \sum_{n=1}^{+\infty} n c_n x^{n-1}$$

Thus, if a function f is defined by a power series, we may use term-by-term differentiation. Furthermore, **Abel's Theorem** states that if the series (1) converges at R, then f is continuous from the left at R, and if the series converges at $-R$, then f is continuous from the right at R.

Uniqueness It follows from Exercise 59 that if $f(x) = \sum_{n=0}^{+\infty} c_n x^n$ and $f(x) = \sum_{n=0}^{+\infty} d_n x^n$ in $(-R, R)$, then $c_n = d_n$ for $n = 0, 1, \ldots$.

Differential Equations Let $y = \sum_{n=0}^{+\infty} c_n x^n$, $y' = \sum_{n=0}^{+\infty} n c_n x^{n-1}$ and equate coefficients of equal powers of x on both sides.

See Ex. 56. For *Bessel's equation* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - n^2)y = 0$, $n = 0, 1$, see Misc. Ex. 75–79.

Odd and Even A power series for an even (odd) function has only even (odd) powers. See Exercises 57.

The result of Example 3 may be stated as a theorem.

Theorem A
$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

8.8.4 Theorem Let $\sum_{n=0}^{+\infty} c_n x^n$ be a power series with radius of convergence R > 0. If f is the function defined by

$$f(x) = \sum_{n=0}^{+\infty} c_n x^n$$

then f is integrable on every closed subinterval of $(-R, R)$, and we evaluate the integral of f by integrating the given power series term by term; that is, if x is in $(-R, R)$, then

$$\int_0^x f(t)\,dt = \sum_{n=0}^{+\infty} \frac{c_n}{n+1} x^{n+1}$$

Furthermore, R is the radius of convergence of the resulting series.

Thus, if a function f is defined by a power series, we may use term by term integration on the power series to find a power-series representation of $F(x) = \int_0^x f(t)\,dt$. And if b is a constant in the interval of convergence of the power-series representation of $F(x)$, we may use the first n terms of this power series to approximate the definite integral $\int_0^b f(t)\,dt$. If the lower limit is not 0 we change variables to make it 0; see Exercise 31.

Truncation Error If each term in a series of terms of the same sign is less than one nth the preceding term, the error in an approximation by a partial sum is less than $1/(n-1)$ times the last term retained. We see this by comparing the rest of the series with a geometric series with $r = 1/n$. See Exercise 35.

We have the following power-series representations which are stated as theorems.

Theorem B $\tan^{-1} x = \sum_{n=0}^{+\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots$ if $|x| \le 1$

Theorem C $\sinh x = \sum_{n=0}^{+\infty} \dfrac{x^{2n+1}}{(2n+1)!}$ and $\cosh x = \sum_{n=0}^{+\infty} \dfrac{x^{2n}}{(2n)!}$ for all x. See Exercise 21.

Exercises 8.8

In Exercises 1–10 do the following: (a) Find the radius of convergence of the power series and the domain of f. (b) Write the power series that defines the function and find its radius of convergence. (c) Find the domain of f'.

1. (a) $f(x) = \sum_{n=1}^{+\infty} \dfrac{x^n}{n^2}$. $\lim_{n\to+\infty} \left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty} \left|\dfrac{x^{n+1}}{(n+1)^2} \cdot \dfrac{n^2}{x^n}\right| = \lim_{n\to+\infty} \left(\dfrac{n}{n+1}\right)^2 |x| = |x|$.

   The power series is absolutely convergent if $|x| < 1$; so its radius of convergence is 1.

   $x = \pm 1$: $\sum_{n=1}^{+\infty} |u_n| = \sum_{n=1}^{+\infty} \dfrac{1}{n^2}$ is a convergent p-series ($p = 2$). Thus the domain of f is $[-1, 1]$.

   (b) $f'(x) = \sum_{n=1}^{+\infty} \dfrac{x^{n-1}}{n}$. $\lim_{n\to+\infty} \left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty} \left|\dfrac{x^n}{n+1} \cdot \dfrac{n}{x^{n-1}}\right| = \lim_{n\to+\infty} \left|\dfrac{n}{n+1}\right| |x| = |x|$

The power series is absolutely convergent if $|x| < 1$; so its radius of convergence is 1.

(c) $f'(1) = \sum_{n=1}^{+\infty} \frac{1}{n}$ which is divergent because it is the harmonic series.

$f'(-1) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ which is convergent by the alternating-series test. Therefore, the domain of f' is $[-1,1)$.

2. (a) $f(x) = \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{x^n}{n}$. $R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{n+1}{n} = 1$. $f(1) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}$ which is convergent by the alternating series test. $f(-1) = -\sum_{n=1}^{+\infty}\frac{1}{n}$ is divergent because it the harmonic series. $\text{Dom}(f) = (-1,1]$.

(b) $f'(x) = \sum_{n=1}^{+\infty}(-1)^{n-1}x^n$. $R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{1}{1} = 1$.

(c) $f'(\pm 1) = \sum_{n=1}^{+\infty}(-1)^{n-1}(\pm 1)^n$ is divergent because the terms do not approach zero. $\text{Dom}(f') = (-1,1)$.

3. (a) $f(x) = \sum_{n=1}^{+\infty}\frac{x^n}{\sqrt{n}}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{\sqrt{n+1}}\right|\cdot\frac{\sqrt{n}}{x^n} = \lim_{n\to+\infty}\sqrt{\frac{n}{n+1}}|x| = |x|$

The power series is absolutely convergent if $|x| < 1$; so its radius of convergence is 1.

$x = 1$: $\sum_{n=1}^{+\infty}\frac{1}{\sqrt{n}}$ is divergent because it is the p-series with $p = \frac{1}{2} < 1$.

$x = -1$: $\sum_{n=1}^{+\infty}\frac{(-1)^n}{\sqrt{n}}$ is convergent by the alternating-series test. Therefore, the domain of f is $[-1,1)$.

(b) $f'(x) = \sum_{n=1}^{+\infty}\sqrt{n}\,x^{n-1}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{\sqrt{n+1}\,x^{n+1}}{\sqrt{n}x^n}\right| = \lim_{n\to+\infty}\sqrt{\frac{n+1}{n}}|x| = |x|$

The power series is absolutely convergent if $|x| < 1$; so its radius of convergence is 1.

(c) $f'(\pm 1) = \sum_{n=1}^{+\infty}(\pm 1)^{n-1}\sqrt{n}$ which is divergent because $\lim_{n\to+\infty}\sqrt{n} \neq 0$. Thus the domain of f' is $(-1,1)$.

4. $f(x) = \sum_{n=2}^{+\infty}\frac{(x-2)^n}{\sqrt{n-1}}$

▷ (a) We use the ratio test to find the radius of convergence.

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(x-2)^{n+1}}{\sqrt{n}}\cdot\frac{\sqrt{n-1}}{(x-2)^n}\right| = |x-2|\lim_{n\to+\infty}\sqrt{1-\tfrac{1}{n}} = |x-2|$$

Because the power series converges for all x such that $|x-2|$ and diverges for all x with $|x-2| > 1$, the radius of convergence is $R = 1$. If $|x-2| < 1$, then $1 < x < 3$. Thus, the domain of f contains all x in the open interval $(1,3)$. We test the endpoints.

$$f(1) = \sum_{n=2}^{+\infty}\frac{(-1)^n}{\sqrt{n-1}}$$

which converges by the alternating series test. Hence, 1 is in the domain of f. Moreover

$$f(3) = \sum_{n=2}^{+\infty}\frac{1}{\sqrt{n-1}}$$

which diverges by comparison with the p-series with $p = \frac{1}{2}$. Thus, 3 is not in the domain of f. Therefore the domain of f is $[1,3)$.

(b) Differentiating f, we obtain

$$f'(x) = \sum_{n=2}^{+\infty}\frac{n(x-2)^{n-1}}{\sqrt{n-1}}$$

We find the radius of convergence of f'. We have

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)(x-2)^n}{\sqrt{n}}\cdot\frac{\sqrt{n-1}}{n(x-2)^{n-1}}\right| = |x-2|\lim_{n\to+\infty}\sqrt{1-\tfrac{1}{n}}\left(1+\tfrac{1}{n}\right) = |x-2|$$

Because the series converges for all x with $|x-2| < 1$ and diverges for all x with $|x-2| > 1$, then $R = 1$.

(c) Because $|x-2| < 1$ if $1 < x < 3$, the domain of f' contains all numbers in $(1,3)$. If $|x-2| = 1$,

$$\lim_{n\to+\infty}|u_n| = \lim_{n\to+\infty}\frac{n}{\sqrt{n-1}} = +\infty$$

Because the terms do not converge to zero, f' is not defined at either endpoint. Hence the domain of f' is $(1,3)$.

5. (a) $f(x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n+1}}{(2n+1)!}\cdot\frac{(2n-1)!}{x^{2n-1}}\right| = \lim_{n\to+\infty}\frac{1}{2n(2n+1)}x^2 = 0 < 1$.
The power series converges for all x, its radius of convergence is $+\infty$ and its domain is $(-\infty, +\infty)$.
(b) $f'(x) = \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n}}{(2n)!}\cdot\frac{(2n-2)!}{x^{2n-1}}\right| = \lim_{n\to+\infty}\frac{1}{(2n-1)(2n)}x^2 = 0 < 1$
The power series for $f'(x)$ converges for all x; so its radius of convergence is $+\infty$. (c) $\text{Dom}(f') = (-\infty, +\infty)$

6. (a) $f(x) = \sum_{n=0}^{+\infty}\frac{x^{2n}}{(n!)^2}$. $R^2 = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{[(n+1)!]^2}{(n!)^2} = \lim_{n\to+\infty}(n+1)^2 = +\infty$. $R = +\infty$.
$\text{Dom}(f) = (-\infty, +\infty)$ (b) $f'(x) = \sum_{n=1}^{+\infty}\frac{2nx^{2n-1}}{(n!)^2}$. $R^2 = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{2n}{(n!)^2}\cdot\frac{[(n+1)!]^2}{2(n+1)} = \lim_{n\to+\infty} n(n+1)$
$= +\infty$. $R = +\infty$. (c) $\text{Dom}(f') = (-\infty, +\infty)$

7. (a) $f(x) = \sum_{n=1}^{+\infty}(n+1)(3x-1)^n$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(n+2)(3x-1)^{n+1}}{(n+1)(3x-1)^n}\right| = \lim_{n\to+\infty}\frac{n+2}{n+1}|3x-1| = |3x-1|$
The power series is absolutely convergent if $|3x-1| < 1$, that is, if $0 < x < \frac{2}{3}$. The radius of convergence is $\frac{1}{3}$.
$3x - 1 = \pm 1$: $\sum_{n=1}^{+\infty}(\pm 1)^n(n+1)$ is divergent because $\lim_{n\to+\infty}(n+1) \neq 0$. Therefore the domain of f is $(0, \frac{2}{3})$.
(b) $f'(x) = \sum_{n=1}^{+\infty} 3n(n+1)(3x-1)^{n-1}$.
$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{3(n+1)(n+2)(3x-1)^n}{3n(n+1)(3x-1)^{n-1}}\right| = \lim_{n\to+\infty}\frac{n+2}{n}|3x-1| = |3x-1|$
The power series is absolutely convergent if $|3x-1| < 1$, that is, if $0 < x < \frac{2}{3}$. The radius of convergence is $\frac{1}{3}$.
(c) If $3x - 1 = \pm 1$, $f'(x) = \sum_{n=1}^{+\infty}(\pm 1)^{n-1}3n(n+1)$ which is divergent because $\lim_{n\to+\infty} 3n(n+1) \neq 1$.
Therefore the domain of f' is $(0, \frac{2}{3})$.

8. $f(x) = \sum_{n=1}^{+\infty}\frac{x^{2n-2}}{(2n-2)!}$

▸ (a) We use the ratio test to find R.
$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n}}{(2n!)}\cdot\frac{(2n-2)!}{x^{2n-2}}\right| = x^2\lim_{n\to+\infty}\frac{(2n-2)!}{(2n)(2n-1)(2n-2)!} = x^2\lim_{n\to+\infty}\frac{1}{(2n)(2n-1)} = 0$$
Thus, the series converges absolutely for all x. We have $R = +\infty$ and the domain of f is $(-\infty, +\infty)$.
(b) $f(x) = 1 + \sum_{n=2}^{+\infty}\frac{x^{2n-2}}{(2n-2)!}$
Differentiating, we have
$$f'(x) = \sum_{n=2}^{+\infty}\frac{(2n-2)x^{2n-3}}{(2n-2)!} = \sum_{n=2}^{+\infty}\frac{x^{2n-3}}{(2n-3)!}$$
$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n-1}}{(2n-1)!}\cdot\frac{(2n-3)!}{x^{2n-3}}\right| = x^2\lim_{n\to+\infty}\frac{(2n-3)!}{(2n-1)(2n-2)(2n-3)!}$$
$$= x^2\lim_{n\to+\infty}\frac{1}{(2n-1)(2n-2)} = 0$$
Thus, the series converges for all x, and hence $R = +\infty$.
(c) Because the power series for f' converges for all x, the domain of f' is $(-\infty, +\infty)$.

9. (a) $f(x) = \sum_{n=1}^{+\infty}\frac{(x-1)^n}{n3^n}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(x-1)^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(x-1)^n}\right| = \lim_{n\to+\infty}\frac{n}{3(n+1)}|x-1| = \frac{1}{3}|x-1|$
The power series is absolutely convergent if $\frac{1}{3}|x-1| < 1$, that is if $-2 < x < 4$. The radius of convergence is 3.
$x = 4$: $\sum_{n=1}^{+\infty}\frac{1}{n}$ is the divergent harmonic series. $x = -2$: $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}$ is convergent by the alternating-series test.
Therefore, the domain of f is $[-2, 4)$.
(b) $f'(x) = \sum_{n=1}^{+\infty}\frac{(x-1)^{n-1}}{3^n}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(x-1)^n}{3^{n+1}}\cdot\frac{3^n}{(x-1)^{n-1}}\right| = \frac{1}{3}|x-1|$
The power series is absolutely convergent if $\frac{1}{3}|x-1| < 1$, that is if $-2 < x < 4$. The radius of convergence is 3.
(c) $f'(-2) = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{3}$ which is divergent. Hence the domain of f' is $(-2, 4)$.

10. (a) $f(x) = \sum_{n=2}^{+\infty} (-1)^n \frac{(x-3)^n}{n(n-1)}$. $R = \lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{(n+1)n}{n(n-1)} = 1$. If $|x-3| = 1$, $f(x) = \sum_{n=2}^{+\infty} \frac{(\pm 1)^n}{n(n-1)}$ which converges by comparison with a p-series, $p = 2$. Therefore, the domain of f is $-1 \le x - 3 \le 1$ or $[2, 4]$.
(b) $f'(x) = \sum_{n=2}^{+\infty} (-1)^n \frac{(x-3)^{n-1}}{n-1}$. $R = \lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{n}{n-1} = 1$. (c) $f'(4) = \sum_{n=2}^{+\infty} \frac{(-1)}{n-1}$ which converges by the alternating series test. $f'(2) = -\sum_{n=2}^{+\infty} \frac{1}{n-1}$ which is a divergent harmonic series. $\mathrm{Dom}(f') = (2, 4]$.

11. Using Example 2 and differentiating on both sides by Theorem 8.8.3, if $|x| < 1$ then
$\frac{1}{(1-x)^2} = \sum_{n=1}^{+\infty} nx^{n-1}$, $\frac{2}{(1-x)^3} = \sum_{n=1}^{+\infty} n(n-1)x^{n-2}$, $\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{n=1}^{+\infty} n(n-1)x^{n-2}$

12. Use the result of Example 3 to find a series representation of e.
▹ In Example 3 we are given that
$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$
If $x \ge 0$ we may replace x by $\sqrt{x}$ and thus obtain

$$e = \sum_{n=0}^{+\infty} \frac{(\sqrt{x})^n}{n!} = \sum_{n=0}^{+\infty} \frac{x^{n/2}}{n!} \quad \text{if } x \ge 0$$

which is the required representation. It is not a power series.

13. Using equation (8.7.4) and differentiating on both sides by Theorem 8.8.3, if $|x| < 1$ then
$\frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n$, $-\frac{1}{(1+x)^2} = \sum_{n=1}^{+\infty} (-1)^n nx^{n+1} = \sum_{n=0}^{+\infty} (-1)^{n-1}(n+1)x^n$, $\frac{1}{(1+x)^2} = \sum_{n=0}^{+\infty} (-1)^n (n+1)x^n$

14. $\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$, $-\frac{2x}{(1+x^2)^2} = \sum_{n=1}^{+\infty} (-1)^n (2n)x^{2n-1}$, $\frac{x}{(1+x^2)^2} = \sum_{n=1}^{+\infty} (-1)^{n-1} nx^{2n-1}$

15. $\sinh x = \int_0^x \cosh t\, dt = \int_0^x \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!}\, dt = \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)(2n)!}\Big]_0^x = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$

16. Find a power-series representation for $\tanh^{-1} x$ by integrating term by term from 0 to x a power-series representation for $(1 - t^2)^{-1}$.
▹ From equation 8.7.5,
$\frac{1}{1-t^2} = \sum_{n=0}^{+\infty} t^{2n}$ if $|t| < 1$
Therefore, if $|x| < 1$,

$$\tanh^{-1} x = \int_0^x \frac{dt}{1-t^2} = \int_0^x \sum_{n=0}^{+\infty} t^{2n}\, dt = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1}$$

17. (a) Using Example 3 and replacing x by x^2 we have, for all x, $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$, $e^{x^2} = \sum_{n=0}^{+\infty} \frac{x^{2n}}{n!}$
(b) Differentiating both sides of the last equation we get, for all x,
$2xe^{x^2} = \sum_{n=1}^{+\infty} \frac{2n}{n!} x^{2n-1}$, $2xe^{x^2} = \sum_{n=1}^{+\infty} \frac{2}{(n-1)!} x^{2n-1}$, $xe^{x^2} = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{n!}$

18. (a) $f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^n}{3^n(n+2)}$. $R = \lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{3^{n+1}(n+3)}{3^n(n+2)} = \lim_{n\to+\infty} 3\frac{n+3}{n+2} = 3$. $f(-3) = \sum_{n=0}^{+\infty} \frac{1}{n+2}$ is a divergent harmonic series. $f(3) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+2}$ converges by the alternating series test. $\mathrm{Dom}(f) = (-3, 3]$
(b) $f'(x) = \sum_{n=1}^{+\infty} (-1)^n \frac{nx^{n-1}}{3^n(n+2)}$. $R = \lim_{n\to+\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{n}{3^n(n+2)} \cdot \frac{3^{n+1}(n+3)}{n+1} = \lim_{n\to+\infty} 3\frac{n(n+3)}{(n+2)(n+1)} = 3$
If $x = \pm 3$, $f(x) = \sum_{n=1}^{+\infty} (\pm 1)^n \frac{n}{n+2}$ which diverges because $\lim_{n\to+\infty} \frac{n}{n+2} \ne 0$. $\mathrm{Dom}(f') = (-3, 3)$.

19. Let $x = \frac{1}{2}$ in the result of Example 4. We need $R_n < \frac{1}{200{,}000}$ for five place accuracy.
$e^{-x} = \sum_{n=0}^{+\infty} (-1)^n \frac{x^n}{n!}$. $\frac{1}{\sqrt{e}} \approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} - \frac{1}{3840} + \frac{1}{46{,}080} = 0.60653$ with $|\text{error}| < \frac{1}{645{,}120}$

20. If $f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{3^n}$ find $f'(\frac{1}{2})$ to four decimal places.

▸ Differentiating f and disregarding the term with value zero, we have

$$f'(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{2nx^{2n-1}}{3^n} = \sum_{n=1}^{+\infty} (-1)^n \frac{2nx^{2n-1}}{3^n}$$

Thus,

$$f'(\tfrac{1}{2}) = \sum_{n=1}^{+\infty} (-1)^n \frac{2n(\frac{1}{2})^{2n-1}}{3^n} = \sum_{n=1}^{+\infty} (-1)^n \frac{n}{3^n 2^{2n-2}} = -\frac{1}{3} + \frac{2}{3^2 \cdot 2^2} - \frac{3}{3^3 \cdot 2^4} + \frac{4}{3^4 \cdot 2^6} - \frac{5}{3^5 \cdot 2^8} + \cdots$$

$$= -0.33333 + 0.05556 - 0.00694 + 0.00077 - 0.00008 + \cdots \qquad (1)$$

Because series (1) satisfies the hypothesis of the alternating series test, if the five terms of (1) shown are used to approximate the sum, the absolute value of the error is less than the absolute value of the sixth term, and so is accurate to four decimal places. Adding, we obtain -0.28402, which we round off to -0.2840. Thus $f'(\frac{1}{2}) = -0.2840$.

21. (a) $\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right)\right]$

$= \frac{1}{2}\left[2x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5 + \cdots + \frac{2}{(2n+1)!}x^{2n+1} + \cdots\right] = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$ for all x.

(b) $\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right)\right]$

$= \frac{1}{2}\left[2 + \frac{2}{2!}x^2 + \frac{2}{4!}x^4 + \cdots + \frac{2}{(2n)!}x^{2n} + \cdots\right] = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$ for all x.

22. $\frac{d}{dx}\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{+\infty} \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$ and $\frac{d}{dx}\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} = \sum_{n=1}^{+\infty} \frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{+\infty} \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$

In Exercises 23–27, show that the power series is a solution of the differential equation.

23. $y = \sum_{n=1}^{+\infty} \frac{2^n}{n!}x^n$. $2y = 2\sum_{n=0}^{+\infty} \frac{2^n}{n!}x^n = \sum_{n=0}^{+\infty} \frac{2^{n+1}}{n!}x^n$ and $\frac{dy}{dx} = \sum_{n=1}^{+\infty} n \cdot \frac{2^n}{n!}x^{n-1} = \sum_{n=1}^{+\infty} \frac{2^n}{(n-1)!}x^{n-1} = \sum_{n=0}^{+\infty} \frac{2^{n+1}}{n!}$.

Therefore $\frac{dy}{dx} = 2y$, $\frac{dy}{dx} - 2y = 0$.

24. $y = \sum_{n=0}^{+\infty} \frac{1}{2^n n!}x^{2n}$ $\quad \frac{dy}{dx} - xy = 0$

▸ We differentiate on both sides of the given power series.

$$\frac{dy}{dx} = \sum_{n=0}^{+\infty} \frac{2n}{2^n n!}x^{2n-1}$$

Because the first term is zero, we can simplify the result as follows:

$$\frac{dy}{dx} = \sum_{n=1}^{+\infty} \frac{2n}{2^n n!}x^{2n-1}$$

If we let $m = n - 1$, we obtain

$$\frac{dy}{dx} = \sum_{m=0}^{+\infty} \frac{1}{2^m m!}x^{2m+1} \qquad (1)$$

Multiplying on both sides of the given power series by x, we obtain

$$xy = \sum_{n=0}^{+\infty} \frac{1}{2^n n!}x^{2n+1}$$

Because the power series for dy/dx, given in Eq. (1) is the same as the power series for xy, given in Eq. (2), we conclude that the given power series is a solution of the given differential equation.

25. $y = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n-1)!}x^{2n-1}$. $\frac{dy}{dx} = \sum_{n=1}^{+\infty} (2n-1)\frac{(-1)^{n+1}}{(2n-1)!}x^{2n-2} = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n-2)!}x^{2n-2}$.

$\frac{d^2y}{dx^2} = \sum_{n=2}^{+\infty} (2n-2)\frac{(-1)^{n+1}}{(2n-2)!}x^{2n-3} = \sum_{n=2}^{+\infty} \frac{(-1)^{n+1}}{(2n-3)!}x^{2n-3} = \sum_{n=1}^{+\infty} \frac{(-1)^{n+2}}{(2n-1)!}x^{2n-1} = -\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n-1)!}x^{2n-1}$

Therefore $\frac{d^2y}{dx^2} = -y$, $\frac{d^2y}{dx^2} + y = 0$.

26. $y = x + \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$; $\frac{dy}{dx} = 1 + \sum_{n=1}^{+\infty} (-1)^n \frac{(2n)x^{2n-1}}{(2n)!} = 1 + \sum_{n=1}^{+\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!}$

$\frac{d^2y}{dx^2} = \sum_{n=1}^{+\infty} (-1)^n \frac{(2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{+\infty} (-1)^n \frac{x^{2n-2}}{(2n-2)!} = -\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} = x - y$. Thus, $\frac{d^2y}{dx^2} + y - x = 0$.

27. $y = \sum_{n=0}^{+\infty} (-1)^n \frac{2^n n!}{(2n+1)!} x^{2n+1}$. $\frac{dy}{dx} = \sum_{n=0}^{+\infty} (2n+1)(-1)^n \frac{2^n n!}{(2n+1)!} x^{2n}$. $x\frac{dy}{dx} = \sum_{n=0}^{+\infty} (2n+1)(-1)^n \frac{2^n n!}{(2n+1)!} x^{2n+1}$

$\frac{d^2y}{dx^2} = \sum_{n=1}^{+\infty} (2n+1)(2n)(-1)^n \frac{2^n n!}{(2n+1)!} x^{2n-1} = \sum_{n=1}^{+\infty} (-1)^n \frac{2^n n!}{(2n-1)!} x^{2n-1}$

$= \sum_{n=0}^{+\infty} (-1)^{n+1} \frac{2^{n+1}(n+1)!}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{+\infty} (-2)(n+1)(-1)^n \frac{2^n n!}{(2n+1)!} x^{2n+1}$

Hence $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sum_{n=0}^{+\infty} [-2(n+1) + (2n+1) + 1](-1)^n \frac{2^n n!}{(2n+1)!} x^{2n+1} = 0.$

28. Use the result of Example 2 to find the sum of the series $\sum_{n=1}^{+\infty} \frac{n}{2^n}$

▸ Example 2 shows that

$\sum_{n=1}^{+\infty} nx^{n-1} = \frac{1}{(1-x)^2}, |x| < 1$

Replacing x with $\frac{1}{2}$ we obtain

$\sum_{n=1}^{+\infty} n(\frac{1}{2})^{n-1} = \frac{1}{(1-\frac{1}{2})^2}$; $\sum_{n=1}^{+\infty} \frac{n}{2^{n-1}} = 4$; $\frac{1}{2}\sum_{n=1}^{+\infty} \frac{n}{2^{n-1}} = 2$; $\sum_{n=1}^{+\infty} \frac{n}{2^n} = 2$

In Exercises 29–32, find a power-series representation of the integral and determine its radius of convergence.

29. $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$ for all x. Hence $R = +\infty$ and $\int_0^x e^t dt = \int_0^x \left(\sum_{n=0}^{+\infty} \frac{t^n}{n!}\right)dt = \sum_{n=0}^{+\infty} \int_0^x \frac{t^n}{n!}\,dt = \sum_{n=0}^{+\infty} \frac{x^{n+1}}{(n+1)!}$.

30. $\frac{1}{t^4+4} = \frac{1}{4}\cdot\frac{1}{1+(\frac{1}{2}t)^2} = \frac{1}{4}\sum_{n=0}^{+\infty}(-1)^n(\frac{1}{2}t)^{2n}$, $\left|\frac{1}{2}t\right| < 1$. Hence $R = 2$.

$\int_0^x \frac{dt}{t^4+4} = \frac{1}{4}\sum_{n=0}^{+\infty}\int_0^x (-1)^n \frac{t^{2n}}{2^{2n}} = \frac{1}{4}\sum_{n=0}^{+\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)2^{2n}} = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)2^{2n}}$

31. Let $t = u+2$. $\int_{t=2}^{x} \frac{dt}{4-t} = \int_{u=0}^{x-2}\frac{du}{2-u} = \frac{1}{2}\int_0^{x-2}\frac{du}{1-(\frac{1}{2}u)} = \frac{1}{2}\sum_{n=0}^{+\infty}\int_0^{x-2}(\frac{1}{2}u)^n = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{u^{n+1}}{(n+1)2^n}\Big]_0^{x-2} = \sum_{n=1}^{+\infty}\frac{(x-2)^n}{n2^n}$

$R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{(n+1)2^{n+1}}{n2^n} = \lim_{n\to+\infty} 2\frac{n+1}{n} = 2$

32. $\int_0^x \tan^{-1}t\,dt$

▸ Applying Theorem 8.8.4 to Theorem B, we have

$\int_0^x \tan^{-1}t\,dt = \sum_{n=0}^{+\infty}\int_0^x (-1)^n \frac{t^{2n+1}}{2n+1}dt = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+2}}{(2n+1)(2n+2)}$

The radius of convergence of this power series is the same as in Theorem B, that is $R = 1$.

In Exercises 33–36, compute accurate to three decimal places the value of the integral by two methods: (a) Use the second fundamental theorem of the calculus; (b) use the result of the indicated exercise.

33. (a) $\int_0^1 e^t dx = e^t\Big]_0^1 = e - 1 \approx 2.718 - 1 = 1.718$. (b) From Exercise 29 with $x = 1$,

$\int_0^1 e^x dx = \sum_{n=0}^{+\infty}\frac{1}{(n+1)!} \approx 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} = 1.718$ with $|\text{error}| < \frac{1}{5040} = 0.0002$.

34. (a) $\int_0^1 \frac{dt}{t^2+4} = \frac{1}{2}\tan^{-1}\frac{t}{2}\Big]_0^1 = \frac{1}{2}\tan^{-1}\frac{1}{2} = 0.2318 \approx 0.232$ (b) From Exercise 30 with $x = 1$.

$\int_0^1 \frac{dt}{t^2+4} = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{(2n-1)2^{2n}} \approx \frac{1}{1(4)} - \frac{1}{3(16)} + \frac{1}{5(64)} - \frac{1}{7(256)} = 0.2317 \approx 0.232$ with $|\text{error}| < \frac{1}{9(1024)} = 0.0001$

35. (a) $\int_2^3 \frac{dt}{4-t} = -\ln|4-t|\Big]_2^3 = -\ln 1 + \ln 2 = \ln 2 = 0.6931 \approx 0.693$

(b) From Ex. 31 with $x = 3$, $\int_2^3 \frac{dt}{4-t} = \sum_{n=1}^{+\infty}\frac{1}{n2^n} \approx \frac{1}{1(2)} + \frac{1}{2(4)} + \frac{1}{3(8)} + \frac{1}{4(16)} + \frac{1}{5(32)} + \frac{1}{6(64)} + \frac{1}{7(128)} + \frac{1}{8(256)}$

$= 0.6928 \approx 0.693$. Because each term is less half the preceding, error $< \frac{1}{8(256)} = 0.0005$.

36. $\displaystyle\int_0^{1/3}\tan^{-1}t\,dt$

▸ (a) We use integration by parts. Let

$$u=\tan^{-1}t \qquad dv=dt$$
$$du=\frac{dt}{1+t^2} \qquad v=t$$

$$\int_0^{1/3}\tan^{-1}t\,dt=\left[t\tan^{-1}t-\int\frac{t\,dt}{1+t^2}\right]_0^{1/3}=\left[t\tan^{-1}t-\tfrac12\ln(1+t^2)\right]_0^{1/3}=\tfrac13\tan^{-1}\tfrac13-\tfrac12\ln\tfrac{10}{9}=0.0546\approx 0.055$$

(b) In Exercise (32) we obtained the power-series representation

$$\int_0^x\tan^{-1}t\,dt=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+2}}{(2n+1)(2n+2)}$$

which converges for $|x|\le 1$. Therefore, we may let $x=\frac13$ and obtain

$$\int_0^{1/3}\tan^{-1}t\,dt=\sum_{n=0}^{+\infty}(-1)^n\frac{(\frac13)^{2n+2}}{(2n+1)(2n+2)}=\frac{1}{1(2)(3^2)}-\frac{1}{3(4)(3^4)}-\cdots=0.0556-0.0010+\cdots=0.0545\approx 0.055$$

37. $f(t)=\begin{cases}(e^t-1)/t & \text{if } t\ne 0\\ 1 & \text{if } t=0\end{cases}=\begin{cases}\left(1+\sum_{n=1}^{+\infty}\frac{t^n}{n!}-1\right)/t & \text{if } t\ne 0\\ 1 & \text{if } t=0\end{cases}=\sum_{n=1}^{+\infty}\frac{t^{n-1}}{n!}$ for all t; so R $=+\infty$ and

$$\int_0^x f(t)dt=\sum_{n=1}^{+\infty}\int_0^x\frac{t^{n-1}}{n!}\,dt=\sum_{n=1}^{+\infty}\frac{x^n}{n\cdot n!}$$

38. $f(t)=\begin{cases}\tan^{-1}t/t & \text{if } t\ne 0\\ 1 & \text{if } t=0\end{cases}=\begin{cases}\left(\sum_{n=0}^{+\infty}(-1)^n\frac{t^{2n+1}}{2n+1}\right)/t & \text{if } t\ne 0\\ 1 & \text{if } t=0\end{cases}=\sum_{n=0}^{+\infty}(-1)^n\frac{t^{2n}}{2n+1}$ for all $t\in(-1,1)$. R $=1$.

$$\int_0^x f(t)dt=\sum_{n=0}^{+\infty}\int_0^x(-1)^n\frac{t^{2n}}{2n+1}dt=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)^2}$$

39. $\displaystyle\int_0^1 f(x)dx=\sum_{n=1}^{+\infty}\frac{1}{n\cdot n!}=\frac{1}{1\cdot 1!}+\frac{1}{2\cdot 2!}+\frac{1}{3\cdot 3!}+\frac{1}{4\cdot 4!}+\frac{1}{5\cdot 5!}+\frac{1}{6\cdot 6!}+\cdots$

$\approx 1+\frac14+\frac1{18}+\frac1{96}+\frac1{600}+\frac1{4320}=1.318$ with $|\text{error}|<\frac{1}{4320}=0.0002$.

40. Use your series in Exercise 38(b) to compute $\int_0^{1/4}f(t)dt$ accurate to three decimal places. Check by NINT.

▸ In Exercise 38 we have

$$\int_0^x f(t)dt=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)^2}$$

If $x=\frac14$ we obtain

$$\int_0^{1/4}f(t)dt=\frac14-\frac{1}{3^2(4^3)}+\frac{1}{5^2(4^5)}-\cdots=0.2500-0.0017+0.00004-\cdots=0.248$$

Because NINT ignores points where f is undefined, we use $f(t)=\tan^{-1}t/t$.

In Exercises 41–46, compute accurate to 3 decimal places the value of the integral by using series. Check by NINT.

41. From equation 13.2.2 with x replaced by x^3

$$\int_0^{1/2}\frac{dx}{1+x^3}=\int_0^{1/2}\sum_{n=0}^{+\infty}(-1)^n x^{3n}dx=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{3n+1}}{3n+1}\Big]_0^{1/2}=\sum_{n=0}^{+\infty}(-1)^n\frac{1}{(3n+1)2^{3n+1}}\approx\frac12-\frac{1}{4\cdot 2^4}+\frac{1}{7\cdot 2^7}$$

$=0.485$ with $|\text{error}|<\frac{1}{10\cdot 2^{10}}=0.0001$. The exact value is $\frac16\ln 3+\frac1{18}\sqrt3\pi\approx 0.4854$.

42. $\displaystyle\int_0^{1/3}\frac{dx}{1+x^4}=\sum_{n=0}^{+\infty}\int_0^{1/3}(-1)^n x^{4n}dx=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{4n+1}}{4n+1}\Big]_0^{1/3}=\sum_{n=0}^{+\infty}(-1)^n\frac{1}{(4n+1)3^{4n+1}}\approx\frac13-\frac{1}{5\cdot 3^5}=0.3325$

≈ 0.332 with $|\text{error}|<\frac{1}{9\cdot 3^9}=0.000006$. The exact value is $\frac14\sqrt2\tan^{-1}\frac38\sqrt2+\frac18\sqrt2\ln\frac1{41}(59+30\sqrt2)$.

43. If $|x|<1$, $\tan^{-1}x=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$; so $\tan^{-1}x^2=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{4n+2}}{2n+1}$.

$$\int_0^{1/2}\tan^{-1}x^2dx=\int_0^{1/2}\sum_{n=0}^{+\infty}(-1)^n\frac{x^{4n+2}}{2n+1}dx=\sum_{n=0}^{+\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)(4n+3)}\Big]_0^{1/2}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)(4n+3)2^{4n+3}}$$

$\approx\frac{1}{1\cdot 3\cdot 2^3}-\frac{1}{2\cdot 7\cdot 2^7}=\frac1{24}-\frac1{2688}=0.0417-0.0004=0.041$ with $|\text{error}|<\frac{1}{5\cdot 11\cdot 2^{11}}=0.00001$. Note: a term

< 0.0005 affected the third decimal place. The exact value is $\frac{1}{2}\tan^{-1}\frac{1}{4} - \frac{1}{2}\sqrt{2}\tan^{-1}\frac{2}{3}\sqrt{2} - \frac{1}{4}\ln\frac{1}{17}(33 - 20\sqrt{2})$

44. $\int_0^{1/2} e^{-x^3}\,dx$

▸ From Theorem A we have $e^x = \sum_{n=0}^{+\infty}\frac{x^n}{n!}$. Replacing x by $-x^3$ we obtain $e^{-x^3} = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{3n}}{n!}dx$. Thus, by Theorem 8.8.4,

$$\int_0^{1/2} e^{-x^3}dx = \sum_{n=0}^{+\infty}\int_0^{1/2}(-1)^n\frac{x^{3n}}{n!}dx = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{3n+1}}{(3n+1)n!}\bigg]_0^{1/2} = \sum_{n=0}^{+\infty}(-1)^n\frac{1}{(3n+1)(2^{3n+1})n!}$$
$$\approx \frac{1}{2} - \frac{1}{4(2^4)1!} + \frac{1}{7(2^7)2!} = 0.48493 \approx 0.485$$

Because the series is alternating, then $|\text{error}| < \frac{1}{10(2^{10})3!} = 0.00002$. Using NINT we get 0.48492.

45. $\sinh x = \sum_{n=0}^{+\infty}\frac{x^{2n+1}}{(2n+1)!}$ for all x; so $x\sinh\sqrt{x} = \sum_{n=0}^{+\infty}\frac{x^{n+3/2}}{(2n+1)!}$ if $x \geq 0$.

$$\int_0^1 x\sinh\sqrt{x}\,dx = \int_0^1\sum_{n=0}^{+\infty}\frac{x^{n+3/2}}{(2n+1)!}dx = \sum_{n=0}^{+\infty}\frac{x^{n+5/2}}{(n+5/2)(2n+1)!}\bigg]_0^1 = \sum_{n=0}^{+\infty}\frac{2}{(2n+5)(2n+1)!}$$

$\approx \frac{2}{5\cdot 1} + \frac{2}{7\cdot 6} + \frac{2}{9\cdot 120} + \frac{2}{11\cdot 5040} = 0.450$ with $|\text{error}| < \frac{2}{11\cdot 5040} = 0.00004$. The exact value is $16e^{-1} - 2e$.

46. $\cosh x = \sum_{n=0}^{+\infty}\frac{x^{2n}}{(2n)!}$ for all x; so $\cosh x^2 = \sum_{n=0}^{+\infty}\frac{x^{4n}}{(2n)!}$. $\int_0^{1/2}\cosh x^2dx = \sum_{n=0}^{+\infty}\int_0^{1/2}\frac{x^{4n}}{(2n)!}dx = \sum_{n=0}^{+\infty}\frac{x^{4n+1}}{(4n+1)(2n)!}\bigg]_0^{1/2}$

$= \sum_{n=0}^{+\infty}\frac{1}{(4n+1)2^{4n+1}(2n)!} \approx \frac{1}{2} + \frac{1}{5(2^5)2!} + \frac{1}{9(2^9)4!} = 0.503134 \approx 0.503$. Because each term is less than half the preceding, $|\text{error}| < \frac{1}{9(2^9)4!} = 0.00004$. The exact value cannot be found but NINT gives 0.503134.

47. From Theorem B, $\tan^{-1}x = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ if $|x| < 1$. If $x = \frac{1}{4}$, have $\tan^{-1}\frac{1}{4} = \frac{1}{4} - \left(\frac{1}{4}\right)^3\frac{1}{3} + \left(\frac{1}{4}\right)^5\frac{1}{5} - \left(\frac{1}{4}\right)^7\frac{1}{7} + \cdots$

$\approx \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} = 0.25000 - 0.00521 + 0.00020 = 0.2450$ with $|\text{error}| < \frac{1}{114{,}688} = 0.000009$.

48. If $f'(x) = \sum_{n=0}^{+\infty}(-1)^n\frac{(x-1)^n}{n!}$ and $f(1) = 0$, find $f(\frac{5}{4})$ accurate to three decimal places.

▸ $f(x) = f(x) - f(1) = \int_1^x f'(t)dt = \int_1^x\sum_{n=0}^{+\infty}(-1)^n\frac{(t-1)^n}{n!}dt = \sum_{n=0}^{+\infty}(-1)^n\frac{(t-1)^{n+1}}{(n+1)n!}\bigg]_1^x = \sum_{n=0}^{+\infty}(-1)^n\frac{(x-1)^{n+1}}{(n+1)!}$

$f(\frac{5}{4}) = \sum_{n=0}^{+\infty}(-1)^n\frac{(\frac{1}{4})^{n+1}}{(n+1)!} = \sum_{n=0}^{+\infty}(-1)^n\frac{1}{(n+1)!4^{n+1}} = \frac{1}{4} + \frac{1}{2!4^2} - \frac{1}{3!4^3} + \frac{1}{4!4^4} - \cdots \approx \frac{1}{4} - \frac{1}{32} + \frac{1}{384} = 0.221$

Because the series is alternating, then $|\text{error}| < \frac{1}{6144} = 0.0002$.

49. $g(1) = g(1) - g(0) = \int_0^1 g'(x)dx = \sum_{n=0}^{+\infty}\int_0^1(-1)^n\frac{x^n}{n^2+3}dx = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{n+1}}{(n+1)(n^2+3)}\bigg]_0^1$

$= \sum_{n=0}^{+\infty}(-1)^n\frac{1}{(n+1)(n^2+3)} \approx \frac{1}{3} - \frac{1}{2(4)} + \frac{1}{3(7)} - \frac{1}{4(12)} + \frac{1}{5(19)} - \frac{1}{6(28)} + \frac{1}{7(39)} = 0.243 \approx 0.24$ with $|\text{error}| < \frac{1}{8(52)} = 0.002$. Using half the last term gives 0.2415. $g(1) = 0.2419$.

50. $xe^x = x\sum_{n=0}^{+\infty}\frac{x^n}{n!} = \sum_{n=0}^{+\infty}\frac{x^{n+1}}{n!}$. $\int_0^1 x\boxed{e^xdx} = \left[xe^x - \int e^xdx\right]_0^1 = [xe^x - e^x]_0^1 = 1 = \int_0^1 x\,dx + \sum_{n=1}^{+\infty}\int_0^1\frac{x^{n+1}}{n!}dx$

$= \frac{1}{2} + \sum_{n=1}^{+\infty}\frac{1}{n!(n+2)}$ and so $\sum_{n=1}^{+\infty}\frac{1}{n!(n+2)} = \frac{1}{2}$

51. (a) From the result of Example 4, for all x, $e^{-x} = \sum_{n=0}^{+\infty}(-1)^n\frac{x^n}{n!}$, $x^2e^{-x} = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{n+2}}{n!}$

(b) Differentiating both sides of the last equation and letting $x = 2$, for all x,

$2xe^{-x} - x^2e^{-x} = \sum_{n=1}^{+\infty}(-1)^n\frac{(n+2)x^{n+1}}{n!}$, $4e^{-2} - 4e^{-2} = \sum_{n=1}^{+\infty}(-1)^n\frac{(n+2)2^{n+1}}{n!}$

$0 = 4 + \sum_{n=1}^{+\infty}(-1)^n\frac{(n+2)2^{n+1}}{n!}$, $(-1)\sum_{n=1}^{+\infty}(-1)^n2^{n+1}\frac{n+2}{n!} = 4$, $\sum_{n=1}^{+\infty}(-2)^{n+1}\frac{n+2}{n!} = 4$

52. (a) Find a power-series representation for $(e^x - 1)/x$.
(b) By differentiating term by term the power series in part (a), show that $\sum_{n=1}^{+\infty} \frac{n}{(n+1)!} = 1$

▸ (a) From Theorem A, we have for all x $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Thus $\quad e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=1}^{+\infty} \frac{x^n}{n!}$

Dividing on both sides by x, we obtain $\quad \frac{e^x - 1}{x} = \sum_{n=1}^{+\infty} \frac{x^{n-1}}{n!} \qquad (1)$

which is the required power-series representation.
(b) Differentiating on both sides of (1), we obtain $\quad \frac{xe^x - e^x + 1}{x^2} = \sum_{n=1}^{+\infty} \frac{(n-1)x^{n-2}}{n!} = \sum_{n=2}^{+\infty} \frac{(n-1)x^{n-2}}{n!} \qquad (2)$

Replacing n by $n+1$ in (2), we obtain $\quad \frac{xe^x - e^x + 1}{x^2} = \sum_{n=1}^{+\infty} \frac{nx^{n-1}}{(n+1)!} \qquad (3)$

If $x = 1$ in (3), we obtain $\quad 1 = \sum_{n=1}^{+\infty} \frac{n}{(n+1)!}$

53. From Theorem B, if $|x| < 1$, $\tan^{-1}x = \sum_{n=1}^{+\infty} (-1)^{n+1}\frac{x^{2n-1}}{2n-1}$; $x\tan^{-1}x = \sum_{n=1}^{+\infty} (-1)^{n+1}\frac{x^{2n}}{2n-1}$.
$\int_0^x t\tan^{-1}t\,dt = \int_0^x \sum_{n=1}^{+\infty} (-1)^{n+1}\frac{t^{2n}}{2n-1}\,dt$; $\frac{1}{2}[(t^2+1)\tan^{-1}t - t]_0^x = \sum_{n=1}^{+\infty} (-1)^{n+1}\frac{t^{2n+1}}{(2n-1)(2n+1)}\Big]_0^x$
$\frac{1}{2}[(x^2+1)\tan^{-1}x - x] = \sum_{n=1}^{+\infty} (-1)^{n+1}\frac{x^{2n+1}}{(2n-1)(2n+1)}$

54. Replace x by $-x^2$ in Theorem A, differentiate twice and let $x = \sqrt{\frac{1}{2}}$ to get: $e^{-x^2} = 1 + \sum_{n=1}^{+\infty}(-1)^n\frac{x^{2n}}{n!}$,

$-2xe^{-x^2} = \sum_{n=1}^{+\infty}(-1)^n\frac{2nx^{2n-1}}{(n-1)!} = 2\sum_{n=1}^{+\infty}(-1)^n\frac{x^{2n-1}}{(n-1)!}$, $(4x^2-2)e^{-x^2} = 2\sum_{n=1}^{+\infty}(-1)^n\frac{(2n-1)x^{2n-2}}{(n-1)!}$

$= -2 + 2\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{(2n+1)x^{2n}}{n!}$, $0 = -2 + 2\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{2n+1}{2^n n!}$, $\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{2n+1}{2^n n!} = 1$

55. We are given $f(x) = \sum_{n=0}^{+\infty} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + \cdots$.
Because $f(0) = 1$, then $c_0 = 1$. Because $f'(x) = f(x)$ then
$\sum_{n=1}^{+\infty} nc_nx^{n-1} = \sum_{n=0}^{+\infty} c_nx^n$, $\sum_{n=1}^{+\infty} nc_nx^{n-1} = \sum_{n=1}^{+\infty} c_{n-1}x^{n-1}$
Therefore, if $n \geq 1$, $nc_n = c_{n-1}$, $c_n = \frac{c_{n-1}}{n}$. Then
$c_1 = \frac{c_0}{1} = 1$; $c_2 = \frac{c_1}{2} = \frac{1}{2}$; $c_3 = \frac{c_2}{3} = \frac{1}{3!}$; $c_4 = \frac{c_3}{4} = \frac{1}{4!}$; ...; $c_n = \frac{1}{n!}$. Hence $f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n}$.

56. Use only properties of power series to find a power-series representation of the function f for which $f(x) > 0$ and $f'(x) = 2xf(x)$ for all x, and $f(0) = 1$. (b) Verify your result in part (a) by solving the differential equation $dy/dx = 2xy$ with the initial condition $y = 1$ when $x = 0$.

▸ (a) Let a power-series representation of f be given by
$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots + c_nx^n + \cdots \qquad (1)$
We find the coefficients $c_i i = 0, 1, 2, \ldots$. Because $f(0) = 1$, by substituting in (1) we have $c_0 = 1$. Differentiating on both sides we obtain
$f'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \cdots + nc_nx^{n-1} + \cdots \qquad (2)$
Multiplying on both sides of (1) by $2x$ with $c_0 = 1$, we obtain
$2xf(x) = 2x + 2c_1x^2 + 2c_2x^3 + 2c_3x^4 + 2c_4x^5 + \cdots + 2c_{n-2}x^{n-1} + \cdots \qquad (3)$
Because $f'(x) = 2xf(x)$ for all x, coefficients of like powers of x in (2) and (3) must be equal. Equating corresponding coefficients in (2) and (3), by induction we have

x^0: $c_1 = 0$ $\qquad$ x^1: $c_2 = 1$

x^2: $c_3 = \frac{2}{3}c_1 = 0$ $\qquad$ x^3: $c_4 = \frac{1}{2}c_2 = \frac{1}{2}\cdot 1 = \frac{1}{2!}$

x^4: $c_5 = \frac{2}{5}c_3 = 0$ $\qquad$ x^5: $c_6 = \frac{1}{3}c_4 = \frac{1}{3}\cdot\frac{1}{2!} = \frac{1}{3!}$

$\vdots$ $\qquad$ $\vdots$

x^{2n}: $c_{2n+1} = 0$ $\qquad$ x^{2n-1}: $c_{2n} = \frac{1}{n!}$

Substituting the values for c_i, $i = 0, 1, 2, \ldots$ in (1), we obtain

$$f(x) = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} + \cdots$$

which is the desired power-series representation of f.

(b) If $y = f(x) > 0$ and if $f'(x) = 2xf(x)$, we have

$$\frac{dy}{dx} = 2xy \tag{4}$$

By separating variables in (4) we have

$$\frac{dy}{y} = 2x\,dx, \quad \int \frac{dy}{y} = \int 2x\,dx, \quad \ln|y| = x^2 + \ln|C|, \quad y = Ce^{x^2} \tag{5}$$

Alternatively, and without the hypothesis that $f(x) > 0$, by multiplying on both sides of (4) by e^{-x^2}, we get

$$\frac{dy}{dx}e^{-x^2} = 2xye^{-x^2}, \quad \frac{dy}{dx}e^{-x^2} - 2xye^{-x^2} = 0, \quad \frac{d}{dx}(ye^{-x^2}) = 0, \quad ye^{-x^2} = C, \quad y = Ce^{-x^2}$$

Because $y = 1$ when $x = 0$, from (5) we have $C = 1$. Thus,

$$y = e^{x^2} \tag{6}$$

$$f(x) = e^{x^2}$$

From Example 3 we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Replacing x by x^2, we get

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} + \cdots \tag{7}$$

Substituting from (7) into (6), we have

$$f(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} + \cdots$$

which agrees with the result in (a).

57. We are given $f(x) = \sum_{n=1}^{+\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$.
Therefore $f(-x) = \sum_{n=1}^{+\infty} c_n(-x)^n = c_0 - c_1 x + c_2 x^2 - c_3 x^3 + \cdots + (-1)^n c_n x^n + \cdots$.
If f is an even function, then $f(x) = f(-x)$. Equating corresponding odd terms, we get
$c_1 = -c_1, c_3 = -c_3, c_5 = -c_5, \ldots, c_{2n+1} = -c_{2n+1}$
Therefore $c_1 = 0, c_3 = 0, c_5 = 0, \ldots, c_{2n+1} = 0$. Hence $c_n = 0$ when n is odd.
If f is an odd function, then $f(x) = -f(-x)$. Equating corresponding even terms, we get
$c_0 = -c_0, c_2 = -c_2, c_4 = -c_4, \ldots, c_{2n} = -c_{2n}$. Hence $c_n = 0$ when n is even.

58. For $|x| < R$, let $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots + c_n x^n + \cdots$.
$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots + nc_n x^{n-1} + \cdots$
Because $f(0) = 0$, then $c_0 = 0$. Because $f'(0) = 1$, then $c_1 = 1 = 1/1!$.
$f''(x) = 2c_2 + 2\cdot 3c_3 + 3\cdot 4c_4 x^2 + 4\cdot 5c_5 x^3 + \cdots + n(n-1)c_n x^{n-2} + \cdots$
Because $f''(x) = -f(x)$,
$2c_2 = -c_0, 2\cdot 3c_3 = -c_1, 3\cdot 4c_4 = -c_2, 4\cdot 5c_5 = -c_3, \ldots, (n-1)nc_n = c_{n-2}, \ldots$

$$c_2 = -\frac{1}{2}c_0 = 0,\ c_3 = -\frac{1}{2\cdot 3}c_1 = -\frac{1}{2\cdot 3} = -\frac{1}{3!},\ c_4 = -\frac{1}{3\cdot 4}c_2 = 0,\ c_5 = -\frac{1}{4\cdot 5}c_3 = \left(-\frac{1}{4\cdot 5}\right)\left(-\frac{1}{3!}\right) = \frac{1}{5!}, \ldots$$

And so on, by induction, $c_{2n} = 0$ and $c_{2n+1} = \dfrac{(-1)^n}{(2n+1)!}$. Therefore $f(x) = \sum_{n=0}^{+\infty} \dfrac{(-1)^n}{(2n+1)!} x^{2n+1}$.

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right| = \lim_{n\to+\infty}\left|\frac{1}{(2n+2)(2n+3)}\right| x^2 = 0 < 1 \text{ for all } x, \text{ so } R = +\infty.$$

59. If $f(x) \equiv 0$, then $f'(x) \equiv 0, f''(x) \equiv 0, \ldots, f^{(n)}(x) \equiv 0, \ldots$.
$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots,\ f(0) = c_0 = 0$
$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + \cdots,\ f'(0) = c_1 = 0$
$f''(x) = 2c_2 + 3\cdot 2c_3 x + \cdots + n(n-1)c_n x^{n-2} + \cdots,\ f''(0) = 2!c_2 = 0,\ c_2 = 0$
$f'''(x) = 3\cdot 2c_3 + \cdots + n(n-1)(n-2)x^{n-3} + \cdots,\ f'''(0) = 3!c_3 = 0,\ c_3 = 0, \cdots, f^{(n)}(0) = n!c_n = 0,\ c_n = 0.$

8.9 TAYLOR SERIES

The *Taylor series* of the function f at the number a is given by

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

If $a = 0$ in the Taylor series of f at a, we have the special case called the *Maclaurin series* given by

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Shortcut Any method that gives a power series in $x - a$ representing f will be the Taylor series of f at a. In particular, we may multiply (Misc. Ex. 74), divide (Ex. 17), and compose two series.

The power series given in Sections 8.7 and 8.8 that represent $1/(1-x)$, e^x, $\tan^{-1}x$, $\sinh x$ and $\cosh x$ are Maclaurin series. We also have the following Maclaurin series valid for all values of x:

Theorem D $\sin x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, $\cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Even if a function f has infinitely many derivatives, its Taylor series may not converge, and if it does converge, it may not converge to f. The following theorem provides a test.

Theorem 8.9.1 Let f be a function such that f and all of its derivatives exist in some interval $(a-r, a+r)$. Then the function is represented by its Taylor series

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all x such that $|x - a| < r$ if and only if

$$\lim_{n \to +\infty} \mathrm{R}_n = \lim_{n \to +\infty} \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-a)^{n+1} = 0$$

where each z_n is between x and a.

Exercises 8.9

1. $\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{+\infty} \frac{(-1)^n(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

2.

n	0	1	2	3	$2n$	$2n+1$
$f^{(n)}(x)$	$\sinh x$	$\cosh x$	$\sinh x$	$\cosh x$	$\sinh x$	$\cosh x$
$f^{(n)}(0)$	0	1	0	1	0	1

$\sinh x = x + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$ and $\mathrm{R}_n = \frac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1}$ where z_n is between 0 and x and $f^{(n+1)}(z_n)$ is either $\sinh z_n$ or $\cosh z_n$. In any case, $\left|f^{(n+1)}(z_n)\right| \le \cosh x$. Thus $0 \le |\mathrm{R}_n| \le \cosh x \cdot \frac{|x^{n+1}|}{(n+1)!}$. Because $\lim_{n \to +\infty} \frac{x^{n+1}}{(n+1)!} = 0$ from equation 8.9.14, it follows from the squeeze theorem that $\lim_{n \to +\infty} \mathrm{R}_n(x) = 0$. Hence the given Maclaurin series represents $\sinh x$ for all x.

3.

n	0	1	2	3	$2n$	$2n+1$
$f^{(n)}(x)$	$\cosh x$	$\sinh x$	$\cosh x$	$\sinh x$	$\cosh x$	$\sinh x$
$f^{(n)}(0)$	1	0	1	0	1	0

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \cdots + \frac{x^{2n}}{(2n)!} + \cdots$ and $\mathrm{R}_n = \frac{f^{(n+1)}(z_n)}{(n+1)!}x^{n+1}$ where z_n is between 0 and x and $f^{(n+1)}(z_n)$ is either $\sinh z_n$ or $\cosh z_n$. In any case, $\left|f^{(n+1)}(z_n)\right| \le \cosh x$. Thus $0 \le |\mathrm{R}_n| \le \cosh x \cdot \frac{|x^{n+1}|}{(n+1)!}$. Because $\lim_{n \to +\infty} \frac{x^{n+1}}{(n+1)!} = 0$ from equation 8.9.14, it follows from the squeeze theorem that $\lim_{n \to +\infty} \mathrm{R}_n(x) = 0$. Hence the given Maclaurin series represents $\cosh x$ for all x.

4. Obtain the Maclaurin series for $\cosh x$ by performing operations on the Maclaurin series for e^x and e^{-x}.

▶ $$\cosh x = \tfrac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots\right)\right]$$

$$= \frac{1}{2}\left[2 + \frac{2}{2!}x^2 + \frac{2}{4!}x^4 + \cdots + \frac{2}{(2n)!}x^{2n} + \cdots\right] = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \text{ for all } x.$$

5. See Exercise 8.8.21.

6. See Exercise 8.8.22

7. $e^x = e^3 e^{x-3} = e^3 \sum_{n=0}^{+\infty} \frac{(x-3)^n}{n!}$

8. Find the Taylor series for e^{-x} at 2 by using the Maclaurin series for e^x.

▶ We have

$$e^{-x} = e^{-2} e^{-(x-2)} \tag{1}$$

Replacing x with $-(x-2)$ in the Maclaurin series for e^x gives

$$e^{-(x-2)} = \sum_{n=0}^{+\infty} \frac{(-1)^n (x-2)^n}{n!} \tag{2}$$

Substituting from (2) into (1) gives the required Taylor series

$$e^{-x} = e^{-2} \sum_{n=0}^{+\infty} \frac{(-1)^n (x-2)^n}{n!}$$

In Exercises 9–14, find a power-series representation for $f(x)$ at the number a, and determine its radius of convergence. Support your answer graphically.

9.

n	0	1	2	3	4	5	n
$f^{(n)}(x)$	$\ln x$	x^{-1}	$-x^{-2}$	$2x^{-3}$	$-3!x^{-4}$	$4!x^{-5}$	$(-1)^{n-1}(n-1)!x^{-n}$
$f^{(n)}(1)$	0	1	-1	$2!$	$-3!$	$4!$	$(-1)^{n-1}(n-1)!$

$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n!} + \cdots$

$R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{n-1}{n} = 1$. The series represents $f(x)$ for $0 < x \le 2$.

10.

n	0	1	2	3	4	n
$f^{(n)}(x)$	$x^{1/4}$	$\frac{1}{4}x^{-3/4}$	$-\frac{3}{4^2}x^{-7/4}$	$\frac{3\cdot 7}{4^3}x^{-11/4}$	$-\frac{3\cdot 7\cdot 11}{4^4}x^{-15/4}$	$(-1)^{n-1}\frac{3\cdot 7\cdot\ldots\cdot(4n-5)}{4^n}x^{(1-4n)/4}$
$f^{(n)}(1)$	1	$\frac{1}{4}$	$-\frac{3}{4^2}$	$\frac{3\cdot 7}{4^3}$	$-\frac{3\cdot 7\cdot 11}{4^4}$	$(-1)^{n-1}\frac{3\cdot 7\cdot\ldots\cdot(4n-5)}{4^n}$

$x^{1/4} = 1 + \frac{x-1}{4} - \frac{3(x-1)^2}{2!(4^2)} + \frac{3\cdot 7(x-1)^3}{3!(4^3)} - \frac{3\cdot 7\cdot 11(x-1)^4}{4!(4^4)} + \cdots + (-1)^{n-1}\frac{3\cdot 7\cdot\ldots\cdot(4n-5)(x-1)^n}{n!(4^n)} + \cdots$

$R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty} \frac{3\cdot 7\cdot\ldots\cdot(4n-5)}{n!(4^n)}\cdot\frac{(n+1)!(4^{n+1})}{3\cdot 7\cdot\ldots\cdot(4n-5)(4n-1)} = \lim_{n\to+\infty}\frac{(n+1)4}{4n-1} = 1$

11.

n	0	1	2	3	4	n
$f^{(n)}(x)$	$x^{1/3}$	$\frac{1}{3}x^{-2/3}$	$-\frac{2}{3^2}x^{-5/3}$	$\frac{2\cdot 5}{3^3}x^{-8/3}$	$-\frac{2\cdot 5\cdot 8}{3^4}x^{-11/3}$	$(-1)^{n-1}\frac{2\cdot 5\cdot\ldots\cdot(3n-4)}{3^n}x^{(1-3n)/3}$
$f^{(n)}(8)$	2	$\frac{2}{3\cdot 8}$	$-\frac{4}{3^2 8^2}$	$\frac{4\cdot 5}{24^3}$	$-\frac{4\cdot 5\cdot 8}{24^4}$	$(-1)^{n-1}\frac{4\cdot 5\cdot\ldots\cdot(3n-4)}{24^n}$

$x^{1/3} = 2 + \frac{2(x-8)}{24} - \frac{4(x-8)^2}{2!(24^2)} + \frac{4\cdot 5(x-8)^3}{3!(24^3)} - \frac{4\cdot 5\cdot 8(x-8)^4}{4!(24^4)} + \cdots + \frac{(-1)^{n-1}4\cdot 5\cdot\ldots(3n-4)(x-8)^n}{n!(24^n)} + \cdots$

$R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{4\cdot 5\cdot\ldots\cdot(3n-4)}{n!(24^n)}\cdot\frac{(n+1)!(24^{n+1})}{4\cdot 5\cdot\ldots\cdot(3n-4)(3n-1)} = \lim_{n\to+\infty}\frac{(n+1)24}{3n-1} = 8$

12. $f(x) = \sin x$; $a = \frac{1}{6}\pi$

▶

n	0	1	2	3	4	n
$f^{(n)}(x)$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	
$f^{(n)}(\frac{1}{6}\pi)$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$		

$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}(x-\frac{1}{6}\pi) - \frac{1}{2\cdot 2!}(x-\frac{1}{6}\pi)^2 - \frac{\sqrt{3}}{2\cdot 3!}(x-\frac{1}{6}\pi)^3 + \frac{1}{2\cdot 4!}(x-\frac{1}{6}\pi)^4 + \cdots$

is the required representation. To find the radius of convergence, note that $\left|f^{(n+1)}(\frac{1}{6}\pi)/f^{(n)}(\frac{1}{6}\pi)\right|$ is either $\sqrt{3}$ or $\frac{1}{3}\sqrt{3}$ and hence $\le \sqrt{3}$. Thus

$0 \le \left|\frac{u_{n+1}}{u_n}\right| \le \frac{\sqrt{3}}{n+1}\left|x - \frac{1}{6}\pi\right|$

By the squeeze theorem, $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = 0$ for all x. Hence the radius of convergence is $+\infty$.

13. $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$. The Taylor series of f at $\frac{1}{3}\pi$ is

$$\cos x = \cos\tfrac{1}{3}\pi - \sin\tfrac{1}{3}\pi(x-\tfrac{1}{3}\pi) - \cos\tfrac{1}{3}\pi\frac{(x-\frac{1}{3}\pi)^2}{2!} + \sin\tfrac{1}{3}\pi\frac{(x-\frac{1}{3}\pi)^3}{3!} + \cos\tfrac{1}{3}\pi\frac{(x-\frac{1}{3}\pi)^4}{4!} + \cdots$$
$$= \frac{1}{2} - \frac{1}{2}\sqrt{3}(x-\tfrac{1}{3}\pi) - \tfrac{1}{4}(x-\tfrac{1}{3}\pi)^2 + \frac{1}{12}\sqrt{3}(x-\tfrac{1}{3}\pi)^3 + \frac{1}{48}(x-\tfrac{1}{3}\pi)^4 - \cdots$$

$\left|f^{(n+1)}(\frac{1}{3}\pi)/f^{(n)}(\frac{1}{3}\pi)\right|$ is either $\tan\frac{1}{3}\pi = \sqrt{3}$ or $\cot\frac{1}{3}\pi = \frac{1}{3}\sqrt{3}$ and hence $\leq \sqrt{3}$. Thus

$$0 \leq \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{f^{(n+1)}(\frac{1}{3}\pi)(x-\frac{1}{3}\pi)^{n+1}}{(n+1)!}\cdot\frac{n!}{f^{(n)}(\frac{1}{3})(x-\frac{1}{3}\pi)^n}\right| \leq \sqrt{3}\frac{1}{n+1}\left|x-\tfrac{1}{3}\pi\right|$$

By the squeeze theorem, $\lim\limits_{n\to+\infty}|u_{n+1}/u_n| = 0$ for all x; so the radius of convergence is $+\infty$.

14. $2^x = e^{(\ln 2)x} = \sum\limits_{n=0}^{+\infty}\dfrac{(\ln 2)^n x^n}{n!}$ for all x

15. $\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2}\left[1-\left(1+\sum\limits_{n=1}^{+\infty}(-1)^n\dfrac{(2x)^{2n}}{(2n)!}\right)\right] = \frac{1}{2}\sum\limits_{n=1}^{+\infty}\dfrac{(-1)^{n-1}2^{2n}x^{2n}}{(2n)!} = \sum\limits_{n=1}^{+\infty}(-1)^{n-1}\dfrac{2^{2n-1}x^{2n}}{(2n)!}$

16. Find the Maclaurin series for $\cos^2 x$. (Hint: Use $\cos^2 x = \frac{1}{2}(1+\cos 2x)$.)

▹ We have for all x

$$\cos x = \sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

Replacing x by $2x$, we obtain

$$\cos 2x = \sum_{n=0}^{+\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!} = \sum_{n=0}^{+\infty}\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!} = 1 + \sum_{n=1}^{+\infty}\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!}$$

Adding 1 to both sides and multiplying both sides by $\frac{1}{2}$, we obtain

$$\tfrac{1}{2}(1+\cos 2x) = \frac{1}{2}\left[2 + \sum_{n=1}^{+\infty}\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!}\right]$$
$$\cos^2 x = 1 + \sum_{n=1}^{+\infty}\frac{(-1)^n 2^{2n-1}}{(2n)!}$$

17. $f(x) = \tan x$; $f'(x) = \sec^2 x$; $f''(x) = 2\sec^2 x\tan x$; $f'''(x) = 4\sec^2 x\tan^2 x + 2\sec^4 x$;
$f^{(4)}(x) = 8\sec^2 x\tan^3 x + 16\sec^4 x\tan x$; $f^{(5)} = 16\sec^2 x\tan^4 x + 88\sec^4 x\tan^2 x + 16\sec^6 x$
$f(0) = 0$; $f'(0) = 1$; $f''(0) = 0$; $f'''(0) = 2$; $f^{(4)}(0) = 0$; $f^{(5)}(0) = 16$

Hence the Maclaurin series of f is $\tan x = 0 + 1\cdot x + 0\cdot\frac{x^2}{2!} + 2\cdot\frac{x^3}{3!} + 0\cdot\frac{x^4}{4!} + 16\cdot\frac{x^5}{5!} + \cdots = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

Alternatively, we may divide the series for $\sin x$ by the series for $\cos x$:

$$\begin{array}{r}
x+\frac{1}{3}x^3+\frac{2}{15}x^5+\cdots \\
1-\frac{1}{2}x^2+\frac{1}{24}x^4+\cdots\,\overline{\big)\,x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots} \\
\underline{x-\frac{1}{2}x^3+\frac{1}{24}x^5-\cdots} \\
\frac{1}{3}x^3-\frac{1}{30}x^5+\cdots \\
\underline{\frac{1}{3}x^3-\frac{1}{6}x^5+\cdots} \\
\frac{2}{15}x^5+\cdots
\end{array}$$

18. $\cot x = \tan(\frac{1}{2}\pi - x) = -\tan(x-\frac{1}{2}\pi) = -(x-\frac{1}{2}\pi) - \frac{1}{3}(x-\frac{1}{2}\pi)^3 - \frac{2}{15}(x-\frac{1}{2}\pi)^5 + \cdots$

19. $\sec^2 x = \dfrac{d}{dx}\tan x \approx 1 + x^2 + \dfrac{2}{3}x^4 + \cdots$

20. Use the answer in Exercise 18 and term by term differentiation to find the first three nonzero terms of the Taylor series for $\csc^2 x$ at $\frac{1}{2}\pi$.

▹ In Exercise 18, we found
$\cot x = -(x-\frac{1}{2}\pi) - \frac{1}{3}(x-\frac{1}{2}\pi)^3 - \frac{2}{15}(x-\frac{1}{2}\pi)^5 + \cdots$
Hence

$$\csc^2 x = -\frac{d}{dx}(\cot x) = \frac{d}{dx}\left[(x-\tfrac{1}{2}\pi) + \tfrac{1}{3}(x-\tfrac{1}{2}\pi)^3 + \tfrac{2}{15}(x-\tfrac{1}{2}\pi)^5 + \cdots\right] = 1 + (x-\tfrac{1}{2}\pi)^2 + \tfrac{2}{3}(x-\tfrac{1}{2}\pi)^4 + \cdots$$

21. $\ln|\sec x| = \int_0^x \tan t\, dt = \int_0^x \left(t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \cdots\right)dt = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \cdots$

22. $\ln|\sin x| = \int_{\pi/2}^x \cot t\, dt = \int_{\pi/2}^x \left[-(t - \frac{1}{2}\pi) - \frac{1}{3}(t - \frac{1}{2}\pi)^3 - \frac{2}{15}(t - \frac{1}{2}\pi)^5 - \cdots\right]dt$

$= -\frac{1}{2}(x - \frac{1}{2}\pi)^2 - \frac{1}{12}(x - \frac{1}{2}\pi)^4 - \frac{1}{45}(x - \frac{1}{2}\pi)^6 + \cdots$

In Exercises 23–28, use a power series to compute the value to the given accuracy. Check with a calculator value.

23. $58^\circ = \left(\frac{1}{3}\pi - \frac{1}{90}\pi\right)$rad. Using the result of Exercise 13 with $x = \frac{1}{3}\pi - \frac{1}{90}\pi$, we have
$\cos 58^\circ = \frac{1}{2} - \frac{1}{2}\sqrt{3}\left(-\frac{1}{90}\pi\right) - \frac{1}{4}\left(-\frac{1}{90}\pi\right)^2 + \frac{1}{12}\sqrt{3}\left(-\frac{1}{90}\pi\right)^3 + \cdots \approx \frac{1}{2} + \frac{\pi}{180}\sqrt{3} - \frac{\pi^2}{32400} - \frac{\pi^3}{8,748,000}\sqrt{3} = 0.5299$
with $|\text{error}| < \frac{\pi^3}{8,748,000}\sqrt{3} = 0.00001$.

24. $\sqrt[5]{e}$; four decimal places.

▹ We have for all x

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

Replacing x by $\frac{1}{5}$, we have

$$\sqrt[5]{e} = \sum_{n=0}^{+\infty} \frac{1}{5^n n!} \approx 1 + \frac{1}{5} + \frac{1}{5^2(2!)} + \frac{1}{5^3(3!)} + \frac{1}{5^4(4!)}$$
$$= 1 + 0.20000 + 0.02000 + 0.00133 + 0.00007 = 1.22140 \approx 1.2214$$

Because each term is less than $\frac{1}{5}$ of the preceding, the error is less than $\frac{1}{4}$ of the last, that is, less than 0.00002. Thus, the approximation is accurate to four decimal places. A calculator gives 1.22140.

25.

n	0	1	2	3	4
$f^{(n)}(x)$	$x^{1/5}$	$\frac{1}{5}x^{-4/5}$	$-\frac{4}{5^2}x^{-7/5}$	$\frac{4\cdot 9}{5^3}x^{-14/5}$	$-\frac{4\cdot 9\cdot 14}{5^4}x^{-19/5}$
$f^{(n)}(2^5)$	2	$\frac{1}{5\cdot 2^4}$	$-\frac{4}{5^2 2^9}$	$\frac{4\cdot 9}{5^3 2^{14}}$	$-\frac{4\cdot 9\cdot 14}{5^4 2^{19}}$

Therefore the Taylor series for f at 32 is
$\sqrt[5]{x} = 2 + \frac{1}{5\cdot 2^4}(x - 32) - \frac{4}{5^2 2^9}\cdot\frac{(x-32)^2}{2} + \frac{4\cdot 9}{5^3 2^{14}}\cdot\frac{(x-32)^3}{6} - \frac{4\cdot 9\cdot 14}{5^4 2^{19}}\cdot\frac{(x-32)^4}{24} + \cdots$
Using this series with $x = 30$, that is, $x - 32 = -2$ we have
$\sqrt[5]{30} = 2 - \frac{1}{5\cdot 2^3} - \frac{4}{5^2 2^7\cdot 2} - \frac{4\cdot 9}{5^3 2^{11}\cdot 6} - \frac{4\cdot 9\cdot 14}{5^4 2^{15}\cdot 24} - \cdots \approx 2 - 0.025 - 0.000625 - 0.000023 = 1.974352 \approx 1.9744$
with $|\text{error}| < 0.000023$. A calculator gives 1.9743505.

26. From Theorem C, $\sinh\frac{1}{2} \approx \frac{1}{2} + \frac{1}{6}(\frac{1}{2})^3 + \frac{1}{120}(\frac{1}{2})^5 = 0.500000 + 0.020833 + 0.000026 = 0.521094 \approx 0.52109$

Because $u_{n+1} \le \frac{1}{6\cdot 7\cdot 2}u_n$, error $\le \frac{1}{83}(0.000026) = 0.000003$. However, $\sinh\frac{1}{2} = 0.5210953 \approx 0.52110$

27. From Exercise 9, $\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$. Thus, $\ln 0.9 \approx -0.1 - \frac{1}{2}(0.1)^2 - \frac{1}{3}(0.1)^3$
$= -0.10533 \approx -0.1053$ with $|\text{error}| < \frac{1}{4}(0.1)^4 = 0.000025$. However $\ln 0.9 = 0.10536 \approx 0.1054$.

28. $\sqrt[3]{29}$ to three decimal places.

▹ Because 27 is the perfect cube nearest to 29, we find the Taylor series for $f(x) = x^{1/3}$ at $x = 27$ and calculate $f(29)$.

$f(x) = x^{1/3}$ $\qquad f(27) = 3$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad f'(27) = \frac{1}{3}\cdot\frac{1}{9} = \frac{1}{27}$

$f''(x) = -\frac{2}{9}x^{-5/3}$ $\qquad f''(27) = -\frac{2}{9}\cdot\frac{1}{3^5} = -\frac{2}{2187}$

$f'''(x) = \frac{10}{27}x^{-8/3}$ $\qquad f'''(27) = \frac{10}{27}\cdot\frac{1}{3^8} = \frac{10}{177147}$

Thus

$$f(x) = 3 + \frac{1}{27}(x - 27) - \frac{2}{2187}\frac{(x-27)^2}{2!} + \frac{10}{177147}\frac{(x-27)^3}{3!} - \cdots = 3 + \frac{x-27}{27} - \frac{(x-27)^2}{2187} + \frac{5(x-27)^3}{531441} - \cdots$$

$f(27) = 3 + \frac{2}{27} - \frac{4}{2187} + \cdots \approx 3.072$

Because the series is alternating, the error is less than the first term omitted namely

$$\frac{5 \cdot 2^3}{531451} = 0.0008 < \frac{1}{2} \cdot 10^{-3}$$

and so the approximation is accurate to three decimal places.

29. From Theorem A, $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$ for all x. Therefore

$e^1 = \sum_{n=0}^{+\infty} \frac{1}{n!} \approx 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \frac{1}{10!} + \frac{1}{11!} = 2.7182818$ with $|\text{error}| < \frac{1}{11!} = 2.5 \cdot 10^{-8}$.

In Exercises 30–33, use series to evaluate the definite integral accurate to three decimal places. Check by NINT.

30. $\int_0^1 \sqrt{x} e^{-x^2} dx = \int_0^1 x^{1/2} \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{n!} dx = \sum_{n=0}^{+\infty} \int_0^1 \frac{(-1)^n x^{(4n+1)/2}}{n!} dx = \sum_{n=0}^{+\infty} \frac{(-1)^n 2}{(4n+3)n!}$

$\approx \frac{2}{3} - \frac{2}{7} + \frac{2}{11 \cdot 2!} - \frac{2}{15 \cdot 3!} + \frac{2}{19 \cdot 4!} - \frac{2}{23 \cdot 5!} = 0.4533 \approx 0.453$ with $|\text{error}| < \frac{2}{27 \cdot 6!} = 0.0001$

31. For all x, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$. Therefore

$\int_0^{1/2} \sin x^2 \, dx = \int_0^{1/2} \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{4!} - \cdots \right) dx = \left[\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \cdots \right]_0^{1/2} \approx \frac{1}{3 \cdot 8} - \frac{1}{42 \cdot 128} = 0.042$

with $|\text{error}| < \frac{1}{1320 \cdot 2048} = 0.000004$.

32. $\int_0^1 \cos\sqrt{x} \, dx$

▷ We have

$$\cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!}$$

Thus,

$$\cos\sqrt{x} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{(2n)!} \quad \text{if } x \geq 0$$

$$\int_0^1 \cos\sqrt{x} \, dx = \sum_{n=0}^{+\infty} \int_0^1 \frac{(-1)^n x^n}{(2n)!} dx = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+1)(2n)!} \approx 1 - \frac{1}{2 \cdot 2!} + \frac{1}{3 \cdot 4!} - \frac{1}{4 \cdot 6!}$$
$$= 1 - 0.25000 + 0.01389 - 0.00035 = 0.76354$$

Because we have an alternating series, the error is less than the absolute value of the first term omitted, namely $1/(5 \cdot 8!) = 5 \times 10^{-6}$. The value of the integral is $2 \cos 1 + 2 \sin 1 - 2 = 0.763547$.

33. Let $f(x) = \ln(1 + \sin x)$; $f'(x) = \frac{\cos x}{1 + \sin x}$; $f''(x) = \frac{-\sin x(1 + \sin x) - \cos x(\cos x)}{(1 + \sin x)^2}$

$= \frac{-1 - \sin x}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x}$. Then $f(0) = \ln 1 = 0$; $f'(0) = 1$; $f''(0) = -1$.

Therefore the Taylor series for $\ln(1 + \sin x)$ at 0 is $x - \frac{x^2}{2} + \cdots$. Hence

$\int_0^{0.1} \ln(1 + \sin x) dx = \int_0^{0.1} \left(x - \frac{x^2}{2} + \cdots \right) dx = \left[\frac{1}{2} x^2 - \frac{1}{6} x^3 \right]_0^{0.1} \approx \frac{1}{200} = 0.005$ with $|\text{error}| < \frac{1}{6000}$.

34. (a) $\int_0^x \frac{\sin t}{t} dt = \sum_{n=0}^{+\infty} \int_0^x \frac{(-1)^n t^{2n}}{(2n+1)!} dt = \sum_{n=0}^{+\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!} \Big]_0^x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$

(c) $\int_0^{1/3} \frac{\sin t}{t} dt \approx \frac{1}{3} - \frac{1}{3^3(3)3!} + \frac{1}{3^5(5)5!} \approx 0.3333 - 0.0021 = 0.3313 \approx 0.331$ with $|\text{error}| < 0.000007$

The integral exists without defining the integrand at 0. It can be shown that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$.

35. (a) $g(x) = \begin{cases} (1 - \cos x)/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} \left[1 - \sum_{n=1}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right] / x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n)!}$ for all x.

(c) $\int_0^1 g(x) dx = \int_0^1 \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n)!} dx = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^{2n}}{2n(2n)!} \Big]_0^1 = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{2n(2n)!} = \frac{1}{2 \cdot 2!} - \frac{1}{4 \cdot 4!} + \frac{1}{6 \cdot 6!} - \cdots$

$\approx \frac{1}{4} - \frac{1}{96} = 0.240$ with $|\text{error}| < \frac{1}{4320} = 0.0002$.

36. The function E defined by $E(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\,dt$ is called the *error function*, and it is important in mathematical statistics. Find the Maclaurin series for the error function.

▸ We have

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

Thus,

$$e^{-t^2} = \sum_{n=0}^{+\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{+\infty} \frac{(-1)^n t^{2n}}{n!}$$

Integrating, we have

$$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$$

Hence,

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$$

37. We divide the following identity repeatedly by $x - 1$ using synthetic division:

$3x^4 - 17x^3 + 35x^2 - 32x + 17 =$
$a_4(x-1)^4 + a_3(x-1)^3 + a_2(x-1)^2 + a_1(x-1) + a_0$

```
3  -17   35  -32   17 (1
   -14   21  -11
3  -14   21  -11  ( 6 = a0
   -11   10
3  -11   10  (-1 = a1
    -8
3   -8  ( 2 = a2

3  (-5 = a3;  a4 = 3
```

38. We divide the following identity repeatedly by $x + 2$ using synthetic division:

$4x^3 - 5x^2 + 2x - 3 =$
$a_3(x+2)^3 + a_2(x+2)^2 + a_1(x+2) + a_0$

```
4  -5    2   -3 (-2
   -8   26  -56
4 -13   28(-59 = a0
   -8   42
4 -21  (70 = a1
   -8
4 -29 = a2;  a3 = 4
```

8.10 POWER SERIES FOR NATURAL LOGARITHMS AND THE BINOMIAL SERIES

We have the following power-series representations which are stated as theorems.

Theorem E $\ln(1+x) = \displaystyle\sum_{n=1}^{+\infty} (-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{if } -1 < x \le 1$

Theorem F $\ln\dfrac{1+x}{1-x} = 2\displaystyle\sum_{n=1}^{+\infty} \frac{x^{2n-1}}{2n-1} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$ for $|x| < 1$. If $y = \dfrac{1+x}{1-x}$ then $x = \dfrac{y-1}{y+1}$.

8.10.1 Theorem (Binomial Theorem) If m is any real number, then

$$(1+x)^m = 1 + \sum_{n=1}^{+\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n$$

for all values of x such that $|x| < 1$. The equation is valid for $-1 < x \le 1$ if $m > -1$ and for $-1 \le x \le 1$ if $m > 0$.

Theorem G (Example 2) $\dfrac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \dfrac{1\cdot 3}{2^2\cdot 2!}x^2 - \cdots + (-1)^n\dfrac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n n!}x^n + \cdots$ if $-1 < x \le 1$

Theorem H (Example 3) $\sin^{-1}x = x + \displaystyle\sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n n!}\cdot\frac{x^{2n+1}}{2n+1}$ if $|x| \le 1$

By the Wallis inequality, when $|x| = 1$, $u_n < 1/(2n+1)^{3/2}$ and the series converges by comparison with a p-series with $p = \frac{3}{2}$. By Abel's theorem, it represents $\sin^{-1}x$ there.

Exercises 8.10

In Exercises 1–4, use Theorem F to compute the natural logarithm to four decimal places. Check with a calculator.

1. To find x we let $\dfrac{1+x}{1-x} = 3$. Then $x = \dfrac{3-1}{3+1} = \frac{1}{2}$. Replacing x by $\frac{1}{2}$ in Theorem F we obtain

$$\ln 3 = 2\left(\frac{1}{2} + \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} + \frac{1}{7\cdot 2^7} + \frac{1}{9\cdot 2^9} + \frac{1}{11\cdot 2^{11}} + \cdots\right)$$
$$= 2(0.5000 + 0.04167 + 0.00625 + 0.00112 + 0.00022 + 0.00004 + \cdots = 1.0986$$

2. If $\frac{1+x}{1-x} = .8$ then $x = \frac{.8-1}{.8+1} = -\frac{1}{9}$. Then $\ln .8 \approx -2\left(\frac{1}{9} + \frac{1}{9^3 \cdot 3} + \frac{1}{9^5 \cdot 5}\right) = -.2231$ with $|\text{error}| < \frac{2}{9^5 \cdot 5} = .000007$.

3. If $\frac{1+x}{1-x} = 1.4$ then $x = \frac{1.4-1}{1.4+1} = \frac{1}{6}$. Then $\ln 1.4 = 2\left(\frac{1}{6} + \frac{1}{3 \cdot 6^3} + \frac{1}{5 \cdot 6^5}\right) = 0.33647$ with error $< \frac{2}{5 \cdot 6^5} = 0.00005$.

4. $\ln 2.5$

▸ To find x we let $\frac{1+x}{1-x} = 2.5$. Then $x = \frac{2.5-1}{2.5+1} = \frac{3}{7}$. Replacing x by $\frac{3}{7}$ in Theorem F we obtain

$$\begin{aligned}\ln 2.5 &= 2\left(\frac{3}{7} + \frac{3^3}{7^3 \cdot 3} + \frac{3^5}{7^5 \cdot 5} + \frac{3^7}{7^7 \cdot 7} + \frac{3^9}{7^9 \cdot 9} + \cdots\right) \\ &= 2(0.42857 + 0.02624 + 0.00289 + 0.00038 + 0.00005 + \cdots = 0.91627\end{aligned}$$

Using a calculator, we obtain $\ln 2.5 = 0.91629$

5. $\ln x = \ln[a + (x-a)] = \ln\left[a\left(1 + \frac{x-a}{a}\right)\right] = \ln a + \ln\left(1 + \frac{x-a}{a}\right) = \ln a + \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{(x-a)^n}{a^n n}$, $0 < x < 2a$

6. (a) $\ln x = \sum_{n=1}^{+\infty} (-1)^n \frac{(x-1)^n}{n}$ (b) $\ln .8 = -.2 - \frac{.2^2}{2} - \frac{.2^3}{3} - \frac{.2^4}{4} - \frac{.2^5}{5} - \cdots = -.22313$ with $|\text{error}| < \frac{1}{4} \cdot \frac{.2^5}{5} = .00002$

7. (a) $\ln x = \ln 2 + \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{(x-2)^n}{2^n n}$ (b) With $|\text{error}| < \frac{1}{2^{11} \cdot 11} = 0.00004$, we have

$$\ln 3 \approx 0.69315 + \frac{1}{2} - \frac{1}{2^2 \cdot 2} + \frac{1}{2^3 \cdot 3} - \frac{1}{2^4 \cdot 4} + \frac{1}{2^5 \cdot 5} - \frac{1}{2^6 \cdot 6} + \frac{1}{2^7 \cdot 7} - \frac{1}{2^8 \cdot 8} + \frac{1}{2^9 \cdot 9} - \frac{1}{2^{10} \cdot 10} = 1.09859$$

8. Find a power-series representation for $\ln(1+ax)$ by integrating term by term from 0 to x a power-series representation for $1/(1+at)$.

▸ From equation 8.7.4 we have

$$\frac{1}{1+x} = 1 + \sum_{n=1}^{+\infty} (-1)^n x^n$$

Replacing x with at and integrating from 0 to x, we obtain

$$\int_0^x \frac{1}{1+at}\,dt = \int_0^x \left[\sum_{n=0}^{+\infty} (-1)^n (at)^n\right] dt$$

$$\frac{\ln(1+at)}{a}\bigg]_0^x = \left[\sum_{n=0}^{+\infty} (-1)^n \frac{a^n t^{n+1}}{n+1}\right]_0^x$$

$$\frac{\ln(1+ax)}{a} = \sum_{n=0}^{+\infty} (-1)^n \frac{a^n x^{n+1}}{n+1}$$

Multiplying on both sides by a, we obtain the required representation

$$\ln(1+ax) = \sum_{n=0}^{+\infty} (-1)^n \frac{a^{n+1} x^{n+1}}{n+1} = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{a^n x^n}{n}$$

In Exercises 9–18 use a binomial series to find the Maclaurin series for the function and determine its radius of convergence R. Support your answer graphically.

9. $f(x) = \sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \cdots + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{1}{2}-n+1)}{n!}x^n + \cdots$

$= 1 + \frac{x}{2} + \sum_{n=2}^{+\infty} \frac{(-1)^{n+1} \cdot 1 \cdot 3 \cdots (2n-3)}{2^n n!} x^n$. The series is absolutely convergent if $|x| \le 1$; thus R is 1.

10. $f(x) = (3-x)^{-2} = \frac{1}{9}[1 + (-\frac{1}{3}x)]^{-2} = \frac{1}{9}\left[1 + \sum_{n=1}^{+\infty} \frac{(-2)(-3)\cdot \ldots \cdot(-n-1)(-x)^n}{3^n n!}\right] = \frac{1}{9}\left[1 + \sum_{n=1}^{+\infty} \frac{(n+1)! x^n}{3^n n!}\right]$

$= \frac{1}{9}\left[1 + \sum_{n=1}^{+\infty} \frac{(n+1)x^n}{3^n}\right] = \sum_{n=0}^{+\infty} \frac{(n+1)x^n}{3^{n+2}}$. The series is absolutely convergent if $\left|\frac{1}{3}x\right| < 1$; Thus R is 3.

11. $f(x) = (4+x)^{-1/2} = \frac{1}{2}(1 + \frac{1}{4}x)^{-1/2}$. Replace x by $\frac{1}{4}x$ in Theorem G to get

$f(x) = \frac{1}{2}\left[-\frac{1}{2}\left(\frac{1}{4}x\right) + \frac{1 \cdot 3}{2^2 2^1}\left(\frac{1}{4}x\right)^2 - \frac{1 \cdot 3 \cdot 5}{2^3 3^1}\left(\frac{1}{4}x\right)^3 + \cdots + (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}\left(\frac{1}{4}x\right)^n + \cdots\right]$

$= \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{+\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{8^n n!} x^n$. The series converges absolutely if $\frac{1}{4}|x| < 1$; $|x| < 4$. R is 4.

12. $f(x) = \sqrt[3]{8+x}$

▹ We have

$\sqrt[3]{8+x} = 2(1 + \frac{1}{8}x)^{1/3}$

By Theorem 8.10.1, with $m = \frac{1}{3} > 0$, we have for $|x| \le 1$

$$(1+x)^{1/3} = 1 + \sum_{n=1}^{+\infty} \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)\cdots(\frac{1}{3}-n+1)}{n!}x^n = 1 + \sum_{n=1}^{+\infty} \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})\cdots(\frac{4}{3}-n)}{n!}x^n$$

$$= 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n(-1)(2)(5)\cdots(3n-4)}{3^n n!}x^n$$

Replacing x by $\frac{1}{8}x$, we obtain

$$(1+\tfrac{1}{8}x)^{1/3} = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n(-1)(2)(5)\cdots(3n-4)}{3^n n!}(\tfrac{1}{8}x)^n \quad \text{if } \left|\tfrac{1}{8}x\right| \le 1$$

$$= 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n(-1)(2)(5)\cdots(3n-4)}{3^n 2^{3n} n!}x^n \qquad \text{if } |x| \le 8$$

Multiplying both sides by 2, we obtain

$$\sqrt[3]{8+x} = 2 + \sum_{n=1}^{+\infty} \frac{(-1)^n(-1)(2)(5)\cdots(3n-4)}{3^n 2^{3n-1} n!}x^n \qquad \text{if } |x| \le 8$$

The radius of convergence is $R = 8$.

13. $f(x) = \sqrt[3]{1-x^3} = (1-x^3)^{1/3}$

$$= 1 + \tfrac{1}{3}(-x^3) + \tfrac{1}{3}\left(-\tfrac{2}{3}\right)\frac{(-x^3)^2}{2!} + \tfrac{1}{3}\left(-\tfrac{2}{3}\right)\left(-\tfrac{5}{3}\right)\frac{(-x^3)^3}{3!} + \cdots + \tfrac{1}{3}\left(-\tfrac{2}{3}\right)\left(-\tfrac{5}{3}\right)\cdots\left(\tfrac{1}{3}-n+1\right)\frac{(-x^3)^n}{n!}$$

$= 1 - \sum_{n=1}^{+\infty} \frac{2\cdot 5\cdot 8\cdots(3n-4)}{3^n n!}x^{3n}$. The series is absolutely convergent if $|x^3| \le 1$; R is 1.

14. $f(x) = (4+x^2)^{-1} = \frac{1}{4}[1 + (\frac{1}{4}x^2)]^{-1} = \frac{1}{4}\left[1 + \sum_{n=1}^{+\infty} \frac{(-1)(-2)(-3)\cdots(-n)(\frac{1}{4}x^2)^n}{n!}\right] = \frac{1}{4}\left[1 + \sum_{n=1}^{+\infty}\frac{(-1)^n x^{2n}}{4^n n!}\right]$

$= \sum_{n=0}^{+\infty} \frac{(-1)nx^{2n}}{2^{2n+2}n!}$. The series is absolutely convergent if $\left|\frac{1}{4}x^2\right| < 1$; Thus R is 2.

15. $f(x) = (9+x^4)^{-1/2} = \frac{1}{3}\left(1 + \frac{1}{9}x^4\right)^{-1/2}$. Replace x by $\frac{1}{9}x^4$ in Theorem G to get

$$f(x) = \tfrac{1}{3}\left[1 - \tfrac{1}{2}\left(\tfrac{1}{9}x^4\right) + \frac{1\cdot 3}{2^2 2!}\left(\tfrac{1}{9}x^4\right)^2 - \frac{1\cdot 3\cdot 5}{2^3 3!}\left(\tfrac{1}{9}x^4\right)^3 + \cdots + (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}\left(\tfrac{1}{9}x^4\right) + \cdots\right]$$

$= \frac{1}{3} + \frac{1}{3}\sum_{n=1}^{+\infty} \frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{18^n n!}x^{4n}$. The series converges absolutely if $\frac{1}{9}x^4 < 1$; $|x| < \sqrt{3}$. R is $\sqrt{3}$.

16. $f(x) = \dfrac{x}{\sqrt{1-x}}$

▹ $f(x) = x(1-x)^{-1/2}$

We apply Theorem 8.10.1 with $m = -\frac{1}{2} > -1$ to obtain for $-1 < x \le 1$

$$(1+x)^{-1/2} = 1 + \sum_{n=1}^{+\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-n+1)}{n!}x^n = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})(n-\frac{1}{2})}{n!}x^n$$

$$= 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n n!}x^n$$

Replacing x with $-x$, we have for $-1 \le x < 1$

$$(1-x)^{-1/2} = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)(-x)^n}{2^n n!} = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^{2n}\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n n!}x^n$$

$$= 1 + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n n!}x^n$$

Multiplying on both sides by x, we obtain

$$f(x) = \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n n!} \qquad \text{if } -1 \le x < 1$$

The radius of convergence is $R = 1$.

17. $f(x) = \dfrac{x^2}{\sqrt{1+x}} = x^2(1+x)^{-1/2}$. From Theorem G, if $|x| < 1$,

$f(x) = x^2\left[1 + \sum_{n=1}^{+\infty} (-1)^n \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} x^n\right] = x^2 + \sum_{n=1}^{+\infty} (-1)^n \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} x^{n+2}$. R is 1.

18. $\dfrac{x}{\sqrt[3]{1+x^2}} = x(1+x^2)^{-1/3} = x\left[1 + \sum_{n=1}^{+\infty} \dfrac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})\cdots(\frac{2}{3}-n)x^{2n}}{n!}\right] = x + \sum_{n=1}^{+\infty} \dfrac{(-1)^n 1\cdot 4\cdot 7\cdot\ldots\cdot(3n-2)x^{2n+1}}{3^n n!}$
R is 1.

19. (a) $\sqrt[4]{1+x} = (1+x)^{1/4} = 1 + \frac{1}{4}x + \frac{1}{4}\left(-\frac{3}{4}\right)\frac{x^2}{2!} + \frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\frac{x^3}{3!} + \cdots = 1 + \frac{1}{4}x + \sum_{n=2}^{+\infty} (-1)^{n+1} \dfrac{3\cdot 7\cdot 11\cdots(4n-5)}{4^n n!} x^n$

Replacing x by x^2 we get $(1+x^2)^{1/4} = 1 + \frac{1}{4}x^2 + \sum_{n=2}^{+\infty} (-1)^{n+1} \dfrac{3\cdot 7\cdot 11\cdots(4n-5)}{4^n n!} x^{2n}$.

(b) $\int_0^{1/2} (1+x^2)^{1/4} dx = \int_0^{1/2}\left(1 + \frac{1}{4}x^2 - \frac{3}{32}x^4 + \frac{7}{128}x^6 - \cdots\right)dx = \left[x + \frac{1}{12}x^3 - \frac{3}{160}x^5 + \frac{1}{128}x^7 - \cdots\right]_0^{1/2}$

$\approx \frac{1}{2} + \frac{1}{96} - \frac{3}{5120} = 0.510$ with $|\text{error}| < \frac{1}{16{,}384} = 0.00006$.

20. (a) Express $(1-x^3)^{-1/2}$ as a power series in x by first obtaining a power series for $(1-x)^{-1/2}$ and then replacing x by x^3. (b) Use the result of part (a) to compute accurate to three decimal places the value of $\int_0^{1/2}(1-x^3)^{-1/2}$. Check using NINT.

▷ (a) From Exercise 16 we have for $-1 < x \le 1$

$$(1-x)^{-1/2} = 1 + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n n!} x^n$$

Replacing x by x^3, we obtain for $-1 < x \le x^3$

$$(1-x^3)^{-1/2} = 1 + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n n!} x^{3n}$$

(b) Integrating the above series term by term, we obtain

$$\int_0^{1/2} (1-x^3)^{-1/2} dx = \left[x + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)x^{3n+1}}{2^n n!(3n+1)}\right]_0^{1/2} = \frac{1}{2} + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^{4n+1} n!(3n+1)}$$

$$= \frac{1}{2} + \frac{1}{2^5(4)} + \frac{1\cdot 3}{2^9(2!)(7)} + \cdots = 0.50823 \approx 0.508$$

Using NINT we find the value 0.50826.

In Exercises 21–26, use series to compute accurate to 3 decimal places the value of the integral. Check by NINT.

21. $\int_0^{1/3} \sqrt{1+x^3}\, dx = \int_0^{1/3} (1+x^3)^{1/2} dx = \int_0^{1/3} \left[1 + \frac{1}{2}x^3 + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^6}{2!} + \cdots\right]dx = \left[x + \frac{1}{8}x^4 - \frac{1}{56}x^7 + \cdots\right]_0^{1/3}$

$\approx \frac{1}{3} + \frac{1}{648} = 0.335$ with $|\text{error}| < \frac{1}{56\cdot 3^7} = 0.000008$.

22. $\int_0^{2/5} \sqrt[3]{1+x^4} dx = \int_0^{.4} (1+x^4)^{1/3} dx = \int_0^{.4}\left[1 + \frac{1}{3}x^4 + \frac{1}{3}\left(-\frac{2}{3}\right)\frac{x^8}{2!} + \cdots\right]dx = \left[x + \frac{x^5}{15} - \frac{x^9}{81} + \cdots\right]_0^{.4}$

$\approx .4 + \frac{.4^5}{15} = 0.400683$ with $|\text{error}| < \frac{.4^9}{81} = 0.000003$. NINT gives 0.400680

23. $\int_0^1 \sqrt[3]{8+x^2}\, dx = \int_0^1 2\left(1+\frac{x^2}{8}\right)^{1/3} dx = 2\int_0^1 \left[1 + \frac{1}{3}\frac{x^2}{8} + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}\left(\frac{x^2}{2}\right)^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}\left(\frac{x^2}{8}\right)^3 + \cdots\right]dx$

$= 2\int_0^1 \left(1 + \frac{x^2}{24} - \frac{x^4}{9\cdot 8^2} + \frac{5x^6}{81\cdot 8^3} - \cdots\right)dx = 2\left[x + \frac{x^3}{72} - \frac{x^5}{45\cdot 8^2} + \frac{5x^7}{567\cdot 8^3} - \cdots\right]_0^1$

$\approx 2\left(1 + \frac{1}{72} - \frac{1}{45\cdot 8^2}\right) = 2.027$ with $|\text{error}| < \frac{10}{567\cdot 8^3} = 0.00003$.

24. $\int_0^{1/2} \sqrt{1-x^3} dx$

▷ By the binomial theorem with $m = \frac{1}{2}$ we have for all $|x| < 1$

$$(1+x)^{1/2} = 1 + \sum_{n=1}^{+\infty} \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{3}{2}-n)x^n}{n!} = 1 + \frac{x}{2} + \sum_{n=2}^{+\infty} \frac{(-1)^{n-1}\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-3)x^n}{2^n n!}$$

Replacing x by $-x^3$, we obtain for all $|x| < 1$

$$(1-x^3)^{1/2} = 1 - \frac{x^3}{2} + \sum_{n=2}^{+\infty} \frac{(-1)^{n-1}\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-3)(-x^3)^n}{2^n n!}$$

$$= 1 - \frac{x^3}{2} + \sum_{n=2}^{+\infty} \frac{(-1)^{2n-1} \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)x^{3n}}{2^n n!} = 1 - \frac{x^3}{2} - \sum_{n=2}^{+\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)x^{3n}}{2^n n!}$$

Therefore, by integrating on both sides, we obtain

$$\int_0^{1/2} \sqrt{1-x^3}\,dx = \left[x - \frac{x^4}{8} - \sum_{n=2}^{+\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)x^{3n+1}}{(3n+1)2^n n!}\right]_0^{1/2} \approx \frac{1}{2} - \frac{1}{8(2^4)} - \frac{1}{7(2^2)2!(2^7)}$$

$$= 0.50000 - 0.00781 - 0.00014 = 0.49205 \approx 0.492$$

Because $|u_{n+1}| < (1/2^3)|u_n|$, the error $< \frac{1}{7}(0.00014) = 0.00002$. NINT gives the value 0.49204.

25. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1+x^4}} = \int_0^{1/2} (1+x^4)^{-1/2}dx = \int_0^{1/2} \left[1 + \left(-\frac{1}{2}\right)x^4 + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(x^4)^2}{2!} + \cdots\right]dx$

$\displaystyle= \int_0^{1/2} \left(1 - \frac{1}{2}x^4 + \frac{3}{8}x^8 - \cdots\right)dx = \left[x - \frac{1}{10}x^5 + \frac{1}{24}x^9 - \cdots\right]_0^{1/2} \approx \frac{1}{2} - \frac{1}{320} = 0.497$

with $|\text{error}| < \dfrac{1}{72 \cdot 2^9} = 0.00008$.

26. $\displaystyle\int_0^{1/3} \frac{dx}{\sqrt[3]{x^2+1}} = \int_0^{1/3}(1+x^2)^{-1/3}dx = \int_0^{1/3}\left[1 + \left(-\frac{1}{3}\right)x^2 + \left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\frac{x^4}{2!} + \cdots\right]dx = \left[x - \frac{x^3}{3 \cdot 3} + \frac{4x^5}{3^2(5)2!} - \cdots\right]_0^{1/3}$

$\displaystyle\approx \frac{1}{3} - \frac{1}{3^5} = 0.3292$ with $|\text{error}| < \dfrac{4}{3^7(5)2!} = 0.0002$. NINT gives 0.3294

27. (a) $\displaystyle\lim_{t\to 0} \frac{\ln(1+t)}{t} \overset{0/0}{=} \lim_{t\to 0} \frac{1/(1+t)}{1} = 1$ (b) $\displaystyle\int_0^x \frac{\ln(1+t)}{t}dt = \int_0^x \left[\frac{1}{t}\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{t^n}{n}\right]dt = \sum_{n=1}^{+\infty}\int_0^x (-1)^{n-1}\frac{t^{n-1}}{n}dt$

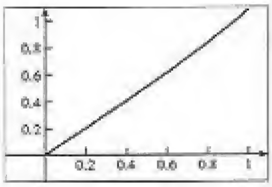
$\displaystyle= \sum_{n=1}^{+\infty}\left[(-1)^{n-1}\frac{t^n}{n^2}\right]_0^x = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{x^n}{n^2}$. $\displaystyle R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{(n+1)^2}{n^2} = 1$

28. Given $f(t) = \begin{cases} \sin^{-1}t/t & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$. (a) Prove that f is continuous at 0. (b) Find a power-series representation of $\int_0^x f(t)dt$ and determine its radius of convergence. (c) Plot in the same windows the graphs of $\int_0^x f(t)dt$ and the polynomial consisting of the first ten nonzero terms of the series in part (b).

▸ (a) We apply L'Hôpital's rule.

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\frac{\sin^{-1}t}{t} \overset{0/0}{=} \lim_{t\to 0}\frac{1}{\sqrt{1-t^2}} = 1$$

f is continuous at 0 because the limit exists and is equal to $f(0)$.



(b) We use the series of Theorem H.

$$\sin^{-1}t = t + \sum_{n=1}^{+\infty}\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n n!} \cdot \frac{t^{2n+1}}{2n+1} \quad \text{for } |t| \leq 1$$

Thus,

$$\frac{\sin^{-1}t}{t} = 1 + \sum_{n=1}^{+\infty}\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n n!} \cdot \frac{t^{2n}}{2n+1} \quad \text{if } 0 < |t| \leq 1$$

Because this sum has value 1 when $t = 0$, we have

$$f(t) = 1 + \sum_{n=1}^{+\infty}\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n n!} \cdot \frac{t^{2n}}{2n+1} \quad \text{if } |t| \leq 1$$

(b) $$\int_0^x f(t)dt = \left[t + \sum_{n=1}^{+\infty}\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)t^{2n+1}}{2^n n!(2n+1)^2}\right]_0^x = x + \sum_{n=1}^{+\infty}\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)x^{2n+1}}{2^n n!(2n+1)^2}$$

(c) The plots of $P_{10}(x) = x + \dfrac{x^3}{2 \cdot 1 \cdot 9} + \dfrac{1 \cdot 3x^5}{4 \cdot 2 \cdot 25} + \dfrac{1 \cdot 3 \cdot 5x^7}{8 \cdot 6 \cdot 49} + \dfrac{1 \cdot 3 \cdot 5 \cdot 7x^9}{16 \cdot 24 \cdot 81}$ and the integral coincide.

29. $\displaystyle\int_0^{1/3}\frac{\ln(1+t)}{t}dt = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{3^n n^2} \approx \frac{1}{3} - \frac{1}{3^2 2^2} + \frac{1}{3^3 3^2} - \frac{1}{3^4 4^2} = .3089$. $|\text{error}| < \dfrac{1}{3^5 5^2} = 0.00016$. NINT: .3090

30. $\displaystyle\int_0^{1/2}\frac{\sin^{-1}t}{t}dt = \frac{1}{2} + \sum_{n=1}^{+\infty}\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n n!(2n+1)^2 2^{2n+1}} = \frac{1}{2} + \sum_{n=1}^{+\infty}\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^{3n+1}n!(2n+1)^2} =$

$\displaystyle\frac{1}{2} + \frac{1}{2^4 \cdot 3^2} + \frac{1 \cdot 3}{2^7(2!)5^2} + \frac{1 \cdot 3 \cdot 5}{2^{10}(3!)7^2} + \cdots = .50000 + .00694 + .00047 + .00005 + \cdots = 0.50746 \approx .507$. NINT: .50747

31. In the series in Theorem G replace x by t^2 and obtain $\displaystyle(1+t^2)^{-1/2} = 1 + \sum_{n=1}^{+\infty}(-1)^n\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!}t^{2n}$ if $|t| \leq 1$. Hence if $|x| \leq 1$, and so the radius of convergence is 1,

$$\sinh^{-1}x = \int_0^x \frac{dt}{\sqrt{1+t^2}} = \int_0^x\left[1 + \sum_{n=1}^{+\infty}(-1)^n\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!}t^{2n}\right]dt = x + \sum_{n=1}^{+\infty}(-1)^n\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!(2n+1)}x^{2n+1}$$

32. Obtain the Maclaurin series for $\tanh^{-1} x$ by integrating term by term the binomial series for $(1-t^2)^{-1}$ and determine the radius of convergence.

▸ Replacing x by $-t^2$ in the binomial series, we obtain

$$(1-t^2)^{-1} = 1 + \sum_{n=1}^{+\infty} \frac{(-1)(-2)(-3)\cdots(-n)(-t^2)^n}{n!} = 1 + \sum_{n=1}^{+\infty} t^{2n} = \sum_{n=0}^{+\infty} t^{2n}$$

Integrating both sides of the above equation we obtain

$$\tanh^{-1} t \Big]_0^x = \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{2n+1}\Big]_0^x$$

$$\tanh^{-1} x = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1}$$

In Exercises 33–36, use a Maclaurin series to compute the quantity accurate to four decimals. Check by calculator.

33. From Th. H, $\sin^{-1} 0.24 = 0.24 + \frac{1}{2} \cdot \frac{0.24^3}{3} + \frac{1\cdot 3}{2^2 2!} \cdot \frac{0.24^5}{5} + \cdots \approx .24000 + .00230 + .00006 = .24236$. Calc: .24237

34. From Theorem H, $\sin^{-1}(-.62)$

$$= -.62 - \frac{1}{2}\cdot\frac{.62^3}{3} - \frac{1\cdot 3}{2^2 2!}\cdot\frac{.62^5}{5} - \frac{1\cdot 3\cdot 5}{2^3 3!}\cdot\frac{.62^7}{7} - \frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}\cdot\frac{.62^9}{9} - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2^5 5!}\cdot\frac{.62^{11}}{11} - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{2^6 6!}\cdot\frac{.62^{13}}{13} - \cdots$$

$\approx -.62000 - .03972 - .00687 - .00157 - .00041 - .00012 - 0.00003 - = -0.66873$. Calculator: -0.66874

35. From Ex. 31., $\sinh^{-1}(-.15) = -.15 + \frac{1}{2}\cdot\frac{.15^3}{3} - \cdots \approx -.15000 + .00056 = -.14944$. Calculator: -0.14944

36. $\tanh^{-1} 0.27$. Use the series found in Exercise 32.

▸ $\tanh^{-1} 0.27 = 0.27 + \frac{0.27^3}{3} + \frac{0.27^5}{5} + \cdots = 0.27000 + 0.00656 + 0.0029 = 0.27685 \approx 0.277$

On a calculator we find $\tanh^{-1} 0.27 = 0.27686$.

37. From Theorem E, $\ln(1+x) = \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{x^n}{n}$ if $|x| < 1$.

Therefore $\ln(1-t) = \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{(-t)^n}{n} = -\sum_{n=1}^{+\infty} \frac{t^n}{n}$ if $|t| < 1$. Thus, if $|x| < 1$,

$$\int_0^x -\ln(1-t)\,dt = \int_0^x \sum_{n=1}^{+\infty} \frac{t^n}{n}\,dt;\ (1-t)\ln(1-t) + t\Big]_0^x = \sum_{n=1}^{+\infty} \frac{x^{n+1}}{n(n+1)};\ x + (1-x)\ln(1-x) = \sum_{n=2}^{+\infty} \frac{x^n}{(n-1)n}$$

38. $\int_0^1 \ln(1+t)\boxed{dt} = \left[(1+t)\ln(1+t) - \int dt\right]_0^1 = \left[(1+t)\ln(1+t) - t\right]_0^1 = 2\ln 2 - 1$ and

$$\int_0^1 \ln(1+t)\,dt = \sum_{n=1}^{+\infty} \int_0^1 (-1)^n \frac{t^n}{n}\,dt = \sum_{n=1}^{+\infty} \frac{(-1)^n t^{n+1}}{n(n+1)}\Big]_0^1 = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n(n+1)}$$

39. $\text{KE} = m_0c^2\left(\frac{1}{\sqrt{1-(v^2/c^2)}} - 1\right) = m_0c^2\left[(1-v^2/c^2)^{-1/2} - 1\right]$

$$= m_0c^2\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{v^2}{c^2}\right)^2\frac{1}{2!} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{v^2}{c^2}\right)^3\frac{1}{3!} + \cdots - 1\right] = \frac{1}{2}m_0v^2 + \frac{3}{8}m_0\frac{v^4}{c^2} + \frac{5}{16}m_0\frac{v^6}{c^4} + \cdots$$

40. Find the Maclaurin series for $\int_0^x \frac{t^p}{\sqrt{1-t^2}}\,dt$ if p is a nonnegative integer. Determine its radius of convergence.

▸ In the series in Theorem G replace x by $-t^2$ and we have for $|t| < 1$

$$t^p(1-t^2)^{-1/2} = t^p\left[1 + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} t^{2n}\right] = t^p + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} t^{2n+p}$$

$$\int_0^x t^p(1-t^2)^{-1/2}\,dt = \int_0^x \left[t^p + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} t^{2n+p}\right]dt$$

$$= \left[\frac{t^{p+1}}{p+1} + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(2n+p+1)n!} t^{2n+p+1}\right]_0^x = \frac{x^{p+1}}{p+1} + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(2n+p+1)n!} x^{2n+p+1}$$

The series converges absolutely if $|x| \leq 1$, like $\sin^{-1} x$; therefore the radius of convergence is 1.

Miscellaneous Exercises for Chapter 8

In Exercises 1–6, find the Taylor or Maclaurin polynomial P of the stated degree at the number for the function f with the Lagrange form of the remainder. Plot the graphs of f and P in the same window.

1. $f(x) = \sin^2 x = \frac{1}{2}(1 - \cos 2x)$; $a = 0$; degree 5

$f'(x) = \sin 2x$, $f''(x) = 2\cos 2x$, $f'''(x) = -4\sin 2x$, $f^{(4)}(x) = -8\cos 2x$, $f^{(5)}(x) = 16\sin 2x$

$f(0) = 0$, $f'(0) = 0$, $\dfrac{f''(0)}{2!} = \dfrac{2}{2} = 1$, $\dfrac{f'''(0)}{3!} = 0$, $\dfrac{f^{(4)}(0)}{4!} = \dfrac{-8}{24} = -\dfrac{1}{3}$, $\dfrac{f^{(5)}(0)}{5!} = 0$

$\mathrm{P}_5(x) = x^2 - \frac{1}{3}x^4$. Because $f^{(6)}(x) = 32\cos x$, then $\mathrm{R}_5(x) = \dfrac{32\cos 2z}{6!}x^6 = \dfrac{2}{45}(\cos 2z)x^6$, z is between 0 and x

2. $f(x) = e^{x^2}$; $a = 0$; degree 4

$f'(x) = 2xe^{x^2}$, $f''(x) = (2 + 4x^2)e^{x^2}$, $f'''(x) = (12x + 8x^3)e^{x^2}$, $f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}$

$f(0) = 1$, $f'(0) = 0$, $\dfrac{f''(0)}{2!} = \dfrac{2}{2} = 1$, $\dfrac{f'''(0)}{3!} = 0$, $\dfrac{f^{(4)}(0)}{4!} = \dfrac{12}{24} = \dfrac{1}{2}$

$\mathrm{P}_4(x) = 1 + x^2 + \frac{1}{2}x^4$. $f^{(5)}(x) = (120x + 160x^3 + 32x^5)$. $\mathrm{R}_4(x) = \dfrac{120z + 160z^3 + 32z^5}{5!}x^5$, z is between 0 and x

3. $f(x) = x^{-1/2}$; $a = 9$; $n = 4$

$f'(x) = -\dfrac{1}{2}x^{-3/2}$, $f''(x) = \dfrac{3}{4}x^{-5/2}$, $f'''(x) = -\dfrac{15}{8}x^{-7/2}$, $f^{(4)}(x) = \dfrac{105}{16}x^{-9/2}$

$f(9) = \frac{1}{3}$, $f'(9) = \dfrac{-1}{2\cdot 27} = -\dfrac{1}{54}$, $\dfrac{f''(9)}{2!} = \dfrac{1}{2}\cdot\dfrac{3}{4\cdot 3^5} = \dfrac{1}{648}$, $\dfrac{f'''(9)}{3!} = \dfrac{1}{6}\cdot\dfrac{-15}{8\cdot 3^7} = -\dfrac{5}{34{,}992}$, $\dfrac{f^{(4)}(9)}{4!} = \dfrac{1}{24}\cdot\dfrac{105}{16\cdot 3^9}$

$= \dfrac{35}{2{,}519{,}424}$. $\mathrm{P}_4(x) = \frac{1}{3} - \frac{1}{54}(x-9) + \frac{1}{648}(x-9)^2 - \frac{5}{34{,}992}(x-9)^3 + \frac{35}{2{,}519{,}424}(x-9)^4$

$f^{(5)}(x) = -\dfrac{945}{32}x^{-11/2}$. $\mathrm{R}_4(x) = \dfrac{1}{120}\left(-\dfrac{945}{32}z^{-11/2}\right)(x-9)^5 = -\dfrac{63}{256}z^{-11/2}(x-9)^5$, z is between 9 and x.

4. $f(x) = (1 + x^2)^{-1}$; $a = 1$; $n = 3$

▹ We have

$$\mathrm{P}_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} \tag{1}$$

and

$$\mathrm{R}_3(x) = \frac{f^{(4)}(z)(x-1)^4}{4!} \text{ where } z \text{ is between 1 and } x. \tag{2}$$

Differentiating the given function, we obtain

$f'(x) = -2x(1 + x^2)^{-2}$ $f''(x) = 2(3x^2 - 1)(1 + x^2)^{-3}$

$f'''(x) = -24(x^3 - x)(1 + x^2)^{-4}$ $f^{(4)}(x) = 24(5x^4 - 10x^2 + 1)(1 + x^2)^{-5}$

Thus,

$f(1) = \frac{1}{2}$ $f'(1) = -\frac{1}{2}$ $f''(1) = \frac{1}{2}$ $f'''(1) = 0$

Substituting in (1) we obtain

$\mathrm{P}_3(x) = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2$

Substituting in (2) we get

$$\mathrm{R}_3(x) = \frac{24(5z^4 - 10z^2 + 1)(1 + z^2)^{-5}}{4!}(x-1)^4 = \frac{5z^4 - 10z^2 + 1}{(1 + z^2)^5}(x-1)^4 \text{ where } z \text{ is between 1 and } x.$$

5. $f(x) = xe^x$; $a = 0$; degree 6

n	0	1	2	3	4	5	6	7
$f^{(n)}(x)$	xe^x	$(1+x)e^x$	$(2+x)e^x$	$(3+x)e^x$	$(4+x)e^x$	$(5+x)e^x$	$(6+x)e^x$	$(7+x)e^x$
$\dfrac{f^{(n)}(0)}{n!}$	0	1	$\dfrac{2}{2!} = 1$	$\dfrac{3}{3!} = \dfrac{1}{2!}$	$\dfrac{4}{4!} = \dfrac{1}{3!}$	$\dfrac{5}{5!} = \dfrac{1}{4!}$	$\dfrac{6}{6!} = \dfrac{1}{5!}$	

$\mathrm{P}_6(x) = x + x^2 + \dfrac{x^3}{2!} + \dfrac{x^4}{3!} + \dfrac{x^5}{4!} + \dfrac{x^6}{5!}$. $\mathrm{R}_6(x) = \dfrac{(7+z)e^z}{7!}x^7$ where z is between 0 and x

6. $f(x) = x\cos x$; $a = \frac{1}{4}\pi$; degree 5. $f(\frac{1}{4}\pi) = \frac{1}{4}\pi \cdot \frac{1}{2}\sqrt{2}$

$f'(x) = \cos x - x\sin x$, $f'(\frac{1}{4}\pi) = (1 - \frac{1}{4}\pi)\frac{1}{2}\sqrt{2}$ $\quad$ $f''(x) = -2\sin x - x\cos x$, $f''(\frac{1}{4}\pi) = -(2 + \frac{1}{4}\pi)\frac{1}{2}\sqrt{2}$

$f'''(x) = -3\cos x + x\sin x$, $f'''(\frac{1}{4}\pi) = (-3 + \frac{1}{4}\pi)\frac{1}{2}\sqrt{2}$ $\quad$ $f^{(4)}(x) = 4\sin x + x\cos x$, $f^{(4)}(\frac{1}{4}\pi) = (4 + \frac{1}{4}\pi)\frac{1}{2}\sqrt{2}$

$f^{(5)}(x) = 5\cos x - x\sin x$, $f^{(5)}(\frac{1}{4}\pi) = (5 - \frac{1}{4}\pi)\frac{1}{2}\sqrt{2}$ $\quad$ $f^{(6)}(z) = -6\sin z - z\cos z$. $P_5(x) =$

$\frac{1}{2}\sqrt{2}[\frac{1}{4}\pi + (1 - \frac{1}{4}\pi)(x - \frac{1}{4}\pi) - \frac{1}{2}(2 + \frac{1}{4}\pi)(x - \frac{1}{4}\pi)^2 + \frac{1}{6}(-3 + \frac{1}{4}\pi)(x - \frac{1}{4}\pi)^3 + \frac{1}{24}(4 + \frac{1}{4}\pi)(x - \frac{1}{4}\pi)^4 + \frac{1}{120}(5 - \frac{1}{4}\pi)(x - \frac{1}{4}\pi)^5]$

$R_5(x) = \frac{1}{720}(-6\sin z - z\cos z)(x - \frac{1}{4}\pi)^6$ where z is between $\frac{1}{4}\pi$ and x

7. $\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2}\left[1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right)\right] = \frac{1}{2}\left[\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \cdots\right]$

$\sin^2 0.3 = \frac{1}{2}\left(\frac{.6^2}{2} - \frac{.6^4}{4!}\right) = 0.0873$ with $|\text{error}| < \frac{1}{2} \cdot \frac{.6^6}{6!} = 0.00003$

8. Compute $\sqrt[3]{e}$ accurate to five decimal places by using a Taylor polynomial, and prove that your answer has the required accuracy. Support your answer graphically.

▶ Let $f(x) = e^x$. Because $f^{(n)}(x) = e^x$ for every integer n, then for ever integer n

$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ and $R_n(x) = \frac{e^z x^{n+1}}{(n+1)!}$ where z is between 0 and x.

Because $e^{1/3} < e < 4$, then $R_n(\frac{1}{3}) = \frac{e^z}{3^{n+1}(n+1)!} < \frac{4}{3^{n+1}(n+1)!}$. We find that $R_6(\frac{1}{3}) < 0.0000004$. Thus

$e^{1/3} \approx P_6(\frac{1}{3}) = 1 + \frac{1}{3} + \frac{1}{3^2 2!} + \frac{1}{3^3 3!} + \frac{1}{3^4 4!} + \frac{1}{3^5 5!} + \frac{1}{3^6 6!} = 1.39561$

is correct to 5 decimal places.

9. (a) $\lim\limits_{x\to 0^+} \frac{e^{-x^2/2} - \cos x}{x^4} \overset{0/0}{=} \lim\limits_{x\to 0^+} \frac{-xe^{-x^2/2} + \sin x}{4x^3} \overset{0/0}{=} \lim\limits_{x\to 0^+} \frac{(x^2 - 1)e^{-x^2/2} + \cos x}{12x^2} \overset{0/0}{=} \lim\limits_{x\to 0^+} \frac{(3x - x^3)e^{-x^2/2} - \sin x}{24x}$

$\overset{0/0}{=} \lim\limits_{x\to 0^+} \frac{(x^4 - 6x^2 + 3)e^{-x/2} - \cos x}{24} = \frac{2}{24} = \frac{1}{12}$

(b) $\lim\limits_{x\to 0^+} \frac{e^{-x^2/2} - \cos x}{x^4} = \lim\limits_{x\to 0^+} \frac{(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \cdots) - (1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots)}{x^4} = \lim\limits_{x\to 0^+} \frac{\frac{1}{12}x^4 + ax^6 + \cdots}{x^4} = \frac{1}{12}$

10. $P(x) = 4x^3 + 5x^2 - 2x + 1$, $P(-2) = -7$; $P'(x) = 12x^2 + 10x - 2$, $P'(-2) = 26$; $P''(x) = 24x + 10$, $P''(-2)/2! = -38/2 = -19$; $P'''(x) = 24$, $P'''(-2)/3! = 24/6 = 4$. $P_3(x) = -7 + 26(x+2) - 19(x+2)^2 + 4(x+2)^3$

In Exercises 11–18, write the first four elements of the sequence and determine whether it is convergent or divergent. If the sequence converges, find its limit and support your answer graphically.

11. $a_n = \frac{3n}{n+2}$, $a_1 = \frac{3}{1+2} = 1$, $a_2 = \frac{6}{2+2} = \frac{3}{2}$, $a_3 = \frac{9}{3+2} = \frac{9}{5}$, $a_4 = \frac{12}{4+2} = 2$. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{3}{1 + 2n^{-1}} = 3$.

12. $\left\{\frac{(-1)^{n-1}}{(n+1)^2}\right\}$

▶ Let $a_n = \frac{(-1)^{n-1}}{(n+1)^2}$. Then

$a_1 = \frac{1}{2^2} = \frac{1}{4}$ $\quad a_2 = \frac{-1}{3^2} = -\frac{1}{9}$ $\quad a_3 = \frac{1}{4^2} = \frac{1}{16}$ $\quad a_4 = \frac{-1}{5^2} = -\frac{1}{25}$. $\qquad \lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \frac{(-1)^{n-1}}{(n+1)^2} = 0$

13. $a_n = \frac{n^2 - 1}{n^2 + 1}$. $a_1 = \frac{1-1}{1+1} = 0$, $a_2 = \frac{4-1}{4+1} = \frac{3}{5}$, $a_3 = \frac{9-1}{9+1} = \frac{4}{5}$, $a_4 = \frac{16-1}{16+1} = \frac{15}{17}$. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1 - n^{-2}}{1 + n^{-2}} = 1$.

14. $a_n = \frac{n^2}{\ln(n+1)}$. $a_1 = \frac{1}{\ln 2} \approx 1.4$, $a_2 = \frac{4}{\ln 3} \approx 3.6$, $a_3 = \frac{9}{\ln 4} \approx 6.5$, $a_4 = \frac{16}{\ln 5} \approx 9.9$. $\lim\limits_{n\to+\infty} a_n = +\infty$

15. $a_n = 2 + (-1)^n$. $a_1 = 2 - 1 = 1$, $a_2 = 2 + 1 = 3$, $a_3 = 2 - 1 = 1$, $a_4 = 2 + 1 = 3$. $\lim\limits_{n\to\infty} a_n$ does not exist.

16. $\left\{\frac{n + 3n^2}{4 + 2n^3}\right\}$

▶ Let $a_n = \frac{n + 3n^2}{4 + 2n^3}$. Then

$a_1 = \frac{2}{3}$ $\quad a_2 = \frac{7}{10}$ $\quad a_3 = \frac{15}{29}$ $\quad a_4 = \frac{13}{33}$. $\qquad \lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \frac{n + 3n^2}{4 + 2n^3} = \lim\limits_{n\to+\infty} \frac{\frac{1}{n^2} + \frac{3}{n}}{\frac{4}{n^3} + 2} = 0$

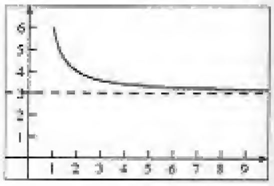
17. $a_n = \left(1+\frac{1}{n}\right)^{2n}$. $a_1 = (1+1)^2 = 4$, $a_2 = \left(1+\frac{1}{2}\right)^4 = \frac{81}{16} \approx 5.06$, $a_3 = \left(1+\frac{1}{3}\right)^6 = \frac{4096}{729} \approx 5.62$,
$a_4 = \left(1+\frac{1}{4}\right)^8 = \frac{390{,}625}{65{,}536} \approx 5.96$. $\lim\limits_{n\to\infty} a_n = \left[\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^2 = e^2 \approx 7.39$.

18. $a_n = \frac{(n+2)^2}{n+4} - \frac{(n+2)^2}{n} = (n+2)^2\frac{n-(n+4)}{(n+4)n} = -\frac{4(n+2)^2}{n(n+4)}$, $a_1 = -\frac{4(9)}{1(5)} = -\frac{36}{5}$, $a_2 = -\frac{4(16)}{2(6)} = -\frac{16}{3}$,
$a_3 = -\frac{4(25)}{3(7)} = -\frac{100}{21}$, $a_4 = -\frac{4(36)}{4(8)} = -\frac{9}{2}$. $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} -\frac{4(1+2/n)}{1+4/n} = -4$

In Exercises 19–22, estimate graphically the limit of the convergent sequence. Confirm your estimate analytically.

19. $\lim\limits_{n\to+\infty} \frac{-5}{n^2+4} = 0$

20. $\left\{\frac{6n}{2n-1}\right\}$



▷ The figure shows a plot of $y = \frac{6x}{2x-1}$ and line $y = 3$. The limit seems to be 3.
Let $a_n = \frac{6n}{2n-1}$. Then
$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{6n}{2n-1} = \lim_{n\to+\infty} \frac{6}{2-\frac{1}{n}} = \frac{6}{2} = 3$$

21. $\lim\limits_{n\to+\infty} \frac{3-4n}{1+2n} = \lim\limits_{n\to+\infty} \frac{3/n-4}{1/n+2} = \frac{-4}{2} = -2$

22. $\lim\limits_{n\to+\infty} \frac{3}{n+1} = 0$

In Exercises 23 and 24, prove that the sequence is convergent by applying Theorem 8.2.10.

23. $a_n = \frac{3n}{2^{n-1}}$. $a_n > 0$ and $\frac{a_{n+1}}{a_n} = \frac{3(n+1)}{2^n} \cdot \frac{2^{n-1}}{3n} = \frac{n+1}{2n} \le 1$ if $n \ge 1$.
a_n converges because it is decreasing and bounded from below. The limit is 0.

24. $\left\{\frac{1\cdot 4\cdot 7\cdot \ldots \cdot(3n-2)}{3\cdot 6\cdot 9\cdot \ldots \cdot(3n)}\right\}$

▷ Theorem 8.2.10 implies that a sequence is convergent if it is decreasing and bounded from below. Let
$a_n = \frac{1\cdot 4\cdot 7\cdot \ldots \cdot(3n-2)}{3\cdot 6\cdot 9\cdot \ldots \cdot(3n)}$
Then $a_n > 0$. Furthermore
$$a_{n+1} = \frac{1\cdot 4\cdot 7\cdot \ldots \cdot(3n-2)(3n+1)}{3\cdot 6\cdot 9\cdot \ldots \cdot(3n)(3n+3)} = \frac{3n+1}{3n+3}a_n < a_n$$
Because the sequence is decreasing and bounded from below, it converges.

In Exercises 25–36, find the interval of convergence I of the power series.

25. $\sum\limits_{n=1}^{+\infty} \frac{x^n}{\sqrt{n}}$. $\lim\limits_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to+\infty}\left|\frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n}\right| = \lim\limits_{n\to+\infty}\sqrt{\frac{n}{n+1}}|x| = \lim\limits_{n\to+\infty}\sqrt{\frac{1}{1+\frac{1}{n}}}|x| = |x|$
Therefore, the power series is absolutely convergent if $|x| < 1$, that is, if $-1 < x < 1$.

$x = -1$: $\sum\limits_{n=1}^{+\infty} (-1)^n\frac{1}{\sqrt{n}}$ is convergent by the alternating-series test.
$x = 1$: $\sum\limits_{n=1}^{+\infty} \frac{1}{\sqrt{n}}$ is divergent because it is the p series with $p = \frac{1}{2} < 1$. Hence I is $[-1,1)$.

26. $\sum\limits_{n=1}^{+\infty} \frac{(x-2)^n}{n}$. $R = \lim\limits_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim\limits_{n\to+\infty} \frac{n+1}{n} = 1$. If $x-2=-1$, $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n}$ is convergent by the alternating series test. If $x-2=1$, $\sum\limits_{n=1}^{+\infty} \frac{1}{n}$ is the divergent harmonic series. Therefore the interval of convergence is $[1,3)$.

27. $\sum_{n=1}^{+\infty} \frac{x^n}{3^n n(n+1)}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{3^{n+1}(n+1)(n+2)}\cdot\frac{3^n n(n+1)}{x^n}\right| = \lim_{n\to+\infty}\frac{n}{3(n+2)}|x| = \frac{1}{3}|x|$

Therefore, the power series is absolutely convergent if $|x| < 3$, that is, if $-3 < x < 3$.

$x = \pm 3$: $\sum_{n=1}^{+\infty} \frac{(\pm 1)^n}{n^2+n}$ is absolutely convergent by comparison with the p series ($p = 2$). Therefore I is $[-3,3]$.

28. $\sum_{n=0}^{+\infty} \frac{x^n}{2^n}$

▷ We apply the ratio test.

$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{x^n}\right| = \lim_{n\to+\infty}\left|\frac{x}{2}\right| = \frac{|x|}{2}$

If $|x|/2 < 1$, then $|x| < 2$, and by the ratio test the series converges absolutely. If $x = 2$, the given series is $\sum_{n=0}^{+\infty} 1^n$ which is divergent because $\lim_{n\to+\infty} 1^n = 1 \neq 0$. If $x = -2$, the given series is $\sum_{n=0}^{+\infty}(-1)^n$ which is divergent because $\lim_{n\to+\infty}(-1)^n \neq 0$. Hence the interval of convergence of the given power series is $(-2, 2)$.
ALTERNATE SOLUTION: For all x the given series is a geometric series with ratio $r = \frac{1}{2}x$. Because a geometric series converges if and only if $|r| < 1$, the given series is convergent if $\left|\frac{1}{2}x\right| < 1$, or equivalently, if $-2 < x < 2$.

29. $\sum_{n=1}^{+\infty} \frac{n!}{2^n}(x-3)^n$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)!(x-3)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n!(x-3)^n}\right| = \lim_{n\to+\infty}\frac{n+1}{2}|x-3| = \begin{cases}+\infty & \text{if } x \neq 3\\ 0 & \text{if } x = 3\end{cases}$
Thus, the series is convergent only if $x = 3$.

30. $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$. $R = \lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{(2n+1)!}{(2n-1)!} = \lim_{n\to+\infty}(2n+1)(2n) = +\infty$. I is $(-\infty, +\infty)$.

31. $\sum_{n=1}^{+\infty} \frac{n^2}{6^n}(x+1)^n$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)^2(x+1)^{n+1}}{6^{n+1}}\cdot\frac{6^n}{n^2(x+1)^n}\right| = \lim_{n\to+\infty}\frac{1}{6}\left(\frac{n+1}{n}\right)^2|x+1| = \frac{1}{6}|x+1|$

Hence the power series is absolutely convergent if $|x+1| < 6$, that is, if $-7 < x < 5$.

$x + 1 = \pm 6$: $\sum_{n=1}^{+\infty}(\pm 1)^n n^2$ is divergent because $\lim_{n\to+\infty} n^2 \neq 0$. Therefore the interval of convergence is $(-7,5)$.

32. $\sum_{n=1}^{+\infty} n(2x-1)^n$

▷ $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)(2x-1)^{n+1}}{n(2x-1)^n}\right| = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)|2x-1| = |2x-1|$

If $|2x-1| < 1$ then $-1 < 2x-1 < 1$, $0 < 2x < 2$, $0 < x < 1$

By the ratio test the series converges absolutely if $0 < x < 1$. Furthermore, if $|2x-1| = 1$, then $|u_n| = n$ and the series diverges because its terms do not approach zero. Therefore, the interval of convergence is $(0,1)$.

33. $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}(x-1)^n}{n2^n}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(x-1)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(x-1)^n}\right| = \lim_{n\to+\infty}\frac{n}{2(n+1)}|x-1| = \frac{1}{2}|x-1|$

Hence, the power series is absolutely convergent if $\frac{1}{2}|x-1| < 1$, that is, if $-1 < x < 3$.

$x = 3$: $\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}2^n}{n2^n} = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}$ is convergent by the alternating-series test.
$x = -1$: $\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}(-2)^n}{n2^n} = -\sum_{n=1}^{+\infty}\frac{1}{n}$ is the divergent harmonic series. Therefore I is $(-1,3]$.

34. $\sum_{n=1}^{+\infty} n^n x^n$. $\lim_{n\to+\infty}|u_n|^{1/n} = \lim_{n\to+\infty}|nx| = +\infty$ unless $x = 0$. The series converges only for $x = 0$.

35. $\sum_{n=1}^{+\infty}(\sin 2n)x^n$. Because $\sin 2n$ varies between -1 and 1, the ratio and root tests fail. However, if $|x| < 1$, the power series is absolutely convergent by comparison with a geometric series: $|(\sin 2n)x^n| < |x|^n$; while if $|x| = 1$, the series diverges because $\lim_{n\to+\infty}\sin 2n \neq 0$. Therefore the interval of convergence is $(-1,1)$.

36. $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{x^n}{(n+1)\ln(n+1)}$

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{(n+2)\ln(n+2)}\cdot\frac{(n+1)\ln(n+1)}{x^n}\right| = \lim_{n\to+\infty}\frac{n+1}{n+2}\cdot\frac{\ln(n+1)}{\ln(n+2)}|x|$$

$$= \lim_{n\to+\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\cdot\lim_{n\to+\infty}\frac{\ln(n+1)}{\ln(n+1)+\ln\left(\frac{n+2}{n+1}\right)}|x| = 1\cdot\frac{1}{1+\dfrac{\ln\left(1+\frac{1}{n+1}\right)}{\ln(n+1)}}|x| = |x|$$

By the ratio test the series converges absolutely if $|x| < 1$. If $x = 1$, the given series is

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{(n+1)\ln(n+1)}$$

which converges by the alternating series test. If $x = -1$, the given series is

$$-\sum_{n=1}^{+\infty}\frac{1}{(n+1)\ln(n+1)}$$

We apply the integral test. Let

$$f(x) = \frac{1}{(x+1)\ln(x+1)}$$

Then f is continuous, decreasing, and positive for $x \geq 1$. And

$$\int_1^{+\infty}\frac{dx}{(x+1)\ln(x+1)} = \lim_{b\to+\infty}\int_1^b\frac{d[\ln(x+1)]}{\ln(x+1)} = \lim_{b\to+\infty}\ln(\ln(x+1))\Big]_1^b = \lim_{b\to+\infty}[\ln(\ln(t+1) - \ln(\ln 2)] = +\infty$$

Thus, by the integral test the power series is divergent when $x = -1$. We conclude that the interval of convergence is $(-1, 1]$.

In Exercises 37–40: (a) Find the radius of convergence of the given power series and the domain of f; (b) write the power series that defines the function f' and state its radius of convergence; (c) find the domain of f'.

37. $f(x) = \displaystyle\sum_{n=1}^{+\infty}(-1)^n\frac{x^{2n}}{2n}$. (a) $\displaystyle\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n+2}}{2n+2}\cdot\frac{2n}{x^{2n}}\right| = |x|$.

The power series is absolutely convergent if $|x| < 1$. Hence $R = 1$.

$x = \pm 1$: $\displaystyle\sum_{n=1}^{+\infty}(-1)^n\frac{1}{2n}$ converges by the alternating series test. Therefore, the domain of f is $[-1, 1]$.

(b) $f'(x) = \displaystyle\sum_{n=1}^{+\infty}(-1)^n x^{2n-1}$. $\displaystyle\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n+1}}{x^{2n-1}}\right| = x^2$. Hence $R = 1$.

(c) $f'(\pm 1) = \displaystyle\sum_{n=1}^{+\infty}\pm(-1)^n$ is divergent because $\displaystyle\lim_{n\to+\infty}1 \neq 0$. Hence the domain of f' is $(-1, 1)$.

38. $f(x) = \displaystyle\sum_{n=1}^{+\infty}\frac{x^n}{n^3}$. (a) $R = \displaystyle\lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{(n+1)^3}{n^3} = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right) = 1$.

$x = \pm 1$: $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^3}$ is a convergent p-series with $p = 3$. Therefore the domain of f is $[-1, 1]$.

(b) $f'(x) = \displaystyle\sum_{n=1}^{+\infty}\frac{x^{n-1}}{n^2}$. $R = \displaystyle\lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to+\infty}\frac{(n+1)^2}{n^2} = 1$

(c) $f'(\pm 1) = \displaystyle\sum_{n=1}^{+\infty}\frac{(\pm 1)}{n^2}$ is a convergent p-series with $p = 2$. Hence the domain of f' is $[-1, 1]$.

39. $f(x) = \displaystyle\sum_{n=0}^{+\infty}\frac{x^n}{(n!)^2}$ (a) $\displaystyle\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{[(n+1)!]^2}\cdot\frac{(n!)^2}{x^n}\right| = \lim_{n\to+\infty}\frac{x}{(n+1)^2} = 0 < 1$

Therefore $R = +\infty$, and the domain of f is $(-\infty, +\infty)$.

(b) $f'(x) = \displaystyle\sum_{n=1}^{+\infty}\frac{x^{n-1}}{n!(n-1)!}$. $\displaystyle\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^n}{(n+1)!n!}\cdot\frac{n!(n-1)!}{x^{n-1}}\right| = \lim_{n\to+\infty}\frac{|x|}{(n+1)n} = 0 < 1$

Therefore $R = +\infty$ and (c) the domain of f' is $(-\infty, +\infty)$.

40. $f(x) = \sum_{n=1}^{+\infty} \frac{(x+1)^n}{n2^n}$

▸ (a) We apply the ratio test.

$$\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{(x+1)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(x+1)^n}\right| = \lim_{n\to+\infty}\frac{|x+1|}{2\left(1+\frac{1}{n}\right)} = \frac{|x+1|}{2}$$

By the ratio test we conclude that the power series converges absolutely if $|x+1| < 2$ and diverges if $|x+1| > 2$. Thus the radius of convergence is $R = 2$. If $|x+1| < 2$, then $-3 < x < 1$. Furthermore,

$$f(1) = \sum_{n=1}^{+\infty} \frac{1}{n}$$

and so $f(1)$ does not exist, because the harmonic series is divergent. Moreover,

$$f(-3) \sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$$

and thus $f(-3)$ does exist by the alternating series test. The domain of f is $[-3, 1)$.

(b) Differentiating the given power series, we obtain

$$f'(x) = \sum_{n=1}^{+\infty} \frac{(x+1)^{n-1}}{2^n}$$

The radius of convergence of f' is $R = 2$ because it is the same as that of f. If $|x+1| = 2$, then

$$|u_n| = \frac{2^{n-1}}{2^n} = \frac{1}{2}$$

Thus $f'(-3)$ and $f'(1)$ do not exist because their terms do not approach zero. The domain of f' is $(-3, 3)$.

In Exercises 41 and 42, find a power-series representation of the integral and determine its radius of convergence.

41. Using equation 8.7.4, where $\left|\frac{1}{4}x\right| < 1$ so that $R = 4$.

$$\int_0^x \frac{dt}{t^2+16} = \frac{1}{16}\int_0^x \frac{dt}{1+\left(\frac{1}{4}t\right)^2} = \frac{1}{16}\int_0^x \left[\sum_{n=0}^{+\infty} (-1)^n\left(\tfrac{1}{4}t\right)^{2n}\right]dt = \frac{1}{4}\sum_{n=0}^{+\infty} (-1)^n\frac{\left(\frac{1}{4}x\right)^{2n+1}}{2n+1} = \sum_{n=1}^{+\infty} \frac{(-1)^n x^{2n+1}}{2^{4n+4}(2n+1)}$$

42. $$\int_3^x \frac{dt}{t-1} = \int_3^x \frac{dt}{(t-3)+2} = \frac{1}{2}\int_3^x \frac{dt}{1+\frac{1}{2}(t-3)} = \frac{1}{2}\sum_{n=0}^{+\infty}\int_3^x \frac{(-1)^n(t-3)^n}{2^n} = \frac{1}{2}\sum_{n=0}^{+\infty} \frac{(-1)^n(t-3)^{n+1}}{2^n(n+1)}\Bigg]_3^x$$

$$= \sum_{n=0}^{+\infty}\frac{(-1)^n(x-3)^{n+1}}{2^{n+1}(n+1)} = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}(x-3)^n}{n2^n}.\ R = 2.$$

In Exercises 43 and 44, compute accurate to 3 decimal places the value of the definite integral by two methods: (a) use the second fundamental theorem of calculus (b) use the series obtained in the indicated exercise.

43. (a) $\int_0^3 \frac{dt}{t^2+16} = \frac{1}{4}\tan^{-1}\frac{t}{4}\Big]_0^3 = \frac{1}{4}\tan^{-1}\frac{3}{4} = 0.1609 \approx 0.161$ (b) Setting $x = 3$ in Exercise 41,

$$\int_0^3 \frac{dt}{t^2+16} = \frac{1}{4}\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}\left(\frac{3}{4}\right)^{2n+1} \approx \frac{1}{4}\left[\frac{3}{4} - \frac{1}{3}\left(\frac{3}{4}\right)^3 + \frac{1}{5}\left(\frac{3}{4}\right)^5 - \frac{1}{7}\left(\frac{3}{4}\right)^7 + \frac{1}{9}\left(\frac{3}{4}\right)^9 - \frac{1}{11}\left(\frac{3}{4}\right)^{11}\right] = 0.1606 = 0.161$$

with $|\text{error}| < \frac{1}{4}\cdot\frac{1}{13}\left(\frac{3}{4}\right)^{13} = 0.0005$.

44. $\int_3^4 \frac{dt}{t-1}$; Exercise 42

▸ (a) By the second fundamental theorem of the calculus,

$$\int_3^4 \frac{dt}{t-1} = \ln|t-1|\Big]_3^4 = \ln 3 - \ln 2 = \ln 1.5 = 0.40547 \approx 0.405$$

(b) The series in Exercise 42 is

$$\int_3^x \frac{dt}{t-1} = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}(x-3)^n}{n2^n}$$

Replacing x with 4, we obtain

$$\int_3^4 \frac{dt}{t-1} = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n2^n} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \frac{1}{5\cdot 2^5} - \frac{1}{6\cdot 2^6} + \frac{1}{7\cdot 2^7} = 0.4058 \approx 0.406$$

with $|\text{error}| < \frac{1}{8\cdot 2^8} = 0.00049$. The error of 1 unit in the last digit is unavoidable.

In Exercises 45 and 46, use the series obtained in Ex. 8.9.34 and 35 to compute the integral accurate to 3 decimals.

45. Setting $x = \frac{1}{2}$ in Exercise 34.

$$\int_0^{1/2} \frac{\sin t}{t} dt = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}\left(\frac{1}{2}\right)^{2n+1} \approx \frac{1}{2} - \frac{1}{8 \cdot 3 \cdot 6} = 0.493 \text{ with } |\text{error}| < \frac{1}{32 \cdot 5} = 0.00005.$$

46. Setting $x = 0.25$ in Exercise 35,

$$\int_0^{.25} \frac{1-\cos t}{t} dx = \int_0^{.25} \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n)!} dx = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^{2n}}{2n(2n)!}\Big]_0^{.25}$$

$$= \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{2n(4^{2n})(2n)!} \approx \frac{1}{2(4^2)2!} = 0.0156 \text{ with } |\text{error}| < \frac{1}{4(4^4)(4!)} = 0.00004$$

47. f is continuous at 0 because $\lim_{t \to 0} f(t) = \lim_{t \to 0} \frac{\cosh t - 1}{t} \stackrel{0/0}{=} \lim_{t \to 0} \frac{\sinh t}{1} = 0 = f(0)$.

$$\int_0^x \frac{\cosh t - 1}{t} dt = \int_0^x \frac{1}{t}\left(1 + \sum_{n=1}^{+\infty} \frac{t^{2n}}{(2n)!} - 1\right) dt = \sum_{n=1}^{+\infty} \int_0^x \frac{t^{2n-1}}{(2n)!} dt = \sum_{n=1}^{+\infty} \frac{t^{2n}}{2n(2n)!}\Big]_0^x = \sum_{n=1}^{+\infty} \frac{x^{2n}}{2n(2n)!}$$

$$\int_0^1 \frac{\cosh t - 1}{t} dt \approx \frac{1}{2 \cdot 2!} + \frac{1}{4 \cdot 4!} + \frac{1}{6 \cdot 6!} = 0.2606 \text{ with error} < \frac{1}{55} \cdot \frac{1}{6 \cdot 6!} = 0.000004$$

In Exercises 48–50, use series to evaluate accurate to three decimal places the definite integral. Check by NINT.

48. $\int_0^{1/2} \frac{dx}{1+x^5}$

▹ Replacing x by x^5 in Equation 8.7.4, we obtain

$$\frac{1}{1+x^5} = \sum_{n=0}^{+\infty} (-1)^n x^{5n}$$

Integrating, we get

$$\int_0^{1/2} \frac{dx}{1+x^5} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{5n+1}}{5n+1}\Big]_0^{1/2} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(5n+1)2^{5n+1}} \approx \frac{1}{2} - \frac{1}{6 \cdot 2^6} = 0.4974 \approx 0.497$$

with $|\text{error}| < \frac{1}{11 \cdot 2^{11}} = 0.00004$. NINT gives the value 0.4974. The exact value is too long to give here.

49. $$\int_0^{1/4} \sqrt{x} \sin x \, dx = \int_0^{1/4} x^{1/2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) dx = \int_0^{1/4} \left(x^{3/2} - \frac{x^{7/2}}{6} + \frac{x^{11/2}}{120} - \cdots\right) dx$$

$$= \left[\frac{2}{5} x^{5/2} - \frac{x^{9/2}}{27} + \frac{x^{13/2}}{60 \cdot 13} - \cdots\right]_0^{1/4} \approx \frac{2}{5 \cdot 2^5} - \frac{1}{27 \cdot 2^9} = .0124 \text{ with } |\text{error}| < \frac{1}{60 \cdot 13 \cdot 2^{13}} = .0000001. \text{ NINT: } .0124$$

50. Substituting $x = 1$ in Example 8.8.6 we get $\int_0^1 e^{-x^2} dx \approx 1 - \frac{1}{3} + \frac{1}{2! \cdot 5} - \frac{1}{3! \cdot 7} + \frac{1}{4! \cdot 9} - \frac{1}{5! \cdot 11} = 0.7467 \approx 0.747$ with $|\text{error}| < \frac{1}{6! \cdot 13} = 0.0001$. NINT gives 0.7468.

In Exercises 51–58, use series to compute the quantity accurate to four decimal places. Compare with calculator.

51. $\sqrt[4]{e} = e^{1/4} \approx 1 + \frac{1}{4} + \frac{1}{4^2 \cdot 2!} + \frac{1}{4^3 \cdot 3!} + \frac{1}{4^4 \cdot 4!} = 1.28402 \approx 1.2840$ with error $< \frac{1}{19} \cdot \frac{1}{4^4 \cdot 4!} = .000009$. Cal: 1.28403

52. $\sin^{-1} 0.1$

▹ From Theorem G we have

$$\sin^{-1} x = x + \sum_{n=1}^{+\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) x^{2n+1}}{2^n n!(2n+1)} \quad \text{if } |x| \le 1 \tag{1}$$

With $x = 0.1$, (1) becomes

$$\sin^{-1} 0.1 = 0.1 + \frac{(0.1)^3}{2 \cdot 3} + \frac{1 \cdot 3(0.1)^5}{2^2 \cdot 2! \cdot 5} + \cdots = 0.100000 + 0.0001667 + \cdots = 0.1001667.$$

The calculator value is 0.1001674.

53. From Theorem 8.8.B we have $\tan^{-1} x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ if $|x| < 1$. Therefore

$$\tan^{-1} \frac{1}{5} = \frac{1}{5} - \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5 - \frac{1}{7}\left(\frac{1}{5}\right)^7 + \cdots \approx \frac{1}{5} - \frac{1}{375} + \frac{1}{15{,}625} = 0.1974 \text{ with } |\text{error}| < \frac{1}{7 \cdot 5^7} = 0.000002.$$

54. $\sqrt[3]{130} = \sqrt{125+5} = 5\left(1+\frac{1}{25}\right)^{1/3} = 5\left[1 + \frac{1}{3}\left(\frac{1}{25}\right) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{1}{25}\right)^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{1}{25}\right)^3 + \cdots\right]$

$\approx 5\left[1 + \frac{1}{3\cdot 25} - \frac{1}{9\cdot 25^2}\right] = 5.0658$ with $|\text{error}| < \frac{5}{81\cdot 25^3} = 0.000004.$

55. $3^\circ = \frac{1}{60}\pi$. Because $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots$ for all x, then

$\cos\frac{\pi}{60} = 1 - \frac{1}{2}\left(\frac{\pi}{60}\right)^2 + \frac{1}{24}\left(\frac{\pi}{60}\right)^4 - \cdots \approx 1 - \frac{\pi^2}{7200} = 0.9986$ with $|\text{error}| < \frac{1}{24}\left(\frac{\pi}{60}\right)^4 = 0.000003.$

56. sin 0.3

▷ From Theorem 8.9.D we have

$$\sin x = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

Substituting $x = 0.3$ gives

$$\sin 0.3 \approx 0.3 - \frac{(0.3)^3}{3!} = 0.2955$$

with $|\text{error}| < \frac{(0.3)^5}{5!} = 0.00002$. The calculator value is 0.29552.

57. From Theorem 8.10.F if $\frac{x+1}{x-1} = 5$, then $x = \frac{5-1}{5+1} = \frac{2}{3}$.

$\ln 5 \approx 2\left[\frac{2}{3} + \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{5}\left(\frac{2}{3}\right)^5 + \frac{1}{7}\left(\frac{2}{3}\right)^7 + \frac{1}{9}\left(\frac{2}{3}\right)^9 + \frac{1}{11}\left(\frac{2}{3}\right)^{11} + \frac{1}{13}\left(\frac{2}{3}\right)^{13} + \frac{1}{15}\left(\frac{2}{3}\right)^{15} + \frac{1}{17}\left(\frac{2}{3}\right)^{17} + \frac{1}{19}\left(\frac{2}{3}\right)^{19}\right] = 1.60941$

with error $< \frac{2}{19}\left(\frac{2}{3}\right)^{19} = 0.00005$

58. $\sinh 0.3 \approx 0.3 + \frac{(0.3)^3}{3!} = 0.3045$

In Exercises 59–62, use two different series to approximate the value to four significant digits.

59. (a) See Example 8.8.9. (b) $\pi = 6\sin^{-1}\frac{1}{2} \approx 6\left(\frac{1}{2} + \frac{1}{2\cdot 3\cdot 2^3} + \frac{1\cdot 3}{2^2\cdot 2!\cdot 5\cdot 2^5} + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!\cdot 7\cdot 2^7} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!\cdot 9\cdot 2^9}\right) = 3.1415$

60. e

▷ Theorem 8.8.A is the series

$$e^x = \sum_{n=0}^{+\infty}\frac{x^n}{n!} \tag{1}$$

Method 1. Let $x = 1$ in (1).

$$e^1 \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} = 2.71825$$

with error $< \frac{1}{7\cdot 7!} = 0.00003$

Method 2. Multiply on both sides of (1) by x^2 and integrate from 0 to 1.

$$x^2e^x = \sum_{n=0}^{+\infty}\frac{x^{n+2}}{n!}$$

$$\int_0^1 x^2e^x\,dx = \sum_{n=0}^{+\infty}\int_0^1\frac{x^{n+2}}{n!}dx$$

$$(x^2-2x-2)e^x\Big]_0^1 = \sum_{n=0}^{+\infty}\frac{x^{n+3}}{(n+3)n!}\Big]_0^1$$

$$e - 2 = \sum_{n=0}^{+\infty}\frac{1}{(n+3)n!} \approx \frac{1}{3} + \frac{1}{4\cdot 1!} + \frac{1}{5\cdot 2!} + \frac{1}{6\cdot 3!} + \frac{1}{7\cdot 4!} + \frac{1}{8\cdot 5!} + \frac{1}{9\cdot 6!} = 0.71826$$

and so $e \approx 2.71826$

61. $1/\sqrt{3} = 0.5773502 \approx 0.5774$ (a) $3^{-1/2} = (4-1)^{-1/2} = \frac{1}{2}(1-\frac{1}{4})^{-1/2}$

$\approx \frac{1}{2}\left(1 + \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2!\cdot 4^2} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{3!\cdot 4^3} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{1}{4!\cdot 4^4} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{1}{5!\cdot 4^5}\right) = 0.57731 \approx 0.5773$

(b) $\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} = \frac{1}{3}(4-3)^{1/2} = \frac{2}{3}(1-\frac{1}{4})^{1/2}$

$\approx \frac{2}{3}\left(1 - \frac{1}{2}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2!\cdot 4^2} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{3!\cdot 4^3} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{4!\cdot 4^4}\right) = 0.57737 \approx 0.5774$

62. (a) $\ln 1.3 \approx 0.26236 \approx 0.2624$ (a) $\ln 1.3 = .3 - \frac{.3^2}{2} + \frac{.3^3}{3} - \frac{.3^4}{4} + \frac{.3^5}{5} - \frac{.3^6}{6} \approx 0.26234$

(b) If $\frac{x+1}{x-1} = 1.3$ then $x = \frac{1.3-1}{1.3+1} = \frac{3}{23}$. $\ln 1.3 \approx 2\left[\frac{3}{23} + \frac{1}{3}\left(\frac{3}{23}\right)^3 + \frac{1}{5}\left(\frac{3}{23}\right)^5\right] = 0.26236$

In Exercises 63–66, find the Maclaurin series for the function and its interval of convergence I. Support graphically.

63. $\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{+\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-3)x^n}{2^n n!}$, $-1 \le x \le 1$

64. $f(x) = \frac{1}{2-x}$

▹ Because

$$\frac{1}{2-x} = \;= \frac{\frac{1}{2}}{1-\frac{1}{2}x}$$

we use the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n \quad \text{if } |x| < 1$$

We replace x by $\frac{1}{2}x$. Thus

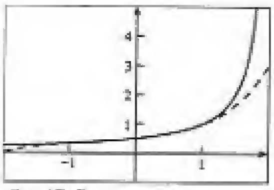
$$\frac{1}{1-\frac{1}{2}x} = \sum_{n=0}^{+\infty}\left(\frac{x}{2}\right)^n \quad \text{if } \left|\tfrac{1}{2}x\right| < 1$$

Multiplying both side by $\frac{1}{2}$ we have

$$\frac{\frac{1}{2}}{1-\frac{1}{2}x} = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{x^n}{2^n} \text{ if } |x| < 2$$

or, equivalently,

$$\frac{1}{2-x} = \sum_{n=0}^{+\infty}\frac{x^n}{2^{n+1}} \quad \text{if } |x| < 2$$



The plot shows $f(x)$ and $P_5(x) = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \frac{x^4}{32} + \frac{x^5}{64}$ (dashed).

65. $f(x) = a^x = e^{x \ln a}$. Because $e^x = \sum_{n=0}^{+\infty}\frac{x^n}{n!}$ for all x, then, for all x, $a^x = \sum_{n=0}^{+\infty}\frac{(\ln a)^n x^n}{n!}$. I is $(-\infty, +\infty)$.

66. Using $\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$ and $\sin x = \sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all x, we get

$f(x) = \sin^3 x = \sin^2 x \sin x = \frac{1}{2}(1-\cos 2x)\sin x = \frac{1}{2}(\sin x - \sin x\cos 2x) = \frac{1}{2}[\sin x - \frac{1}{2}(\sin 3x - \sin x)]$

$= \frac{3}{4}\sin x - \frac{1}{4}\sin 3x = \frac{3}{4}\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} - \frac{1}{4}\sum_{n=0}^{+\infty}(-1)^n\frac{3^{2n+1}x^{2n+1}}{(2n+1)!} = \frac{3}{4}\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{3^{2n}-1}{(2n+1)!}x^{2n+1}$

The interval of convergence is $(-\infty, +\infty)$.

In Exercises 67–70, find the Taylor series for f at the number and its interval of convergence I. Support graphically.

67.

n	0	1	2	3	4	5
$f^{(n)}(x)$	$\sin 3x$	$3\cos 3x$	$-3^2\sin 3x$	$-3^3\cos 3x$	$3^4\sin 3x$	$3^5\cos x$
$f^{(n)}\left(-\frac{\pi}{3}\right)$	0	-3	0	3^3	0	-3^5

$\sin 3x = -3\left(x+\frac{\pi}{3}\right) + \frac{3^3}{3!}\left(x+\frac{\pi}{3}\right)^3 - \frac{3^5}{5!}\left(x+\frac{\pi}{3}\right)^5 + \cdots = -(3x+\pi) + \frac{1}{3!}(3x+\pi)^3 - \frac{1}{5!}(3x+5)^5 + \cdots$

$= \sum_{n=1}^{+\infty}(-1)^n\frac{(3x+\pi)^{2n-1}}{(2n-1)!}$

68. Find the Taylor series for the function at the indicated number.

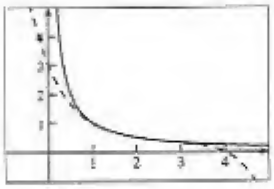
$f(x) = \frac{1}{x}$; at 2

▹ Because

$$\frac{1}{x} = \frac{1}{2+(x-2)} = \frac{\frac{1}{2}}{1+\frac{1}{2}(x-2)}$$

we use the geometric series

$$\frac{1}{1+x} = \sum_{n=0}^{+\infty}(-1)^n x^n \quad \text{if } |x| < 1$$



Replacing x by $\frac{1}{2}(x-2)$, we obtain

$$\frac{1}{1+\frac{1}{2}(x-2)} = \sum_{n=0}^{+\infty} \frac{(-1)^n(x-2)^n}{2^n} \text{ if } \left|\tfrac{1}{2}(x-2)\right| < 1$$

Multiplying both sides by $\frac{1}{2}$, we get

$$\frac{\frac{1}{2}}{1+\frac{1}{2}(x-2)} = \sum_{n=0}^{+\infty} \frac{(-1)^n(x-2)^n}{2^{n+1}} \text{ if } |x-2| < 2$$

The plot shows $f(x)$ and $P_5(x) = \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \frac{(x-2)^4}{32} - \frac{(x-2)^5}{64}$ (dashed).

69. $f(x) = \ln|x|$; $f'(x) = \frac{1}{x}$; $f''(x) = -\frac{1}{x^2}$; $f'''(x) = \frac{2}{x^3}$; $f^{(4)}(x) = -\frac{3!}{x^4}$; $f^{(5)}(x) = \frac{4!}{x^5}$

$f(-1) = \ln 1 = 0$; $f'(-1) = -1$; $f''(-1) = -1$; $f'''(-1) = -2$; $f^{(4)}(-1) = -3!$; $f^{(5)}(-1) = -4!$

$\ln|x| = 0 - (x+1) - \frac{1}{2}(x+1)^2 - \frac{1}{3}(x+1)^3 - \frac{1}{4}(x+1^4 - \frac{1}{5}(x+1)^5 - \cdots = -\sum_{n=1}^{+\infty} \frac{(x+1)^n}{n}$

70. $e^{x-2} = \sum_{n=0}^{+\infty} \frac{(x-2)^n}{n!}$ for all x

71. $\lim_{n\to+\infty} \int_0^1 \frac{x^n}{x+1}dx = \lim_{n\to+\infty} \int_0^1 \left[x^{n-1} - x^{n-2} + \cdots + (-1)^n \frac{1}{x+1}\right]dx$

$= \lim_{n\to+\infty} (-1)^n\left[\ln 2 - \{1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)\frac{1}{k}\}\right] = 0$. Note that $\int_0^1 \lim_{n\to+\infty}\left(\frac{x^n}{x+1}\right)dx = \int_0^1 0\,dx = 0$ also.

72. Find the Maclaurin series for $(1+x)^n$ where n is a positive integer and show it reduces to the binomial theorem.

▸ The binomial theorem (8.10.1) states that

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)\cdot\ldots\cdot(m-n+1)}{n!}x^n + \cdots \qquad (1)$$

If m is the integer n, then every term after the last shown has the factor $m-n=0$ and vanishes. Furthermore, the kth coefficient becomes

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\cdot\frac{(n-k)!}{(n-k)!} = \frac{n!}{k!(n-k)!}$$

and so (1) becomes

$$(1+x)^n = 1 + nx + \cdots + \frac{n!}{k!(n-k)!}x^k + \cdots + x^n \qquad (2)$$

Finally, we replace x by b/a and multiply on both sides of (2) by a^n to get the binomial theorem

$$(a+b)^n = a^n + na^{n-1}b + \cdots + \frac{n!}{k!(n-k)!}a^{n-k}b^k + \cdots + b^n$$

73. See Exercise 8.9.16.

74. $\sin 2x = 2(\sin x)(\cos x) = 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$

$= 2\left[x - \left(1\cdot\frac{1}{2!} + \frac{1}{3!}\cdot 1\right)x^3 + \left(1\cdot\frac{1}{4!} + \frac{1}{3!}\cdot\frac{1}{2!} + \frac{1}{5!}\cdot 1\right)x^5 - \cdots\right] = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \cdots$

In Exercises 75–79, $J_0 = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n}}{n!n!2^{2n}}$ and $J_1 = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{n!(n+1)!2^{2n+1}}$ are Bessel functions of order 0 and 1.

75. $J_0(x) = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n}}{n!n!2^{2n}}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n+2}}{(n+1)!(n+1)!2^{2n+2}}\cdot\frac{n!n!2^{2n}}{x^{2n}}\right| = \lim_{n\to+\infty}\frac{x^2}{4(n+1)^2} = 0$

for all x. Hence J_0 converges for all x.

$J_1(x) = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{n!(n+1)!2^{2n+1}}$. $\lim_{n\to+\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{2n+3}}{(n+1)!(n+2)2^{2n+3}}\cdot\frac{n!(n+1)!2^{2n+1}}{x^{2n+1}}\right|$

$= \lim_{n\to+\infty}\frac{x^2}{4(n+2)(n+1)} = 0$. Hence J_1 converges for all x.

76. Show that $y = J_0(x)$ is a solution of the differential equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$

▶ We have

$$y = J_0(x) = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n}}{n!n!2^{2n}} = 1 + \sum_{n=1}^{+\infty}\frac{(-1)^n x^{2n}}{n!n!2^{2n}}$$

Differentiating (1) successively we have

$$\frac{dy}{dx} = \sum_{n=1}^{+\infty}\frac{(-1)^n 2nx^{2n-1}}{n!n!2^{2n}} \tag{2}$$

$$\frac{d^2y}{dx^2} = \sum_{n=1}^{+\infty}\frac{(-1)^n 2n(2n-1)x^{2n-2}}{n!n!2^{2n}}$$

Multiplying both sides by x, we obtain

$$x\frac{d^2y}{dx^2} = \sum_{n=1}^{+\infty}\frac{(-1)^n 2n(2n-1)x^{2n-1}}{n!n!2^{2n}} \tag{3}$$

Adding the corresponding members of (2) and (3), we obtain

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sum_{n=1}^{+\infty}\frac{(-1)^n(2n)[(2n-1)+1]x^{2n-1}}{n!n!2^{2n}} = \sum_{n=1}^{+\infty}\frac{(-1)^n x^{2n-1}}{(n-1)!(n-1)!} \tag{4}$$

Replacing n by $n+1$ in (4), we obtain

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sum_{n=0}^{+\infty}\frac{(-1)^{n+1}x^{2n+1}}{n!n!2^{2n}} = -\sum_{n=1}^{+\infty}\frac{(-1)^n x^{2n+1}}{n!n!2^{2n}} \tag{5}$$

Multiplying both sides of (1) by x, we get

$$xy = \sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{n!n!2^{2n}} \tag{6}$$

Adding the corresponding members of (5) and (6), we get $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$

77. $J_0'(x) = \frac{d}{dx}\left[\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n}}{n!n!2^{2n}}\right] = \sum_{n=1}^{+\infty}\frac{(-1)^n 2nx^{2n-1}}{n!n!2^{2n}} = \sum_{n=1}^{+\infty}\frac{(-1)^n x^{2n-1}}{(n-1)!n!2^{2n-1}} = -\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}} = -J_1(x)$

78. $D_x(xJ_1(x)) = D_x\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+2}}{n!(n+1)!2^{2n+1}} = \sum_{n=0}^{+\infty}\frac{(-1)^n(2n+2)x^{2n+1}}{n!(n+1)!2^{2n+1}} = x\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n}}{n!n!2^{2n}} = xJ_0(x)$

79. $y = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{n!(n+1)!2^{2n+1}}$; $\frac{dy}{dx} = \sum_{n=0}^{+\infty}(-1)^n\frac{(2n+1)x^{2n}}{n!(n+1)!2^{2n+1}}$; $\frac{d^2y}{dx^2} = \sum_{n=0}^{+\infty}(-1)\frac{2n(2n+1)x^{2n-1}}{n!(n+1)!2^{2n+1}}$

$$\begin{aligned}
&x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y \\
&= x^2\sum_{n=0}^{+\infty}(-1)^n\frac{2n(2n+1)x^{2n-1}}{n!(n+1)!2^{2n+1}} + x\sum_{n=0}^{+\infty}(-1)^n\frac{(2n+1)x^{2n}}{n!(n+1)!2^{2n+1}} + (x^2-1)\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{n!(n+1)!2^{2n+1}} \\
&= \sum_{n=0}^{+\infty}(-1)^n\frac{2n(2n+1)x^{2n+1}}{n!(n+1)!2^{2n+1}} + \sum_{n=0}^{+\infty}(-1)^n\frac{(2n+1)x^{2n+1}}{n!(n+1)2^{2n+1}} + \sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+3}}{n!(n+1)!2^{2n+1}} - \sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}} \\
&= \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{n!(n+1)!2^{2n+1}}[2n(2n+1)+(2n+1)-1] + \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+3}}{n!(n+1)!2^{2n+1}} \\
&= \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{n!(n+1)!2^{2n+1}}[4n(n+1)] + \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+3}}{n!(n+1)!2^{2n+1}} \\
&= 0 + \sum_{n=1}^{+\infty}(-1)^n\frac{x^{2n+1}}{n!(n+1)!2^{2n+1}}[2^2n(n+1)] - \sum_{n=1}^{+\infty}(-1)^n\frac{x^{2n+1}}{(n-1)!n!2^{2n-1}} \\
&= \sum_{n=1}^{+\infty}(-1)^n\frac{x^{2n+1}}{(n-1)!n!2^{2n-1}} - \sum_{n=1}^{+\infty}(-1)^n\frac{x^{2n+1}}{(n-1)!n!2^{2n-1}} = 0
\end{aligned}$$

Thus $y = J_1(x)$ is a solution of the differential equation.

80. In quantum mechanics $\mathrm{E} = \left(\sum_{n=0}^{+\infty} \mathrm{N}_0 nh\mathrm{F} e^{-nhF/kT}\right) / \left(\sum_{n=1}^{+\infty} \mathrm{N}_0 e^{-nhF/kT}\right)$ (1)

(a) Let $x = e^{-hF/kT}$, and show that $\mathrm{E} = h\mathrm{F}\dfrac{1 + 2x + 3x^2 + 4x^3 + \cdots}{1 + x + x^2 + x^3 + \cdots}$

(b) Use the result of part (a) and power series for $(1-x)^{-1}$ and $(1-x)^{-2}$ to show that $\mathrm{E} = \dfrac{h\mathrm{F}}{e^{hF/kT} - 1}$

▸ (a) We factor in (1) and make the substitution $x = e^{-hF/kT}$.

$$\mathrm{E} = \frac{\mathrm{N}_0 h\mathrm{F} \sum_{n=0}^{+\infty} n e^{(-hF/kT)n}}{\mathrm{N}_0 \sum_{n=1}^{+\infty} e^{(-hF/kT)n}} = h\mathrm{F}\frac{\sum_{n=0}^{+\infty} n x^n}{\sum_{n=0}^{+\infty} x^n} = h\mathrm{F}\frac{x + 2x^2 + 3x^3 + 4x^4 + \cdots}{1 + x + x^2 + x^3 + \cdots} = h\mathrm{F}x\frac{1 + 2x + 3x^2 + 4x^3 + \cdots}{1 + x + x^2 + x^3 + \cdots}$$

(b) We express the numerator and denominator as powers of $1 - x$, simplify and substitute back for x.

$$\mathrm{E} = h\mathrm{F}x\frac{(1-x)^{-2}}{(1-x)^{-1}} = \frac{h\mathrm{F}x}{1-x} = \frac{h\mathrm{F}}{x^{-1} - 1} = \frac{h\mathrm{F}}{e^{hF/kT} - 1}$$

81. (a) $-\dfrac{d}{d\alpha}\ln \sum_{n=0}^{+\infty} x^n = -\dfrac{d}{d\alpha}\ln \sum_{n=0}^{+\infty} e^{-\alpha nhF} = \dfrac{\sum_{n=0}^{+\infty} nh\mathrm{F} e^{-\alpha nhF}}{\sum_{n=0}^{+\infty} e^{-\alpha nhF}}$. Replace α with $1/k\mathrm{T}$ to get E.

(b) $-\dfrac{d}{d\alpha}\ln(1-x)^{-1} = \dfrac{d}{d\alpha}\ln(1 - e^{-\alpha hF}) = \dfrac{h\mathrm{F}e^{-\alpha hF}}{1 - e^{-\alpha hF}} = \dfrac{h\mathrm{F}}{e^{\alpha hF} - 1}$. Replace α with $1/k\mathrm{T}$ to get E.

NINE

PARAMETRIC EQUATIONS, PLANE CURVES, AND POLAR GRAPHS

9.1 PARAMETRIC EQUATIONS AND PLANE CURVES

Parametric Equations Let f and g be two real-valued functions of a real variable t. Then for every number t in the domain common to f and g, the curve C is the set of all points (x, y) such that

$$x = f(t) \quad \text{and} \quad y = g(t) \tag{1}$$

Equations (1) are called *parametric equations* of the curve C, and t is called a *parameter*. If we eliminate t from Eqs. (1), we have a Cartesian equation of the curve C (and possibly additional points). For this equation we have

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{if } \frac{dx}{dt} \neq 0 \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'/dt}{dx/dt}$$

Thus if $dy/dt = 0$ and $dx/dt \neq 0$ at some point on the curve C, there is a horizontal tangent at that point. Moreover, if $dx/dt = 0$ and $dy/dt \neq 0$ at some point on C, there is a vertical tangent at that point. If $dy/dt = 0$ and $dx/dt = 0$, further investigation is needed to determine the inclination of the tangent line, if it exists.

Inflection Point Where the direction of rotation of the tangent line changes.

Smooth, Simple A curve is *smooth* on $[a, b]$ if f' and g' are continuous and never simultaneously 0 in (a, b). A curve is *closed* if $(f(a), g(a)) = (f(b), g(b))$. A curve is *simple* if it does not cross itself.

Trochoid traced by point (x, y) b units from center of circle of radius a rolling on x axis: $x = at - b\sin t$, $y = a - b\cos t$. *Cycloid* is when $b = a$: $x = a(t - \sin t)$, $y = a(1 - \cos t)$

Hypocycloid traced by point (x, y) on circle of radius b rolling inside a circle of radius $a > b$:

$$x = (a-b)\cos t + b \cos \frac{a-b}{b}t, \; y = (a-b)\sin t - b \sin \frac{a-b}{b}t$$

If $a/b = m/n$ in lowest terms, the full curve requires an interval length $2n\pi$ and has m cusps. If $a/b = 4$, we have an *astroid*.

Epicycloid traced by point (x, y) on circle of radius b rolling outside a circle of radius a (Misc. Ex. 73):

$$x = (a+b)\cos t - b \cos \frac{a+b}{b}t, \; y = (a+b)\sin t - b \sin \frac{a+b}{b}t$$

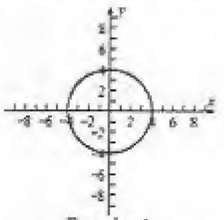
Exercises 9.1

In Exercises 1–10, sketch the graph of the parametric equation and find a cartesian equation.

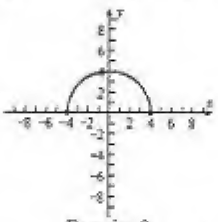
1. $x = 4\cos t$, $y = 4\sin t$, $t \in [0, 2\pi]$ ▷ $x^2 + y^2 = 16\cos^2 t + 16\sin^2 t = 16$

2. $x = 4\cos t$, $y = 4\sin t$, $t \in [0, \pi]$ ▷ $x^2 + y^2 = 16\cos^2 t + 16\sin^2 t = 16$, $y \geq 0$

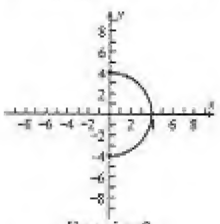
3. $x = 4\cos t$, $y = 4\sin t$, $t \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ ▷ $x^2 + y^2 = 16\cos^2 t + 16\sin^2 t = 16$, $x \geq 0$



Exercise 1



Exercise 2

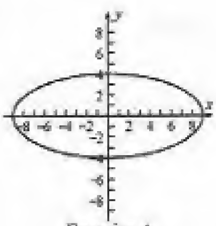


Exercise 3

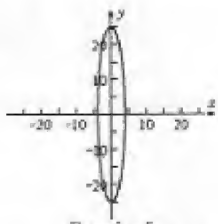
4. $x = 9\cos t$, $y = 4\sin t$, $t \in [0, 2\pi]$ ▷ $\frac{x^2}{9^2} + \frac{y^2}{4^2} = \cos^2 t + \sin^2 t = 1$

5. $x = 4\cos t$, $y = 25\sin t$, $t \in [0, 2\pi]$ ▷ $\frac{x^2}{4^2} + \frac{y^2}{25^2} = \cos^2 t + \sin^2 t = 1$

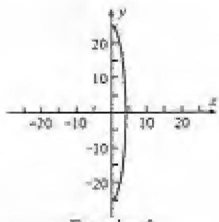
6. $x = 4\cos t$, $y = 25\sin t$, $t \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ ▷ $\frac{x^2}{4^2} + \frac{y^2}{25^2} = \cos^2 t + \sin^2 t = 1$, $x \geq 0$



Exercise 4



Exercise 5



Exercise 6

7. $x = 4\sec^2 t,\ y = 9\tan t,\ t \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ ▷ $\frac{x^2}{4^2} - \frac{y^2}{9^2} = \sec^2 t - \tan^2 t = 1,\ x > 0$

8. $x = 4\tan t,\ y = 9\sec t,\ t \in [0, \frac{1}{2}\pi) \cup [\pi, \frac{3}{2}\pi)$ ▷ $\frac{y^2}{9^2} - \frac{x^2}{4^2} = \sec^2 t - \tan^2 t = 1,\ x > 0$

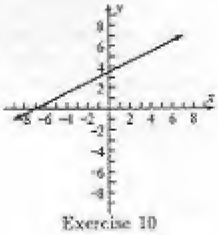
9. $x = 3 - 2t,\ y = 4 + t$ ▷ $x + 2y = (3 - 2t) + (8 + 2t) = 11$

10. $x = 2t - 5,\ y = t + 1$ ▷ $x - 2y = (2t - 5) - (2t + 2) = -7$

Exercise 7

Exercise 8

Exercise 9



Exercise 10

In Exercises 11–16 find dy/dx and d^2y/dx^2 without eliminating the parameter.

11. $x = 3t,\ y = 2t^2$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t}{3} = \frac{4}{3}t$; $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{4/3}{3} = \frac{4}{9}$.

12. $x = 1 - t^2,\ y = 1 + t$

▷ $\frac{dx}{dt} = -2t$ and $\frac{dy}{dt} = 1$

Hence,

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{1}{2t}$

Let $y' = dy/dx$. Then

$\frac{d(y')}{dt} = \frac{1}{2}t^{-2}$

Thus,

$$\frac{d^2y}{dx^2} = \frac{d(y')}{dx} = \frac{d(y')/dt}{dx/dt} = \frac{\frac{1}{2}t^{-2}}{-2t} = -\frac{1}{4t^3}$$

13. $x = t^2e^t,\ y = t\ln t$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\ln t + 1}{te^t(2+t)}$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{e^t(2+t) - (\ln t + 1)[e^t(2+t) + te^t(2+t) + te^t]}{t^2e^{2t}(2+t)^2}}{te^t(2+t)} = \frac{(2+t) - (\ln t + 1)(t^2 + 4t + 2)}{t^3e^{2t}(2+t)^3}$$

14. $x = e^{2t},\ y = 1 + \cos t$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{2e^{2t}} = -\frac{1}{2}e^{-2t}\sin t$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{e^{-2t}\sin t - \frac{1}{2}e^{-2t}\cos t}{2e^{2t}} = (\tfrac{1}{2}\sin t - \tfrac{1}{4}\cos t)e^{-4t}$$

15. $x = a\cos t$, $y = b\sin t$. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{b\cos t}{-a\sin t} = -\dfrac{b}{a}\cot t$. $\dfrac{d^2y}{dx^2} = \dfrac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \dfrac{\frac{b}{a}\csc^2 t}{-a\sin t} = -\dfrac{b}{a^2}\csc^3 t$.

16. $x = a\cosh t$, $y = b\sinh t$

▷ $\dfrac{dx}{dt} = a\sinh t$ and $\dfrac{dy}{dt} = b\cosh t$

Thus,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cosh t}{a\sinh t} = \frac{b}{a}\coth t$$

Because $y' = dy/dx$, then

$$\frac{d(y')}{dt} = -\frac{b}{a}\operatorname{csch}^2 t$$

So

$$\frac{d^2y}{dx^2} = \frac{d(y')/dt}{dx/dt} = \frac{-\frac{b}{a}\operatorname{csch}^2 t}{a\sinh t} = -\frac{b}{a^2}\operatorname{csch}^3 t$$

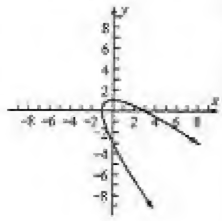
In Exercises 17–21, for the graph of the parametric equations, (a) find the horizontal and vertical tangent lines, (b) determine the concavity, and (c) sketch the graph. (d) Support your graph on your calculator.

17. $x = 4t^2 - 4t$, $y = 1 - 4t^2$ (parabola). $dx/dt = 8t - 4$, $dy/dt = -8t$.

Setting $dy/dt = 0$, we get $t = 0$. When $t = 0$, $dx/dt = -4 \neq 0$ and $y = 1$; hence $y = 1$ is an equation of the horizontal tangent line.

Setting $dx/dt = 0$, we get $t = \frac{1}{2}$. When $t = \frac{1}{2}$, $dy/dt = -4 \neq 0$ and $x = -1$; thus $x = -1$ is an equation of the vertical tangent line.

$\dfrac{dy}{dx} = \dfrac{dy/dy}{dx/dt} = \dfrac{-8t}{8t-4}$. $\dfrac{d^2y}{dx^2} = \dfrac{d(y')/dt}{dx/dt} = \dfrac{32/(8t-4)^2}{8t-4} = \dfrac{1}{16(t-\frac{1}{2})^3}$.



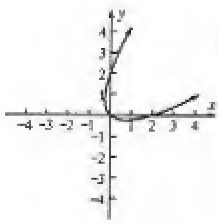
The graph is concave downward if $t < \frac{1}{2}$ and upward if $t > \frac{1}{2}$. Because $dx/dt = -1 + \frac{1}{2}t^{-1}$ is decreasing at $t = \frac{1}{2}$, it is not a point of inflection.

18. $x = t^2 + t$, $y = t^2 - t$ (parabola). $dx/dt = 2t + 1$, $dy/dt = 2t - 1$.

Setting $dy/dt = 0$, we get $t = \frac{1}{2}$. When $t = \frac{1}{2}$, $dx/dt = 2 \neq 0$ and $y = -\frac{1}{4}$; hence $y = -\frac{1}{4}$ is an equation of the horizontal tangent line.

Setting $dx/dt = 0$, we get $t = -\frac{1}{2}$. When $t = -\frac{1}{2}$, $dy/dt = -2 \neq 0$ and $x = -\frac{1}{4}$; thus $x = -\frac{1}{4}$ is an equation of the vertical tangent line.

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t-1}{2t+1}$. $\dfrac{d^2y}{dt^2} = \dfrac{d(y')/dt}{dx/dt} = \dfrac{4/(2t+1)^2}{2t+1} = \dfrac{\frac{1}{2}}{(t+\frac{1}{2})^3}$



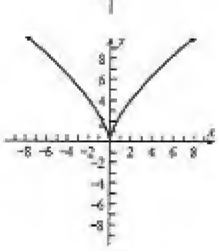
The graph is concave downward if $t < -\frac{1}{2}$ and upward if $t > -\frac{1}{2}$. Because $dx/dt = 1 + (t-\frac{1}{2})^{-1}$ is decreasing at $t = -\frac{1}{2}$, it is not an inflection point.

19. $x = 2t^3$, $y = 4t^2$ (semicubical parabola). $dx/dt = 6t^2$, $dy/dt = 8t$.

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{8t}{6t^2} = \dfrac{4}{3t}$ and so $t = 0$ gives the vertical tangent $x = 0$.

$\dfrac{d^2y}{dt^2} = \dfrac{d(y')/dt}{dx/dt} = \dfrac{-4/3t^2}{6t^2} = -\dfrac{2}{9t^4}$.

The graph is concave downward for all t.



20. $x = 2t^2$, $y = 3t^3$

▷ $\dfrac{dx}{dt} = 4t$ and $\dfrac{dy}{dt} = 9t^2$

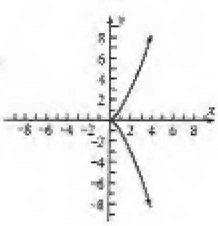
Thus

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9t^2}{4t} = \frac{9}{4}t$$

When $t = 0$, then $dy/dx = 0$ and $y = 0$. Thus $y = 0$ is an equation of the horizontal tangent line. There is no vertical tangent line.

Because $y' = dy/dx$, then

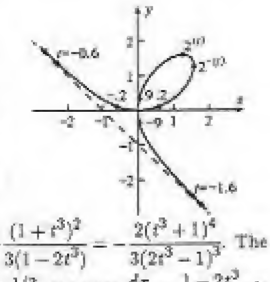
$$\frac{d(y')}{dt} = \frac{9}{4}$$



So

$$\frac{d^2y}{dx^2} = \frac{d(y')/dt}{dx/dt} = \frac{\frac{9}{4}}{4t} = \frac{9}{16t}$$

Hence $y'' < 0$ and the graph is concave downward if $t < 0$ and $y'' > 0$ if $t > 0$. Because $dy/dx = \frac{9}{4}t$ is increasing, $t = 0$ is not an inflection point. The graph, shown above right, is a semicubical parabola.



21. $x = \frac{3t}{1+t^3}$, $y = \frac{3t^2}{1+t^3}$ (folium of Descartes). $\frac{dx}{dt} = \frac{3(1-2t^3)}{(1+t^3)^2}$, $\frac{dy}{dt} = \frac{3t(2-t^3)}{(1+t^3)^2}$

Setting $dy/dt = 0$, we get $t = 0, 2^{1/3}$. When $t = 0$, $dx/dt = 3 \neq 0$ and $y = 0$; when $t = 2^{1/3}$, $dx/dt = -\frac{9}{25} \neq 0$ and $y = 2^{2/3}$. These are the horizontal tangent lines. Setting $dx/dt = 0$, we get $t = 2^{-1/3}$. Then $dy/dt = 2^{2/3} \neq 0$ and so $x = 2^{2/3}$ is a vertical tangent line. As $t \to \pm\infty$, $x \to 0$, $y \to 0$ and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t(2-t^3)}{1-2t^3} \to \pm\infty$. Thus $x = 0$ is a vertical tangent line if we add to the curve the point $(0,0)$. $\frac{d(y')}{dt} = \frac{2(t^3+1)^2}{(2t^3-1)^2}$ so $\frac{d^2y}{dx^2} = \frac{d(y')/dt}{dx/dt} = \frac{2(t^3+1)^2}{(2t^3-1)^2} \cdot \frac{(1+t^3)^2}{3(1-2t^3)} = -\frac{2(t^3+1)^4}{3(2t^3-1)^3}$. The graph is concave upward if $t < -1$ or $-1 < t < 2^{-1/3}$ and downward if $t > 2^{-1/3}$. Because $\frac{dx}{dy} = \frac{1-2t^3}{t(2-t^3)}$ is negative if $t < 2^{-1/3}$, 0 if $t = 2^{-1/3}$, and positive if $t > 2^{-1/3}$, it is increasing at $t = 2^{-1/3}$ so there is no inflection point. Also, $x^3 + y^3 = \frac{27t^3(1+t^3)}{(1+t^3)^3} = 3xy$.

Finally, we show that the line $x + y + 1 = 0$ is an asymptote. From the given equations it follows that the distance of a point P on the curve approaches $+\infty$ as t approaches -1 and the limit of the vertical distance between the point $P(x,y)$ on the curve (1) and the point $(x, -1-x)$ on the line is

$$\lim_{t\to-1} y-(-1-x) = \lim_{t\to-1} (y+x+1) = \lim_{t\to-1} \left(\frac{3t^2}{1+t^3} + \frac{3t}{1+t^3} + 1\right) = \lim_{t\to-1} \left[\frac{3t(1+t)}{(1+t)(1-t+t^2)} + 1\right]$$
$$= \lim_{t\to-1} \left(\frac{3t}{1-t+t^2} + 1\right) = -1 + 1 = 0$$

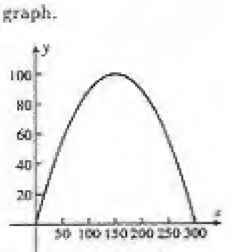
Because the distance from P to the line is less than the vertical distance, then the distance also approaches 0.

22. The graph for Ex. 21 is labeled with t values. When $t < -1$, we get the part in the fourth quadrant; $-1 < t \leq 0$ is in Q2; $t > 0$ is in Q1.

23. $x^3 + y^3 = \frac{27t^3 + 27t^6}{(1+t^3)^3} = \frac{27t^3}{(1+t^3)^2} = 3 \cdot \frac{3t}{1+t^3} \cdot \frac{3t^2}{1+t^3} = 3xy$. $(0,0)$ is a point of this graph.

24. A projectile moves so that the coordinates of its position at any time t are given by the parametric equations $x = 60t$ and $y = 80t - 16t^2$. Sketch the path of the projectile and check your graph on your graphics calculator.

▸ The path, a parabola, is shown at the right.



25. $x = 2\sin t$, $y = 5\cos t$ (ellipse) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-5\sin t}{2\cos t} = -\frac{5}{2}\tan t$. When $t = \frac{1}{3}\pi$, $x = 2\sin\frac{1}{3}\pi = \sqrt{3}$, $y = 5(\frac{1}{2}) = \frac{5}{2}$, $\frac{dy}{dx} = -\frac{5}{2}\sqrt{3}$. The tangent line is $y - \frac{5}{2} = -\frac{5}{2}\sqrt{3}(x - \sqrt{3})$; $2y - 5 = -5\sqrt{3}\,x + 15$; $5\sqrt{3}x + 2y = 20$

26. $x = 1 + 3\sin t$, $y = 2 - 5\cos t$ (ellipse) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{5\sin t}{3\cos t} = \frac{5}{3}\tan t$. When $t = \frac{1}{6}\pi$, $x = 1 + 3(\frac{1}{2}) = \frac{5}{2}$, $y = 2 - 5(\frac{1}{2}\sqrt{3})$, $\frac{dy}{dx} = \frac{\frac{5}{2}}{\frac{3}{2}\sqrt{3}} = \frac{5}{9}\sqrt{3}$. The tangent line is $y = (2 - \frac{5}{2}\sqrt{3}) + \frac{5}{9}\sqrt{3}(x - \frac{5}{2}) = \frac{5}{9}\sqrt{3}x + (2 - \frac{35}{9}\sqrt{3})$.

27. If x is in $[0, 2\pi a]$ then t is in $[0, 2\pi]$. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1-\cos t}$.

$$\frac{d^2y}{dx^2} = \frac{d(y')/dt}{dx/dt} = \frac{-\frac{1}{1-\cos t}}{a(1-\cos t)} = -\frac{1}{a(1-\cos t)^2}, \quad \frac{d^3y}{dx^3} = \frac{d(y'')/dt}{dx/dt} = \frac{\frac{2\sin t}{a(1-\cos t)^3}}{a(1-\cos t)} = \frac{2\sin t}{a^2(1-\cos t)^3}$$

y has its largest value when $\cos t = -1$. Then $t = \pi$ and $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = -\frac{1}{4a}$, $\frac{d^3y}{dx^3} = 0$.

28. Show that the slope of the tangent line at $t = t_1$ to the cycloid having the given equations is $\cot(\frac{1}{2}t_1)$. Deduce then that the tangent line is vertical when $t = 2n\pi$, where n is any integer.

$x = a(t - \sin t)$ and $y = a(1 - \cos t)$

▸ $\dfrac{dx}{dt} = a(1 - \cos t)$ and $\dfrac{dy}{dt} = a \sin t$

Therefore,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t} = \cot \tfrac{1}{2}t$$

from trigonometry. Hence the slope of the tangent line at $t = t_1$ to the cycloid is $\cot(t_1)$. For any integer n,

$\lim\limits_{t \to 2n\pi} \cos \frac{1}{2}t = \pm 1$ and $\lim\limits_{t \to 2n\pi} \sin \frac{1}{2}t = 0$

Thus,

$$\lim_{t \to 2n\pi} \cot \tfrac{1}{2}t = \lim_{t \to 2n\pi} \frac{\cos \frac{1}{2}t}{\sin \frac{1}{2}t} = \pm\infty$$

Therefore, the tangent line is vertical when $t = 2n\pi$.

29. A square units is the area of one arch of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

$$A = \int_{x=0}^{2\pi a} y\,dx = \int_{t=0}^{2\pi} a(1 - \cos t)d[a(t - \sin t)] = a^2 \int_0^{2\pi} (1 - \cos t)^2 dt$$
$$= a^2 \int_0^{2\pi} [1 - 2\cos t + \tfrac{1}{2}(1 + \cos 2t)]dt = a^2\Big[t - 2\sin t + \tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\Big]_0^{2\pi} = 3\pi a^2$$

30. $\bar{x} = \pi a$ by symmetry. $M_x = \frac{1}{2}\int_0^{2\pi a} y^2 dx = \frac{1}{2}a^3 \int_0^{2\pi} (1 - \cos t)^3 dt = \frac{1}{2}a^3 \int_0^{2\pi} (1 - 3\cos t + 3\cos^2 t - \cos^3 t)dt$

$$= \tfrac{1}{2}a^3 \int_0^{2\pi} [1 - 3\cos t + \tfrac{3}{2}(1 + \cos 2t) - \cos^3 t]dt = \tfrac{1}{2}a^3 \int_0^{2\pi} \tfrac{5}{2}\,dt = \tfrac{5}{2}\pi a^3.\ \bar{y} = \frac{M_x}{A} = \frac{\frac{5}{2}\pi a^3}{3\pi a^2} = \frac{5}{6}a.$$

31. $x = at - b\sin t$, $y = a - b\cos t$, $[-\pi, \pi]$. Because $\cos t \le 1$, if $a > b > 0$, $D_t x = a - b\cos t \ge a - b > 0$ and the trochoid has no vertical tangent line. The solid line shows $a = 3$, $b = 1$; the dashed line shows $a = 1$, $b = 3$.

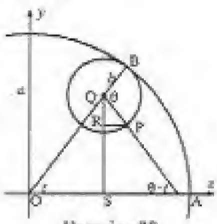
32. A hypocycloid is the curve traced by a point P on a circle of radius b that is rolling inside a fixed circle of radius a, $a > b$. If origin is at the center of the fixed circle, $A(a, 0)$ is one of the points at which the point P comes in contact with the fixed circle, B is the moving point of tangency of the two circles, and the parameter t is the number of radians in the angle AOB, derive the parametric equations of the hypocycloid.

▸ See the figure below. Because the measure of the length of arc AB equals the measure of the length of arc BP, it follows that $at = b\theta$, and hence $\theta = \frac{a}{b}t$, $\theta - t = \frac{a - b}{b}t$. Furthermore $|\overline{OQ}| = a - b$. Let $P = (x, y)$. We see from the figure that

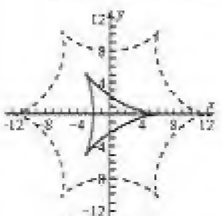
$$x = |\overline{OS}| + |\overline{RP}| = (a - b)\cos t + b\cos(\theta - t) = (a - b)\cos t + b\cos\frac{a - b}{b}t$$

$$y = |\overline{SQ}| - |\overline{QR}| = (a - b)\sin t - b\sin(\theta - t) = (a - b)\sin t - b\sin\frac{a - b}{b}t$$

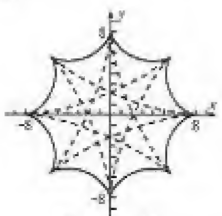
Exercise 31



Exercise 32



Exercise 33



Exercise 34

33. Hypocycloid, $t \in [-\pi, \pi]$. Solid line shows $a = 6$, $b = 2$ with 3 cusps, dashed shows $a = 12$, $b = 2$ with 6 cusps.

34. Hypocycloids of 8 cusps. Solid shows $a = 8$, $b = 7$, $t \in [-7\pi, 7\pi]$, dashed shows $a = 8$, $b = 3$, $t \in [-3\pi, 3\pi]$.

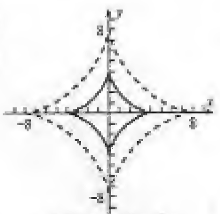
35. If $a = 4b$, $x = b(3\cos t + \cos 3t) = b[3\cos t + (4\cos^3 t - 3\cos t)] = 4b\cos^3 t = a\cos^3 t$,
$y = b(3\sin t - \sin 3t) = b[3\sin t - (3\sin t - 4\sin^3 t)] = 4b\sin^3 t = a\sin^3 t$, $t \in [-\pi, \pi]$
The solid line shows $a = 4$, dashed shows $a = 8$.

36. From the parametric equations of Exercise 35, find a Cartesian equation of the hypocycloid of four cusps. (b) Use your Cartesian equation in part (a) to sketch the graph of this hypocycloid.

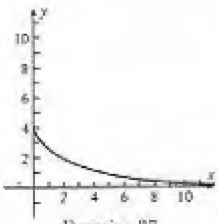
▸ The equations of Exercise 35 are $x = a\cos^3 t$ and $y = a\sin^3 t$. Then

$$x^{2/3} + y^{2/3} = a^{2/3}\cos^2 t + a^{2/3}\sin^2 t$$
$$= a^{2/3}(\sin^2 t + \cos^2 t) = a^{2/3}$$

An equation is $x^{2/3} + y^{2/3} = a^{2/3}$. The figure shows the graph for $a = 4$ solid and $a = 8$ dashed.



Exercises 35-36



Exercise 37

37. (a) $y = a\operatorname{sech}\frac{t}{a}$, $\sqrt{a^2 - y^2} = \sqrt{a^2 - a^2\operatorname{sech}^2\frac{t}{a}} = a\sqrt{1 - \operatorname{sech}^2\frac{t}{a}} = a\sqrt{\tanh^2\frac{t}{a}} = a\tanh\frac{t}{a}$

$$x = a\ln\frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2} = a\ln\frac{a + a\tanh(t/a)}{a\operatorname{sech}(t/a)} - a\tanh\frac{t}{a} = a\ln(\cosh\frac{t}{a} + \sinh\frac{t}{a}) - a\tanh\frac{t}{a}$$

$= a\ln e^{t/a} - a\tanh\frac{t}{a} = t - a\tanh\frac{t}{a}$ (b) The figure shows the tractrix for $a = 4$.

38. Parametric equations of the tractrix are $x = t - a\tanh\frac{t}{a}$ and $y = a\operatorname{sech}\frac{t}{a}$.

$\frac{dx}{dt} = 1 - \operatorname{sech}^2\frac{t}{a} = \tanh^2\frac{t}{a}$. From Newton's method, the x intercept of the tangent line is

$$x_1 = x - y/\frac{dy}{dx} = x - y\frac{dx/dt}{dy/dt} = \left(t - a\tanh\frac{t}{a}\right) - \left(a\operatorname{sech}\frac{t}{a}\right)\frac{\tanh^2(t/a)}{-\operatorname{sech}(t/a)\tanh(t/a)} = t$$

9.2 LENGTH OF ARC OF A PLANE CURVE

9.2.2 Theorem Duhamel's Principle If the functions F and G are continuous on the closed interval $[a, b]$, and if Δ is a partition of $[a, b]$ and z_i and w_i are any numbers in (t_{i-1}, t_i), then

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{[F(z_i)]^2 + [G(w_i)]^2}\,\Delta_i t = \int_a^b \sqrt{[F(t)]^2 + [G(t)]^2}\,dt$$ and

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} F(z_i)G(w_i)\Delta_i t = \int_a^b F(t)G(t)\,dt$$

The second part may used to prove the Definition 6.5.1. The first part is used to prove:

9.2.3 Theorem Let the curve C have parametric equations $x = f(t)$ and $y = g(t)$, and suppose that f' and g' are continuous on the closed interval $[a, b]$. If L units is the length of arc of the curve C from the point $(f(a), g(a))$ to the point $(f(b), g(b))$, then

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

We may also write the formula of Theorem 9.2.3 as follows.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

When using the formulas for length of arc, you must remember that

$$\sqrt{z^2} = |z| = \begin{cases} z & \text{if } z \geq 0 \\ -z & \text{if } z < 0 \end{cases}$$

A Product Rule Define θ by $\cos\theta = a/\sqrt{a^2 + b^2}$, $\sin\theta = b/\sqrt{a^2 + b^2}$. Then $\frac{d}{dx}e^{ax}\sin bx$
$= ae^{ax}\sin bx + be^{ax}\cos bx = \sqrt{a^2 + b^2}e^{ax}(\sin bx\cos\theta + \cos bx\sin\theta) = \sqrt{a^2 + b^2}e^{ax}\sin(bx + \theta)$.
Similarly, $\frac{d}{dx}e^{ax}\cos bx = \sqrt{a^2 + b^2}e^{ax}\cos(bx + \theta)$. Thus

$$\int e^{ax}\sin bx\,dx = e^{ax}\sin(bx - \theta)/\sqrt{a^2 + b^2} + C = e^{ax}(a\sin bx - b\cos bx)/(a^2 + b^2) + C$$

$$\int e^{ax}\cos bx\,dx = e^{ax}\cos(bx - \theta)/\sqrt{a^2 + b^2} + C = e^{ax}(a\cos bx + b\sin bx)/(a^2 + b^2) + C$$

Exercises 9.2

In Exercises 1–14 find the exact length L of arc of the parametric curve. Check by observing a plot.

1. $x = \frac{1}{2}t^2 + t,\ y = \frac{1}{2}t^2 - t$; from $t = 0$ to $t = 1$.

▹ $L = \int_0^1 \sqrt{x'^2 + y'^2}\,dt = \int_0^1 \sqrt{(t+1)^2 + (t-1)^2}\,dt = \int_0^1 \sqrt{2t^2+2}\,dt = \sqrt{2}\int_0^1 \sqrt{t^2+1}\,dt$

$= \sqrt{2}\left[\frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln(t + \sqrt{t^2+1})\right]_0^1 = \sqrt{2}[\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1+\sqrt{2})] = 1 + \frac{1}{2}\sqrt{2}\ln(1+\sqrt{2})$

2. $x = 3t^2,\ y = 2t^3$; from $t = 0$ to $t = 3$. $L = \int_0^3 \sqrt{x'^2 + y'^2}dt = \int_0^3 \sqrt{(6t)^2 + (6t^2)^2}dt = \int_0^3 6t\sqrt{1+t^2}dt$

$= 3\int_0^3 (1+t^2)^{1/2}d(1+t^2) = 2(1+t^2)^{3/2}\Big]_0^3 = 2(10)^{3/2} - 2 = 20\sqrt{10} - 2$

3. $x = t^2 + 2t,\ y = t^2 - 2t$; from $t = 0$ to $t = 2$.

$L = \int_0^2 \sqrt{x'^2 + y'^2}\,dt = \int_0^2 \sqrt{(2t+2)^2 + (2t-2)^2}\,t = \int_0^2 \sqrt{8t^2+8}\,dt = 2\sqrt{2}\int_0^2 \sqrt{t^2+1}\,dt$

$= 2\sqrt{2}\left[\frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln(t+\sqrt{t^2+1})\right]_0^2 = \sqrt{2}[2\sqrt{5} + \ln(2+\sqrt{5})] = 2\sqrt{10} + \sqrt{2}\ln(2+\sqrt{5})$

4. $x = t^3,\ y = 3t^2$; from $t = -2$ to $t = 0$.

▹ We apply Theorem 9.2.3. We have

$f(t) = t^3 \qquad g(t) = 3t^2$
$f'(t) = 3t^2 \qquad g'(t) = 6t$

The length of arc is given by

$$L = \int_{-2}^0 \sqrt{[f'(t)]^2 + [g'(t)]^2}dt = \int_{-2}^0 \sqrt{(3t^2)^2 + (6t)^2}dt = \int_{-2}^0 3|t|\sqrt{t^2+4}dt$$

Because $-2 \le t \le 0$, then $|t| = -t$. Thus,

$$L = -3\int_{-2}^0 t\sqrt{t^2+4}dt$$

We let $u = t^2 + 4$. Then $du = 2t\,dt$. When $t = -2$, $u = 8$. When $t = 0$, $u = 4$. Hence,

$$L = -\frac{3}{2}\int_8^4 u^{1/2}du = u^{3/2}\Big]_4^8 = 16\sqrt{2} - 8$$

• The length of arc is $16\sqrt{2} - 8$ units.

5. $x = 2t^2,\ y = 2t^3$; from $t = 1$ to $t = 2$. $x'(t) = 4t,\ y'(t) = 6t^2$.

$L = \int_0^2 \sqrt{x'^2 + y'^2}dt = \int_0^2 \sqrt{16t^2 + 36t^4}\,dt = 2\int_0^2 t\sqrt{4+9t^2}dt = \frac{2}{27}(4+9t^2)^{3/2}\Big]_0^2 = \frac{2}{27}[(40)^{3/2} - (13)^{3/2}]$

6. $x = t,\ y = \cosh t$; from $t = 0$ to 3. $L = \int_0^3 \sqrt{x'^2 + y'^2}dt = \int_0^3 \sqrt{1 + \sinh^2 t}dt = \int_0^3 \cosh dt = \sinh t\Big]_0^1 = \sinh 3$

7. $x = 3e^{2t},\ y = -4e^{2t}$; from $t = 0$ to $t = \ln 5$. $x'(t) = 6e^{2t},\ y'(t) = -8e^{2t}$. $L = \int_0^{\ln 5} \sqrt{x'^2 + y'^2}dt$

$= \int_0^{\ln 5} \sqrt{36e^{4t} + 64e^{4t}}\,dt = \int_0^{\ln 5} \sqrt{100e^{4t}}\,dt = 10\int_0^{\ln 5} e^{2t}dt = 5e^{2t}\Big]_0^{\ln 5} = 5(e^{2\ln 5} - 1) = 5(5^2 - 1) = 120$

8. $x = t^2 + 3,\ y = 3t^2$; from $t = 1$ to $t = 4$.

▹ Because t is in $[1, 4]$, then $\sqrt{t^2} = t$. Therefore

$$L = \int_1^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \int_1^4 \sqrt{(2t)^2 + (6t)^2}dt = \int_1^4 \sqrt{40t^2}dt = 2\sqrt{10}\int_1^4 t\,dt = \sqrt{10}t^2\Big]_1^4 = 15\sqrt{10}$$

9. $x = e^t\cos t,\ y = e^t\sin t$; from $t = 0$ to $t = 1$. $x'(t) = (\cos t - \sin t)e^t,\ y'(t) = (\sin t + \cos t)e^t$.

$L = \int_0^1 \sqrt{x'^2 + y'^2}dt = \int_0^1 \sqrt{(\cos^2 t - 2\cos t\sin t + \sin^2 t)e^{2t} + (\sin^2 t + 2\sin t\cos t + \cos^2 t)e^{2t}}\,dt$

$= \int_0^1 \sqrt{2(\cos^2 t + \sin^2 t)e^{2t}}\,dt = \int_0^1 \sqrt{2}e^t dt = \sqrt{2}e^t\Big]_0^1 = \sqrt{2}(e - 1)$

10. $x = \ln\sin t,\ y = t + 1$; from $t = \frac{1}{6}\pi$ to $\frac{1}{2}\pi$

$L = \int_{\pi/6}^{\pi/2} \sqrt{x'^2 + y'^2}dt = \int_{\pi/6}^{\pi/2} \sqrt{(\cot t)^2 + 1}dt = \int_{\pi/6}^{\pi/2} \csc t\,dt = -\ln(\csc t + \cot t)\Big]_{\pi/6}^{\pi/2} = \ln(2+\sqrt{3})$

11. $x = \tan^{-1} t$, $y = \frac{1}{2}\ln(t^2+1)$; from $t = 0$ to $t = 1$. $x'(t) = \dfrac{1}{1+t^2}$, $y'(t) = \dfrac{t}{1+t^2}$. $\mathrm{L} = \int_0^1 \sqrt{x'^2 + y'^2}\,dt$

$$= \int_0^1 \sqrt{\left(\frac{1}{1+t^2}\right)^2 + \left(\frac{t}{1+t^2}\right)^2}\,dt = \int_0^1 \sqrt{\frac{1+t^2}{(1+t^2)^2}}\,dt = \int_0^1 \frac{dt}{\sqrt{1+t^2}} = \ln\left|t + \sqrt{1+t^2}\right|\Big]_0^1 = \ln(1+\sqrt{2})$$

12. $x = 2(\cos t + t \sin t)$, $y = 2(\sin t - t\cos t)$ from $t = 0$ to $t = \frac{1}{3}\pi$.

▹ $$\mathrm{L} = \int_0^{\pi/3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_0^{\pi/3} \sqrt{[2(-\sin t + t\cos t + \sin t)]^2 + [2(\cos t - \cos t + t \sin t)]^2}\,dt$$
$$= \int_0^{\pi/3} 2\sqrt{t^2(\cos^2 t + \sin^2 t)}\,dt = \int_0^{\pi/3} 2|t|\,dt = \int_0^{\pi/3} 2t\,dt = t^2\Big]_0^{\pi/3} = \tfrac{1}{9}\pi^2$$

13. $x = 4\sin 2t$, $y = 4\cos 2t$; from $t = 0$ to $t = \pi$. $\mathrm{L} = \int_0^{\pi} \sqrt{x'^2 + y'^2}\,dt = \int_0^{\pi} \sqrt{(8\cos 2t)^2 + (-8\sin 2t)^2}\,dt$

$$= \int_0^{\pi} \sqrt{64\cos^2 2t + 64 \sin^2 2t}\,dt = \int_0^{\pi} 8\sqrt{\cos^2 2t + \sin^2 2t}\,dt = 8\int_0^{\pi} dt = 8t\Big]_0^{\pi} = 8\pi$$

14. $x = e^{-t}\cos t$, $y = e^{-t}\sin t$; from $t = 0$ to $t = \pi$. By a product rule, $\mathrm{L} = \int_0^{\pi} \sqrt{x'^2 + y'^2}\,dt$

$$= \int_0^{\pi} \sqrt{[\sqrt{2}e^{-t}\cos(t + \tfrac{3}{4}\pi)]^2 + [\sqrt{2}e^{-t}\sin(t + \tfrac{3}{4}\pi)]^2}\,dt = \int_0^{\pi} \sqrt{2}e^{-t}\,dt = -\sqrt{2}e^{-t}\Big]_0^{\pi} = \sqrt{2}(1 - e^{-\pi})$$

In Exercises 15–22, use NINT to find to four significant digits the length L of arc of the parametric curve.

15. $x = t + 2$, $y = 4t^2 + t$; from $t = 0$ to $t = 3$. $\mathrm{L} = \int_0^3 \sqrt{x'^2 + y'^2}\,dt = \int_0^3 \sqrt{1^2 + (8t+1)^2}\,dt = 39.194 \approx 39.19$

16. $x = 2t^2 + 3t$, $y = 2t - 1$; from $t = 1$ to $t = 2$

▹ $$\mathrm{L} = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_1^2 \sqrt{(4t+3)^2 + 2^2}\,dt$$

Using NINT, we find $\mathrm{L} = 9.223$ to four significant digits.

17. $x = 3\cos t$, $y = 2\sin t$; from $t = 0$ to $t = \frac{1}{2}\pi$. $\mathrm{L} = \int_0^{\pi/2} \sqrt{(-3\sin t)^2 + (2\cos t)^2}\,dt = 3.9664 \approx 3.966$

18. $x = 2\sec t$, $y = 3\tan t$; from $t = 0$ to $t = \frac{1}{4}\pi$. $\mathrm{L} = \int_0^{\pi/4} \sqrt{(2\sec t\tan t)^2 + (3\sec^2 t)^2}\,dt = 3.1381 \approx 3.138$

19. $x = 8\tan t$, $y = 6\sec t$; from $t = \frac{3}{4}\pi$ to $t = \pi$. $\mathrm{L} = \int_{3\pi/4}^{\pi} \sqrt{(8\sec^2 t)^2 + (6\sec t\tan t)^2}\,dt = 8.4622 \approx 8.462$

20. $x = e^t$, $y = \ln t$; from $t = 1$ to $t = 5$

▹ $$\mathrm{L} = \int_1^5 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_1^5 \sqrt{(e^t)^2 + \left(\frac{1}{t}\right)^2}\,dt$$

Using NINT, we find $\mathrm{L} = 145.8$ to four significant digits.

21. $x = 4 - t^2$, $y = t^2 + 4t$; from $t = -4$ to $t = 4$. $\mathrm{L} = \int_{-4}^{4} \sqrt{(-2t)^2 + (2t+4)^2}\,dt = 55.314 \approx 55.31$

22. $x = 3t$, $y = 4t^3$; from $t = -1$ to $t = 1$. $\mathrm{L} = \int_{-1}^{1} \sqrt{9 + (12t^2)^2}\,dt = 10.960 \approx 10.96$

23. Parametric equations of the hypocycloid of four cusps are $x = a\cos^3 t$ and $y = a\sin^3 t$.

$$\mathrm{L} = \int_0^{\pi/2} \sqrt{x'^2 + y'^2}\,dt = 4\int_0^{\pi/2} \sqrt{(-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2}\,dt$$
$$= 4\int_0^{\pi/2} \sqrt{9a^2\cos^2 t\sin^2 t}\sqrt{\cos^2 t + \sin^2 t}\,dt = 12a\int_0^{\pi/2} \sin t\cos t\,dt = 6a\sin^2 t\Big]_0^{\pi/2} = 6a$$

24. One arch of the cycloid: $x = a(t - \sin t)$, $y = a(1 - \cos t)$

▹ We use Theorem 9.2.3. Thus,

$$\frac{dx}{dt} = a(1 - \cos t) \quad\text{and}\quad \frac{dy}{dt} = a\sin t$$

Hence,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2(1 - 2\cos t + \cos^2 t) + a^2\sin^2 t} = a\sqrt{2 - 2\cos t} \qquad (1)$$

Because

$$\sin^2 \tfrac{1}{2}t = \tfrac{1}{2}(1 - \cos t)$$

then

$$4\sin^2 \tfrac{1}{2}t = 2 - 2\cos t \tag{2}$$

Substituting from (2) into (1), we obtain

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = a\sqrt{4\sin^2 \tfrac{1}{2}t} = 2a\left|\sin \tfrac{1}{2}t\right| \tag{3}$$

Because one arch of the cycloid is defined for $0 \le t \le 2\pi$, we have by (3)

$$\mathrm{L} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2a\int_0^{2\pi} \left|\sin \tfrac{1}{2}t\right| dt \tag{4}$$

Because $0 \le t \le 2\pi$, then $0 \le \frac{1}{2}t \le \pi$. Thus, $\sin \frac{1}{2}t \ge 0$. Hence, from (4)

$$\mathrm{L} = 2a\int_0^{2\pi} \sin \tfrac{1}{2}t\,dt = -4a\cos \tfrac{1}{2}t\Big]_0^{2\pi} = 8a$$

25. $x = t - a\tanh \frac{t}{a}$, $y = a\operatorname{sech} \frac{t}{a}$; from $t = a$ to $t = 2a$. Let $u = \frac{t}{a}$. Then $x = au - a\tanh u$ and $u = a\operatorname{sech} u$ from $u = 1$ to $u = 2$.

$$\mathrm{L} = \int_1^2 \sqrt{(\mathrm{D}_u x)^2 + (\mathrm{D}_u y)^2}\,du = \int_1^2 \sqrt{(a - a\operatorname{sech}^2 u)^2 + (-a\operatorname{sech} u \tanh u)^2}\,du$$
$$= a\int_1^2 \sqrt{(1 - \operatorname{sech}^2 u)^2 + \operatorname{sech}^2 u \tanh^2 u}\,du = a\int_1^2 \sqrt{\tanh^4 u + (1 - \tanh^2 u)\tanh^2 u}\,du = a\int_1^2 \sqrt{\tanh^2 u}\,du$$
$$= a\int_1^2 \tanh u\,du = a\ln|\cosh u|\Big]_1^2 = a(\ln\cosh 2 - \ln\cosh 1)$$

26. Using meters, the path of the thumbtack is a cycloid having the parametric equations $x = 0.4(t - \sin t)$ and $y = 0.4(1 - \cos t)$. If the bicycle goes a distance of 50π m and the circumference of the tire is 0.8π m, then the number of revolutions made by the thumbtack is $\frac{50\pi}{0.8\pi} = 62.5$. If L m is the distance traveled by the thumbtack during 62.5 revolutions, then

$$\mathrm{L} = 62.5\int_0^{2\pi} \sqrt{x'^2 + y'^2}\,dt = 62.5\int_0^{2\pi} \sqrt{[0.4(1 - \cos t)]^2 + (0.4\sin t)^2}\,dt$$
$$= 62.5(0.4)\int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\,dt = 25\int_0^{2\pi} \sqrt{2 - 2\cos t}\,dt = 25(2)\int_0^{2\pi} \sqrt{\tfrac{1}{2}(1 - \cos t)}\,dt$$
$$= 50\int_0^{2\pi} \sin\tfrac{1}{2}t\,dt = 50\Big[-2\cos\tfrac{1}{2}t\Big]_0^{2\pi} = -100(-1 - 1) = 200$$

27. (a) $x = a\sin t$ and $y = b\cos t$. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \sin^2 t + \cos^2 t$; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, an ellipse.

(b) $\mathrm{C} = 4\int_0^{\pi/2} \sqrt{(\mathrm{D}_t x)^2 + (\mathrm{D}_t y)^2}\,dt = 4\int_0^{\pi/2} \sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt = 4\int_0^{\pi/2} \sqrt{a^2(1 - \sin^2 t) + b^2\sin^2 t}\,dt$

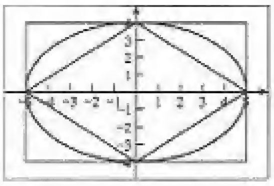
$= 4\int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2)\sin^2 t}\,dt = 4\int_0^{\pi/2} a\sqrt{1 - k^2\sin^2 t}\,dt$ where $k^2 = (a^2 - b^2)/a^2 < 1$, an elliptic integral

28. (a) Use the formula of Exercise 27(b) and NINT to determine the circumference C of the ellipse defined by $x = 5\sin t$ and $y = 4\cos t$. (b) Plot the ellipse. support your answer in part (a) by finding the perimeters of the inscribed rhombus and circumscribed rectangle and showing that C is between these two perimeters.

▸ (a) From Exercise 27, we have $\mathrm{C} = 4\int_0^{\pi/2} a\sqrt{1 - k^2\sin^2 t}\,dt$ where $k^2 = (a^2 - b^2)/a^2$. Because $a = 5$ and $b = 4$, then $k^2 = (5^2 - 4^2)/5^2 = \frac{9}{25}$ and

$$\mathrm{C} = 4\int_0^{\pi/2} 5\sqrt{1 - \tfrac{9}{25}\sin^2 t}\,dt = 28.36$$

using NINT.



(b) From the figure we see the part of the inscribed rhombus in the first quadrant has length $\sqrt{4^2 + 5^2} = \sqrt{41}$ and so its perimeter is $4\sqrt{41} \approx 25.6$ while the part of the circumscribed rectangle in the first quadrant has length $4 + 5 = 9$ and so its perimeter is $4 \times 9 = 36$, and s is between these values.

9.3 POLAR COORDINATES AND POLAR GRAPHS

Pole, Polar axis The pole is the origin, the polar axis is the positive x axis

Polar Coordinates $P(r,\theta)$ is located r units from the pole on the ray with direction angle θ.

There is not a unique pair of polar coordinates for a point. If (r,θ) is a polar coordinate representation of point P, then we may add any even multiple of π to θ and obtain another polar coordinate representation of P. That is, for any integer n, $(r,\theta) = (r,\theta + 2n\pi)$.

If we add an odd multiple of π to θ and also replace r by $-r$, we obtain another polar coordinate representation of P. That is, for any integer n, $(r,\theta) = (-r, \theta + (2n-1)\pi)$.

If $(x,y)^C$ is the Cartesian coordinate representation of P and $(r,\theta)^P$ is a polar coordinate representation of P, then
$x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$

You should be able to identify the graph and make a quick sketch by plotting only a few points for any polar equation of a type listed below. The letters a and b are any constants, but n is a positive integer in each of the following equations.

Special Graphs

TYPE OF EQUATION	NAME OF CURVE	See Exercise
1. $r = a$	Circle, center at pole, radius r	16(b)
2. $\theta = a$	Line, of angle a with polar axis	16(a)
3. $r = a\theta$	Spiral of Archimedes	40
4. $r = a/\cos(\theta - b)$	Line through $P(a,b)^P$ and $\perp$ OP	
5. $r = 2a\cos\theta + 2b\sin\theta$	Circle through O, center at $(a,b)^C$	
6. $r^2 = a^2\cos 2\theta$ or $r = a^2\sin 2\theta$	Lemniscate	44
7. $r = a\cos n\theta$ or $r = a\sin n\theta$	Rose, $2n$ leaves if n is even	35
8. $r = a\cos n\theta$ or $r = a\sin n\theta$	Rose, n leaves if n is odd	36
9. $r = a + b\cos\theta$ or $a + b\sin\theta$	Limaçon, points to furthest point	
$\lvert a\rvert \geq 2\lvert b\rvert$	Convex	29
$2\lvert b\rvert > \lvert a\rvert > \lvert b\rvert$	With a dent	27
$\lvert a\rvert = \lvert b\rvert$	Cardioid	24
$0 < \lvert a\rvert < \lvert b\rvert$	With a loop	28

Symmetry Tests Let n be any integer. A polar graph is symmetric with respect to

(i) the polar axis if an equivalent equation is obtained when (r,θ) is replaced by either $(r, -\theta + 2n\pi)$ or $(-r, \pi - \theta + 2n\pi)$.

(ii) $\frac{1}{2}\pi$ axis if an equivalent equation is obtained when (r,θ) is replaced by either $(r, \pi - \theta + 2n\pi)$ or $(-r, -\theta + 2n\pi)$.

(iii) the pole if an equivalent equation is obtained when (r,θ) is replaced by either $(r, \theta + \pi + 2n\pi)$ or $(-r, \theta + 2n\pi)$.

Intersection Often we may find all the points of intersection of two graphs by solving the equations simultaneously. However, we must treat the pole as a special case. We test each equation to see whether there is a replacement for θ (not necessarily the same replacement) that gives r the value 0. If so, then each graph contains the pole and hence the intersection contains the pole.

Because two or more distinct polar equations may represent the same graph, we must sometimes take additional steps to find all points of intersection. If the curve C is the graph of the polar equation $r = f(\theta)$, then C is also the graph of any equation of the form $r = (-1)^n f(\theta + n\pi)$, where n is any integer. Thus, to find all the points of intersection of a curve C_1 and a curve C_2, do the following:

1. Find all the distinct equations of curve C_2.
2. Solve the equation of curve C_1 simultaneously with each equation of C_2.
3. Test to see whether the pole is a point of intersection.

Tangents The slope of the tangent line to $r = f(\theta)$ at the point (r,θ) is $m = \dfrac{\sin\theta(dr/d\theta) + r\cos\theta}{\cos\theta(dr/d\theta) - r\sin\theta}$.

The angle α between the tangent line and the radius satisfies $\tan\alpha = r/(dr/d\theta)$.

If α is constant, then $r = r_0 e^{\theta/r}$, a logarithmic spiral. See Exercises 37 and 38.

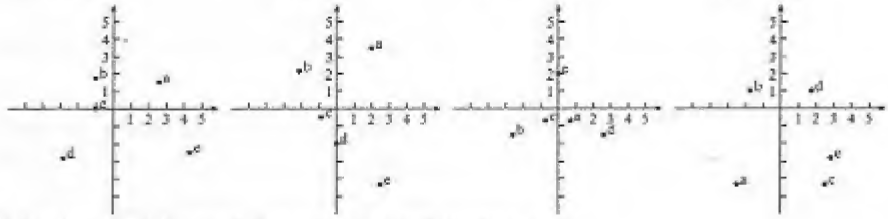
To find an equation of a tangent line to a curve at the pole, let $r = 0$ in the equation and solve for θ. If $\theta = \theta_1$ is a solution, then the line $\theta = \theta_1$ is a tangent line to the curve at the pole

Parametric Plot The polar graph of $r = f(\theta)$ yields the cartesian graph $x = f(t)\cos t$, $y = f(t)\sin t$.

Exercises 9.3

In Exercises 1–4, locate the point having the given set of polar coordinates.

1. (a) $(3, \frac{1}{6}\pi)$ (b) $(2, \frac{2}{3}\pi)$ (c) $(1, \pi)$ (d) $(4, \frac{5}{4}\pi)$ (e) $(5, \frac{11}{6}\pi)$
2. (a) $(4, \frac{1}{3}\pi)$ (b) $(3, \frac{3}{4}\pi)$ (c) $(1, \frac{7}{6}\pi)$ (d) $(2, \frac{3}{2}\pi)$ (e) $(5, \frac{5}{3}\pi)$
3. (a) $(1, -\frac{1}{4}\pi)$ (b) $(3, -\frac{5}{6}\pi)$ (c) $(-1, -\frac{1}{4}\pi)$ (d) $(-3, \frac{5}{6}\pi)$ (e) $(-2, -\frac{1}{2}\pi)$
4. (a) $(5, -\frac{2}{3}\pi)$ (b) $(2, -\frac{7}{6}\pi)$ (c) $(-5, \frac{2}{3}\pi)$ (d) $(-2, \frac{7}{6}\pi)$ (e) $(-4, -\frac{5}{4}\pi)$



In Exercises 5 and 6, convert from polar to rectangular coordinates.

5. (a) $(3, \pi) = (3\cos\pi, 3\sin\pi)^c = (-3, 0)^c$

   (b) $(\sqrt{2}, -\frac{3}{4}\pi) = (\sqrt{2}\cos(-\frac{3}{4}\pi), \sqrt{2}\sin(-\frac{3}{4}\pi))^c = (-\sqrt{2}\cdot 1/\sqrt{2}, -\sqrt{2}\cdot 1/\sqrt{2})^c = (-1, 1)^c$

   (c) $(-4, \frac{2}{3}\pi) = (-4\cos\frac{2}{3}\pi, -4\sin\frac{2}{3}\pi)^c = (-4(-\frac{1}{2}), -3(\frac{1}{2}\sqrt{3}))^c = (2, -2\sqrt{3})^c$

   (d) $(-1, -\frac{7}{6}\pi) = (1, -\frac{1}{6}\pi) = (\cos(-\frac{1}{6}\pi), \sin(-\frac{1}{6}\pi))^c = (\frac{1}{2}\sqrt{3}, -\frac{1}{2})^c$

6. (a) $(-2, -\frac{1}{2}\pi) = (2, \frac{1}{2}\pi) = (2\cos\frac{1}{2}\pi, 2\sin\frac{1}{2}\pi)^c = (0, 2)^c$

   (b) $(-1, \frac{1}{4}\pi) = (-\cos\frac{1}{4}\pi, -\sin\frac{1}{4}\pi)^c = (-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})^c$

   (c) $(2, -\frac{7}{6}\pi) = (2, \frac{5}{6}\pi) = (2\cos\frac{5}{6}\pi, 2\sin\frac{5}{6}\pi)^c = (2\cdot -\frac{1}{2}\sqrt{3}, 2\cdot\frac{1}{2})^c = (-\sqrt{3}, 1)^c$

   (d) $(2, \frac{7}{4}\pi) = (2\cos\frac{7}{4}\pi, 2\sin\frac{7}{4}\pi)^c = (2\cdot\frac{1}{2}\sqrt{2}, 2\cdot -\frac{1}{2}\sqrt{2})^c = (\sqrt{2}, -\sqrt{2})^c$

In Exercises 7 and 8, convert from rectangular to polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

7. (a) $(1, -1)^c$. $r = \sqrt{2}$. Q4, $\theta = \tan^{-1}(-1/1) + 2\pi = \frac{7}{4}\pi$. $(\sqrt{2}, \frac{7}{4}\pi)$

   (b) $(-\sqrt{3}, 1)^c$. $r = 2$. Q2, $\theta = \tan^{-1}(1/-\sqrt{3}) + \pi = \frac{5}{6}\pi$. $(2, \frac{5}{6}\pi)$

   (c) $(2, 2)^c$. $r = 2\sqrt{2}$. Q1, $\theta = \tan^{-1}(2/2) = \frac{1}{4}\pi$. $(2\sqrt{2}, \frac{1}{4}\pi)$

   (d) $(-5, 0)^c$. quadrantal. $(5, \pi)$

8. (a) $(3, -3)$; (b) $(-1, \sqrt{3})$; (c) $(0, -2)$; (d) $(-2, -2\sqrt{3})$

▷ (a) $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$ and $\tan\theta = \frac{y}{x} = \frac{-3}{3} = -1$

Because $\tan\frac{1}{4}\pi = 1$, the reference angle is $\frac{1}{4}\pi$. Because $x > 0$ and $y < 0$, the point is in the fourth quadrant. Thus, $\theta = 2\pi - \frac{1}{4} = \frac{7}{4}\pi$. The polar coordinates of the point are $(3\sqrt{2}, \frac{7}{4}\pi)$

(b) $r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2$ $\tan\theta = \frac{y}{x} = -\sqrt{3}$

Because $x < 0$ and $y > 0$, the point is in the second quadrant. Because $\tan\frac{1}{3}\pi = \sqrt{3}$, then $\theta = \pi - \frac{1}{3}\pi = \frac{2}{3}\pi$. Thus, the polar coordinates are $(2, \frac{2}{3}\pi)$.

(c) $r = \sqrt{x^2 + y^2} = \sqrt{4} = 2$. Because $x = 0$, then $\tan\theta$ is not defined. However, because the point is on the negative y axis, then $\theta = \frac{3}{2}\pi$. Thus, the polar coordinates are $(2, \frac{3}{2}\pi)$.

(d) $r = \sqrt{x^2 + y^2} = \sqrt{4 + 12} = 4$ $\tan\theta = \frac{y}{x} = \sqrt{3}$

The point is in the third quadrant, and $\tan\frac{1}{3}\pi = \sqrt{3}$, so $\theta = \pi + \frac{1}{3}\pi = \frac{4}{3}\pi$. The polar coordinates are $(4, \frac{4}{3}\pi)$.

In Exercises 9–12, find a Cartesian equation of the graph having the given polar equation.

9. $r^2 = 2\sin 2\theta = 4\sin\theta\cos\theta$ ▷ $r^4 = 4(r\sin\theta)(r\cos\theta)$. $(x^2 + y^2)^2 = 4xy$

   (b) $r^2 = \cos\theta$ ▷ (pole for $\theta = \frac{1}{2}\pi$) $r^3 = r\cos\theta$, $(r^2)^3 = (r\cos\theta)^2$. $(x^2 + y^2)^3 = x^2$

10. (a) $r^2\cos 2\theta = 10$ ▹ $r^2(\cos^2\theta - \sin^2\theta) = 10$, $x^2 - y^2 = 10$

(b) $r^2 = 4\cos 2\theta$ ▹ (pole for $\theta = \frac{1}{4}\pi$) $r^4 = 4r^2(\cos^2\theta - \sin^2\theta)$, $(x^2+y^2)^2 = 4x^2 - 4y^2$

11. (a) $r\cos\theta = -1$ ▹ $x = -1$

(b) $r = 6/(2 - 3\sin\theta)$ ▹ $2r - 3r\sin\theta = 6$, $2r = 3r\sin\theta + 6$, $4r^2 = (3r\sin\theta + 6)^2$,
$4(x^2+y^2) = (3y+6)^2$, $4x^2 + 4y^2 = 9y^2 + 36y + 36$, $4x^2 - 5y^2 - 36y - 36 = 0$

12. (a) $r = 2\sin 3\theta$; (b) $r = \dfrac{4}{3 - 2\cos\theta}$

▹ (a) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
Thus, the given equation may be written
$r = 6\sin\theta - 8\sin^3\theta$
Because the graph of this equation contains the pole, we may multiply both sides by r^3 without affecting the graph. Thus,
$$r^4 = 6r^3\sin\theta - 8r^3\sin^3\theta$$
$$(r^2)^2 = 6r^2(r\sin\theta) - 8(r\sin\theta)^3$$
Because $r^2 = x^2 + y^2$ and $r\sin\theta = y$, this becomes
$$(x^2+y^2)^2 = 6(x^2+y^2)y - 8y^3$$
$$x^4 + y^4 + 2x^2y^2 - 6x^2y + 2y^3 = 0$$
(b) Eliminating the fraction, we have
$$3r - 2r\cos\theta = 4$$
$$3r = 2(r\cos\theta + 2)$$
$$9r^2 = 4(r\cos\theta + 2)^2$$
Because $r^2 = x^2 + y^2$ and $r\cos\theta = x$, we have
$$9(x^2+y^2) = 4(x+2)^2$$
$$5x^2 + 9y^2 - 16x - 16 = 0$$

In Exercises 13–20, sketch the graph of the equation.

13. (a) $\theta = \frac{1}{3}\pi$ ▹ line through the pole with direction 60°

(b) $r = \frac{1}{3}\pi$ ▹ circle centered at the pole of radius $\frac{1}{3}\pi \approx 1.05$

14. (a) $\theta = \frac{3}{4}\pi$ ▹ line through the pole with direction 135°

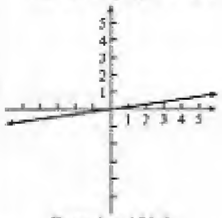
(b) $r = \frac{3}{4}\pi$ ▹ circle centered at the pole of radius $\frac{3}{4}\pi \approx 2.36$

15. (a) $\theta = 2$ ▹ line through the pole with direction $2 \cdot 180°/\pi \approx 114.6°$

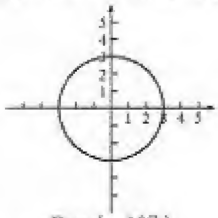
(b) $r = 2$ ▹ circle centered at the pole of radius 2

16. (a) $\theta = -3$ (b) $r = -3$

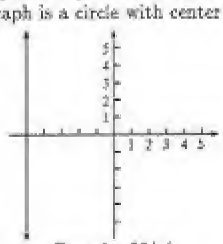
▹ (a) This equation is of Type 2. The graph is a line through the pole that makes an angle of -3 radians with the polar axis. A sketch of the graph is shown below.
(b) This equation is of Type 1. Because we may replace r by $-r$ and θ by $\theta + \pi$, then the equation $r = -3$ has the same graph as the equation $r = 3$. Thus, the graph is a circle with center at the pole and radius 3.



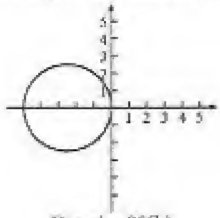
Exercise 16(a)



Exercise 16(b)



Exercise 20(a)



Exercise 20(b)

17. (a) $r\cos\theta = 4$ ▹ line $x = 4$

(b) $r = 4\cos\theta$ ▹ circle centered at $(2, 0)$ of radius 2

18. (a) $r\sin\theta = 2$ ▹ line $y = 2$

(b) $r = 2\sin\theta$ ▹ circle centered at $(1, \frac{1}{2}\pi)$ of radius 1

19. (a) $r\sin\theta = -4$ ▹ line $y = -4$

(b) $r = -4\sin\theta$ ▹ circle centered at $(2, \frac{3}{2}\pi)$ of radius 2

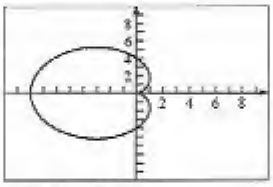
20. (a) $r\cos\theta = -5$ (b) $r = -5\cos\theta$

▷ (a) Because $r\cos\theta = x$, then a Cartesian equation is $x = -5$. The graph is the line shown above.
(b) This is a Type 5 equation with $a = -\frac{5}{2}$ and $b = 0$. Thus the graph is the circle passing through the origin and centered at $(-\frac{5}{2}, 0)^c$ shown above.

In Exercises 21–30, determine the type of limaçon, its symmetry S, and the direction it points. Plot the limaçon.

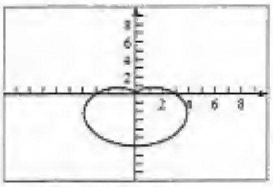
21. $r = 4(1 - \cos\theta)$

▷ $\frac{a}{b} = \frac{4}{4} = 1$, cardioid
S: polar axis, points left



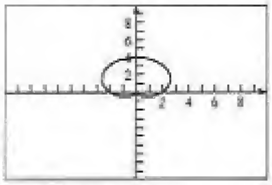
22. $r = 3(1 - \sin\theta)$

▷ $\frac{a}{b} = \frac{3}{3} = 1$, cardioid
S: $\frac{1}{2}\pi$ axis, points downward



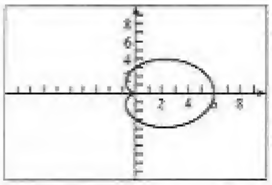
23. $r = 2(1 + \sin\theta)$

▷ $\frac{a}{b} = \frac{2}{2} = 1$, cardioid
S: $\frac{1}{2}\pi$ axis, points upward



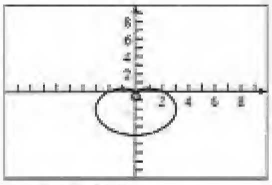
24. $r = 3(1 + \cos\theta)$

▷ $\frac{a}{b} = \frac{3}{3} = 1$, cardioid
S: polar axis, points right



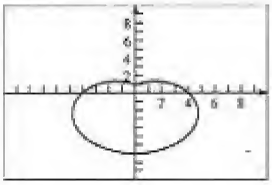
25. $r = 2 - 3\sin\theta$

▷ $\frac{a}{b} = \frac{2}{3} \in (0, 1)$, a loop
S: $\frac{1}{2}\pi$ axis, points downward



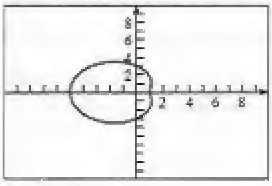
26. $r = 4 - 3\sin\theta$

▷ $\frac{a}{b} = \frac{4}{3} \in (1, 2)$, dented
S: $\frac{1}{2}\pi$ axis, points downward



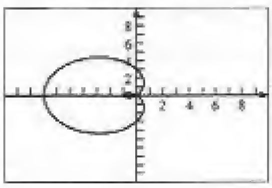
27. $r = 3 - 2\cos\theta$

▷ $\frac{a}{b} = \frac{3}{2} \in (1, 2)$, dented
S: polar axis, points left



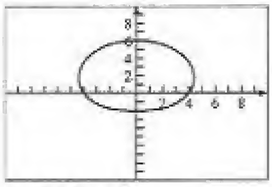
28. $r = 3 - 4\cos\theta$

▷ $\frac{a}{b} = \frac{3}{4} \in (0, 1)$, with a loop
S: polar axis, points left



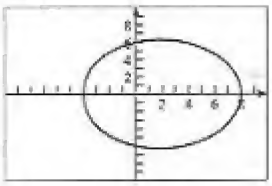
29. $r = 4 + 2\sin\theta$

▷ $\frac{a}{b} = \frac{4}{2} \geq 2$, convex
S: $\frac{1}{2}\pi$ axis, points upward



30. $r = 6 + 2\cos\theta$

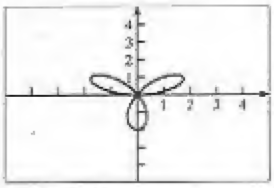
▷ $\frac{a}{b} = \frac{6}{2} \geq 2$, convex
S: polar axis, points right



In Exercises 31–50, describe and plot the graph of the equation.

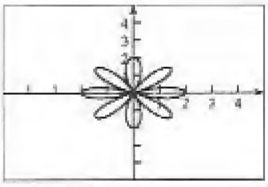
31. $r = 2\sin 3\theta$

▷ 3-leafed rose



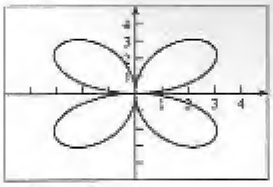
33. $r = 2\cos 4\theta$

▷ 8-leafed rose



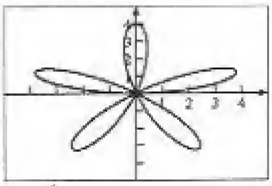
35. $r = 4\sin 2\theta$

▷ 4-leafed rose



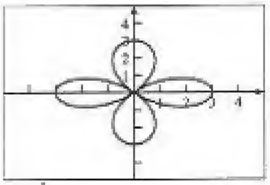
32. $r = 4\sin 5\theta$

▷ 5-leafed rose



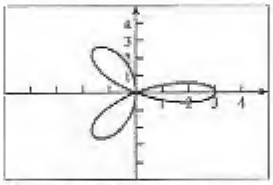
34. $r = 3\cos 2\theta$

▷ 4-leafed rose



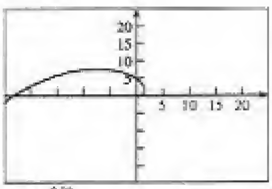
36. $r = 3\cos 3\theta$

▷ 3-leafed rose



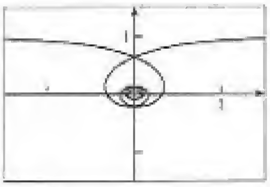
37. $r = e^{\theta}$

▷ logarithmic spiral



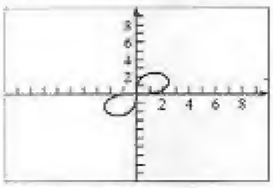
39. $r = \frac{1}{\theta}$

▷ reciprocal spiral



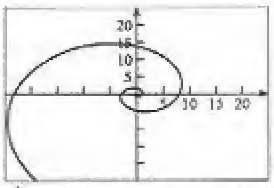
41. $r^2 = 9\sin 2\theta$ (lemniscate)

▷ $X_1 = 3\cos t\sqrt{\sin 2t}$,
$Y_1 = 3\sin t\sqrt{\sin 2t}$, $0 \le t \le 6.3$



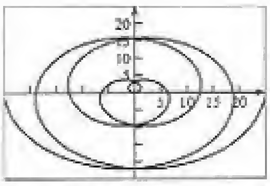
38. $r = e^{\theta/3}$

▷ logarithmic spiral



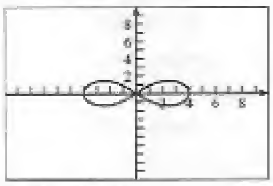
40. $r = 2\theta$

▷ spiral of Archimedes



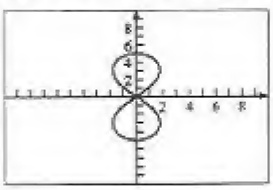
42. $r^2 = 16\cos 2\theta$ (lemniscate)

▷ $X_1 = 4\cos t\sqrt{\cos 2t}$,
$Y_1 = 4\sin t\sqrt{\cos 2t}$, $0 \le t \le 6.3$



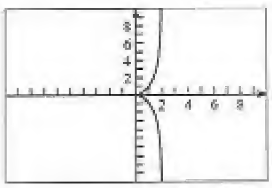
43. $r^2 = -25\cos 2\theta$ (lemniscate)

▷ $X_1 = 5\cos t\sqrt{-\cos 2t}$,
$Y_1 = 5\sin t\sqrt{-\cos 2t}$, $0 \le t \le 6.3$



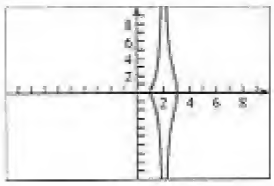
45. $r = 2\sin\theta\tan\theta$

▷ cissoid

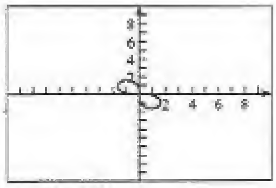


47. $r = 2\sec\theta - 1$

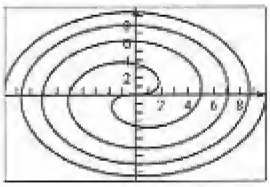
▷ conchoid of Nicomedes
coefficient (2) > constant (1), no loop



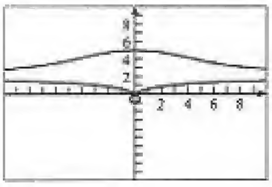
44. $r^2 = -4 \sin 2\theta$ (lemniscate)
▷ $X_1 = 2 \cos t \sqrt{-\sin 2t}$,
$Y_1 = 2 \sin t \sqrt{-\sin 2t}$,
$0 \le t \le 6.3$



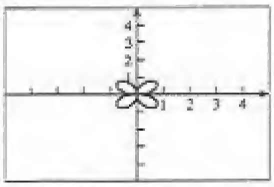
46. $r^2 = 8\theta$ (Fermat spiral)
▷ $X_1 = \cos t \sqrt{8t}$, $Y_1 = \sin t \sqrt{8t}$
$X_2 = -\cos t \sqrt{8t}$, $Y_1 = -\sin t \sqrt{8t}$
$0 \le t \le 6.3$



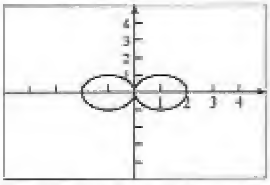
48. $r = 2 \csc \theta + 3$
conchoid of Nicomedes
coefficient (2) < constant (3), loop



49. $r = |\sin 2\theta|$
▷ 4-leafed rose, same as $r = \sin 2\theta$

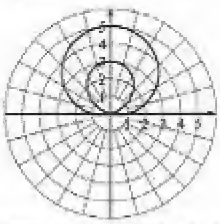


50. $r = 2|\cos \theta|$
▷ two circles

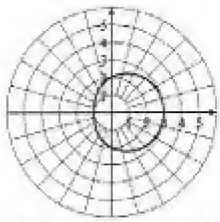


In Exercises 51 and 52, construct a polar coordinate system and sketch on it the graph of the function.

51. $r = 1 + 4 \sin \theta$
▷ limaçon with a loop



52. $r = 2 + \cos \theta$
▷ convex limaçon



In Exercises 53 through 60, determine the horizontal HT and vertical tangents VT of the graph, and sketch them.

53. $r = 4 + 3 \sin \theta$; $\dfrac{dr}{d\theta} = 3 \cos \theta$. $m = \dfrac{\sin \theta(3 \cos \theta) + (4 + 3 \sin \theta)\cos \theta}{\cos \theta(3 \cos \theta) - (4 + 3 \sin \theta)\sin \theta} = \dfrac{\cos \theta(4 + 6 \sin \theta)}{3 - 4 \sin \theta - 6 \sin^2\theta}$

Horizontal tangents. $-2 \cos \theta(3 \sin \theta + 2) = 0$. $\cos \theta = 0$; $\theta = \frac{1}{2}\pi$, $\theta = \frac{3}{2}\pi$ and $4 + 6 \sin \theta = 0$; $\sin \theta = -\frac{2}{3}$;

$\theta = \pi - \sin^{-1}(-\frac{2}{3}) \approx 3.87$, $\theta = 2\pi + \sin^{-1}(-\frac{2}{3}) \approx 5.55$

There are horizontal tangent lines at the points $(7, \frac{1}{2}\pi)$, $(1, \frac{3}{2}\pi)$, $(2, 3.87)$, $(2, 5.55)$.

Vertical tangents. $6 \sin^2\theta + 4 \sin \theta - 3 = 0$; $\sin \theta = \frac{1}{12}(-4 \pm \sqrt{16 + 72}) = \frac{1}{12}(-4 \pm 2\sqrt{22}) = \frac{1}{6}(-2 \pm \sqrt{22})$

$\sin \theta \approx 0.448$; $\theta = \sin^{-1} 0.448 = 0.46$, $\theta = \pi - \sin^{-1} 0.448 = 2.68$. $\sin \theta \approx -1.115$; no solution.

There are vertical tangent lines at the points $(5.35, 0.46)$ and $(5.35, 2.68)$.

54. $r = 2 + \cos \theta$. $\dfrac{dr}{d\theta} = -\sin \theta$. $m = \dfrac{\sin \theta(-\sin \theta) + (2 + \cos \theta)\cos \theta}{\cos \theta(-\sin \theta) - (2 + \cos \theta)\sin \theta} = \dfrac{\cos^2\theta - 1 + 2 \cos \theta + \cos^2\theta}{-2 \sin \theta(1 + \cos \theta)}$

HT: $2 \cos^2\theta + 2 \cos \theta - 1 = 0$, $\cos \theta = \frac{1}{4}(-2 \pm \sqrt{12}) = \frac{1}{2}(-1 \pm \sqrt{3})$. $\cos \theta = \frac{1}{2}(-1 - \sqrt{3})$, no solution.

$\theta = \cos^{-1}[\frac{1}{2}(\sqrt{3} - 1)] = 1.196$ or $2\pi - 1.196 = 5.087$, $r = 2 + \frac{1}{2}(\sqrt{3} - 1) = 2.366$

Vertical tangents: $\sin \theta = 0$, $r = 3$ and $\theta = \pi$, $r = 1$. $\cos \theta = -1$, $\theta = \pi$, duplicate

55. $r = 4 - 2\cos\theta$; $\dfrac{dr}{d\theta} = 2\sin\theta$. $m = \dfrac{\sin\theta(2\sin\theta) + (4 - 2\cos\theta)\cos\theta}{\cos\theta(2\sin\theta) - (4 - 2\cos\theta)\sin\theta} = \dfrac{2\sin^2\theta + 4\cos\theta - 2\cos^2\theta}{2\sin\theta\cos\theta - 4\sin\theta + 2\sin\theta\cos\theta}$

$= \dfrac{2 - 4\cos^2\theta + 4\cos\theta}{4\sin\theta(\cos\theta - 1)}$. HT: $2\cos^2\theta - 2\cos\theta - 1 = 0$; $\cos\theta = \frac{1}{4}(2 \pm \sqrt{4 + 8}) = \frac{1}{4}(2 \pm 2\sqrt{3}) = \frac{1}{2}(1 \pm \sqrt{3})$.

$\cos\theta \approx 1.366$, no solution. $\cos\theta \approx = -0.366$; $\theta = \cos^{-1}(-0.366) = 1.95$. $\theta = 2\pi - \cos^{-1}(-0.366) = 4.34$

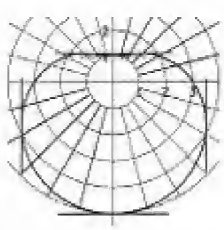
There are horizontal tangent lines at the points $(4.73, 1.95)$ and $(4.73, 4.34)$.

Vertical tangents. $-2\sin\theta(\cos\theta - 1) = 0$. $\sin\theta = 0$; $\theta = 0$, $\theta = \pi$ and $\cos\theta = 1$; $\theta = 0$.

There are vertical tangent lines at the points $(2, 0)$ and $(6, \pi)$.

56. $r = 3 - 2\sin\theta$

▷ We have



$\dfrac{dr}{d\theta} = -2\cos\theta$

Therefore, if m is the slope of the tangent line to the curve, then

$$m = \frac{\sin\theta(dr/d\theta) + r\cos\theta}{\cos\theta(dr/d\theta) - r\sin\theta} = \frac{\sin\theta(-2\cos\theta) + (3 - 2\sin\theta)\cos\theta}{\cos\theta(-2\cos\theta) - (3 - 2\sin\theta)\sin\theta}$$
$$= \frac{-2\sin\theta\cos\theta + 3\cos\theta - 2\sin\theta\cos\theta}{-2\cos^2\theta - 3\sin\theta + 2\sin^2\theta}$$
$$= \frac{-4\sin\theta\cos\theta + 3\cos\theta}{-2(1 - \sin^2\theta) + 2\sin^2\theta - 3\sin\theta} = \frac{\cos\theta(3 - 4\sin\theta)}{4\sin^2\theta - 3\sin\theta - 2} \tag{1}$$

Because $m = 0$ at a point where the tangent line is horizontal, we set the numerator to 0 and solve for θ. Thus $\cos\theta = 0$, $\theta = \frac{1}{2}\pi$ or $\theta = \frac{3}{2}\pi$; $\sin\theta = \frac{3}{4}$, $\theta = \sin^{-1}\frac{3}{4} = 0.848$ or $\theta = \pi - \sin^{-1}\frac{3}{4} = 2.294$

There is a horizontal tangent line at the points with polar coordinates $(1, \frac{1}{2}\pi)$, $(5, \frac{3}{2}\pi)$, $(\frac{3}{2}, 0.848)$, and $(\frac{3}{2}, 2.294)$.

Because m is not defined at a point where the tangent line is vertical, we set the denominator of (1) to 0 and solve for θ. Thus,

$4\sin^2\theta - 3\sin\theta - 2 = 0$

$\sin\theta = \frac{1}{8}(3 + \sqrt{41}) = 1.18$, no solution or $\sin\theta = \frac{1}{8}(3 - \sqrt{41}) \approx -0.4254$,

$r = \frac{1}{4}(9 + \sqrt{41}) = 3.85$, $\theta = \pi + \sin^{-1}(0.4254) = 3.58$, $\theta = 2\pi - \sin^{-1}(0.4254) = 5.84$

Therefore, the curve has a vertical tangent at the points with polar coordinates $(3.85, 3.58)$ and $(3.85, 5.84)$.

Because $a/b = 1.5$ is close to 2, the limçon is nearly convex and the 3 tangents near the dent are hard to see.

57. $r = \cos 2\theta = \cos^2\theta - \sin^2\theta$; $\dfrac{dr}{d\theta} = -2\sin 2\theta = -4\sin\theta\cos\theta$.

$$m = \frac{\sin\theta(-4\sin\theta\cos\theta) + (\cos^2\theta - \sin^2\theta)\cos\theta}{\cos\theta(-4\sin\theta\cos\theta) - (\cos^2\theta - \sin^2\theta)\sin\theta} = \frac{\cos\theta(-4\sin^2\theta + \cos^2\theta - \sin^2\theta)}{\sin\theta(-4\cos^2\theta - \cos^2\theta + \sin^2\theta)} = \frac{\cos\theta(1 - 6\sin^2\theta)}{\sin\theta(1 - 6\cos^2\theta)}$$

Horizontal tangents: $\cos\theta = 0$; $\theta = \frac{1}{2}\pi$, $\theta = \frac{3}{2}\pi$ and $1 - 6\sin^2\theta = 0$

$\sin\theta = \pm\frac{1}{6}\sqrt{6}$; $\theta = \sin^{-1}(\frac{1}{6}\sqrt{6}) \approx 0.42$, $\theta = \pi - \sin^{-1}(-\frac{1}{6}\sqrt{6}) \approx 3.56$, $\theta = 2\pi + \sin^{-1}(-\frac{1}{6}\sqrt{6}) \approx 5.86$

Horizontal tangents are at $(-1, \frac{1}{2}\pi)$, $(-1, \frac{3}{2}\pi)$, $(\frac{2}{3}, 0.42)$, $(\frac{2}{3}, 2.72)$, $(\frac{2}{3}, 3.56)$, $(\frac{2}{3}, 5.86)$.

Vertical tangents: $\sin\theta = 0$; $\theta = 0$, $\theta = \pi$ and $1 - 6\cos^2\theta = 0$; $\cos\theta = \pm\frac{1}{6}\sqrt{6}$

$\theta = \cos^{-1}(\frac{1}{6}\sqrt{6}) \approx 1.15$, $\theta = \cos^{-1}(-\frac{1}{6}\sqrt{6}) \approx 1.99$, $\theta = 2\pi - \cos^{-1}(-\frac{1}{6}\sqrt{6})$, $\theta = 2\pi - \cos^{-1}(\frac{1}{6}\sqrt{6}) \approx 5.13$

Vertical tangents are at $(1, 0)$, $(1, \pi)$, $(-\frac{2}{3}, 1.15)$, $(-\frac{2}{3}, 1.99)$, $(-\frac{2}{3}, 4.29)$, $(-\frac{2}{3}, 5.13)$.

58. $r = 2\sin 3\theta$. $\dfrac{dr}{d\theta} = 6\cos 3\theta$.

$$m = \frac{\sin\theta(6\cos 3\theta) + (2\sin 3\theta)\cos\theta}{\cos\theta(6\cos 3\theta) - (2\sin 3\theta)\sin\theta} = \frac{6\sin\theta(4\cos^3\theta - 3\cos\theta) + 2(3\sin\theta - 4\sin^3\theta)\cos\theta}{6\cos\theta(4\cos^3\theta - 3\cos\theta) - 2(3\sin\theta - 4\sin^3\theta)\sin\theta}$$

HT: $0 = 2\sin\theta\cos\theta[(12\cos^2\theta - 9) + (3 - 4\sin^2\theta)] = 2\sin\theta\cos\theta(6 - 16\sin^2\theta)$. $\sin\theta = 0$, $\theta = 0$, $r = 0$.

$\cos\theta = 0$, $\theta = \frac{1}{2}\pi$, $r = -3$. $\sin^2\theta = \frac{6}{16}$, $\sin\theta = \pm\frac{1}{4}\sqrt{6}$, $\theta = \sin^{-1}\frac{1}{4}\sqrt{6} = 0.659$, $r = 1.837$.

$\theta = \pi - \sin^{-1}\frac{1}{4}\sqrt{6} = 2.483$, $r = 1.837$

VT: $0 = 2[3\cos^2\theta(4\cos^2\theta - 3) - (3\sin^2\theta - 4\sin^4\theta)] = 2[(3 - 3\sin^2\theta)(1 - 4\sin^2\theta) - (3\sin^2\theta - 4\sin^4\theta)]$
$= 2(16\sin^4\theta - 18\sin^2\theta + 3)$. $\sin^2\theta = \frac{1}{16}(9 \pm \sqrt{33})$. $\sin\theta = 0.9600$, $\theta = 1.286$, $r = -1.33$.
$\theta = \pi - 1.286 = 1.855$, $r = -1.33$. $\sin\theta = 0.4511$, $\theta = 0.467$, $r = 1.96$. $\theta = \pi - 0.467 = 2.674$, $r = 1.96$

59. $r^2 = 4 \sin 2\theta = 8 \sin\theta \cos\theta$, $\theta \in [0, \frac{1}{2}\pi]$; $r\dfrac{dr}{d\theta} = 8\cos^2\theta - 8\sin^2\theta$

$$m = \frac{\sin\theta(dr/d\theta) + r\cos\theta}{\cos\theta(dr/d\theta) - r\sin\theta} = \frac{\sin\theta(r\,dr/d\theta) + r^2\cos\theta}{\cos\theta(r\,dr/d\theta) - r^2\sin\theta} = \frac{\sin\theta(4\cos^2\theta - 4\sin^2\theta + 8\cos^2\theta)}{\cos\theta(4\cos^2\theta - 4\sin^2\theta + 8\sin^2\theta)} = \frac{\sin\theta(4\cos^2\theta - 1)}{\cos\theta(1 - 4\sin^2\theta)}$$

Horizontal tangents. $\sin\theta = 0$; $\theta = 0$ and $4\cos^2\theta - 1 = 0$; $\cos\theta = \pm\frac{1}{2}$; $\theta = \frac{1}{3}\pi$.

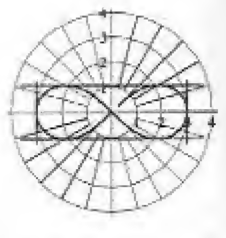
There are horizontal tangent lines at the points $(0,0)$, $(\sqrt[4]{12}, \frac{1}{3}\pi)$, $(-\sqrt[4]{12}, \frac{1}{3}\pi)$.

Vertical tangents. $\cos\theta = 0$; $\theta = \frac{1}{2}\pi$ and $1 - 4\sin^2\theta = 0$; $\sin\theta = \pm\frac{1}{2}$; $\theta = \frac{1}{6}\pi$.

There are vertical tangent lines at the points $(0, \frac{1}{2}\pi)$, $(\sqrt{2}, \frac{1}{6}\pi)$, $(-\sqrt{2}, \frac{1}{6}\pi)$.

60. $r^2 = 9\cos 2\theta$

▹ The graph is a lemniscate. We have



$$2r\frac{dr}{d\theta} = -18\sin 2\theta; \qquad r\frac{dr}{d\theta} = -9\sin 2\theta$$

If m is the slope of the tangent line to the curve, then

$$m = \frac{\sin\theta(dr/d\theta) + r\cos\theta}{\cos\theta(dr/d\theta) - r\sin\theta} = \frac{\sin\theta(r\,dr/d\theta) + r^2\cos\theta}{\cos\theta(r\,dr/d\theta) - r^2\sin\theta}$$
$$= \frac{\sin\theta(-9\sin 2\theta) + (9\cos 2\theta)\cos\theta}{\cos\theta(-9\sin 2\theta) - (9\cos 2\theta)\sin\theta} = \frac{\cos\theta\cos 2\theta - \sin\theta\sin 2\theta}{-(\sin\theta\cos 2\theta + \cos\theta\sin 2\theta)}$$
$$= -\frac{\cos 3\theta}{\sin 3\theta}$$

Setting the numerator equal to zero, and bearing in mind that we need $\cos 2\theta \geq 0$, we get
$\cos 3\theta = 0$, $3\theta = \frac{1}{2}\pi$, $\theta = \frac{1}{6}\pi$, $r = \pm\frac{3}{2}\sqrt{2}$; $3\theta = \frac{3}{2}\pi$, $\theta = \frac{1}{2}\pi$, no r; $3\theta = \frac{5}{2}\pi$, $\theta = \frac{5}{6}\pi$, $r = \pm\frac{3}{2}\sqrt{2}$

Thus, there is a horizontal tangent line at the points $(\frac{3}{2}\sqrt{2}, \frac{1}{6}\pi)$, $(-\frac{3}{2}\sqrt{2}, \frac{1}{6}\pi)$, $(\frac{3}{2}\sqrt{2}, \frac{5}{6}\pi)$ and $(-\frac{3}{2}\sqrt{2}, \frac{5}{6}\pi)$.
Setting the denominator equal to zero, we get
$\sin 3\theta = 0$, $3\theta = 0$, $\theta = 0$, $r = \pm 3$; $3\theta = \pi$, $\theta = \frac{1}{3}\pi$, no r.
Therefore the curve has a vertical tangent at the points with polar coordinates $(3, 0)$ and $(-3, 0)$.

In Exercises 61–64, find the Cartesian coordinates of the intersection graphically to two significant digits. Then find the polar coordinates algebraically and compare.

61. The circle $r = 3$ and the cardioid $r = 2(1 + \cos\theta)$. Solving the equations simultaneously, we have $2 + 2\cos\theta = 3$; $\cos\theta = \frac{1}{2}$, $\theta = \frac{1}{3}\pi$, $-\frac{1}{3}\pi$

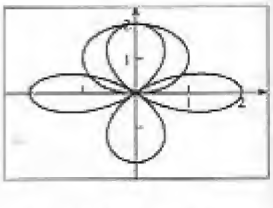
We have the points of intersection $(3, \frac{1}{3}\pi)^p = (\frac{3}{2}, \frac{3}{2}\sqrt{3})^c \approx (1.5, 2.6)$ and $(3, -\frac{1}{3}\pi)^p = (\frac{3}{2}, -\frac{3}{2}\sqrt{3})^c \approx (1.5, -2.6)$.

62. The circles $r = 2\cos\theta$ and $r = 2\sin\theta$. Dividing corresponding members of the two equations, we obtain $\dfrac{2\sin\theta}{2\cos\theta} = \dfrac{r}{r}$; $\tan\theta = 1$. Hence, $\theta = \frac{1}{4}\pi$ and $\frac{5}{4}\pi$. Substituting these values of θ into one of the two given equations we obtain the corresponding values of r. We then have the following points of intersection:
$(\sqrt{2}, \frac{1}{4}\pi)^p = (2,2)^c$ and $(-\sqrt{2}, \frac{5}{4}\pi)^p = (2,2)^c$, the same point. The pole lies on the graph of $r = 2\cos\theta$ because $r = 0$ when $\theta = \frac{1}{2}\pi$. The pole also lies on the graph of $r = 2\sin\theta$ because $r = 0$ when $\theta = 0$. Thus, the two curves intersect at the pole.

63. The 3-leafed rose $r = 2\sin 3\theta$ and the circle $r = 4\sin\theta$. $2(3\sin\theta - 4\sin^3\theta) = 4\sin\theta$,
$0 = 8\sin^3\theta - 2\sin\theta = 8\sin\theta(\sin^2\theta - \frac{1}{4})$. $\sin\theta = 0$, $\theta = 0$, $r = 0$. $\sin\theta = \frac{1}{2}$, $\theta = \frac{1}{6}\pi$, $r = 2$. $\sin\theta = \pi - \frac{1}{6}\pi = \frac{5}{6}\pi$, $r = 2$. $(0,0)^P = (0,0)^c$, $(2, \frac{1}{6}\pi)^p = (\sqrt{3}, 1) \approx (1.7, 1)$, $(2, \frac{5}{6}\pi) = (-\sqrt{3}, 1) \approx (-1.7, 1)$

64. $r = 2\cos 2\theta$ and $r = 2\sin\theta$

▹ The graph of $r = 2\cos 2\theta$ is a four-leafed rose, and the graph of $r = 2\sin\theta$ is a circle as shown. First we solve the given equations simultaneously. Thus,



$$\begin{aligned} 2\cos 2\theta &= 2\sin\theta \\ 2(1 - 2\sin^2\theta) &= 2\sin\theta \\ 2\sin^2\theta + \sin\theta - 1 &= 0 \\ (\sin\theta + 1)(2\sin\theta - 1) &= 0 \end{aligned}$$

$\sin\theta = -1$ or $\sin\theta = \frac{1}{2}$
$\theta = \frac{3}{2}\pi$, $r = -2$ or $\theta = \frac{1}{6}\pi$, $r = 1$ or $\theta = \frac{5}{6}\pi$, $r = 1$

Hence $(-2, \frac{3}{2}\pi)^p = (0,2)^c$, $(1, \frac{1}{6}\pi)^p = (\frac{1}{2}\sqrt{3}, \frac{1}{2})^c \approx (0.87, 0.5)^c$ and $(1, \frac{5}{6}\pi)^p = (-\frac{1}{2}\sqrt{3}, \frac{1}{2})^c \approx (-0.87, 0.5)$ are points of intersection. Because r can be 0 in each of the given equations, the pole is also point of intersection.

9.4 LENGTH OF ARC AND AREA OF A REGION FOR POLAR GRAPHS

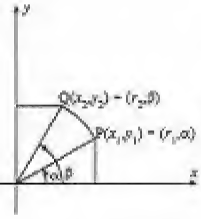
Arc Length The formula for the number of units in the length of arc of a curve defined by the polar equation $r = f(\theta)$ from the point where $\theta = \alpha$ to the point where $\theta = \beta$ is given by

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

9.4.1 Theorem Area Let R be the region bounded by the lines $\theta = \alpha$ and $\theta = \beta$ and the curve whose polar equation is $r = r(\theta)$, where r is continuous and nonnegative on the closed interval $[\alpha, \beta]$. If A square units is the area of region R, then

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[r(w_i)]^2 \Delta_i\theta = \tfrac{1}{2}\int_\alpha^\beta [r(\theta)]^2\, d\theta$$

Proof Because area has already been defined, the limit of this sum is not the definition; rather, it is a consequence of the integral formula. Refer to the figure. Let $m_1 = y_1/x_1$ and $m_2 = y_2/x_2$ and let $y = y(x)$ and $x = x(y)$ be the cartesian equations of the curve. (If necessary, we divide $[\alpha, \beta]$ into subintervals on which the curve is the cartesian graph of a function.) By Definition 4.5.4 (vertical elements of area), we have



$$\begin{aligned} A &= \int_0^{x_2} m_2 x\, dx + \int_{x_2}^{x_1} y\, dx - \int_0^{x_1} m_1 x\, dx \\ &= \tfrac{1}{2}x_2y_2 + \int_{x_2}^{x_1} y\, dx - \tfrac{1}{2}x_1y_1 \end{aligned} \qquad (1)$$

By the theorem of Section 4.8 (horizontal elements of area), we have

$$\begin{aligned} A &= \int_0^{y_1} \frac{1}{m_1}\, dy + \int_{y_1}^{y_2} x\, dy - \int_0^{y_2} \frac{1}{m_2}\, dy \\ &= \tfrac{1}{2}x_1y_1 + \int_{y_1}^{y_2} x\, dy - \tfrac{1}{2}x_2y_2 \end{aligned} \qquad (2)$$

Adding Eqs. (1) and (2) and dividing by 2, we get

$$\begin{aligned} A &= \tfrac{1}{2}\left[\int_{x_2}^{x_1} y\, dx + \int_{y_1}^{y_2} x\, dy\right] \\ &= \tfrac{1}{2}\left[\int_{y_1}^{y_2} x\, dy - \int_{x_1}^{x_2} y\, dx\right] \end{aligned} \qquad (3)$$

Now $x = r\cos\theta$, $y = r\sin\theta$ and so

$$dx = (r'\cos\theta - r\sin\theta)d\theta \quad \text{and} \quad dy = (r'\sin\theta + r\cos\theta)d\theta$$

Furthermore, when $x = x_1$, then $y = y_1$ and $\theta = \alpha$; when $x = x_2$, then $y = y_2$ and $\theta = \beta$. Making these replacements in (3), we obtain

$$\begin{aligned} A &= \tfrac{1}{2}\int_\alpha^\beta [(r\cos\theta)(r'\sin\theta + r\cos\theta) - (r\sin\theta)(r'\cos\theta - r\sin\theta)]\, d\theta \\ &= \tfrac{1}{2}\int_\alpha^\beta r^2(\sin^2\theta + \cos^2\theta)\, d\theta \\ &= \tfrac{1}{2}\int_\alpha^\beta r^2\, d\theta \end{aligned}$$

Area Between Curves If $f(\theta) \ge g(\theta)$ for $\alpha \le \theta \le \beta$, then the area of the region bounded by the curves $r = f(\theta)$ and $r = g(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$\begin{aligned} A &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}\{[f(w_i)]^2 - [g(w_i)]^2\}\Delta_i\theta \\ &= \tfrac{1}{2}\int_\alpha^\beta \{[f(\theta)]^2 - [g(\theta)]^2\}\, d\theta \end{aligned}$$

Exercises 9.4

In Exercises 1–4, use the length of arc formula to find the circumference of the circle having the polar equation.

1. If L units is the circumference of the circle $r = 5\cos\theta$, then
$$L = \int_0^{\pi} \sqrt{r'^2 + r^2}\, d\theta = \int_0^{\pi} \sqrt{(-5\sin\theta)^2 + (5\cos\theta)^2}\, d\theta = \int_0^{\pi} 5\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = 5\int_0^{\pi} d\theta = 5\theta\Big]_0^{\pi} = 5\pi$$

2. $r = 4\sin\theta$. $L = \int_0^{\pi}\sqrt{r'^2 + r^2}\, d\theta = \int_0^{\pi}\sqrt{(4\cos\theta)^2 + (4\sin\theta)^2}d\theta = \int_0^{\pi} 4\sqrt{\cos^2\theta + \sin^2\theta}d\theta = 4\int_0^{\pi} d\theta = 4\pi$

3. If L units is the circumference of the circle $r = a$, then
$$L = \int_0^{2\pi}\sqrt{r'^2 + r^2}\, d\theta = \int_0^{2\pi}\sqrt{0^2 + a^2}\, d\theta = \int_0^{2\pi} a\, d\theta = a\theta\Big]_0^{2\pi} = 2\pi a$$

4. $r = a\sin\theta$

▷ We use the formula
$$L = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}d\theta$$
Because $r = 0$ when $\theta = 0$; $r = 0$ when $\theta = \pi$; and $r > 0$ for $0 < \theta < \pi$, we have $\alpha = 0$ and $\beta = \pi$. That is, the entire circle is traced out when θ takes on all values in the closed interval $[0, \pi]$. Thus,
$$L = \int_0^{\pi}\sqrt{(a\cos\theta)^2 + (a\sin\theta)^2}d\theta = a\int_0^{\pi} d\theta = \pi a$$
We note that the diameter of the circle is a units. Thus, the circumference is π times the diameter.

In Exercises 5–12, find the exact length of arc of the given polar graph.

5. The entire cardioid $r = 4 + 4\cos\theta$.
$$L = 2\int_0^{\pi}\sqrt{r'^2 + r^2}d\theta = 2\int_0^{\pi}\sqrt{(-4\sin\theta)^2 + (4 + 4\cos\theta)^2}d\theta = 8\int_0^{\pi}\sqrt{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}\, d\theta$$
$$= 8\int_0^{\pi}\sqrt{2 + 2\cos\theta}d\theta = 16\int_0^{\pi}\sqrt{\tfrac{1}{2}(1 + \cos\theta)}d\theta = 16\int_0^{\pi}\cos\tfrac{1}{2}\theta\, d\theta = 32\sin\tfrac{1}{2}\theta\Big]_0^{\pi} = 32$$

6. The entire cardioid $r = 1 - \sin\theta$. $L = 2\int_{-\pi/2}^{\pi/2}\sqrt{r'^2 + r^2}d\theta = 2\int_{-\pi/2}^{\pi/2}\sqrt{(-\cos\theta)^2 + (1 - \sin\theta)^2}d\theta$
$$= 2\int_{-\pi/2}^{\pi/2}\sqrt{\cos^2\theta + 1 - 2\sin\theta + \sin^2\theta}d\theta = 2\int_{-\pi/2}^{\pi/2}\sqrt{2 - 2\sin\theta}d\theta = 2\int_{-\pi/2}^{\pi/2}\sqrt{2 + 2\cos(\theta + \tfrac{1}{2}\pi)}d\theta$$
$$= 4\int_{-\pi/2}^{\pi/2}\sqrt{\tfrac{1}{2}[1 + \cos(\theta + \tfrac{1}{2}\pi)]}d\theta = 4\int_{-\pi/2}^{\pi/2}\cos(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi)d\theta = 8\sin(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi)\Big]_{-\pi/2}^{\pi/2} = 8$$

7. If L units is the length of the entire cardioid $r = 3\cos^2\frac{1}{2}\theta = \frac{3}{2}(1 + \cos\theta)$, then
$$L = 2\int_0^{\pi}\sqrt{r'^2 + r^2}\, d\theta = \int_0^{\pi}\sqrt{(-3\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta)^2 + 9\cos^4\tfrac{1}{2}\theta}\, d\theta = 2\int_0^{\pi}\sqrt{9\cos^2\tfrac{1}{2}\theta}\sqrt{\sin^2\tfrac{1}{2}\theta + \cos^2\tfrac{1}{2}\theta}\, d\theta$$
$$= 6\int_0^{\pi}\cos\tfrac{1}{2}\theta\, d\theta = 12\sin\tfrac{1}{2}\theta\Big]_0^{\pi} = 12$$

8. $r = 3\theta$, from $\theta = 0$ to $\theta = 2\pi$

▷
$$L = \int_0^{2\pi}\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}d\theta = \int_0^{2\pi}\sqrt{3^2 + (3\theta)^2}d\theta = 3\int_0^{2\pi}\sqrt{1 + \theta^2}d\theta$$
$$= \tfrac{3}{2}\theta\sqrt{1 + \theta^2} + \ln(\theta + \sqrt{1 + \theta^2})\Big]_0^{2\pi} = \tfrac{3}{2}[2\pi\sqrt{1 + 4\pi^2} + \ln(2\pi + \sqrt{1 + 4\pi^2})]$$
with the help of Integral Formula 28.

9. If L units is the length of the arc of the curve $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 4$, then
$$L = \int_0^4\sqrt{r'^2 + r^2}d\theta = \int_0^4\sqrt{(2e2^{\theta})^2 + 4e^{4\theta}}d\theta = \int_0^4\sqrt{4e^{4\theta} + e^{4\theta}}\, d\theta = \sqrt{5}\int_0^4 e^{2\theta}d\theta = \tfrac{1}{2}\sqrt{5}[e^{2\theta}]_0^4 = \tfrac{1}{2}\sqrt{5}(e^8 - 1)$$

10. $r = 3\theta^2$. $L = \int_0^{\pi}\sqrt{r'^2 + r^2}d\theta = \int_0^{\pi}\sqrt{(6\theta)^2 + (3\theta^2)^2}d\theta = \int_0^{\pi}3\theta\sqrt{4 + \theta^2}d\theta = (4 + \theta^2)^{3/2}\Big]_0^{\pi} = (4 + \pi^2)^{3/2} - 8$

11. If L units is the length of the arc of the curve $r = 2\sin^3\frac{1}{3}\theta$ from $\theta = 0$ to $\theta = 6\pi$, then
$$L = \int_0^{6\pi}\sqrt{r'^2 + r^2}\, d\theta = \int_0^{6\pi}\sqrt{(2\sin^2\tfrac{1}{3}\theta\cos\tfrac{1}{3}\theta)^2 + 2^2\sin^6\tfrac{1}{3}\theta}\, d\theta = 2\int_0^{6\pi}\sqrt{\sin^4\tfrac{1}{3}\theta(\cos^2\tfrac{1}{3}\theta + \sin^2\tfrac{1}{3}\theta)}d\theta$$
$$= 2\int_0^{6\pi}\sin^2\tfrac{1}{3}\theta\, d\theta = \int_0^{6\pi}(1 - \cos\tfrac{2}{3}\theta)d\theta = \left[\theta - \tfrac{3}{2}\sin\tfrac{2}{3}\theta\right]_0^{6\pi} = 6\pi$$

12. $r = \sin^2 \frac{1}{2}\theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$

▹ Because $r = \frac{1}{2}(1 - \cos\theta)$, then

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = \int_0^{\pi/2} \sqrt{(\tfrac{1}{2}\sin\theta)^2 + [\tfrac{1}{2}(1-\cos\theta)]^2}\,d\theta = \int_0^{\pi/2} \sqrt{\tfrac{1}{4}(\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta)}\,d\theta$$

$$= \int_0^{\pi/2} \sqrt{\tfrac{1}{2}(1-\cos\theta)}\,d\theta = \int_0^{\pi/2} \sin\tfrac{1}{2}\theta\,d\theta = -2\cos\tfrac{1}{2}\theta\Big]_0^{\pi/2} = -2(\cos\tfrac{1}{4}\pi - \cos 0) = -2(\tfrac{1}{2}\sqrt{2} - 1) = 2 - \sqrt{2}$$

In Exercises 13–20, use NINT to approximate to four significant digits the length of arc of the given polar graph.

13. $r = 3 + \cos\theta$. $L = 2\int_0^{\pi} \sqrt{r'^2 + r^2}\,d\theta = 2\int_0^{\pi/2} \sqrt{(-\sin\theta)^2 + (3+\cos\theta)^2}\,d\theta = 2\int_0^{\pi} \sqrt{10 + 6\cos\theta}\,d\theta = 19.377$

14. $r = 3 - 2\sin\theta$.

$L = 2\int_{-\pi/2}^{\pi/2} \sqrt{r'^2 + r^2}\,d\theta = 2\int_{-\pi/2}^{\pi/2} \sqrt{(-2\cos\theta)^2 + (3 - 2\sin\theta)^2}\,d\theta = 2\int_{-\pi/2}^{\pi/2} \sqrt{13 - 12\sin\theta}\,d\theta = 21.01$

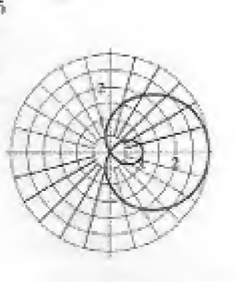
15. The loop of the limaçon $r = 2 - 3\sin\theta$. Let $a = \sin^{-1}\frac{2}{3}$.

$L = 2\int_a^{\pi/2} \sqrt{(-3\cos\theta)^2 + (2 - 3\sin\theta)^2}\,d\theta = 2\int_a^{\pi/2} \sqrt{13 - 12\sin\theta}\,d\theta = 2.505$

16. The loop of the limaçon $r = 1 + 2\cos\theta$.

▹ See the figure. The midpoint of the loop is at $\theta = \pi$. The loop begins when

$$r = 0$$
$$2\cos\theta = -1$$
$$\cos\theta = -\tfrac{1}{2}$$
$$\theta = \tfrac{2}{3}\pi$$



By symmetry,

$$L = 2\int_{2\pi/3}^{\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = 2\int_{2\pi/3}^{\pi} \sqrt{(-2\sin\theta)^2 + (1 + 2\cos\theta)^2}\,d\theta$$

$$= 2\int_{2\pi/3}^{\pi} \sqrt{5 + 4\cos\theta}\,d\theta$$

Using NINT we find $L = 2.6824 = 2.682$ to four significant digits.

17. One leaf of rose $r = 2\sin 3\theta$. $L = 2\int_0^{\pi/6} \sqrt{(6\cos 3\theta)^2 + (2\sin 3\theta)^2}\,d\theta = 4\int_0^{\pi/6} \sqrt{9\cos^2 3\theta + \sin^2 3\theta}\,d\theta = 4.455$

18. 1 leaf of rose $r = 3\cos 4\theta$. $L = 2\int_0^{\pi/8} \sqrt{(-12\sin 4\theta)^2 + (3\cos 4\theta)^2}\,d\theta = 6\int_0^{\pi/8} \sqrt{16\sin^2 4\theta + 9\cos^2 4\theta}\,d\theta = 7.203$

19. The lemniscate $r^2 = 25\cos 2\theta$. $2rr' = -50\sin 2\theta$. $r' = \dfrac{-25\sin 2\theta}{r}$. $r'^2 = \dfrac{25^2\sin^2 2\theta}{25\cos 2\theta}$

$r'^2 + r^2 = 25\left(\dfrac{\sin^2 2\theta}{\cos 2\theta} + \cos 2\theta\right) = \dfrac{25}{\cos 2\theta}$. $L = 4\int_0^{\pi/4} \dfrac{5\,d\theta}{\sqrt{\cos 2\theta}} = 26.22$, an improper integral.

20. The lemniscate $r^2 = 4\sin 2\theta$.

▹ We have

$$r = 2\sqrt{\sin 2\theta}$$
$$\frac{dr}{d\theta} = \frac{2\cos 2\theta}{\sqrt{\sin 2\theta}}$$
$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{4\cos^2 2\theta}{\sin 2\theta} + 4\sin 2\theta = \frac{4}{\sin 2\theta}$$

By symmetry

$$L = 4\int_0^{\pi/4} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = 8\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

Some calculators can evaluate this improper integral to get $10.488 \approx 10.49$. The substitution $\theta = x^2$, $d\theta = 2x\,dx$ yields the proper integral

$$L = 16\int_0^{\sqrt{\pi/2}} \frac{x\,dx}{\sqrt{\sin 2x^2}}$$

In Exercises 21–26, find the exact area of the region enclosed by the graph of the equation.

21. The graph of the equation $r = 3\cos\theta$ is a circle. We obtain the region enclosed by a semicircle if $\theta \in [0, \frac{1}{2}\pi]$.

$A = 2\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2}(3\cos w_i)^2 \Delta_i\theta = 9\int_0^{\pi/2} \cos^2\theta\,d\theta = 9\int_0^{\pi/2} \dfrac{1 + \cos 2\theta}{2}\,d\theta = \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta\Big]_0^{\pi/2} = \frac{9}{4}\pi$

22. $r = 2 - \sin\theta$, a convex limaçon. $A = 2\cdot\frac{1}{2}\int_{-\pi/2}^{\pi/2} (2 - \sin\theta)^2\,d\theta = \int_{-\pi/2}^{\pi/2} (4 - 4\sin\theta + \sin^2\theta)\,d\theta =$

$$\int_{-\pi/2}^{\pi/2}[4 - 4\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)]d\theta = \int_{-\pi/2}^{\pi/2}(\tfrac{9}{2} - 4\sin\theta - \tfrac{1}{2}\cos 2\theta)d\theta = \tfrac{9}{2}\pi + 0 + 0 = \tfrac{9}{2}\pi$$

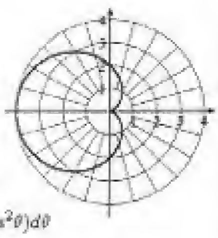
23. The graph of $r = 4\cos 3\theta$ is a three-leafed rose. We get the region enclosed by one half of a leaf if $\theta \in [0, \frac{1}{6}\pi]$.

$$A = 6\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}(4\cos 3w_i)^2\Delta_i\theta = 48\int_0^{\pi/6}\cos^2 3\theta\, d\theta = 24\int_0^{\pi/6}(1+\cos 6\theta)d\theta = 24\Big[\theta + \tfrac{1}{6}\sin 6\theta\Big]_0^{\pi/6} = 4\pi$$

24. $r = 4\sin^2\frac{1}{2}\theta$.

▸ First, we simplify the given equation.

$r = 4\sin^2\frac{1}{2}\theta = 2(1-\cos\theta)$



The graph is the cardioid shown in the figure. Because the region R is symmetric with respect to the polar axis, the area of the region R_1 above the horizontal axis is one-half the area of the entire region R. Because $r = 0$ when $\theta = 0$, we take the line $\theta = 0$ as one of the boundaries or R_1. Thus R_1 is bounded by the lines $\theta = 0$, $\theta = \pi$, and the curve $r = 2(1-\cos\theta)$. Hence, the area of R is given by

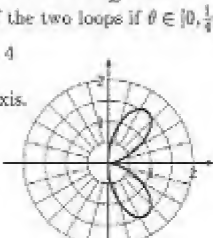
$$A = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}[2(1-\cos w_i)]^2 d\theta = \int_0^{\pi}[2(1-\cos\theta(]^2 d\theta = 4\int_0^{\pi}(1 - 2\cos\theta + \cos^2\theta)d\theta$$

$$= 4\int_0^{\pi}[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)]d\theta = \int_0^{\pi}(6 - 8\cos\theta + 2\cos 2\theta)d\theta = \Big[6\theta - 8\sin\theta + \sin 2\theta\Big]_0^{\pi} = 6\pi$$

25. The graph of $r^2 = 4\sin 2\theta$ is a lemniscate. We obtain the area of one-half of one of the two loops if $\theta \in [0, \frac{1}{4}\pi]$.

$$A = 4\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}(4\sin 2w_i)\Delta_i\theta = 8\int_0^{\pi/4}\sin 2\theta\, d\theta = -4\cos 2\theta\Big]_0^{\pi/4} = -4(0-1) = 4$$

26. $r = 4\sin^2\theta\cos\theta$. The folium (see figure) is symmetrical with respect to the polar axis.



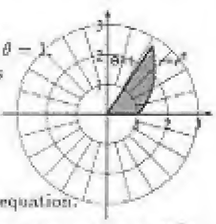
$$A = 2\cdot\tfrac{1}{2}\int_0^{\pi/2}16\sin^4\theta\cos^2\theta\, d\theta = \int_0^{\pi/2}4(2\sin\theta\cos\theta)^2\sin^2\theta\, d\theta$$

$$= \int_0^{\pi/2}2\sin^2 2\theta(1-\cos 2\theta)d\theta = \int_0^{\pi/2}(1 - \cos 4\theta - 2\sin^2 2\theta\cos 2\theta)d\theta$$

$$= \Big[\theta - \tfrac{1}{4}\sin 4\theta - \tfrac{1}{3}\sin^3 2\theta\Big]_0^{\pi/2} = \tfrac{1}{2}\pi$$

27. $A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\frac{1}{2}(w_i)^2\Delta_i\theta = \frac{1}{2}\int_0^{3\pi/2}\theta^2 d\theta = \frac{1}{6}\theta^3\Big]_0^{3\pi/2} = \frac{1}{6}\left(\frac{3}{2}\pi\right)^3 = \frac{9}{16}\pi^3$

28. Find the area of the region enclosed by the graph of $r = e^{\theta}$ and the lines $\theta = 0$ and $\theta = 1$.



▸ The curve is an equiangular spiral with endpoints $(1, 0)$ and $(e, 1)$. The figure shows the region. If A square units is the area of the region, then

$$A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}[e^{w_i}]^2\Delta_i\theta = \tfrac{1}{2}\int_0^1 e^{2\theta}d\theta = \tfrac{1}{4}e^{2\theta}\Big]_0^1 = \tfrac{1}{4}(e^2 - 1)$$

▪ The area of the region is $\frac{1}{4}(e^2-1)$ square units.

In Exercises 29–32, find the area of the region enclosed by one loop of the graph of the equation.

29. The graph of $r = 3\cos 2\theta$ is a four-leafed rose. We get the region enclosed by one-half of a leaf if $\theta \in [0, \frac{1}{4}\pi]$.

$$A = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}(3\cos 2w_i)^2\Delta_i\theta = 9\int_0^{\pi/4}\cos^2 2\theta\, d\theta = 9\int_0^{\pi/4}\frac{1+\cos 4\theta}{2}d\theta = \frac{9}{2}\theta + \frac{9}{8}\sin 4\theta\Big]_0^{\pi/4} = \frac{9}{8}\pi$$

30. $r = 4(1 - 2\cos\theta)$ is a limaçon with a loop. $A = 2\cdot\frac{1}{2}\int_0^{\pi/3}[4(1-2\cos\theta)]^2 d\theta = 16\int_0^{\pi/3}(1 - 4\cos\theta + 4\cos^2\theta)$

$$= 16\int_0^{\pi/3}(1 - 4\cos\theta + 2 + 2\cos 2\theta)d\theta = 16\Big[3\theta - 4\sin\theta + \sin 2\theta\Big]_0^{\pi/3} = 16(\pi - \tfrac{3}{2}\sqrt{3})16\pi - 24\sqrt{3}$$

31. The graph of $r = 1 + 3\sin\theta$ is a limaçon with a loop. The graph is symmetric with respect to the $\frac{1}{2}\pi$ axis. We get the region enclosed by half of the loop if $\theta \in [-\frac{1}{2}\pi, b]$, where $\sin b = -\frac{1}{3}$, $\cos b = \frac{2}{3}\sqrt{2}$, and $b = \sin^{-1}(-\frac{1}{3})$.

$$A = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}(1 + 3\sin w_i)^2\Delta_i\theta = \int_{-\pi/2}^{b}(1 + 3\sin\theta)^2 d\theta = \int_{-\pi/2}^{b}(1 + 6\sin\theta + 9\sin^2\theta)d\theta$$

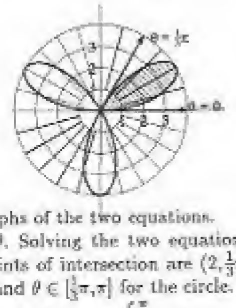
$$= \int_{-\pi/2}^{b}\left(1 + 6\sin\theta + \tfrac{9}{2} - \tfrac{9}{2}\cos 2\theta\right)d\theta = \tfrac{11}{2}\theta - 6\cos\theta - \tfrac{9}{4}\sin 2\theta\Big]_{-\pi/2}^{b}$$

$$= \tfrac{11}{2}\sin^{-1}\left(-\tfrac{1}{3}\right) - 6\left(\tfrac{2}{3}\sqrt{2}\right) - \tfrac{9}{2}(2)\left(-\tfrac{1}{3}\right)\left(\tfrac{2}{3}\sqrt{2}\right) + \tfrac{11}{4}\pi = \tfrac{11}{4}\pi - \tfrac{11}{2}\sin^{-1}\tfrac{1}{3} - 3\sqrt{2}$$

32. $r = 4 \sin 3\theta$.

▸ The graph is a three-leafed rose. We take the first quadrant loop. Because $r = 0$ when $3\theta = \pi$, or equivalently, $\theta = \frac{1}{3}\pi$, we take the lines $\theta = 0$ and $\theta = \frac{1}{3}\pi$ as boundaries. See the figure. Thus, the area of the loop is given by

$$\begin{aligned} A &= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[4 \sin 3w_i]^2 \Delta_i\theta \\ &= \frac{1}{2}\int_0^{\pi/3} [4 \sin 3\theta]^2 d\theta \\ &= 8\int_0^{\pi/3} \sin^2 3\theta \, d\theta \\ &= 8\int_0^{\pi/3} (1 - \cos 6\theta) d\theta = 8\Big[\theta - \tfrac{1}{6}\sin 6\theta\Big]_0^{\pi/3} = \tfrac{4}{3}\pi \end{aligned}$$



▪ The area of the loop is $\frac{4}{3}\pi$ square units.

In Exercises 33–36, find the area of the intersection of the regions enclosed by the graphs of the two equations.

33. The equations are the circle $r = 2$ and the limaçon with a dent $r = 3 - 2\cos\theta$. Solving the two equations simultaneously, we get $3 - 2\cos\theta = 2$; $\cos\theta = \frac{1}{2}$; so $\theta = \frac{1}{3}\pi, \frac{5}{3}\pi$. Therefore the points of intersection are $(2, \frac{1}{3}\pi)$ and $(2, \frac{5}{3}\pi)$. We obtain one-half of the required area if $\theta \in [0, \frac{1}{3}\pi]$ for the limaçon and $\theta \in [\frac{1}{3}\pi, \pi]$ for the circle.

$$A = 2\Big[\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(3 - 2\cos w_i)^2 \Delta_i\theta + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(2)^2 \Delta_i\theta\Big] = \int_0^{\pi/3} (9 - 12\cos\theta + 4\cos^2\theta) d\theta + 4\int_{\pi/3}^{\pi} d\theta$$

$$= \Big[9\theta - 12\sin\theta + 4\int \frac{1 + \cos 2\theta}{2} d\theta\Big]_0^{\pi/3} + 4\theta\Big]_0^{\pi/3} = 9\theta - 12\sin\theta + 2\theta + \sin 2\theta\Big]_0^{\pi/3} + \Big(4\pi - \tfrac{4}{3}\pi\Big)$$

$$= \Big[\tfrac{11}{3}\pi - 12\Big(\tfrac{1}{2}\sqrt{3}\Big) + \tfrac{1}{2}\sqrt{3}\Big] + \tfrac{8}{3}\pi = \tfrac{19}{3}\pi - \tfrac{11}{2}\sqrt{3}$$

34. $r = 4\sin\theta$, $r = 4\cos\theta$ are congruent circles point upward and to the right. By symmetry,

$$A = 2 \cdot \tfrac{1}{2}\int_0^{\pi/4} (4\sin\theta)^2 d\theta = 8\int_0^{\pi/4} (1 - \cos 2\theta) d\theta = 8\Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/4} = 8(\tfrac{1}{4}\pi - \tfrac{1}{2}) = 2\pi - 4$$

35. The given equations are the four-leafed roses $r = 3\sin 2\theta$ and $r = 3\cos 2\theta$. We obtain one-eighth of the area if $\theta \in [0, \frac{1}{8}\pi]$ for the first graph and $\theta \in [\frac{1}{8}\pi, \frac{1}{4}\pi]$ for the second.

$$A = 8\Big[\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(3\sin 2w_i)^2 \Delta_i\theta + \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(3\cos 2w_i)^2 \Delta_i\theta\Big] = 4\int_0^{\pi/8} 9\sin^2 2\theta \, d\theta + 4\int_{\pi/8}^{\pi/4} 9\cos^2 2\theta \, d\theta$$

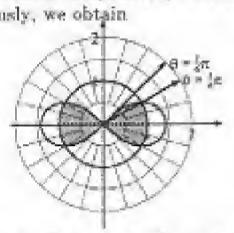
$$= 18\int_0^{\pi/8} (1 - \cos 4\theta) d\theta + 18\int_{\pi/8}^{\pi/4} (1 + \cos 4\theta) d\theta = 18\Big[\theta - \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/8} + 18\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_{\pi/8}^{\pi/4}$$

$$= 18\Big(\tfrac{\pi}{8} - \tfrac{1}{4}\Big) + 18\Big(\tfrac{\pi}{4} + 0 - \tfrac{\pi}{8} - \tfrac{1}{4}\Big) = \tfrac{9}{4}\pi - \tfrac{9}{2} + \tfrac{9}{2}\pi - \tfrac{9}{4}\pi - \tfrac{9}{2} = \tfrac{9}{2}\pi - 9$$

36. $r^2 = 2\cos 2\theta$ and $r = 1$.

▸ The region is the intersection of the regions bounded by the lemniscate $r^2 = 2\cos 2\theta$ and the circle $r = 1$, as shown in the figure. Let R be the first quadrant portion of the required region. Because of symmetry, the area of the entire region is four times the area of R. Solving the equations simultaneously, we obtain

$$1 = 2\cos 2\theta, \quad \cos 2\theta = \tfrac{1}{2}, \qquad 2\theta = \tfrac{1}{3}\pi, \qquad \theta = \tfrac{1}{6}\pi$$



Thus, the line $\theta = \frac{1}{6}\pi$ contains the first-quadrant intersection of the lemniscate and the circle. Let R_1 be the part of region R that is between the lines $\theta = 0$ and $\theta = \frac{1}{6}\pi$. Because the region R_1 is bounded by the circle $r = 1$ and the lines $\theta = 0$ and $\theta = \frac{1}{6}\pi$, we have

$$A_1 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}\Delta_i\theta = \frac{1}{2}\int_0^{\pi/6} d\theta = \tfrac{1}{2}\theta\Big]_0^{\pi/6} = \frac{1}{12}\pi \qquad (1)$$

Let R_2 be the part of the region R that is not in region R_1. Setting $r = 0$ in the equation $r^2 = 2\cos 2\theta$, we get $\theta = \frac{1}{4}\pi$. Thus, R_2 is bounded by the lines $\theta = \frac{1}{6}\pi$ and $\theta = \frac{1}{4}\pi$ and the lemniscate $r^2 = 2\cos 2\theta$. Hence,

$$A_2 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[2\cos 2w_i]\Delta_i\theta = \frac{1}{2}\int_{\pi/6}^{\pi/4} (2\cos 2\theta) d\theta = \tfrac{1}{2}\sin 2\theta\Big]_{\pi/6}^{\pi/4} = \tfrac{1}{2}(1 - \tfrac{1}{2}\sqrt{3}) \qquad (2)$$

Using (1) and (2), the area of the entire region, is given by

$$A = 4(A_1 + A_2) = 4(\tfrac{1}{12}\pi + \tfrac{1}{2} - \tfrac{1}{4}\sqrt{3}) = \tfrac{1}{3}\pi + 2 - \sqrt{3}$$

▪ The area of the region is $\frac{1}{3}\pi + 2 - \sqrt{3}$ square units.

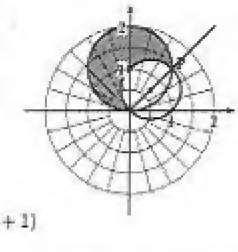
In Exercises 37–44, find the area inside the graph of the first equation and outside the graph of the second.
In Exercises 41–44, (a) find all points of intersection, then (b) find the required area.

37. The given equations are the circle $r = 3$ and the cardioid $r = 3(1 - \cos\theta)$. Each graph is symmetric with respect to the polar axis. We get one-half the required area if $\theta \in [0, \frac{1}{2}\pi]$.

$$A = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[9 - 9(1 - \cos w_i)^2]\Delta_i\theta = \int_0^{\pi/2} [9 - 9(1 - 2\cos\theta + \cos^2\theta)]d\theta$$
$$= 9\int_0^{\pi/2}\left(2\cos\theta - \frac{1+\cos 2\theta}{2}\right)d\theta = 9\left[2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\cos 2\theta\right]_0^{\pi/2} = 9(2 - \tfrac{1}{4}\pi + \tfrac{1}{4} - \tfrac{1}{4}) = 9(2 - \tfrac{1}{4}\pi)$$

38. The lemniscate $r^2 = 4\sin 2\theta$ and the circle $r = \sqrt{2}$ are symmetric with respect to $\theta = \frac{1}{4}\pi$ meet when $2 = 4\sin 2\theta$, $\sin 2\theta = \frac{1}{2}$, $2\theta = \frac{1}{6}\pi$, $\theta = \frac{1}{12}\pi$. $A = 4\cdot\frac{1}{2}\int_{\pi/12}^{\pi/4}(4\sin 2\theta - 2)d\theta = 2\left[-2\cos 2\theta - 2\theta\right]_{\pi/12}^{\pi/4} = 2\sqrt{3} - \frac{2}{3}\pi$

39. The given equations are the circle $r = 2\sin\theta$ with center at $(1, \frac{1}{2}\pi)$ and radius 1, and the circle $r = \sin\theta + \cos\theta$ with center at $(\frac{1}{2}\sqrt{2}, \frac{1}{4}\pi)$ and radius $\frac{1}{2}\sqrt{2}$. The two circles intersect at the pole and at $P = (\sqrt{2}, \frac{1}{4}\pi)$. The required region is that enclosed by the first circle except for two regions, one of which is one-half of the region enclosed by the second circle and the other is a segment of the first circle formed by the diameter of the second circle from the pole to P. See the figure.



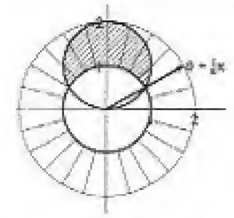
$$A = \pi - \left[\frac{1}{4}\pi + \lim_{\|\Delta\| \to 0}\sum_{i=1}^{n}\tfrac{1}{2}(2\sin w_i)^2\Delta_i\theta\right] = \pi - \frac{1}{4}\pi - 2\int_0^{\pi/4}\sin^2\theta\, d\theta$$
$$= \frac{3}{4}\pi - \int_0^{\pi/4}(1 - \cos 2\theta)d\theta = \frac{3}{4}\pi - \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{3}{4}\pi - \left(\frac{1}{4}\pi - \frac{1}{2}\sin\frac{\pi}{2}\right) = \frac{1}{2}(\pi + 1)$$

40. $r = 4\sin\theta$ and $r = 2$.

▸ The figure shows the region R. Let R_1 be the first-quadrant part of R. Solving the equations simultaneously, we have

$$4\sin\theta = 2, \quad \sin\theta = \tfrac{1}{2}, \qquad \theta = \tfrac{1}{6}\pi$$

Thus, the region R_1 is bounded by the lines $\theta = \frac{1}{6}\pi$ and $\theta = \frac{1}{2}\pi$ and by the curves $r = 2$ and $r = 4\sin\theta$, where $2 \le 4\sin\theta$ when $\frac{1}{6}\pi \le \theta \le \frac{1}{2}\pi$. Thus, the area of the entire region is given by



$$A = 2\lim_{\|\Delta\| \to 0}\sum_{i=1}^{n}\tfrac{1}{2}[(4\sin w_i)^2 - 2^2]\Delta_i\theta$$
$$= \int_{\pi/6}^{\pi/2}[(4\sin\theta)^2 - 2^2]d\theta$$
$$= 4\int_{\pi/6}^{\pi/2}[2(1 - \cos 2\theta) - 1]d\theta = 4\int_{\pi/6}^{\pi/2}(1 - 2\cos 2\theta)d\theta$$
$$= 4\left[\theta - \sin 2\theta\right]_{\pi/6}^{\pi/2} = 4[\tfrac{1}{2}\pi - (\tfrac{1}{6}\pi - \tfrac{1}{2}\sqrt{3})] = 4(\tfrac{1}{3}\pi + \tfrac{1}{2}\sqrt{3})$$

▪ The area of the region is $4(\frac{1}{3}\pi + \frac{1}{2}\sqrt{3})$ square units.

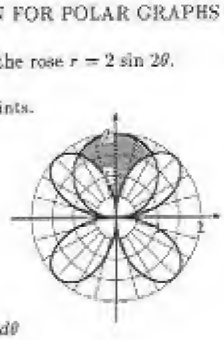
41. $r = 1 + 4\cos\theta$ (loop) and $r = 1$. (a) $1 + 4\cos\theta = 1$, $4\cos\theta = 0$, $\theta = \pm\frac{1}{2}\pi$ and $1 + 4\cos\theta = -1$, $4\cos\theta = -2$, $\cos\theta = -\frac{1}{2}$, $\theta = \pm\frac{2}{3}\pi$. (b) $A = 2\cdot\frac{1}{2}\int_{2\pi/3}^{\pi}[(1 + 4\cos\theta)^2 - 1]d\theta = 8\int_{2\pi/3}^{\pi}(\cos\theta + 2\cos^2\theta)d\theta$
$= 8\int_{2\pi/3}^{\pi}(\cos\theta + 1 + \cos 2\theta)d\theta = 8\left[\sin\theta + \theta + \frac{1}{2}\sin 2\theta\right]_{2\pi/3}^{\pi} = \frac{8}{3}\pi - 2\sqrt{3}$

42. $r = 1 - 3\sin\theta$ (loop) and $r = 1$. (a) $1 - 3\sin\theta = 1$, $-3\sin\theta = 0$, $\theta = 0$, π and $1 - 3\sin\theta = -1$, $3\sin\theta = 2$, $\sin\theta = \frac{2}{3}$, $\theta = \alpha = \sin^{-1}\frac{2}{3}$, $\theta = \pi - \sin^{-1}\frac{2}{3}$. (b) $\cos\alpha = \sqrt{1 - \frac{4}{9}} = \frac{1}{3}\sqrt{5}$, $\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{4}{9}\sqrt{5}$
$A = 2\cdot\frac{1}{2}\int_{\alpha}^{\pi/2}[(1 - 3\sin\theta)^2 - 1]d\theta = 3\int_{\alpha}^{\pi/2}(3\sin^2\theta - 2\sin\theta)d\theta = 3\int_{\alpha}^{\pi/2}(\frac{3}{2} - \frac{3}{2}\cos 2\theta - 2\sin\theta)d\theta$
$= 3\left[\frac{3}{2}\theta - \frac{3}{4}\sin 2\theta + 2\cos\theta\right]_{\alpha}^{\pi/2} = 3(\frac{3}{4}\pi - \frac{3}{2}\sin^{-1}\frac{2}{3} + \frac{1}{3}\sqrt{5} - \frac{2}{3}\sqrt{5}) = 3(\frac{3}{4}\pi - \frac{3}{2}\sin^{-1}\frac{2}{3} - \frac{1}{3}\sqrt{5})$

43. $r = 4\cos 2\theta$ (4 leaves) and $r = 2$. (a) $4\cos 2\theta = 2$; $\cos 2\theta = \frac{1}{2}$; $2\theta = \pm\frac{1}{3}\pi$, $\pm\frac{7}{3}\pi$; $\theta = \pm\frac{1}{6}\pi$, $\pm\frac{7}{6}\pi$ and $4\cos 2\theta = -2$; $\cos 2\theta = -\frac{1}{2}$; $2\theta = \pm\frac{2}{3}\pi$, $\pm\frac{4}{3}\pi$; $\theta = \pm\frac{1}{3}\pi$, $\pm\frac{2}{3}\pi$. $A = 8\cdot\frac{1}{2}\int_0^{\pi/6}[(4\cos 2\theta)^2 - 2^2]d\theta$
$= 16\int_0^{\pi/6}(1 + 2\cos 4\theta)d\theta = 16\left[\theta + \frac{1}{2}\sin 4\theta\right]_0^{\pi/6} = 16(\frac{1}{6}\pi + \frac{1}{4}\sqrt{3}) = \frac{8}{3}\pi + 4\sqrt{3}$

44. (a) Find the coordinates of all points of intersection of the circle $r = 2 \sin \theta$ and the rose $r = 2 \sin 2\theta$.
(b) Find the area of the region inside the circle and outside the rose.

▷ (a) From the figure we see that the curves intersect in the pole and two other points.
We use $\sin 2\theta = 2 \sin \theta \cos \theta$ and solve simultaneously.



$$4 \sin \theta \cos \theta = 2 \sin \theta$$
$$4 \sin \theta \cos \theta - 2 \sin \theta = 0$$
$$4 \sin \theta(\cos \theta - \tfrac{1}{2}) = 0$$

$\sin \theta = 0$, $\theta = 0, \pi$ give the pole
$\cos \theta = \frac{1}{2}$, $\theta = \frac{1}{3}\pi, \frac{2}{3}\pi$
Thus, the points of intersection are the pole, $(\sqrt{3}, \frac{1}{3}\pi)$ and $(\sqrt{3}, \frac{2}{3}\pi)$.
(b) By symmetry, we find

$$A = 2 \cdot \tfrac{1}{2} \int_{\pi/3}^{\pi/2} [(2 \sin \theta)^2 - (2 \sin 2\theta)^2] d\theta = 2 \int_{\pi/3}^{\pi/2} [(1 - \cos 2\theta) - (1 - \cos 4\theta)] d\theta$$
$$= 2\Big[\tfrac{1}{4} \sin 4\theta - \tfrac{1}{2} \sin 2\theta\Big]_{\pi/3}^{\pi/2} = \tfrac{1}{2}(\sin 2\pi - \sin \tfrac{4}{3}\pi) - (\sin \pi - \sin \tfrac{2}{3}\pi) = \tfrac{1}{2} \cdot \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{3} = \tfrac{3}{4}\sqrt{3}$$

▪ The area of the region is $\frac{3}{4}\sqrt{3}$ square units.

45. We obtain one-fourth of the area of the face of the bow tie when $\theta \in [0, \frac{1}{4}\pi]$.

$$A = 4 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(4 \cos 2w_i)\Delta_i\theta = 8 \int_0^{\pi/4} \cos 2\theta \, d\theta = 4 \sin 2\theta\Big]_0^{\pi/4} = 4$$

46. $9\pi = 2 \cdot \frac{1}{2} \int_0^{\pi} [a(1 - \cos \theta)]^2 d\theta = a^2 \int_0^{\pi} [1 - 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta)] d\theta = a^2\Big[\frac{3}{2}\theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta\Big]_0^{\pi} = \frac{3}{2}\pi a^2$
$a^2 = 6$, $a = \sqrt{6}$

47. $A = \frac{1}{2} \int_0^{2\pi} [a(\theta + 2\pi)^2 - (a\theta)^2] d\theta = \frac{1}{2}a^2 \int_0^{2\pi} (4\pi\theta + 4\pi^2) d\theta = \frac{1}{2}a^2\Big[2\pi\theta^2 + 4\pi^2\theta\Big]_0^{2\pi} = 8\pi^3 a^2$

48. Find the area of the region swept out by the radius vector of the spiral $r = a\theta$ during its third revolution that was not swept out during its second revolution.

▷ If the radius vector sweeps out a region of area A_3 square units during its third revolution when $\theta \in [4\pi, 6\pi]$ and A_2 square units during its second revolution when $\theta \in [2\pi, 4\pi]$, then

$$A = A_3 - A_2 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(aw_i)^2\Delta_i\theta - \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(aw_i)^2\Delta_i\theta = \frac{1}{2} a^2 \int_{4\pi}^{6\pi} \theta^2 d\theta - \frac{1}{2} a^2 \int_{2\pi}^{4\pi} \theta^2 d\theta$$
$$= \frac{1}{6} a^2\theta^3\Big]_{4\pi}^{6\pi} - \frac{1}{6} a^2\theta^3\Big]_{2\pi}^{4\pi} = \frac{1}{6} a^2(216\pi^3 - 64\pi^3 - 64\pi^3 + 8\pi^3) = 16a^2\pi^3$$

49. We are given the cardioid $r = a(1 + \cos\theta)$ and the circle $r = 2a\cos\theta$ of radius a. The circle is contained in the cardioid. We get top half of the cardioid if $\theta \in [0, \pi]$ and the top half of the circle if $\theta \in [0, \frac{1}{2}\pi]$.

$$A = 2\Big[\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}(a + a \cos w_i)^2\Delta_i\theta - \tfrac{1}{2}\pi a^2\Big] = a^2 \int_0^{\pi} (1 + 2 \cos \theta + \cos^2\theta) d\theta - \pi a^2$$
$$= a^2 \int_0^{\pi} \Big(\frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta\Big) d\theta - \pi a^2 = a^2\Big[\frac{3}{2}\theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta\Big]_0^{\pi} - \pi a^2 = \frac{3}{2}\pi a^2 - \pi a^2 = \frac{1}{2}\pi a^2$$

50. $[F'(\theta) \cos \theta - F(\theta) \sin \theta]^2 + [F'(\theta) \sin \theta + F(\theta) \cos \theta]^2$
$= [F'(\theta)^2\cos^2\theta - 2F'(\theta)F(\theta)\sin \theta \cos \theta + F(\theta)^2\sin^2\theta] + [F'(\theta)^2\sin^2\theta + 2F'(\theta)F(\theta)\sin \theta \cos \theta + F(\theta)^2\cos^2\theta]$
$= F'(\theta)^2(\cos^2\theta + \sin^2\theta) + F(\theta)^2(\sin^2\theta + \cos^2\theta) = F'(\theta)^2 + F(\theta)^2$

9.5 A UNIFIED TREATMENT OF CONIC SECTIONS AND POLAR EQUATIONS OF CONICS

9.5.1 Theorem A conic section can be defined as the set of all points P in a plane such that the ratio of the undirected distance of P from a fixed point to the undirected distance of P from a fixed line that does not contain the fixed point is a positive constant e. Furthermore, if $e = 1$, the conic is a parabola; if $0 < e < 1$, it is an ellipse; and if $e > 1$, it is a hyperbola.

The positive constant of Theorem 9.5.1 is called the *eccentricity* of the conic. (It has nothing to do with 2.718....) The fixed point is a focus of the conic, and the fixed line is called a *directrix* of the conic. A parabola has only one focus and one directrix. An ellipse and a hyperbola, the *central conics*, have two foci and two corresponding directrices. A circle has eccentricity 0.

9.5.2 Theorem The central conic having the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

where $a > 0$, has a focus at $(-ae, 0)$, whose corresponding directrix is $x = -a/e$, and a focus at $(ae, 0)$, whose corresponding directrix is $x = a/e$.

Note that if $0 < e < 1$, then $1 - e^2 > 0$ and the central conic of Theorem 9.5.2 is an ellipse. However, if $e > 1$, then $1 - e^2 < 0$, and the conic is a hyperbola. If x and y are interchanged in the equation of Theorem 9.5.2, then there will be a focus at $(0, -ae)$ with corresponding directrix $y = -a/e$, and a focus at $(0, ae)$ with corresponding directrix $y = a/e$.

Standard Form For a central conic in standard form, $e = \frac{c}{a}$ and the directrix is $\delta = \frac{a}{e} = \frac{a^2}{c}$ units from C.

Asymptotes Lines through center in two directions for which r is not defined.

For polar equations of the conics let e be the eccentricity of a conic and let d be the undirected distance from a focus to the corresponding directrix. If the focus is at the pole and the axis is the line $\theta = \theta_0$ then the vertex nearest the pole is in the direction θ, the equation of the conic is

$$r = \frac{ed}{1 + e\cos(\theta - \theta_0)}$$

and the equation of the directrix nearest the pole is $r\cos(\theta - \theta_0) = d$, obtained by setting the constant of the denominator to 0. We have the following four special cases.

θ	Equation of conic	Directrix nearest pole	Vertex nearest pole
0	$r = \frac{ed}{1 + e\cos\theta}$	$r\cos\theta = d$	To the right of the pole
$\frac{1}{2}\pi$	$r = \frac{ed}{1 + e\sin\theta}$	$r\sin\theta = d$	Above the pole
π	$r = \frac{ed}{1 - e\cos\theta}$	$r\cos\theta = -d$	To the left of the pole
$\frac{3}{2}\pi$	$r = \frac{ed}{1 - e\sin\theta}$	$r\sin\theta = -d$	Below the pole

Exercises 9.5

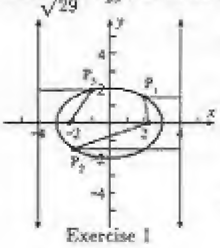
In Exercises 1–8, (a) find the eccentricity e, foci F, and directrices ℓ of the central conic. (b) Sketch the conic and show foci and directrices. Also choose a point P in each of three quadrants and draw the line segments to a focus and corresponding directrix. Observe that the ratio of their lengths is e.

▹ XA means the x axis is the principal axis; YA means the y axis is the principal axis.

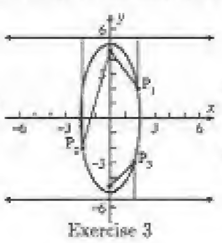
1. $4x^2 + 9y^2 = 36$ ▹ $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Ellipse, XA. $a = 3$, $b = 2$, $c = \sqrt{9-4} = \sqrt{5}$. $e = \frac{\sqrt{5}}{3}$, $\delta = \frac{9}{\sqrt{5}} = \frac{9}{5}\sqrt{5}$. Foci $(\pm\sqrt{5}, 0)$, directrices: $x = \pm\frac{9}{5}\sqrt{5}$

3. $25x^2 + 4y^2 = 100$ ▹ $\frac{x^2}{4} + \frac{y^2}{25} = 1$. Ellipse, YA. $a = 5$, $b = 2$, $c = \sqrt{25-4} = \sqrt{21}$. $e = \frac{\sqrt{21}}{5}$, $\delta = \frac{25}{\sqrt{21}} = \frac{25}{21}\sqrt{21}$. Foci: $(0, \pm\sqrt{21})$, directrices: $y = \pm\frac{25}{21}\sqrt{21}$

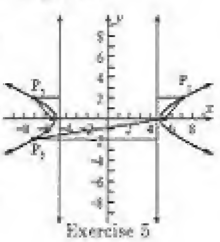
5. $4x^2 - 25y^2 = 100$ ▹ $\frac{x^2}{25} - \frac{y^2}{4} = 1$. Hyperbola, XA. $a = 5$, $b = 2$, $c = \sqrt{25+4} = \sqrt{29}$. $e = \frac{\sqrt{29}}{5}$, $\delta = \frac{25}{\sqrt{29}} = \frac{25}{29}\sqrt{29}$. Foci: $(\pm\sqrt{29}, 0)$, directrices: $x = \pm\frac{25}{29}\sqrt{29}$



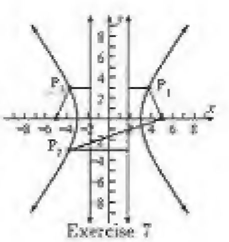
Exercise 1



Exercise 3



Exercise 5



Exercise 7

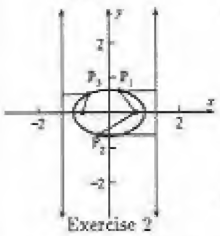
7. $16x^2 - 9y^2 = 144$ ▹ $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Hyperbola, XA. $a = 3$, $b = 4$, $c = \sqrt{9+16} = 5$. $e = \frac{5}{3}$. $\delta = \frac{9}{5}$.
Foci: $(\pm 5, 0)$, directrices: $x = \pm\frac{9}{5}$

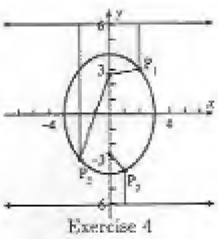
2. $4x^2 + 9y^2 = 4$ ▹ $\frac{x^2}{1} + \frac{y^2}{\frac{4}{9}} = 1$. Ellipse, XA. $a = 1$, $b = \frac{2}{3}$, $c = \sqrt{1 - \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{1}{3}\sqrt{5}$. $e = \frac{1}{3}\sqrt{5}$.
$\delta = \frac{1}{\sqrt{5/3}} = \frac{3}{5}\sqrt{5}$. Foci: $(\pm\frac{1}{3}\sqrt{5}, 0)$, directrices: $x = \pm\frac{3}{5}\sqrt{5}$

4. $16x^2 + 9y^2 = 144$ ▹ $\frac{x^2}{9} + \frac{y^2}{16} = 1$. Ellipse, YA. $a = 4$, $b = 2$, $c = \sqrt{16 - 9} = \sqrt{7}$. $e = \frac{1}{4}\sqrt{7}$.
$\delta = \frac{16}{\sqrt{7}} = \frac{16}{7}\sqrt{7}$. Foci: $(0, \pm\sqrt{7})$, directrices: $y = \pm\frac{16}{7}\sqrt{7}$

6. $x^2 - 9y^2 = 9$ ▹ $\frac{x^2}{9} - \frac{y^2}{1} = 1$. Hyperbola, XA. $a = 3$, $b = 1$, $c = \sqrt{9+1} = \sqrt{10}$. $e = \frac{1}{3}\sqrt{10}$.
$\delta = \frac{9}{\sqrt{10}} = \frac{9}{10}\sqrt{10}$. Foci: $(\pm\sqrt{10}, 0)$, directrices: $x = \pm\frac{9}{10}\sqrt{10}$

8. $4y^2 - x^2 = 16$ ▹ $\frac{y^2}{4} - \frac{x^2}{16} = 1$. Hyperbola, YA. $a = 2$, $b = 4$, $c = \sqrt{4+16} = 2\sqrt{5}$. $e = \frac{2\sqrt{5}}{2} = \sqrt{5}$.
$\delta = \frac{4}{2\sqrt{5}} = \frac{2}{5}\sqrt{5}$. Foci: $(0, \pm 2\sqrt{5})$, directrices: $y = \pm\frac{2}{5}\sqrt{5}$.


Exercise 2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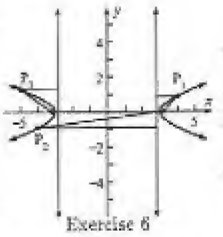
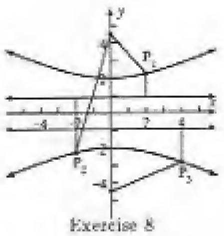

Exercise 6


Exercise 8

In Exercises 9 and 10, the conic has a focus F at the pole. Identify the conic.

9. (a) $r = \frac{3}{1 - \cos\theta}$ ▹ $e = 1$: parabola (b) $r = \frac{6}{4 + 5\sin\theta}$ ▹ $e = \frac{5}{4}$: hyperbola

(c) $r = \frac{5}{4 - \cos\theta}$ ▹ $e = \frac{1}{4}$: ellipse (d) $r = \frac{4}{1 + \sin\pi}$ ▹ circle of radius 4

10. (a) $r = \frac{1}{1 - \sin\theta}$ ▹ $e = 1$: parabola (b) $r = \frac{2}{3 + \sin\theta}$ ▹ $e = \frac{1}{3}$: ellipse

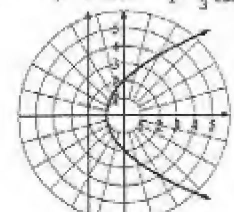
(c) $r = \frac{3}{2 + 4\cos\theta}$ ▹ $e = 2$: hyperbola (d) $5 = \frac{5}{1 - \cos\pi}$ ▹ circle of radius $\frac{5}{2}$

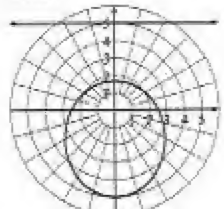
In Exercises 11–22, the conic has a focus F at the pole. Find (a) the eccentricity e, (b) the type of conic, (c) an equation of the directrix ℓ of F. (d) Sketch the conic and check by plotting.

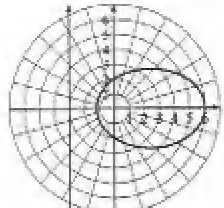
11. $r = \frac{2}{1 - \cos\theta} = \frac{ed}{1 - e\cos\theta}$ with $e = 1$ and $d = \frac{2}{1} = 2$. A parabola. ℓ is to the left of F: $r\cos\theta = -2$

13. $r = \frac{5}{2 + \sin\theta} = \frac{\frac{5}{2}}{1 + \frac{1}{2}\sin\theta} = \frac{ed}{1 + e\sin\theta}$ with $e = \frac{1}{2}$ and $d = \frac{5}{1} = 5$. An ellipse. ℓ is above F: $r\sin\theta = 5$

15. $r = \frac{6}{3 - 2\cos\theta} = \frac{2}{1 - \frac{2}{3}\cos\theta} = \frac{ed}{1 - e\cos\theta}$ with $e = \frac{2}{3}$ and $d = \frac{6}{2} = 3$. Ellipse. ℓ is left of F: $r\cos\theta = -3$


Exercise 11

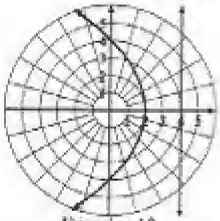

Exercise 13


Exercise 15

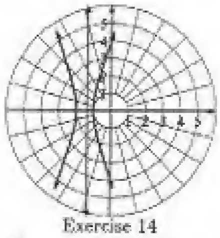
12. $r = \frac{4}{1+\cos\theta} = \frac{ed}{1+e\cos\theta}$ with $e = 1$ and $d = \frac{4}{1} = 4$. A parabola. ℓ is to the right of F: $r\cos\theta = 4$

14. $r = \frac{4}{1-3\cos\theta} = \frac{ed}{1-e\cos\theta}$ with $e = 3$ and $d = \frac{4}{3}$. A hyperbola. ℓ is to the left of F: $r\cos\theta = -\frac{4}{3}$

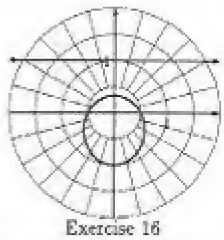
16. $r = \frac{1}{2+\sin\theta} = \frac{\frac{1}{2}}{1+\frac{1}{2}\sin\theta} = \frac{ed}{1+e\sin\theta}$ with $e = \frac{1}{2}$ and $d = 1$. An ellipse. ℓ is above F: $r\sin\theta = 1$



Exercise 12



Exercise 14

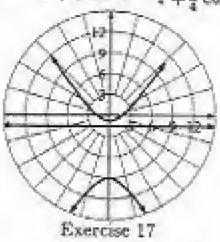


Exercise 16

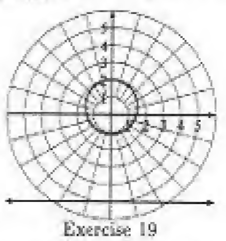
17. $r = \frac{9}{5-6\sin\theta} = \frac{\frac{9}{5}}{1-\frac{6}{5}\sin\theta} = \frac{ed}{1-e\sin\theta}$ with $e = \frac{6}{5}$ and $d = \frac{9}{6} = \frac{3}{2}$. Hyperbola. ℓ is below F: $r\sin\theta = -\frac{3}{2}$

19. $r = \frac{10}{7-2\sin\theta} = \frac{\frac{10}{7}}{1-\frac{2}{7}\sin\theta} = \frac{ed}{1-e\sin\theta}$ with $e = \frac{2}{7}$ and $d = \frac{10}{2} = 5$. Ellipse. ℓ is below F: $r\sin\theta = -5$

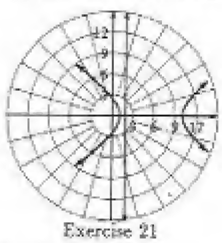
21. $r = \frac{10}{4+5\cos\theta} = \frac{\frac{5}{2}}{1+\frac{5}{4}\cos\theta} = \frac{ed}{1+e\cos\theta}$ with $e = \frac{5}{4}$ and $d = \frac{10}{5} = 2$. Hyperbola. ℓ is right of F: $r\cos\theta = 2$



Exercise 17



Exercise 19

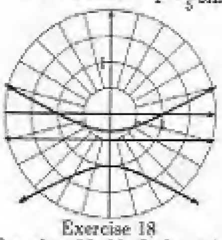


Exercise 21

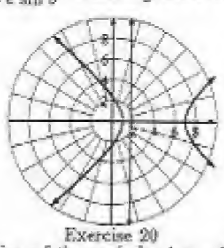
18. $r = \frac{1}{1-2\sin\theta} = \frac{ed}{1-e\sin\theta}$ with $e = 2$ and $d = \frac{1}{2}$. Hyperbola. ℓ is below F: $r\sin\theta = -\frac{1}{2}$

20. $r = \frac{7}{3+4\cos\theta} = \frac{\frac{7}{3}}{1+\frac{4}{3}\cos\theta} = \frac{ed}{1+e\cos\theta}$ with $e = \frac{4}{3}$ and $d = \frac{7}{4}$. Hyperbola. ℓ is right of F: $r\cos\theta = \frac{7}{4}$

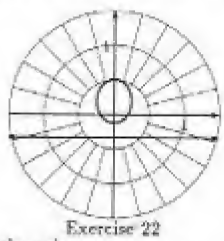
22. $r = \frac{1}{5-3\sin\theta} = \frac{\frac{1}{5}}{1-\frac{3}{5}\sin\theta} = \frac{ed}{1-e\sin\theta}$ with $e = \frac{3}{5}$ and $d = \frac{1}{3}$. Ellipse. ℓ is below F: $r\sin\theta = -\frac{1}{3}$



Exercise 18



Exercise 20



Exercise 22

In Exercises 23–30, find a polar equation of the conic having a focus at the pole.

23. Parabola; vertex at $(4, \frac{3}{2}\pi)$. ▸ $e = 1$, ℓ is horizontal $4 + 4 = 8$ units below F. $r = \frac{8}{1-\sin\theta}$

24. Ellipse; $e = \frac{1}{2}$; the corresponding vertex at $(4, \pi)$.

▸ ℓ is vertical, $4 + 4/\frac{1}{2} = 12$ units left of F. $r = \frac{6}{1-\frac{1}{2}\cos\theta} = \frac{12}{2-\cos\theta}$

25. Hyperbola; $e = \frac{4}{3}$; $r \cos\theta = 9$ is the corresponding directrix.

▷ ℓ is vertical 9 units right of pole. $r = \dfrac{\frac{4}{3}\cdot 9}{1 + \frac{4}{3}\cos\theta} = \dfrac{36}{3 + 4\cos\theta}$

26. Hyperbola; vertices at $\left(1, \frac{1}{2}\pi\right)$ and $\left(3, \frac{1}{2}\pi\right) = \left(-3, \frac{3}{2}\pi\right)$.

▷ ℓ is horizontal, above F. $\theta = \frac{1}{2}\pi$: $1 = \dfrac{ed}{1+e}$. A: $1 + e = ed$. $\theta = \frac{3}{2}\pi$: $-3 = \dfrac{ed}{1-e}$. B: $1 - e = -\frac{1}{3}ed$

A + 3B: $4 - 2e = 0$, $e = 2$, $ed = 3$. $r = \dfrac{3}{1 + 2\sin\theta}$

27. Ellipse; vertices at $(3, 0)$ and $(1, \pi)$.

ℓ is vertical, left of F. $r = \dfrac{ed}{1 - e\cos\theta}$. $\theta = 0$: $3 = \dfrac{ed}{1-e}$. A: $1 - e = \frac{1}{3}ed$. $\theta = \pi$: $1 = \dfrac{ed}{1+e}$. B: $1 + e = ed$

B − 3A: $-2 + 4e = e$, $e = \frac{1}{2}$, $ed = \frac{3}{2}$. $r = \dfrac{\frac{3}{2}}{1 - \frac{1}{2}\cos\theta} = \dfrac{3}{2 - \cos\theta}$

28. Parabola; vertex at $\left(6, \frac{1}{2}\pi\right)$. ▷ $e = 1$, ℓ is horizontal $6 + 6 = 12$ units above F. $r = \dfrac{12}{1 + \sin\theta}$

29. (a) $e = 3$, ℓ is left of the pole and $\left(2, \frac{4}{3}\pi\right)$ is on the hyperbola. (b) Write an equation of ℓ.

▷ (a) $r = \dfrac{3d}{1 - 3\cos\theta}$. $2 = \dfrac{3d}{1 - 3\cos\frac{4}{3}\pi} = \dfrac{3d}{1 + \frac{3}{2}}$. $3d = 5$, $d = \frac{5}{3}$. $r = \dfrac{5}{1 - 3\cos\theta}$ (b) $r\cos\theta = -\frac{5}{3}$

30. (a) $e = 3$, ℓ is $r\sin\theta = 3$. (b) Find equations of the lines through the pole parallel to the asymptotes.

▷ (a) $d = 3$. $r = \dfrac{3(3)}{1 + 3\sin\theta} = \dfrac{9}{1 + 3\sin\theta}$ (b) $\sin\theta = -\frac{1}{3}$, $\theta = -0.34$ and $\theta = 3.48$

31. A units is the area inside the ellipse $r = 6/(2 - \sin\theta)$ and above the parabola $r = 3/(1 + \sin\theta)$. The two curves intersect at the points $(3, 0)$ and $(3, \pi)$ and each is symmetric with respect to the $\frac{1}{2}\pi$ axis. See the figure for Exercise 32. Then

$$A = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\frac{1}{2}\left[\left(\frac{6}{2 - \sin w_i}\right)^2 - \left(\frac{3}{1 + \sin w_i}\right)^2\right]\Delta_i\theta = \int_0^{\pi/2}\left[\left(\frac{6}{2 - \sin\theta}\right)^2 - \left(\frac{3}{1 + \sin\theta}\right)^2\right]d\theta$$

Because cosines are easier to integrate, let $\theta = \frac{1}{2}\pi - x$, $\sin\theta = \cos x$. Therefore

$A = \displaystyle\int_0^{\pi/2}\left[\frac{36}{(2 - \cos x)^2} - \frac{9}{(1 + \cos x)^2}\right]dx$. Now let $z = \tan\frac{1}{2}\theta$ so $\cos\theta = \dfrac{1 - z^2}{1 + z^2}$ and $d\theta = \dfrac{2\,dz}{1 + z^2}$. Thus

$$A = \int_0^1\left[\frac{36\cdot\frac{2}{1+z^2}}{\left(2 - \frac{1 - z^2}{1 + z^2}\right)^2} - \frac{9\cdot\frac{2}{1+z^2}}{\left(1 + \frac{1 - z^2}{1 + z^2}\right)^2}\right]dz = 72\int_0^1\frac{1 + z^2}{(3z^2 + 1)^2}\,dz - \frac{9}{2}\int_0^1(1 + z^2)dz = I - J$$

In I, let $z = \frac{1}{3}\sqrt{3}\tan u$. Then $dz = \frac{1}{3}\sqrt{3}\sec^2 u\,du$, and $u = \frac{1}{3}\pi$ when $z = 1$. Hence

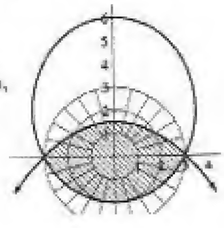
$$I = 72\int_0^{\pi/3}\frac{1 + \frac{1}{3}\tan^2 u}{\sec^4 u}\left(\frac{1}{3}\sqrt{3}\sec^2 u\,du\right) = 24\sqrt{3}\int_0^{\pi/3}\left(\cos^2 u + \frac{1}{3}\sin^2 u\right)du$$

$$= 24\sqrt{3}\int_0^{\pi/3}\left[\frac{1 + \cos 2u}{2} + \frac{1 - \cos 2u}{6}\right]du = 4\sqrt{3}\int_0^{\pi/3}(4 + 2\cos 2u)du = 4\sqrt{3}\Big[4u + \sin 2u\Big]_0^{\pi/3} = \frac{16}{3}\sqrt{3}\pi + 6$$

$J = \frac{9}{2}\Big[z + \frac{1}{3}z^3\Big]_0^1 = \frac{9}{2}\cdot\frac{4}{3} = 6$. Therefore $A = \left(\frac{16}{3}\sqrt{3}\pi + 6\right) - 6 = \frac{16}{3}\sqrt{3}\pi$.

32. Find the area of the region inside the ellipse $r = 6/(2 - \sin\theta)$ and below the parabola $r = 3/(1 + \sin\theta)$.

▷ The two curves intersect at the points $(3, 0)$ and $(3, \pi)$ and each is symmetric with respect to the $\frac{1}{2}\pi$ axis. The figure shows the region R. Let R_1 be that part of the region R in the fourth quadrant. Then R_1 is bounded by the ellipse $r = 6/(2 - \sin\theta)$ and the lines $\theta = -\frac{1}{2}\pi$ and $\theta = 0$. Thus,



$$A_1 = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\frac{1}{2}\left[\frac{6}{2 - \sin w_i}\right]^2\Delta_i\theta = \frac{1}{2}\int_{-\pi/2}^{0}\frac{36\,d\theta}{(2 - \sin\theta)^2}$$

Because cosines lead to simpler integrals, we let $\theta = x - \frac{1}{2}\pi$ so $\sin\theta = -\cos x$. Thus,

$$A_1 = 18\int_0^{\pi/2}\frac{dx}{(2 + \cos x)^2}$$

Now we let $z = \tan\frac{1}{2}\theta$, $\cos\theta = \dfrac{1 - z^2}{1 + z^2}$ and $d\theta = \dfrac{2\,dz}{1 + z^2}$. Thus

$$A_1 = 18\int_0^1\frac{\frac{2\,dz}{1 + z^2}}{\left(2 + \frac{1 - z^2}{1 + z^2}\right)^2} = 36\int_0^1\frac{1 + z^2}{(3 + z^2)^2}\,dz$$

We let $z = \sqrt{3}\tan t$ and $dz = \sqrt{3}\sec^2 t\,dt$; when $z = 0$, $t = 0$; when $z = 1$, $t = \pi/6$. Thus

$$
\begin{aligned}
A_1 &= 36\int_0^{\pi/6} \frac{1 + 3\tan^2 t}{9\sec^4 t}\sqrt{3}\sec^2 dt = 4\sqrt{3}\int_0^{\pi/6} (\cos^2 t + 3\sin^2 t)\,dt \\
&= 2\sqrt{3}\int_0^{\pi/6} [(1 + \cos 2t) + 3(1 - \cos 2t)]\,dt = 2\sqrt{3}\int_0^{\pi/6} (4 - 2\cos 2t)\,dt \\
&= 2\sqrt{3}\Big[4t - \sin 2t\Big]_0^{\pi/6} = 2\sqrt{3}[\tfrac{2}{3}\pi - \tfrac{1}{2}\sqrt{3}] = \tfrac{4}{3}\sqrt{3}\pi - 3 \qquad (1)
\end{aligned}
$$

Next, let R_2 be the part of the region R in first quadrant. Then R_2 is bounded by the parabola $r = 3/(1 + \sin\theta)$ and the lines $\theta = 0$ and $\theta = \frac{1}{2}\pi$. Thus,

$$A_2 = \frac{1}{2}\int_0^{\pi/2} \frac{9\,d\theta}{(1 + \sin\theta)^2} \qquad (2)$$

We let $\theta = \frac{1}{2}\pi - t$. Then

$$(1 + \sin\theta)^2 = [1 + \sin(\tfrac{1}{2}\pi - t)]^2 = (1 + \cos t)^2 = (2\cos^2 \tfrac{1}{2}t)^2 = 4\cos^4 \tfrac{1}{2}t$$

Furthermore, $d\theta = -dt$; $t = \frac{1}{2}\pi$ when $\theta = 0$; and $t = 0$ when $\theta = \frac{1}{2}\pi$. Thus, from (2) we have

$$A_2 = \frac{1}{2}\int_{\pi/2}^0 \frac{-9\,dt}{4\cos^4 \frac{1}{2}t} = \frac{9}{8}\int_0^{\pi/2} \sec^4 \tfrac{1}{2}t\,dt = \frac{9}{4}\int_0^{\pi/2} (1 + \tan^2 \tfrac{1}{2}t)\sec^2 \tfrac{1}{2}t(\tfrac{1}{2}dt) = \tfrac{9}{4}\Big[\tan\tfrac{1}{2}t + \tfrac{1}{3}\tan^3 \tfrac{1}{2}t\Big]_0^{\pi/2} = 3 \qquad (3)$$

Because the region R is symmetric with respect to the line $\theta = \frac{1}{2}\pi$, then the area of R is given by $A = 2(A_1 + A_2)$. Substituting from (1) and (3), we get

$$A = 2[(-3 + \tfrac{4}{3}\sqrt{3}\pi) + 3] = \tfrac{8}{3}\sqrt{3}\pi$$

Thus, the area of region R is $\frac{8}{3}\sqrt{3}\pi$ square units.

33. Derive the third form of the polar equation. ▹ Rotate π: $r = \dfrac{ed}{1 - e\cos(\theta + \pi)} = \dfrac{ed}{1 + e\cos\theta}$

34. Derive the fourth form of the polar equation. ▹ Rotate $\frac{3}{2}\pi$: $r = \dfrac{ed}{1 - e\cos(\theta + \frac{3}{2}\pi)} = \dfrac{ed}{1 + e\sin\theta}$

35. Derive the second form of the polar equation. ▹ Rotate $\frac{1}{2}\pi$: $r = \dfrac{ed}{1 - e\cos(\theta - \frac{1}{2}\pi)} = \dfrac{ed}{1 - e\sin\theta}$

36. Show that the equation $r = k\csc^2 \frac{1}{2}\theta$, where k is a constant, is a polar equation of a parabola.

▹ $r = \dfrac{k}{\sin^2 \frac{1}{2}\theta} = \dfrac{k}{\frac{1}{2}(1 - \cos\theta)} = \dfrac{2k}{1 - \cos\theta}$, a parabola

37. The orbit of a comet is a parabola with the sun as its focus. When the comet is 80 million miles from the sun, the line from the sun makes an angle of $\frac{1}{3}\pi$ with axis of the orbit. (a) Find an equation of the orbit. (b) How close does it come to the sun?

▹ (a) Choose the focus at the pole and the polar axis along the axis of the parabola. Let the unit be 1 million miles. Because $e = 1$, then $r = \dfrac{d}{1 - \cos\theta}$. Set $\theta = \frac{1}{3}\pi$: $80 = \dfrac{d}{1 - \frac{1}{2}} = 2d$, $d = 40$. $r = \dfrac{40}{1 - \cos\theta}$

(b) r is least when $\cos\theta = -1$. Then $r = \frac{40}{2} = 20$. The comet comes within 20 million miles of the sun.

38. Let $z = \tan\frac{1}{2}\theta$. $\bar{r} = \dfrac{1}{\pi}\displaystyle\int_{\theta=0}^{\pi} \frac{p\,d\theta}{1 + e\cos\theta} = \frac{p}{\pi}\int_{z=0}^{\infty} \frac{2\,dz/(1 + z^2)}{1 + e(1 - z^2)/(1 + z^2)} = \frac{p}{\pi}\int_0^{\infty} \frac{2\,dz}{(1 + e) + (1 - e)z^2}$

$$= \frac{2p}{(1 - e)\pi}\int_0^{\infty} \frac{dz}{z^2 + (1 + e)/(1 - e)} = \frac{2p}{(1 - e)\pi}\sqrt{\frac{1 - e}{1 + e}}\tan^{-1}\sqrt{\frac{1 - e}{1 + e}}z\,\Bigg]_0^{\infty} = \frac{2p}{\pi\sqrt{1 - e^2}}\cdot\frac{\pi}{2} = \frac{p}{\sqrt{1 - e^2}}$$

39. The orbit of a satellite is an ellipse with the center of the earth as a focus and $e = \frac{1}{3}$. The closest it gets to the earth is 300 mi. Find the farthest it gets from the earth. Assume the earth's radius is 4000 mi.

▹ From the center, $300 + 4000 = 4300 = a - c = 3c - c = 2c$. $a + c = (a - c) + 2c = 4300 + 4300 = 8600$, $8600 - 4000 = 4600$. The greatest distance from the surface is 4600 miles.

40. Show that the point F mentioned in the proof of Theorem 9.5.1 is one of the foci when the conic is either an ellipse or a hyperbola.

▷ We are asked to show that if $h = \frac{e^2 d}{1-e^2}$ and $a = \frac{ed}{1-e^2}$, then the origin is a focus of $\frac{(x-h)^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$. The focus is c units from the center $(h, 0)$ where $c = ae = \frac{ed}{1-e^2}e = h$.

41. Find the angle α between the asymptotes of a hyperbola in terms of its eccentricity e.

▷ From the auxiliary rectangle we have $\tan \frac{1}{2}\alpha = \frac{b}{a} = \frac{\sqrt{c^2 - a^2}}{a} = \frac{\sqrt{a^2e^2 - a^2}}{a} = \sqrt{e^2 - 1}$.

42. Describe how the shape of a conic changes as the eccentricity e increases from 0 to 2.

▷ Start with a circle of diameter 1" and keep the left edge fixed. Because $a = b/\sqrt{1-e^2}$, when $e = .01$, the major axis is 1.00005; when $e = 0.10$, it is 1.005"; when $e = 0.5$ it is 1.15", a noticeable elongation; when $e = .99$ it is 7.1". As e approaches 1, the right edge moves off the paper and when $e = 1$, there is no right edge: we have a parabola. When $e = 1.01$ we have a hyperbola the angle between whose asymptotes is $\alpha = 2\tan^{-1}\sqrt{0.0201} = 16.1°$. When $e = \sqrt{2}$, $\alpha = 2\tan^{-1} 1 = 90°$ and we have a rectangular (equilateral) hyperbola. When $e = 2$, α has increased to $2\tan^{-1}\sqrt{3} = 120°$.

Miscellaneous Exercises for Chapter 9

In Exercises 1–4, (a) sketch the parametric equations; check by plotting. (b) Find a cartesian equation.

1. $x = 2 - t$, $y = 2t$

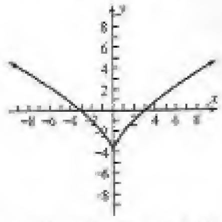
▷ $2x + y = 4 - 2t + 2t = 4$

2. $x = t^2$, $y = t^3 + 1$

▷ $t = (y-1)^{1/3}$, $x = (y-1)^{2/3}$

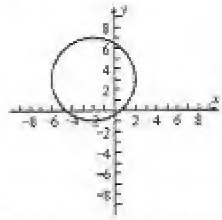
3. $x = 2t^3$, $y = 3t^2 - 4$

▷ $t = (\frac{1}{2}x)^{1/3}$, $y = 3(\frac{1}{2}x)^{2/3} - 4$



4. $x = 4\cos t - 2$, $y = 4\sin t + 3$

▷ $(x+2)^2 + (y-3)^2 = 16\cos^2 t + 16\sin^2 t = 16$



In Exercises 5 and 6, find dy/dx and d^2y/dx^2 without eliminating the parameter.

5. $x = 9t^2 - 1$ and $y = 3t + 1$. Therefore $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{18t} = \frac{1}{6t}$ and $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{-\frac{1}{6t^2}}{18t} = -\frac{1}{108t}$

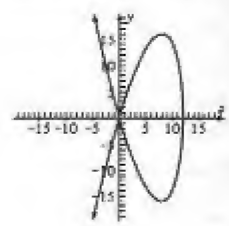
6. $x = e^{2t}$ and $y = e^{-3t}$. Hence $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3e^{-3t}}{2e^{2t}} = -\frac{3}{2}e^{-5t}$ and $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\frac{15}{2}e^{-5t}}{2e^{2t}} = \frac{15}{4}e^{-7t}$

In Exercises 7 and 8, find the horizontal and vertical tangent lines. Sketch the graph of the parametric equations.

7. $x = 12 - t^2$ and $y = 12t - t^3$. $\frac{dx}{dt} = -2t$ and $\frac{dy}{dt} = 12 - 3t^2$.

Set $\frac{dy}{dt} = 0$: $12 - 3t^2 = 0$; $t = \pm 2$; $y = \pm 16$ are the horizontal tangent lines.

Set $\frac{dx}{dt} = 0$: $-2t = 0$; $t = 0$; $x = 12$ is the vertical tangent line.



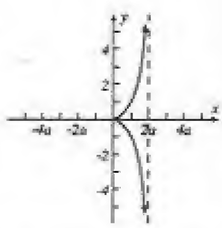
8. $x = \frac{2at^2}{1+t^2}$, $y = \frac{2at^3}{1+t^2}$, $a > 0$ (the cissoid of Diocles)

▸ Differentiating with respect to t, we obtain

$$\frac{dx}{dt} = \frac{4at}{(1+t^2)^2} \quad \text{and} \quad \frac{dy}{dt} = \frac{2at^2(t^2+3)}{(1+t^2)^2}$$

Thus,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2at^2(t^2+3)}{(1+t^2)^2}}{\frac{4at}{(1+t^2)^2}} = \tfrac{1}{2}t(t^2+3),\ t \neq 0$$



Although $\frac{dy}{dx}$ is not defined for $t = 0$, because $\lim_{t \to 0} \frac{dy}{dx} = 0$, there is a horizontal tangent at $t = 0$. Moreover, $0 \leq \frac{2at^2}{1+t^2} < 2a$ for all t. Thus, x is in the interval $(0, 2a)$. Furthermore,

$$\lim_{t \to +\infty} x = \lim_{t \to +\infty} \frac{2at^2}{1+t^2} = 2a \quad \text{and} \quad \lim_{t \to +\infty} y = \lim_{t \to +\infty} \frac{2at^3}{1+t^2} = +\infty$$

Therefore, the line $x = 2a$ is a vertical asymptote. If t is replaced by $-t$, then x is unchanged and y is replaced by $-y$. Thus, for every point (x, y) on the curve we have the point $(x, -y)$, and hence the graph is symmetric with respect to the x axis. We use the above information to sketch the graph, as shown in the figure.

In Exercises 9 and 10, locate the point in polar coordinates. Rename twice, once with r, once with $-r$.

9. (a) $(2, \frac{3}{4}\pi)$, $(-2, \frac{7}{4}\pi)$, $(2, \frac{11}{4}\pi)$ (b) $(-3, \frac{7}{6}\pi)$, $(3, \frac{13}{6}\pi)$, $(-3, \frac{19}{6}\pi)$

10. (a) $(4, -\frac{1}{3}\pi)$, $(-4, \frac{2}{3}\pi)$, $(4, \frac{5}{3}\pi)$ (b) $(-1, \frac{1}{4}\pi)$, $(1, \frac{5}{4}\pi)$, $(-1, \frac{9}{4}\pi)$

In Exercises 11 and 12, convert from polar coordinates to rectangular cartesian coordinates.

11. (a) $(1, \frac{1}{2}\pi) = (0, 1)^c$ (b) $(2, -\frac{1}{3}\pi) = (2\cos(-\frac{1}{3}\pi), 2\sin(-\frac{1}{3}\pi))^c = (2 \cdot \frac{1}{2}, 2(-\frac{1}{2}\sqrt{3}))^c = (1, -\sqrt{3})^c$

(c) $(4, \frac{5}{4}\pi) = (4\cos\frac{5}{4}\pi, 4\sin\frac{5}{4}\pi)^c = (4(-\frac{1}{2}\sqrt{2}), 4(-\frac{1}{2}\sqrt{2}))^c = (-2\sqrt{2}, -2\sqrt{2})^c$

(d) $(-3, \frac{1}{6}\pi) = (-3\cos\frac{1}{6}\pi, -3\sin\frac{1}{6}\pi)^c = (-3 \cdot \frac{1}{2}\sqrt{3}, -3 \cdot \frac{1}{2})^c = (-\frac{3}{2}\sqrt{3}, -\frac{3}{2})^c$

12. (a) $(5, \pi) = (-5, 0)^c$ (b) $(-2, \frac{5}{6}\pi) = (-2\cos\frac{5}{6}\pi, -2\sin\frac{5}{6}\pi)^c = (-2 \cdot -\frac{1}{2}\sqrt{3}, -2 \cdot \frac{1}{2})^c = (\sqrt{3}, -1)^c$

(c) $(-\sqrt{2}, \frac{1}{4}\pi) = (-\sqrt{2}\cos\frac{1}{4}\pi, -\sqrt{2}\sin\frac{1}{4}\pi)^c = (-\sqrt{2} \cdot \frac{1}{2}\sqrt{2}, -\sqrt{2} \cdot \frac{1}{2}\sqrt{2})^c = (-1, -1)^c$

(d) $(1, \frac{4}{3}\pi) = (\cos\frac{4}{3}\pi, \sin\frac{4}{3}\pi)^c = (-\frac{1}{2}, -\frac{1}{2}\sqrt{3})^c$

In Exercises 13 and 14, convert from rectangular cartesian to polar coordinates with $r > 0$ and $0 \leq \theta < 2\pi$.

13. (a) $(-4, 4)^c$. $r = 4\sqrt{2}$. Q2, $\theta = \tan^{-1}(-4/4) + \pi = \frac{3}{4}\pi$. $(4\sqrt{2}, \frac{3}{4}\pi)$

(b) $(1, -\sqrt{3})^c$. $r = \sqrt{1+3} = 2$. Q4, $\theta = \tan(-\sqrt{3}/1) + 2\pi = \frac{5}{3}\pi$. $(2, \frac{5}{3}\pi)$

(c) $(0, 6)^c = (6, \frac{1}{2}\pi)$ (d) $(-2\sqrt{3}, -2)^c$. $r = \sqrt{12+4} = 4$. Q3, $\theta = \tan(-2/-2\sqrt{3}) + \pi = \frac{7}{6}\pi$. $(4, \frac{7}{6}\pi)$

14. (a) $(-4, 0)^c = (4, \pi)$ (b) $(\sqrt{3}, 1)^c$. $r = \sqrt{3+1} = 2$. Q1, $\theta = \tan^{-1}(1/\sqrt{3}) = \frac{1}{6}\pi$. $(2, \frac{1}{6}\pi)$

(c) $(-2, -2)^c$. $r = 2\sqrt{2}$. Q3. $\theta = \tan(-2/-2) + \pi = \frac{5}{4}\pi$. $(2\sqrt{2}, \frac{5}{4}\pi)$

(d) $(3, -3\sqrt{3})^c$. $r = \sqrt{9+27} = 6$. Q4, $\theta = \tan^{-1}(-3\sqrt{3}/3) + 2\pi = \frac{5}{3}\pi$. $(6, \frac{5}{3}\pi)$

In Exercises 15–18, find a polar equation of the graph having the given cartesian equation.

15. $4x^2 - 9y^2 = 36$ ▸ $4(r\cos\theta)^2 - 9(r\sin\theta)^2 = 36$. $r^2(4\cos^2\theta - 9\sin^2\theta) = 36$

16. $2xy = 1$ ▸ $2(r\cos\theta)(r\cos\theta) = r^2(2\sin\theta\cos\theta) = 1$, $r^2\sin 2\theta = 1$

17. $x^2 + y^2 - 9x + 8y = 0$ ▸ $r^2 - 9r\cos\theta + 8r\sin\theta = 0$. $r = 9\cos\theta - 8\sin\theta$ (pole at $\tan^{-1}\frac{9}{8}$)

18. $y^4 = x^2(a^2 - y^2)$
▹ $y^2(y^2 + x^2) = a^2x^2$, $r^2\sin^2\theta \cdot r^2 = a^2r^2\cos^2\theta$, $r^2\sin^2\theta = a^2\cos^2\theta$ (pole at $\frac{1}{2}\pi$), $r = a\cot\theta$

In Exercises 19–22, find a cartesian equation of the graph having the given polar equation.

19. $r^2\sin 2\theta = 4$ ▹ $2(r\sin\theta)(r\cos\theta) = 4$, $2xy = 4$, $xy = 2$

20. $r(1 - \cos\theta) = 2$ ▹ $r = 2 + r\cos\theta$, $r^2 = (2 + r\cos\theta)^2$, $x^2 + y^2 = (2 + x)^2$, $y^2 = 4x + 4$

21. $r^2 = \sin^2\theta$ ▹ $(r^2)^2 = (r\sin\theta)^2$, $(x^2 + y^2)^2 = y^2$. $x^4 + 2x^2y^2 + y^4 - y^2 = 0$

22. $r = a\tan^2\theta$
▹ $r\cos^2\theta = a\sin^2\theta$, $r \cdot r^2\cos^2\theta = ar^2\sin^2\theta$ (pole at 0), $r^2(r\cos\theta)^4 = a^2(r\sin\theta)^4$, $(x^2 + y^2)x^4 = a^2y^4$

In Exercises 23–26, sketch the graph of the equation.

23. (a) $\theta = \frac{1}{4}\pi$ ▹ line through the pole with direction 45°

(b) $r = 4$ ▹ circle centered at the pole of radius 4

24. (a) $\theta = \frac{2}{3}$ ▹ line through the pole with direction $\frac{2}{3} \cdot 180°/\pi = 38.2°$

(b) $r = \frac{3}{2}$ ▹ circle centered at the pole of radius $\frac{3}{2}$

25. (a) $r\cos\theta = 3$ ▹ line $x = 3$

(b) $r = 3\cos\theta$ ▹ circle centered at $(\frac{3}{2}, 0)$ of radius $\frac{3}{2}$

26. (a) $r\sin\theta = 6$ ▹ line $y = 6$

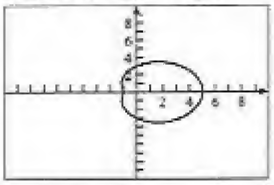
(b) $r = 6\sin\theta$ ▹ circle centered at $(3, \frac{1}{2}\pi)$ of radius 3

In Exercises 27–32, determine the type of limaçon, its symmetry S, and the direction it points. Plot it.

27. $r = 3 + 2\cos\theta$

▹ $\frac{a}{b} = \frac{3}{2} \in (1, 2)$, dented

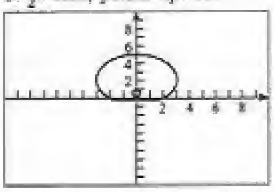
S: polar axis, points right



28. $r = 2 + 3\sin\theta$

▹ $\frac{a}{b} = \frac{2}{3} \in (0, 1)$, with a loop

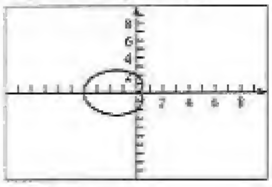
S: $\frac{1}{2}\pi$ axis, points upward



29. $r = 2(1 - \cos\theta)$

▹ $\frac{a}{b} = \frac{2}{2} = 1$, a cardioid

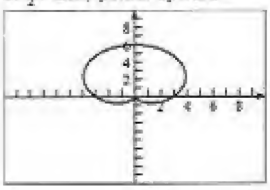
S: polar axis, points left



30. $r = 3(1 + \sin\theta)$

▹ $\frac{a}{b} = \frac{3}{3} = 1$, cardioid

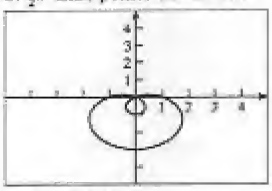
S: $\frac{1}{2}\pi$ axis, points upward



31. $r = 1 - 2\sin\theta$

▹ $\frac{a}{b} = \frac{1}{2} \in (0, 1)$, with a loop

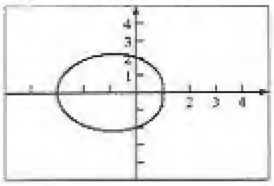
S: $\frac{1}{2}\pi$ axis, points downward



32. $r = 2 - \cos\theta$

▹ $\frac{a}{b} = \frac{2}{1} \geq 2$, 1 convex

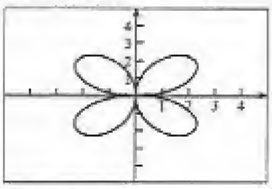
S: polar axis, points left



In Exercises 33–38, describe and plot the graph of the equation.

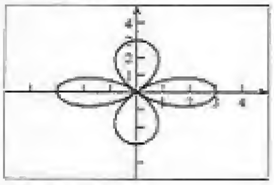
33. $r = 3 \sin 2\theta$

▷ 4-leafed rose

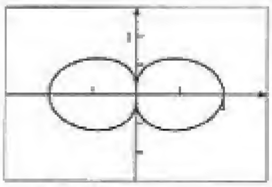


34. $r = 3 \cos 2\theta$

▷ 4-leafed rose

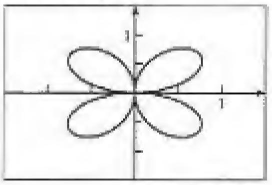


35. $r = \sqrt{|\cos \theta|}$



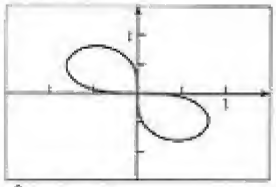
36. $r = |\sin 2\theta|$

▷ 4-leafed rose



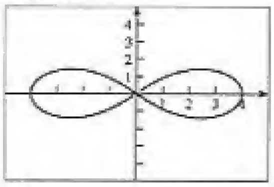
37. $r^2 = -\sin 2\theta$

▷ lemniscate



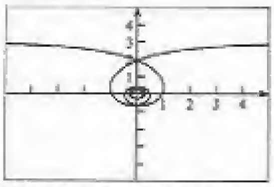
38. $r^2 = 16 \cos \theta$

▷ lemniscate



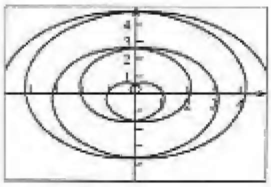
39. (a) $r\theta = 3$

▷ reciprocal spiral



(b) $3r = \theta$

▷ spiral of Archimedes



40. Show that $r = 1 + \sin \theta$ and $r = \sin \theta - 1$ have the same graph.

▷ $r = 1 + \sin \theta \Leftrightarrow -r = 1 + \sin(\theta + \pi) \Rightarrow r = \sin \theta - 1$

In Exercises 41–44, find the exact length of arc of the parametric or polar graph.

41. $x = 2 - t,\ y = t^2.\ L = \int_0^3 \sqrt{x'^2 + y'^2}dt = \int_0^3 \sqrt{(-1)^2 + (2t)^2} = \int_0^3 \sqrt{1 + 4t^2}dt = 2\int_0^3 \sqrt{\frac{1}{4} + t^2}dt$

$= \left[t\sqrt{\frac{1}{4} + t^2} + \frac{1}{4}\ln(t + \sqrt{\frac{1}{4} + t^2})\right]_0^3 = 3\sqrt{\frac{1}{4} + 9} + \frac{1}{4}\ln(3 + \sqrt{\frac{1}{4} + 9}) - \frac{1}{4}\ln\sqrt{\frac{1}{4}} = \frac{3}{2}\sqrt{37} + \frac{1}{4}\ln(6 + \sqrt{37})$

42. $x = t^2,\ y = t^3.\ L = \int_1^2 \sqrt{x'^2 + y'^2}dt = \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}dt = \int_0^1 t\sqrt{4 + 9t^2}\,dt = \frac{1}{27}(4 + 9t^2)^{3/2}\Big]_1^2$

$= \frac{1}{27}(40^{3/2} - 13^{3/2})$

43. $r = 4(1 - \sin \theta).\ L = 2\int_{-\pi/2}^{\pi/2} \sqrt{r'^2 + r^2}d\theta = 2\int_{-\pi/2}^{\pi/2} \sqrt{(-4\cos \theta)^2 + [4(1 - \sin \theta)]^2}d\theta = 8\int_{-\pi/2}^{\pi/2} \sqrt{2 - 2\sin \theta}d\theta$

$= 16\int_{-\pi/2}^{\pi/2} \sqrt{\frac{1}{2}[1 - \cos(\theta - \frac{1}{2}\pi)]}d\theta = 16\int_{-\pi/2}^{\pi/2} -\sin(\frac{1}{2}\theta - \frac{1}{4}\pi)d\theta = 32\cos(\frac{1}{2}\theta - \frac{1}{4}\pi)\Big]_{-\pi/2}^{\pi/2} = 32$

44. $r = 3\sec \theta$ from $\theta = 0$ to $\theta = \frac{1}{4}\pi$.

▷ We have

$$L = \int_0^{\pi/4} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}d\theta = \int_0^{\pi/4} \sqrt{(3\sec \theta \tan \theta)^2 + (3\sec \theta)^2}d\theta = \int_0^{\pi/4} 3\sqrt{\sec^2\theta(\tan^2\theta + 1)}d\theta$$

$$= \int_0^{\pi/4} 3\sqrt{\sec^4\theta}d\theta = \int_0^{\pi/4} 3\sec^2\theta\, d\theta = 3\tan \theta\Big]_0^{\pi/4} = 3$$

45. The given limaçon has the equation $r = 4(1 + 2\cos\theta)$.

(a) Half of the region enclosed by the loop is obtained when $\theta \in [\frac{2}{3}\pi, \pi]$.

$$A = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}[4(1+2\cos w_i)]^2\Delta_i\theta = 16\int_{2\pi/3}^{\pi}(1+2\cos\theta)^2 d\theta = 16\int_{2\pi/3}^{\pi}(1+4\cos\theta+4\cos^2\theta)d\theta$$

$$= 16\int_{2\pi/3}^{\pi}[1+4\cos\theta+2(1+\cos 2\theta)]d\theta = 16\int_{2\pi/3}^{\pi}(3+4\cos\theta+2\cos 2\theta)d\theta$$

$$= 16\Big[3\theta+4\sin\theta+\sin 2\theta\Big]_{2\pi/3}^{\pi} = 16[3\pi-(2\pi+2\sqrt{3}-\tfrac{1}{2}\sqrt{3})] = 16\pi-24\sqrt{3}$$

(b) One half of the region enclosed by the outer part of the limaçon is obtained when $\theta \in [0, \frac{2}{3}\pi]$.

$$A = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}[4(1+2\cos w_i)]^2\Delta_i\theta = 16\int_{0}^{2\pi/3}(1+2\cos\theta)^2 d\theta$$

$$= 16\Big[3\theta+4\sin\theta+\sin 2\theta\Big]_{0}^{2\pi/3} = 16(2\pi+2\sqrt{3}-\tfrac{1}{2}\sqrt{3}) = 32\pi+24\sqrt{3}$$

46. See Exercise 9.4.17

47. The given equations are $r = 2a\sin\theta$ and $r = a$. The points of intersection of the two graphs are $(a, \frac{1}{6}\pi)$ and $(a, \frac{5}{6}\pi)$. The required region is symmetric with respect to the $\frac{1}{2}\pi$ axis. Thus, if we take $\theta \in [\frac{1}{6}\pi, \frac{1}{2}\pi]$, we obtain one-half of the required area. Therefore,

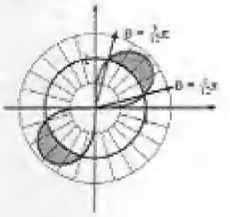
$$A = 2\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}[\tfrac{1}{2}(2a\sin w_i)^2-\tfrac{1}{2}a^2]\Delta_i\theta = a^2\int_{\pi/6}^{\pi/2}(4\sin^2\theta-1)d\theta = \int_{\pi/6}^{\pi/2}(1-2\cos 2\theta)d\theta$$

$$= \Big[\theta-\sin 2\theta\Big]_{\pi/3}^{\pi/2} = a^2(\tfrac{1}{2}\pi-0-\tfrac{1}{6}\pi-\tfrac{1}{2}\sqrt{3}) = a^2(\tfrac{1}{3}\pi+\tfrac{1}{2}\sqrt{3})$$

48. Find the area of the region inside the lemniscate $r^2 = 2\sin 2\theta$ and outside the circle $r = 1$.

▸ The figure shows the region. Solving the equations simultaneously, we get

$$2\sin 2\theta = 1$$
$$\sin 2\theta = \tfrac{1}{2}$$
$$2\theta = \tfrac{1}{6}\pi \quad\text{or}\quad 2\theta = \tfrac{5}{6}\pi$$
$$\theta = \tfrac{1}{12}\pi \quad\text{or}\quad \theta = \tfrac{5}{12}\pi$$



By symmetry, the region bounded by the lines $\theta = \frac{1}{12}\pi$ and $\theta = \frac{1}{4}\pi$ and the lemniscate and the circle is one-fourth the entire area. Therefore,

$$A = 4\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\tfrac{1}{2}[2\sin 2w_i-1]\Delta_i\theta$$

$$= 2\int_{\pi/12}^{\pi/4}(2\sin 2\theta-1)d\theta = 2\Big[-\cos 2\theta-\theta\Big]_{\pi/12}^{\pi/4} = 2[(-\tfrac{1}{4}\pi)-(-\tfrac{1}{2}\sqrt{3}-\tfrac{1}{12}\pi)] = \sqrt{3}-\tfrac{1}{3}\pi$$

Thus, the area is $\sqrt{3}-\frac{1}{3}\pi$ square units.

49. $A = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\frac{1}{2}\left(e^{kw_i}\right)^2\Delta_i\theta = \frac{1}{2}\int_0^{2\pi}e^{2k\theta}\,d\theta = \frac{1}{4k}e^{2k\theta}\Big]_0^{2\pi} = \frac{e^{4k\pi}-1}{4k}$

50. $r = a\cos\theta$ (circle center $(\frac{1}{2}a, 0)$), $r = a(1-\cos\theta)$ (cardioid pointing left). $a\cos\theta = a(1-\cos\theta)$, $2a\cos\theta = a$, $\cos\theta = \frac{1}{2}$, $\theta = \pm\frac{1}{3}\pi$. $A = 2\left[\frac{1}{2}\int_0^{\pi/3}[a(1-\cos\theta)]^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}[a\cos\theta]^2 d\theta\right]$

$$= a^2\int_0^{\pi/3}[1-2\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)]d\theta + a^2\int_{\pi/3}^{\pi/2}\tfrac{1}{2}(1+\cos 2\theta)d\theta$$

$$= a^2\Big[\tfrac{3}{2}\theta-2\sin\theta+\tfrac{1}{4}\sin 2\theta\Big]_0^{\pi/3} + \tfrac{1}{2}a^2\Big[\theta+\tfrac{1}{2}\sin 2\theta\Big]_{\pi/3}^{\pi/2} = a^2(\tfrac{1}{2}\pi-\sqrt{3}+\tfrac{1}{8}\sqrt{3}) + \tfrac{1}{2}a^2(\tfrac{1}{6}\pi-\tfrac{1}{4}\sqrt{3}) = a^2(\tfrac{7}{12}\pi-\sqrt{3})$$

51. We apply the law of cosines to the triangle with sides r, r_0, and a where the angle opposite a is $\theta-\theta_0$. Thus, $a^2 = {r_0}^2 + r^2 - 2r_0 r\cos(\theta-\theta_0)$

52. Find the area enclosed by one loop of the curve $r = a \sin n\theta$.

▹ The graph is a $2n$-leafed rose if n is even, a circle if $n = 1$, and an n-leafed rose if n is odd and greater than 1. We take the first loop. Because $r = 0$ when $n\theta = \pi$, or equivalently, $\theta = \frac{1}{n}\pi$, we take the lines $\theta = 0$ and $\theta = \frac{1}{n}\pi$ as boundaries. Thus, the area of the loop is given by

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \tfrac{1}{2}[a \sin n w_i]^2 \Delta_i\theta = \frac{1}{2}\int_0^{\pi/n} a \sin n\theta]^2 d\theta = \frac{1}{2}a^2 \int_0^{\pi/n} \sin^2 n\theta \, d\theta$$
$$= \frac{1}{4}a^2 \int_0^{\pi/n} (1 - \cos 2n\theta) d\theta = \frac{1}{4}a^2 \Big[\theta - \frac{1}{2n}\sin 2n\theta\Big]_0^{\pi/n} = \frac{1}{4n}\pi a^2$$

■ The area of the loop is $\frac{1}{4n}\pi a^2$ square units.

In Exercises 53–56, the equation is of a conic having a focus at the pole. (a) Find the eccentricity e; (b) identify the conic; (c) write an equation of the directrix corresponding to the focus at the pole; (d) sketch the curve.

53. $r = \dfrac{2}{2 - \sin\theta} = \dfrac{1}{1 - \frac{1}{2}\sin\theta}$ is of the form $r = \dfrac{ed}{1 - e\sin\theta}$ with (a) $e = \frac{1}{2} < 1$; (b) the conic is an ellipse.

(c) $d = \frac{2}{1} = 2$ and the directrix is below the focus at the pole; so its equation is $r \sin\theta = -2$.

54. $r = \dfrac{5}{3 + 3\sin\theta} = \dfrac{\frac{5}{3}}{1 + \sin\theta}$ is of the form $r = \dfrac{ed}{1 + e\sin\theta}$ with (a) $e = 1$; (b) the conic is a parabola.

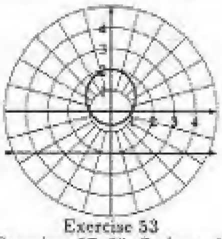
(c) $d = \dfrac{\frac{5}{3}}{1} = \frac{5}{3}$ and the equation of the directrix is $r \sin\theta = \frac{5}{3}$.

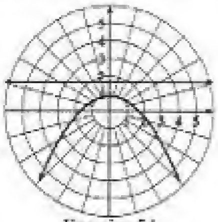
55. $r = \dfrac{4}{2 + 3\cos\theta} = \dfrac{2}{1 + \frac{3}{2}\cos\theta}$ is of the form $r = \dfrac{ed}{1 + e\cos\theta}$ with (a) $e = \frac{3}{2} > 1$; (b) the conic is a hyperbola.

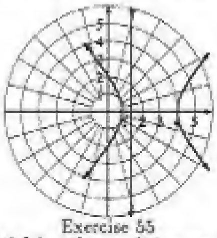
(c) $d = \frac{4}{3}$ and the directrix is to the right of the focus, so its equation is $r\cos\theta = \frac{4}{3}$.

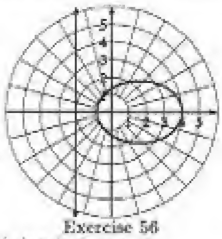
56. $r = \dfrac{4}{3 - 2\cos\theta}$ ▹ (a) Dividing the numerator and denominator by 3, we obtain $r = \dfrac{\frac{4}{3}}{1 - \frac{2}{3}\cos\theta}$.

Thus $e = \frac{2}{3} < 1$ and so (b) the conic is an ellipse. (c) We get the equation of the directrix by setting the constant of the denominator to 0. Thus $r\cos\theta = -\frac{4}{2}$. $r\cos\theta = -2$. (d) A sketch is shown below right.


Exercise 53


Exercise 54


Exercise 55


Exercise 56

In Exercises 57–60, find a polar equation of the conic satisfying the conditions and sketch the graph.

57. The directrix corresponding to the focus at the pole is to the left of the focus. Thus an equation of the conic is of the form $r = \dfrac{ed}{1 - e\cos\theta}$. Because a vertex is at $(2, \pi)$, $2 = \dfrac{ed}{1 - e\cos\pi}$; $2 = \dfrac{ed}{1 + e}$; $ed = 2 + 2e$

Because a vertex is at $(4, \pi)$ or, equivalently $(-4, 0)$, then $-4 = \dfrac{ed}{1 - e\cos 0}$; $-4 = \dfrac{ed}{1 - e}$; $ed = -4 + 4e$

Solving simultaneously, we get $e = 3$ and $ed = 8$. The conic is a hyperbola; an equation is $r = \dfrac{8}{1 - 3\cos\theta}$.

58. A focus at the pole, a vertex at $(6, \frac{1}{2}\pi)$, and $e = \frac{3}{4}$.

▹ There are two ellipses possible. Because the focus is at the pole and a vertex is above the pole, one polar equation is of the form. $r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{ed}{1 + \frac{3}{4}\sin\theta}$. We let $(r, \theta) = (6, \frac{1}{2}\pi)$ to obtain $6 = \dfrac{ed}{1 + \frac{3}{4}} = \dfrac{ed}{\frac{7}{4}}$

and so $ed = \frac{21}{2}$. Thus, an equation is $r = \dfrac{\frac{21}{2}}{1 + \frac{3}{4}\sin\theta} = \dfrac{42}{4 + 3\sin\theta}$.

Another polar equation is of the form $r = \dfrac{ed}{1 - \frac{3}{4}\sin\theta}$. We let $(r, \theta) = (6, \frac{3}{2}\pi)$ to get $6 = \dfrac{ed}{1 - \frac{3}{4}} = 4ed$

and so $ed = \frac{3}{2}$. Thus an equation is $r = \dfrac{\frac{3}{2}}{1 - \frac{3}{4}\sin\theta} = \dfrac{6}{4 - 3\sin\theta}$

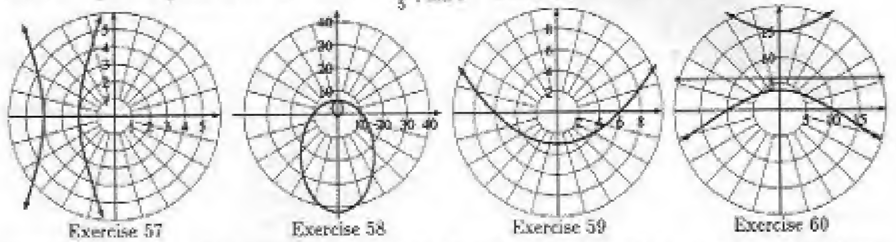
59. Because $e = 1$, the conic is a parabola. Because a focus is at the pole and the vertex is at $(3, \frac{3}{2}\pi)$, the directrix is below the focus. Thus an equation of the parabola is of the form $r = \frac{d}{1 - \cos\theta}$. Because d is the undirected distance between the focus and the directrix $d = 2(3) = 6$. Hence an equation of the parabola is $r = \frac{6}{1 - \sin\theta}$.

60. The line $r \sin\theta = 6$ is the directrix corresponding to the focus at the pole and $e = \frac{5}{3}$.

▹ Because the directrix is obtained by setting the constant to 0, the equation is of the form $r = \frac{6}{a + \sin\theta}$

Dividing numerator and denominator by a, we get the form $r = \frac{6/a}{1 + (1/a)\sin\theta}$

Therefore $\frac{1}{a} = e = \frac{5}{3}$ and so $a = \frac{3}{5}$. Hence, $r = \frac{6}{\frac{3}{5} + \sin\theta} = \frac{30}{3 + 5\sin\theta}$. A sketch appears below right.



Exercise 57 Exercise 58 Exercise 59 Exercise 60

In Exercises 61–64, use NINT to approximate to four digits the length L of the parametric or polar curve.

61. $x = 2t^3$, $y = t - 2$. $L = \int_2^3 \sqrt{x'^2 + y'^2}\,dt = \int_2^3 \sqrt{(6t^2)^2 + 1^2}\,dt = \int_2^3 \sqrt{36t^4 + 1}\,dt = 38.014 \approx 38.02$

62. $x = e^t$, $y = \cos t$. $L = \int_{-\pi/2}^{\pi/2} \sqrt{x'^2 + y'^2}\,dt = \int_{-\pi/2}^{\pi/2} \sqrt{e^{2t} + \sin^2 t}\,dt = 5.3476 \approx 5.348$

63. $r = 4 - 2\sin\theta$.

$L = 2\int_{-\pi/2}^{\pi/2} \sqrt{r'^2 + r^2}\,d\theta = 2\int_{-\pi/2}^{\pi/2} \sqrt{(-2\cos\theta)^2 + (4 - 2\sin\theta)^2}\,d\theta = 4\int_{-\pi/2}^{\pi/2} \sqrt{5 - 4\sin\theta}\,d\theta = 26.730 \approx 26.73$

64. One leaf of the rose $r = 4\cos 3\theta$.

▹ Setting $r = 0$, we get

$4\cos 3\theta = 0$, $3\theta = \pm\frac{1}{2}\pi$, $\theta = \pm\frac{1}{6}\pi$

By symmetry, and with the help of NINT, we get

$$L = 2\int_0^{\pi/6} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = 2\int_0^{\pi/6} \sqrt{(-12\sin 3\theta)^2 + (4\cos 3\theta)^2}\,d\theta = 8\int_0^{\pi/6} \sqrt{9\sin^2 3\theta + \cos^2 3\theta}\,d\theta = 8.910$$

▪ The length of arc is 9.910 units.

65. The distance from the center to the focus is $ae = 36(0.206) = 7.4$. Thus Mercury gets as near as $36.0 - 7.4 = 28.6$ and as far as $36.0 + 7.4 = 43.4$ million miles from the sun.

66. From the center, $200 + 4000 = 4200 = a - c = 2c - c = c$. $a + c = 3c = 12600$, $12600 - 4000 = 8600$. The greatest distance from the surface is 8600 miles.

67. Choose the coordinate system with the pole at the focus. The conic is a parabola; so $e = 1$ and an equation is $r = \frac{d}{1 - \cos\theta}$. Let $(15, \theta)$ and $(5, \theta + \frac{1}{2}\pi)$ be the given points. Then $15 = \frac{d}{1 - \cos\theta}$; $15 - 15\cos\theta = d$;

$\cos\theta = \frac{15 - d}{15}$; $5 = \frac{d}{1 - \cos(\theta + \frac{1}{2}\pi)} = \frac{d}{1 + \sin\theta}$; $5 + 5\sin\theta = d$; $\sin\theta = \frac{d - 5}{5}$

$\sin^2\theta + \cos^2\theta = \left(\frac{d - 5}{5}\right)^2 + \left(\frac{15 - d}{15}\right)^2$; $1 = \frac{d^2 - 10d + 25}{25} + \frac{d^2 - 30d + 25}{225}$

$225 = 9d^2 - 90d + 225 + d^2 - 30d + 225$; $10d^2 - 120d + 225 = 0$; $d^2 - 12d + 2.5 = 0$

$d = \frac{1}{2}(12 \pm \sqrt{144 - 90}) = \frac{1}{2}(12 \pm \sqrt{54}) = 6 \pm \frac{3}{2}\sqrt{6}$

▪ The closest distances of the two orbits are $\frac{1}{2}d = (3 \pm \frac{3}{4}\sqrt{6})$ million miles.

68. If the distance between the two directrices of an ellipse is three times the distance between the foci, find the eccentricity e.

▹ The distance from the center to the focus is ae, and to the directrix is a/e.
$2(a/e) = 3 \cdot 2ae$, $\quad e^2 = \frac{1}{3}$, $\quad e = \frac{1}{3}\sqrt{3}$

69. Because the conic is a parabola, $e = 1$. Because the directrix is to the left of the focus, an equation of the parabola is of the form $r = \dfrac{d}{1-\cos\theta}$. The point $(2, \frac{1}{3}\pi)$ is on the parabola; so $2 = \dfrac{d}{1-\cos\frac{1}{3}\pi}$; $2 = \dfrac{d}{1-\frac{1}{2}}$; $d = 1$. An equation of the parabola is $r = \dfrac{1}{1-\cos\theta}$.

70. $A = 4 \cdot \frac{1}{2}\displaystyle\int_0^{\pi/2} \frac{d\theta}{[\frac{1}{2}(1+\cos\theta)]^2} = 2\int_0^{\pi/2} \frac{d\theta}{\cos^4\frac{1}{2}\theta} = 2\int_0^{\pi/2} \sec^4\tfrac{1}{2}\theta\, d\theta = 2\int_0^{\pi/2} (\tan^2\tfrac{1}{2}\theta + 1)\sec^2\tfrac{1}{2}\theta\, d\theta$
$= 4\left[\frac{1}{3}\tan^3\frac{1}{2}\theta + \tan\frac{1}{2}\theta\right]_0^{\pi/2} = \frac{16}{3}$

In Exercises 71 and 72, a focal chord of a conic is a segment through a focus with endpoints on the conic.

▹ A focal chord of a conic is divided by the focus into two segments whose directions differ by π.

71. Show: the sum of the reciprocals of the lengths s of perpendicular focal chords of a parabola is a constant.

▹ $s(\theta) = \dfrac{d}{1-\cos\theta} + \dfrac{d}{1+\cos\theta} = \dfrac{2d}{1-\cos^2\theta} = \dfrac{2d}{\sin^2\theta}$. $\dfrac{1}{s(\theta)} + \dfrac{1}{s(\theta+\frac{1}{2}\pi)} = \dfrac{\sin^2\theta}{2d} + \dfrac{\cos^2\theta}{2d} = \dfrac{1}{2d}$

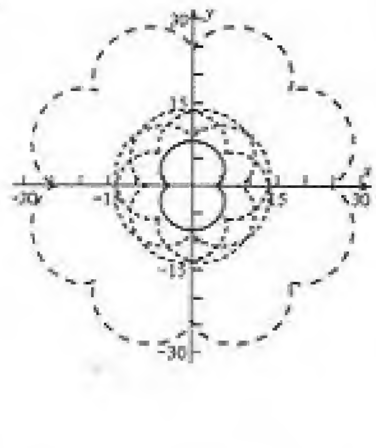
72. Show that the sum of the reciprocals of their length is the same for any chord.

▹ An equation of the conic is
$$r(\theta) = \frac{ed}{1-e\cos\theta}$$
Because $\cos(\theta+\pi) = -\cos\theta$, the sum of the reciprocals of the lengths is
$$\frac{1}{r(\theta)} + \frac{1}{r(\theta+\pi)} = \frac{1-e\cos\theta}{ed} + \frac{1+e\cos\theta}{ed} = \frac{2}{ed}$$

73. Derive the formula for an epicycloid. (Section 9.1)

▹ Let the centers of the fixed and moving circles be the origin and C. Let $A(a,0)$ and B be the original and moving points of contact. t is the $m(\angle AOB)$. Let $\alpha = m(\angle BCP)$. Then $\alpha + t$ is the angle between CP and the horizontal. Let a vertical line through C meet a horizontal line through $P(x,y)$ in D and the x axis in E. Rolling means arc BP = arc BA, $b\alpha = at$, $\alpha = \frac{a}{b}t$, $\alpha + t = \left(\frac{a}{b}+1\right)t = \dfrac{a+b}{b}t$.
$x = OE - PD = (a+b)\cos t - b\cos(\alpha+t) = (a+b)\cos\theta - b\cos\dfrac{a+b}{b}t$
$y = CE - CD = (a+b)\sin t + \sin(\alpha+t) = (a+b)\sin t - b\sin\dfrac{a+b}{b}t$. If $b = a$, then
$x - a = a[2\cos t - (\cos 2t + 1)] = a(2\cos t - 2\cos^2 t) = 2a\cos t(1-\cos t)$
$y = a(2\sin t - \sin 2t) = a(2\sin t - 2\sin t\cos t) = 2a\sin t(1-\cos t)$
which is the cardioid $r = 2a(1-\cos t)$. Similarly, a hypocycloid with $b = a$ is a limaçon with a loop.

74. Plot ans sketch the epicycloids.
(a) $a = 4$, $b = 2$, $t \in [-\pi, \pi]$ solid
(b) $a = 24$, $b = 3$, $t \in [-\pi, \pi]$ long dashes
(c) $a = 8$, $b = 3$, $t \in [-3\pi, 3\pi]$ short dashes



T E N

VECTORS AND PLANES, LINES, AND SURFACES IN SPACE

10.1 VECTORS IN THE PLANE

10.1.1 Definition A *vector in the plane* is an ordered pair of real numbers $\langle x, y\rangle$. The numbers x and y are called the *components* of the vector $\langle x, y\rangle$. Vectors $\langle x, y\rangle$ and $\langle u, v\rangle$ are *equal* if and only if $x = u$ and $y = v$. $\mathbf{0} = \langle 0, 0\rangle$ is the *zero* vector. In contrast, we call a number a *scalar*.

If $\mathbf{A} = \langle a_1, a_2\rangle$, then the directed line segment $\overrightarrow{OA}$, where O is the origin and A is the point (a_1, a_2), is called the *position representation* of the vector $\mathbf{A}$. If $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, then the directed line segment $\overrightarrow{PQ}$ is a *representation* of the vector $\langle q_1 - p_1, q_2 - p_2\rangle$. We use the abbreviation $\mathbf{a} = \mathbf{V}(\overrightarrow{OA})$.

10.1.2 Definition The *magnitude* of a vector is the length of any of its representations, and the *direction* of a nonzero vector is the direction of any of its representations.

The magnitude of the vector $\mathbf{A}$ is denoted by $\|\mathbf{A}\|$. $\|\mathbf{A}\| = 0$ if and only if $\mathbf{A} = \mathbf{0}$.

10.1.3 Theorem If $\mathbf{A}$ is the vector $\langle a_1, a_2\rangle$, then $\|\mathbf{A}\| = \sqrt{a_1{}^2 + a_2{}^2}$.
If $\mathbf{A} \neq \mathbf{0}$, the direction of $\mathbf{A}$ is the radian measure of the angle $\theta \in [0, 2\pi)$ such that

$\cos\theta = \dfrac{a_1}{\|\mathbf{A}\|}$ and $\sin\theta = \dfrac{a_2}{\|\mathbf{A}\|}$

Thus, if $a_1 \neq 0$, we have

$\tan\theta = \dfrac{a_2}{a_1}$

In navigation, the *course* or *heading* ϕ of a ship or airplane is the angle measured in degrees clockwise from the north to the direction in which the vessel is traveling. The angle is considered positive. Thus $\theta = 90^\circ - \phi$ if $0 \leq \phi \leq 90^\circ$, or $450^\circ - \phi$ if $\phi > 90^\circ$.

10.1.4 Definition The *sum* of two vectors $\mathbf{A} = \langle a_1, a_2\rangle$ and $\mathbf{B} = \langle b_1, b_2\rangle$ is the vector $\mathbf{A} + \mathbf{B}$, defined by
$\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2\rangle$
The rule for addition of vectors is sometimes referred to as the *parallelogram law*.

Force Force is a vector quantity. Any set of forces acting at a single point can be replaced by a single force, the *resultant*, which is their vector sum.

10.1.5 Definition If $\mathbf{A} = \langle a_1, a_2\rangle$, then the *negative* of $\mathbf{A}$, denoted by $-\mathbf{A}$, is defined to be the vector $\langle -a_1, -a_2\rangle$.

10.1.6 Definition The *difference* of the two vectors $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} - \mathbf{B}$, is the vector obtained by adding $\mathbf{A}$ to the negative of $\mathbf{B}$: that is

$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$

By Definition 10.1.6, it follows that the vector represented by the directed line segment $\overrightarrow{PQ}$ is the difference of the vectors whose position representations are $\overrightarrow{OQ}$ and $\overrightarrow{OP}$. In symbols $\mathbf{V}(\overrightarrow{PQ}) = \mathbf{q} - \mathbf{p} = \langle q_1 - p_1, q_2 - p_2\rangle$. Furthermore, ABCD is a parallelogram if and only if $\mathbf{V}(\overrightarrow{AB}) = \mathbf{V}(\overrightarrow{DC})$.

10.1.7 Definition If c is a scalar and $\mathbf{A}$ is the vector $\langle a_1, a_2\rangle$, then the *product of* c and $\mathbf{A}$, denoted by $c\mathbf{A}$, is the vector given by $c\mathbf{A} = c\langle a_1, a_2\rangle = \langle ca_1, ca_2\rangle$.

If $c \neq 0$, then $\dfrac{\mathbf{A}}{c}$ is defined to be $\frac{1}{c}\mathbf{A}$.

V_2 is the set of vectors together with addition and scalar multiplication.
A sum of terms of the form $c_i\mathbf{A}_i$ is called a *linear combination* of the vectors $\mathbf{A}_i$. The linear combination is *trivial* if all the scalars c_i are zero. The set of vectors is *dependent* if some nontrivial combination is the zero vector.

10.1.8 Theorem If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are any vectors in V_2, and c and d are any scalars, then vector addition and scalar multiplication satisfy the following properties:

(i) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (commutative law)
(ii) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ (associative law)
(iii) There is a vector $\mathbf{0}$ in V_2 for which $\mathbf{A} + \mathbf{0} = \mathbf{A}$ (existence of additive identity)
(iv) There is a vector $-\mathbf{A}$ in V_2 such that $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$ (existence of negative)
(v) $(cd)\mathbf{A} = c(d\mathbf{A})$ (associative law)
(vi) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$ (distributive law)
(vii) $(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$ (distributive law)
(viii) $1(\mathbf{A}) = \mathbf{A}$ (existence of scalar multiplicative identity)

10.1.9 Definition A *real vector space* V is a set of elements, called *vectors*, together with the set of real numbers, called *scalars*, with two operations called *vector addition* and *scalar multiplication* such that for every pair of vectors $\mathbf{A}$ and $\mathbf{B}$ in V and for ever scalar c, a vector $\mathbf{A} + \mathbf{B}$ and a vector $c\mathbf{A}$ are defined so that properties (i)–(viii) of Theorem 10.1.8 are satisfied.

Any vector that has magnitude one is called a *unit* vector. We define the vectors $\mathbf{i}$ and $\mathbf{j}$ which form a basis for the vector space V_2 as follows:

$\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$

Thus, for any vector $\langle a_1, a_2\rangle$ in V_2, we have

$\langle a_1, a_2\rangle = a_1\mathbf{i} + a_2\mathbf{j}$

10.1.10 Theorem If the nonzero vector $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$, then the unit vector $\mathbf{U}$ having the same direction as $\mathbf{A}$ is given by

$$\mathbf{U} = \frac{\mathbf{A}}{\|\mathbf{A}\|} = \frac{a_1}{\|\mathbf{A}\|}\mathbf{i} + \frac{a_2}{\|\mathbf{A}\|}\mathbf{j}$$

Following are two important inequalities. See Exercise 56.

The Cauchy-Schwarz Inequality $\quad (a_1b_1 + a_2b_2)^2 \le (a_1{}^2 + a_2{}^2)(b_1{}^2 + b_2{}^2)$

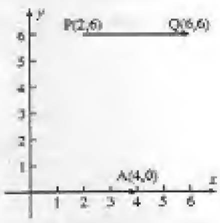
The Triangle Inequality for Vectors $\quad \|\mathbf{A} + \mathbf{B}\| \le \|\mathbf{A}\| + \|\mathbf{B}\|$

Equality holds if and only if $\mathbf{A}$ and $\mathbf{B}$ have the same direction or either is the zero vector.

Exercises 10.1

In Exercises 1–4, (a) draw the position representation of the vector $\mathbf{A}$ and also the particular representation $\overrightarrow{PQ}$ through the given point P. (b) Find the magnitude of $\mathbf{A}$.

1. $\mathbf{A} = \langle 3, 4\rangle$; $P = (2, 1)$. $\mathbf{q} = \mathbf{p} + \mathbf{a} = \langle 2 + 3, 1 + 4\rangle = \langle 5, 5\rangle$. $\|\langle 3,4\rangle\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$
2. $\mathbf{A} = \langle -2, 5\rangle$; $P = (-3, 4)$. $\mathbf{q} = \mathbf{p} + \mathbf{a} = \langle -3 - 2, 4 + 5\rangle = \langle -5, 9\rangle$. $\|\langle -2, 5\rangle\| = \sqrt{2^2 + 5^2} = \sqrt{29}$
3. $\mathbf{A} = \langle c, -\frac{1}{2}\rangle$; $P = (-2, -c)$. $\mathbf{q} = \mathbf{p} + \mathbf{a} = \langle c - 2, -c - \frac{1}{2}\rangle$. $\|\langle c, -\frac{1}{2}\rangle\| = \sqrt{c^2 + (-\frac{1}{2})^2} = \sqrt{c^2 + \frac{1}{4}}$
4. $\mathbf{A} = \langle 4, 0\rangle$; $P = (2, 6)$.



▸ Let A be the point (4, 0). The directed line segment $\overline{OA}$ is the position representation of vector $\mathbf{A}$, as shown in the figure. Let $\overrightarrow{PQ}$ be the particular representation of vector $\mathbf{A}$ through point P. Then

$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \mathbf{a}$

and so

$\mathbf{q} = \mathbf{p} + \mathbf{a} = \langle 2 + 4, 6 + 0\rangle = \langle 6, 6\rangle$

Thus $Q = (6, 6)$. The representation $\overrightarrow{PQ}$ is also shown in the figure. By Theorem 10.1.3, the magnitude of vector $\mathbf{A}$ is given by

$\|\mathbf{A}\| = \sqrt{4^2 + 0^2} = 4$

In Exercises 5 and 6, find the exact (in (c) also approximate) radian measure of the direction angle of the vector.

5. (a) $\langle 1, -1\rangle$. $\tan\theta = \frac{-1}{1} = -1$; $\theta = \frac{7}{4}\pi$
   (b) $\langle -3, 0\rangle$. $\tan\theta = \frac{0}{-3} = 0$; $\theta = \pi$
   (c) $\langle 5, 2\rangle$. $\tan\theta = \frac{2}{5} = 0.4$; $\theta = \tan^{-1}(\frac{2}{5}) \approx 0.38$
6. (a) $\langle\sqrt{3}, 1\rangle$. $\tan\theta = \frac{1}{\sqrt{3}}$; $\theta = \frac{1}{6}\pi$
   (b) $\langle 0, 4\rangle$. $\tan\theta = \frac{4}{0}$, undefined. $\theta = \frac{1}{2}\pi$
   (c) $\langle -3, 2\rangle$. $\tan\theta = \frac{2}{-3}$; $\theta = \pi - \tan^{-1}\frac{2}{3} \approx 2.55$

In Exercises 7–10, find the vector **A** having $\overrightarrow{PQ}$ as a representation. Draw $\overrightarrow{PQ}$ and the position representation of **A**.

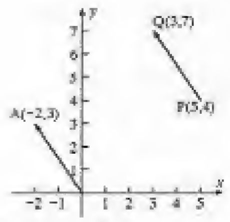
7. $P = (3,7)$; $Q = (5,4)$. $\mathbf{v}(\overrightarrow{PQ}) = \langle 5-3, 4-7\rangle = \langle 2,-3\rangle$.

8. $P = (5,4)$; $Q = (3,7)$

▷ Since $\overrightarrow{PQ}$ is a representation of **A**, we have

$\mathbf{A} = \mathbf{q} - \mathbf{p} = \langle 3-5, 7-4\rangle = \langle -2, 3\rangle$

The representation $\overrightarrow{PQ}$ and the position representation **a** of the vector **A** are shown in the figure.



9. $P = (-5,-3)$; $Q = (0,3)$. $\mathbf{v}(\overrightarrow{PQ}) = \langle 0+5, 3+3\rangle = \langle 5,6\rangle$.

10. $P = (-\sqrt{2}, 0)$; $Q = (0,0)$. $\mathbf{v}(\overrightarrow{PQ}) = \langle 0+\sqrt{2}, 0-0\rangle = \langle \sqrt{2}, 0\rangle$

In Exercises 11–14, find the point S so that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are each representations of the same vector.

▷ We have $\mathbf{q} - \mathbf{p} = \mathbf{s} - \mathbf{r}$ and so $\mathbf{s} = \mathbf{q} - \mathbf{p} + \mathbf{r}$.

11. $P = (2,5)$; $Q = (1,6)$; $R = (-3,2)$; $\mathbf{s} = \langle 1,6\rangle - \langle 2,5\rangle + \langle -3,2\rangle = \langle -4,3\rangle$

12. $P = (-2,0)$; $Q = (-3,-4)$; $R = (4,2)$

▷ Because $\overrightarrow{PQ} = \overrightarrow{RS}$ then $\mathbf{q} - \mathbf{p} = \mathbf{s} - \mathbf{r}$ and so

$\mathbf{s} = \mathbf{q} - \mathbf{p} + \mathbf{r} = \langle -3-(-2)+4, -4-0+2\rangle = \langle 3,-2\rangle$

Thus $S = -(3,-2)$.

13. $P = (0,3)$; $Q = (5,-2)$; $R = (7,0)$; $\mathbf{s} = \langle 5,-2\rangle - \langle 0,3\rangle + \langle 7,0\rangle = \langle 12,-5\rangle$

14. $P = (-1,4)$; $Q = (2,-3)$; $R = (-5,-2)$. $\mathbf{s} = \langle 2,-3\rangle - \langle -1,4\rangle + \langle -5,-2\rangle = \langle -2,-9\rangle$

In Exercises 15 and 16, find the sum of the pair of vectors and illustrate geometrically.

15. (a) $\langle 2,4\rangle + \langle -3,5\rangle = \langle 2-3, 4+5\rangle = \langle -1,9\rangle$ (b) $\langle -3,0\rangle + \langle 4,-5\rangle = \langle -3+4, 0-5\rangle = \langle 1,-5\rangle$

16. (a) $\langle 0,3\rangle, \langle -2,3\rangle$; (b) $\langle 2,3\rangle, \langle -\sqrt{2},-1\rangle$

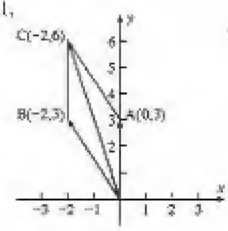
▷ (a) Let $\mathbf{A} = \langle 0,3\rangle$ and $\mathbf{B} = \langle -2,3\rangle$. By Definition 10.1.1,

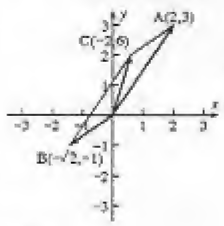
$\mathbf{A} + \mathbf{B} = \langle 0,3\rangle + \langle -2,3\rangle = \langle 0+(-2), 3+3\rangle = \langle -2,6\rangle$

(b) Let $\mathbf{A} = \langle 2,3\rangle$ and $\mathbf{B} = \langle -\sqrt{2},-1\rangle$. Then

$\mathbf{A} + \mathbf{B} = \langle 2,3\rangle + \langle -\sqrt{2},-1\rangle = \langle 2-\sqrt{2}, 2\rangle$

In the figures, OACB is a parallelogram, $\overrightarrow{OA}$ is the position representation of vector **A** and $\overrightarrow{OB}$ is the position representation of vector **B**. The diagonal $\overrightarrow{OC}$ is the position representation of $\mathbf{A} + \mathbf{B}$.





In Exercises 17 and 18, subtract the second vector from the first and illustrate geometrically.

17. (a) $\langle -3,-4\rangle - \langle 6,0\rangle = \langle -3-6, -4-0\rangle = \langle -9,-4\rangle$ (b) $\langle 1,e\rangle - \langle -3,2e\rangle = \langle 1+3, e-2e\rangle = \langle 4,-e\rangle$

18. (a) $\langle 0,5\rangle - \langle 2,8\rangle = \langle 0-2, 5-8\rangle = \langle -2,-3\rangle$ (b) $\langle 3,7\rangle - \langle 3,7\rangle = \mathbf{0}$

In Exercises 19 and 20, find the vector or scalar if $\mathbf{A} = \langle 2,4\rangle$, $\mathbf{B} = \langle 4,-3\rangle$, and $\mathbf{C} = \langle -3,2\rangle$.

19. $\mathbf{A} = \langle 2,4\rangle$, $\mathbf{B} = \langle 4,-3\rangle$, $\mathbf{C} = \langle -3,2\rangle$. (a) $\mathbf{A} + \mathbf{B} = \langle 2,4\rangle + \langle 4,-3\rangle = \langle 6,1\rangle$

(b) $\|\mathbf{C} - \mathbf{B}\| = \|\langle -3,2\rangle - \langle 4,-3\rangle\| = \|\langle -7,5\rangle\| = \sqrt{(-7)^2 + 5^2} = \sqrt{49+25} = \sqrt{74}$

(c) $\|7\mathbf{A} - \mathbf{B}\| = \|7\langle 2,4\rangle - \langle 4,-3\rangle\| = \|\langle 14,28\rangle - \langle 4,-3\rangle\| = \|\langle 10,31\rangle\| = \sqrt{10^2 + 31^2} = \sqrt{100+961} = \sqrt{1061}$

20. (a) $\mathbf{A} - \mathbf{B}$; (b) $\|\mathbf{C}\|$; (c) $2\mathbf{A} + 3\mathbf{B}$

▷ (a) $\mathbf{A} - \mathbf{B} = \langle 2,4\rangle - \langle 4,-3\rangle = \langle 2-4, 4-(-3)\rangle = \langle -2,7\rangle$

(b) $\|\mathbf{C}\| = \|\langle -3,2\rangle\| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$

(c) $2\mathbf{A} + 3\mathbf{B} = 2\langle 2,4\rangle + 3\langle 4,-3\rangle = \langle 4,8\rangle + \langle 12,-9\rangle = \langle 16,-1\rangle$

In Exercises 21–24, find the vector or scalar if $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = 4\mathbf{i} - \mathbf{j}$.

21. (a) $5\mathbf{A} = 5(2\mathbf{i} + 3\mathbf{j}) = 10\mathbf{i} + 15\mathbf{j}$ (b) $-6\mathbf{B} = -6(4\mathbf{i} - \mathbf{j}) = -24\mathbf{i} + 6\mathbf{j}$

(c) $\mathbf{A} + \mathbf{B} = (2\mathbf{i} + 3\mathbf{j}) + (4\mathbf{i} - \mathbf{j}) = 6\mathbf{i} + 2\mathbf{j}$ (d) $\|\mathbf{A} + \mathbf{B}\| = \|6\mathbf{i} + 2\mathbf{j}\| = \sqrt{6^2 + 2^2} = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10}$

22. (a) $-2\mathbf{A} = -2(2\mathbf{i} + 3\mathbf{j}) = -4\mathbf{i} - 6\mathbf{j}$ (b) $3\mathbf{B} = 3(4\mathbf{i} - \mathbf{j}) = 12\mathbf{i} - 3\mathbf{j}$

(c) $\mathbf{A} - \mathbf{B} = (2\mathbf{i} + 3\mathbf{j}) - (4\mathbf{i} - \mathbf{j}) = -2\mathbf{i} + 4\mathbf{j}$ (d) $\|\mathbf{A} - \mathbf{B}\| = \|-2\mathbf{i} + 4\mathbf{j}\| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$

23. (a) $\|\mathbf{A}\| + \|\mathbf{B}\| = \|2\mathbf{i} + 3\mathbf{j}\| + \|4\mathbf{i} - \mathbf{j}\| = \sqrt{2^2 + 3^2} + \sqrt{4^2 + (-1)^2} = \sqrt{13} + \sqrt{17}$

(b) $5\mathbf{A} - 6\mathbf{B} = 5(2\mathbf{i} + 3\mathbf{j}) - 6(4\mathbf{i} - \mathbf{j}) = 10\mathbf{i} + 15\mathbf{j} - 24\mathbf{i} + 6\mathbf{j} = -14\mathbf{i} + 21\mathbf{j}$

(c) $\|5\mathbf{A} - 6\mathbf{B}\| = \|-14\mathbf{i} + 21\mathbf{j}\| = \sqrt{(-14)^2 + 21^2} = \sqrt{96 + 441} = \sqrt{637} = 7\sqrt{13}$

(d) $\|5\mathbf{A}\| - \|6\mathbf{B}\| = 5\|\mathbf{A}\| - 6\|\mathbf{A}\| = 5\sqrt{2^2 + 3^2} - 6\sqrt{4^2 + (-1)^2} = 5\sqrt{13} - 6\sqrt{17}$

24. (a) $\|\mathbf{A}\| - \|\mathbf{B}\|$; (b) $3\mathbf{B} - 2\mathbf{A}$; (c) $\|3\mathbf{B} - 2\mathbf{A}\|$; (d) $\|3\mathbf{B}\| - \|2\mathbf{A}\|$

▷ (a) $\|\mathbf{A}\| - \|\mathbf{B}\| = \|2\mathbf{i} + 3\mathbf{j}\| - \|4\mathbf{i} - \mathbf{j}\| = \sqrt{2^2 + 3^2} - \sqrt{4^2 + (-1)^2} = \sqrt{13} - \sqrt{17}$

(b) $3\mathbf{B} - 2\mathbf{A} = 3(4\mathbf{i} - \mathbf{j}) - 2(2\mathbf{i} + 3\mathbf{j}) = (12\mathbf{i} - 3\mathbf{j}) - (4\mathbf{i} + 6\mathbf{j}) = 8\mathbf{i} - 9\mathbf{j}$

(c) $\|3\mathbf{B} - 2\mathbf{A}\| = \|8\mathbf{i} - 9\mathbf{j}\| = \sqrt{8^2 + 9^2} = \sqrt{145}$

(d) $\|3\mathbf{B}\| - \|2\mathbf{A}\| = \|12\mathbf{i} - 3\mathbf{j}\| - \|4\mathbf{i} + 6\mathbf{j}\| = \sqrt{12^2 + 3^2} - \sqrt{4^2 + 6^2} = \sqrt{153} - \sqrt{52} = 3\sqrt{17} - 2\sqrt{13}$

In Exercises 25 and 26, $\mathbf{A} = -4\mathbf{i} + 2\mathbf{j}$, $\mathbf{B} = -\mathbf{i} + 3\mathbf{j}$, and $\mathbf{C} = 5\mathbf{i} - \mathbf{j}$.

25. (a) $5\mathbf{A} - 2\mathbf{B} - 2\mathbf{C} = 5(-4\mathbf{i} + 2\mathbf{j}) - 2(-\mathbf{i} + 3\mathbf{j}) - 2(5\mathbf{i} - \mathbf{j}) = (-20\mathbf{i} + 10\mathbf{j}) + (2\mathbf{i} - 6\mathbf{j}) + (-10\mathbf{i} + 2\mathbf{j})$
$= (-20 + 2 - 10)\mathbf{i} + (10 - 6 + 2)\mathbf{j} = -28\mathbf{i} + 6\mathbf{j}$

(b) $\|5\mathbf{A} - 2\mathbf{B} - 2\mathbf{C}\| = \sqrt{(-28)^2 + 6^2} = \sqrt{784 + 36} = \sqrt{820} = 2\sqrt{205}$

26. (a) $3\mathbf{B} - 2\mathbf{A} - \mathbf{C} = 3(-\mathbf{i} + 3\mathbf{j}) - 2(-4\mathbf{i} + 2\mathbf{j}) - (5\mathbf{i} - \mathbf{j}) = (-3\mathbf{i} + 9\mathbf{j}) + (8\mathbf{i} - 4\mathbf{j}) + (-5\mathbf{i} + \mathbf{j}) = 6\mathbf{j}$

(b) $\|3\mathbf{B} - 2\mathbf{A} - \mathbf{C}\| = \|6\mathbf{j}\| = 6$

In Exercises 27 and 28, let $\mathbf{A} = 8\mathbf{i} + 5\mathbf{j}$ and $\mathbf{B} = 3\mathbf{i} - \mathbf{j}$. find a unit vector $\mathbf{U}$ having the same direction as:

27. $\mathbf{A} + \mathbf{B} = (8\mathbf{i} + 5\mathbf{j}) + (3\mathbf{i} - \mathbf{j}) = 11\mathbf{i} + 4\mathbf{j}$; $\|\mathbf{A} + \mathbf{B}\| = \sqrt{11^2 + 4^2} = \sqrt{121 + 16} = \sqrt{137}$; $\mathbf{U} = \dfrac{11}{\sqrt{137}}\mathbf{i} + \dfrac{4}{\sqrt{137}}\mathbf{j}$

28. $\mathbf{A} - \mathbf{B}$

▷ $$\mathbf{A} - \mathbf{B} = (8\mathbf{i} + 5\mathbf{j}) - (3\mathbf{i} - \mathbf{j}) = 5\mathbf{i} + 6\mathbf{j}$$

We apply Theorem 10.1.10. Because

$$\|\mathbf{A} - \mathbf{B}\| = \|5\mathbf{i} + 6\mathbf{j}\| = \sqrt{5^2 + 6^2} = \sqrt{61}$$

then the unit vector having the same direction as $\mathbf{A} - \mathbf{B}$ is

$$\mathbf{U} = \frac{5}{\sqrt{61}}\mathbf{i} + \frac{6}{\sqrt{61}}\mathbf{j}$$

In Exercises 29–32, write the given vector in the form $r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$, where r is the magnitude and θ is the direction angle. Also find a unit vector having the same direction.

29. (a) $3\mathbf{i} - 4\mathbf{j}$. $r = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$; $3\mathbf{i} - 4\mathbf{j} = 5(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j})$; $\mathbf{U} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

(b) $2\mathbf{i} + 2\mathbf{j}$. $r = 2\|\mathbf{i} + \mathbf{j}\| = 2\sqrt{1^2 + 1^2} = 2\sqrt{2}$; $2\mathbf{i} + 2\mathbf{j} = 2\sqrt{2}(\frac{1}{2}\sqrt{2}\mathbf{i} + \frac{1}{2}\sqrt{2}\mathbf{j}) = 2\sqrt{2}(\cos\frac{1}{4}\pi\mathbf{i} + \sin\frac{1}{4}\pi\mathbf{j})$.
$\mathbf{U} = \frac{1}{2}\sqrt{2}\mathbf{i} + \frac{1}{2}\sqrt{2}\mathbf{j}$

30. (a) $8\mathbf{i} + 6\mathbf{j}$. $r = \sqrt{8^2 + 6^2} = 10$. $8\mathbf{i} + 6\mathbf{j} = 10(\frac{8}{10}\mathbf{i} + \frac{6}{10}\mathbf{j})$. $\mathbf{U} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$

(b) $2\sqrt{5}\mathbf{i} + 4\mathbf{j}$. $r = \sqrt{20 + 16} = 6$. $2\sqrt{5}\mathbf{i} + 4\mathbf{j} = 6(\frac{1}{3}\sqrt{5}\mathbf{i} + \frac{2}{3}\mathbf{j})$. $\mathbf{U} = \frac{1}{3}\sqrt{5}\mathbf{i} + \frac{2}{3}\mathbf{j}$

31. (a) $r = \|-4\mathbf{i} + 4\sqrt{3}\mathbf{j}\| = \sqrt{(-4)^2 + (4\sqrt{3})^2} = \sqrt{16 + 48} = \sqrt{64} = 8$
$-4\mathbf{i} + 4\sqrt{3}\mathbf{j} = 8(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}) = 8(\cos\frac{2}{3}\pi\mathbf{i} + \sin\frac{2}{3}\pi\mathbf{j})$; $\mathbf{U} = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$

(b) $r = \|-16\mathbf{i}\| = 16\|\mathbf{i}\| = 16$; $-16\mathbf{i} = 16(-\mathbf{i} + 0\mathbf{j}) = 16(\cos\pi\mathbf{i} + \sin\pi\mathbf{j})$; $\mathbf{U} = -\mathbf{i}$

32. (a) $3\mathbf{i}-3\mathbf{j}$; (b) $2\mathbf{j}$

▶ $r=\|3\mathbf{i}-3\mathbf{j}\|=\sqrt{3^2+(-3)^2}=3\sqrt{2}$

Because $\cos\theta=\frac{a_1}{r}=\frac{3}{3\sqrt{2}}=\frac{1}{\sqrt{2}}$ and $\sin\theta=\frac{a_2}{r}=\frac{-3}{3\sqrt{2}}=-\frac{1}{\sqrt{2}}$ then $\theta=\frac{7}{4}\pi$. Therefore, we have

$3\mathbf{i}-3\mathbf{j}=3\sqrt{2}(\cos\frac{7}{4}\pi\mathbf{i}+\sin\frac{7}{4}\pi\mathbf{j})$

A unit vector having the same direction as the given vector is $\cos\frac{7}{4}\pi\mathbf{i}+\sin\frac{7}{4}\pi\mathbf{j}$ or, equivalently, $\frac{1}{2}\sqrt{2}\mathbf{i}-\frac{1}{2}\sqrt{2}\mathbf{j}$

(b) $r=\|2\mathbf{j}\|=\sqrt{2^2}=2$

$\cos\theta=\frac{a_1}{r}=0$ and $\sin\theta=\frac{a_2}{r}=\frac{2}{2}=1$

Thus, $\theta=\frac{1}{2}\pi$ and

$2\mathbf{j}=2(\cos\frac{1}{2}\pi\mathbf{i}+\sin\frac{1}{2}\pi\mathbf{j})$

A unit vector having the same direction as the given vector $2\mathbf{j}$ is $\mathbf{j}$. Note that this result is also given by $\cos\frac{1}{2}\pi\mathbf{i}+\sin\frac{1}{2}\pi\mathbf{j}$.

33. $\mathbf{A}=-2\mathbf{i}+\mathbf{j}$; $\mathbf{B}=3\mathbf{i}-2\mathbf{j}$; $\mathbf{C}=5\mathbf{i}-4\mathbf{j}$; $\mathbf{C}=h\mathbf{A}+k\mathbf{B}\Rightarrow 5\mathbf{i}-4\mathbf{j}=h(-2\mathbf{i}+\mathbf{j})+k(3\mathbf{i}-2\mathbf{j})=(-2h+3k)\mathbf{i}+(h-2k)\mathbf{j}$

Thus $\begin{matrix}-2h+3k=5\\ h-2k=-5\end{matrix}\Rightarrow\begin{matrix}h=2\\ k=3\end{matrix}$.

34. $\mathbf{A}=5\mathbf{i}-2\mathbf{j}$; $\mathbf{B}=-4\mathbf{i}+3\mathbf{j}$; $\mathbf{C}=-6\mathbf{i}+8\mathbf{j}$. $\mathbf{B}=h\mathbf{C}-k\mathbf{A}\Rightarrow -4\mathbf{i}+3\mathbf{j}=h(-6\mathbf{i}+8\mathbf{j})-k(5\mathbf{i}-2\mathbf{j})$. Therefore,

$\begin{matrix}-6h-5k=-4\\ 8h+2k=3\end{matrix}\Rightarrow\begin{matrix}h=\frac{1}{4}\\ k=\frac{1}{2}\end{matrix}$

35. $\mathbf{A}=\mathbf{i}-2\mathbf{j}$; $\mathbf{B}=-2\mathbf{i}+4\mathbf{j}$; $\mathbf{C}=7\mathbf{i}-5\mathbf{j}$. $\mathbf{C}=h\mathbf{A}+k\mathbf{B}\Rightarrow 7\mathbf{i}-5\mathbf{j}=h(\mathbf{i}-2\mathbf{j})+k(-2\mathbf{i}+4\mathbf{j})=(h-2k)\mathbf{i}+(-2h+4k)\mathbf{j}$

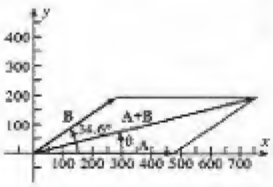
Therefore $\begin{matrix}h-2k=7\\ -2h+4k=-5\end{matrix}\Rightarrow\begin{matrix}-2h+4k=-14\\ -2h+4k=-5\end{matrix}$.

Because this system has no solution, $\mathbf{C}$ cannot be written in the form $h\mathbf{A}+k\mathbf{B}$.

36. Two forces of magnitudes 340 lb and 475 lb make an angle of 34.6° with each other and are applied to an object at the same point. Find (a) the magnitude of the resultant force and (b) to the nearest tenth of a degree the angle it makes with the force of 475 lb.

▶ (a) Refer to the figure. Let $\mathbf{A}=\langle 475,0\rangle$ represent the force of magnitude 475 lb. If $\mathbf{B}=\langle b_1,b_2\rangle$ represents the force of magnitude 340 lb, then because the angle between $\mathbf{A}$ and $\mathbf{B}$ is 34.6°, we have

$b_1=340\cos 34.6°=280$ and $b_2=340\sin 34.6°=193$



Thus, $\mathbf{B}=\langle 280,193\rangle$. The resultant force is $\mathbf{A}+\mathbf{B}$ and

$\mathbf{A}+\mathbf{B}=\langle 475,0\rangle+\langle 280,193\rangle=\langle 755,193\rangle$

Then

$\|\mathbf{A}+\mathbf{B}\|=\sqrt{755^2+193^2}=779$

The magnitude of the resultant force is 779 lb.

(b) If θ is the angle the vector $\mathbf{A}+\mathbf{B}$ makes with $\mathbf{A}$, then

$\tan\theta=\frac{193}{755}=0.2556 \qquad \theta=14.3°$

The angle that the resultant force makes with the force of 475 lb is 14.3°.

37. Choose the axes so that the position representation of the 60 lb force is along the positive x axis. Then the vector $\mathbf{A}=\langle 60,0\rangle$ represents this force. Let the vector $\mathbf{B}=\langle b_1,b_2\rangle$ represent the 80 lb. force. Then $b_1=80\cos 30°\approx 69.3$ and $b_2=80\sin 30°=40$. The resultant force is $\mathbf{A}+\mathbf{B}=\langle 60,0\rangle+\langle 69.3,40\rangle=\langle 129.3,40\rangle$.

(a) $\|\mathbf{A}+\mathbf{B}\|=\sqrt{(129.3)^2+(40)^2}\approx 135.3$

(b) If θ is the angle between $\mathbf{A}$ and $\mathbf{A}+\mathbf{B}$ then $\tan\theta=\frac{40}{129.3}\approx 0.309$; $\theta\approx 17°$.

38. Let $\mathbf{A}=\langle 34,0\rangle$. Then $\mathbf{B}=\langle 22\cos\theta,22\sin\theta\rangle$. $46^2=\|\mathbf{A}+\mathbf{B}\|^2=(34+22\cos\theta)^2+(22\sin\theta)^2$

$=34^2+2\cdot 34\cdot 22\cos\theta+22^2$, $\cos\theta=\frac{46^2-34^2-22^2}{2\cdot 34\cdot 22}=0.3182$, $\theta=71.4°\approx 71°$

39. Draw a figure similar to Figure 14 in the text. Let $a = \|\mathbf{A}\| = 112$, $b = \|\mathbf{B}\| = 84$, $c = \|\mathbf{A} - \mathbf{B}\| = 162$. From the law of cosines, $\cos\theta = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{112^2 + 162^2 - 84^2}{2 \cdot 112 \cdot 162} = 0.874$; $\theta \approx 29.0^\circ$.

40. A plane has an air speed of 350 mi/h. In order for the actual course of the plane to be due north, the compass heading is 340°. If the wind is blowing from the west, (a) what is its speed? (b) What is the plane's ground speed?

▷ Refer to the Figure. Let $\overrightarrow{OB} = \langle b_1, b_2\rangle$ be the plane's velocity. Because the compass heading is 340°, the direction of $\overrightarrow{OB}$ is $450^\circ - 340^\circ = 110^\circ$. Thus

$$b_1 = 350 \cos 110^\circ = -119.7 \quad b_2 = 350 \sin 110^\circ = 119.7$$

Therefore C is the point $(0, 328.9)$.

(a) The magnitude of the wind's velocity is $\|\overrightarrow{BC}\| = |b_1| = 119.7$.
(b) The number of miles per hour in the plane's ground speed is

$$\|\overrightarrow{OC}\| = |b_2| = 328.9$$

41. (a) heading: $360^\circ - \sin^{-1}(60/250) = 360^\circ - 13.9^\circ = 346.1^\circ$ (b) $v = \sqrt{250^2 - 60^2} = 242.7$ (mi/hr)

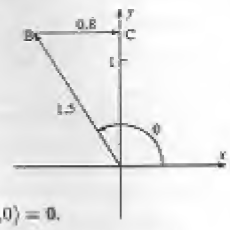
42. $v = \sqrt{15^2 + 3^2} = 15.30$ knots, course $= 180^\circ + \tan^{-1}\frac{3}{15} = 191.31^\circ$

43. (a) heading: $\tan^{-1}\frac{0.8}{1.5} = 28.1^\circ$ (b) $v = \sqrt{1.5^2 + .8^2} = 1.7$ (c) in $\frac{1}{1.5}$hr he moves $\frac{0.8}{1.5} = 0.53$ mi

44. A swimmer who can swim at a speed of 1.5 mi/h relative to the water leaves the south bank of a river whose current is toward the east at 0.8 mi/h. He wishes to reach a point directly north across the river. (a) In what direction should he head? (b) What will be his speed relative to the land if this direction is taken?

▷ Refer to the figure. Let $\overrightarrow{OB} = \langle b_1, b_2\rangle$ be his 's velocity and θ his direction. Then

$$b_1 = -0.8 = 1.5 \cos\theta$$
$$\cos\theta = -\frac{0.8}{1.5} = -0.5333$$
$$b_2 = 1.5 \sin 122.2 = 1.27$$



(a) The swimmer's heading should be $450^\circ - 122.2^\circ = 327.8^\circ$.

(b) His speed relative to the land will be 1.27 mi/h.

45. Let $\mathbf{A} = \langle a_1, a_2\rangle$. The $0\mathbf{A} = 0\langle a_1, a_2\rangle = \langle 0a_1, 0a_2\rangle = \mathbf{0}$ and $c\mathbf{0} = c\langle 0,0\rangle = \langle c0, c0\rangle = \langle 0,0\rangle = \mathbf{0}$.

46. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \langle a_1, a_2\rangle + (\langle b_1, b_2\rangle + \langle c_1, c_2\rangle) = \langle a_1, a_2\rangle + \langle b_1 + c_1, b_2 + c_2\rangle = \langle a_1 + (b_1 + c_1), a_2 + (b_2 + c_2)\rangle$
$= \langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2\rangle = \langle a_1 + b_1, a_2 + b_2\rangle + \langle c_1, c_2\rangle = (\langle a_1, a_2\rangle + \langle b_1, b_2\rangle) + \langle c_1, c_2\rangle = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

47. Let $\mathbf{A} = \langle a_1, a_2\rangle$. Then $\mathbf{A} + \mathbf{0} = \langle a_1, a_2\rangle + \langle 0,0\rangle = \langle a_1 + 0, a_2 + 0\rangle = \langle a_1, a_2\rangle = \mathbf{A}$. Also
$1\mathbf{A} = 1\langle a_1, a_2\rangle = \langle 1a_1, 1a_2\rangle = \langle a_1, a_2\rangle = \mathbf{A}$

48. Prove Theorem 10.1.8(iv).

▷ We are asked to prove that if $\mathbf{A}$ is any vector then there is a vector $-\mathbf{A}$ such that $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$. Let $\mathbf{A} = \langle a_1, a_2\rangle$ and let $-\mathbf{A} = \langle -a_1, -a_2\rangle$ be the vector of Definition 10.1.5. Then, by Definition 10.1.4

$$\mathbf{A} + (-\mathbf{A}) = \langle a_1 + (-a_1), a_2 + (-a_2)\rangle = \langle 0,0\rangle = \mathbf{0}$$

49. Let $\mathbf{A} = \langle a_1, a_2\rangle$. Then $(cd)\mathbf{A} = (cd)\langle a_1, a_2\rangle = \langle (cd)a_1, (cd)a_2\rangle = \langle c(da_1), c(da_2)\rangle = c\langle da_1, da_2\rangle = c(d\mathbf{A})$

50. $(c + d)\mathbf{A} = (c + d)\langle a_1, a_2\rangle = \langle (c + d)a_1, (c + d)a_2\rangle = \langle ca_1 + da_1, ca_2 + da_2\rangle = \langle ca_1, ca_2\rangle + \langle da_1, da_2\rangle$
$= c\langle a_1, a_2\rangle + d\langle a_1, a_2\rangle = c\mathbf{A} + d\mathbf{A}$

51. (a) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \langle 2, -5\rangle + (\langle 3,1\rangle + \langle -4,2\rangle) = \langle 2, -5\rangle + \langle -1, 3\rangle = \langle 1, -2\rangle$
(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = (\langle 2, -5\rangle + \langle 3, 1\rangle) + \langle -4, 2\rangle = \langle 5, -4\rangle + \langle -4, 2\rangle = \langle 1, -2\rangle$

52. Two vectors are said to be *independent* if and only if their position representations are not collinear. Furthermore, two vectors **A** and **B** are said to form a *basis* for the vector space V_2 if and only if any vector in V_2 can be written as a linear combination of **A** and **B**. A theorem can be proved which states that two vectors form a basis for the vector space V_2 if they are independent. Show that this theorem holds for the two vectors $\langle 2,5\rangle$ and $\langle 3,-1\rangle$ by doing the following: (a) Verify that the vectors are independent by showing that the position representations are not collinear; (b) verify that the vectors form a basis by showing that any vector $a_1\mathbf{i}+a_2\mathbf{j}$ can be written as $c(2\mathbf{i}+5\mathbf{j})+d(3\mathbf{i}-\mathbf{j})$, where c and d are scalars. (Hint. Find c and d in terms of a_1 and a_2.)

▸ (a) If $A=\langle 2,5\rangle$ then $\overrightarrow{OA}$ is the position representation of vector $\langle 2,5\rangle$. If $B=\langle 3,-1\rangle$, then $\overrightarrow{OB}$ is the position representation of vector $\langle 3,-1\rangle$. Because the slope of $\overrightarrow{OA}$ is $\frac{5}{2}$ and the slope of $\overrightarrow{OB}$ is $-\frac{1}{3}\neq\frac{5}{2}$, the position representations are not collinear and thus the vectors $\langle 2,5\rangle$ and $\langle 3,-1\rangle$ are independent.

(b) If $a_1\mathbf{i}+b_1\mathbf{j}=c(2\mathbf{i}+5\mathbf{j})+d(3\mathbf{i}-\mathbf{j})=(2c+3d)\mathbf{i}+(5c-d)\mathbf{j}$ then

$$a_1=2c+3d$$
$$a_2=5c-d$$

Therefore,

$$a_1+3a_2=2c+3(5c)=17c \qquad 5a_1-2a_2=5(3d)+2d=17d$$
$$c=\tfrac{1}{17}(a_1+3a_2 \qquad d=\tfrac{1}{17}(5a_1-2a_2)$$

Thus any vector can be written as a linear combination of the vectors $\langle 2,5\rangle$ and $\langle 3,-1\rangle$. That is, with the values of c and d above,

$\langle a_1,a_2\rangle=c\langle 2,5\rangle+d\langle 3,-1\rangle$

53. (a) Let $\mathbf{A}=\langle 3,-2\rangle$ and $\mathbf{B}=\langle -6,4\rangle$. Because $\mathbf{B}=-2\mathbf{A}$, **B** is a scalar multiple of **A**. Hence representations of the two vectors are parallel. Thus the position representations of the two vectors are collinear.

(b) Let $\mathbf{C}=\mathbf{i}+\mathbf{j}$. If $\mathbf{C}=c\mathbf{A}+d\mathbf{B}$ then $\mathbf{i}+\mathbf{j}=c(3\mathbf{i}-2\mathbf{j})+d(-6\mathbf{i}+4\mathbf{j})=(3c-6d)\mathbf{i}+(-2c+4d)\mathbf{j}$.

$\begin{matrix}3c-6d=1\\-2c+4d=1\end{matrix}\Rightarrow\begin{matrix}6c-12d=2\\6c-12d=-3\end{matrix}$. Since this system has no solution, **A** and **B** are not a basis.

54. $a(3\mathbf{i}-2\mathbf{j})+b(\mathbf{i}+4\mathbf{j})+(2\mathbf{i}+5\mathbf{j})=(3a+b+2)\mathbf{i}+(-2a+4b+5)\mathbf{j}=\mathbf{0}\Rightarrow\begin{matrix}3a+b=-2\\-2a+4b=-5\end{matrix}\Rightarrow a=-\frac{3}{14},\ b=-\frac{19}{14}$

55. We are given $\mathbf{A}=\mathbf{V}(\overrightarrow{PQ})$, $\mathbf{B}=\mathbf{V}(\overrightarrow{QR})$, $\mathbf{C}=\mathbf{V}(\overrightarrow{RS})$. Because $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{RS}$ are sides of a triangle, S is the same as point P. From Figure 13 of the text we see that $\mathbf{V}(\overrightarrow{PQ})+\mathbf{V}(\overrightarrow{QR})=\mathbf{V}(\overrightarrow{PR})$. Hence

$$\mathbf{A}+\mathbf{B}+\mathbf{C}=\mathbf{V}(\overrightarrow{PQ})+\mathbf{V}(\overrightarrow{QR})+\mathbf{V}(\overrightarrow{RS})=\mathbf{V}(\overrightarrow{PR})+\mathbf{V}(\overrightarrow{RS})=\mathbf{V}(\overrightarrow{PS})=\mathbf{0}$$

56. Prove analytically the triangle inequality for vectors $\|\mathbf{A}+\mathbf{B}\|\leq\|\mathbf{A}\|+\|\mathbf{B}\|$.

▸ Let $\mathbf{A}=\langle a_1,a_2\rangle$ and $\mathbf{B}=\langle b_1,b_2\rangle$. Then $\mathbf{A}+\mathbf{B}=\langle a_1+b_1,a_2+b_2\rangle$.

$$\begin{aligned}\|\mathbf{A}+\mathbf{B}\|^2&=(a_1+b_1)^2+(a_2+b_2)^2=(a_1{}^2+2a_1b_1+b_1{}^2)+(a_2{}^2+2a_2b_2+b_2{}^2)\\&=(a_1{}^2+a_2{}^2)+(b_1{}^2+b_2{}^2)+2(a_1b_1+a_2b_2)=\|\mathbf{A}\|^2+\|\mathbf{B}\|^2+2(a_1b_1+a_2b_2)\\&\leq\|\mathbf{A}\|^2+\|\mathbf{B}\|^2+2|a_1b_1+a_2b_2|\end{aligned}\tag{1}$$

We prove the Cauchy-Schwarz inequality. Consider the quadratic equation

$$(a_1x-b)^2+(a_2x-b_2)^2=0\tag{2}$$

or, equivalently,

$$(a_1{}^2+a_2{}^2)x^2-2(a_1b_1+a_2b_2)x+(b_1{}^2+b_2{}^2)=0\tag{3}$$

From (2) we see that if $x=k$ is a solution then we must have

$$a_1k-b_1=0\quad a_2k-b_2=0$$

or, equivalently,

$$b_1=ka_1\quad b_2=ka_2\tag{4}$$

that is, **B** is a scalar multiple of **A**, and otherwise there is no real solution. Therefore, the discriminant of the quadratic equation (3) is either negative or 0, that is

$$[-2(a_1b_1+a_2b_2)]^2-4(a_1{}^2+a_2{}^2)(b_1{}^2+b_2{}^2)\leq 0$$

or, equivalently,

$$(a_1b_1 + a_2b_2)^2 \le (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

with equality if and only if (4) holds. Taking the square root of both sides, we have

$$|a_1b_1 + a_2b_2| \le \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2} \tag{5}$$

Substituting from (5) into (1), we obtain

$$\|\mathbf{A} + \mathbf{B}\|^2 \le \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + 2\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2} = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 + 2\|\mathbf{A}\|\|\mathbf{B}\| = (\|\mathbf{A}\| + \|\mathbf{B}\|)^2$$

Taking the square root of both sides gives

$$\|\mathbf{A} + \mathbf{B}\| \le \|\mathbf{A}\| + \|\mathbf{B}\|$$

with equality if and only if (4) holds for some $k \ge 0$, that is, if and only if **A** and **B** have the same direction or either is the zero vector.

10.2 VECTORS IN THREE DIMENSIONAL SPACE

10.2.1 Definition The set of all ordered triples of real numbers is called the *three-dimensional number space* and is denoted by R^3. Each ordered triple (x, y, z) is called a point in the three-dimensional number space. The x axis is $\{(x, y, z) \mid y = 0, z = 0\}$; similarly for the y and z axes. The *first octant* is $\{(x, y, z) \mid x > 0, y > 0, z > 0\}$.

We use a *right-handed system*: if the x axis is pointing to us, the positive y axis rotates in the positive direction (counterclockwise) to coincide with the positive z axis.

10.2.2-3 Theorem

A line is parallel to	if and only if all points on the line have
the yz plane	equal x coordinates
the xz plane	equal y coordinates
the xy plane	equal z coordinates
the x axis	equal y coordinates and equal z coordinates
the y axis	equal x coordinates and equal z coordinates
the z axis	equal x coordinates and equal y coordinates

10.2.5 Theorem The undirected distance between the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$|\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

10.2.7 Definition The *graph of an equation* in R^3 is the set of all points (x, y, z) whose coordinates are numbers satisfying the equation.

A surface is the graph of an equation in R^3. (A graph like $[\![x]\!] = 1$ is not a surface. In §12.7 we formally define "surface" similar to "curve" in §11.1.) One particular surface is the sphere.

10.2.8 Definition A *sphere* is the set of all points in three-dimensional space equidistant from a fixed point. The fixed point is called the *center* of the sphere and the measure of the constant distance is called the *radius* of the sphere.

10.2.9 Theorem An equation of the sphere of radius r and center at (h, k, l) is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

An equation of the sphere with diameter endpoints $A(h_1, k_1, l_1)$ and $B(h_2, k_2, l_2)$ is (Ex. 26)

$$(x - h_1)(x - h_2) + (y - k_1)(y - k_2) + (z - l_1)(z - l_2) = 0$$

10.2.10 Theorem The graph of a second-degree equation in x, y, and z, of the form

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

is either a sphere, a point, or the empty set.

10.2.11 Definition A *vector in three-dimensional space* is an ordered triple of real numbers $\langle x, y, z\rangle$. The numbers x, y, and z are called the *components* of the vector $\langle x, y, z\rangle$.

Vectors $\langle x, y, z\rangle$ and $\langle u, v, w\rangle$ are *equal* if and only if $x = u$, $y = v$, and $z = w$. The *position representation* of the vector $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ is the directed line segment from the origin to the

point (a_1, a_2, a_3). The *zero vector* is the vector $\langle 0, 0, 0\rangle$ and is denoted by $\mathbf{0}$.

The magnitude of the vector $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ is given by $\|\mathbf{A}\| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$. $\|\mathbf{A}\| = 0$ if and only if $\mathbf{A} = \mathbf{0}$.

10.2.12 Definition The *direction angles* of a nonzero vector are the three angles that have the smallest nonnegative radian measure α, β, γ measured from the positive x, y, and z axes, respectively, to the position representation of the vector.

Thus the direction angles for the vector $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ are α, β and γ with $0 \le \alpha \le \pi$, $0 \le \beta \le \pi$, and $0 \le \gamma \le \pi$, such that

$$\cos\alpha = \frac{a_1}{\|\mathbf{A}\|} \quad \cos\beta = \frac{a_2}{\|\mathbf{A}\|} \quad \cos\gamma = \frac{a_3}{\|\mathbf{A}\|}$$

The three numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of vector $\mathbf{A}$.

10.2.13 Theorem If $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines of a vector, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

The remaining definitions and theorems in this section are extensions of the corresponding definitions for vectors in V_2.

Definition If $\mathbf{A} = \langle a_1, a_2, a_3\rangle$, $\mathbf{B} = \langle b_1, b_2, b_3\rangle$, and c is a scalar, then

$$\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$$
$$-\mathbf{A} = \langle -a_1, -a_2, -a_3\rangle$$
$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle$$
$$c\mathbf{A} = c\langle a_1, a_2, a_3\rangle = \langle ca_1, ca_2, ca_3\rangle$$

Theorem $c\mathbf{A}$ has the same direction as $\mathbf{A}$ if $c > 0$; the opposite direction if $c < 0$. Also, $\|c\mathbf{A}\| = |c|\,\|\mathbf{A}\|$.

10.2.6 Theorem The coordinates of the midpoint of the line segment having endpoints $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are given by

$$\bar{x} = \tfrac{1}{2}(x_1 + x_2) \quad \bar{y} = \tfrac{1}{2}(y_1 + y_2) \quad \bar{z} = \tfrac{1}{2}(z_1 + z_2)$$

Theorem 10.2.6 says that the position representation of the midpoint P of segment P_1P_2 is given by $\mathbf{p} = \frac{1}{2}\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_2$. More generally, if P is the fraction w of the way from P_1 to P_2, then $\mathbf{p} = (1 - w)\mathbf{p}_1 + w\mathbf{p}_2$.

V_3 is the set of vectors together with addition and scalar multiplication and satisfies the properties given in Theorem 10.2.8. Thus V_3 is a real vector space. The three unit vectors

$\mathbf{i} = \langle 1, 0, 0\rangle$ $\mathbf{j} = \langle 0, 1, 0\rangle$ $\mathbf{k} = \langle 0, 0, 1\rangle$

form a basis for V_3 because any vector can be expressed in terms of them as follows:

$\mathbf{A} = \langle a_1, a_2, a_3\rangle = a_1\langle 1, 0, 0\rangle + a_2\langle 0, 1, 0\rangle + a_3\langle 0, 0, 1\rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

10.2.14 Theorem If the nonzero vector $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then the unit vector $\mathbf{U}$ having the same direction as $\mathbf{A}$ is given by

$$\mathbf{U} = \frac{a_1}{\|\mathbf{A}\|}\mathbf{i} + \frac{a_2}{\|\mathbf{A}\|}\mathbf{j} + \frac{a_3}{\|\mathbf{A}\|}\mathbf{k}$$

Exercises 10.2

In Exercises 1–5, points A and B are opposite vertices of a rectangular parallelopiped, having its faces parallel to the coordinate planes. In each exercise, (a) sketch the figure, (b) find the coordinates of the other six vertices, (c) Find the length of the diagonal AB.

▷ If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ are opposite vertices of a parallepiped having its faces parallel to the coordinate planes, then any vertex has the form (c_1, c_2, c_3) where $c_i = a_i$ or b_i, $i = 1, 2, 3$.

1. A(0, 0, 0), B(7, 2, 3) (b) (7, 2, 0), (0, 0, 3), (0, 2, 0), (0, 2, 3), (7, 0, 3), (7, 0, 0)

   (c) $|\overline{AB}| = \sqrt{(7-0)^2 + (2-0)^2 + (3-0)^2} = \sqrt{49 + 4 + 9} = \sqrt{62}$

2. A(1, 1, 1); B(3, 4, 2) (b) (1, 1, 2), (1, 4, 1), (1, 4, 2), (3, 1, 1), (3, 1, 2), (3, 4, 1)

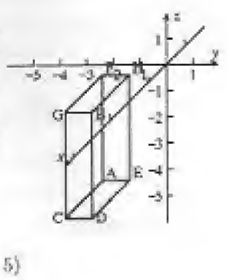
   (c) $|\overline{AB}| = \sqrt{(3-1)^2 + (4-1)^2 + (2-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$

3. A = (−1, 1, 2), B = (2, 3, 5). (b) (2, 1, 2), (−1, 3, 2), (−1, 1, 5), (2, 3, 2), (−1, 3, 5), (2, 1, 5)

(c) $|\overline{AB}| = \sqrt{(2+1)^2 + (3-1)^2 + (5-2)^2} = \sqrt{9+4+9} = \sqrt{22}$

4. A(2, −1, −3); B(4, 0, 1)

▹ (a) The figure shows the parallelopiped. Rectangles ACDE and FGBH are parallel faces.



(b) We use Theorems 10.2.2-3 to find the coordinates of vertex C. Because line BC is parallel to the yz plane, then points B and C have equal x coordinates. Because line AC is parallel to the x axis, then points A and C have equal y coordinates and equal z coordinates. Thus C = (4, −1, −3). By similar reasoning, we find the coordinates of the remaining vertices. Therefore, D = (4, 0, 3); E = (2, 0, −3); F = (2, −1, −1); G = (4, −1, −1); and H = (2, 0, −1).

(c) We use the distance formula. Thus,

$|\overline{AB}| = \sqrt{(4-2)^2 + (0+1)^2 + (-1+3)^2} = 3$

5. A = (1, −1, 0), B = (3, 3, 5). (b) (3, −1, 0), (3, 3, 0), (1, 3, 0), (1, 3, 5), (1, −1, 5), (3, −1, 5)

(c) $|\overline{AB}| = \sqrt{(3-1)^2 + (3+1)^2 + (5-0)^2} = \sqrt{4+16+25} = \sqrt{45} = 3\sqrt{5}$

6. (b) $d = \sqrt{18^2 + 15^2 + 12^2} = 3\sqrt{6^2 + 5^2 + 4^2} = 3\sqrt{77}$

In Exercises 7–11, find (a) the undirected distance between points A and B and (b) the midpoint of segment AB.

7. A(3, 4, 2), B(1, 6, 3). (a) $|\overline{AB}| = \sqrt{(3-1)^2 + (4-6)^2 + (2-3)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

(b) The midpoint of segment AB is $\left(\frac{1}{2}(3+1), \frac{1}{2}(4+6), \frac{1}{2}(2+3)\right) = \left(2, 5, \frac{5}{2}\right)$.

8. A(4, −3, 2); B(−2, 3, −5)

▹ (a) Applying the distance formula, we obtain

$|AB| = \sqrt{(-2-4)^2 + (3+3)^2 + (-5-2)^2} = \sqrt{6^2 + 6^2 + 7^2} = 11$

(b) We apply Theorem 10.2.6 to find $(\bar{x}, \bar{y}, \bar{z})$, the midpoint of segment AB. Thus,

$\bar{x} = \frac{1}{2}(4-2) = 1 \quad \bar{y} = \frac{1}{2}(-3+3) = 0 \quad \bar{z} = \frac{1}{2}(2-5) = -\frac{3}{2}$

Hence the midpoint of segment AB is $\left(1, 0, -\frac{3}{2}\right)$.

9. A(2, −4, 1), B$\left(\frac{1}{2}, 2, 3\right)$. (a) $|\overline{AB}| = \sqrt{\left(2-\frac{1}{2}\right)^2 + (-4-2)^2 + (1-3)^2} = \sqrt{\frac{9}{4} + 36 + 4} = \frac{1}{2}\sqrt{169} = \frac{13}{2}$

(b) The midpoint of segment AB is $\left(\frac{1}{2}\left(2+\frac{1}{2}\right), \frac{1}{2}(-4+2), \frac{1}{2}(1+3)\right) = \left(\frac{5}{4}, -1, 2\right)$.

10. A$\left(-2, -\frac{1}{2}, 5\right)$; B(5, 1, −4). (a) $|\overline{AB}| = \sqrt{(5+2)^2 + \left(1+\frac{1}{2}\right)^2 + (-4-5)^2} = \sqrt{49 + \frac{9}{4} + 81} = \frac{1}{2}\sqrt{529} = \frac{23}{2}$

(b) The midpoint of segment AB is $\left(\frac{1}{2}(-2+5), \frac{1}{2}\left(-\frac{1}{2}+1\right), \frac{1}{2}(5-4)\right) = \left(\frac{3}{2}, \frac{1}{4}, \frac{1}{2}\right)$

11. A(−5, 2, 1), B(3, 7, −2). (a) $|\overline{AB}| = \sqrt{(-5-3)^2 + (2-7)^2 + (1+2)^2} = \sqrt{64+25+9} = \sqrt{98} = 7\sqrt{2}$

(b) The midpoint of segment AB is $\left(\frac{1}{2}(-5+3), \frac{1}{2}(2+7), \frac{1}{2}(1-2)\right) = \left(-1, \frac{9}{2}, -\frac{1}{2}\right)$.

12. Prove that the three points A(1, −1, 3), B(2, 1, 7), C(4, 2, 6) are the vertices of a right triangle. Find its area.

▹ Applying the distance formula, we get

$|AB| = \sqrt{(2-1)^2 + (1+1)^2 + (7-3)^2} = \sqrt{21}$

$|BC| = \sqrt{(4-2)^2 + (2-1)^2 + (6-7)^2} = \sqrt{6}$

$|AC| = \sqrt{(4-1)^2 + (2+1)^2 + (6-3)^2} = \sqrt{27}$

Because $|AB|^2 + |BC|^2 = 21 + 6 = 27 = |AC|^2$, then triangle ABC is a right triangle with hypotenuse side AC. To find the area, we use the area formula $\frac{1}{2}bh$. We take $b = |AB| = \sqrt{21}$ and $h = |BC| = \sqrt{6}$. Thus, the number of square units in the area is $\frac{1}{2}\sqrt{21}\sqrt{6} = \frac{3}{2}\sqrt{14}$.

13. Because the line is perpendicular to the yz plane, it is parallel to the x axis. Thus any point on this line differs from (6, 4, 2) only in its x coordinate and so has the form $(x, 4, 2)$. Because the distance from $(x, 4, 2)$ to the point (0, 4, 0) is 10 units then $\sqrt{(x-0)^2 + (4-4)^2 + (2-0)^2} = 10$; $x^2 + 4 = 100$, $x^2 = 96$; $x = \pm 4\sqrt{6}$

Therefore the required points are $(4\sqrt{6}, 4, 2)$ and $(-4\sqrt{6}, 4, 2)$.

14. The point $(6,4,z)$ is 10 units from $(0,4,0)$. Thus, $\sqrt{(6-0)^2+(4-4)^2+(z-0)^2}=10^2$; $36+z^2=100$; $z^2=64$; $z=\pm 8$. The points are $(6,4,8)$ and $(6,4,-8)$.

15. Let $A=(-3,2,4)$, $B=(6,1,2)$, $C=(-12,3,6)$. Then
$|\overline{AB}|=\sqrt{(6+3)^2+(1-2)^2+(2-4)^2}=\sqrt{81+1+4}=\sqrt{86}$
$|\overline{AC}|=\sqrt{(-12+3)^2+(3-2)^2+(6-4)^2}=\sqrt{81+1+4}=\sqrt{86}$
$|\overline{BC}|=\sqrt{(-12-6)^2+(3-1)^2+(6-2)^2}=\sqrt{324+4+16}=\sqrt{344}=2\sqrt{86}$
If A, B, C are vertices of a triangle then $|AB|+|AC|>|BC|$. But $|AB|+|AC|=\sqrt{86}+\sqrt{86}=|BC|$. Hence A, B, and C are not the vertices of a triangle, and so they are collinear.

16. Find the vertices of the triangle whose sides have midpoints at $D(3,2,3)$, $E(-1,1,5)$, and $F(0,3,4)$.
▸ Let A, B, C be the vertices of the triangle, with D the midpoint of BC, E the midpoint of AC and F the midpoint of AB. Then DEAF is a parallelogram, and and using position vectors we have

$$\overrightarrow{FA}=\overrightarrow{DE}$$
$$\mathbf{a}-\mathbf{f}=\mathbf{e}-\mathbf{d}$$
$$\mathbf{a}=-\mathbf{d}+\mathbf{e}+\mathbf{f}=-\langle 3,2,3\rangle+\langle 0,3,4\rangle+\langle -1,1,5\rangle=\langle -4,2,6\rangle$$

Similarly,

$$\mathbf{b}=\mathbf{d}-\mathbf{e}+\mathbf{f}=\langle 3,2,3\rangle-\langle 0,3,4\rangle+\langle -1,1,5\rangle=\langle 2,0,4\rangle$$
$$\mathbf{c}=\mathbf{d}+\mathbf{e}-\mathbf{f}=\langle 3,2,3\rangle+\langle 0,3,4\rangle-\langle -1,1,5\rangle=\langle 4,4,2\rangle$$

Thus, $A=(-4,2,6)$ $B=(2,0,4)$ $C=(4,4,2)$

17. $A=(2,-5,3)$, $B=(-1,7,0)$, $C=(-4,9,7)$
(a) $|\overline{AB}|=\sqrt{(2+1)^2+(-5-7)^2+(3-0)^2}=\sqrt{9+144+9}=\sqrt{162}=9\sqrt{2}$
$|\overline{AC}|=\sqrt{(2+4)^2+(-5-9)^2+(3-7)^2}=\sqrt{36+196+16}=\sqrt{248}=2\sqrt{62}$
$|\overline{BC}|=\sqrt{(-1+4)^2+(7-9(^2+(0-7)^2}=\sqrt{9+4+49}=\sqrt{62}$
(b) The midpoint of side AB is $\left(\frac{1}{2}(2-1),\frac{1}{2}(-5+7),\frac{1}{2}(3+0)\right)=\left(\frac{1}{2},1,\frac{3}{2}\right)$.
The midpoint of side AC is $\left(\frac{1}{2}(2-4),\frac{1}{2}(-5+9),\frac{1}{2}(3+7)\right)=(-1,2,5)$.
The midpoint of side BC is $\left(\frac{1}{2}(-1-4),\frac{1}{2}(7+9),\frac{1}{2}(0+7)\right)=\left(-\frac{5}{2},8,\frac{7}{2}\right)$.

18. If P is the fraction w of the way from P_1 to P_2 then $\overrightarrow{P_1P}=w\overrightarrow{P_1P_2}$ and so $\mathbf{p}-\mathbf{p}_1=w(\mathbf{p}_2-\mathbf{p}_1)$ and so $\mathbf{p}=(1-w)\mathbf{p}_1+w\mathbf{p}_2$. If $w=\frac{1}{2}$, we have $\mathbf{p}=\frac{1}{2}(\mathbf{p}_1+\mathbf{p}_2)=\left(\frac{1}{2}(x_1+x_2),\frac{1}{2}(y_1+y_2),\frac{1}{2}(z_1,z_2)\right)$

19.
$$x^2+y^2+z^2+Gx+Hy+Iz+J=0$$
$$(x^2+Gx+\tfrac{1}{4}G^2)+(y^2+Hy+\tfrac{1}{4}H^2)+(z^2+Iz+\tfrac{1}{4}I^2)=\tfrac{1}{4}(G^2+H^2+I^2-4J)$$
$$(x+\tfrac{1}{2}G)^2+(y+\tfrac{1}{2}H)^2+(z+\tfrac{1}{2}I)^2=\tfrac{1}{4}(G^2+H^2+I^2-4J)$$
$$(x-h)^2+(y-k)^2+(z-l)^2=K$$
where $h=-\frac{1}{2}G$, $k=-\frac{1}{2}H$, $l=-\frac{1}{2}I$, and $K=\frac{1}{4}(G^2+H^2+I^2-4J)$.

In Exercises 20-25, determine the graph of the equation.

20. $x^2+y^2+z^2-8y+6z-25=0$
▸ We complete the squares on the terms in y and z. Thus,

$$x^2+(y^2-8y+\underline{16})+(z^2+6z+\underline{9})=25+\underline{16}+\underline{9}$$
$$x^2+(y-4)^2+(z+3)^2=50$$

By Theorem 10.2.9, we find that the graph is a sphere with center at $(0,4,-3)$ and radius $\sqrt{50}=5\sqrt{2}$ units.

21.
$$x^2+y^2+z^2-8x+4y+2z-4=0$$
$$(x^2-8x+16)+(y^2+4y+4)+(z^2+2z+1)=4+16+4+1$$
$$(x-4)^2+(y+2)^2+(z+1)^2=25$$
The graph is the sphere with center at $(4,-2,-1)$ and $r=5$.

22.
$$x^2 + y^2 + z^2 - x - y - 3z + 2 = 0$$
$$(x^2 - x + \tfrac{1}{4}) + (y^2 - y + \tfrac{1}{4}) + (z^2 - 3z + \tfrac{9}{4}) = -2 + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{9}{4}$$
$$(x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 + (z - \tfrac{3}{2})^2 = \tfrac{3}{4}$$

The graph is the sphere with center at $(\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ and $r = \frac{1}{2}\sqrt{3}$.

23. $x^2 + y^2 + z^2 - 6z + 9 = 0$; $x^2 + y^2 + (z-3)^2 = 0$. The graph is the point $(0, 0, 3)$.

24. $x^2 + y^2 + z^2 - 8x + 10y - 4z + 13 = 0$

▸ We complete the squares.

$$(x^2 - 8x + \underline{16}) + (y^2 + 10y + \underline{25}) + (z^2 - 4z + \underline{4}) = -13 + \underline{16} + \underline{4}$$
$$(x-4)^2 + (y+5)^2 + (z-2)^2 = 32$$

By Theorem 10.2.9, we find that the graph is a sphere with center at $(4, -5, 2)$ and radius $\sqrt{32} = 4\sqrt{2}$ units.

25.
$$x^2 + y^2 + z^2 - 6x + 2y - 4z + 19 = 0$$
$$(x^2 - 6x + 9) + (y^2 + 2y + 1) + (z^2 - 4z + 4) = -19 + 9 + 1 + 4$$
$(x-3)^2 + (y+1)^2 + (z-2)^2 = -5$. The graph is the empty set.

In Exercises 26–28, find an equation of the sphere satisfying the conditions.

26. (a) A diameter has endpoints $A(h_1, k_1, l_1)$ and $B(h_2, k_2, l_2)$.

▸ If $P(x, y, z)$ is on the sphere, then APB is a right triangle and so $AP^2 + BP^2 = AB^2$, that is
$$(x-h_1)^2 + (y-k_1)^2 + (z-l_1)^2 + (x-h_2)^2 + (y-k_2)^2 + (z-l_2)^2 = (h_2-h_1)^2 + (k_2-k_1)^2 + (l_2-l_1)^2$$
$$2\{[x^2 - (h_1+h_2)x + h_1h_2] + [y^2 - (k_1+k_2)y + k_1k_2] + [z^2 - (l_1+l_2)z + l_1l_2]\} = 0$$
(b) A diameter has endpoints $(6, 2, -5)$ and $(-4, 0, 7)$. $(x-6)(x+4) + (y-2)(y-0) + (z+5)(z-7) = 0$

27.
$$x^2 + y^2 + z^2 - 2y + 8z - 9 = 0$$
$$x^2 + (y^2 - 2y + 1) + (z^2 + 8z + 16) = 9 + 1 + 16$$
$$x^2 + (y-1)^2 + (z+4)^2 = 26$$

The center of the given sphere is $(0, 1, -4)$ so an equation of the required sphere is
$x^2 + (y-1)^2 + (z+4)^2 = 9$

28. It contains the points $(0, 0, 4)$, $(2, 1, 3)$, and $(0, 2, 6)$ and has its center in the yz plane.

▸ From Theorem 10.2.10 we have
$$Gx + Hy + Iz + J = -(x^2 + y^2 + z^2)$$
Because the center is in the yz plane, we have $G = 0$. Thus
$$Hy + Iz + J = -(x^2 + y^2 + z^2)$$
We substitute the coordinates of the given points:

$(0,0,4)$: $\quad 4I + J = -16 \qquad (1)$

$(2,1,6)$: $\quad H + 3I + J = -14 \qquad (2)$

$(0,2,6)$: $\quad 2H + 6I + J = -40 \qquad (3)$

$2(2) - (3)$: $\quad J = 12$

From (1): $\quad 4I = -16 - 12 = -28$

$\quad I = -7$

From (2): $\quad H = -14 - 12 - 3(-7) = -5$

The equation is

$x^2 + y^2 + z^2 = -5y - 7z + 12 = 0$

In Exercises 29–34, $\mathbf{A} = \langle 1, 2, 3\rangle$, $\mathbf{B} = \langle 4, -3, -1\rangle$, $\mathbf{C} = \langle -5, -3, 5\rangle$, $\mathbf{D} = \langle -2, 1, 6\rangle$

29. (a) $\mathbf{A} + 5\mathbf{B} = \langle 1,2,3\rangle + 5\langle 4,-3,-1\rangle = \langle 1,2,3\rangle + \langle 20,-15,-5\rangle = \langle 21,-13,-2\rangle$

(b) $7\mathbf{C} - 5\mathbf{D} = 7\langle -5,-3,5\rangle - 5\langle -2,1,6\rangle = \langle -35,-21,35\rangle + \langle 10,-5,-30\rangle = \langle -25,-26,5\rangle$

(c) $\|7\mathbf{C}\| - \|5\mathbf{D}\| = 7\|\langle -5,-3,5\rangle\| - 5\|\langle -2,1,6\rangle\| = 7\sqrt{25+9+25} - 5\sqrt{4+1+36} = 7\sqrt{59} - 5\sqrt{41}$

(d) $\|7\mathbf{C} - 5\mathbf{D}\| = \|\langle -25,-26,5\rangle\| = \sqrt{625+676+25} = \sqrt{1326}$

30. (a) $2\mathbf{A} - \mathbf{C} = 2\langle 1,2,3\rangle - \langle -5,-3,5\rangle = \langle 2,4,6\rangle + \langle 5,3,-5\rangle = \langle 7,7,1\rangle$

(b) $\|2\mathbf{A}\| - \|\mathbf{C}\| = 2\sqrt{1^2+2^2+3^2} - \sqrt{5^2+3^2+5^2} = 2\sqrt{14} - \sqrt{59}$. (c) $4\mathbf{B} + 6\mathbf{C} - 2\mathbf{D}$

$= 4\langle 4,-3,-1\rangle + 6\langle -5,-3,5\rangle - 2\langle -2,1,6\rangle = \langle 16,-12,-4\rangle + \langle -30,-18,30\rangle + \langle 4,-2,-12\rangle = \langle -10,-32,14\rangle$

(d) $\|4\mathbf{B}\| + \|6\mathbf{C}\| - \|2\mathbf{D}\| = 4\sqrt{4^2+3^2+1^2} + 6\sqrt{5^2+3^2+5^2} - 2\sqrt{2^2+1^2+6^2} = 4\sqrt{26} + 6\sqrt{59} - 2\sqrt{41}$

31. (a) $\mathbf{C} + 3\mathbf{D} - 8\mathbf{A} = \langle -5,-3,5\rangle + 3\langle -2,1,6\rangle - 8\langle 1,2,3\rangle$
$= \langle -5,-3,5\rangle + \langle -6,3,18\rangle + \langle -8,-16,-24\rangle = \langle -19,-16,-1\rangle$

(b) $\|\mathbf{A}\|\|\mathbf{B}\|(\mathbf{C}-\mathbf{D}) = \|\langle 1,2,3\rangle\|\|\langle 4,-3,-1\rangle\|(\langle -5,-3,5\rangle - \langle -2,1,6\rangle) = \sqrt{1+4+9}\sqrt{16+9+1}\langle -3,-4,-1\rangle$
$= \sqrt{14}\sqrt{26}\langle -3,-4,-1\rangle = \sqrt{2^2\cdot 7\cdot 13}\langle -3,-4,-1\rangle = 2\sqrt{91}\langle -3,-4,-1\rangle = \langle -6\sqrt{91},-8\sqrt{91},-2\sqrt{91}\rangle$

32. Find: (a) $3\mathbf{A} - 2\mathbf{B} + \mathbf{C} - 12\mathbf{D}$; (b) $\|\mathbf{A}\|\mathbf{C} - \|\mathbf{B}\|\mathbf{D}$

(a)$3\mathbf{A} - 2\mathbf{B} + \mathbf{C} - 12\mathbf{D} = 3\langle 1,2,3\rangle - 2\langle 4,-3,-1\rangle + \langle -5,-4,5\rangle - 12\langle -2,1,6\rangle$
$= \langle 3,6,9\rangle + \langle -8,6,2\rangle + \langle -5,-3,5\rangle + \langle 24,-12,-72\rangle = \langle 14,-3,-56\rangle$

(b) $\|\mathbf{A}\| = \|\langle 1,2,3\rangle\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$, $\|\mathbf{B}\| = \|\langle 4,-3,-1\rangle\| = \sqrt{4^2+(-3)^2+(-1)^2} = \sqrt{26}$

Thus,

$$\|\mathbf{A}\|\mathbf{C} - \|\mathbf{B}\|\mathbf{D} = \sqrt{14}\langle -5,-3,5\rangle - \sqrt{26}\langle -2,1,6\rangle = \langle -5\sqrt{14}+2\sqrt{26}, -3\sqrt{14}-\sqrt{26}, 5\sqrt{15}-6\sqrt{26}\rangle$$

33. $a(\mathbf{A}+\mathbf{B}) + b(\mathbf{C}+\mathbf{D}) = \mathbf{0}$; $a(\langle 1,2,3\rangle + \langle 4,-3,-1\rangle) + b(\langle -5,-3,5\rangle + \langle -2,1,6\rangle) = \langle 0,0,0\rangle$
$a\langle 5,-1,2\rangle + b\langle -7,-2,11\rangle = \langle 0,0,0\rangle$. Hence $5a - 7b = 0$, $-a - 2b = 0$, and $2a + 11b = 0$.
The only solution to this system of three equations is $a = 0$ and $b = 0$.

34. $a\mathbf{A} + b\mathbf{B} + c\mathbf{C} = \mathbf{D}$. $a\langle 1,2,3\rangle + b\langle 4,-3,-1\rangle + c\langle -5,-3,5\rangle = \langle -2,1,6\rangle$
$a + 4b - 5c = -2$, $2a - 3b - 3c = 1$, $3a - 3b + 5c = 6$. $a = \frac{141}{129}$, $b = -\frac{16}{129}$, $c = \frac{67}{129}$

In Exercises 35–38, find the direction cosines of $\mathbf{V}(\overrightarrow{P_1P_2})$ and check by verifying that the sum of their squares is 1.

35. $P_1(3,-1,-4)$; $P_2(7,2,4)$. $\mathbf{V}(\overrightarrow{P_1P_2}) = \langle 7-3, 2+1, 4+4\rangle = \langle 4,3,8\rangle$; $\left\|\mathbf{V}(\overrightarrow{P_1P_2})\right\| = \sqrt{16+9+64} = \sqrt{89}$.
Therefore $\cos\alpha = \dfrac{4}{\sqrt{89}}$, $\cos\beta = \dfrac{3}{\sqrt{89}}$, $\cos\gamma = \dfrac{8}{\sqrt{89}}$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{16}{89} + \frac{9}{89} + \frac{64}{89} = 1$.

36. $P_1(-2,6,5)$; $P_2(2,4,1)$

▷ We have

$$\mathbf{V} = \langle 2+2, 4-6, 1-5\rangle = \langle 4,-2,-4\rangle \qquad \|\mathbf{V}\| = \sqrt{4^2+(-2)^2+(-4)^2} = 6$$

Thus,

$$\cos\alpha = \frac{a_1}{\|\mathbf{V}\|} = \frac{4}{6} = \frac{2}{3} \qquad \cos\beta = \frac{a_2}{\|\mathbf{V}\|} = \frac{-2}{6} = -\frac{1}{3} \qquad \cos\gamma = \frac{a_3}{\|\mathbf{V}\|} = \frac{-4}{6} = -\frac{2}{3}$$

Furthermore,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (\frac{2}{3})^2 + (-\frac{1}{3})^2 + (-\frac{2}{3})^2 = 1$

37. $P_1(4,-3,-1)$; $P_2(-2,-4,-8)$. $\mathbf{V} = \langle -2-4, -4+3, -8+1\rangle = \langle -6,-1,-7\rangle$; $\|\mathbf{V}\| = \sqrt{36+1+49} = \sqrt{86}$.
Thus $\cos\alpha = \dfrac{-6}{\sqrt{86}}$, $\cos\beta = \dfrac{-1}{\sqrt{86}}$, $\cos\gamma = \dfrac{-7}{\sqrt{86}}$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{36}{86} + \frac{1}{86} + \frac{49}{86} = 1$.

38. $P_1(1,3,5)$; $P_2(2,-1,4)$. $\mathbf{V} = \langle 2-1, -1-3, 4-5\rangle = \langle 1,-4,-1\rangle$; $\|\mathbf{V}\| = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$
$\cos\alpha = \dfrac{1}{3\sqrt{2}} = \frac{1}{6}\sqrt{2}$, $\cos\beta = \dfrac{-4}{3\sqrt{2}} = -\frac{2}{3}\sqrt{2}$, $\cos\gamma = \dfrac{-1}{3\sqrt{2}} = -\frac{1}{6}\sqrt{2}$; $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{2}{36} + \frac{32}{36} + \frac{2}{36} = 1$

39. $P_1 = (3,-1,-4)$, $P_2 = (7,2,4)$. Let $Q = (x,y,z)$. Then $\mathbf{V}(\overrightarrow{P_1P_2}) = 3\mathbf{V}(\overrightarrow{P_1Q})$. $\langle 4,3,8\rangle = 3\langle x-3, y+1, z+4\rangle$;
$3x - 9 = 4$, $x = \frac{13}{3}$; $3y + 3 = 3$, $y = 0$; and $3z + 2 = 8$, $z = -\frac{4}{3}$. Therefore, Q is the point $(\frac{13}{3}, 0, -\frac{4}{3})$.

40. Given $P_1(1,3,5)$ and $P_2(2,-1,4)$ find the point R such that $\mathbf{V}(\overrightarrow{P_1R}) = -2\mathbf{V}(\overrightarrow{P_2R})$.

▹ Expressing the vectors in terms of position vectors, we have

$$\mathbf{r} - \mathbf{p}_1 = -2(\mathbf{r} - \mathbf{p}_2) = -2\mathbf{r} + 2\mathbf{p}_2$$

$$3\mathbf{r} = \mathbf{p}_1 + 2\mathbf{p}_2 = \langle 1,3,5\rangle + 2\langle 2,-1,4\rangle = \langle 5,1,13\rangle$$

$$\mathbf{r} = \langle \tfrac{5}{3}, \tfrac{1}{3}, \tfrac{13}{3}\rangle$$

Thus,

$$R = (\tfrac{5}{3}, \tfrac{1}{3}, \tfrac{13}{3})$$

41. $P_1 = (3,2,-4)$, $P_2 = (-5,4,2)$. Let $P_3 = (x,y,z)$. Then $4\mathbf{V}(\overrightarrow{P_1P_2}) = -3\mathbf{V}(\overrightarrow{P_2P_3})$.
$4\langle -5-3, 4-2, 2+4\rangle = -3\langle x+5, y-4, z-2\rangle$; $\langle -32, 8, 24\rangle = \langle -3x-15, -3y+2, -3z+6\rangle$
Thus $-3x - 15 = -32$, $x = \frac{17}{3}$; $-3y + 12 = 8$, $y = \frac{4}{3}$; $-3z + 6 = 24$, $z = -6$. Hence, $P_3 = (\frac{17}{3}, \frac{4}{3}, -6)$.

42. $P_1(7,0,-2)$, $P_2(2,-3,5)$. $\overrightarrow{P_1P_3} = 5\overrightarrow{P_2P_3}$, $\mathbf{p}_3 - \mathbf{p}_1 = 5(\mathbf{p}_3 - \mathbf{p}_2)$, $4\mathbf{p}_3 = 5\mathbf{p}_2 - \mathbf{p}_1$, $\mathbf{p}_3 = \frac{1}{4}(5\mathbf{p}_2 - \mathbf{p}_1) = \frac{1}{4}\langle 3,-15,27\rangle$

$P_3 = (\frac{3}{4}, -\frac{15}{4}, \frac{27}{4})$

In Exercises 43 and 44, express the vector in terms of its magnitude and direction cosines.

43. (a) Let $\mathbf{A} = -6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Then $\|\mathbf{A}\| = \sqrt{36+4+9} = \sqrt{49} = 7$. $\mathbf{A} = 7(-\frac{6}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k})$

(b) Let $\mathbf{A} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. Then $\|\mathbf{A}\| = \sqrt{4+1+9} = \sqrt{14}$. $\mathbf{A} = \sqrt{14}\left(-\frac{2}{\sqrt{14}}\mathbf{i} + \frac{1}{\sqrt{14}}\mathbf{j} - \frac{3}{\sqrt{14}}\mathbf{k}\right)$

44. (a) $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$; (b) $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$

▹ If α, β, and γ are the direction angles for the vector $\mathbf{V} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\cos\alpha = \frac{a}{\|\mathbf{V}\|} \quad \cos\beta = \frac{b}{\|\mathbf{V}\|} \quad \cos\gamma = \frac{c}{\|\mathbf{V}\|} \quad \text{and} \quad \mathbf{V} = \|\mathbf{V}\|(\cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k})$$

(a) $\|2\mathbf{i} - 2\mathbf{j} + \mathbf{k}\| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

$\cos\alpha = \frac{2}{3}$ $\cos\beta = \frac{-2}{3}$ $\cos\gamma = \frac{1}{3}$

Therefore,

$$2\mathbf{i} - 2\mathbf{j} + \mathbf{k} = 3(\tfrac{2}{3}\mathbf{i} - \tfrac{2}{3}\mathbf{j} + \tfrac{1}{3}\mathbf{k})$$

(b) $\|3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}\| = \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$

$\cos\alpha = \frac{3}{5\sqrt{2}} = \frac{3}{10}\sqrt{2}$ $\quad \cos\beta = \frac{4}{5\sqrt{2}} = \frac{2}{5}\sqrt{2}$ $\quad \cos\gamma = \frac{-5}{5\sqrt{2}} = -\frac{1}{2}\sqrt{2}$

Therefore,

$$3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} = 5\sqrt{2}(\tfrac{3}{10}\sqrt{2}\mathbf{i} + \tfrac{2}{5}\sqrt{2}\mathbf{j} - \tfrac{1}{2}\sqrt{2}\mathbf{k})$$

In Exercises 45 and 46, find the unit vector $\mathbf{U}$ having the same direction as $\mathbf{V}(\overrightarrow{P_1P_2})$.

45. (a) $P_1 = (4,-1,-6)$, $P_2 = (5,7,-2)$. $\mathbf{V}(\overrightarrow{P_1P_2}) = \langle 5-4, 7+1, -2+6\rangle = \langle 1,8,4\rangle$.
$\left\|\mathbf{V}(\overrightarrow{P_1P_2})\right\| = \sqrt{1+64+16} = \sqrt{81} = 9$. $\mathbf{U} = \langle \frac{1}{9}, \frac{8}{9}, \frac{4}{9}\rangle$.

(b) $P_1 = (-2,5,3)$, $P_2 = (-4,7,5)$. $\mathbf{V}(\overrightarrow{P_1P_2}) = \langle -4+2, 7-5, 5-3\rangle = \langle -2,2,2\rangle$.
$\left\|\mathbf{V}(\overrightarrow{P_1P_2})\right\| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$. $\mathbf{U} = \langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$.

46. (a) $P_1(3,0,-1)$, $P_2(-3,8,-1)$. $\mathbf{V} = \langle -3-3, 8-0, -1+1\rangle = \langle -6,8,0\rangle$. $\|\mathbf{V}\| = \sqrt{6^2+8^2} = 10$. $\mathbf{U} = \langle -\frac{3}{5}, \frac{4}{5}, 0\rangle$

(b) $P_1(-8,-5,2)$, $P_2(-3,-9,4)$. $\mathbf{V} = \langle -3+8, -9+5, 4-2\rangle = \langle 5,-4,2\rangle$. $\|\mathbf{V}\| = \sqrt{5^2+4^2+2^2} = \sqrt{45} = 15/\sqrt{5}$

$\mathbf{U} = \langle \frac{1}{3}\sqrt{5}, -\frac{4}{15}\sqrt{5}, \frac{2}{15}\sqrt{5}\rangle$

In Exercises 47 and 48, prove the property if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are any vectors of V_3 and c and d are any scalars.

47. Let $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle$. Use the same property of the real numbers.
(a) $\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle = \langle b_1 + a_1, b_2 + a_2, b_3 + a_3\rangle = \mathbf{B} + \mathbf{A}$
(b) Let $\mathbf{0} = \langle 0,0,0\rangle$. Then $\mathbf{A} + \mathbf{0} = \langle a_1 + 0, a_2 + 0, a_3 + 0\rangle = \langle a_1, a_2, a_3\rangle = \mathbf{A}$.
(c) Let $-\mathbf{A} = \langle -a_1, -a_2, -a_3\rangle$. Then $\mathbf{A} + (-\mathbf{A}) = \langle a_1 + (-a_1), a_2 + (-a_2), a_3 + (-a_3)\rangle = \langle 0,0,0\rangle = \mathbf{0}$.
(d) $c(\mathbf{A} + \mathbf{B}) = c\langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle = \langle c(a_1 + b_1), c(a_2 + b_2), c(a_3 + b_3)\rangle$
$= \langle ca_1 + cb_1, ca_2 + cb_2, ca_3 + cb_3\rangle = \langle ca_1, ca_2, ca_3\rangle + \langle cb_1, cb_2, cb_3\rangle = c\mathbf{A} + c\mathbf{B}$

48. (a) $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$ (associative law)
(b) $(cd)\mathbf{A}=c(d\mathbf{A})$ (associative law)
(c) $(c+d)\mathbf{A}=c\mathbf{A}+d\mathbf{A}$ (distributive law)

▹ Let $\mathbf{A}=\langle a_1,a_2,a_3\rangle$, $\mathbf{B}=\langle b_1,b_2,b_3\rangle$, and $\mathbf{C}=\langle c_1,c_2,c_3\rangle$. We apply the definitions for addition of vectors and scalar multiplication and the associative and distributive laws for real numbers.

$$\begin{aligned}(a)\ \mathbf{A}+(\mathbf{B}+\mathbf{C}) &= \langle a_1,a_2,a_3\rangle+(\langle b_1,b_2,b_3\rangle+\langle c_1,c_2,c_3\rangle)=\langle a_1,a_2,a_3\rangle+\langle b_1+c_1,b_2+c_2,b_3+c_3\rangle\\ &=\langle a_1+(b_1+c_1),a_2+(b_2+c_2),a_3+(b_3+c_3)\rangle=\langle (a_1+b_1)+c_1,(a_2+b_2)+c_2,(a_3+b_3)+c_3\rangle\\ &=\langle a_1+b_1,a_2+b_2,a_3+b_3\rangle+\langle c_1,c_2,c_3\rangle=(\langle a_1,a_2,a_3\rangle+\langle b_1,b_2,b_3\rangle)+\langle c_1,c_2,c_3\rangle\\ &=(\mathbf{A}+\mathbf{B})+\mathbf{C}\end{aligned}$$

$$\begin{aligned}(b)\qquad (cd)\mathbf{A} &= (cd)\langle a_1,a_2,a_3\rangle=\langle (cd)a_1,(cd)a_2,(cd)a_3\rangle=\langle c(da_1),c(da_2),c(da_3)\rangle=c\langle da_1,da_2,da_3\rangle\\ &=c(d\mathbf{A})\end{aligned}$$

$$\begin{aligned}(c)\qquad (c+d)\mathbf{A} &= (c+d)\langle a_1,a_2,a_3\rangle=\langle (c+d)a_1,(c+d)a_2,(c+d)a_3\rangle=\langle ca_1+da_1,ca_2+da_2,ca_3+da_3\rangle\\ &=\langle ca_1,ca_2,ca_3\rangle+\langle da_1,da_2,da_3\rangle=c\langle a_1,a_2,a_3\rangle+d\langle a_1,a_2,a_3\rangle=c\mathbf{A}+d\mathbf{A}\end{aligned}$$

In Exercises 49–51, prove the theorem by analytic geometry.

49. The four diagonals of a rectangular parallelpiped bisect each other.

▹ Choose the coordinate planes as in Figure 3 of the text so the vertices of the parallelepiped are A(0,0,0), B(a,b,c); C(a,b,0), H(0,0,c); D(a,0,0), G(0,b,c); E(a,0,c) and F(0,b,0). Because each of the diagonals AB, CH, DG, EF have $(\frac{1}{2}a,\frac{1}{2}b,\frac{1}{2}c)$ as midpoint, they bisect each other.

50. If P, Q, R, S are four points of R^3, then the midpoints A, B, C, D of PQ, QR, RS, SP form a parallelogram.

▹ We have $\mathbf{a}=\frac{1}{2}(\mathbf{p}+\mathbf{q})$, $\mathbf{b}=\frac{1}{2}(\mathbf{q}+\mathbf{r})$, $\mathbf{c}=\frac{1}{2}(\mathbf{r}+\mathbf{s})$, $\mathbf{d}=\frac{1}{2}(\mathbf{s}+\mathbf{p})$. Therefore, $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\frac{1}{2}(\mathbf{r}-\mathbf{p})=\mathbf{c}-\mathbf{d}=\overrightarrow{DC}$.

51. The four diagonals of a rectangular parallelopiped have the same length.

▹ Choose the coordinate planes as in Figure 3 of the text so the vertices of the rectangular parallepiped are at A(0,0,0), B(a,b,c); C(a,b,0), H(0,0,c); D(a,0,0), G(0,b,c); E(a,0,c) and F(0,b,0). The four diagonals have the same length:

$$|\overline{AB}|=\sqrt{(a-0)^2+(b-0)^2+(c-0)^2}=\sqrt{a^2+b^2+c^2}$$
$$|\overline{CH}|=\sqrt{(a-0)^2+(0-b)^2+(0-c)^2}=\sqrt{a^2+b^2+c^2}$$
$$|\overline{DG}|=\sqrt{(a-0)^2+(b-0)^2+(0-c)^2}=\sqrt{a^2+b^2+c^2}$$
$$|\overline{EF}|=\sqrt{(0-a)^2+(b-0)^2+(0-c)^2}=\sqrt{a^2+b^2+c^2}$$

52. Three vectors in V_3 are said to be *independent* if and only if their position representations do not lie in a plane, and three vectors $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ are said to form a *basis* for the vector space V_3 if and only if any vector in V_3 can be written as a linear combination of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$. A theorem can be proved which states that two vectors form a basis for the vector space V_3 if they are independent. Show that this theorem holds for the three vectors $\langle 1,0,0\rangle$, $\langle 1,1,0\rangle$ and $\langle 1,1,1\rangle$ by doing the following: (a) Verify that the vectors are independent by showing that the position representations are not coplanar; (b) verify that the vectors form a basis by showing that any vector $\mathbf{A}$ can be written

$$\mathbf{A}=r\langle 1,0,0\rangle+s\langle 1,1,0\rangle+t\langle 1,1,1\rangle \qquad (1)$$

where r, s, and t are scalars. (c) If $\mathbf{A}=\langle 6,-2,5\rangle$, find the particular values of r, s, and t, so that (1) holds.

▹ (a) The position representations of the vectors $\langle 1,0,0\rangle$ and $\langle 1,1,0\rangle$ are in the xy plane because their z components are both zero; because they are not collinear, they do not lie in any other plane. The position representation of the vector $\langle 1,1,1\rangle$ is not in the xy plane because its z component is not zero. Therefore, the position representations are not coplanar and the vectors are independent.

(b) Let $\mathbf{A}=\langle a,b,c\rangle$. We show that there exist scalars r, s, and t such that

$$\langle a,b,c\rangle=r\langle 1,0,0\rangle+s\langle 1,1,0\rangle+t\langle 1,1,1\rangle=\langle r,0,0\rangle+\langle s,s,0\rangle+\langle t,t,t\rangle=\langle r+s+t,s+t,t\rangle$$

Corresponding components of equal vectors are equal numbers. Thus,

$$a=r+s+t \quad b=s+t \quad c=t$$

Solving for r, s, and t in terms of a, b, and c, we obtain

$$r = a - b \quad s = b - c \quad t = c \tag{2}$$

Thus, any vector **A** can be written

$$\mathbf{A} = r\langle 1,0,0\rangle + s\langle 1,1,0\rangle + t\langle 1,1,1\rangle$$

and the vectors $\langle 1,0,0\rangle$, $\langle 1,1,0\rangle$ and $\langle 1,1,1\rangle$ form a basis for V_3.
(c) If $\mathbf{A} = \langle 6,-2,5\rangle$ then we have $a = 6$, $b = -2$, and $c = 5$. Substituting these values for a, b, and c into Eqs. (2), we obtain

$$r = 8 \quad s = -7 \quad t = 5$$

53. (a) Let $\mathbf{A} = \langle a,b,c\rangle$. We solve $r\langle 2,0,1\rangle + s\langle 0,-1,0\rangle + t\langle 1,-1,0\rangle = \langle a,b,c\rangle$ for r, s, t.
$2r + t = a$, $-s - t = b$, $r = c$; $t = a - 2c$, $s = 2c - a - b$.
Thus $\langle a,b,c\rangle = c\langle 2,0,1\rangle + (2c - a - b)\langle 0,-1,0\rangle + (a - 2c)\langle 1,-1,0\rangle$ so the vectors form a basis.
(b) Set $a = -2$, $b = 3$, $c = 5$ in (a). Then $r = 5$, $s = 2(5) - (-2) - 3 = 9$, $t = (-2) - 2(5) = -12$.

54. Let $\mathbf{F}_1 = \langle 1,0,1\rangle$, $\mathbf{F}_2 = \langle 1,1,1\rangle$, $\mathbf{F}_3 = \langle 2,1,2\rangle$. (a) Because $\mathbf{F}_3 = \mathbf{F}_1 + \mathbf{F}_2$, the vectors represent the sides of a triangle and so are coplanar. (b) If $r\langle 1,0,1\rangle + s\langle 1,1,1\rangle + t\langle 2,1,2\rangle = \langle a,b,c\rangle$, then $r + s + 2t = a$, $s + t = b$, and $r + s + 2t = c$. Hence, if $a \neq c$ there is no solution.

55. $\mathbf{U} = \dfrac{a_1}{\|\mathbf{A}\|}\mathbf{i} + \dfrac{a_2}{\|\mathbf{A}\|}\mathbf{j} + \dfrac{a_3}{\|\mathbf{A}\|}\mathbf{k} = \dfrac{1}{\|\mathbf{A}\|}\mathbf{A}$. $\|\mathbf{U}\| = \dfrac{1}{\|\mathbf{A}\|}\|\mathbf{A}\| = 1$. Because $\dfrac{1}{\|\mathbf{A}\|} > 0$, then **U** has the same direction as **A**.

56. If the radian measure of each direction angle is the same, what is it?

▸ Let α be the radian measure of each direction angle of a vector. By Theorem 10.2.13 we have
$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$; $\cos^2\alpha = \frac{1}{3}$; $\cos\alpha = \pm\dfrac{1}{\sqrt{3}}$; $\alpha = \cos^{-1}\dfrac{1}{\sqrt{3}}$ or $\alpha = \cos^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

10.3 DOT PRODUCT

10.3.1 Definition The *dot product* of two vectors **A** and **B**, denoted by $\mathbf{A}\cdot\mathbf{B}$ is defined as follows:
If $\mathbf{A} = \langle a_1,a_2\rangle = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = \langle b_1,b_2\rangle = b_1\mathbf{i} + b_2\mathbf{j}$ are two vectors in V_2
$\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2$
If $\mathbf{A} = \langle a_1,a_2,a_3\rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = \langle b_1,b_2,b_3\rangle = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are two vectors in V_3
$\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$

We note that the dot product, or *inner product*, of two vectors is a scalar.

Work We use the dot product to find the work W done by a force **F** that causes a displacement **D** of an object. We have $W = \mathbf{F}\cdot\mathbf{D}$.

10.3.2 Theorem If **A**, **B**, and **C** are any vectors in V_2 or V_3, then

(i) $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$ (commutative law)
(ii) $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ (distributive law)

10.3.3 Theorem If **A** and **B** are any vectors in V_2 or V_3 and c is any scalar, then

(i) $c(\mathbf{A}\cdot\mathbf{B}) = (c\mathbf{A})\cdot\mathbf{B}$
(ii) $\mathbf{0}\cdot\mathbf{A} = 0$
(iii) $\mathbf{A}\cdot\mathbf{A} = \|\mathbf{A}\|^2$

10.3.4 Definition Let **A** and **B** be two nonzero vectors such that **A** is not a scalar multiple of **B**. If $\overrightarrow{OP}$ is the position representation of **A** and $\overrightarrow{OQ}$ is the position representation of **B**, then *the angle between the vectors* **A** and **B**, denoted by $\langle\mathbf{A},\mathbf{B}\rangle$, is defined to be the angle of positive measure between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ interior to the triangle determined by the points O, P, and Q. If $\mathbf{A} = c\mathbf{B}$, where c is a scalar, then if $c > 0$, the angle between the vectors has radian measure 0; if $c < 0$, the angle between the vectors has radian measure π.

10.3.5 Theorem If α is the radian measure of the angle between the two nonzero vectors **A** and **B**, then
$\mathbf{A}\cdot\mathbf{B} = \|\mathbf{A}\|\|\mathbf{B}\|\cos\alpha$
Thus,
$\mathbf{A}\cdot\mathbf{B} > 0$ if $0 \le \alpha < \frac{1}{2}\pi$ $\quad$ $\mathbf{A}\cdot\mathbf{B} = 0$ if $\alpha = \frac{1}{2}\pi$ $\quad$ $\mathbf{A}\cdot\mathbf{B} < 0$ if $\frac{1}{2}\pi < \alpha \le \pi$
We see that the dot product of two vectors does not depend on the coordinate system.

10.3.6 Definition Two vectors called parallel if and only if one of the vectors is a scalar multiple of the other.

10.3.7 Definition Two vectors $\mathbf{A}\cdot\mathbf{B}$ are said to be *orthogonal* (*perpendicular*) if and only if $\mathbf{A}\cdot\mathbf{B}=0$.

The *scalar* and *vector projection* of the vector $\mathbf{A}$ onto the nonzero vector $\mathbf{B}$ are given by

$$A_B=\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|} \quad\text{and}\quad \mathbf{A}_B=\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B}$$

$\mathbf{A}-\mathbf{A}_B$ is orthogonal to $\mathbf{B}$. In fact, $(\mathbf{A}-\mathbf{A}_B)\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{B}-\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}(\mathbf{B}\cdot\mathbf{B})=\mathbf{A}\cdot\mathbf{B}-\mathbf{A}\cdot\mathbf{B}=0$

If $\mathbf{U}$ is a unit vector, then $A_U=\mathbf{A}\cdot\mathbf{U}$, and the scalar projection of $\mathbf{A}$ onto $\mathbf{U}$ is called the *component* of the vector $\mathbf{A}$ in the direction of $\mathbf{U}$. As a special case, for the vector $a_1\mathbf{i}+a_2\mathbf{j}$ the numbers a_1 and a_2 are the components of the vector $\mathbf{A}$ in the directions of $\mathbf{i}$ and $\mathbf{j}$, respectively.

Exercises 10.3

In Exercises 1-4, find $\mathbf{A}\cdot\mathbf{B}$.

1. (a) $\langle -1,2\rangle\cdot\langle -4,3\rangle=(-1)(-4)+2(3)=10$ (b) $(2\mathbf{i}-\mathbf{j})\cdot(\mathbf{i}+3\mathbf{j})=2(1)+(-1)3=-1$

2. (a) $\langle \frac13,-\frac12\rangle\cdot\langle\frac52,\frac43\rangle=\frac13(\frac52)+(-\frac12)\frac43=\frac16$ (b) $(-2\mathbf{i})\cdot(-\mathbf{i}+\mathbf{j})=(-2)(-1)+0(1)=2$

3. (a) $\langle \frac25,\frac14,-\frac32\rangle\cdot\langle\frac12,\frac35,\frac12\rangle=\frac25(\frac12)+\frac14(\frac35)+(-\frac32)\frac12=-\frac25$ (b) $(3\mathbf{j}-2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-3\mathbf{k})=0(1)+3(1)+(-2)(-3)=9$

4. (a) $\mathbf{A}=\langle 4,0,2\rangle$, $\mathbf{B}=\langle 5,2,-1\rangle$; (b) $\mathbf{A}=3\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{B}=6\mathbf{i}+7\mathbf{j}+2\mathbf{k}$

▹ We apply Definition 10.3.1.
(a) $\mathbf{A}\cdot\mathbf{B}=\langle 4,0,2\rangle\cdot\langle 5,2,-1\rangle=4(5)+0(2)+2(-1)=18$
(b) $\mathbf{A}\cdot\mathbf{B}=(3\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot(6\mathbf{i}+7\mathbf{j}+2\mathbf{k})=3(6)+(-2)7+1(2)=6$

5. $\mathbf{i}\cdot\mathbf{i}=\langle 1,0,0\rangle\cdot\langle 1,0,0\rangle=1(1)+0(0)+0(0)=1$
$\mathbf{i}\cdot\mathbf{k}=\langle 1,0,0\rangle\cdot\langle 0,0,1\rangle=1(0)+0(0)+0(1)=0$
$\mathbf{j}\cdot\mathbf{k}=\langle 0,1,0\rangle\cdot\langle 0,0,1\rangle=0(0)+1(0)+0(1)=0$

6. $\mathbf{j}\cdot\mathbf{j}=\langle 0,1,0\rangle\cdot\langle 0,1,0\rangle=0(0)+1(1)+0(0)=1$
$\mathbf{k}\cdot\mathbf{k}=\langle 0,0,1\rangle\cdot\langle 0,0,1\rangle=0(0)+0(0)+1(1)=1$
$\mathbf{i}\cdot\mathbf{j}=\langle 1,0,0\rangle\cdot\langle 0,1,0\rangle=1(0)+0(1)+0(0)=0$

In Exercises 7–10, prove the theorem for vectors in V_3.

▹ Let $\mathbf{A}=\langle a_1,a_2,a_3\rangle$, $\mathbf{B}=\langle b_1,b_2,b_3\rangle$, $\mathbf{C}=\langle c_1,c_2,c_3\rangle$.

7. $\mathbf{A}\cdot\mathbf{B}=\langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle=a_1b_1+a_2b_2+a_3b_3=b_1a_1+b_2a_2+b_3a_3=\langle b_1,b_2,b_3\rangle\cdot\langle a_1,a_2,a_3\rangle=\mathbf{B}\cdot\mathbf{A}$

8. Theorem 10.3.2(ii)

▹
$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})&=\langle a_1,a_2,a_3\rangle\cdot\langle b_1+c_1,b_2+c_2,b_3+c_3\rangle\\&=a_1(b_1+c_1)+a_2(b_2+c_2)+a_3(b_3+c_3)\\&=(a_1b_1+a_1c_1)+(a_2b_2+a_2c_2)+(a_3b_3+a_3c_3) &&\text{(distributive law for real numbers)}\\&=(a_1b_1+a_2b_2+a_3b_3)+(a_1c_1+a_2c_2+a_3c_3)\\&=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}\end{aligned}$$

9. $c(\mathbf{A}\cdot\mathbf{B})=c(\langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle)=c(a_1b_1+a_2b_2+a_3b_3)=c(a_1b_1)+c(a_2b_2)+c(a_3b_3)$
$\underset{\text{associative for numbers}}{=}(ca_1)b_1+(ca_2)b_2+(ca_3)b_3=\langle ca_1,ca_2,ca_3\rangle\cdot\langle b_1,b_2,b_3\rangle=(c\mathbf{A})\cdot\mathbf{B}$

10. $\mathbf{0}\cdot\mathbf{A}=\langle 0,0,0\rangle\cdot\langle a_1,a_2,a_3\rangle=0a_1+0a_2+0a_3=0$ $\mathbf{A}\cdot\mathbf{A}=\langle a_1,a_2,a_3\rangle\cdot\langle a_1,a_2,a_3\rangle={a_1}^2+{a_2}^2+{a_3}^2=\|\mathbf{A}\|^2$

In Exercises 11 and 12, if θ is the angle between $\mathbf{A}$ and $\mathbf{B}$, find $\cos\theta$.

11. (a) $\mathbf{A}=\langle 4,3\rangle$; $\mathbf{B}=\langle -1,-1\rangle$. $\mathbf{A}\cdot\mathbf{B}=4-3=1$
$\|\mathbf{A}\|=\sqrt{16+9}=5$; $\|\mathbf{B}\|=\sqrt{1+1}=\sqrt2$
$\cos\theta=\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}=\frac{1}{5\sqrt2}=\frac{1}{10}\sqrt2$
(b) $\mathbf{A}=5\mathbf{i}-12\mathbf{j}$; $\mathbf{B}=4\mathbf{i}+3\mathbf{j}$. $\mathbf{A}\cdot\mathbf{B}=20-36=-16$
$\|\mathbf{A}\|=\sqrt{25+144}=13$; $\|\mathbf{B}\|=\sqrt{16+9}=5$
$\cos\theta=\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}=\frac{-16}{13\cdot 5}=-\frac{16}{65}$

12. (a) $\mathbf{A}=\langle -2,-3\rangle$, $\mathbf{B}=\langle 3,2\rangle$; (b) $\mathbf{A}=2\mathbf{i}+4\mathbf{j}$, $\mathbf{B}=-5\mathbf{j}$

▹ (a) $\mathbf{A}\cdot\mathbf{B}=\langle -2,-3\rangle\cdot\langle 3,2\rangle=(-2)3+(-3)2=-6$
$\|\mathbf{A}\|=\sqrt{2^2+3^2}=\sqrt{13}$, $\|\mathbf{B}\|=\sqrt{3^2+2^2}=\sqrt{13}$
$\cos\theta=\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}=\frac{-6}{\sqrt{13}\sqrt{13}}=-\frac{6}{13}$
(b) $\mathbf{A}\cdot\mathbf{B}=(2\mathbf{i}+4\mathbf{j})\cdot(-5\mathbf{j})=2(0)+4(-5)=-20$
$\|\mathbf{A}\|=\sqrt{2^2+4^2}=\sqrt{20}=2\sqrt5$, $\|\mathbf{B}\|=\sqrt{0^2+5^2}=5$
$\cos\theta=\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|}=\frac{-20}{2\sqrt5(5)}=-\frac25\sqrt5$

13. $\mathbf{A}=3\mathbf{i}+2\mathbf{j}$; $\mathbf{B}=2\mathbf{i}+k\mathbf{j}$. We wish to find k so that $\mathbf{A}\cdot\mathbf{B}=\|\mathbf{A}\|\|\mathbf{B}\|\cos\frac14\pi$; $6+2k=\sqrt{13}\sqrt{4+k^2}(\frac12\sqrt2)$; $72+48k+8k^2=52+3k^2$; $5k^2-48k-20=0$; $(5k+2)(k-10)=0$; $k=-\frac25$ or $k=10$

14. $\mathbf{A}$ and $\mathbf{B}$ are orthogonal $\Leftrightarrow 0=(k\mathbf{i}-2\mathbf{j})\cdot(k\mathbf{i}+6\mathbf{j})=k^2-12$, $k=\pm\sqrt{12}=\pm2\sqrt3$

15. $\mathbf{A} = 5\mathbf{i} - k\mathbf{j}$ and $\mathbf{B} = k\mathbf{i} + 6\mathbf{j}$, where k is a scalar.
(a) $\mathbf{A}$ and $\mathbf{B}$ are orthogonal $\Leftrightarrow \mathbf{A} \cdot \mathbf{B} = 0 \Leftrightarrow 5k - 6k = 0 \Leftrightarrow k = 0$
(b) $\mathbf{A}$ and $\mathbf{B}$ are parallel if and only if there is a nonzero scalar a such that
$a\mathbf{A} = \mathbf{B}$; $a(5\mathbf{i} - k\mathbf{j}) = k\mathbf{i} + 6\mathbf{j}$; $5a = k$ and $-ak = 6$; $-5a^2 = 6$. This equation has no real solution.

16. Find k such that $\mathbf{A} = k\mathbf{i} - 2\mathbf{j}$ and $\mathbf{B} = k\mathbf{i} + 6\mathbf{j}$ have opposite directions.
▸ $\mathbf{A}$ and $\mathbf{B}$ have opposite directions if $\mathbf{A} = c\mathbf{B}$ for some negative scalar c.
$k\mathbf{i} - 2\mathbf{j} = c(k\mathbf{i} + 6\mathbf{j}) = ck\mathbf{i} + 6c\mathbf{j}$
Because $6c = -2$, then $c = -\frac{1}{3}$. Because $k = ck$ then $(c-1)k = 0$. Because $c - 1 \neq 0$, then $k = 0$.

17-18. $\mathbf{A} = -8\mathbf{i} + 4\mathbf{j}$; $\mathbf{B} = 7\mathbf{i} - 6\mathbf{j}$. $\mathbf{A} \cdot \mathbf{B} = (-8)(7) + 4(-6) = -80$, $\|\mathbf{A}\| = \sqrt{8^2 + 4^2} = 4\sqrt{5}$, $\|\mathbf{B}\| = \sqrt{7^2 + 6^2} = \sqrt{85}$.
$A_B = \dfrac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|} = -\dfrac{80}{\sqrt{85}}$, $\mathbf{A}_B = \dfrac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B} = -\frac{80}{85}(7\mathbf{i} - 6\mathbf{j}) = -\frac{112}{17}\mathbf{i} + \frac{96}{17}\mathbf{j}$.
$B_A = \dfrac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{A}\|} = \dfrac{-80}{4\sqrt{5}} = -4\sqrt{5}$, $\mathbf{B}_A = \dfrac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{A}\|^2}\mathbf{A} = \frac{-80}{80}(-8\mathbf{i} + 4\mathbf{j}) = 8\mathbf{i} - 4\mathbf{j}$

19. The component of $\mathbf{A} = 5\mathbf{i} - 6\mathbf{j}$ in the direction of $\mathbf{B} = 7\mathbf{i} + \mathbf{j}$ is $\dfrac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|} = \dfrac{35 - 6}{\sqrt{49 + 1}} = \dfrac{29}{\sqrt{50}} = \frac{29}{10}\sqrt{2}$.

20. For the vectors $\mathbf{A} = 5\mathbf{i} - 6\mathbf{j}$ and $\mathbf{B} = 7\mathbf{i} + \mathbf{j}$, find the component of the vector $\mathbf{B}$ in the direction of vector $\mathbf{A}$.
▸ he component of the vector $\mathbf{B}$ in the direction of vector $\mathbf{B}$ is B_A, the scalar projection of $\mathbf{B}$ onto $\mathbf{A}$. Thus,

$$B_A = \frac{\mathbf{B} \cdot \mathbf{A}}{\|\mathbf{A}\|} = \frac{(5\mathbf{i} - 6\mathbf{j}) \cdot (7\mathbf{i} + \mathbf{j})}{\|5\mathbf{i} - 6\mathbf{j}\|} = \frac{5(7) - (-6)1}{\sqrt{5^2 + 6^2}} = \frac{29}{\sqrt{61}}$$

In Exercises 21–26, $\mathbf{A} = \langle -4, -2, 4 \rangle$; $\mathbf{B} = \langle 2, 7, -1 \rangle$; $\mathbf{C} = \langle 6, -3, 0 \rangle$, and $\mathbf{D} = \langle 5, 4, -3 \rangle$.

21. (a) $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \langle -4, -2, 4 \rangle \cdot (\langle 2, 7, -1 \rangle + \langle 6, -3, 0 \rangle) = \langle -4, -2, 4 \rangle \cdot \langle 8, 4, -1 \rangle = -32 - 8 - 4 = -44$
(b) $(\mathbf{A} \cdot \mathbf{B})(\mathbf{C} \cdot \mathbf{D}) = (\langle -4, -2, 4 \rangle \cdot \langle 2, 7, -1 \rangle)(\langle 6, -3, 0 \rangle \cdot \langle 5, 4, -3 \rangle) = (8 - 14 - 4)(30 - 12 + 0) = (-26)(18) = -468$
(c) $\mathbf{A} \cdot \mathbf{D} - \mathbf{B} \cdot \mathbf{C} = (\langle -4, -2, 4 \rangle \cdot \langle 5, 4, -3 \rangle) - (\langle 2, 7, -1 \rangle \cdot \langle 6, -3, 0 \rangle) = (-20 - 8 - 12) - (12 - 21 + 0)$
$= -40 - (-9) = -31$
(d) $(\mathbf{B} \cdot \mathbf{D})\mathbf{A} - (\mathbf{D} \cdot \mathbf{A})\mathbf{B} = (\langle 2, 7, -1 \rangle \cdot \langle 5, 4, -3 \rangle)\langle -4, -2, 4 \rangle - (\langle 5, 4, -3 \rangle \cdot \langle -4, -2, 4 \rangle)\langle 2, 7, -1 \rangle$
$= (10 + 28 + 3)\langle -4, -2, 4 \rangle - (-20 - 8 - 12)\langle 2, 7, -1 \rangle = 41\langle -4, -2, 4 \rangle + 40\langle 2, 7, -1 \rangle =$
$= \langle -164, -82, 164 \rangle + \langle 80, 280, -40 \rangle = \langle -84, 198, 124 \rangle$

22. (a) $\mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} = \langle -4, -2, 4 \rangle \cdot \langle 2, 7, -1 \rangle + \langle -4, -2, 4 \rangle \cdot \langle 6, -3, 0 \rangle = -8 - 14 - 4 - 24 + 6 + 0 = -44$
(b) $(\mathbf{A} \cdot \mathbf{B})(\mathbf{B} \cdot \mathbf{C}) = (\langle -4, -2, 4 \rangle \cdot \langle 2, 7, -1 \rangle)(\langle 2, 7, -1 \rangle \cdot \langle 6, -3, 0 \rangle) = (-8 - 14 - 4)(12 - 21 + 0) = (-26)(-9) = 234$
(c) $(\mathbf{A} \cdot \mathbf{B})\mathbf{C} + (\mathbf{B} \cdot \mathbf{C})\mathbf{D} = (\langle -4, -2, 4 \rangle \cdot \langle 2, 7, -1 \rangle)\mathbf{C} + (\langle 2, 7, -1 \rangle \cdot \langle 6, -3, 0 \rangle)\mathbf{D} = -26\langle 6, 3, 0 \rangle - 9\langle 5, 4, -3 \rangle$
$= \langle -201, 42, 27 \rangle$
(d) $(2\mathbf{A} + 3\mathbf{B}) \cdot (4\mathbf{C} - \mathbf{D}) = (\langle -8, -4, 8 \rangle + \langle 6, 21, -3 \rangle) \cdot (\langle 24, -12, 0 \rangle - \langle 5, 4, -3 \rangle) = \langle -2, 17, 5 \rangle \cdot \langle 19, -16, 3 \rangle = -295$

23. $\mathbf{A} \cdot \mathbf{C} = \langle -4, -2, 4 \rangle \cdot \langle 6, -3, 0 \rangle = -24 + 6 + 0 = -18$, $\|\mathbf{A}\| = \sqrt{16 + 4 + 16} = 6$, $\|\mathbf{C}\| = \sqrt{36 + 9 + 0} = \sqrt{45} = 3\sqrt{5}$
(a) $\cos\theta = \dfrac{\mathbf{A} \cdot \mathbf{C}}{\|\mathbf{A}\|\|\mathbf{C}\|} = \dfrac{-18}{6(3\sqrt{5})} = -\dfrac{1}{\sqrt{5}}$ (b) component of $\mathbf{C}$ in the direction of $\mathbf{A}$ is $\|\mathbf{C}\|\cos\theta = 3\sqrt{5}\left(-\dfrac{1}{\sqrt{5}}\right) = -3$
(c) The vector projection of $\mathbf{C}$ onto $\mathbf{A}$ is $\dfrac{\mathbf{A} \cdot \mathbf{C}}{\|\mathbf{A}\|^2}\mathbf{A} = \frac{-18}{36}\langle -4, -2, 4 \rangle = \langle 2, 1, -2 \rangle$.

24. Find: (a) $\cos\theta$ if θ is the angle between $\mathbf{B}$ and $\mathbf{D}$;
(b) the component of $\mathbf{B}$ in the direction of $\mathbf{D}$;
(c) the vector projection of $\mathbf{B}$ onto $\mathbf{D}$.
▸
$$\|\mathbf{B}\| = \|\langle 2, 7, -1 \rangle\| = \sqrt{2^2 + 7^2 + (-1)^2} = \sqrt{54}$$
$$\|\mathbf{D}\| = \|\langle 5, 4, -3 \rangle\| = \sqrt{5^2 + 4^2 + (-3)^2} = \sqrt{50}$$
$$\mathbf{B} \cdot \mathbf{D} = \langle 2, 7, -1 \rangle \cdot \langle 5, 4, -3 \rangle = 2(5) + 7(4) + 2(-1)(-1) = 41$$

(a) $\cos\theta = \dfrac{\mathbf{B} \cdot \mathbf{D}}{\|\mathbf{B}\|\|\mathbf{D}\|} = \dfrac{41}{\sqrt{54}\sqrt{50}} = \frac{41}{90}\sqrt{3}$

(b) The component of $\mathbf{B}$ in the direction of $\mathbf{D}$ is $B_D = \dfrac{\mathbf{B} \cdot \mathbf{D}}{\|\mathbf{D}\|} = \dfrac{41}{\sqrt{50}} = \frac{41}{10}\sqrt{2}$

(c) $\mathbf{B}_D = \dfrac{\mathbf{B} \cdot \mathbf{D}}{\|\mathbf{D}\|^2}\mathbf{D} = \frac{41}{50}\langle 5, 4, -3 \rangle = \langle \frac{41}{10}, \frac{82}{25}, -\frac{123}{50} \rangle$

25. $\mathbf{A}\cdot\mathbf{B} = \langle -4,-2,4\rangle\cdot\langle 2,7,-1\rangle = -8-14-4 = -26$, $\|\mathbf{B}\| = \sqrt{4+49+1} = \sqrt{54} = 3\sqrt{6}$.

(a) The scalar projection of **A** onto **B** is $\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|} = \dfrac{-26}{3\sqrt{6}} = -\frac{13}{9}\sqrt{6}$.

(b) The vector projection of **A** onto **B** is $\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B} = \frac{-26}{54}\langle 2,7,-1\rangle = \langle -\frac{26}{27}, -\frac{91}{27}, \frac{13}{27}\rangle$.

26. $\mathbf{C}\cdot\mathbf{D} = \langle 6,-3,0\rangle\cdot\langle 5,4,-3\rangle = 30-12+0 = 18$, $\|\mathbf{C}\| = \sqrt{6^2+3^2+0^2} = 3\sqrt{5}$

(a) $\mathrm{D}_C = \dfrac{\mathbf{C}\cdot\mathbf{D}}{\|\mathbf{C}\|} = \dfrac{18}{3\sqrt{5}} = \frac{6}{5}\sqrt{5}$ (b) $\mathbf{D}_C = \dfrac{\mathbf{C}\cdot\mathbf{D}}{\|\mathbf{C}\|^2}\mathbf{C} = \frac{18}{45}\langle 6,-3,0\rangle = \langle \frac{12}{5}, -\frac{6}{5}, 0\rangle$

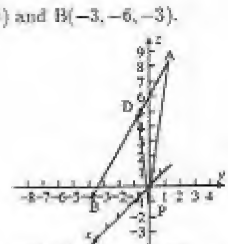
27. d units is the distance from $\mathrm{P}(2,-1,-4)$ to the line through $\mathrm{A}(3,-2,2)$ and $\mathrm{B}(-9,-6,6)$.

$\mathbf{V}(\overrightarrow{\mathrm{AP}}) = \langle -1,1,-6\rangle$, $\|\mathbf{V}(\overrightarrow{\mathrm{AP}})\|^2 = 1+1+36 = 38$, $\mathbf{V}(\overrightarrow{\mathrm{AB}}) = \langle -12,-4,4\rangle$, $\|\mathbf{V}(\overrightarrow{\mathrm{AB}})\|^2 = 144+16+16 = 176$

$\mathbf{V}(\overrightarrow{\mathrm{AP}})\cdot\mathbf{V}(\overrightarrow{\mathrm{AB}}) = 12-4-24 = -16$. $d = \sqrt{\|\mathbf{V}(\overrightarrow{\mathrm{AP}})\|^2 - \dfrac{[\mathbf{V}(\overrightarrow{\mathrm{AP}})\cdot\mathbf{V}(\overrightarrow{\mathrm{AB}})]}{\|\mathbf{V}(\overrightarrow{\mathrm{AB}})\|^2}} = \sqrt{38-\frac{256}{176}} = \sqrt{\frac{402}{11}} = \frac{1}{11}\sqrt{4422}$

28. Find the distance d from the point $\mathrm{P}(3,2,1)$ to the line through the points $\mathrm{A}(1,2,9)$ and $\mathrm{B}(-3,-6,-3)$.

▹ Let D be the foot of the perpendicular from P on AB. See the figure.



$\overrightarrow{\mathrm{AP}} = \langle 3,2,1\rangle - \langle 1,2,9\rangle = \langle 2,0,-8\rangle$. Let $c = \|\overrightarrow{\mathrm{AP}}\| = \sqrt{2^2+0^2+8^2} = \sqrt{68}$.

$\overrightarrow{\mathrm{AB}} = \langle -3,-6,-3\rangle - \langle 1,2,9\rangle = \langle -4,-8,-12\rangle$.

Let $a = \|\overrightarrow{\mathrm{AD}}\| = |\mathrm{AP}_{AB}| = \dfrac{|\overrightarrow{\mathrm{AP}}\cdot\overrightarrow{\mathrm{AB}}|}{\|\overrightarrow{\mathrm{AB}}\|} = \dfrac{|-8+0+96|}{\sqrt{4^2+8^2+12^2}} = \dfrac{88}{4\sqrt{14}} = \dfrac{22}{\sqrt{14}}$. Then

$d = \sqrt{c^2-a^2} = \sqrt{68-\frac{484}{14}} = \sqrt{\frac{234}{7}} = \frac{1}{7}\sqrt{1638}$

29. Let $\mathrm{A} = (2,2,2)$, $\mathrm{B} = (2,0,1)$, $\mathrm{C} = (4,1,-1)$, $\mathrm{D} = (4,3,0)$. $\mathbf{V}(\overrightarrow{\mathrm{BC}}) = \langle 2,1,-2\rangle = \mathbf{V}(\overrightarrow{\mathrm{AD}})$ so ABCD is a parallelogram. $\mathbf{V}(\overrightarrow{\mathrm{AB}})\cdot\mathbf{V}(\overrightarrow{\mathrm{AD}}) = \langle 0,-2,-1\rangle\cdot\langle 2,1,-2\rangle = 0$ so there is a right angle at A. Therefore the parallelogram is a rectangle.

30. $\mathrm{A} = (2,2,2)$, $\mathrm{B} = (0,1,2)$, $\mathrm{C} = (-1,3,3)$, $\mathrm{D} = (3,0,1)$. $\overrightarrow{\mathrm{AD}} = \langle 1,-2,-1\rangle = \overrightarrow{\mathrm{CB}} \Rightarrow$ ADBC is a parallelogram.

31. Let $\mathrm{A} = (-2,3,1)$, $\mathrm{B} = (1,2,3)$, $\mathrm{P} = (3,-1,2)$. $\|\mathbf{V}(\overrightarrow{\mathrm{AB}})\|^2 = \|\langle 3,-1,2\rangle\|^2 = 9+1+4 = 14$, $\|\mathbf{V}(\overrightarrow{\mathrm{AP}})\|^2 = \|\langle 5,-4,1\rangle\|^2 = 25+16+1 = 42$, $\mathbf{V}(\mathrm{AP})\cdot\mathbf{V}(\mathrm{AB}) = 15+4+2 = 21$. If A square units is the area,

$$\mathrm{A} = \tfrac{1}{2}\mathrm{bh} = \tfrac{1}{2}\|\mathbf{V}(\overrightarrow{\mathrm{AB}})\|\sqrt{\|\mathbf{V}(\overrightarrow{\mathrm{AP}})\|^2 - \frac{[\mathbf{V}(\overrightarrow{\mathrm{AP}})\cdot\mathbf{V}(\overrightarrow{\mathrm{AB}})]^2}{\|\mathbf{V}(\overrightarrow{\mathrm{AB}})\|^2}} = \tfrac{1}{2}\sqrt{\|\mathbf{V}(\overrightarrow{\mathrm{AP}})\|^2\|\mathbf{V}(\overrightarrow{\mathrm{AB}})\|^2 - [\mathbf{V}(\overrightarrow{\mathrm{AP}})\cdot\mathbf{V}(\overrightarrow{\mathrm{AB}})]^2}$$

$$= \tfrac{1}{2}\sqrt{(14)(42)-(21)^2} = \tfrac{7}{2}\sqrt{(2)(6)-9} = \tfrac{7}{2}\sqrt{3}$$

32. Prove using vectors that the points $\mathrm{A}(-2,1,6)$, $\mathrm{B}(2,4,5)$, and $\mathrm{C}(-1,-2,1)$ form a right triangle. Find its area.

▹ We have $\overrightarrow{\mathrm{AB}} = \langle 2,4,5\rangle - \langle -2,1,6\rangle = \langle 4,3,-1\rangle$ and $\overrightarrow{\mathrm{AC}} = \langle -1,-2,1\rangle - \langle -2,1,6\rangle = \langle 1,-3,-5\rangle$.

Because $\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{AC}} = 4(1)+3(-3)+(-1)(-5) = 0$, then ABC is a right angle.

Area $= \frac{1}{2}\|\overrightarrow{\mathrm{AB}}\|\|\overrightarrow{\mathrm{AC}}\| = \frac{1}{2}\sqrt{4^2+3^2+1^2}\sqrt{1^2+3^2+5^2} = \frac{1}{2}\sqrt{26}\sqrt{35} = \frac{1}{2}\sqrt{910}$

33. $y = x^2$, $y' = 2x$, $y'(2) = 4$. Thus $\langle 1,4\rangle$ is a tangent vector. $\mathbf{U} = \pm\langle 1,4\rangle/\sqrt{1^2+4^2} = \pm\langle 1,4\rangle/\sqrt{17}$

34. For the parabola $y = x^2$, $y' = 2x$. Hence a normal to the parabola at the point $(2,4)$ has a slope of $-\frac{1}{4}$. Thus $\tan\theta = -\frac{1}{4}$. The required unit vectors are $\pm(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$, that is

$-\dfrac{4}{\sqrt{17}}\mathbf{i} + \dfrac{1}{\sqrt{17}}\mathbf{j}$ and $\dfrac{4}{\sqrt{17}}\mathbf{i} - \dfrac{1}{\sqrt{17}}\mathbf{j}$ or equivalently $-\frac{4}{17}\sqrt{17}\,\mathbf{i} + \frac{1}{17}\sqrt{17}\,\mathbf{j}$ and $\frac{4}{17}\sqrt{17}\,\mathbf{i} - \frac{1}{17}\sqrt{17}\,\mathbf{j}$.

35. $\mathbf{A} = 3\mathbf{i}+5\mathbf{j}-3\mathbf{k}$, $\mathbf{B} = -\mathbf{i}-2\mathbf{j}+3\mathbf{k}$, $\mathbf{C} = 2\mathbf{i}-\mathbf{j}+4\mathbf{k}$. $\mathbf{A}-2\mathbf{C} = 3\mathbf{i}+5\mathbf{j}-3\mathbf{k}-2(2\mathbf{i}-\mathbf{j}+4\mathbf{k}) = -\mathbf{i}+7\mathbf{j}-11\mathbf{k}$.

The component of **B** in the direction of $\mathbf{A}-2\mathbf{C}$ is $\dfrac{\mathbf{B}\cdot(\mathbf{A}-2\mathbf{C})}{\|\mathbf{A}-2\mathbf{C}\|} = \dfrac{1-14-33}{\sqrt{1+49+121}} = \dfrac{-46}{\sqrt{171}} = -\frac{46}{57}\sqrt{19}$.

36. Find the cosines of the angles of the triangle having vertices at $A(0,0,0)$, $B(4,-1,3)$, and $C(1,2,3)$.

▹ Theorem 10.3.5 requires the calculation of three magnitudes and three dot products, while a direct application of the law of cosines needs only the three magnitudes. We find

$$a = \|\overrightarrow{BC}\| = \|\langle -3,3,0\rangle\| = \sqrt{3^2+3^2+0^2} = 3\sqrt{2}$$
$$b = \|\overrightarrow{AC}\| = \|\langle 1,2,3\rangle\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$
$$c = \|\overrightarrow{AB}\| = \|\langle 4,1,-3\rangle\| = \sqrt{4^2+1^2+3^2} = \sqrt{26}$$
$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{14+26-18}{2\sqrt{14}\sqrt{26}} = \frac{11}{\sqrt{14}\sqrt{26}} = \frac{11}{182}\sqrt{91}$$
$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{18+26-14}{2(3\sqrt{2})\sqrt{26}} = \frac{5}{2\sqrt{13}} = \frac{5}{26}\sqrt{13}$$
$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{18+14-26}{2(3\sqrt{2})\sqrt{14}} = \frac{1}{2\sqrt{7}} = \frac{1}{14}\sqrt{7}$$

37. $\mathbf{F} = 8(\cos\frac{1}{3}\pi\mathbf{i} + \sin\frac{1}{3}\pi\mathbf{j}) = 4\mathbf{i} + 4\sqrt{3}\mathbf{j}$.
(a) If W_1 ft-lb is the work done by $\mathbf{F}$ in moving an object along the x axis from the origin to the point $(6,0)$, then $W_1 = \mathbf{F}\cdot\langle 6,0\rangle = \langle 4,4\sqrt{3}\rangle\cdot\langle 6,0\rangle = 24$.
(b) If W_2 ft-lb is the work done by $\mathbf{F}$ in moving an object along the y axis from the origin to the point $(0,6)$, then $W_2 = \mathbf{F}\cdot\langle 0,6\rangle = \langle 4,4\sqrt{3}\rangle\cdot\langle 0,6\rangle = 24\sqrt{3}$.

38. $W = \mathbf{F}\cdot\mathbf{D} = 10(\cos\frac{1}{4}\pi\mathbf{i} + \sin\frac{1}{4}\pi\mathbf{j})\cdot[5-(-2)]\mathbf{j} = 10(\frac{1}{2}\sqrt{2})7 = 35\sqrt{2}$. The work done is $35\sqrt{2}$ foot-pounds.

39. $\mathbf{F} = 9(\cos\frac{2}{3}\pi\mathbf{i} + \sin\frac{2}{3}\pi\mathbf{j}) = -\frac{9}{2}\mathbf{i} + \frac{9}{2}\sqrt{3}\,\mathbf{j}$.
If W ft-lb is the work done by $\mathbf{F}$ in moving an object from the origin to $(-4,-2)$, then
$W = \mathbf{F}\cdot\langle -4,-2\rangle = \langle -\frac{9}{2},\frac{9}{2}\sqrt{3}\rangle\cdot\langle -4,-2\rangle = 18 - 9\sqrt{3} \approx 2.41$

40. Two forces represented by the vectors $\mathbf{F}_1 = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{F}_2 = -4\mathbf{i} + 5\mathbf{j}$ act on a particle and cause it to move along a straight line from point $A(2,5)$ to point $B(7,3)$. If the magnitudes of the forces are measured in pounds and distance is measured in feet, find the work done by the two forces acting together.

▹ The displacement vector is given by
$\mathbf{D} = \mathbf{b} - \mathbf{a} = \langle 7,3\rangle - \langle 2,5\rangle = 5\mathbf{i} - 2\mathbf{j}$
The number of foot-pounds in the work done by the two forces is given by
$$W = (\mathbf{F}_1 + \mathbf{F}_2)\cdot\mathbf{D} = [(3\mathbf{i}-\mathbf{j}) + (-4\mathbf{i}+5\mathbf{j})]\cdot\mathbf{D} = (-\mathbf{i}+4\mathbf{j})\cdot(5\mathbf{i}-2\mathbf{j}) = (-1)5 + 4(-2) = -13$$
The work done is -13 foot-pounds.

41. If W ft-lb is the work done by the force $\mathbf{F} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in moving an object from $P_1(-2,4,3)$ to $P_2(1,-3,5)$, then $W = \mathbf{F}\cdot\mathbf{V}(\overrightarrow{P_1P_2})(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})(3\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}) = 9 + 14 + 2 = 25$.

42. $W = \mathbf{F}\cdot\mathbf{D} = (5\mathbf{i} - 3\mathbf{k})\cdot(\langle -5,6,2\rangle - \langle 4,1,3\rangle) = \langle 5,0,-3\rangle\cdot\langle -9,5,-1\rangle = -42$. The work done is -42 foot-pounds.

43. For force $\mathbf{F}$, $\cos\alpha = \frac{1}{6}\sqrt{6}$ and $\cos\beta = \frac{1}{3}\sqrt{6}$. Therefore
$(\frac{1}{6}\sqrt{6})^2 + (\frac{1}{3}\sqrt{6})^2 + \cos^2\gamma = 1$; $\frac{1}{6} + \frac{2}{3} + \cos^2\gamma = 1$; $\cos^2\gamma = \frac{1}{6}$; $\cos\gamma = \frac{1}{6}\sqrt{6}$. $\mathbf{F} = 10(\frac{1}{6}\sqrt{6}\,\mathbf{i} + \frac{1}{3}\sqrt{6}\,\mathbf{j} + \frac{1}{6}\sqrt{6}\,\mathbf{k})$
If W ft-lb is the work done by $\mathbf{F}$ in moving an object from the origin to $(7,-4,2)$, then
$$W = \mathbf{F}\cdot\mathbf{V}(\overrightarrow{OP}) = (\tfrac{5}{3}\sqrt{6}\,\mathbf{i} + \tfrac{10}{3}\sqrt{6}\,\mathbf{j} + \tfrac{5}{3}\sqrt{6}\,\mathbf{k})\cdot(7\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = \tfrac{35}{3}\sqrt{6} - \tfrac{40}{3}\sqrt{6} + \tfrac{10}{3}\sqrt{6} = \tfrac{5}{3}\sqrt{6}$$

44. If $\mathbf{A}$ and $\mathbf{B}$ are nonzero vectors, prove that the vector $\mathbf{A} - c\mathbf{B}$ is orthogonal to $\mathbf{B}$ if $c = \mathbf{A}\cdot\mathbf{B}/\|\mathbf{B}\|^2$.

▹ The vectors $\mathbf{A} - c\mathbf{B}$ and $\mathbf{B}$ are orthogonal if
$$(\mathbf{A} - c\mathbf{B})\cdot\mathbf{B} = 0 \qquad (1)$$
Equation (1) is true if and only if
$$\mathbf{A}\cdot\mathbf{B} - c\mathbf{B}\cdot\mathbf{B} = 0$$
$$\mathbf{A}\cdot\mathbf{B} - c\|\mathbf{B}\|^2 = 0 \qquad (2)$$
Because $\|\mathbf{B}\| \neq 0$, we may solve Eq. (2), which contains all scalars, for c. Thus, (1) is true if
$$c = \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|^2}$$

45. $\mathbf{A} = 12\mathbf{i} + 9\mathbf{j} - 5\mathbf{k}$, $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$. From Exercise 44, $\mathbf{B} - c\mathbf{A}$ is orthogonal to $\mathbf{A}$ if
$$c = \frac{\mathbf{B}\cdot\mathbf{A}}{\|\mathbf{A}\|^2} = \frac{(4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})\cdot(12\mathbf{i} + 9\mathbf{j} - 5\mathbf{k})}{144 + 81 + 25} = \frac{48 + 27 + 25}{250} = \frac{100}{250} = \frac{2}{5}$$

46. $\mathbf{A} = 12\mathbf{i} + 9\mathbf{j} - 5\mathbf{k}$, $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$. From Exercise 44, $\mathbf{A} - d\mathbf{B}$ is orthogonal to $\mathbf{B}$ if
$$c = \frac{\mathbf{B}\cdot\mathbf{A}}{\|\mathbf{B}\|^2} = \frac{(4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})\cdot(12\mathbf{i} + 9\mathbf{j} - 5\mathbf{k})}{16 + 9 + 25} = \frac{48 + 27 + 25}{50} = \frac{100}{50} = 2$$

47. $(\|\mathbf{B}\|\mathbf{A} + \|\mathbf{A}\|\mathbf{B})\cdot(\|\mathbf{B}\|\mathbf{A} - \|\mathbf{A}\|\mathbf{B}) = \|\mathbf{B}\|\mathbf{A}\cdot\|\mathbf{B}\|\mathbf{A} - \|\mathbf{A}\|\mathbf{B}\cdot\|\mathbf{A}\|\mathbf{B} = \|\mathbf{B}\|^2\|\mathbf{A}\|^2 - \|\mathbf{A}\|^2\|\mathbf{B}\|^2 = 0$
Therefore, the vectors $\|\mathbf{B}\|\mathbf{A} + \|\mathbf{A}\|\mathbf{B}$ and $\|\mathbf{B}\|\mathbf{A} - \|\mathbf{A}\|\mathbf{B}$ are orthogonal.

48. Prove that if $\mathbf{A}$ and $\mathbf{B}$ are any nonzero vectors and $\mathbf{C} = \|\mathbf{B}\|\mathbf{A} + \|\mathbf{A}\|\mathbf{B}$, then the angle θ_1 between $\mathbf{A}$ and $\mathbf{C}$ has the same measure as the angle θ_2 between $\mathbf{B}$ and $\mathbf{C}$.

▸ Let $\mathbf{U} = \dfrac{\mathbf{A}}{\|\mathbf{A}\|}$ and $\mathbf{V} = \dfrac{\mathbf{B}}{\|\mathbf{B}\|}$ be the directions of $\mathbf{A}$ and $\mathbf{B}$. Then
$$\mathbf{D} = \frac{\mathbf{C}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \frac{\mathbf{A}}{\|\mathbf{A}\|} + \frac{\mathbf{B}}{\|\mathbf{B}\|}$$
has the same direction as $\mathbf{C}$. Thus,
$$\cos\theta_1 = \frac{\mathbf{U}\cdot\mathbf{D}}{\|\mathbf{U}\|\|\mathbf{D}\|} = \frac{\mathbf{U}\cdot(\mathbf{U}+\mathbf{V})}{1\|\mathbf{D}\|} = \frac{1 + \mathbf{U}\cdot\mathbf{V}}{\|\mathbf{D}\|}$$
$$\cos\theta_2 = \frac{\mathbf{V}\cdot\mathbf{D}}{\|\mathbf{V}\|\|\mathbf{D}\|} = \frac{\mathbf{V}\cdot(\mathbf{U}+\mathbf{V})}{1\|\mathbf{D}\|} = \frac{1 + \mathbf{U}\cdot\mathbf{V}}{\|\mathbf{D}\|}$$
Because $\cos\theta_1 = \cos\theta_2$, the angle between $\mathbf{A}$ and $\mathbf{C}$ has the same measure as the angle between $\mathbf{B}$ and $\mathbf{C}$.

49. If $\mathbf{A}$ and $\mathbf{B}$ are two nonzero parallel vectors, then $\mathbf{B} = k\mathbf{A}$. If α is the radian measure of the angle between them, $\cos\alpha = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \dfrac{\mathbf{A}\cdot(k\mathbf{A})}{\|\mathbf{A}\|\|k\mathbf{A}\|} = \dfrac{k(\mathbf{A}\cdot\mathbf{A})}{|k|\|\mathbf{A}\|^2} = \dfrac{k\|\mathbf{A}\|^2}{|k|\|\mathbf{A}\|^2} = \pm 1$. Thus $\alpha = 0$ or π.

Conversely, if $\alpha = 0$ then $\cos\alpha = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = 1$. Therefore
$$\left(\frac{\mathbf{A}}{\|\mathbf{A}\|} - \frac{\mathbf{B}}{\|\mathbf{B}\|}\right)^2 = \frac{\mathbf{A}\cdot\mathbf{A}}{\|\mathbf{A}\|^2} - 2\frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} + \frac{\mathbf{B}\cdot\mathbf{B}}{\|\mathbf{B}\|^2} = 1 - 2 + 1 = 0; \text{ so } \frac{\mathbf{A}}{\|\mathbf{A}\|} - \frac{\mathbf{B}}{\|\mathbf{B}\|} = \mathbf{0};\ \mathbf{B} = \frac{\|\mathbf{B}\|}{\|\mathbf{A}\|}\mathbf{A}.$$
Thus there is a scalar $k = \|\mathbf{B}\|/\|\mathbf{A}\|$ such that $\mathbf{B} = k\mathbf{A}$ and so $\mathbf{A}$ and $\mathbf{B}$ are parallel.
Similarly, if $\alpha = \pi$, then $\cos\alpha = -1$ and $\mathbf{B} = -(\|\mathbf{B}\|/\|\mathbf{A}\|)\mathbf{A}$ and again $\mathbf{A}$ and $\mathbf{B}$ are parallel.

In Exercises 50–52, prove the theorem by vector analysis.

50. The medians of triangle ABC meet in a point.

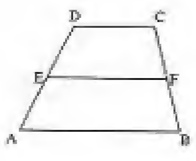
▸ Let G be the point $\frac{2}{3}$ of the way from A to the midpoint of BC. Then $\mathbf{g} = (1 - \frac{2}{3})\mathbf{a} + \frac{2}{3}(\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}) = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c}$.
By symmetry, we see that G also lies on the other two medians.

51. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.

▸ If P is the midpoint of AB and Q the midpoint of AC, then $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}(\mathbf{c} - \mathbf{b}) = \frac{1}{2}\overrightarrow{BC}$
Thus PQ is parallel to AB and half its length.

52. The line joining the midpoints of the nonparallel sides of a trapezoid is parallel to the parallel sides and that its length is one-half the sum of the lengths of the parallel sides.

▸ ABCD is a trapezoid with parallel sides AB and DC. Let E be the midpoint of side AD, and F the midpoint of side BC. Refer to the figure. To simplify the notation, we let $\overrightarrow{AB}$ denote the vector having the directed line segment $\overrightarrow{AB}$ as a representation, and similarly for other directed line segments. The origin O is not shown. Because E is the midpoint of segment AD, and F is the midpoint of BC, then
$$\overrightarrow{OE} = \tfrac{1}{2}\overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{OD}$$
and
$$\overrightarrow{OF} = \tfrac{1}{2}\overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{OC}$$



It follows that
$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = (\tfrac{1}{2}\overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{OC}) - (\tfrac{1}{2}\overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{OD}) = \tfrac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA}) + \tfrac{1}{2}(\overrightarrow{OC} - \overrightarrow{OD})$$

$$= \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{DC} \tag{3}$$

Because $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are parallel and the vectors have the same direction, there is a positive scalar k such that

$$\overrightarrow{DC} = k\overrightarrow{AB} \tag{4}$$

Substituting from (4) into (3) we have

$$\overrightarrow{EF} = \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}k\overrightarrow{AB} = \tfrac{1}{2}(1+k)\overrightarrow{AB}$$

Because $\overrightarrow{EF}$ is a scalar multiple of $\overrightarrow{AB}$, then $\overrightarrow{EF}$ is parallel to $\overrightarrow{AB}$. Furthermore,

$$\begin{aligned} \|\overrightarrow{EF}\| &= \left\|\tfrac{1}{2}(1+k)\overrightarrow{AB}\right\| = \tfrac{1}{2}(1+k)\|\overrightarrow{AB}\| && (1+k>0) \\ &= \tfrac{1}{2}(\|\overrightarrow{AB}\| + k\|\overrightarrow{AB}\|) = \tfrac{1}{2}(\|\overrightarrow{AB}\| + \|k\overrightarrow{AB}\|) && (k>0) \\ &= \tfrac{1}{2}(\|\overrightarrow{AB}\| + \|\overrightarrow{DC}\|) && (\text{from } (4)) \end{aligned}$$

53. Let $\mathbf{F}$ be a unit vector in the interface pointing to the left, let α_1 be the complement of θ_1 and let α_2 be the supplement of θ_2. Then $\mathbf{A}\cdot\mathbf{F} + \mu\mathbf{B}\cdot\mathbf{F} = \cos\alpha_1 + \mu\cos\alpha_2 = \sin\theta_1 - \mu\sin\theta_2 = 0$

54. We have $\|x\mathbf{A} - \mathbf{B}\| > 0$ unless $\mathbf{B} = x\mathbf{A}$. Then
$0 < \|x\mathbf{A} - \mathbf{B}\|^2 = (x\mathbf{A} - \mathbf{B})\cdot(x\mathbf{A} - \mathbf{B}) = x^2\mathbf{A}\cdot\mathbf{A} - 2x\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B} = x^2\|\mathbf{A}\|^2 - 2x\mathbf{A}\cdot\mathbf{B} + \|\mathbf{B}\|^2$
Because the quadratic has no roots, its discriminant is negative, that is,
$(2\mathbf{A}\cdot\mathbf{B})^2 - 4\|\mathbf{A}\|^2\|\mathbf{B}\|^2 < 0 \Leftrightarrow (\mathbf{A}\cdot\mathbf{B})^2 < \|\mathbf{A}\|^2\|\mathbf{B}\|^2 \Leftrightarrow |\mathbf{A}\cdot\mathbf{B}| < \|\mathbf{A}\|\|\mathbf{B}\|$

55. $\|\mathbf{A}+\mathbf{B}\|^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B} = \|\mathbf{A}\|^2 + 2\mathbf{A}\cdot\mathbf{B} + \|\mathbf{B}\|^2$

56. $\mathbf{A}$ and $\mathbf{B}$ are orthogonal $\Leftrightarrow \mathbf{A}\cdot\mathbf{B} = 0 \Leftrightarrow \|\mathbf{A}\|^2 + 2\mathbf{A}\cdot\mathbf{B} + \|\mathbf{B}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 \Leftrightarrow \|\mathbf{A}+\mathbf{B}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2$

57. Prove the *Parallelogram law*. $\|\mathbf{A}+\mathbf{B}\|^2 + \|\mathbf{A}-\mathbf{B}\|^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) + (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})$
$= (\mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B}) + (\mathbf{A}\cdot\mathbf{A} - 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B}) = 2\mathbf{A}\cdot\mathbf{A} + 2\mathbf{B}\cdot\mathbf{B} = 2\|\mathbf{A}\|^2 + 2\|\mathbf{B}\|^2$.
The geometric interpretation is that a parallelogram's perimeter equals the sum of the length of its diagonals. The converse is also true: If a quadrilateral Q in E^3 is such that its perimeter equals the sum of the lengths of its diagonals, then Q is a plane parallelogram.

58. Prove the *polarization identity*: $\tfrac{1}{4}(\|\mathbf{A}+\mathbf{B}\|^2 - \|\mathbf{A}-\mathbf{B}\|^2) = \tfrac{1}{4}[(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) - (\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})]$
$= \tfrac{1}{4}[(\mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{A} - 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B})] = \tfrac{1}{4}(4\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B}$

59. $\mathbf{E}_1 = \mathbf{E}_{//} = \dfrac{\mathbf{E}\cdot\mathbf{H}}{\|\mathbf{H}\|^2}\mathbf{H}$ is parallel to $\mathbf{H}$ and $\mathbf{E}_2 = \mathbf{E} - \mathbf{E}_{//}$ is orthogonal to $\mathbf{H}$.

60. Find the total revenue $R = \mathbf{A}\cdot\mathbf{S}$ for each day of the week.

▹ Mon.	$\langle 250, 180, 310\rangle\cdot\langle 25.50, 16.80, 54.55\rangle = \$26{,}309.50$
Tues.	$\langle 185, 210, 215\rangle\cdot\langle 27.50, 14.60, 61.25\rangle = \$21{,}322.25$
Wed.	$\langle 400, 120, 180\rangle\cdot\langle 21.20, 21.50, 66.50\rangle = \$23{,}030.00$
Thurs.	$\langle 355, 165, 200\rangle\cdot\langle 23.40, 18.50, 62.30\rangle = \$23{,}819.50$
Fri.	$\langle 370, 145, 240\rangle\cdot\langle 22.60, 19.10, 61.75\rangle = \$25{,}951.50$

10.4 PLANES AND LINES IN R^3

10.4.1 Definition If $\mathbf{N}$ is a given nonzero vector and P_0 is a given point, then the set of all points P for which $\mathbf{V}(\overrightarrow{P_0P})$ and $\mathbf{N}$ are orthogonal is defined to be a plane through P_0 having $\mathbf{N}$ as a *normal vector*.

10.4.2 Definition If $P_0(x_0, y_0, z_0)$ is a point in a plane and a normal vector to the plane is $\mathbf{N} = \langle a, b, c\rangle$, then an equation of the plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Theorem 10.4.3 If a, b, and c are not all zero, the graph of an equation of the form $ax + by + cz + d = 0$ is a plane, and $\langle a, b, c\rangle$ is a normal vector to the plane.

If one of the coefficients a, b, and c in Theorem 10.4.3 is zero, then the plane is parallel to a coordinate axis. If two of the coefficients a, b, and c are zero, then the plane is parallel to two coordinate axes and, hence, perpendicular to one of the coordinate axes. We summarize the possible cases.

$a = 0$	Plane is parallel to the x axis.
$b = 0$	Plane is parallel to the y axis.
$c = 0$	Plane is parallel to the z axis.
$a = 0$ and $b = 0$	Plane is perpendicular to the z axis.
$a = 0$ and $c = 0$	Plane is perpendicular to the y axis.
$b = 0$ and $c = 0$	Plane is perpendicular to the x axis.

10.4.4 Definition An *angle between two planes* is defined to be the angle between normal vectors of the planes. There are two angles between two planes. If one of those angles is θ, the other is the supplement of θ.

10.4.5 Definition Two planes are *parallel* if and only if their normal vectors are parallel.

10.4.6 Definition Two planes are *perpendicular* if and only if their normal vectors are orthogonal.

Line A line L that contains the point (x_0, y_0, z_0) and is parallel to representations of the vector (a, b, c) has *parametric equations*

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct \tag{I}$$

It has the *symmetric equations*

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \text{ if } a \neq 0,\ b \neq 0,\ c \neq 0 \tag{II}$$

$\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b}$ and $z = z_0$ if $a \neq 0$, $b \neq 0$, $c = 0$, and similar equations if $a = 0$ or $b = 0$.

The vector $\mathbf{V} = (a, b, c)$ is called a *direction vector* of the line L; we also say that $[a, b, c]$ is a set of *direction numbers* for L. $k\mathbf{V}$, $k \neq 0$, is also a direction vector of L.

Let (a_1, b_1, c_1) and (a_2, b_2, c_2) be direction vectors for lines L_1 and L_2. L_1 and L_2 are parallel if $(a_1, b_1, c_1) = k(a_2, b_2, c_2)$ and L_1 and L_2 are perpendicular if $(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = 0$.

Let M be the plane $ax + by + cz + d = 0$ with normal vector (a, b, c). If line L is perpendicular to plane M, then (a, b, c) is a direction vector for line L. If line L lies in plane M or if L is parallel to M, then (a', b', c') is a direction vector of line L if and only if $(a, b, c) \cdot (a', b', c') = 0$.

Two lines that do not lie in one plane are *skew*. See Exercises 57 and 58.

Determinant A second order determinant is defined by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

A third-order determinant may defined by expanding along the first row:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Similarly, a fourth-order determinant is defined by expanding along its first row:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_3 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_3 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

A determinant of any order has following properties:

(i) A determinant is zero if and only if some row is a linear combination of the others. In particular, a determinant is zero if two rows are equal.

(ii) Multiplying the elements of a row by a number c multiplies the value of the determinant by c.

The following result is useful because after expanding along the first row we can use our graphics calculators to evaluate the four numerical determinants. See Exercise 8.

Determinant Form An equation of the plane through the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Proof Expanding by the first row, we see from Theorem 10.4.3 that this is the equation of a plane. Furthermore, if $(x, y, z) = (x_1, y_1, z_1)$, then two rows of the determinant are equal. Thus, P_1 satisfies the equation, similarly, P_2 and P_3 satisfy the equation.

Intercept Form If a, b, and c are nonzero and are the x intercept, y intercept, and z intercept, respectively of a plane, then an equation of the plane is $x/a + y/b + z/c = 1$. See Exercise 28.

Distance to a Plane The distance from the plane $ax + by + cz + d = 0$ to the point $P_0(x_0, y_0, z_0)$ is given by

$\dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$. See Exercise 60.

Exercises 10.4

In Exercises 1–6, find an equation of the plane containing the point P and having the vector **N** as a normal vector.

1. $P(3,1,2)$; $\mathbf{N} = \langle 1,2,-3\rangle$: $1(x-3) + 2(y-1) - 3(z-2) = 0$; $x - 3 + 2y - 2 - 3z + 6 = 0$; $x + 2y - 3z + 1 = 0$

2. $P(-3,2,5)$; $\mathbf{N} = \langle 6,-3,-2\rangle$: $6(x+3) - 3(y-2) - 2(z-5) = 6x + 18 - 3y + 6 - 2z + 10 = 0$; $6x - 3y - 2z + 34 = 0$

3. $P(0,-1,2)$; $\mathbf{N} = \langle 0,1,-1\rangle$: $0(x-0) + 1(y+1) - 1(z-2) = 0$; $y + 1 - z + 2 = 0$; $y - z + 3 = 0$

4. $P(-1,8,3)$; $\mathbf{N} = \langle -7,-1,1\rangle$

▷ We apply Theorem 10.4.2. Thus, an equation of the plane is
$-7(x+1) - (y-8) + (z-3) = 0 \qquad\qquad 7x + y - z + 2 = 0$

5. $P(2,1,-1)$; $\mathbf{N} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$: $-1(x-2) + 3(y-1) + 4(z+1) = 0$; $x + 2 + 3y - 3 + 4z + 4 = 0$; $x - 3y - 4z - 3 = 0$

6. $P(1,0,0)$; $\mathbf{N} = \mathbf{i} + \mathbf{k}$: $1(x-1) + 0(y-0) + 1(z-0) = 0$; $x + z - 1 = 0$

In Exercises 7 and 8, find an equation of the plane containing the three given points.

7. By Theorem 10.4.3 an equation of the plane is of the form $ax + by + cz + d = 0$ (1)

Because the plane contains $(3,4,1)$, $(1,7,1)$, and $(-1,-2,5)$, we have the three equations

E_1:		$3a + 4b + c = -d$
$F_1 = E_1 + 3E_2$:		$-2b + 16c = -4d$
E_2: $a + 7b + c = -d$	$5F_1 + 2F_2$:	$92c = -24d$
$F_2 = E_2 + E_3$:		$5b + c = -2d$
E_3:		$-a - 2b + 5c = -d$

Hence $c = -\frac{6d}{23}$, $b = -\frac{2d}{23}$, $a = -\frac{3d}{23}$. Substituting the values into (1) we get

$-\frac{3d}{23}x - \frac{2d}{23}y - \frac{6d}{23}z + d = 0$; $3x + 2y + 6z - 23 = 0$ (2)

Alternatively, $\begin{vmatrix} x & y & z & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ -1 & -2 & 5 & 1 \end{vmatrix} = x\begin{vmatrix} 4 & 1 & 1 \\ 7 & 1 & 1 \\ -2 & 5 & 1 \end{vmatrix} - y\begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & 1 \end{vmatrix} + z\begin{vmatrix} 3 & 4 & 1 \\ 1 & 7 & 1 \\ -1 & -2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 4 & 1 \\ 1 & 7 & 1 \\ -1 & -2 & 5 \end{vmatrix}$

Evaluating the determinants, we get the equation $12x + 8y + 24z - 92 = 0$, which is equivalent to (2).

8. $(0,0,2)$, $(2,4,1)$, $(-2,3,3)$

▷ By Theorem 10.4.3, an equation of the plane is of the form
$$ax + by + cz = -d$$
We substitute the coordinates of the points and solve the resulting system for a, b, and c in terms of d.

$A(0,0,2)$:	$2c = -d$
$B(2,4,1)$:	$2a + 4b + c = -d$
$C(-2,3,3)$:	$-2a + 3b + 3c = -d$
$D = B + C$:	$7b + 4c = -2d$
$E = D - 2A$:	$7b \quad = 0$
$F = \frac{1}{7}E$:	$b = 0$
$G = \frac{1}{2}A$:	$c = -\frac{1}{2}d$
$\frac{1}{2}(B - 4F - C)$:	$a = -\frac{1}{4}d$

Setting $d = -4$, we get $a = 1$, $b = 0$, $c = 2$, and so an equation is
$$x + 2z - 4 = 0 \qquad (1)$$
Alternatively, we apply the determinant form to the given points.

$\begin{vmatrix} x & y & z & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 4 & 1 & 1 \\ -2 & 3 & 3 & 1 \end{vmatrix} = x\begin{vmatrix} 0 & 2 & 1 \\ 4 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix} - y\begin{vmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ -2 & 3 & 1 \end{vmatrix} + z\begin{vmatrix} 0 & 0 & 1 \\ 2 & 4 & 1 \\ -2 & 3 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 2 \\ 2 & 4 & 1 \\ -2 & 3 & 3 \end{vmatrix}$

Evaluating the determinants, we get the equation $7x + 14z - 28 = 0$ which is equivalent to (1).

In Exercises 9–14, sketch the plane and find two unit vectors normal to the plane.

9. A normal vector to the plane $2x - y + 2z - 6 = 0$ is $\langle 2, -1, 2\rangle$. $\|\langle 2, -1, 2\rangle\| = \sqrt{4+1+4} = 3$.
Hence two unit vectors normal to the plane are $\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\rangle$ and $\langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$.

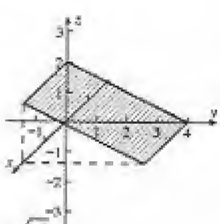
10. A normal vector to the plane $4x - 4y + 2z - 9 = 0$ is $\langle 4, -4, 2\rangle$. $\|\langle 4, -4, 2\rangle\| = \sqrt{4^2 + 4^2 + 2^2} = 6$
Hence two unit vectors normal to the plane are $\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\rangle$ and $\langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$.

11. A normal vector to the plane $4x + 3y - 12z = 0$ is $\langle 4, 3, -12\rangle$. $\|\langle 4, 3, -12\rangle\| = \sqrt{16 + 9 + 144} = 13$.
Hence two unit vectors normal to the plane are $\langle \frac{4}{13}, \frac{3}{13}, -\frac{12}{13}\rangle$ and $\langle -\frac{4}{13}, -\frac{3}{13}, \frac{12}{13}\rangle$.

12. $y + 2z - 4 = 0$

▸ Because the coefficient of x is zero, the plane is parallel to the x axis and perpendicular to the yz plane. The yz trace is the line in the yz plane determined by $y + 2z - 4 = 0$. This line intersects the y axis at $(0, 4, 0)$ and the z axis at $(0, 0, 2)$. The figure shows a sketch of the plane. By Theorem 10.4.3, a normal vector of the plane is $\mathbf{N} = \langle 0, 1, 2\rangle$. Because $\|\mathbf{N}\| = \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5}$, unit normal vectors are

$$\pm\frac{1}{\sqrt{5}}\langle 0, 1, 2\rangle = \pm\langle 0, \tfrac{1}{5}\sqrt{5}, \tfrac{2}{5}\sqrt{5}\rangle$$



13. A normal vector to the plane $3x + 2z - 6 = 0$ is $\langle 3, 0, 2\rangle$. $\|\langle 3, 0, 2\rangle\| = \sqrt{9 + 0 + 4} = \sqrt{13}$.
Hence two unit vectors normal to the plane are $\langle \frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}\rangle$ and $\langle -\frac{3}{\sqrt{13}}, 0, -\frac{2}{\sqrt{13}}\rangle$.

14. Unit normals to the plane $z = 5$ are $\langle 0, 0, 1\rangle$ and $\langle 0, 0, -1\rangle$.

In Exercises 15–20, find an equation of the plane satisfying the conditions.

15. Let $A = (2, 2, -4)$, $B = (7, -1, 3)$. Then $\mathbf{V}(\overrightarrow{AB}) = \langle 5, -3, 7\rangle$ is a normal vector to the plane.
Because the plane contains the point $(-5, 1, 2)$, an equation of the plane is
$(x + 5) - 3(y - 1) + 7(z - 2) = 0$; $5x - 3y + 7z + 14 = 0$

16. Parallel to the plane $4x - 2y + z - 1 = 0$ and containing the point $(2, 6, -1)$.

▸ Any plane parallel to the given plane is of the form
$4x - 2y + z + d = 0$
Substituting the coordinates of the given point, we find
$4(2) - 2(6) + (-1) + d = 0$
$d = 5$
Thus
$4x - 2y + z + 5 = 0$

17. A normal vector to the plane $x + 3y - z - 7 = 0$ is $\langle 1, 3, -1\rangle$. A normal vector $\langle a, b, c\rangle$ to the required plane will be orthogonal to $\langle 1, 3, -1\rangle$ so $\langle a, b, c\rangle \cdot \langle 1, 3, -1\rangle = 0$; $a + 3b - c = 0$. (1) Because $A(2, 0, 5)$ and $B(0, 2, -1)$ lie in the required plane, $\mathbf{V}(\overrightarrow{AB}) = \langle -2, 2, -6\rangle$ will be orthogonal to $\langle a, b, c\rangle$ so $\langle a, b, c\rangle \cdot \langle -2, 2, -6\rangle = 0$; $-2a + 2b - 6c = 0$ (2). Solving equations (1) and (2) simultaneously we get $a = -2c$ and $b = c$. An equation of the required plane is
$-2c(x - 2) + c(y - 0) + c(z - 5) = 0$; $2x - y - z + 1 = 0$

18. Perpendicular to $x - y + z = 0$ and $2x + y - 4z - 5 = 0$ and containing $(4, 0, -2)$. Let $\mathbf{N} = \langle a, b, c\rangle$.
$\langle 1, -1, 1\rangle \cdot \langle a, b, c\rangle = 0 \Rightarrow$ A: $a - b + c = 0$. $\langle 2, 1, -4\rangle \cdot \langle a, b, c\rangle = 0 \Rightarrow$ B: $2a + b - 4c = 0$. A + B: $3a - 3c = 0$, B − 2A: $3b - 6c = 0$. Choose $c = 1$ so $a = 1$, $b = 2$. $1(x - 4) + 2(y - 0) + 1(z + 2) = 0$, $x + 2y + z - 2 = 0$

19. Because the required plane is perpendicular to the yz plane, a normal vector has 0 as its first component. Hence it has the form $\mathbf{N} = \langle 0, b, c\rangle$. Because the required plane makes an angle of measure $\cos^{-1}(\frac{2}{3})$ with the plane $2x - y + 2z - 3$ their normal vectors make the same angle. Therefore

$$\frac{\langle 0, b, c\rangle \cdot \langle 2, -1, 2\rangle}{\|\langle 0, b, c\rangle\| \, \|\langle 2, -1, 2\rangle\|} = \frac{2}{3};\ -b + 2c = 2\sqrt{b^2 + c^2};\ b^2 - 4bc + 4c^2 = 4b^2 + 4c^2;\ 0 = 3b^2 + 4bc$$

from which we obtain $b = 0$ and $b = -\frac{4}{3}c$. The plane contains the point $(2, 1, 1)$.
When $b = 0$, $\mathbf{N} = \langle 0, 0, c\rangle$ and an equation is $0(x - 2) + 0(y - 1) + c(z - 1) = 0$; $z = 1$.
When $b = -\frac{4}{3}c$, $\mathbf{N} = \langle 0, -\frac{4}{3}c, c\rangle$. An equation is $0(x - 2) - \frac{4}{3}c(y - 1) + c(z - 1) = 0$; $4y - 3z - 1 = 0$.

20. Containing the point $P(-3,5,-2)$ and perpendicular to the vector $\mathbf{V}(\overrightarrow{OP})$.

▹ We apply Theorem 10.4.2 with the point $P(-3,5,-2)$ and the normal vector $\mathbf{N} = \langle -3,5,-2\rangle$. An equation of the plane is

$-3(x+3)+5(y-5)-2(z+2)=0 \qquad 3x-5y+2z+38=0$

In Exercises 21–23, find the acute angle θ between the two planes to the nearest tenth of a degree.

21. $2x-y-2z-5$, $6x-2y+3z+8=0$. $\cos\theta = \dfrac{\langle 2,-1,-2\rangle\cdot\langle 6,-2,3\rangle}{\sqrt{4+1+4}\sqrt{36+4+9}} = \dfrac{12+2-6}{\sqrt{9}\sqrt{49}} = \dfrac{8}{3\cdot 7} = \dfrac{8}{21}$; $\theta = 67.6^\circ$

22. $2x-5y+3z-1=0$, $y-5z+3=0$. $\cos\theta = \dfrac{|\langle 2,-5,3\rangle\cdot\langle 0,1,-5\rangle|}{\sqrt{2^2+5^2+3^2}\sqrt{0^2+1^2+5^2}} = \dfrac{20}{\sqrt{38}\sqrt{26}}$; $\theta = 50.5^\circ$

23. $3x+4y=0$ and $4x-7y+4z-6=0$. $\cos\theta = \dfrac{\langle 3,4,0\rangle\cdot\langle 4,-7,4\rangle}{\sqrt{9+16+0}\sqrt{16+49+16}} = \dfrac{12-28+0}{\sqrt{25}\sqrt{81}} = \dfrac{16}{5\cdot 9} = \dfrac{16}{45}$; $\theta = 69.2^\circ$

24. Find the distance from the plane $2x+2y-z-6=0$ to the point $(2,2,-4)$.

▹ We use the formula of Exercise 60.

$$\text{distance} = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|2(2)+2(2)-(-4)-6|}{\sqrt{2^2+2^2+1^2}} = \frac{6}{3} = 2$$

25. A normal vector to the plane $5x+11y+2z-30=0$ is $\mathbf{N} = \langle 5,11,2\rangle$. Choose any point in the given plane, say $Q=(6,0,0)$. If $P=(-2,6,3)$ then $\mathbf{V}(\overrightarrow{PQ}) = \langle -8,6,3\rangle$. If d units is the distance from the plane to point P, it follows by Example 10.4.5 that

$$d = \frac{\mathbf{N}\cdot\mathbf{V}(\overrightarrow{QP})}{\|\mathbf{N}\|} = \frac{\langle 5,11,2\rangle\cdot\langle -8,6,3\rangle}{\|\langle 5,11,23\rangle\|} = \frac{-40+66+6}{\sqrt{25+121+4}} = \frac{32}{\sqrt{150}} = \frac{16}{15}\sqrt{6}$$

26. The point $(0,0,9)$ lies in $4x-8y-z+9=0$. Distance to $4x-8y-z-6=0$ is $\dfrac{|4(0)-8(0)-9-6|}{\sqrt{4^2+8^2+1^2}} = \dfrac{15}{9} = \dfrac{5}{3}$

27. Choose a point in the plane $8y-6z-27=0$, say $P=(0,0,-\frac{9}{2})$, and a point in the plane $4y-3z-6=0$, say $Q=(0,0,-2)$. $\mathbf{N}=\langle 0,4,-3\rangle$ is normal to the parallel planes. If d units is the distance between the planes then

$$d = \frac{\mathbf{N}\cdot\mathbf{V}(QP)}{\|\mathbf{N}\|} = \frac{\langle 0,4,-3\rangle\cdot\langle 0,0,\frac{5}{2}\rangle}{\|\langle 0,4,-3\rangle\|} = \frac{15/2}{5} = \frac{3}{2}$$

28. Prove the intercept form of the equation of a plane.

▹ We are given that the x, y, and z intercepts of a plane are a, b, and c. Thus the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ lie on the plane. Substituting these points into the given equation $x/a+y/b+z/c=1$, we get the identities $1+0+0=1$, $0+1+0=1$ and $0+0+1=1$. Thus the given equation is that of the plane with the given intercepts.

In Exercises 29–36, find parametric and symmetric equations for the line L satisfying the conditions.

▹ $\mathbf{V}$ is a direction vector.

29. Let $P_1=(1,2,1)$ and $P_2=(5,-1,1)$. Then $\mathbf{V}=\mathbf{V}(\overrightarrow{P_1P_2}) = \langle 4,-3,0\rangle$. Choosing $P_0=P_1$ we get parametric equations $x=1+4t$, $y=2-3t$, $z=1$. Symmetric equations of L are $\dfrac{x-1}{4}=\dfrac{y-2}{-3}$ and $z=1$.

30. $P(5,3,2)$, $\mathbf{V}=\langle 4,1,-1\rangle$. $x=5+4t$, $y=3+t$, $z=2-t$; $\dfrac{x-5}{4}=\dfrac{y-3}{1}=\dfrac{z-2}{-1}$

31. Through the origin and perpendicular to so the line $\frac{1}{4}(x-10)=\frac{1}{3}y=\frac{1}{2}z=t$ at their intersection. A set of parametric equations for the given line is $x=10+4t$, $y=3t$, $z=2t$. Let $P=(4t+10,3t,2t)$ on the given line be the point of intersection. Because L contains the origin and the two lines are perpendicular, it follows that the two vectors $\langle 4,3,2\rangle$ and $\mathbf{V}(\overrightarrow{OP}) = \langle 4t+10,3t,2t\rangle$ are orthogonal. Therefore $\langle 4,3,2\rangle\cdot\langle 4t+10,3t,2t\rangle=0$; $16t+10+9t+4t=0$; $t=-\frac{40}{29}$; $P=(\frac{130}{29},-\frac{120}{29},-\frac{80}{29})$. Hence $\left[\frac{130}{29},-\frac{120}{29},-\frac{80}{29}\right]$ is a set of direction numbers of L, and so is the proportional set $[13,-12,-8]$. Therefore symmetric equations of L are $\dfrac{x}{13}=\dfrac{y}{-12}=\dfrac{z}{-8}$ and parametric equations are $x=13t$, $y=-12t$, $z=-8t$.

32. Through the origin and perpendicular to the lines having direction numbers $[4,2,1]$ and $[-3,-2,1]$.

▷ Let $[a,b,c]$ be a set of direction numbers of the required line L. Because line L is perpendicular to the line with direction numbers $[4,2,1]$, the vector $\langle a,b,c\rangle$ is orthogonal to the vector $\langle 4,2,1\rangle$. Thus

$$\langle a,b,c\rangle \cdot \langle 4,2,1\rangle = 0$$
$$4a + 2b + c = 0 \qquad (1)$$

Because line L is perpendicular to the line with direction numbers $[-3,-2,1]$, we have

$$\langle a,b,c\rangle \cdot \langle -3,-2,1\rangle = 0$$
$$-3a - 2b + c = 0 \qquad (2)$$

$$(1)+(2): \qquad a + 2c = 0$$
$$3(1)+4(2): \qquad -2b + 7c = 0$$

Thus, $a = -2c$, $b = \frac{7}{2}c$. We take $c = 2$ to eliminate the fraction and obtain $[-4,7,2]$ as a set of direction numbers for L. Because L contains the origin, we use parametric equations (I) with $(x_0, y_0, z_0) = (0,0,0)$. Thus, parametric equations for L are

$x = -4t \quad y = 7t \quad z = 2t$

We use symmetric equations (II) to obtain

$\frac{x}{-4} = \frac{y}{7} = \frac{z}{2}$

33. Let $[a,b,c]$ be a set of direction numbers of L. Because L is perpendicular to the lines having direction numbers $[-5,1,2]$ and $[2,-3,-4]$, then

$\langle a,b,c\rangle \cdot \langle -5,1,2\rangle = 0$; $-5a + b + 2c = 0$; $b + 2c = 5a$

$\langle a,b,c\rangle \cdot \langle 2,-3,-4\rangle = 0$; $2a - 3b - 4c = 0$; $3b + 4c = 2a$

Solving for b and c, we obtain $b = -8a$, $c = \frac{13}{2}a$. Thus $\left[a, -8a, \frac{13}{2}a\right]$ is a set of direction numbers of L, and so is the proportional set $[2,-16,13]$. Because L contains the point $(-2,0,3)$, parametric equations of L are $x = -2 + 2t$, $y = -16t$, $z = 3 + 13t$ and symmetric equations are $\frac{x+2}{2} = \frac{y}{-16} = \frac{z-3}{13}$.

34. $P(-3,1,-5)$, $\mathbf{V} = \langle 4,-2,1\rangle$. $x = -3 + 4t$, $y = 1 - 2t$, $z = -5 + t$; $\frac{x+3}{4} = \frac{y-1}{-2} = \frac{z+5}{1}$

35. A normal vector to the plane $x + 3y - 6z - 8 = 0$ is $\langle 1,3,-6\rangle$. Hence a set of direction numbers of L is $[1,3,-6]$. Because $(4,-5,20)$ is on L, a set of parametric equations of L is $x = 4 + t$, $y = -5 + 3t$, $z = 20 - 6t$. Symmetric equations are $\frac{x-4}{1} = \frac{y+5}{3} = \frac{z-20}{-6}$.

36. Through the point $(2,0,-4)$ and parallel to each of the planes $2x + y - z = 0$ and $x + 3y + 5z = 0$.

▷ Let $\langle a,b,c\rangle$ be a direction vector of the line. Because the vector $\langle 2,1,-1\rangle$ is normal to the plane $2x + y - z = 0$, and the vector $\langle a,b,c\rangle$ is parallel to the plane $2x + y - z = 0$, then $\langle 2,1,-1\rangle$ is orthogonal to $\langle a,b,c\rangle$. Thus

$$\langle 2,1,-1\rangle \cdot \langle a,b,c\rangle = 0$$
$$2a + b - c = 0 \qquad (1)$$

Similarly, the vector $\langle 1,3,5\rangle$ is normal to the plane $x + 3y + 5z = 0$, and thus $\langle 1,3,5\rangle$ is orthogonal to $\langle a,b,c\rangle$. Hence

$$\langle 1,3,5\rangle \cdot \langle a,b,c\rangle = 0$$
$$a + 3b + 5c = 0 \qquad (2)$$

$$3(1) - 2: \qquad 5a - 8c = 0$$
$$(1) - 2(2): \qquad -5b - 11c = 0$$

If $c = 5$, then $a = 8$ and $b = -11$. Thus $\langle a,b,c\rangle = \langle 8,-11,5\rangle$ is a direction vector for the line. We use parametric equations (I) with the given point $(2,0,-4)$ and the direction vector $\langle 8,-11,5\rangle$ to obtain

$x = 2t + 8 \quad y = -11t \quad z = -4 + 5t$

which are parametric equations of the line. We use symmetric equations (II) to obtain

$\frac{x-2}{8} = \frac{y}{-11} = \frac{z+4}{5}$

37. We wish to find symmetric equations for the line $\begin{cases} 4x - 3y + z - 2 = 0 \\ 2x + 5y - 3z + 4 = 0 \end{cases}$; $\begin{cases} 4x + z = 3y + 2 \\ 2x - 3z = -5y - 4 \end{cases}$. Solving simultaneously for x and z in terms of y, we get $x = \frac{2}{7}y + \frac{1}{7}$, $z = \frac{13}{7}y + \frac{10}{7}$. Hence symmetric equations are $\frac{x - \frac{1}{7}}{\frac{2}{7}} = y = \frac{z - \frac{10}{7}}{\frac{13}{7}}$; $\frac{x - \frac{1}{7}}{2} = \frac{y}{7} = \frac{z - \frac{10}{7}}{13}$; $\frac{x-1}{2} = \frac{y-3}{7} = \frac{z-7}{13}$, subtracting $\frac{3}{7}$ from each.

38. $\frac{x+1}{2} = \frac{y+4}{-5} = \frac{z-2}{3} = s$, $\frac{x-3}{-2} = \frac{y+14}{5} = \frac{z-8-3}{2} = t$

$x = 2s - 1 = -2t + 3$, $2s + 2t = 4$; $y = -5s - 4 = 5t - 14$, $5s + 5t = 10$; $z = 3s + 2 = -3t + 8$, $3s + 3t = 6$

The lines coincide because replacing t with $2 - s$ yields the same parametric equations.

39. The line $\frac{1}{2}(x-3) = \frac{1}{3}(y+2) = \frac{1}{4}(z+1) = t$ has parametric equations $x = 3 + 2t$, $y = -2 + 3t$, $z = -1 + 4t$.
Substituting in the equation of the plane $x - 2y + z = 6$, we get the identity
$(3+2t) - 2(-2+3t) + (-1+4t) \equiv 6$. Therefore the given line lies in the plane.

40. Prove that the line $x + 1 = -\frac{1}{2}(y-6) = z$ lies in the plane $3x + y - z = 3$.

▷ We write the equations of the line in parametric form by setting $z = t$. Thus

$$x = t - 1 \qquad y = -2t + 6 \qquad z = t$$

Substituting into the equation of the plane, we have

$$3(t-1) + (-2t+6) - t = 3 \qquad\qquad 3 = 3$$

Because the equation of the plane is satisfied for all points of the line, the line lies in the plane.

In Exercises 41–44, the planes through a line that are perpendicular to the coordinate planes are called the *projecting planes* of the line. Find equations of the projecting planes of the given line and sketch the line.

41. The given line has equations $\begin{cases} 3x - 2y + 5z - 30 = 0 \\ 2x + 3y - 10z - 6 = 0 \end{cases}$.

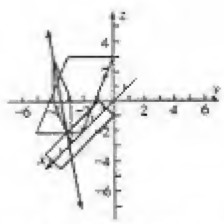
Eliminating x between the two equations we get $13y - 40z + 42 = 0$.
Eliminating y between the two equations we get $13x - 5z - 102 = 0$.
Eliminating z between the two equations we get $8x - y - 66 = 0$.
These are equations of the projecting planes.

42. The given line has equations $\begin{cases} \text{A: } x + y - 3z + 1 = 0 \\ \text{B: } 2x - y - 3z + 14 = 0 \end{cases}$. The equations of the projecting planes are
$2A - B$: $3y - 3z - 12 = 0$; $y - z - 4 = 0$. $A + B$: $3x - 6z + 15 = 0$, $x - 2z + 5 = 0$; $A - B$: $-x + 2y - 13 = 0$

43. Equations of the given line are $\begin{cases} x - 2y - 3z + 6 = 0 \\ x + y + z - 1 = 0 \end{cases}$.

Eliminating x between the two equations we obtain $3y + 4z - 7 = 0$.
Eliminating y between the two equations we obtain $3x - z + 4 = 0$.
Eliminating z between the two equations we obtain $4x + y + 3 = 0$.
These are equations of the projecting planes.

44. A: $2x - y + z - 7 = 0$
B: $4x - y + 3z - 13 = 0$



▷ Let L be the given line. If point P lies on L, (the coordinates of) P satisfies (1) and P satisfies (2). Thus, P satisfies a linear combination of (1) and (2). We form combinations of (1) and (2) to eliminate each of x, y, and z to get the projecting planes perpendicular to the yz-, xz- and xy-planes, respectively.

$B - 2A$: $\quad y + z + 1 = 0$

$B - A$: $\quad 2x + 2z - 6 = 0$ or, equivalently, $x + z - 3 = 0$

$3A - B$: $\quad 2x - 2y - 8 = 0$ or, equivalently $x - y - 4 = 0$

The figure shows the line and the three projecting plane.

45. Choosing y as parameter, the lines $x = 2y + 4$, $z = -y + 4$ and $x = y + 7$, $2z = y + 2$ have parametric equations $x = 2y + 4$, $y = y$, $z = -y + 4$ and $x = y + 7$, $y = y$, $z = \frac{1}{2}y + 1$. Hence vectors whose representations are parallel to these lines are $\langle 2, 1, -1 \rangle$ and $\langle 1, 1, \frac{1}{2} \rangle$. The angle θ between the two vectors is the angle between the two lines. Therefore

$$\cos\theta = \frac{\langle 2,1,-1\rangle \cdot \langle 1,1,\frac{1}{2}\rangle}{\left\| \langle 2,1,-1\rangle \cdot \langle 1,1,\frac{1}{2}\rangle \right\|} = \frac{2 + 1 - \frac{1}{2}}{\sqrt{4+1+1}\sqrt{1+1+\frac{1}{4}}} = \frac{\frac{5}{2}}{\sqrt{6}\cdot\frac{3}{2}} = \frac{5}{18}\sqrt{6}$$

46. We have $x = 1 + 5t$, $y = -2 + 6t$, $z = 3 + 7t$. With $t = 0$, $t = 1$, and the given $(6, 2, 4)$, we get

$$\begin{vmatrix} x & y & z & 1 \\ 1 & -2 & 3 & 1 \\ 6 & 4 & 10 & 1 \\ 6 & 2 & 4 & 1 \end{vmatrix} = -22x + 30y - 10z + 112 = 0 \text{ or } 11x - 15y + 5z - 56 = 0$$

In Exercises 47 and 48, find an equation of the plane containing the given intersecting lines.

47. $\frac{x-2}{4} = \frac{y+3}{-1} = \frac{z+2}{3} = t$ and $\begin{cases} 3x+2y+z+2=0 \\ x-y+2z-1=0 \end{cases}$

Parametric equations of the first line are $x = 4t+2$, $y = -t-3$, $z = 3t-2$. $t = 0$ gives the point $(2,-3,-2)$ which also belongs to the second line; $t = 1$ gives the point $(6,-4,1)$ which does not. For any value of k, $(3x+2y+z+2)+k(x-y+2z-1)=0$ is a plane containing the second line. Because plane C contains the point $(6,-4,1)$ of the second line, we have $[3(6)+2(-4)+1+2]+k[6-(-4)+2(1)-1]=0$; $13-11k=0$; $k=-\frac{13}{11}$. Hence an equation of C is $(3x+2y+z+2)-\frac{13}{11}(x-y+2z-1)=0$; $4x+7y-3z+7=0$.

48. $\frac{x}{2} = \frac{y-2}{3} = \frac{z-1}{1}$ and $\frac{x}{1} = \frac{y-2}{-1} = \frac{z-1}{1}$

▶ Because the numerators are same in both lines, we see that $P(0,2,1)$ is on both lines. Let $\mathbf{N} = \langle a,b,c\rangle$ be a normal vector of the required plane. A direction vector of the first line is $\mathbf{A} = \langle 2,3,1\rangle$ and a direction vector of the second line is $\mathbf{B} = \langle 1,-1,1\rangle$. Because the plane contains the lines, $\mathbf{N}$ is orthogonal to both $\mathbf{A}$ and $\mathbf{B}$. Thus,

$$\langle 2,3,1\rangle \cdot \langle a,b,c\rangle = 0$$
$$2a+3b+c=0 \tag{1}$$

and

$$\langle 1,-1,1\rangle \cdot \langle a,b,c\rangle = 0$$
$$a-b+c=0 \tag{2}$$
$$(1)+3(2):\quad 5a+4c=0$$
$$(1)-2(2):\quad 5b-c=0$$

Choosing $c=5$, we obtain $\mathbf{N} = \langle -4,1,5\rangle$. Applying Theorem 10.4.3, with $\mathbf{N}$ and the point $P(0,2,1)$, we get an equation of the plane

$$-4x+(y-2)+5(z-1)=0$$
$$4x-y-5z=0$$

49. The given lines have equations $\begin{cases} 3x-y-z=0 \\ 8x-2y-3z+1=0 \end{cases}$ and $\begin{cases} x-3y+z+3=0 \\ 3x-y-z+5=0 \end{cases}$. Solving the equations of each line for x and y in terms of z we get the symmetric equations $\frac{x+\frac{1}{2}}{\frac{1}{2}} = \frac{y+\frac{3}{2}}{\frac{1}{2}} = \frac{z-0}{1}$ and $\frac{x+\frac{3}{2}}{\frac{1}{2}} = \frac{y-\frac{1}{2}}{\frac{1}{2}} = \frac{z-0}{1}$.

Because $[\frac{1}{2},\frac{1}{2},1]$ is a set of direction numbers for each line, the lines are parallel. For any value of k, $(8x-2y-3z+1)+k(3x-y-z)=0$ is a plane containing the first line. Because the required plane must contain the point $(-\frac{3}{2},\frac{1}{2},0)$ of the second line, we must have

$[8(-\frac{3}{2})-2(\frac{1}{2})-3(0)+1]+k[3(-\frac{3}{2})-\frac{1}{2}-0]=0$; $-12-5k=0$; $k=-\frac{12}{5}$.

Hence an equation of the plane is $(8x-2y-3z+1)-\frac{12}{5}(3x-y-z)=0$; $4x+2y-3z+5=0$.

50. The lines $\frac{x+2}{5} = \frac{y-1}{-2} = \frac{z+4}{1}$ and $\frac{x-3}{-5} = \frac{y+4}{2} = \frac{z-3}{-1}$ have $[5,-2,1]$ for direction numbers and are parallel. Planes containing the first line are $x+2=5z+20$; $x-5z-18=0$ and $y-1=-2z-8$, $y+2z+7=0$ and hence $(x-5z-18)+k(y+2z+7)=0$. Because the required plane C must contain the point $(3,-4,3)$ of the second line, we have $[3-5(3)-18]+k[-4+2(3)+7]=0$; $-30+9k=0$; $k=\frac{10}{3}$. Hence an equation of C is $(x-5z-18)+\frac{10}{3}(y+2z+7)=0$; $3x+10y+5z+16=0$

51. We wish to find the intersection of the plane $5x-y+2z-12=0$ and the line $\frac{x-2}{4} = \frac{y+3}{-2} = \frac{z-1}{7} = t$:

$x=4t+2$, $y=-2t-3$, $z=7t+1$. Substituting into the equation of the plane we get $5(4t+2)-(-2t-3)+2(7t+1)-12=0$; $36t+3=0$; $t=-\frac{1}{12}$. The point of intersection is $(-\frac{1}{3}+2, \frac{1}{6}-3, -\frac{7}{12}+1) = (\frac{5}{3}, -\frac{17}{6}, \frac{5}{12})$.

52. Find equations of the line through the point $(1,-1,1)$ perpendicular to the line $3x=2y=z$, and parallel to the plane $x+y-z=0$.

▶ Let $\langle a,b,c\rangle$ be a direction vector of the required line. Because symmetric equations of the given line are

$$\frac{x}{\frac{1}{3}} = \frac{y}{\frac{1}{2}} = \frac{z}{1}$$

then the vector $\langle \frac{1}{3},\frac{1}{2},1\rangle$ is a direction vector for the given line. Because $\langle \frac{1}{3},\frac{1}{2},1\rangle$ is orthogonal to $\langle a,b,c\rangle$, then

$$\langle \tfrac{1}{3},\tfrac{1}{2},1\rangle \cdot \langle a,b,c\rangle = 0$$

$$\tfrac{1}{3}a + \tfrac{1}{2}b + c = 0$$

$$2a + 3b + 6c = 0 \tag{1}$$

Because the vector $\langle 1,1,-1\rangle$ is normal to the plane $x + y - z = 0$, then $\langle 1,1,-1\rangle$ is orthogonal to $\langle a,b,c\rangle$ and thus

$$\langle 1,1,-1\rangle \cdot \langle a,b,c\rangle = 0$$

$$a + b - c = 0 \tag{2}$$

$$(1) - 3(2): \quad -a + 9c = 0$$

$$(1) - 2(2): \quad b + 8c = 0$$

Choosing $c = 1$, we find $\langle a,b,c\rangle = \langle 9,-8,1\rangle$ is a direction vector of the required line. Parametric equations of the line are obtained from (I) with the given point $(1,-1,1)$ and the direction vector $\langle 9,-8,1\rangle$. Thus

$x = 1 + 9t \quad y = -1 - 8t \quad z = 1 + t$

53. Because the line L intersects the z axis it contains the point $(0,0,z_0)$. Because the point $(3,6,4)$ is also on L a set of direction numbers of L is $[3,6,4-z_0]$. A normal vector for the plane $x - 3y + 5z - 6 = 0$ is $\langle 1,-3,5\rangle$. Because L is parallel to this plane, $\langle 3,6,4-z_0\rangle \cdot \langle 1,-3,5\rangle = 0$; $3 - 18 + 5(4 - z_0) = 0$; $z_0 = 1$. A set of direction numbers of L is then $[3,6,3]$, and another set is $[1,2,1]$. Because $(3,6,4)$ is on L, symmetric equations of L are

$\dfrac{x-3}{1} = \dfrac{y-6}{2} = \dfrac{z-4}{1}$.

54. $Q(-2,7,4)$ is a point of the line. $\mathbf{V} = \langle 6,-2,3\rangle$ is a vector in the line. $\left\|\overrightarrow{OQ}\right\| = c = \sqrt{2^2 + 7^2 + 4^2} = \sqrt{69}$.

$a = OQ_V = \dfrac{\overrightarrow{OQ} \cdot \mathbf{V}}{\|\mathbf{V}\|} = \dfrac{\langle -2,7,4\rangle \cdot \langle 6,-2,3\rangle}{\sqrt{6^2 + 2^2 + 3^2}} = \dfrac{-14}{7} = -2$. $d = \sqrt{c^2 - a^2} = \sqrt{69 - 4} = \sqrt{65}$

55. $Q(7,1,0)$ is a point of the line $\frac{x-7}{2} = \frac{z}{1}$. $\mathbf{V} = \langle 2,0,1\rangle$ is a vector in the line. $P(-1,3,-1)$ is the given point.

$c = \left\|\overrightarrow{PQ}\right\| = \|\langle 8,-2,1\rangle\| = \sqrt{64 + 4 + 1} = \sqrt{69}$. $a = PQ_V = \dfrac{\overrightarrow{PQ} \cdot \mathbf{V}}{\|\mathbf{V}\|} = \dfrac{\langle 8,-2,1\rangle \cdot \langle 2,0,1\rangle}{\sqrt{4+0+1}} = \dfrac{17}{\sqrt{5}}$

$d = \sqrt{c^2 - a^2} = \sqrt{69 - \frac{289}{5}} = \sqrt{\frac{56}{5}} = \frac{2}{5}\sqrt{70}$

56. Find equations of the line through the origin, perpendicular to the line $x = y - 5$, $z = 2y - 3$, and intersecting the line $y = 2x + 1$, $z = x + 2$.

▶ Let L be the required line and let L_1 be the line $y = 2x + 1$, $z = x + 2$ that intersects L. If $P(a,b,c)$ is the point of intersection, then the coordinates of P satisfy the equations of L_1. Thus, we let $x = a$, $y = b$, and $z = c$ in the equations of L_1, obtaining

$$b = 2a + 1 \tag{1}$$

$$c = a + 2 \tag{2}$$

Because line L contains the origin and point P, then the vector $\mathbf{V}(\overrightarrow{OP}) = \langle a,b,c\rangle$ is a direction vector of line L. Let L_2 be the line $x = y - 5$, $z = 2y - 3$. We find symmetric equations of L_2. Solving each equation for y, we have

$x + 5 = y \quad \text{and} \quad y = \frac{1}{2}(z + 3)$

Thus, symmetric equations of L_2 are

$\dfrac{x+5}{1} = \dfrac{y}{1} = \dfrac{z+3}{2}$

Hence, a direction vector of line L_2 is $\langle 1,1,2\rangle$. Because line L is perpendicular to line L_2, any direction vector of L is orthogonal to a direction vector of L_2. Thus,

$$\langle 1,1,2\rangle \cdot \langle a,b,c\rangle = 0$$

$$a + b + 2c = 0 \tag{3}$$

Substituting from (1) and (2) into (3) we obtain

$$a + (2a + 1) + 2(a + 2) = 0$$

$a = -1 \quad b = -1 \quad c = 1$

We use the direction vector $\langle -1,-1,1\rangle$ and the origin to write the symmetric equations of L, which are

$\dfrac{x}{-1} = \dfrac{y}{-1} = \dfrac{z}{1}$

57. The given lines are $\frac{x-1}{5} = \frac{y-2}{-2} = \frac{z+1}{-3} = t$ and $\frac{x-2}{1} = \frac{y+1}{-3} = \frac{z+3}{2} = s$. The two sets of direction numbers $[5,-2,-3]$ and $[1,-3,2]$ are not proportional, and so the lines are not parallel. Parametric equations of the lines are L_1: $x = 5t+1$, $y = -2t+2$, $z = -3t-1$ and L_2: $x = s+2$, $y = -3s-1$, $z = 2s-3$
If (x,y,z) is a point of intersection, we must have $5t+1 = s+2$, $-2t+2 = -3s-1$, $-3t-1 = 2s-3$
The first two equations have the unique solution $t = 0$ and $s = -1$. These values do not satisfy the third equation, so the two lines do not intersect. Thus they are skew.

58. With $t = 0$ and 1 in L_1, and the given, we get the plane $\begin{vmatrix} x & y & z & 1 \\ 1 & 2 & -1 & 1 \\ 6 & 0 & -4 & 1 \\ 3 & -4 & -5 & 1 \end{vmatrix} = -10x + 14y - 26z - 44 = 0$ or $5x - 7y + 13z + 22 = 0$. Substituting from L_2, $5(s+2) - 7(-3s-1) + 13(2s-3) + 22 = 0$, $52s = 0$, $s = 0$ and so the plane meets L_2 in $Q(2,-1,-3)$. $\mathbf{V} = \overrightarrow{PQ} = \langle -1,3,2\rangle$. The line is $\frac{x-3}{-1} = \frac{y+4}{5} = \frac{z+5}{2}$.

59. Choose a point in the plane $ax+by+cz+d_1 = 0$, say $P = (0,0,-d_1/c)$ and a point in plane $ax+by+cz+d_2 = 0$, say $Q = (0,0,d_2/c)$. $\mathbf{N} = \langle a,b,c\rangle$ is normal to the parallel planes. If d units is the distance between the planes then

$$d = \frac{\mathbf{N}\cdot\mathbf{V}(QP)}{\|\mathbf{N}\|} = \frac{\langle a,b,c\rangle\cdot\langle 0,0,(d_1-d_2)/c\rangle}{\|\langle a,b,c\rangle\|} = \frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}$$

60. Prove that the undirected distance from the plane $ax+by+cz+d = 0$ to the point $P_0(x_0,y_0,z_0)$ is given by

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

▸ If $P_1(x_1,y_1,z_1)$ is any point in the plane, then $ax_1+by_1+cz_1 = -d$. The required distance is the absolute value of the scalar projection

$$|PQ_N| = \frac{|\mathbf{N}\cdot\overrightarrow{P_0P_1}|}{\|\mathbf{N}\|} = \frac{|\langle a,b,c\rangle\cdot\langle x_1-x_0,y_1-y_0,z_1-z_0\rangle|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_1+by_1+cz_1-ax_0-by_0-cz_0|}{\sqrt{a^2+b^2+c^2}}$$
$$= \frac{|-d-ax_0-by_0-cz_0|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

61. If the two direction numbers a and b are zero, then $x = x_0$ and $y = y_0$.

10.5 CROSS PRODUCT

10.5.1 Definition If $\mathbf{A} = \langle a_1,a_2,a_3\rangle$ and $\mathbf{B} = \langle b_1,b_2,b_3\rangle$, then the *cross product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A}\times\mathbf{B}$, is given by

$$\mathbf{A}\times\mathbf{B} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$
$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

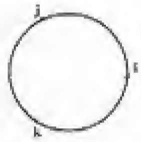
The operation of cross product for vectors is neither commutative nor associative, and depends on the orientation of the coordinate system. However, cross multiplication of vectors is distributive with respect to vector addition. Furthermore, scalar multiplication and cross multiplication are associative. We formally state these properties in the following theorems.

10.5.2 Theorem If $\mathbf{A}$ is any vector in V_3, then
(i) $\mathbf{A}\times\mathbf{A} = \mathbf{0}$
(ii) $\mathbf{0}\times\mathbf{A} = \mathbf{0}$
(iii) $\mathbf{A}\times\mathbf{0} = \mathbf{0}$

10.5.3 Theorem If $\mathbf{A}$ and $\mathbf{B}$ are any vectors in V_3,
$\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A})$

10.5.4 Theorem If **A**, **B**, and **C** are any vectors in V_3, then
$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$ and $(\mathbf{B}+\mathbf{C})\times\mathbf{A} = \mathbf{B}\times\mathbf{A}+\mathbf{C}\times\mathbf{A}$

10.5.5 Theorem If **A** and **B** are any two scalars in V_3 and c is a scalar, then
(i) $(c\mathbf{A})\times\mathbf{B} = \mathbf{A}\times(c\mathbf{B})$
(ii) $(c\mathbf{A})\times\mathbf{B} = c(\mathbf{A}\times\mathbf{B})$



$\mathbf{i}\times\mathbf{i}=\mathbf{0}$	$\mathbf{i}\times\mathbf{j}=\mathbf{k}$	$\mathbf{i}\times\mathbf{k}=-\mathbf{j}$
$\mathbf{j}\times\mathbf{i}=-\mathbf{k}$	$\mathbf{j}\times\mathbf{j}=\mathbf{0}$	$\mathbf{j}\times\mathbf{k}=\mathbf{i}$
$\mathbf{k}\times\mathbf{i}=\mathbf{j}$	$\mathbf{k}\times\mathbf{j}=-\mathbf{i}$	$\mathbf{k}\times\mathbf{k}=\mathbf{0}$

The cross product of each unit vector with itself is the vector **0**. The figure illustrates a way to remember the signs of the remaining six cross products of the unit vectors. The cross product of two consecutive vectors in the positive (counterclockwise) direction, is the next vector. The cross product of two vectors in the negative direction is the negative of the next vector.

10.5.6 Theorem (Scalar Triple Product) If **A**, **B**, and **C** are vectors in V_3, then
$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$
If $\mathbf{A} = \langle a_1, a_2, a_3\rangle$, $\mathbf{B} = \langle b_1, b_2, b_3\rangle$, and $\mathbf{C} = \langle c_1, c_2, c_3\rangle$, then the scalar triple product is given by

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

10.5.7 Theorem (Vector Triple Product) If **A**, **B**, and **C** are vectors in V_3, then
$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ and $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{C}\cdot\mathbf{A})\mathbf{B}-(\mathbf{C}\cdot\mathbf{B})\mathbf{A}$
Remember both as : (outer dot remote)adjacent − (outer dot adjacent)remote

10.5.8 Theorem If **A** and **B** are two vectors in V_3 and θ is the measure of the angle between **A** and **B**, then
$\|\mathbf{A}\times\mathbf{B}\| = \|\mathbf{A}\|\|\mathbf{B}\|\sin\theta$

10.5.9 Theorem If **A** and **B** are two vectors in V_3, **A** and **B** are parallel if and only if $\mathbf{A}\times\mathbf{B} = \mathbf{0}$.

10.5.10 Theorem If **A** and **B** are two vectors in V_3, then the vector $\mathbf{A}\times\mathbf{B}$ is orthogonal to both **A** and **B**. Thus,
$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{A} = 0$ and $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{B} = 0$

Torque of a force **F** applied at a point P with respect to O has magnitude $\left\|\overrightarrow{OP}\times\mathbf{F}\right\|$.

Area and Volume If representations of the vectors **A** and **B** are two adjacent sides of a parallelogram P or a triangle T then the measure of the area of P is $\|\mathbf{A}\times\mathbf{B}\|$ and the measure of the area of T is $\frac{1}{2}\|\mathbf{A}\times\mathbf{B}\|$. If representations of the vectors **A**, **B**, and **C** are three coterminal edges of a parallelopiped P or a tetrahedron T, then the measure of the volume of P is $|\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}|$ and the measure of the volume of T is $\frac{1}{6}|\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}|$.
Let L be a line with direction vector **L** and let P be a point. If Q is any point on the line L, then the undirected distance between the point P and the line L is given by

$$d = \frac{\left\|\mathbf{L}\times\mathbf{V}(\overrightarrow{PQ})\right\|}{\|\mathbf{L}\|}$$

Exercises 10.5

In Exercises 1–12, let $\mathbf{A} = \langle 1,2,3\rangle$, $\mathbf{B} = \langle 4,-3,-1\rangle$, $\mathbf{C} = \langle -5,-3,5\rangle$, $\mathbf{D} = \langle -2,1,6\rangle$, $\mathbf{E} = \langle 4,0,-7\rangle$, and $\mathbf{F} = \langle 0,2,1\rangle$.

1. $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & -3 & -1 \end{vmatrix} = [-2-(-9)]\mathbf{i}-(-1-12)\mathbf{j}+(-3-8)\mathbf{k} = \langle 7,13,-11\rangle$

2. $\mathbf{D}\times\mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 6 \\ 4 & 0 & -7 \end{vmatrix} = -7\mathbf{i}-(14-24)\mathbf{j}+(-4)\mathbf{k} = \langle -7,10,-4\rangle$

3. $(\mathbf{C}\times\mathbf{D})\cdot(\mathbf{E}\times\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -3 & 5 \\ -2 & 1 & 6 \end{vmatrix} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -7 \\ 0 & 2 & 1 \end{vmatrix}$

$= [(-18-5)\mathbf{i} - (-30+10)\mathbf{j} + (-5-6)\mathbf{k}] \cdot (14\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}) = \langle -23, 20, 11\rangle \cdot \langle 14, -4, 8\rangle = -322 - 80 - 88 = -490$

4. Find $(\mathbf{C}\times\mathbf{E})\cdot(\mathbf{D}\times\mathbf{F})$

▹ $\mathbf{C}\times\mathbf{E} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -3 & 5 \\ 4 & 0 & -7 \end{vmatrix} = \begin{vmatrix} -3 & 5 \\ 0 & -7 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -5 & 5 \\ 4 & -7 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -5 & -3 \\ 4 & 0 \end{vmatrix}\mathbf{k}$

$= [(-3)(-7) - 0(5)]\mathbf{i} - [(-5)(-7) - 4(5)]\mathbf{j} + [(-5)0 - 4(-3)]\mathbf{k} = 21\mathbf{i} - 15\mathbf{j} + 12\mathbf{k}$

$\mathbf{D}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 6 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 6 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2 & 6 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2 & 1 \\ 0 & 2 \end{vmatrix}\mathbf{k} = -11\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

Thus,

$(\mathbf{C}\times\mathbf{E})\cdot(\mathbf{D}\times\mathbf{F}) = (21\mathbf{i} - 15\mathbf{j} + 12\mathbf{k})\cdot(-11\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) = 21(-11) + (-15)2 + 12(-4) = -309$

5. $-(\mathbf{B}\times\mathbf{A}) = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & -1 \\ 1 & 2 & 3 \end{vmatrix} = -\langle -7, -13, 11\rangle = \langle 7, 13, -11\rangle = \mathbf{A}\times\mathbf{B}$, from the result of Exercise 1.

6. $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \langle 1, 2, 3\rangle \times \langle -1, -6, 4\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & -6 & 4 \end{vmatrix} = 26\mathbf{i} - 7\mathbf{j} - 4\mathbf{k}$

$\mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & -3 & -1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -5 & -3 & 5 \end{vmatrix} = (7\mathbf{i} + 13\mathbf{j} - 11\mathbf{k}) + (19\mathbf{i} - 20\mathbf{j} + 7\mathbf{k}) = 26\mathbf{i} - 7\mathbf{j} - 4\mathbf{k}$

7. If $c = 3$, $(c\mathbf{A})\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 6 & 9 \\ 4 & -3 & -1 \end{vmatrix} = (-6+27)\mathbf{i} - (-3-36)\mathbf{j} + (-9-24)\mathbf{k} = \langle 21, 39, -33\rangle$

$\mathbf{A}\times(c\mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 12 & -9 & -3 \end{vmatrix} = (-6+27)\mathbf{i} - (-3-36)\mathbf{j} + (-9-24)\mathbf{k} = \langle 21, 39, -33\rangle$

8. Verify Theorem 10.5.5(ii) for vectors $\mathbf{A}$ and $\mathbf{B}$ and $c = 3$.

▹ $(c\mathbf{A})\times\mathbf{B} = 3\langle 1, 2, 3\rangle \times \langle 4, -3, -1\rangle = \langle 3, 6, 9\rangle \times \langle 4, -3, -1\rangle$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 6 & 9 \\ 4 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 6 & 9 \\ -3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 9 \\ 4 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 6 \\ 4 & -3 \end{vmatrix}\mathbf{k} = 21\mathbf{i} + 39\mathbf{j} - 33\mathbf{k}$

$c(\mathbf{A}\times\mathbf{B}) = 3(\langle 1, 2, 3\rangle \times \langle 4, -3, -1\rangle)$

$= 3\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & -3 & -1 \end{vmatrix} = 3\left(\begin{vmatrix} 2 & 3 \\ -3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & -3 \end{vmatrix}\mathbf{k}\right)$

$= 3(7\mathbf{i} + 13\mathbf{j} - 11\mathbf{k}) = 21\mathbf{i} + 39\mathbf{j} - 33\mathbf{k} = (c\mathbf{A})\times\mathbf{B}$

9. $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \langle 1, 2, 3\rangle \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & -1 \\ -5 & -3 & 5 \end{vmatrix} = \langle 1, 2, 3\rangle \cdot \langle -15-3, -(20-5), -12-15\rangle = \langle 1, 2, 3\rangle \cdot \langle -18, -15, -27\rangle$

$= -18 - 30 - 81 = -129$. From Exercise 1, $\mathbf{A}\times\mathbf{B}\cdot\mathbf{C} = \langle 7, 13, -11\rangle \cdot \langle -5, -3, 5\rangle = -35 - 39 - 55 = -129$

10. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & -1 \\ -5 & -3 & 5 \end{vmatrix} = \langle 1, 2, 3\rangle \times \langle -18, -15, -27\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -18 & -15 & -27 \end{vmatrix} = \langle -9, -27, 21\rangle$

$(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} = (\langle 1, 2, 3\rangle \cdot \langle -5, -3, 5\rangle)\mathbf{B} - (\langle 1, 2, 3\rangle \cdot \langle 4, -3, -1\rangle) = 4\langle 4, -3, -1\rangle + 5\langle -5, -3, 5\rangle = \langle -9, -27, 21\rangle$

11. $(\mathbf{A}+\mathbf{B})\times(\mathbf{C}-\mathbf{D}) = (\langle 1,2,3\rangle + \langle 4,-3,-1\rangle)\times(\langle -5,-3,5\rangle - \langle -2,1,6\rangle)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 2 \\ -3 & -4 & -1 \end{vmatrix} = \langle 1+8, -(-5+6), -20-3\rangle = \langle 9,-1,-23\rangle$$

$$(\mathbf{D}-\mathbf{C})\times(\mathbf{A}+\mathbf{B}) = (\langle -2,1,6\rangle - \langle -5,-3,5\rangle)\times\langle 5,-1,2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 1 \\ 5 & -1 & 2 \end{vmatrix} = \langle 8+1, -(6-5), -3-20\rangle = \langle 9,-1,-23\rangle$$

12. Find $\|\mathbf{A}\times\mathbf{B}\|\|\mathbf{C}\times\mathbf{D}\|$

▷ $$\mathbf{A}\times\mathbf{B} = \langle 1,2,3\rangle\times\langle 4,-3,-1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -3 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & -3 \end{vmatrix}\mathbf{k} = 7\mathbf{i} + 13\mathbf{j} - 11\mathbf{k}$$

$$\mathbf{C}\times\mathbf{D} = \langle -5,-3,5\rangle\times\langle -2,1,6\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -3 & 5 \\ -2 & 1 & 6 \end{vmatrix} = \begin{vmatrix} -3 & 5 \\ 1 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -5 & 5 \\ -2 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -5 & -3 \\ -2 & 1 \end{vmatrix}\mathbf{k} = -23\mathbf{i} + 20\mathbf{j} - 11\mathbf{k}$$

$$\|\mathbf{A}\times\mathbf{B}\|\|\mathbf{C}\times\mathbf{D}\| = \sqrt{7^2+13^2+11^2}\sqrt{23^2+20^2+11^2} = \sqrt{339}\sqrt{1050} = 15\sqrt{1582}$$

13. Let $\mathbf{A} = \langle a_1, a_2, a_3\rangle$. Then
$\mathbf{0}\times\mathbf{A} = \langle 0,0,0\rangle\times\langle a_1,a_2,a_3\rangle = \langle 0\cdot a_3 - 0\cdot a_2, 0\cdot a_1 - 0\cdot a_3, 0\cdot a_2 - 0\cdot a_1\rangle = \langle 0,0,0\rangle = \mathbf{0}$
$\mathbf{A}\times\mathbf{0} = \langle a_1,a_2,a_3\rangle\times\langle 0,0,0\rangle = \langle a_2\cdot 0 - a_3\cdot 0, a_3\cdot 0 - a_1\cdot 0, a_1\cdot 0 - a_2\cdot 0\rangle = \langle 0,0,0\rangle = \mathbf{0}$

14. (a) $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{4}{9} & \frac{7}{9} & -\frac{4}{9} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{15}{27}\mathbf{i} + \frac{4}{27}\mathbf{j} + \frac{22}{27}\mathbf{k}$. $\sin(\mathbf{A},\mathbf{B}) = \|\mathbf{A}\times\mathbf{B}\| = \frac{1}{27}\sqrt{225+16+484} = \frac{1}{27}\sqrt{725} = \frac{5}{27}\sqrt{29}$

(b) $\cos(\mathbf{A},\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} = \frac{1}{27}[4(-2)+7(2)-4(1)] = \frac{2}{27}$. $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(\frac{2}{27})^2} = \frac{5}{27}\sqrt{29}$

15. $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ and $\mathbf{B} = \frac{1}{3\sqrt{3}}\mathbf{i} + \frac{5}{3\sqrt{3}}\mathbf{j} + \frac{1}{3\sqrt{3}}\mathbf{k}$. Because $\mathbf{A}$ and $\mathbf{B}$ are unit vectors

(a) $$\sin\theta = \frac{\|\mathbf{A}\times\mathbf{B}\|}{\|\mathbf{A}\|\|\mathbf{B}\|} = \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{3\sqrt{3}} & \frac{5}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{matrix}\right\| = \left\|-\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{k}\right\| = \sqrt{\frac{4}{9}+\frac{4}{9}} = \frac{2}{3}\sqrt{2}|$$

(b) $\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \mathbf{A}\cdot\mathbf{B} = \frac{1}{9} - \frac{5}{9} + \frac{1}{9} = -\frac{1}{3}$. Because $0 \le \theta \le \pi$, $\sin\theta = +\sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{9}} = \frac{2}{3}\sqrt{2}$

16. Show that the quadrilateral having vertices at $P(1,1,3)$, $Q(-2,1,-1)$, $R(-5,4,0)$, and $S(-8,4,-4)$ is a parallelogram and find its area.

▷ We use position representations.

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \langle -2,1,-1\rangle - \langle 1,1,3\rangle = \langle -3,0,4\rangle$$
$$\overrightarrow{PR} = \mathbf{p} - \mathbf{r} = \langle -5,4,0\rangle - \langle 1,1,3\rangle = \langle -6,3,-3\rangle$$
$$\overrightarrow{RS} = \mathbf{s} - \mathbf{r} = \langle -8,4,-4\rangle - \langle -5,4,0\rangle = \langle -3,0,-4\rangle$$

Because $\overrightarrow{PQ} = \overrightarrow{RS}$, then PQSR is a parallelogram. Moreover

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 12\mathbf{i} + 15\mathbf{j} - 9\mathbf{k}$$

and

$$\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\| = \|12\mathbf{i} + 15\mathbf{j} - 9\mathbf{k}\| = \sqrt{144+225+81} = \sqrt{450} = 15\sqrt{2}$$

Therefore the area of parallelogram PQSR is $15\sqrt{2}$ square units.

17. Let P = (1, −2, 3), Q = (4, 3, −1), R = (2, 2, 1), S = (5, 7, −3). Because $\mathbf{V}(\overrightarrow{PQ}) = \langle 3, 5, 4\rangle = \mathbf{V}(\overrightarrow{RS})$, PQRS is a parallelogram. The measure of the area of PQRS is

$$\left\|\mathbf{V}(\overrightarrow{PQ}) \times \mathbf{V}(\overrightarrow{PR})\right\| = \left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & -4 \\ 1 & 4 & -2 \end{vmatrix}\right\| = \|6\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}\| = \sqrt{36+4+49} = \sqrt{89}$$

18. $$\text{area} = \left\|\overrightarrow{PQ} \times \overrightarrow{PS}\right\| = \left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 0 & 3 & 4 \end{vmatrix}\right\| = \|-8\mathbf{i} - 12\mathbf{j} + 9\mathbf{k}\| = \sqrt{64+144+81} = 17$$

19. Let A = (0, 2, 2), B = (8, 8, −2), C = (9, 12, 6). The measure of the area of triangle ABC is

$$\tfrac{1}{2}\left\|\mathbf{V}(\overrightarrow{AB}) \times \mathbf{V}(\overrightarrow{BC})\right\| = \tfrac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & -4 \\ 1 & 4 & -2 \end{vmatrix}\right\| = \tfrac{1}{2}\|\langle 64, -68, 26\rangle\| = \|\langle 32, -34, 13\rangle\| = \sqrt{1024+1156+169} = \sqrt{2349} = 9\sqrt{29}$$

20. Find the area of the triangle having vertices at P(4, 5, 6), Q(4, 4, 5), and R(3, 5, 5).

▷ The area of triangle PQR is $\frac{1}{2}\left\|\overrightarrow{QP} \times \overrightarrow{QR}\right\|$

$$\overrightarrow{QP} \times \overrightarrow{QR} = (\mathbf{p}-\mathbf{q}) \times (\mathbf{r}-\mathbf{q}) = \langle 0, 1, 1\rangle \times \langle -1, 1, 0\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\left\|\overrightarrow{QP} \times \overrightarrow{QR}\right\| = \sqrt{1+1+1} = \sqrt{3}$$

Hence the area of triangle PQR is $\frac{1}{2}\sqrt{3}$ square units.

In Exercises 21 and 22, use the cross product to find an equation of the plane containing the three points.

21. Let A = (−2, 2, 2), B = (−8, 1, 6), C = (3, 4, −1). A normal vector for the plane of A, B, C is

$$\mathbf{V}(\overrightarrow{AB}) \times \mathbf{V}(\overrightarrow{AC}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -1 & 4 \\ 5 & 2 & -3 \end{vmatrix} = \langle -5, 2, -7\rangle.$$ Because point A lies in the plane, an equation is

$-5(x+2) + 2(y-2) - 7(z-2) = 0$; $5x - 2y + 7z = 0$

22. A(2, 3, 0), B(2, 0, 4), C(0, 3, 4). $$\mathbf{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & 4 \\ -2 & 0 & 4 \end{vmatrix} = \langle -12, -8, -6\rangle. \quad -\tfrac{1}{2}\mathbf{N} = \langle 6, 4, 3\rangle$$

$6(x-2) + 4(y-3) + 3(z-0) = 0$; $6x + 4y + 3z - 24 = 0$

23. The given planes $x - y + z = 0$ and $2x + y - 4z - 5 = 0$ have normal vectors $\mathbf{A} = \langle 1, -1, 1\rangle$ and $\mathbf{B} = \langle 2, 1, -4\rangle$. The required plane C has normal vector orthogonal to these. Such a vector is

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & -4 \end{vmatrix} = \langle 3, 6, 3\rangle.$$ Because C contains the point (4, 0, −2), an equation is

$3(x-4) + 6(y-0) + 3(z+2) = 0$; $3x + 6y + 3z - 6 = 0$; $x + 2y + z - 2 = 0$

24. Find a unit vector perpendicular to the plane containing $\overrightarrow{PQ} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{PR} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

▷ Let

$$\mathbf{N} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 2 & -1 & -1 \end{vmatrix} = -5\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$$

By Theorem 10.5.10 the vector **N** is normal to the plane containing $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Unit vectors are

$$\mathbf{U} = \pm\frac{\mathbf{N}}{\|\mathbf{N}\|} = \pm\frac{1}{\sqrt{25+9+49}}(5\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}) = \pm\frac{1}{\sqrt{83}}(5\mathbf{i} + 3\mathbf{j} + 7\mathbf{k})$$

In Exercises 25–27, find a unit vector perpendicular to the plane through the points P, Q, and R.

25. P(5, 2, −1), Q(2, 4, −2), R(11, 1, 4). A normal vector to the plane of P, Q, and R is

$$\mathbf{N} = \mathbf{V}(\overrightarrow{PQ}) \times \mathbf{V}(\overrightarrow{PR}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -1 \\ 6 & -1 & 5 \end{vmatrix} = 9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k};\ \|\mathbf{N}\| = 9\|\mathbf{i} + \mathbf{j} - \mathbf{k}\| = 9\sqrt{1+1+1} = 9\sqrt{3}$$

Hence unit normal vectors are $\pm\frac{1}{9\sqrt{3}}(9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k}) = \pm\frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$.

26. $P(-2,1,0)$, $Q(2,-2,-1)$, $R(-5,0,2)$. $\mathbf{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & -1 \\ -3 & -1 & 2 \end{vmatrix} = \langle -7, -5, -13 \rangle$.

$\|\mathbf{N}\| = \sqrt{49 + 25 + 169} = \sqrt{243} = 9\sqrt{3}$. $\mathbf{U} = \pm \langle \frac{7}{27}\sqrt{3}, \frac{5}{27}\sqrt{3}, \frac{13}{27}\sqrt{3} \rangle$

27. $P = (1,4,2)$, $Q = (3,2,4)$, $R = (4,3,1)$. A normal vector to the plane of P, Q, and R is

$\mathbf{N} = \mathbf{V}(\overrightarrow{PQ}) \times \mathbf{V}(\overrightarrow{PR}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 2 \\ 3 & -1 & -1 \end{vmatrix} = 4\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$; $\|\mathbf{N}\| = 4\|\mathbf{i} + 2\mathbf{j} + \mathbf{k}\| = 4\sqrt{1+4+1} = 4\sqrt{6}$

Hence unit normal vectors are $\pm \dfrac{1}{4\sqrt{6}}(4\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}) = \pm \dfrac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

28. Find the volume of the parallelopiped having edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$ for points $P(1,3,4)$, $Q(3,5,3)$, $R(2,1,6)$ and $S(2,2,5)$.

▷ $$\mathbf{A} = \overrightarrow{PQ} = \langle 3,5,3 \rangle - \langle 1,3,4 \rangle = \langle 2,2,-1 \rangle$$
$$\mathbf{B} = \overrightarrow{PR} = \langle 2,1,6 \rangle - \langle 1,3,4 \rangle = \langle 1,-2,2 \rangle$$
$$\mathbf{C} = \overrightarrow{PS} = \langle 2,2,5 \rangle - \langle 1,3,4 \rangle = \langle 1,-1,1 \rangle$$

The number of cubic units is the volume of the parallelepiped is $|\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}|$.

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \begin{vmatrix} 2 & 2 & -1 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 1$$

Thus the volume of the parallelepiped is 1 cubic unit.

29. Let $\mathbf{A} = \mathbf{V}(\overrightarrow{PQ}) = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. $\mathbf{B} = \mathbf{V}(\overrightarrow{PR}) = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$. $\mathbf{C} = \mathbf{V}(\overrightarrow{PS}) = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. The measure of the volume of parallelepiped PQRS is

$$|\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 2 \\ 2 & 1 & -1 \end{vmatrix} \cdot \mathbf{C} \right| = |(-5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + \mathbf{k})| = |-5 - 10 - 5| = 20$$

30. The plane has equation $\begin{vmatrix} x & y & z & 1 \\ 2 & -1 & 3 & 1 \\ -1 & 1 & 2 & 1 \\ 5 & 1 & -1 & 1 \end{vmatrix} = -6x - 15y - 12z + 33 = 0$; $2x + 5y + 4z - 11 = 0$

In Exercises 31 and 32, find the perpendicular distance between the two given skew lines.

31. The given lines are $\frac{x-1}{5} = \frac{y-2}{3} = \frac{z+1}{2}$ and $\frac{x+2}{4} = \frac{y+1}{2} = \frac{z-3}{-3}$. A vector normal to them is

$$\mathbf{N} = \langle 5,3,2 \rangle \times \langle 4,2,-3 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 3 & 2 \\ 4 & 2 & -3 \end{vmatrix} = \langle -13, 23, -2 \rangle. \ \|\mathbf{N}\| = \sqrt{169 + 529 + 4} = \sqrt{702} = 3\sqrt{78}$$

$A = (1,2,-1)$ is a point of the first line and $B = (-2,-1,3)$ is a point of the second.
The measure of the distance between the lines is the scalar projection of $\mathbf{V}(\overrightarrow{AB})$ on $\mathbf{N}$

$$= \left| \frac{\mathbf{V}(\overrightarrow{AB}) \cdot \mathbf{N}}{\|\mathbf{N}\|} \right| = \left| \frac{\langle -3,-3,4 \rangle \cdot \langle -13, 23, -2 \rangle}{3\sqrt{78}} \right| = \frac{38}{3\sqrt{78}}$$

32. $\frac{x+1}{2} = \frac{y+2}{-4} = \frac{z-1}{-3}$ and $\frac{x-1}{5} = \frac{y-1}{3} = \frac{z+1}{2}$

▷ $\mathbf{L}_1 = \langle 2,-4,-3 \rangle$ and $\mathbf{L}_2 = \langle 5,3,2 \rangle$ are direction vectors of the first and second lines. The vector

$$\mathbf{N} = \mathbf{L}_1 \times \mathbf{L}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ 5 & 3 & 2 \end{vmatrix} = \langle 1, -19, 26 \rangle$$

is perpendicular to each of the lines. The first line contains the point $P_1(-1,-2,1)$ and the second line contains the point $P_2(1,1,-1)$. The measure of the perpendicular distance between the lines is the absolute value of the scalar projection of $\overrightarrow{P_1P_2}$ on $\mathbf{N}$. Because $\overrightarrow{P_1P_2} = \langle 2,3,-2 \rangle$, we take

$$\frac{|\mathbf{N} \cdot \overrightarrow{P_1P_2}|}{\|\mathbf{N}\|} = \frac{|\langle 1,-19,26 \rangle \cdot \langle 2,3,-2 \rangle|}{\sqrt{1^2 + 19^2 + 26^2}} = \frac{107}{\sqrt{1038}}$$

Thus, the perpendicular distance between the lines is $107/\sqrt{1038}$ units.

33. $25(8)\sin 70^\circ = 187.9$ in-lb **34.** $30(6)\sin 80^\circ = 177.3$ in-lb **35.** $\dfrac{\|\mathbf{A}\times\mathbf{B}\|}{\mathbf{A}\cdot\mathbf{B}} = \dfrac{\|\mathbf{A}\|\|\mathbf{B}\|\sin\theta}{\|\mathbf{A}\|\|\mathbf{B}\|\cos\theta} = \tan\theta$

36. If $\mathbf{A}$ and $\mathbf{B}$ are vectors in V_3, prove that $\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = 0$.

▷ Let $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle$. Because a determinant is 0 if two rows are equal, then

$$\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

37. By Th 10.5.4, $(\mathbf{A}-\mathbf{B})\times(\mathbf{A}+\mathbf{B}) = (\mathbf{A}-\mathbf{B})\times\mathbf{A} + (\mathbf{A}-\mathbf{B})\times\mathbf{B} = \mathbf{A}\times\mathbf{A} - \mathbf{B}\times\mathbf{A} + \mathbf{A}\times\mathbf{B} - \mathbf{B}\times\mathbf{B}$
$= \mathbf{0} + \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{B} + \mathbf{0} = 2(\mathbf{A}\times\mathbf{B})$

38. $\mathbf{A} = \mathbf{V}(\overrightarrow{OP})$, $\mathbf{B} = \mathbf{V}(\overrightarrow{OQ})$, and $\mathbf{C} = \mathbf{V}(\overrightarrow{OR})$. Then $\mathbf{V}(\overrightarrow{PQ}) = \mathbf{B}-\mathbf{A}$ and $\mathbf{V}(\overrightarrow{PR}) = \mathbf{C}-\mathbf{A}$. A normal vector to the plane of P, Q and R is $\mathbf{V}(\overrightarrow{PQ})\times\mathbf{V}(\overrightarrow{PR})$
$= (\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A}) = (\mathbf{B}-\mathbf{A})\times\mathbf{C} - (\mathbf{B}-\mathbf{A})\times\mathbf{A} = \mathbf{B}\times\mathbf{C} - \mathbf{A}\times\mathbf{C} - \mathbf{B}\times\mathbf{A} + \mathbf{A}\times\mathbf{A} = \mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A} + \mathbf{A}\times\mathbf{B}$

In Exercises 39–42, let $\mathbf{A} = \langle a_1,a_2,a_3\rangle$, $\mathbf{B} = \langle b_1,b_2,b_3\rangle$, and $\mathbf{C} = \langle c_1,c_2,c_3\rangle$.

39. $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \langle a_1,a_2,a_3\rangle\times\langle b_1+c_1, b_2+c_2, b_3+c_3\rangle$
$= \langle a_2(b_3+c_3) - a_3(b_2+c_2), a_3(b_1+c_1) - a_1(b_3+c_3), a_1(b_2+c_2) - a_2(b_1+c_1)\rangle$
$= \langle a_2b_3 - a_3b_2 + a_2c_3 - a_3c_2, a_3b_1 + a_3c_1 - a_1b_3 + a_1c_3, a_1b_2 + a_1c_2 - a_2b_1 + a_2c_1\rangle$
$= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle + \langle a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1\rangle$
$= (\mathbf{A}\times\mathbf{B}) + (\mathbf{A}\times\mathbf{C})$ and
$(\mathbf{B}+\mathbf{C})\times\mathbf{A} = -[\mathbf{A}\times(\mathbf{B}+\mathbf{C})] = -[\mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}] = -(\mathbf{A}\times\mathbf{B}) - (\mathbf{A}\times\mathbf{C}) = \mathbf{B}\times\mathbf{B} + \mathbf{C}\times\mathbf{A}$

40. Prove Theorem 10.5.5.

▷ Let $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle$ be any two vectors in V_3 and let c be any scalar. We use the fact that multiplying the elements of a row by a number c multiplies the value of the determinant by c. Thus,

$$(c\mathbf{A})\times\mathbf{B} = \langle ca_1, ca_2, ca_3\rangle\times\langle b_1, b_2, b_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ ca_1 & ca_2 & ca_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = c\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad (1)$$

From (1) we get immediately

$(c\mathbf{A})\times\mathbf{B} = c(\langle a_1, a_2, a_3\rangle\times\langle b_1, b_2, b_3\rangle) = c(\mathbf{A}\times\mathbf{B})$

Applying the determinant rule to the third row of (1), we get

$$(c\mathbf{A})\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ cb_1 & cb_2 & cb_3 \end{vmatrix} = \langle a_1, a_2, a_3\rangle\times\langle cb_1, cb_2, cb_3\rangle = \mathbf{A}\times(c\mathbf{B})$$

41. $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle\times\langle c_1,c_2,c_3\rangle = \langle a_1,a_2,a_3\rangle\cdot\langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1\rangle$
$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$
$\mathbf{A}\times\mathbf{B}\cdot\mathbf{C} = \langle a_1,a_2,a_3\rangle\times\langle b_1,b_2,b_3\rangle\cdot\langle c_1,c_2,c_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle\cdot\langle c_1,c_2,c_3\rangle$
$= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3$
$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = \mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$

42. Choose the axes so that $\mathbf{B} = b_1\mathbf{i}$, $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j}$, $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Then $\mathbf{B}\times\mathbf{C} = b_1c_2\mathbf{k}$ and
$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times b_1c_2\mathbf{k} = -a_1b_1c_2\mathbf{j} + a_2b_1c_2\mathbf{i}$ while
$(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} = (a_1c_1 + a_2c_2)b_1\mathbf{i} - (a_1b_1)(c_1\mathbf{i} + c_2\mathbf{j}) = a_2b_1c_2\mathbf{i} - a_1b_1c_2\mathbf{j}$, proving (i).

43. For a parallelpiped, volume = area of base × height and so height $= \dfrac{\text{volume}}{\text{area of base}} = \dfrac{|\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}|}{\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\|}$

44. $\mathbf{A} = \mathbf{V}(\overrightarrow{OP})$, $\mathbf{B} = \mathbf{V}(\overrightarrow{OQ})$, $\mathbf{C} = \mathbf{V}(\overrightarrow{OR})$. The area of triangle PQR is half the area of a parallelogram with PQ and PR as adjacent sides, that is $\frac{1}{2}\left\|\mathbf{V}(\overrightarrow{PR})\times\mathbf{V}(\overrightarrow{PQ})\right\| = \frac{1}{2}\|(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A})\|$.

45. $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = -[\mathbf{C}\times(\mathbf{A}\times\mathbf{B})] = -[(\mathbf{C}\cdot\mathbf{B})\mathbf{A} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B}] = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\mathbf{A}$

46. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = [(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}] + [(\mathbf{B}\cdot\mathbf{A})\mathbf{C} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B}] + [(\mathbf{C}\cdot\mathbf{B})\mathbf{A} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B}]$
$= (-\mathbf{B}\cdot\mathbf{C} + \mathbf{C}\cdot\mathbf{B})\mathbf{A} + (\mathbf{A}\cdot\mathbf{C} - \mathbf{C}\cdot\mathbf{A})\mathbf{B} + (-\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A})\mathbf{C} = 0\mathbf{A} + 0\mathbf{B} + 0\mathbf{C} = \mathbf{0}$, proving Jacobi's identity.

47. From Ex. 47, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = -[\mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B})] = (\mathbf{A}\times\mathbf{B})\times\mathbf{C} - \mathbf{B}\times(\mathbf{C}\times\mathbf{A})$
The given identity is true if and only if the last term is the zero vector.

10.6 SURFACES

10.6.1 Definition A *cylinder* (*cone*) is a surface that is generated by a line moving along a given plane curve $\mathcal{C}$ in such a way that it always remains parallel to a fixed line (passes through a fixed point, the *vertex*) not lying in the plane of $\mathcal{C}$. The moving line is called a *generator* of the surface and $\mathcal{C}$ is called a *directrix*. Any position of a generator is a *ruling*. If $\mathcal{C}$ has a center C, the line through C parallel to the fixed line (passing through the vertex) is the *axis* of the surface.

10.6.2 Theorem In three-dimensional space, the graph of an equation in two of the three variables x, y, and z is a cylinder whose rulings are parallel to the axis associated with the missing variable and whose directrix is a curve in the plane associated with the two variables appearing in the equation. The graph of a homogeneous equation is a cone with vertex at the origin.

The intersection of a plane with a surface is called a *cross section* of the surface.

10.6.3 Definition If a plane curve is revolved about a fixed line lying in the plane of the curve, the surface generated is called a *surface of revolution*. The fixed line is called the *axis* of the surface of revolution, and the plane curve is called the *generating curve*.

The following table describes three types of surfaces of revolution.

Curve in the	revolved about	replace	by
xy plane	x axis	y^2	$y^2 + z^2$
	y axis	x^2	$x^2 + z^2$
xz plane	x axis	z^2	$y^2 + z^2$
	z axis	x^2	$x^2 + y^2$
yz plane	y axis	z^2	$x^2 + z^2$
	z axis	y^2	$x^2 + y^2$

Quadric Surfaces If $A \neq 0$, $B \neq 0$, and $C \neq 0$, the graph of the equation

$$Ax^2 + By^2 + Cz^2 = 1 \tag{I}$$

is symmetric with respect to each of the coordinate planes and is called a *central quadric*. If A, B, and C are all positive, the graph of Eq. (I) is an *ellipsoid*. If two of the numbers A, B, and C are positive and one is negative, the graph of (I) is an *elliptic hyperboloid of one sheet* with axis corresponding to the variable whose coefficient is negative. If one of the numbers A, B, and C is positive and two are negative, the graph of (I) is an *elliptic hyperboloid of two sheets* with axis corresponding to the variable whose coefficient is positive. If A, B, and C are all negative, the graph of (I) is the empty set.

If $A \neq 0$ and $B \neq 0$, the graphs of the equations

$$z = Ax^2 + By^2 \quad y = Ax^2 + Bz^2 \quad x = Ay^2 + Bz^2 \tag{II}$$

contain the origin and are symmetric with respect to two of the coordinate plane but are not symmetric with respect to the coordinate plane corresponding to the two squared variables. If A and B are either both positive or both negative, the graphs of Eqs. (II) are *elliptic paraboloids* with axis corresponding to the variable that is not squared. If A and B have opposite signs, the graphs of (II) are *hyperbolic paraboloids* with axis corresponding to the variable that is not squared.

If $A > 0$ and $B > 0$, the graphs of the homogeneous equations

$$z^2 = Ax^2 + By^2 \quad y^2 = Ax^2 + Bz^2 \quad x^2 = Ay^2 + Bz^2 \tag{III}$$

are *elliptic cones* with axis corresponding to the isolated variable. If $A = B$, we have a right-circular cone and sections perpendicular to the axis are circles. If $A \neq B$, there are two directions with circular sections.

The hyperboloid of one sheet and the hyperbolic paraboloid are doubly ruled surfaces. See Exercise 48. The volume of an ellipsoid of semiaxes a, b, and c is $\frac{4}{3}\pi abc$; thus, the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. See Exercise 50.

Exercises 10.6

In Exercises 1–4, sketch the cross section of the given cylinder in the indicated plane

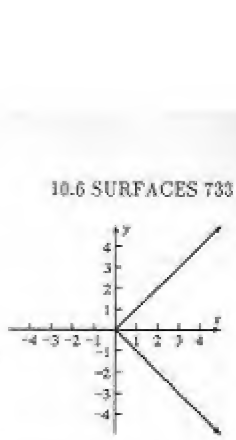
1. The cross section of the cylinder $4x^2 + y^2 = 16$ in the xy plane is the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$.
2. The cross section of the cylinder $4z^2 - y^2 = 4$ in the yz plane is the hyperbola $z^2 - y^2/4 = 1$.
3. The cross section of $z = e^x$ in the xz plane is the graph of the exponential function.

4. $x = |y|$; xy plane

▷ Because

$$x = \begin{cases} y & \text{if } y \ge 0 \\ -y & \text{if } y < 0 \end{cases}$$

the cross section of the cylinder in the xy plane consists of two half-lines. See the figure. The cylinder itself consists of two half-planes perpendicular to the xy plane.



In Exercises 5–12, sketch the cylinder having the given equation.

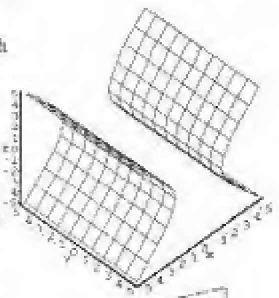
5. $4x^2 + 9y^2 = 36$ has rulings parallel to the z axis; the directrix in the xy plane is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

6. The cylinder $z = \sin y$ has rulings parallel to the x axis; the directrix in the yz plane is the sine curve.

7. $y = |z|$ has rulings parallel to the x axis; the directrix in the yz plane consists of the lines $y = z$ and $y = -z$.

8. $x^2 - z^2 = 4$

▷ The directrix in the xz plane is the hyperbola whose equation is $x^2 - z^2 = 4$. The rulings of the cylinder are parallel to the y axis. The figure shows a sketch of the cylinder.



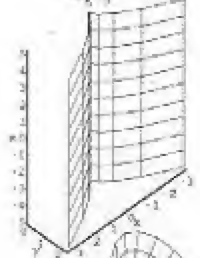
9. The cylinder $z = 2x^2$ has its rulings parallel to the y axis, and its directrix in the xz plane is the parabola $x^2 = \frac{1}{2}z$.

10. The cylinder $z^2 = 4y^2$ has its rulings parallel to the x axis, and its directrix in the yz plane consists of the lines $z = \pm 2y$.

11. The cylinder $y = \cosh x$ has its rulings parallel to the z axis, and the directrix in the xy plane is the graph of the hyperbolic cosine function.

12. $x^2 = y^3$

▷ The rulings are parallel to the z axis. The directrix is the curve $x^2 = y^3$ in the xy plane. The figure shows a sketch of the cylinder.



In Exercises 13–20, find an equation of the surface of revolution generated by revolving the given plane curve about the indicated axis. Sketch the surface.

13. $x^2 = 4y$, about the y axis. We replace x^2 with $x^2 + z^2$: $x^2 + z^2 = 4y$

14. $x^2 + 4z^2 = 16$, about the z axis. We replace x^2 with $x^2 + y^2$: $x^2 + y^2 + 4z^2 = 16$

15. $x^2 + 4z^2 = 16$, about the x axis. We replace z^2 with $z^2 + y^2$: $x^2 + 4y^2 + 4z^2 = 16$

16. $x^2 = 4y$ in the xy plane, about the x axis.

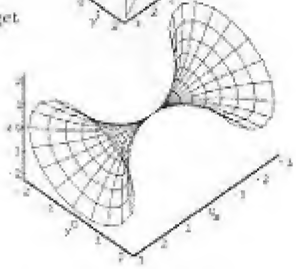
▷ Because we wish to replace y^2 with $y^2 + z^2$, we first square the equation to get

$x^4 = 16y^2$

The equation of the surface of revolution is

$x^4 = 16(y^2 + z^2)$

The figure shows a sketch.



17. $y = 3z$, about the y axis. We square to get $y^2 = 9z^2$, then replace z with $z^2 + x^2$ to get $y^2 = 9x^2 + 9z^2$

18. $9y^2 - z^2 = 144$, about the z axis. Replace y^2 with $x^2 + y^2$ to get $9x^2 + 9y^2 - z^2 = 144$

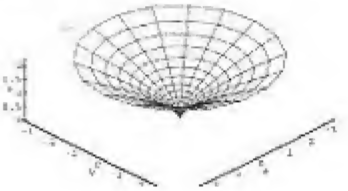
19. $y = \sin x$, about the x axis. Square to get $y^2 = \sin^2 x$, then replace y^2 with $y^2 + z^2$ to get $y^2 + z^2 = \sin^2 x$

20. $y^2 = z^3$ in the yz plane, about the z axis.

▷ We replace y^2 by $x^2 + y^2$. Thus,

$x^2 + y^2 = z^3$

is an equation of the surface of revolution. The figure shows a sketch of the surface.



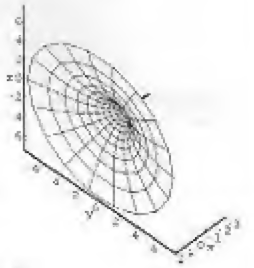
In Exercises 21–28, find a generating curve and the axis for the surface of revolution. Sketch the surface.

21. The sphere $x^2 + y^2 + z^2 = 16$ can be obtained as a surface of revolution in six ways:
by revolving $x^2 + y^2 = 16$ in the xy plane about the x axis or the y axis,
by revolving $x^2 + z^2 = 16$ in the xz plane about the x axis or the z axis, and
by revolving $y^2 + z^2 = 16$ in the yz plane about the y axis or the z axis.

22. $x^2 + z^2 = y$. Revolve $x^2 = y$ in xy plane or $z^2 = y$ in zy plane about y axis.

23. $x^2 + y^2 - z^2 = 4$

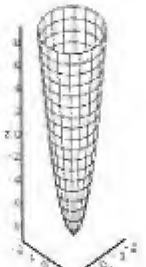
▸ Revolve $x^2 - z^2 = 4$ in xz plane or $y^2 - z^2 = 4$ in yz plane, about z axis



24. $y^2 + z^2 = e^{2x}$

▸ The axis of revolution is the x axis. We can start with either $y^2 = e^{2x}$ or $y = e^x$ or $y = -e^x$ in the xy plane; or $z^2 = e^{2x}$ or $z = e^x$ or $z = -e^x$ in the xz plane. The figure shows a sketch of the surface.

25. The equation $x^2 + z^2 = |y|$ has the y axis as its axis of revolution and as a generating curve either $x^2 = |y|$ or in the xy plane or $z^2 = |y|$ in the yz plane.



26. $4x^2 + 9y^2 + 4z^2 = 9$. The axis of revolution is the y axis. We can start with $4x^2 + 9y^2 = 9$ or $9y^2 + 4z^2 = 9$.

27. $x^2 + z^2 = \frac{1}{9}y^2$. The axis of revolution is the y axis. The generating curve is $x^2 = \frac{1}{9}y^2$ or $y = 3x$ or $y = -3x$ in the xy plane; or $z^2 = \frac{1}{9}y^2$ or $y = 3z$ or $y = -3z$ in the yz plane.

28. $4x^2 + 4y^2 - z = 9$

▸ We can write the equation as

$4(x^2 + y^2) - z = 9$

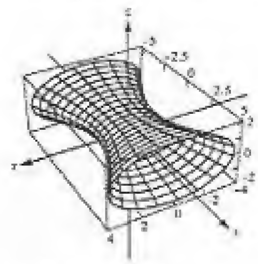
The axis of revolution is the z axis. We can start with $4x^2 - z = 9$ or $4y^2 - z = 9$. The figure shows a sketch of the surface.

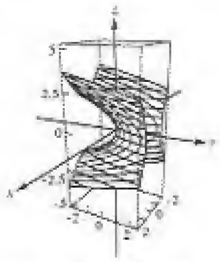
In Exercises 29 and 30, match the equation with one of the figures and name the surface.

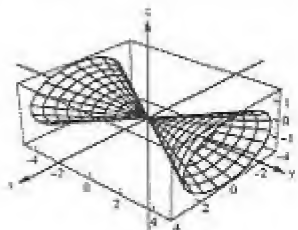
29. (a) $9x^2 - 4y^2 + 36z^2 = 36$ (v) hyperboloid of one sheet

(b) $5z^2 - 2z^2 = 3y$ (iii) hyperbolic paraboloid

(c) $9x^2 - 4y^2 + 36z^2 = 0$ (vi) elliptic cone



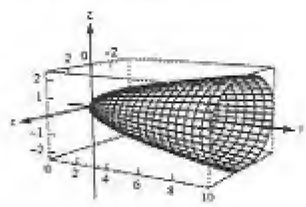


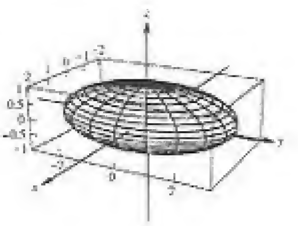


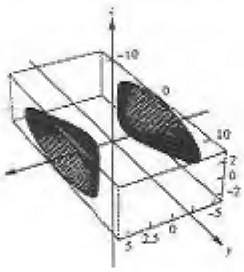
(d) $5x^2 + 2z^2 = 3y$ (i) elliptic paraboloid

(e) $9x^2 + 4y^2 + 36z^2 = 36$ (ii) ellipsoid

(f) $9x^2 - 4y^2 - 36z^2 = 36$ (iv) hyperboloid of two sheets



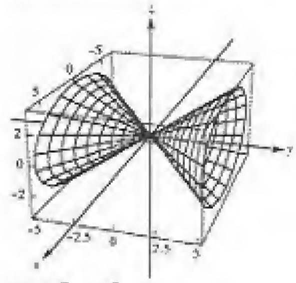


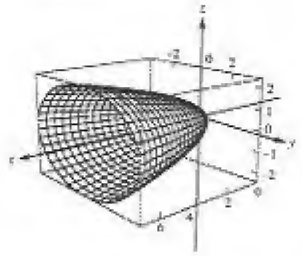


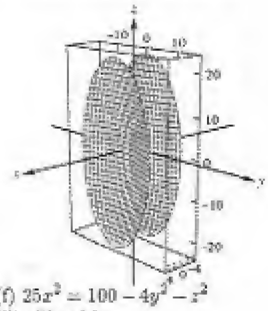
30. (a) $4x^2 - 16y^2 + 9z^2 = 0$
(iii) elliptic cone

(b) $3y^2 + 7z^2 = 6x$
(v) elliptic paraboloid

(c) $25x^2 = 4y^2 + z^2 + 100$
(i) hyperboloid of two sheets



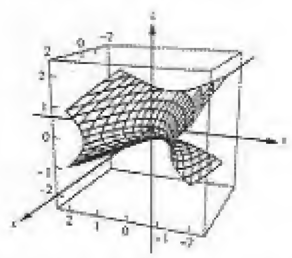


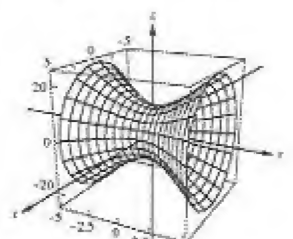


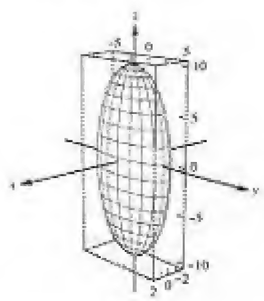
(d) $3y^2 - 7z^2 = 6x$
(vi) hyperbolic paraboloid

(e) $25x^2 = 4y^2 - z^2 + 100$
(iv) hyperboloid of one sheet

(f) $25x^2 = 100 - 4y^2 - z^2$
(ii) ellipsoid







In Exercises 31–42, sketch the graph of the equation and name the surface.

31. $4x^2 + 9y^2 + z^2 = 36$; $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{36} = 1$. The surface is an ellipsoid with semi-axes 3, 2, and 6.

32. $4x^2 - 9y^2 - z^2 = 36$

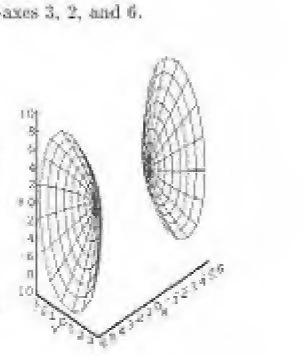
▷ Dividing on both sides of the given equation by 36, we obtain

$$\frac{x^2}{9} - \frac{y^2}{4} - \frac{z^2}{36} = 1$$

Comparing the equation with (I), we conclude the graph is an elliptic hyperboloid of two sheets with the x axis the transverse axis. Because $x = \pm 3$ when $(y, z) = (0, 0)$, the vertices of the hyperboloid are $(\pm 3, 0, 0)$. If $x = k$, $|k| > 3$, we have

$$\frac{y^2}{4} + \frac{z^2}{36} = \frac{k^2}{9} - 1;\quad \frac{y^2}{4(k^2/9 - 1)} + \frac{z^2}{36(k^2/9 - 1)} = 1$$

an ellipse of semiaxes $2\sqrt{k^2/9 - 1}$ and $6\sqrt{k^2/9 - 1}$. The figure shows the graph.



33. $4x^2 + 9y^2 - z^2 = 36$; $\frac{x^2}{9} + \frac{y^2}{4} - \frac{z^2}{36} = 1$

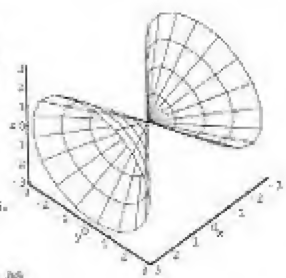
Comparing with (I), we conclude the surface is an elliptic hyperboloid of one sheet whose axis is the z axis.

34. $4x^2 - 9y^2 + z^2 = 36$. From (I), the surface is an elliptic hyperboloid of one sheet whose axis is the y axis.

35. $x^2 - y^2 + z^2 = 0$ is of type (III) with A = B = 1. The surface is a right-circular cone whose axis is the y axis.

36. $x^2 = y^2 + z^2$

▷ The equation is of type (III) with $A = B = 1$ and hence, its graph is a right-circular cone. Each cross section in the plane $x = k$ with $k \neq 0$ is the circle $y^2 + z^2 = k^2$. The figure shows the graph.



37. $\frac{x^2}{36} + \frac{z^2}{25} = 4y$ is of type (II), an elliptic paraboloid whose axis is the y axis.

38. $\frac{y^2}{25} + \frac{x^2}{36} = 4z$ is of type (II), an elliptic paraboloid whose axis is the z axis.

39. $\frac{x^2}{36} - \frac{z^2}{25} = 9y$ is of type (II), a hyperbolic paraboloid whose axis is the y axis.

40. $x^2 = 2y + 4z$

▷ Because $\sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$, we may write the equation of the surface as

$$x^2 = 2\sqrt{5}\left(\frac{1}{\sqrt{5}}y + \frac{2}{\sqrt{5}}z\right) \tag{1}$$

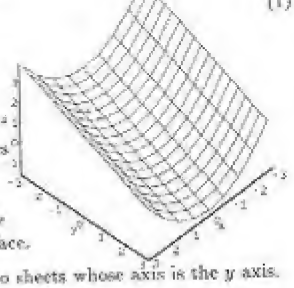
If $\cos\alpha = \frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{2}{\sqrt{5}}$, the substitutions

$$\bar{y} = \frac{1}{\sqrt{5}}y + \frac{2}{\sqrt{5}}z \qquad \bar{z} = -\frac{2}{\sqrt{5}}y + \frac{1}{\sqrt{5}}z$$

represent a rotation of the y and z axes in the yz plane of 63°. Eq (1) becomes

$$x^2 = 2\sqrt{5}\bar{y}$$

which is a parabolic cylinder. The directrix is the parabola $x^2 = 2\sqrt{5}\bar{y}$ in the $x\bar{y}$ plane and the rulings are parallel to the $\bar{z}$ axis. The figure shows the surface.



41. $x^2 + 16z^2 = 4y^2 - 1$; $\frac{y^2}{4} - \frac{x^2}{16} - \frac{z^2}{1} = 1$. Type (I), an elliptic hyperboloid of two sheets whose axis is the y axis.

42. $9y^2 - 4z^2 + 18x = 0$; $\frac{z^2}{9/2} - \frac{y^2}{2} = x$ is of type (II), a hyperbolic paraboloid whose axis is the x axis.

43. Substituting the equation $x = 1 - ky$ of the plane into the equation $y^2 - x^2 - z^2 = 1$ of the hyperboloid we obtain the projection of the intersection on the zy plane:

$y^2 - (1 - ky)^2 - z^2 = 1$; $(k^2 - 1)y^2 - 2ky + z^2 = -2$

(a) The intersection will be an ellipse if its projection is, that is, if $k^2 - 1 > 0$. Then an equivalent equation is

$(k^2 - 1)\left[y^2 - \frac{2k}{k^2 - 1}y + \frac{k^2}{(k^2 - 1)^2}\right] + z^2 = \frac{k^2}{k^2 - 1} - 2$. This graph will have real points if $\frac{k^2}{k^2 - 1} - 2 > 0$; $k^2 > 2k^2 - 2$; $k^2 \langle 2$; $|k| < \sqrt{2}$. Hence the intersection is an ellipse if $1 < |k| < \sqrt{2}$. If $|k| = 1$ it is a parabola; if $|k| = \sqrt{2}$, it is a single point. If $|k| > \sqrt{2}$, the intersection is empty.

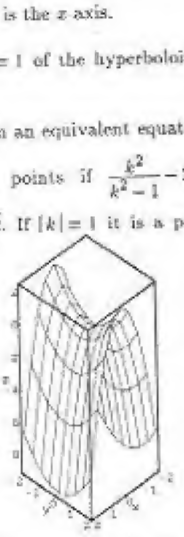
(b) The intersection is a hyperbola if $k^2 - 1 < 0$; $k^2 < 1$; $|k| < 1$.

44. Find the vertex and the focus of the parabola that is the intersection of the plane $y = 2$ with the hyperbolic paraboloid $y^2/16 - x^2/4 = z/9$.

▷ We let $y = 2$ in the given equation and write the resulting equation in standard form. Thus, we have

$$\frac{1}{4} - \frac{x^2}{4} = \frac{z}{9}; \qquad x^2 = 4(-\tfrac{1}{9})(z - \tfrac{9}{4})$$

The cross section of the given surface in the plane $y = 2$ is a parabola with vertex $(0, 2, \frac{9}{4})$. The parabola has axis parallel to the z axis and opens downward. Because $p = -\frac{1}{9}$, then the focus of the parabola is $(0, 2, \frac{9}{4} - \frac{1}{9}) = (0, 2, \frac{77}{36})$. The figure shows the given hyperbolic paraboloid. The right face is the cross section in the plane $y = 2$.



45. Substituting the equation of the plane $x = 1$ into the equation of the hyperbolic paraboloid $\frac{x^2}{4} - \frac{z^2}{9} = \frac{y}{3}$, we get $\frac{z^2}{4} - \frac{1}{9} = \frac{y}{3}$; $x^2 = 4\left(\frac{1}{3}\right)\left(y + \frac{1}{3}\right)$. Because $x = 1$ is parallel to the the projection, we find the vertex is at $(1, -\frac{1}{3}, 0)$. The axis is parallel to the y axis and the parabola opens in the direction of the positive side of the y axis. $p = \frac{1}{3}$ so the focus is at $(1, 0, 0)$.

46. $y = 3$ meets $\frac{x^2}{9} + \frac{y^2}{25} + \frac{z^2}{4} = 1$ in $\frac{x^2}{9} + \frac{9}{25} + \frac{z^2}{4} = 1$, $\frac{x^2}{9} + \frac{z^2}{4} = \frac{16}{25}$, $\frac{x^2}{\frac{144}{25}} + \frac{z^2}{\frac{64}{25}} = 1$. Area $= \pi(\frac{12}{5})\frac{8}{5} = \frac{96}{25}\pi$

47. Any point on $x^2 - 4y^2 - 9z^2 = 36$: $4x^2 + 4y^2 + 4z^2 = 5x^2 - 5z^2 - 36 = 5(x+z)(x-z) - 36$ and a plane of the form $x + z = k$ must lie on the sphere $4x^2 + 4y^2 + 4z^2 = 5k(x - y)$, and the intersection of a plane and a sphere is a circle. Similarly for planes $x - z = k$.

48. Show that each point of (a) the hyperbolic paraboloid $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$ and (b) the elliptic hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is the intersection of two lines lying entirely in the surface.

▷ (a) We factor the given equation.

$\left(\frac{y}{b} - \frac{x}{a}\right)\left(\frac{y}{b} + \frac{x}{a}\right) = \frac{z}{c}$

For any value of k, each point of the line of intersection of the planes

$$\frac{y}{b} - \frac{x}{a} = k\frac{z}{c} \quad \text{and} \quad \frac{y}{b} + \frac{x}{a} = \frac{1}{k} \tag{1}$$

satisfies the equation of the hyperbolic paraboloid, and so the line lies entirely in it. Conversely, each point of the paraboloid determines a value of k. Similarly, for any k, each point of the line of intersection of the planes

$$\frac{y}{b} + \frac{x}{a} = m\frac{z}{c} \quad \text{and} \quad \frac{y}{b} - \frac{x}{a} = \frac{1}{m} \tag{2}$$

lies in the hyperbolic paraboloid.

(b) We arrange the given equation so that each side is the difference of two squares and then factor. Thus,

$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$; $\left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\left(1 + \frac{y}{b}\right)$

For any value of k, each point of the line of intersection of the planes

$\frac{x}{a} - \frac{z}{c} = k\left(1 - \frac{y}{b}\right)$ and $\frac{x}{a} + \frac{z}{c} = \frac{1}{k}\left(1 + \frac{y}{b}\right)$

satisfies the equation of the hyperboloid of one sheet, and so the line lies entirely in it. Conversely, each point of the hyperboloid determines a value of k. Similarly, for any value of m, each point of the line of intersection of the planes

$\frac{x}{a} + \frac{z}{c} = m\left(1 - \frac{y}{b}\right)$ and $\frac{x}{a} - \frac{z}{c} = \frac{1}{m}\left(1 + \frac{y}{b}\right)$

lies in the paraboloid.

49. The ellipsoid is $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$; $\frac{y^2}{4} + \frac{z^2}{9} = 1 - x^2$. A plane section of the ellipsoid at $x = w_i$, $0 \le w_i \le 1$, is a region enclosed by an ellipse whose semiaxes have lengths $2\sqrt{1 - w_i^2}$ and $3\sqrt{1 - w_i^2}$. Therefore if V cubic units is the volume of the ellipsoid,

$$V = 2 \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi(2\sqrt{1 - w_i^2})(3\sqrt{1 - w_i^2})\Delta_i x = 12\pi \int_0^1 (1 - x^2)\,dx = 12\pi\left[1 - \tfrac{1}{3}y^3\right]_0^1 = 12\pi\left(1 - \tfrac{1}{3}\right) = 8\pi$$

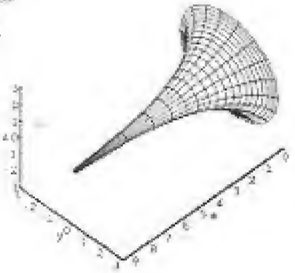
50. Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$. A z-slice is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2} \Rightarrow \frac{x^2}{a^2(1 - z^2/c^2)} + \frac{y^2}{b^2(1 - z^2/c^2)} = 1$, ellipse of semiaxes $a' = a\sqrt{1 - z^2/c^2}$ and $b' = b\sqrt{1 - z^2/c^2}$ and area $A = \pi a'b' = \pi ab(1 - z^2/c^2)$.

$$V = 2\pi ab \int_0^c \left(1 - \frac{z^2}{c^2}\right)dz = 2\pi ab\left[z - \frac{z^3}{3c^2}\right]_0^c = \frac{4}{3}\pi abc$$

51. The elliptic paraboloid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ $(c > 0)$. A plane section of the surface at $z = w_i$, $0 \le w_i \le h$, is an ellipse having semiaxes of lengths $a\sqrt{\frac{w_i}{c}}$ and $b\sqrt{\frac{w_i}{c}}$.

If V cubic units is the volume of the solid,

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \pi\left(a\sqrt{\frac{w_i}{c}}\right)\left(b\sqrt{\frac{w_i}{c}}\right)\Delta_i z = \frac{\pi ab}{c}\int_0^h z\,dz = \frac{\pi ab}{c}\left[\frac{z^2}{2}\right]_0^h = \frac{abh^2}{2c}\pi.$$



52. Sketch the surface of revolution generated by revolving the tractrix

$x = 3\ln\left(\frac{3 + \sqrt{9 - y^2}}{y}\right) - \sqrt{9 - y^2}$ about the x axis.

▷ When $y = 3$, $x = 0$ and as $y \to 0^+$, $x \to +\infty$. The figure shows the surface, called a pseudosphere, which has application in noneuclidean geometry.

Miscellaneous Exercises for Chapter 10

In Exercises 1–18, $\mathbf{A} = 4\mathbf{i} - 6\mathbf{j}$, $\mathbf{B} = \mathbf{i} + 7\mathbf{j}$, and $\mathbf{C} = 9\mathbf{i} - 5\mathbf{j}$

1. $3\mathbf{B} - 7\mathbf{A} = 3(\mathbf{i} + 7\mathbf{j}) - 7(4\mathbf{i} - 6\mathbf{j}) = 3\mathbf{i} + 21\mathbf{j} - 28\mathbf{i} + 42\mathbf{j} = -25\mathbf{i} + 63\mathbf{j}$

2. $5\mathbf{B} - 3\mathbf{C} = 5(\mathbf{i} + 7\mathbf{j}) - 3(9\mathbf{i} - 5\mathbf{j}) = 5\mathbf{i} + 35\mathbf{j} - 27\mathbf{i} + 15\mathbf{j} = -22\mathbf{i} + 50\mathbf{j}$

3. $\|3\mathbf{B} - 7\mathbf{A}\| = \|-25\mathbf{i} + 63\mathbf{j}\| = \sqrt{(-25)^2 + (63)^2} = \sqrt{625 + 3969} = \sqrt{4594}$

4. Find $\|5\mathbf{B} - 3\mathbf{C}\|$

▹ $$5\mathbf{B} - 3\mathbf{C} = 5(\mathbf{i} + 7\mathbf{j}) - 3(9\mathbf{i} - 5\mathbf{j}) = 5\mathbf{i} + 35\mathbf{j} - 27\mathbf{i} + 15\mathbf{j} = -22\mathbf{i} + 50\mathbf{j} = 2(-11\mathbf{i} + 25\mathbf{j})$$

Then
$$\|5\mathbf{B} - 3\mathbf{C}\| = 2\sqrt{(-11)^2 + 25^2} = 2\sqrt{746}$$

5. $\|3\mathbf{B}\| - \|7\mathbf{A}\| = 3\|\mathbf{B}\| - 7\|\mathbf{A}\| = 3\sqrt{1 + 49} - 7\sqrt{16 + 36} = 3\sqrt{50} - 7\sqrt{52} = 15\sqrt{2} - 14\sqrt{13}$

6. $\|5\mathbf{B}\| - \|3\mathbf{C}\| = 5\|\mathbf{B}\| - 3\|\mathbf{C}\| = 5\sqrt{1 + 49} - 3\sqrt{81 + 25} = 5\sqrt{50} - 3\sqrt{106} = 25\sqrt{2} - 3\sqrt{106}$

7. $(\mathbf{A} - \mathbf{B}) \cdot \mathbf{C} = [(4\mathbf{i} - 6\mathbf{j}) - (\mathbf{i} + 7\mathbf{j})] \cdot (9\mathbf{i} - 5\mathbf{j}) = (3\mathbf{i} - 3\mathbf{j}) \cdot (9\mathbf{i} - 5\mathbf{j}) = 27 + 65 = 92$

8. Find $(\mathbf{A} \cdot \mathbf{B})\mathbf{C}$.

▹ $$(\mathbf{A} \cdot \mathbf{B})\mathbf{C} = [(4\mathbf{i} - 6\mathbf{j}) \cdot (\mathbf{i} + 7\mathbf{j})]\mathbf{C} = [4(1) + (-6)7]\mathbf{C} = -38(9\mathbf{i} - 5\mathbf{j}) = -342\mathbf{i} + 190\mathbf{j}$$

9. $2\mathbf{A} + \mathbf{B} = 9\mathbf{i} - 5\mathbf{j}$. $\|2\mathbf{A} + \mathbf{B}\| = \sqrt{9^2 + (-5)^2} = \sqrt{106}$. The required unit vector is $\frac{9}{\sqrt{106}}\mathbf{i} - \frac{5}{\sqrt{106}}\mathbf{j}$.

10. Swap components, change a sign. $\mathbf{N} = 7\mathbf{i} - \mathbf{j}$. $\|\mathbf{N}\| = \sqrt{49 + 1} = \sqrt{50}$. $\mathbf{U} = \pm\frac{1}{\sqrt{50}}(7\mathbf{i} - \mathbf{j}) = \pm(\frac{7}{10}\sqrt{2}\mathbf{i} - \frac{1}{10}\sqrt{2}\mathbf{j}$

11. $\mathbf{A} = h\mathbf{B} + k\mathbf{C}$. $4\mathbf{i} - 6\mathbf{j} = h(\mathbf{i} + 7\mathbf{j}) + k(9\mathbf{i} - 5\mathbf{j}) = (h + 9k)\mathbf{i} + (7h - 5k)\mathbf{j}$; $h + 9k = 4$ and $7h - 5k = -6$. Thus, $h = -\frac{1}{2}$ and $k = \frac{1}{2}$.

12. Find scalars h and k such that $h\mathbf{A} + k\mathbf{B} = -\mathbf{C}$.

▹ Substituting the given vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ into the equation $h\mathbf{A} + k\mathbf{B} = -\mathbf{C}$, we have
$$h(4\mathbf{i} - 6\mathbf{j}) + k(\mathbf{i} + 7\mathbf{j}) = -(9\mathbf{i} - 5\mathbf{j})$$
$$(4h + k)\mathbf{i} + (-6h + 7k)\mathbf{i} = -9\mathbf{i} + 5\mathbf{j}$$

Therefore
$$4h + k = -9 \qquad (1)$$
$$-6h + 7k = 5 \qquad (2)$$

Solving for h and k, we obtain

$7(1) - (2)$: $\quad 34h = -68$, $h = -2$

$3(1) + 2(2)$: $\quad 17h = -17$, $k = -1$

13. The scalar projection of $\mathbf{A}$ onto $\mathbf{B}$ is $\frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|} = \frac{(4\mathbf{i} - 6\mathbf{j}) \cdot (\mathbf{i} + 7\mathbf{j})}{\sqrt{1 + 49}} = \frac{-38}{\sqrt{50}} = -\frac{19}{5}\sqrt{2}$.

14. $C_A = \frac{\mathbf{C} \cdot \mathbf{A}}{\|\mathbf{A}\|} = \frac{(9\mathbf{i} - 5\mathbf{j}) \cdot (4\mathbf{i} - 6\mathbf{j})}{\|4\mathbf{i} - 6\mathbf{j}\|} = \frac{66}{\sqrt{16 + 36}} = \frac{66}{\sqrt{52}} = \frac{33}{\sqrt{13}} = \frac{33}{13}\sqrt{13}$

15. The vector projection of $\mathbf{A}$ onto $\mathbf{B}$ is $\frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{B}\|^2}\mathbf{B} = \frac{(4\mathbf{i} - 6\mathbf{j}) \cdot (\mathbf{i} + 7\mathbf{j})}{1 + 49}(\mathbf{i} + 7\mathbf{j}) = \frac{-38}{50}(\mathbf{i} + 7\mathbf{j}) = -\frac{19}{25}\mathbf{i} - \frac{133}{25}\mathbf{j}$

16. Find the vector projection of $\mathbf{C}$ onto $\mathbf{A}$.

▹ The the vector projection of $\mathbf{C}$ onto $\mathbf{A}$ is given by
$$\mathbf{C}_A = \frac{\mathbf{C} \cdot \mathbf{A}}{\|\mathbf{A}\|^2}\mathbf{A} = \frac{(9\mathbf{i} - 5\mathbf{j}) \cdot (4\mathbf{i} - 6\mathbf{j})}{4^2 + 6^2}\mathbf{A} = \frac{66}{52}(4\mathbf{i} - 6\mathbf{j}) = \frac{66}{13}\mathbf{i} - \frac{99}{13}\mathbf{j}$$

17. The component of $\mathbf{B}$ in the direction of $\mathbf{A}$ is $\frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{A}\|} = \frac{(4\mathbf{i} - 6\mathbf{j}) \cdot (\mathbf{i} + 7\mathbf{j})}{\sqrt{4^2 + 6^2}} = \frac{-38}{\sqrt{52}} = -\frac{19}{13}\sqrt{13}$.

18. $\cos(\mathbf{A}, \mathbf{C}) = \frac{\mathbf{A} \cdot \mathbf{C}}{\|\mathbf{A}\|\|\mathbf{C}\|} = \frac{(4\mathbf{i} - 6\mathbf{j}) \cdot (9\mathbf{i} - 5\mathbf{j})}{\sqrt{4^2 + 6^2}\sqrt{9^2 + 5^2}} = \frac{66}{\sqrt{52}\sqrt{106}} = \frac{33}{\sqrt{1378}} = \frac{33}{1378}\sqrt{1378}$

19–20. Let $\mathbf{A} = -2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{B} = h\mathbf{i} - 2\mathbf{j}$ and let θ be the angle between $\mathbf{A}$ and $\mathbf{B}$. Determine h so that $\theta = \frac{2}{3}\pi$ and show that no h exists such that $\theta = \frac{1}{3}\pi$.

▷ We have
$\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \dfrac{-2h-10}{\sqrt{29}\sqrt{h^2+4}}$
Furthermore,
$(\cos\frac{2}{3}\pi)^2 = (-\frac{1}{2})^2 = \frac{1}{4}$ and $(\cos\frac{1}{3}\pi)^2 = (\frac{1}{2})^2 = \frac{1}{4}$
Thus, in either case,

$$\begin{aligned}
\cos^2\theta &= \frac{1}{4}\\
\frac{4h^2-40h+100}{29(h^2+4)} &= \frac{1}{4}\\
4(4h^2-40h+100) &= 29(h^2+4)\\
13h^2-160h-284 &= 0\\
h = \tfrac{1}{13}(80 \pm 58\sqrt{3}) &\approx 13.88 \text{ or } -1.57
\end{aligned}$$

Because both values of h make $\cos\theta$ negative, both are solutions for $\theta = \frac{2}{3}\pi$ and $\theta = \frac{1}{3}\pi$ has no solution.

In Exercises 21–30 $\mathbf{A} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $\mathbf{C} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{D} = 3\mathbf{j} - \mathbf{k}$, $\mathbf{E} = 5\mathbf{i} - 2\mathbf{j}$. Find the vector or scalar.

21. $6\mathbf{C} + 4\mathbf{D} - \mathbf{E} = 6(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) + 4(3\mathbf{j}-\mathbf{k}) - 5\mathbf{i} - 2\mathbf{j} = 6\mathbf{i} + 2\mathbf{j} - 12\mathbf{k} + 12\mathbf{j} - 4\mathbf{k} - 5\mathbf{i} + 2\mathbf{j} = \mathbf{i} + 26\mathbf{j} - 16\mathbf{k}$

22. $3\mathbf{A} - 2\mathbf{B} + \mathbf{C} = 3(-\mathbf{i}+3\mathbf{j}+2\mathbf{k}) - 2(2\mathbf{i}+\mathbf{j}-4\mathbf{k}) + (\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = -3\mathbf{i} + 9\mathbf{j} + 6\mathbf{k} - 4\mathbf{i} - 2\mathbf{j} + 8\mathbf{k} + \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
$= -6\mathbf{i} + 9\mathbf{j} + 12\mathbf{k}$

23. $\mathbf{D}\cdot\mathbf{B}\times\mathbf{C} = (3\mathbf{j}-\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-4\mathbf{k})\times(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = (3\mathbf{j}-\mathbf{k})\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & -4\\ 1 & 2 & -2 \end{vmatrix} = (3\mathbf{j}-\mathbf{k})\cdot(6\mathbf{i}+3\mathbf{k}) = -3$

24. $(\mathbf{A}\times\mathbf{C}) - (\mathbf{D}\times\mathbf{E})$

▷ $\mathbf{A}\times\mathbf{C} = (-\mathbf{i}+3\mathbf{j}+2\mathbf{k})\times(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 3 & 2\\ 1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 2\\ 2 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 2\\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 3\\ 1 & 2 \end{vmatrix}\mathbf{k} = -10\mathbf{i} - 5\mathbf{k}$

$\mathbf{D}\times\mathbf{E} = (3\mathbf{j}-\mathbf{k})\times(5\mathbf{i}-2\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 3 & -1\\ 5 & -2 & 0 \end{vmatrix} = -2\mathbf{i} - 5\mathbf{j} - 15\mathbf{k}$

Thus
$(\mathbf{A}\times\mathbf{C}) - (\mathbf{D}\times\mathbf{E}) = (-10\mathbf{i} - 5\mathbf{k}) - (-2\mathbf{i} - 5\mathbf{j} - 15\mathbf{k}) = -8\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$

25. $\|\mathbf{A}\times\mathbf{B}\|\|\mathbf{D}\times\mathbf{E}\| = \left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 3 & 2\\ 2 & 1 & -4 \end{vmatrix}\right\|\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 3 & -1\\ 5 & -2 & 0 \end{vmatrix}\right\| = \|-14\mathbf{i} - 7\mathbf{k}\|\|-2\mathbf{i} - 5\mathbf{j} - 15\mathbf{k}\| = \sqrt{14^2+7^2}\sqrt{2^2+5^2+15^2}$
$= 7\sqrt{5}\sqrt{254} = 7\sqrt{1270}$

26. $2\mathbf{B}\cdot\mathbf{C} + 3\mathbf{D}\cdot\mathbf{E} = 2(2\mathbf{i}+\mathbf{j}-4\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) + 3(3\mathbf{j}-\mathbf{k})\cdot(5\mathbf{i}-2\mathbf{j}) = 2(12) + 3(-6) = 6$

27. $\mathrm{A}_B = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{B}\|} = \dfrac{(-\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-4\mathbf{k})}{\sqrt{2^2+1^2+4^2}} = \dfrac{-2+3-8}{\sqrt{21}} = \dfrac{-7}{\sqrt{21}} = -\frac{1}{3}\sqrt{21}.$

28. Find the scalar projection of $\mathbf{C}$ onto $\mathbf{D}$.

▷ The scalar projection of $\mathbf{C}$ onto $\mathbf{D}$ is given by

$$\mathrm{C}_D = \frac{\mathbf{C}\cdot\mathbf{D}}{\|\mathbf{D}\|} = \frac{(1,2,-2)\cdot(0,3,-1)}{\|(0,3,1)\|} = \frac{8}{\sqrt{10}} = \tfrac{4}{5}\sqrt{10}$$

29. $\mathbf{E}_C = \dfrac{\mathbf{E}\cdot\mathbf{C}}{\|\mathbf{C}\|^2}\mathbf{C} = \dfrac{(5\mathbf{i}-2\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k})}{1^2+2^2+2^2}(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = \dfrac{5-4+0}{9}(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = \frac{1}{9}(\mathbf{i}+2\mathbf{j}-2\mathbf{k})$

30. $\mathbf{D}_E = \dfrac{\mathbf{D}\cdot\mathbf{E}}{\|\mathbf{E}\|^2}\mathbf{E} = \dfrac{(3\mathbf{j}-\mathbf{k})\cdot(5\mathbf{i}-2\mathbf{j})}{\|5\mathbf{i}-2\mathbf{j}\|^2}(5\mathbf{i}-2\mathbf{j}) = \dfrac{-6}{25+4}(5\mathbf{i}-2\mathbf{j}) = -\frac{30}{29}\mathbf{i} + \frac{12}{29}\mathbf{j}$

In Exercises 31–36, there is only one way that a meaningful expression can be obtained by inserting parentheses. Insert the parentheses and find the indicated vector or scalar if $\mathbf{A} = \langle 3,-2,4\rangle$, $\mathbf{B} = \langle -5,7,2\rangle$, $\mathbf{C} = \langle 4,6,-1\rangle$.

31. $\mathbf{AB}\cdot\mathbf{C} = \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) = \langle 3,-2,4\rangle(\langle -5,7,2\rangle\cdot\langle 4,6,-1\rangle) = \langle 3,-2,4\rangle(-20+42-2) = \langle 60,-40,80\rangle$

32. $\mathbf{A}\cdot\mathbf{BC}$

▸ Because $\mathbf{B}$ and $\mathbf{C}$ are vectors, $\mathbf{BC}$ is not defined. On the other hand, $\mathbf{A}\cdot\mathbf{B}$ is a scalar whose product with the vector $\mathbf{C}$ is defined. Thus, we take

$$(\mathbf{A}\cdot\mathbf{B})\mathbf{C} = (\langle 3,-2,4\rangle\cdot\langle -5,7,2\rangle)\mathbf{C} = -21\langle 4,6,1\rangle = \langle -84,-126,21\rangle$$

33. $\mathbf{A}+\mathbf{B}\cdot\mathbf{C} = (\mathbf{A}+\mathbf{B})\cdot\mathbf{C} = (\langle 3,-2,4\rangle+\langle -5,7,2\rangle)\cdot\langle 4,6,-1\rangle = \langle -2,5,6\rangle\cdot\langle 4,6,-1\rangle = -8+30-6 = 16$

34. $\mathbf{B}\cdot\mathbf{A}-\mathbf{C} = \mathbf{B}\cdot(\mathbf{A}-\mathbf{C}) = \langle -5,7,2\rangle\cdot(\langle 3,-2,4\rangle-\langle 4,6,-1\rangle) = \langle -5,7,2\rangle\cdot\langle -1,-8,5\rangle = -41$

35. $\mathbf{A}\times\mathbf{B}\cdot\mathbf{A}+\mathbf{B}-\mathbf{C} = (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}-\mathbf{C}) = (\langle 3,-2,4\rangle\times\langle -5,7,2\rangle)\cdot(\langle 3,-2,4\rangle+\langle -5,7,2\rangle-\langle 4,6,-1\rangle)$
$= \langle -32,-26,11\rangle\cdot\langle -6,-1,7\rangle = 192+26+77 = 295$

36. $\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}\times\mathbf{A}$

▸ Both $\mathbf{A}\times\mathbf{B}$ and $\mathbf{C}\times\mathbf{A}$ are vectors, and the dot product of two vectors is defined. Thus, we have

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ -5 & 7 & 2 \end{vmatrix} = -32\mathbf{i}-26\mathbf{j}+11\mathbf{k} \qquad \mathbf{C}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & -1 \\ 3 & -2 & 4 \end{vmatrix} = 22\mathbf{i}-19\mathbf{j}-26\mathbf{k}$$

$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{A}) = \langle -32,-26,11\rangle\cdot\langle 22,-19,-26\rangle = -496$

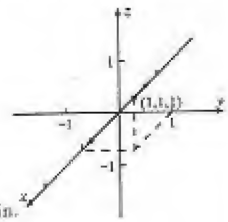
37. The graph of $x = 3$ in R is a point; in R^2 it is a straight line; in R^3 it is a plane.

38. The graph of $x = 6$ and $y = 3$ in R^2 is a point; in R^3 it is a line.

Exercises 39–48, describe in words the set in R^3 satisfying the equation or pair of equations. Sketch the graph.

39. The graph of $\begin{cases} y = 0 \\ z = 0 \end{cases}$ is the x axis.

40. $\begin{cases} x = z \\ y = z \end{cases}$



▸ Because

$x = y = z$

these are symmetric equations of the line that contains the origin and representations of the vector $\langle 1,1,1\rangle$. The figure shows the line.

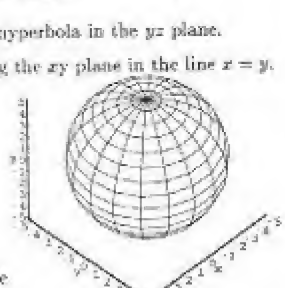
41. The graph of $\begin{cases} x^2+z^2 = 4 \\ y = 0 \end{cases}$ is a circle in the xz plane of radius 2 centered at the origin.

42. $y^2 - z^2 = 0$ is a cylinder with rulings parallel to the x axis. Its generator is a hyperbola in the yz plane.

43. The graph of $x = y$ is the plane perpendicular to the xy plane and intersecting the xy plane in the line $x = y$.

44. $x^2+y^2+z^2 = 25$

▸ The graph is a sphere with center at the origin and radius 5. See the figure.



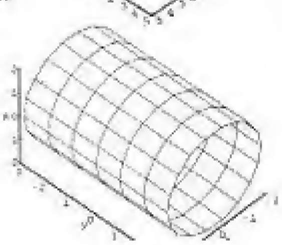
45. The graph of $x^2+y^2 = 9z$ is the paraboloid of revolution generated by revolving $x^2 = 9z$ or $y^2 = 9z$ about the z axis.

46. $x^2+y^2 = z^2$ is a right-circular cone whose axis is the z axis.

47. The graph of $x^2-y^2 = z^2$; $x^2 = y^2+z^2$ is the right circular cone generated by revolving about the x axis the lines $x = y$ or $x = -y$ in the xy plane or the lines $x = z$ or $x = -z$ in the xz plane.

48. $x^2+z^2 = 4$

▸ Because there is no term containing y, the graph is a cylinder with rulings parallel to the y axis. The directrix is a circle in the xz plane with center at the origin and radius 2. See the figure.



49. $c = \sqrt{50^2+70^2-2\cdot 50\cdot 70\cos 120^\circ} = 104.40$

(b) $\sin\alpha = 70\sin 120^\circ/104.4 = .5807$, $\alpha = 35.50^\circ$

50. $\cos\theta = \dfrac{168^2-112^2-136^2}{2\cdot 112\cdot 136} = 95.3^\circ$

51. If W ft-lb is the work, $W = \langle 30\cos\frac{3}{4}\pi, 30\sin\frac{3}{4}\pi\rangle \cdot \langle -2-3, 7-6\rangle = -150\cos\frac{3}{4}\pi + 30\sin\frac{3}{4}\pi$
$= -150(-\frac{1}{2}\sqrt{2}) + 30(\frac{1}{2}\sqrt{2}) = 90\sqrt{2} \approx 127.28$

52. The compass heading of an airplane is 107° and its airspeed is 210 mi/hr. If a wind is blowing from the west at 36 mi/hr, what are (a) the plane's ground speed and (b) its course?

▸ Let $\overrightarrow{OB} = \langle b_1, b_2\rangle$ be the plane's air velocity. Because the compass heading is 107°, the direction of $\overrightarrow{OB}$ is $90° - 107° = -17°$. Thus
$b_1 = 210\cos(-17°) = 200.82 \quad b_2 = 210\sin(-17°) = -61.40$
The wind vector is $\langle 36, 0\rangle$. Thus the plane's ground velocity is $\mathbf{V} = \langle 236.82, -61.40\rangle$
(a) Because
$\|\mathbf{V}\| = \sqrt{236.82^2 + 61.40^2} = 244.65$
the plane's ground speed is about 245 mi/hr
(b) Because
$90° - \tan^{-1}(-61.40/236.82) = 104.53°$
the plane's course is about 105°.

53. $P(-3, y, 1)$ is on the line through $(-3, 5, 1)$ perpendicular to the xz plane. If $A = (-2, 0, 0)$, $\|\overline{PA}\| = 13$;
$\sqrt{(-3+2)^2 + (y-0)^2 + (1-0)^2} = 13$; $1 + y^2 + 1 = 169$; $y^2 = 167$; $y = \pm\sqrt{167}$
Thus, the required points are $(-3, \sqrt{167}, 1)$ and $(-3, -\sqrt{167}, 1)$.

54. Sphere whose diameter has endpoints $(3, 5, -4)$ $(-1, 7, 4)$: $(x-3)(x+1) + (y-5)(y-7) + (z+4)(z-4) = 0$.

55. The given sphere has the equation $x^2 + y^2 + z^2 + 4x + 2y - 6z + 10 = 0$, that is
$(x^2 + 4x + 4) + (y^2 + 2y + 1) + (z^2 - 6z + 9) = -10 + 4 + 1 + 9$; $(x+2)^2 + (y+1)^2 + (z-3)^2 = 4$
Thus the required sphere S is $(x+2)^2 + (y+1)^2 + (z-3)^2 = r^2$. Because $(-4, 2, 5)$ is on S,
$(-4+2)^2 + (2+1)^2 + (5-3)^2 = r^2$; $r^2 = 17$. An equation is $(x+2)^2 + (y+1)^2 + (z-3)^2 = 17$.

56. Prove that the points $P(4, 1, -1)$, $Q(2, 0, 1)$, $R(4, 3, 0)$ are the vertices of a right triangle and find the area of the triangle.

▸ $$\mathbf{V}(\overrightarrow{PQ}) = \langle 2, 0, 1\rangle - \langle 4, 1, -1\rangle = \langle -2, -1, 2\rangle$$
$$\mathbf{V}(\overrightarrow{PR}) = \langle 4, 3, 0\rangle - \langle 4, 1, -1\rangle = \langle 0, 2, 1\rangle$$
Thus,
$$\mathbf{V}(\overrightarrow{PQ}) \cdot \mathbf{V}(\overrightarrow{PR}) = \langle -2, -1, 2\rangle \cdot \langle 0, 2, 1\rangle = 0$$
Thus vector $\mathbf{V}(\overrightarrow{PQ})$ is orthogonal to vector $\mathbf{V}(\overrightarrow{PR})$, and hence the triangle has a right angle at P. Also,
$$\left\|\mathbf{V}(\overrightarrow{PQ})\right\| = \|\langle -2, -1, 2\rangle\| = \sqrt{2^2 + 1^2 + 2^2} = 3$$
$$\left\|\mathbf{V}(\overrightarrow{PR})\right\| = \|\langle 0, 2, 1\rangle\| = \sqrt{5}$$
Applying the formula $\frac{1}{2}bh$ for the area of a triangle, we find the area to be $\frac{1}{2}(3)\sqrt{5} = \frac{3}{2}\sqrt{5}$ square units.

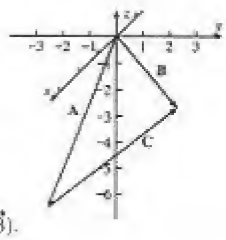
57. The surface $x^2 + z^2 = e^{4y}$ has the y axis as its axis of revolution and a generating curve is either $x^2 = e^{4y}$ or $x = e^{2y}$ or $x = -e^{2y}$ in the xy plane or $z^2 = e^{4y}$ or $z = e^{2y}$ or $z = -e^{2y}$ in the yz plane.

58. Revolve $9x^2 + 4z^2 = 36$ about x axis: replace z^2 with $y^2 + z^2$ to get $9x^2 + 4y^2 + 4z^2 = 36$.

59. If $\langle 3, c, -3\rangle$ and $\langle 5, -4, 1\rangle$ are orthogonal then $\langle 3, c, -3\rangle \cdot \langle 5, -4, 1\rangle = 0$; $15 - 4c - 3 = 0$; $c = 3$.

60. Show that there are representations of the three vectors $\mathbf{A} = 5\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, and $\mathbf{C} = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ which form a triangle.

▸ Representations of the vectors form a triangle if their sum is the zero vector or if one of the vectors is the sum of the other two. Because
$$\mathbf{A} + \mathbf{C} = (5\mathbf{i} + \mathbf{j} - 3\mathbf{k}) + (-4\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} = \mathbf{B}$$
there are representations of the vectors that form a triangle. See the figure.



61. $A = (5, 9, -3)$, $B = (-2, 4, -5)$. $\mathbf{V}(\overrightarrow{AB}) = -7\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$; $\left\|\mathbf{V}(\overrightarrow{AB})\right\| = \sqrt{49 + 25 + 4}$
$= \sqrt{78}$. (a) $\cos\alpha = -\frac{7}{\sqrt{78}}$, $\cos\beta = -\frac{5}{\sqrt{78}}$, $\cos\gamma = -\frac{2}{\sqrt{78}}$
(b) $-\frac{7}{\sqrt{78}}\mathbf{i} - \frac{5}{\sqrt{78}}\mathbf{j} - \frac{2}{\sqrt{78}}\mathbf{k}$ is the unit vector having the same direction as $\mathbf{V}(\overrightarrow{AB})$.

In Exercises 31–36, there is only one way that a meaningful expression can be obtained by inserting parentheses. Insert the parentheses and find the indicated vector or scalar if $\mathbf{A} = \langle 3, -2, 4\rangle$, $\mathbf{B} = \langle -5, 7, 2\rangle$, $\mathbf{C} = \langle 4, 6, -1\rangle$.

31. $\mathbf{AB}\cdot\mathbf{C} = \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) = \langle 3,-2,4\rangle(\langle -5,7,2\rangle\cdot\langle 4,6,-1\rangle) = \langle 3,-2,4\rangle(-20+42-2) = \langle 60,-40,80\rangle$

32. $\mathbf{A}\cdot\mathbf{BC}$

▸ Because $\mathbf{B}$ and $\mathbf{C}$ are vectors, $\mathbf{BC}$ is not defined. On the other hand, $\mathbf{A}\cdot\mathbf{B}$ is a scalar whose product with the vector $\mathbf{C}$ is defined. Thus, we take

$$(\mathbf{A}\cdot\mathbf{B})\mathbf{C} = (\langle 3,-2,4\rangle\cdot\langle -5,7,2\rangle)\mathbf{C} = -21\langle 4,6,1\rangle = \langle -84,-126,21\rangle$$

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34. $\mathbf{B}\cdot\mathbf{A}-\mathbf{C} = \mathbf{B}\cdot(\mathbf{A}-\mathbf{C}) = \langle -5,7,2\rangle\cdot(\langle 3,-2,4\rangle - \langle 4,6,-1\rangle) = \langle -5,7,2\rangle\cdot\langle -1,-8,5\rangle = -41$

35. $\mathbf{A}\times\mathbf{B}\cdot\mathbf{A}+\mathbf{B}-\mathbf{C} = (\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}-\mathbf{C}) = (\langle 3,-2,4\rangle\times\langle -5,7,2\rangle)\cdot(\langle 3,-2,4\rangle + \langle -5,7,2\rangle - \langle 4,6,-1\rangle)$
$= \langle -32,-26,11\rangle\cdot\langle -6,-1,7\rangle = 192+26+77 = 295$

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▸ Both $\mathbf{A}\times\mathbf{B}$ and $\mathbf{C}\times\mathbf{A}$ are vectors, and the dot product of two vectors is defined. Thus, we have

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ -5 & 7 & 2 \end{vmatrix} = -32\mathbf{i} - 26\mathbf{j} + 11\mathbf{k} \qquad \mathbf{C}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & -1 \\ 3 & -2 & 4 \end{vmatrix} = 22\mathbf{i} - 19\mathbf{j} - 26\mathbf{k}$$

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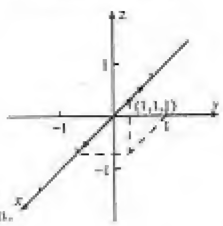

40. $\begin{cases} x = z \\ y = z \end{cases}$

▸ Because
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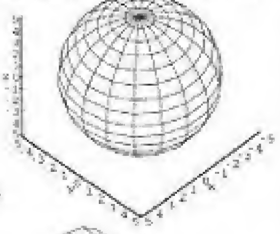

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▸ The graph is a sphere with center at the origin and radius 5. See the figure.

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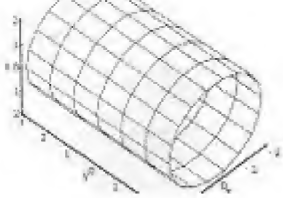

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51. If W ft-lb is the work, $W = \langle 30\cos\frac{3}{4}\pi, 30\sin\frac{3}{4}\pi\rangle \cdot \langle -2-3, 7-6\rangle = -150\cos\frac{3}{4}\pi + 30\sin\frac{3}{4}\pi$
$= -150(-\frac{1}{2}\sqrt{2}) + 30(\frac{1}{2}\sqrt{2}) = 90\sqrt{2} \approx 127.28$

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▸ Let $\overrightarrow{OB} = \langle b_1, b_2\rangle$ be the plane's air velocity. Because the compass heading is 107°, the direction of $\overrightarrow{OB}$ is $90° - 107° = -17°$. Thus

$b_1 = 210\cos(-17°) = 200.82 \quad b_2 = 210\sin(-17°) = -61.40$

The wind vector is $\langle 36, 0\rangle$. Thus the plane's ground velocity is $\mathbf{V} = \langle 236.82, -61.40\rangle$

(a) Because

$\|\mathbf{V}\| = \sqrt{236.82^2 + 61.40^2} = 244.65$

the plane's ground speed is about 245 mi/hr

(b) Because

$90° - \tan^{-1}(-61.40/236.82) = 104.53°$

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53. $P(-3, y, 1)$ is on the line through $(-3, 5, 1)$ perpendicular to the xz plane. If $A = (-2, 0, 0)$, $\|\overline{PA}\| = 13$;
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55. The given sphere has the equation $x^2 + y^2 + z^2 + 4x + 2y - 6z + 10 = 0$, that is
$(x^2 + 4x + 4) + (y^2 + 2y + 1) + (z^2 - 6z + 9) = -10 + 4 + 1 + 9$; $(x+2)^2 + (y+1)^2 + (z-3)^2 = 4$
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$(-4+2)^2 + (2+1)^2 + (5-3)^2 = r^2$; $r^2 = 17$. An equation is $(x+2)^2 + (y+1)^2 + (z-3)^2 = 17$.

56. Prove that the points $P(4, 1, -1)$, $Q(2, 0, 1)$, $R(4, 3, 0)$ are the vertices of a right triangle and find the area of the triangle.

▸
$$\mathbf{V}(\overrightarrow{PQ}) = \langle 2, 0, 1\rangle - \langle 4, 1, -1\rangle = \langle -2, -1, 2\rangle$$
$$\mathbf{V}(\overrightarrow{PR}) = \langle 4, 3, 0\rangle - \langle 4, 1, -1\rangle = \langle 0, 2, 1\rangle$$

Thus,
$$\mathbf{V}(\overrightarrow{PQ}) \cdot \mathbf{V}(\overrightarrow{PR}) = \langle -2, -1, 2\rangle \cdot \langle 0, 2, 1\rangle = 0$$

Thus vector $\mathbf{V}(\overrightarrow{PQ})$ is orthogonal to vector $\mathbf{V}(\overrightarrow{PR})$, and hence the triangle has a right angle at P. Also,
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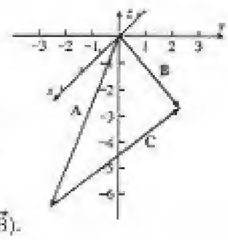
Applying the formula $\frac{1}{2}bh$ for the area of a triangle, we find the area to be $\frac{1}{2}(3)\sqrt{5} = \frac{3}{2}\sqrt{5}$ square units.

57. The surface $x^2 + z^2 = e^{4y}$ has the y axis as its axis of revolution and a generating curve is either $x^2 = e^{4y}$ or $x = e^{2y}$ or $x = -e^{2y}$ in the xy plane or $z^2 = e^{4y}$ or $z = e^{2y}$ or $z = -e^{2y}$ in the yz plane.

58. Revolve $9x^2 + 4z^2 = 36$ about x axis: replace z^2 with $y^2 + z^2$ to get $9x^2 + 4y^2 + 4z^2 = 36$.

59. If $\langle 3, c, -3\rangle$ and $\langle 5, -4, 1\rangle$ are orthogonal then $\langle 3, c, -3\rangle \cdot \langle 5, -4, 1\rangle = 0$; $15 - 4c - 3 = 0$; $c = 3$.

60. Show that there are representations of the three vectors $\mathbf{A} = 5\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{B} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, and $\mathbf{C} = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ which form a triangle.



▸ Representations of the vectors form a triangle if their sum is the zero vector or if one of the vectors is the sum of the other two. Because
$$\mathbf{A} + \mathbf{C} = (5\mathbf{i} + \mathbf{j} - 3\mathbf{k}) + (-4\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} = \mathbf{B}$$
there are representations of the vectors that form a triangle. See the figure.

61. $A = (5, 9, -3)$, $B = (-2, 4, -5)$. $\mathbf{V}(\overrightarrow{AB}) = -7\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$; $\left\|\mathbf{V}(\overrightarrow{AB})\right\| = \sqrt{49 + 25 + 4}$
$= \sqrt{78}$. (a) $\cos\alpha = -\frac{7}{\sqrt{78}}$, $\cos\beta = -\frac{5}{\sqrt{78}}$, $\cos\gamma = -\frac{2}{\sqrt{78}}$
(b) $-\frac{7}{\sqrt{78}}\mathbf{i} - \frac{5}{\sqrt{78}}\mathbf{j} - \frac{2}{\sqrt{78}}\mathbf{k}$ is the unit vector having the same direction as $\mathbf{V}(\overrightarrow{AB})$.

62. $a(\mathbf{i}+\mathbf{j}-\mathbf{k})+b(2\mathbf{i}-\mathbf{j}+\mathbf{k})+c(3\mathbf{i}-2\mathbf{j}+4\mathbf{k})=5\mathbf{i}+6\mathbf{j}-8\mathbf{k}$

$$\begin{aligned} \text{I:}&\quad a+2b+3c=5\\ \text{J:}&\quad a-b-2c=6\\ \text{K:}&\quad -a+b+4c=-8 \end{aligned}$$

$$\begin{aligned} \text{L}=\text{I}-\text{J:}&\quad 3b+5c=-1\\ \text{M}=\text{I}+\text{K:}&\quad 3b+7c=-3\\ \text{M}-\text{L:}&\quad 2c=-2 \end{aligned}$$

Then $c=-1$, $b=\frac{4}{3}$, $a=\frac{16}{3}$

63. $\mathbf{A}=\langle 7,-1,5\rangle$, $\mathbf{B}=\langle -2,3,1\rangle$. $\mathbf{A}\cdot\mathbf{B}=-14-3+5=-12$; $\|\mathbf{A}\|=\sqrt{49+1+25}=\sqrt{75}=5\sqrt{3}$.

(a) The scalar projection of $\mathbf{B}$ onto $\mathbf{A}$ is $\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|}=\dfrac{-12}{5\sqrt{3}}=-\frac{4}{5}\sqrt{3}$.

(b) The vector projection of $\mathbf{B}$ onto $\mathbf{A}$ is $\dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|^2}\mathbf{A}=\frac{-12}{75}\langle 7,-1,5\rangle=\langle -\frac{28}{25},\frac{4}{25},-\frac{4}{5}\rangle$.

64. Find an equation of the plane containing points $(1,7,-3)$ and $(3,1,2)$ and which does not intersect the x axis.

▸ Let $P=(1,7,-3)$ and $Q=(3,1,2)$. We have the vector

$\mathbf{V}(\overrightarrow{PQ})=2\mathbf{i}-6\mathbf{j}+5\mathbf{k}$

whose representations are parallel to the plane. Because the plane is parallel to the x axis, representations of the unit vector $\mathbf{i}$ are parallel to the plane. Therefore $\mathbf{i}\times\mathbf{V}(\overrightarrow{PQ})$ is normal to the plane. We take

$$\mathbf{N}=\mathbf{i}\times\mathbf{V}(\overrightarrow{PQ})=\mathbf{i}\times(2\mathbf{i}-6\mathbf{j}+5\mathbf{k})=-5\mathbf{j}-6\mathbf{k}$$

Applying Theorem 10.4.2 with the normal vector $\mathbf{N}$ and the point P, we obtain

$$\begin{aligned} 0(x-1)-5(y-7)-6(z+3)&=0\\ 5y+6z-17&=0 \end{aligned}$$

65. Let L be the plane through the points $P(-1,2,1)$, $Q(1,4,0)$, $R(1,-1,3)$. (a) $\mathbf{V}(\overrightarrow{PQ})=\langle 2,2,-1\rangle$ and $\mathbf{V}(\overrightarrow{PR})=\langle 2,-3,2\rangle$. A normal vector to L is $\mathbf{V}(\overrightarrow{PQ})\times\mathbf{V}(\overrightarrow{PR})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 2&2&-1\\ 2&-3&2\end{vmatrix}=\mathbf{i}-6\mathbf{j}-10\mathbf{k}$

L contains P; so an equation is $1(x+1)-6(y-2)-10(z-1)=0$; $x-6y-10z+23=0$.

(b) Because L contains P, the equation has the form $a(x+1)+b(y-2)+c(z-1)=0$.

Because Q and R lie on L, $\begin{cases}2a+2b+c=0\\ 2a-3b-2c=0\end{cases}$; so $a=-\frac{1}{10}c$, $b=\frac{3}{5}c$. Hence an equation is

$-\frac{1}{10}c(x+1)+\frac{3}{5}c(y-2)+c(z+1)=0$; $(x+1)-6(y-2)-10(z+1)=0$; $x-6y-10z+23=0$

66. $\mathbf{N}=\langle 3,2,1\rangle-\langle -3,0,4\rangle=\langle 6,2,-3\rangle$. $6(x-7)+2(y+2)-3(z+5)=0$; $6x+2y-3z-53=0$

67. $\mathbf{N}=\langle 5,-3,4\rangle$ is a normal vector to the plane and it contains the point $P(-6,3,-2)$. If d units is the distance from the origin O to the plane, then $d=\dfrac{\mathbf{N}\cdot\mathbf{V}(\overrightarrow{OP})}{\|\mathbf{N}\|}=\dfrac{\langle 5,-3,4\rangle\cdot\langle -6,3,-2\rangle}{\sqrt{25+9+16}}=\dfrac{-30-9-8}{5\sqrt{2}}=\frac{47}{10}\sqrt{2}$

68. Find two unit vectors orthogonal to $\mathbf{i}-3\mathbf{j}+4\mathbf{k}$ and whose representations are parallel to the yz plane.

▸ If the representations of a vector are parallel to the yz plane, then the vector is orthogonal to the unit vector $\mathbf{i}$. Thus we want unit vectors that are orthogonal to both $\mathbf{i}$ and the given vector $\mathbf{i}-3\mathbf{j}+4\mathbf{k}$. We find the cross product

$\mathbf{N}=\mathbf{i}\times(\mathbf{i}-3\mathbf{j}+4\mathbf{k})=-4\mathbf{j}-3\mathbf{k}$

Then the unit vectors are

$\mathbf{U}=\pm\dfrac{\mathbf{N}}{\|\mathbf{N}\|}=\pm\dfrac{-4\mathbf{j}-3\mathbf{k}}{\sqrt{4^2+3^2}}=\pm(\frac{4}{5}\mathbf{j}+\frac{3}{5})$

69. $A=(2,2,1)$, $B=(4,3,-1)$, $P=(4,6,-4)$. $\mathbf{V}(\overrightarrow{AP})=\langle 2,4,-5\rangle$, $\mathbf{V}(\overrightarrow{AB})=\langle 2,1,-2\rangle$, $\left\|\mathbf{V}(\overrightarrow{AB})\right\|=\sqrt{4+1+4}=3$. If d units is the distance from P to line AB and θ is the angle between lines AB and AP, then

$d=|AP|\sin\theta=\dfrac{\left\|\mathbf{V}(\overrightarrow{AP})\times\mathbf{V}(\overrightarrow{AB})\right\|}{\left\|\mathbf{V}(\overrightarrow{AB})\right\|}=\frac{1}{3}\left\|\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ 2&4&-5\\ 2&1&-2\end{vmatrix}\right\|=\frac{1}{3}\|-3\mathbf{i}-6\mathbf{j}-6\mathbf{k}\|=\|-\mathbf{i}-2\mathbf{j}-2\mathbf{k}\|=\sqrt{1+4+4}=3$

70. Distance from $9x-2y+6z+44=0$ to $(-3,2,0)$ is $\dfrac{|9(-3)-2(2)+6(0)+44|}{\sqrt{81+4+36}}=\frac{13}{11}$

71. $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{B} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$. $\|\mathbf{A}\| = \sqrt{4+1+1} = \sqrt{6}$, $\|\mathbf{B}\| = \sqrt{16+9+5} = \sqrt{50} = 5\sqrt{2}$.

(a) If θ is the angle between $\mathbf{A}$ and $\mathbf{B}$, $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{\|\mathbf{A}\|\|\mathbf{B}\|} = \dfrac{\langle 2,1,1\rangle \cdot \langle 4,-3,5\rangle}{\sqrt{6}\cdot 5\sqrt{2}} = \dfrac{8-3+5}{10\sqrt{3}} = \frac{1}{3}\sqrt{3}$.

(b) $\|\mathbf{A}\times\mathbf{B}\| = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 4 & -3 & 5 \end{vmatrix} \right\| = \|8\mathbf{i} - 6\mathbf{j} - 10\mathbf{k}\| = \sqrt{64+36+100} = \sqrt{200} = 10\sqrt{2}$

$\sin\theta = \dfrac{\|\mathbf{A}\times\mathbf{B}\|}{\|\mathbf{A}\|\|\mathbf{B}\|} = \dfrac{10\sqrt{2}}{\sqrt{6}\cdot 5\sqrt{2}} = \dfrac{2}{\sqrt{6}}$; $\cos\theta = \pm\sqrt{1-\sin\theta} = \pm\sqrt{1-\frac{2}{3}} = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{3}\sqrt{3}$. The sign is unknown.

72. Prove that the lines

$\dfrac{x-1}{1} = \dfrac{y+2}{2} = \dfrac{z-2}{2}$ and $\dfrac{x-2}{2} = \dfrac{y-5}{3} = \dfrac{z-5}{1}$

are skew lines and find the distance between them.

▷ Let L_1 be the first line and L_2 be the second line. Because the direction vector $\langle 1,2,2\rangle$ for L_1 is not a multiple of the direction vector $\langle 2,3,1\rangle$ for L_2, the lines are not parallel, and when the distance between then is shown to be positive, this proves the lines are skew. We have point $P_1(1,-2,2)$ in line L_1 and point $P_2(2,5,5)$ in line L_2. The distance between the lines is the absolute value of the scalar projection of vector $\mathbf{V}(\overrightarrow{P_1P_2})$ on a vector $\mathbf{N}$ whose representations are perpendicular to both line L_1 and line L_2. We have

$\mathbf{V}(\overrightarrow{P_1P_2}) = \mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$

We take for $\mathbf{N}$ the cross product of the direction vectors for lines L_1 and L_2. Thus, we take

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

Then the distance between the lines is given by

$$\frac{\left|\mathbf{N}\cdot\mathbf{V}(\overrightarrow{P_1P_2})\right|}{\|\mathbf{N}\|} = \frac{|(-4\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+7\mathbf{j}+3\mathbf{k})|}{\sqrt{4^2+3^2+1^2}} = \frac{14}{\sqrt{26}}$$

73. The lines $\dfrac{x-1}{1} = \dfrac{y+2}{2} = \dfrac{z-2}{2}$ and $\dfrac{x-2}{2} = \dfrac{y-5}{3} = \dfrac{z-5}{1}$ have directions $\mathbf{A} = \langle 1,2,2\rangle$ and $\langle 2,3,1\rangle$.

The direction of a line L perpendicular to these is $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 2 & 3 & 1 \end{vmatrix} = \langle -4,3,-1\rangle$.

Because L contains the origin, symmetric and parametric equations are

$\dfrac{x}{-4} = \dfrac{y}{3} = \dfrac{z}{-1}$ and $x = -4t$, $y = 3t$, $z = -t$

74. $P(-3,5,2)$, $Q(-1,-3,4)$. $\overrightarrow{PQ} = \langle 2,-8,2\rangle = 2\langle 1,-4,1\rangle$. $\dfrac{x+3}{1} = \dfrac{y-5}{-4} = \dfrac{z-2}{1}$; $x = -3+t$, $y = 5-4t$, $z = 2+t$

75. The given lines are L: $\dfrac{x-2}{3} = \dfrac{y+3}{-1} = \dfrac{z-5}{4}$ and M: $\dfrac{x+1}{-6} = \dfrac{y+2}{2} = \dfrac{z-1}{-8}$. Because the point $(2,-3,-5)$ of L lies on M and the point $(-1,-2,1)$ of M lies on L, the lines coincide.

76. Find an equation of the plane containing the line $\frac{1}{2}(x-3) = -(y+5) = -\frac{1}{3}(z+2)$ and the point $(5,0,-4)$.

▷ Because parametric equations of the given line are

$x = 3+2t \quad y = -5-t \quad z = -2-3t$

with $t = 0$ and $t = 1$ we get the points $(3,-5,-2)$ and $(5,-6,-5)$ of the plane. Using these with the given point $(5,0,-4)$, we get the equation

$$\begin{vmatrix} x & y & z & 1 \\ 3 & -5 & -2 & 1 \\ 5 & -6 & -5 & 1 \\ 5 & 0 & -4 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} -5 & -2 & 1 \\ -6 & -5 & 1 \\ 0 & -4 & 1 \end{vmatrix} x - \begin{vmatrix} 3 & -2 & 1 \\ 5 & -5 & 1 \\ 5 & -4 & 1 \end{vmatrix} y + \begin{vmatrix} 3 & -5 & 1 \\ 5 & -6 & 1 \\ 5 & 0 & 1 \end{vmatrix} z - \begin{vmatrix} 3 & -5 & -2 \\ 5 & -6 & -5 \\ 5 & 0 & -4 \end{vmatrix} = 0$$

$17x - 2y + 12z - 37 = 0$

77. The cross section of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1$ in the plane $z=4$ is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}+\frac{16}{25}=1$;

$\frac{x^2}{4}+\frac{y^2}{9}=\frac{9}{25}$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a^2=\frac{36}{25}$ and $b^2=\frac{81}{25}$. The measure of the area is $\pi ab=\pi\left(\frac{6}{5}\right)\left(\frac{9}{5}\right)=\frac{54}{25}\pi$.

78. Area $=\left|(2\mathbf{j}-3\mathbf{k})\times(5\mathbf{i}+4\mathbf{k})\right|=\left|8\mathbf{i}-15\mathbf{j}-10\mathbf{k}\right|=\sqrt{64+225+100}=\sqrt{389}$ sq units

79. Let $P=(1,3,0)$, $Q=(2,-1,3)$, $R=(-2,2,-1)$, $S=(-1,1,2)$. $\mathbf{V}(\overrightarrow{PQ})=\langle 1,-4,3\rangle$, $\mathbf{V}(\overrightarrow{PR})=\langle -3,-1,-1\rangle$, $\mathbf{V}(\overrightarrow{PS})=\langle -2,-2,2\rangle$. The measure of the volume of the parallepiped is

$$\left|\mathbf{V}(\overrightarrow{PQ})\times\mathbf{V}(\overrightarrow{PR})\cdot\mathbf{V}(\overrightarrow{PS})\right|=\left\|\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-4&3\\-3&-1&-1\end{vmatrix}\cdot\langle -2,-2,2\rangle\right\|=\left|\langle 7,-8,-13\rangle\cdot\langle -2,-2,2\rangle\right|=\left|-14+16-26\right|=24$$

80. Prove by vector analysis that the diagonals of a parallelogram bisect each other.

▹ Let OABC be a parallelogram with $\mathbf{V}(\overrightarrow{OA})=\mathbf{V}(\overrightarrow{CB})=\mathbf{A}$ and $\mathbf{V}(\overrightarrow{OC})=\mathbf{V}(\overrightarrow{AB})=\mathbf{C}$. The position representations of the midpoints of diagonals OB and AC are

$$\tfrac{1}{2}\mathbf{V}(\overrightarrow{OB})=\tfrac{1}{2}[\mathbf{V}(\overrightarrow{OA})+\mathbf{V}(\overrightarrow{AB})]=\tfrac{1}{2}(\mathbf{A}+\mathbf{C})$$
$$\mathbf{V}(\overrightarrow{OA})+\tfrac{1}{2}\mathbf{V}(\overrightarrow{AC})=\mathbf{A}+\tfrac{1}{2}[\mathbf{V}(\overrightarrow{OC})-\mathbf{V}(\overrightarrow{AC})]=\mathbf{A}+\tfrac{1}{2}(\mathbf{C}-\mathbf{A})=\tfrac{1}{2}(\mathbf{A}+\mathbf{C})$$

Thus the two midpoint coincide; so the diagonals of a parallelogram bisect each other.

In Exercises 81 and 82, let $\mathbf{A}=\cos\alpha\mathbf{i}+\sin\alpha\mathbf{j}$ and $\mathbf{B}=\cos\beta\mathbf{i}+\sin\beta\mathbf{j}$. $\mathbf{A}$ and $\mathbf{B}$ are unit vectors.

81. $\cos(\alpha-\beta)=\cos(\mathbf{A},\mathbf{B})=\mathbf{A}\cdot\mathbf{B}=\langle\cos\alpha,\sin\alpha\rangle\cdot\langle\cos\beta,\sin\beta\rangle=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

82. $\left|\sin(\alpha-\beta)\right|=\left|\sin(\mathbf{A},\mathbf{B})\right|=\|\mathbf{A}\times\mathbf{B}\|=\|\langle\cos\alpha,\sin\alpha,0\rangle\times\langle\cos\beta,\sin\beta,0\rangle\|=\|\langle 0,0,\sin\alpha\cos\beta-\cos\alpha\sin\beta\rangle\|$
$=\left|\sin\alpha\cos\beta-\cos\alpha\sin\beta\right|$

83. If $\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ then $\mathbf{A}\cdot\mathbf{i}=a_1$, $\mathbf{A}\cdot\mathbf{j}=a_2$, $\mathbf{A}_3\cdot\mathbf{k}=a_3$ and so $\mathbf{A}=(\mathbf{A}\cdot\mathbf{i})\mathbf{i}+(\mathbf{A}\cdot\mathbf{j})\mathbf{j}+(\mathbf{A}\cdot\mathbf{k})\mathbf{k}$

84. Let $\mathbf{A}$ and $\mathbf{B}$ be vectors in V_3, c_1, c_2, and c_3 be direction cosines of $\mathbf{A}$, and d_1, d_2, and d_3 be direction cosines of $\mathbf{B}$. If

$$\frac{c_1}{d_1}=\frac{c_2}{d_2}=\frac{c_3}{d_3}$$

prove that $\mathbf{A}$ and $\mathbf{B}$ are parallel.

▹ Let

$$\frac{c_1}{d_1}=\frac{c_2}{d_2}=\frac{c_3}{d_3}=t$$

Then

$$c_1=td_1\quad c_2=td_2\quad c_3=td_3$$

and so

$$\langle c_1,c_2,c_3\rangle=t\langle d_1,d_2,d_3\rangle \tag{1}$$

Because c_1, c_2, and c_3 are direction cosines of $\mathbf{A}$ and d_1, d_2, and d_3 are direction cosines of $\mathbf{B}$ then

$$\mathbf{A}=\|\mathbf{A}\|\langle c_1,c_2,c_3\rangle \text{ and } \mathbf{B}=\|\mathbf{B}\|\langle d_1,d_2,d_3\rangle \tag{2}$$

From (1) and (2) it follows that $\mathbf{A}$ is a scalar multiple of $\mathbf{B}$ and hence they are parallel.

85. Lagrange's Identity. We use Theorems 10.5.6 and 10.5.7.

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=\mathbf{A}\cdot[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})]=\mathbf{A}\cdot[(\mathbf{B}\cdot\mathbf{D})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{D}]=(\mathbf{A}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})=\begin{vmatrix}\mathbf{A}\cdot\mathbf{C}&\mathbf{A}\cdot\mathbf{D}\\\mathbf{B}\cdot\mathbf{C}&\mathbf{B}\cdot\mathbf{D}\end{vmatrix}$$

86. Using position representations, the given implies

$$\mathbf{d}-\mathbf{a}=\tfrac{1}{3}(\mathbf{b}-\mathbf{a})\quad\mathbf{e}-\mathbf{b}=\tfrac{1}{3}(\mathbf{c}-\mathbf{b})\quad\mathbf{f}-\mathbf{c}=\tfrac{1}{3}(\mathbf{a}-\mathbf{c})$$

or, equivalently,

$$\mathbf{d}=\tfrac{2}{3}\mathbf{a}+\tfrac{1}{3}\mathbf{b}\quad\mathbf{e}=\tfrac{2}{3}\mathbf{b}+\tfrac{1}{3}\mathbf{c}\quad\mathbf{f}=\tfrac{1}{3}\mathbf{a}+\tfrac{2}{3}\mathbf{c}$$

Therefore,

$$\overrightarrow{AE}+\overrightarrow{BF}+\overrightarrow{CD}=(\mathbf{e}-\mathbf{a})+(\mathbf{f}-\mathbf{b})+(\mathbf{d}-\mathbf{c})=(-\mathbf{a}+\tfrac{2}{3}\mathbf{b}+\tfrac{1}{3}\mathbf{c})+(\tfrac{1}{3}\mathbf{a}-\mathbf{b}+\tfrac{2}{3}\mathbf{c})+(\tfrac{2}{3}\mathbf{a}+\tfrac{1}{3}\mathbf{b}-\mathbf{c})=\mathbf{0}$$

E L E V E N

VECTOR-VALUED FUNCTIONS

11.1 VECTOR-VALUED FUNCTIONS AND CURVES IN R^3

11.1.1 Definition Let f, g, and h be real-valued functions of a real variable t. Then there is a *vector-valued function* $\mathbf{R}$, defined by

$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

where t is any number in the domain common to f, g, and h. If $h \equiv 0$, we are in the plane. Operations on vector-valued functions are defined in the obvious way.

11.1.3 Definition Let $\mathbf{R}$ be a vector-valued function whose function values are given by

$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

Then the *limit of* $\mathbf{R}(t)$ *as t approaches a* is defined by

$\lim\limits_{t\to a}\mathbf{R}(t) = [\lim\limits_{t\to a}f(t)]\mathbf{i} + [\lim\limits_{t\to a}g(t)]\mathbf{j} + [\lim\limits_{t\to a}h(t)]\mathbf{k}$

if $\lim\limits_{t\to a}f(t)$, $\lim\limits_{t\to a}g(t)$, $\lim\limits_{t\to a}h(t)$ all exist.

11.1.4 Definition The vector-valued function $\mathbf{R}$ is *continuous* at the number a if and only if the following three conditions are satisfied:

(i) $\mathbf{R}(a)$ exists;

(ii) $\lim\limits_{t\to a}\mathbf{R}(t)$ exists;

(iii) $\lim\limits_{t\to a}\mathbf{R}(t) = \mathbf{R}(a)$

Definition A *curve* is the graph of a continuous vector-valued function.

Exercises 11.1

In Exercises 1–8, find the domain of the vector-valued function.

1. $\text{Dom}[(1/t)\mathbf{i} + \sqrt{4-t}\mathbf{j}] = \text{Dom}(1/t) \cap \text{Dom}\sqrt{4-t} = \{t \neq 0\} \cap \{t \leq 4\} = (-\infty, 0) \cup (0, 4]$

2. $\text{Dom}[(t^2+3)\mathbf{i} + (t-1)^{-1}\mathbf{j}] = \text{Dom}(t^2+3) \cap \text{Dom}(t-1)^{-1} = R \cap \{t \neq 1\} = \{t \neq 1\}$

3. $\text{Dom}[\sin^{-1}t\mathbf{i} + \ln(t+1)\mathbf{j}] = \text{Dom}(\sin^{-1}t) \cap \text{Dom}[\ln(t+1)] = [-1,1] \cap \{t > -1\} = (-1, 1]$

4. $\mathbf{R}(t) = (\cos^{-1}t)\mathbf{i} + (\sec^{-1}t)\mathbf{j}$

▸ Because

$\text{Dom}(\cos^{-1}t) = [-1, 1]$

and

$\text{Dom}(\sec^{-1}t) = (-\infty, -1] \cup [1, +\infty)$

then $\text{Dom}(\mathbf{R})$ consist of the two numbers -1 and 1.

5. $\text{Dom}[\sqrt{t+2}\mathbf{i} + \sqrt{4-t}\mathbf{j} + \cot t\mathbf{k}] = [-2, +\infty) \cap (-\infty, 4] \cap \{t \neq k\pi\} = [-2, 0) \cup (0, \pi) \cup (\pi, 4]$

6. $\text{Dom}[\sqrt{t^2-9}\mathbf{i} + \ln|t-3|\mathbf{j} + (t^2+2t-8)\mathbf{k}] = \{(-\infty, -3] \cup [3, +\infty)\} \cap \{t \neq 3\} \cap R = (-\infty, -3] \cup (3, +\infty)$

7. $\text{Dom}[\ln|\sin t|\mathbf{i} + \sqrt{16-t^2}\mathbf{j} + \ln|t+4|\mathbf{k}] = \{t \neq k\pi\} \cap [-4, 4] \cap \{t \neq -4\} = (-4, -\pi) \cup (-\pi, 0) \cup (0, \pi) \cup (\pi, 4]$

8. $\mathbf{R}(t) = \tan t\mathbf{i} + \sqrt{4-t^2}\mathbf{j} + [1/(2+t)]\mathbf{k}$

▸ We have

$\text{Dom}(\tan t) = \{t \neq (k + \frac{1}{2})\pi\}$ where k is any integer

and

$\text{Dom}(\sqrt{4-t^2}) = [-2, 2]$

and

$\text{Dom}\left(\frac{1}{2+t}\right) = \{t \neq -2\}$

Because $\frac{1}{2}\pi \approx 1.57$, then

$\text{Dom}(\mathbf{R}) = (-2, -\frac{1}{2}\pi) \cup (-\frac{1}{2}\pi, \frac{1}{2}\pi) \cup (\frac{1}{2}\pi, 2]$

In Exercises 9–12, find (a) $(\mathbf{F}+\mathbf{G})(t)$; (b) $(\mathbf{F}-\mathbf{G})(t)$; (c) $(\mathbf{F}\cdot\mathbf{G})(t)$; (d) $(\mathbf{F}\times\mathbf{G})(t)$
In Exercises 13–16, find: (a) $(f\mathbf{F})(t)$; (b) $(f\mathbf{G})(t)$; (c) $(\mathbf{F}\circ g)(t)$; (d) $(\mathbf{G}\circ g)(t)$

9. $\mathbf{F}=\langle t+1, t^2-1, t-1\rangle$, $\mathbf{G}=\langle t-1, 1, t+1\rangle$

▷ (a) $\mathbf{F}+\mathbf{G}=\langle 2t, t^2, 2t\rangle$ (b) $\mathbf{F}-\mathbf{G}=\langle 2, t^2-2, -2\rangle$
(c) $\mathbf{F}\cdot\mathbf{G}=(t^2-1)+(t^2-1)+(t^2-1)=3t^2-3$
(d) $\mathbf{F}\times\mathbf{G}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t+1 & t^2-1 & t-1\\ t-1 & 1 & t+1\end{vmatrix}$
$=(t^3+t^2-2t)\mathbf{i}-4t\mathbf{j}+(2t-t^3+t^2)\mathbf{k}$

10. $\mathbf{F}=\langle 4-t^2, 4, t^2-4\rangle$, $\mathbf{G}=\langle t^2, t^2-4, -4\rangle$

▷ (a) $\mathbf{F}+\mathbf{G}=\langle 4, t^2, t^2-8\rangle$ (b) $\mathbf{F}-\mathbf{G}=\langle 4-2t^2, 8-t^2, t^2\rangle$
(c) $\mathbf{F}\cdot\mathbf{G}=4t^2-t^4+4t^2-16-4t^2+16=4t^2-t^4$
(d) $\mathbf{F}\times\mathbf{G}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4-t^2 & 4 & t^2-4\\ t^2 & t^2-4 & -4\end{vmatrix}$
$=\langle -t^4+8t^2-32, t^4-8t^2+16, -t^4+4t^2-16\rangle$

13. $f(t)=t-1$; $g(t)=t+1$

▷ (a) $f(t)\mathbf{F}(t)=\langle t^2-1, t^3-t^2-t+1, t^2-2t+1\rangle$
(b) $f(t)\mathbf{G}(t)=\langle t^2-2t+1, t-1, t^2-1\rangle$
(c) $\mathbf{F}(g(t))=\langle t+2, t^2+2t, t\rangle$
(d) $\mathbf{G}(g(t))=\langle t, 1, t+2\rangle$

14. $f(t)=1/(2-t)$; $g(t)=2-t$

▷ (a) $f(t)\mathbf{F}(t)=\langle 2+t, \frac{4}{2-t}, -(t+2)\rangle$
(b) $f(t)\mathbf{G}(t)=\langle \frac{t^2}{2-t}, -(t+2), \frac{4}{t-2}\rangle$
(c) $\mathbf{F}(g(t))=\langle 4t-t^2, 4, t^2-4t\rangle$
(d) $\mathbf{G}(g(t))=\langle t^2-4t+4, t^2-4t, -4\rangle$

11. $\mathbf{F}=\cos t\mathbf{i}-\sin t\mathbf{j}+t\mathbf{k}$, $\mathbf{G}=\sin t\mathbf{i}+\cos t\mathbf{j}-t\mathbf{k}$

▷ (a) $\mathbf{F}+\mathbf{G}=(\cos t+\sin t)\mathbf{i}+(\cos t-\sin t)\mathbf{j}$
(b) $\mathbf{F}-\mathbf{G}=(\cos t-\sin t)\mathbf{i}-(\cos t+\sin t)\mathbf{j}+2t\mathbf{k}$
(c) $\mathbf{F}\cdot\mathbf{G}=\cos t\sin t-\sin t\cos t-t^2=-t^2$
(d) $\mathbf{F}\times\mathbf{G}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos t & -\sin t & t\\ \sin t & \cos t & -t\end{vmatrix}$
$=t(\sin t-\cos t)\mathbf{i}+t(\sin t+\cos t)\mathbf{j}+\mathbf{k}$

12. $\mathbf{F}=\sec t\mathbf{i}+\tan t\mathbf{j}-2\mathbf{k}$; $\mathbf{G}=\sec t\mathbf{i}-\tan t\mathbf{j}+t\mathbf{k}$

▷ (a) $\mathbf{F}+\mathbf{G}=2\sec t\mathbf{i}+(t-2)\mathbf{k}$
(b) $\mathbf{F}-\mathbf{G}=2\tan t\mathbf{i}-(t+2)\mathbf{k}$
(c) $\mathbf{F}\cdot\mathbf{G}=\sec^2 t-\tan^2 t-2t=-2t+1$
(d) $\mathbf{F}\times\mathbf{G}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sec t & \tan t & -2\\ \sec t & -\tan t & t\end{vmatrix}$
$=(t-2)\tan t\mathbf{i}-(t+2)\sec t\mathbf{j}-2\tan t\sec t\mathbf{k}$

15. $f(t)=\sin t$; $g(t)=\sin^{-1}t$

▷ (a) $f(t)\mathbf{F}(t)=\sin t\cos t\mathbf{i}-\sin^2 t\mathbf{j}+t\sin t\mathbf{k}$
(b) $f(t)\mathbf{G}(t)=\sin^2 t\mathbf{i}+\sin t\cos t\mathbf{j}-t\sin t\mathbf{k}$
(c) $\mathbf{F}(g(t))=\sqrt{1-t^2}\mathbf{i}-t\mathbf{j}+\sin^{-1}t\mathbf{k}$
(d) $\mathbf{G}(g(t))=t\mathbf{i}+\sqrt{1-t^2}\mathbf{j}-\sin^{-1}t\mathbf{k}$

16. $f(t)=\cos t$; $g(t)=\cos^{-1}t$

▷ (a) $f(t)\mathbf{F}(t)=\mathbf{i}+\sin t\mathbf{j}-2\cos t\mathbf{k}$, $t\neq(k+\frac{1}{2})\pi$
(b) $f(t)\mathbf{G}(t)=\mathbf{i}-\sin t\mathbf{j}+t\cos t\mathbf{k}$, $t\neq(k+\frac{1}{2})\pi$
(c) $\mathbf{F}(g(t))=\sec(\cos^{-1}t)\mathbf{i}+\tan(\cos^{-1}t)\mathbf{j}-2\mathbf{k}$
$=(1/t)\mathbf{i}+(\sqrt{1-t^2}/t)\mathbf{j}-2\mathbf{k}$
(d) $\mathbf{G}(g(t))=(1/t)\mathbf{i}-(\sqrt{1-t^2}/t)\mathbf{j}+\cos^{-1}t\mathbf{k}$

In Exercises 17–24, find the indicated limit, if it exists.

17. $\lim\limits_{t\to 2}\left[(t-2)\mathbf{i}+\frac{t^2-4}{t-2}\mathbf{j}+t\mathbf{k}\right]=0\mathbf{i}+\lim\limits_{t\to 2}(t+2)\mathbf{j}+2t\mathbf{k}=4\mathbf{j}+2\mathbf{k}$

18. $\lim\limits_{t\to -1}\left[\frac{t^2-1}{t+1}\mathbf{i}+\frac{t+1}{t-1}\mathbf{j}+|t+1|\mathbf{k}\right]=\lim\limits_{t\to -1}(t-1)\mathbf{i}+0\mathbf{j}+0\mathbf{k}=-2\mathbf{i}$

19. $\lim\limits_{t\to 0}\left[\sin t\mathbf{i}+\cos t\mathbf{j}+\frac{\sin t}{t}\mathbf{k}\right]=0\mathbf{i}+1\mathbf{j}+1\mathbf{k}=\mathbf{j}+\mathbf{k}$

20. $\mathbf{R}(t)=\frac{1-\cos t}{t}\mathbf{i}+e^t\mathbf{j}+e^{-t}\mathbf{k}$; $\lim\limits_{x\to 0}\mathbf{R}(t)$

▷ Because
$\lim\limits_{t\to 0}\frac{1-\cos t}{t}\overset{0/0}{=}\lim\limits_{t\to 0}\frac{\sin t}{1}=0$
$\lim\limits_{t\to 0}e^t=e^0=1$
$\lim\limits_{t\to 0}e^{-t}=e^0=1$
then
$\lim\limits_{t\to 0}\mathbf{R}(t)=0\mathbf{i}+1\mathbf{j}+1\mathbf{k}=\mathbf{j}+\mathbf{k}$

21. $\lim\limits_{t\to 1}\left[\frac{|t-2|}{t-2}\mathbf{i}+\frac{\sin\pi t}{t^2-1}\mathbf{j}+\frac{\tan\pi t}{t-1}\mathbf{k}\right]\overset{0/0}{=}-1\mathbf{i}+\lim\limits_{t\to 1}\frac{\pi\cos\pi t}{2t}\mathbf{j}+\lim\limits_{t\to 1}\frac{\pi\sec^2\pi t}{1}=-\mathbf{i}-\frac{1}{2}\pi\mathbf{j}+\pi\mathbf{k}$

22. $\lim\limits_{t\to 0}\left[\frac{1+\cos t}{1-\sin t}\mathbf{i}+\frac{1-\cos^2 t}{1-\cos t}\mathbf{j}+t\cdot\frac{t}{\sin t}\mathbf{k}\right]=2\mathbf{i}+\lim\limits_{t\to 0}(1+\cos t)\mathbf{j}+0\cdot 1\mathbf{k}=2\mathbf{i}+2\mathbf{j}$

23. $\lim\limits_{t\to 0}\left[e^{t+1}\mathbf{i}+e^{1-t}\mathbf{j}+(1+t)^{1/t}\mathbf{k}\right]=e(\mathbf{i}+\mathbf{j}+\mathbf{k})$

24. $\mathbf{R}(t) = \dfrac{\ln(t+1)}{t}\mathbf{i} + \sinh t\mathbf{j} + \cosh t\mathbf{k}$; $\lim\limits_{t\to 0}\mathbf{R}(t)$

▷ Because

$$\lim_{t\to 0}\frac{\ln(t+1)}{t} \overset{0/0}{=} \lim_{t\to 0}\frac{1/(t+1)}{1} = 1$$

then

$$\lim_{t\to 0}\mathbf{R}(t) = 1\mathbf{i} + \sinh 0\mathbf{j} + \cosh 0\mathbf{k} = \mathbf{i} + \mathbf{k}$$

In Exercises 25–30, determine the numbers at which the vector-valued function is continuous.

25. $t^2 + \ln(t-1)\mathbf{j} + \dfrac{1}{t-2}\mathbf{k}$

▷ $(1,2)\cup(2,+\infty)$

26. $(t-1)\mathbf{i} + \dfrac{1}{e^t-1}\mathbf{j} + \dfrac{|t-1|}{t-1}\mathbf{k}$

▷ $\{t \neq 0, 1\}$

27. $\cos t\mathbf{i} + \sec t\mathbf{j} + \tan t\mathbf{k}$

▷ $\{t \neq (k+\frac{1}{2})\pi$, k is any integer$\}$

28. $\mathbf{R}(t) = \sin \pi t\mathbf{i} - \tan \pi t\mathbf{j} + \cot \pi t\mathbf{k}$

▷ The tangent function is continuous except at the odd multiples of $\frac{1}{2}\pi$, the cotangent is continuous except at the even multiples of $\frac{1}{2}\pi$. Thus $\mathbf{R}$ is continuous everywhere except at $\frac{1}{2}k$, where k is any integer.

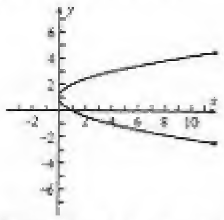
29. $\begin{cases} e^{1/t^2}\mathbf{i} + t^2\mathbf{j} + t\mathbf{k} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$ all real numbers

30. $\begin{cases} \frac{\sin t}{t}\mathbf{i} + \frac{1-\cos t}{t}\mathbf{j} + \frac{1-e^t}{t}\mathbf{k} & \text{if } t \neq 0 \\ \mathbf{i} - \mathbf{k} & \text{if } t = 0 \end{cases}$ all numbers

In Exercises 31–42, sketch the graph of the vector-valued function.

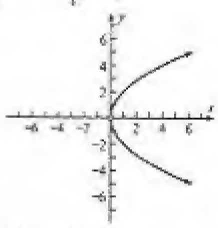
31. $\mathbf{R}(t) = t^2\mathbf{i} + (t+1)\mathbf{j}$

▷



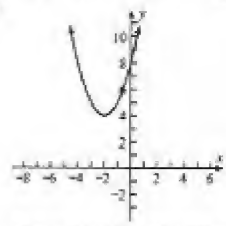
32. $\mathbf{R}(t) = \frac{4}{t^2}\mathbf{i} + \frac{4}{t}\mathbf{j}$

▷



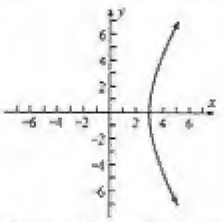
33. $\mathbf{R}(t) = (t-2)\mathbf{i} + (t^2+4)\mathbf{j}$

▷



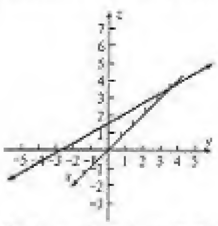
34. $\mathbf{R}(t) = 3\cosh t\mathbf{i} + 5\sinh t\mathbf{j}$

▷



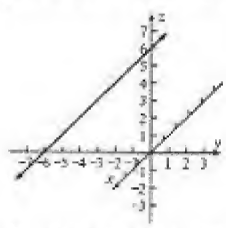
35. $\mathbf{R}(t) = t\mathbf{i} + (6-4t)\mathbf{j} + (5-2t)\mathbf{k}$

▷



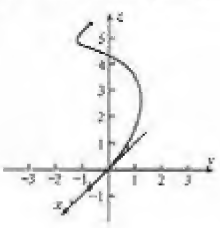
36. $\mathbf{R}(t) = (t+1)\mathbf{i} + (2t-3)\mathbf{j} + (2t+3)\mathbf{k}$

▷



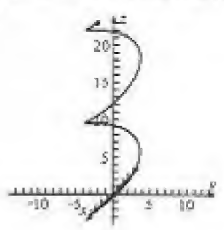
37. $\mathbf{R} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $[0, 2\pi]$

▷



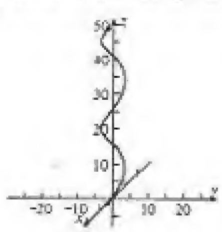
38. $\mathbf{R} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 2t\mathbf{k}$, $[0, 4\pi]$

▷

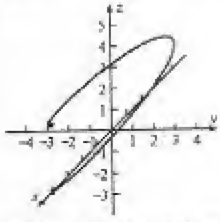


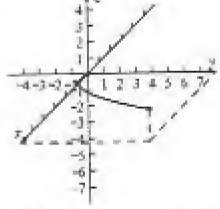
39. $\mathbf{R} = 2\cos t\mathbf{i} + 3\sin t\mathbf{j} + 4t\mathbf{k}$, $[0, 4\pi]$

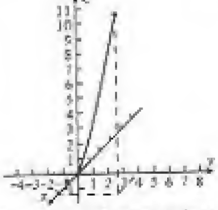
▷



40. $\mathbf{R}(t) = 4\cos t\mathbf{i} + \sin t\mathbf{j} + \frac{1}{2}t\mathbf{k}$, $0 \le t \le 2\pi$ **41.** $\mathbf{R}(t) = 3t\mathbf{i} + 2t^2\mathbf{j} + t\mathbf{k}$, $[0,2]$ **42.** $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}t^3\mathbf{k}$, $[0,2]$

▷ 

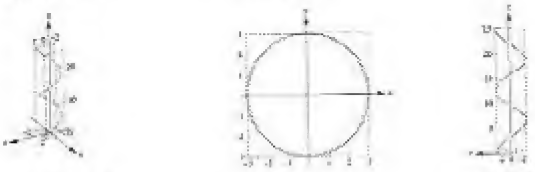
▷ 

▷ 

In Exercises 43–46, match each graph (a), (b), (c) of the same curve with one of the given viewing points.

43. Exercise 37 ▷ (a) $(0,8,0)$ (b) $(4,8,4)$ (c) $(0,0,8)$

44. Exercise 38



▷ (a) $(14,28,14)$ (b) $(0,0,28)$ (c) $(0,28,0)$

45. Exercise 41 ▷ (a) $(0,0,10)$ (b) $(-10,0,0)$ (c) $(10,0,0)$

46. Exercise 42 ▷ (a) $(-15,0,0)$ (b) $(0,0,15)$ (c) $(15,0,0)$

In Exercises 47–49, prove the limit theorem if $\mathbf{U}(t)$ and $\mathbf{V}(t)$ are functions such that $\lim\limits_{t\to a}\mathbf{U}(t)$ and $\lim\limits_{t\to a}\mathbf{V}(t)$ exist.

▷ Let $\mathbf{U}(t) = U_1(t)\mathbf{i} + U_2(t)\mathbf{j} + U_3(t)\mathbf{k}$ and $\mathbf{V}(t) = V_1(t)\mathbf{i} + V_2(t)\mathbf{j} + V_3(t)\mathbf{k}$.

47.
$$\begin{aligned}
\lim_{t\to a}[\mathbf{U}+\mathbf{V}] &= \lim_{t\to a}[(U_1+V_1)\mathbf{i} + (U_2+V_2)\mathbf{j} + (U_3+V_3)\mathbf{k}] && \text{(Definition 11.1.3)}\\
&= [\lim_{t\to a}(U_1+V_1)]\mathbf{i} + [\lim_{t\to a}(U_2+V_2)]\mathbf{j} + [\lim_{t\to a}(U_3+V_3)]\mathbf{k} && \text{(LT 4)}\\
&= [\lim_{t\to a}U_1 + \lim_{t\to a}V_1]\mathbf{i} + [\lim_{t\to a}U_2 + \lim_{t\to a}V_2] + [\lim_{t\to a}U_3 + \lim_{t\to a}V_3]\\
&= [(\lim_{t\to a}U_1)\mathbf{i} + (\lim_{t\to a}U_2)\mathbf{j} + (\lim_{t\to a}U_3)\mathbf{k}] + [(\lim_{t\to a}V_1)\mathbf{i} + (\lim_{t\to a}V_2)\mathbf{j} + (\lim_{t\to a}V_3)\mathbf{k}] && \text{(Definition 11.1.3)}\\
&= \lim_{t\to a}\mathbf{U} + \lim_{t\to a}\mathbf{V}
\end{aligned}$$

48. $\lim\limits_{t\to a}[\mathbf{U}(t)\cdot\mathbf{V}(t)] = \lim\limits_{t\to a}\mathbf{U}(t)\cdot\lim\limits_{t\to a}\mathbf{V}(t)$

▷
$$\begin{aligned}
\lim_{t\to a}[\mathbf{U}(t)\cdot\mathbf{V}(t)] &= \lim_{t\to a}[U_1(t)V_1(t) + U_2(t)V_2(t) + U_3(t)V_3(t)] && \text{(LT 4)}\\
&= \lim_{t\to a}[U_1(t)V_1(t)] + \lim_{t\to a}[U_2(t)V_2(t)] + \lim_{t\to a}[U_3(t)V_3(t)] && \text{(LT 6)}\\
&= \lim_{t\to a}U_1(t)\lim_{t\to a}V_1(t) + \lim_{t\to a}U_2(t)\lim_{t\to a}V_2(t) + \lim_{t\to a}U_3(t)\lim_{t\to a}V_3(t)\\
&= [\lim_{t\to a}U_1(t)\mathbf{i} + \lim_{t\to a}U_2(t)\mathbf{j} + \lim_{t\to a}U_3(t)\mathbf{k}]\cdot[\lim_{t\to a}V_1(t)\mathbf{i} + \lim_{t\to a}V_2(t)\mathbf{j} + \lim_{t\to a}V_3(t)\mathbf{k}] && \text{(Definition 11.1.3)}\\
&= \lim_{t\to a}\mathbf{U}(t)\cdot\lim_{t\to a}\mathbf{V}(t)
\end{aligned}$$

49.
$$\begin{aligned}
\lim_{t\to a}[\mathbf{U}\times\mathbf{V}] &= \lim_{t\to a}[(U_2V_3 - U_3V_2)\mathbf{i} + (U_3V_1 - U_1V_3)\mathbf{j} + (U_1V_2 - U_2V_1)\mathbf{k}] && \text{(Definition 10.5.1)}\\
&= [\lim_{t\to a}(U_2V_3 - U_3V_2)]\mathbf{i} + [\lim_{t\to a}(U_3V_1 - U_1V_3)]\mathbf{j} + [\lim_{t\to a}(U_1V_2 - U_2V_1)]\mathbf{k} && \text{(Definition 11.1.3)}\\
&= [\lim_{t\to a}U_2\lim_{t\to a}V_3 - \lim_{t\to a}U_3\lim_{t\to a}V_2]\mathbf{i} + [\lim_{t\to a}U_3\lim_{t\to a}V_1 - \lim_{t\to a}U_1\lim_{t\to a}V_3]\mathbf{j} + [\lim_{t\to a}U_1\lim_{t\to a}V_2 - \lim_{t\to a}U_2\lim_{t\to a}V_1]\mathbf{k} && \text{(LT4,6)}\\
&= [(\lim_{t\to a}U_1)\mathbf{i} + (\lim_{t\to a}U_2)\mathbf{j} + (\lim_{t\to a}U_3)\mathbf{j}]\times[(\lim_{t\to a}V_1)\mathbf{i} + (\lim_{t\to a}V_2)\mathbf{j} + (\lim_{t\to a}V_3)\mathbf{k}] && \text{(Definition 10.5.1)}\\
&= \lim_{t\to a}\mathbf{U}\times\lim_{t\to a}\mathbf{V} && \text{(Definition 11.1.3)}
\end{aligned}$$

50. $\lim\limits_{t\to a}(f\mathbf{V}) = \lim\limits_{t\to a}[(fV_1)\mathbf{i} + (fV_2)\mathbf{j} + (fV_3)\mathbf{k}] \overset{11.1.3}{=} [\lim\limits_{t\to a}(fV_1)]\mathbf{i} + [\lim\limits_{t\to a}(fV_2)]\mathbf{j} + [\lim\limits_{t\to a}(fV_3)]\mathbf{k}$
$\overset{\text{LT 6}}{=} (\lim\limits_{t\to a}f)(\lim\limits_{t\to a}V_1)\mathbf{i} + (\lim\limits_{t\to a}f)(\lim\limits_{t\to a}V_2)\mathbf{j} + (\lim\limits_{t\to a}f)(\lim\limits_{t\to a}V_3)\mathbf{k} = \lim\limits_{t\to a}f[(\lim\limits_{t\to a}V_1)\mathbf{i} + (\lim\limits_{t\to a}V_2)\mathbf{j} + (\lim\limits_{t\to a}V_3)\mathbf{k}]$
$\overset{11.1.3}{=} [\lim\limits_{t\to a}f][\lim\limits_{t\to a}\mathbf{V}]$

51. If $\mathbf{V}$ is continuous, then by Exercise 48, $\mathbf{V}\cdot\mathbf{V}$ is continuous, and by LT 10, so is $\|\mathbf{V}\| = \sqrt{\mathbf{V}\cdot\mathbf{V}}$.

11.2 CALCULUS OF VECTOR-VALUED FUNCTIONS

The definition of the derivative of a vector-valued function, differentiability of a vector-valued function, and the antiderivative of a vector-valued function are similar to the corresponding definitions for real-valued functions.

11.2.1 Definition If $\mathbf{R}$ is a vector-valued function, then the *derivative* of $\mathbf{R}$ is another vector-valued function, denoted by $\mathbf{R}'$ and defined by

$$\mathbf{R}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{R}(t + \Delta t) - \mathbf{R}(t)}{\Delta t}$$

if this limit exists.

The notation $D_t\mathbf{R}(t)$ is sometimes used in place of $\mathbf{R}'(t)$.

11.2.2 Theorem If $\mathbf{R}$ is a vector-valued function defined by

$$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \tag{1}$$

then

$$\mathbf{R}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

if $f'(t)$, $g'(t)$, and $h'(t)$ exist. For each replacement of t, the direction of the vector $\mathbf{R}'(t)$ is along the tangent line to the curve whose vector equation is (1).

11.2.3 Definition A vector-valued function is said to be *differentiable* on a interval if $\mathbf{R}'(t)$ exists for all values of t in the interval.

While the mean-value theorem does not extend to vector-valued functions, the differentiation formulas for a vector-valued function are similar to the corresponding differentiation formulas for a real-valued function.

11.2.4 Theorem If $\mathbf{R}$ and $\mathbf{Q}$ are differentiable vector-valued functions on an interval, then $\mathbf{R} + \mathbf{Q}$ is differentiable on the interval, and

$$D_t[\mathbf{R}(t) + \mathbf{Q}(t)] = D_t\mathbf{R}(t) + D_t\mathbf{Q}(t)$$

11.2.5 Theorem If $\mathbf{R}$ and $\mathbf{Q}$ are differentiable vector-valued functions on an interval, then $\mathbf{R} \cdot \mathbf{Q}$ is differentiable on the interval, and

$$D_t[\mathbf{R}(t) \cdot \mathbf{Q}(t)] = [D_t\mathbf{R}(t)] \cdot \mathbf{Q}(t) + \mathbf{R}(t) \cdot [D_t\mathbf{Q}(t)]$$

11.2.6 Theorem If $\mathbf{R}$ is a differentiable vector-valued function on an interval and f is a differentiable real-valued function on the interval, then

$$D_t[f(t)\mathbf{R}(t)] = [D_t f(t)]\mathbf{R}(t) + f(t)D_t\mathbf{R}(t)$$

11.2.7 Theorem If $\mathbf{R}$ and $\mathbf{Q}$ are differentiable vector-valued functions on an interval, then $\mathbf{R} \times \mathbf{Q}$ is differentiable on the interval, and

$$D_t[\mathbf{R}(t) \times \mathbf{Q}(t)] = [D_t\mathbf{R}(t)] \times \mathbf{Q}(t) + \mathbf{R}(t) \times [D_t\mathbf{Q}(t)]$$

11.2.8 Theorem Suppose that $\mathbf{F}$ is a vector-valued function and h is a real-valued function. If $\mathbf{F}'(h(t))$ exists and $h'(t)$ exists, then $D_t\mathbf{F}(h(t))$ exists and is given by

$$D_t\mathbf{F}(h(t)) = \mathbf{F}'(h(t))h'(t)$$

11.2.9 Theorem If $\mathbf{R}$ is a differentiable vector-valued function on an interval and $\|\mathbf{R}(t)\|$ is constant for all t in the interval, then the vectors $\mathbf{R}(t)$ and $D_t\mathbf{R}(t)$ are orthogonal.

11.2.10 Definition If $\mathbf{Q}$ is the vector-valued function given by

$$\mathbf{Q}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

then the *indefinite integral* and the *definite integral* from a to b of $\mathbf{Q}(t)$ are defined by

$$\int \mathbf{Q}(t)\,dt = \mathbf{i}\int f(t)\,dt + \mathbf{j}\int g(t)\,dt + \mathbf{k}\int h(t)\,dt \quad \text{and}$$

$$\int_a^b \mathbf{Q}(t)\,dt = \mathbf{i}\int_a^b f(t)\,dt + \mathbf{j}\int_a^b g(t)\,dt + \mathbf{k}\int_a^b h(t)\,dt$$

Further properties of integrals are given in Exercises 60–64.

11.2.11 Theorem Let the curve C have the vector equation $\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f', g' and h' are continuous on the closed interval $[a, b]$. If L is the length of arc of C from the point $\mathbf{R}(a)$ to the point $\mathbf{R}(b)$ then

$$L = \int_a^b \|\mathbf{R}'(t)\|\,dt$$

Exercises 11.2

In Exercises 1–10, find $\mathbf{R}'(t)$ and $\mathbf{R}''(t)$.

1. $\mathbf{R}(t) = t\mathbf{i} + t^{-1}\mathbf{j}$. $\mathbf{R}'(t) = \mathbf{i} - t^{-2}\mathbf{j}$. $\mathbf{R}''(t) = 2t^{-3}\mathbf{j}$

2. $\mathbf{R}(t) = (t^2 - 3)\mathbf{i} + (2t + 1)\mathbf{j}$. $\mathbf{R}'(t) = 2t\mathbf{i} + 2\mathbf{j}$. $\mathbf{R}''(t) = 2\mathbf{i}$

3. $\mathbf{R}(t) = \frac{t-1}{t+1}\mathbf{i} + \frac{t-2}{t}\mathbf{j} = [1 - 2(t+1)^{-1}]\mathbf{i} + (1 - 2t^{-1})\mathbf{j}$. $\mathbf{R}'(t) = 2(t+1)^{-2}\mathbf{i} + 2t^{-2}\mathbf{j}$. $\mathbf{R}''(t) = -4(t+1)^{-3}\mathbf{i} - 4t^{-3}\mathbf{j}$

4. $\mathbf{R}(t) = (t^2 + 4)^{-1} + \sqrt{1 - 5t}\mathbf{j}$

▹ $\mathbf{R}'(t) = -2t(t^2 + 4)^{-2}\mathbf{i} - \frac{5}{2}(1 - 5t)^{-1/2}\mathbf{j}$

$$\mathbf{R}'' = [-2(t^2+4)^{-2} + (-2t)(-4t)(t^2+4)^{-3}]\mathbf{i} + \left[-\tfrac{5}{2}\left(-\tfrac{1}{2}\right)(1-5t)^{-3/2}(-5)\right]\mathbf{j}$$
$$= (t^2+4)^{-3}[-2(t^2+4) + 8t^2]\mathbf{i} - \frac{25}{4}(1-5t)^{-3/2}\mathbf{j} = (6t^2 - 8)(t^2+4)^{-3}\mathbf{i} - \frac{25}{4}(1-5t)^{-3/2}\mathbf{j}$$

5. $\mathbf{R}(t) = e^{2t}\mathbf{i} + \ln t\mathbf{j} + t^2\mathbf{k}$. $\mathbf{R}'(t) = 2e^{2t}\mathbf{i} + t^{-1}\mathbf{j} + 2t\mathbf{k}$, $t > 0$. $\mathbf{R}''(t) = 4e^{2t} - t^{-2}\mathbf{j} + 2\mathbf{k}$, $t > 0$.

6. $\mathbf{R}(t) = \cos 2t\mathbf{i} + \tan t\mathbf{j} + t\mathbf{k}$. $\mathbf{R}'(t) = -2\sin 2t\mathbf{i} + \sec^2 t\mathbf{j} + \mathbf{k}$. $\mathbf{R}''(t) = -2\cos 2t\mathbf{i} + 2\sec^2 t \tan t\mathbf{j}$

7. $\mathbf{R} = \tan^{-1} t\mathbf{i} + \sin^{-1} t\mathbf{j} + \cos^{-1} t\mathbf{k}$. $\mathbf{R}' = \frac{1}{1+t^2}\mathbf{i} + \frac{1}{\sqrt{1-t^2}}\mathbf{j} - \frac{1}{\sqrt{1-t^2}}\mathbf{k}$. $\mathbf{R}'' = -\frac{2t}{(1+t^2)^2}\mathbf{i} + \frac{t}{(1-t^2)^{3/2}}\mathbf{j} - \frac{t}{(1-t^2)^{3/2}}$

8. $\mathbf{R}(t) = (e^{3t} + 2)\mathbf{i} + 2e^{3t}\mathbf{j} + 3 \cdot 2^t\mathbf{k}$

▹ $\mathbf{R}'(t) = 3e^{3t}\mathbf{i} + 6e^{3t}\mathbf{j} + 3(\ln 2)2^t\mathbf{k}$

$\mathbf{R}''(t) = 9e^{3t}\mathbf{i} + 18e^{3t}\mathbf{j} + 3(\ln 2)^2 2^t\mathbf{k}$

9. $\mathbf{R}(t) = 5\sin 2t\mathbf{i} - \sec 4t\mathbf{j} + 4\cos 2t\mathbf{k}$

▹ $\mathbf{R}''(t) = 10\cos 2t\mathbf{i} - 4\sec 4t\tan 4t\mathbf{j} - 8\sin 2t\mathbf{k}$. $\mathbf{R}''(t) = -20\sin 2t\mathbf{i} + (16\sec 4t - 32\sec^3 4t)\mathbf{j} - 16\cos 2t\mathbf{k}$

10. $\mathbf{R}(t) = \tan 3t\mathbf{i} + \ln \sin t\mathbf{j} - t^{-1}\mathbf{k}$. $\mathbf{R}''(t) = 3\sec^2 3t + \cot t\mathbf{j} + t^{-2}\mathbf{k}$. $\mathbf{R}''(t) = 18\sec^2 3t\tan 3t\mathbf{i} - \csc^2 t\mathbf{j} - 2t^{-3}\mathbf{k}$

In Exercises 11–14, find $D_t\|\mathbf{R}(t)\|$.

11. $\mathbf{R}(t) = (t-1)\mathbf{i} + (2-t)\mathbf{j}$

▹ $\|\mathbf{R}(t)\| = \sqrt{(t-1)^2 + (2-t)^2} = \sqrt{2t^2 - 6t + 5}$. $D_t\|\mathbf{R}(t)\| = \frac{4t-6}{2\sqrt{2t^2-6t+5}} = \frac{2t-3}{\sqrt{2t^2-6t+5}}$

12. $\mathbf{R}(t) = (e^t + 1)\mathbf{i} + (e^t - 1)\mathbf{j}$

▹ $\|\mathbf{R}(t)\| = \sqrt{(e^t+1)^2 + (e^t-1)^2} = \sqrt{e^{2t} + 2e^t + 1 + e^{2t} - 2e^t + 1} = \sqrt{2e^{2t} + 2}$

Thus,

$$D_t\|\mathbf{R}(t)\| = \tfrac{1}{2}(2e^{2t}t + 2)^{-1/2}(4e^{2t}) = \frac{2e^{2t}}{\sqrt{2e^{2t}+2}}$$

13. $\mathbf{R}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + 2e^{3t}\mathbf{k}$. $\|\mathbf{R}(t)\| = \sqrt{\sin^2 3t + \cos^2 3t + 4e^{6t}} = \sqrt{1 + 4e^{6t}}$. $D_t\|\mathbf{R}(t)\| = \frac{12e^{6t}}{\sqrt{1+4e^{6t}}}$

14. $\mathbf{R}(t) = \sqrt{t^2+1}\mathbf{i} + \sqrt{t^2-1}\mathbf{j} + t\mathbf{k}$. $\|\mathbf{R}(t)\| = \sqrt{t^2 + 1 + t^2 - 1 + t^2} = \sqrt{3}|t|$. $D_t\|\mathbf{R}(t)\| = \sqrt{3}\,\operatorname{sgn}(t)$

In Exercises 15 and 19, $\mathbf{R}(t) = (t^2 + e^t)\mathbf{i} + (t - e^{2t})\mathbf{j}$; $\mathbf{Q}(t) = (t^3 + 2e^t)\mathbf{i} - (3t + e^{2t})\mathbf{j}$

15. $D_t[\mathbf{R}(t) + \mathbf{Q}(t)] = D_t[(t^2 + t^3 + 3e^t)\mathbf{i} + (-2t - 2e^{2t})\mathbf{j}] = (2t + 3t^2 + 3e^t)\mathbf{i} + (-2 - 4e^{2t})\mathbf{j}$ and

$D_t\mathbf{R}(t) + D_t\mathbf{Q}(t) = [(2t + e^t)\mathbf{i} + (1 - 2e^{2t})\mathbf{j}] + [(3t^2 + 2e^t)\mathbf{i} - (3 + 2e^{2t})\mathbf{j}] = (2t + 3t^2 + 3e^t)\mathbf{i} + (-2 - 4e^{2t})\mathbf{j}$

19. $D_t[\mathbf{R}(t) \cdot \mathbf{Q}(t)] = D_t[(t^2 + e^t)(t^3 + 2e^t) - (t - e^{2t})(3t + e^{2t})]$

$= (2t + e^t)(t^3 + 2e^t) + (t^2 + e^t)(3t^2 + 2e^t) - (1 - 2e^{2t})(3t + e^{2t}) - (t - e^{2t})(3 + 2e^{2t})$ and

$D_t\mathbf{R}(t) \cdot \mathbf{Q}(t) + \mathbf{R}(t) \cdot D_t\mathbf{Q}(t)$

$= [(2t + e^t)\mathbf{i} + (1 - 2e^{2t})\mathbf{j}] \cdot [(t^3 + 2e^t)\mathbf{i} - (3t + e^{2t})\mathbf{j}] + [(t^2 + e^t)\mathbf{i} + (t - e^{2t})\mathbf{j}] \cdot [(3t^2 + 2e^t)\mathbf{i} - (3 + 2e^{2t})\mathbf{j}]$

$= [(2t + e^t)(t^3 + 2e^t) - (1 - 2e^{2t})(3t + e^{2t})] + [(t^2 + e^t)(3t^2 + 2e^t) - (1 - e^{2t})(3 + 2e^{2t})]$

In Exercises 16 and 20, $\mathbf{R}(t) = \cos 2t\mathbf{i} - \sin 2t\mathbf{j}$ and $\mathbf{Q}(t) = \sin^2 t\mathbf{i} + \cos 2t\mathbf{j}$.

16. Verify Theorem 11.2.4 for the given $\mathbf{R}$ and $\mathbf{Q}$.

▷
$$\begin{aligned} D_t[\mathbf{R}(t) + \mathbf{Q}(t)] &= D_t[(\cos 2t\mathbf{i} - \sin 2t\mathbf{j}) + (\sin^2 t\mathbf{i} + \cos 2t\mathbf{j})] \\ &= D_t[(\cos 2t + \sin^2 t)\mathbf{i} + (-\sin 2t + \cos 2t)\mathbf{j}] \\ &= (-2\sin 2t + 2\sin t\cos t)\mathbf{i} + (-2\cos 2t - 2\sin 2t)\mathbf{j} \end{aligned} \tag{1}$$

and

$$\begin{aligned} D_t\mathbf{R}(t) + D_t\mathbf{Q}(t) &= D_t(\cos 2t\mathbf{i} - \sin 2t\mathbf{j}) + D_t(\sin^2 t\mathbf{i} + \cos 2t\mathbf{j}) \\ &= (-2\sin 2t\mathbf{i} - 2\cos 2t\mathbf{j}) + (2\sin t\cos t\mathbf{i} - 2\sin 2t\mathbf{j}) \\ &= (-2\sin 2t + 2\sin t\cos t)\mathbf{i} + (-2\cos 2t - 2\sin 2t)\mathbf{j} \end{aligned} \tag{2}$$

and by comparing Eqs. (1) and (2) we verify Theorem 11.2.4. That is,

$$D_t[\mathbf{R}(t) + \mathbf{Q}(t)] = D_t\mathbf{R}(t) + D_t\mathbf{Q}(t)$$

20. Verify Theorem 11.2.5 for the vectors

▷
$$\begin{aligned} \mathbf{R}(t)\cdot\mathbf{Q}(t) &= (\cos 2t\mathbf{i} - \sin 2t\mathbf{j})\cdot(\sin^2 t\mathbf{i} + \cos 2t\mathbf{j}) \\ &= \sin^2 t\cos 2t - \sin 2t\cos 2t \end{aligned}$$

$$\begin{aligned} D_t[\mathbf{R}(t)\cdot\mathbf{Q}(t)] &= \sin^2 t(-2\sin 2t) + \cos 2t(2\sin t\cos t) - [\sin 2t(-2\sin 2t) + \cos 2t(2\cos 2t)] \\ &= -2\sin^2 t\sin 2t + 2\sin t\cos t\cos 2t + 2\sin^2 2t - 2\cos^2 2t \end{aligned} \tag{1}$$

Because

$$\begin{aligned} D_t\mathbf{R}(t) &= D_t(\cos 2t\mathbf{i} - \sin 2t\mathbf{j}) \\ &= -2\sin 2t\mathbf{i} - 2\cos 2t\mathbf{j} \end{aligned}$$

then

$$\begin{aligned} [D_t\mathbf{R}(t)]\cdot\mathbf{Q}(t) &= (-2\sin 2t\mathbf{i} - 2\cos 2t\mathbf{j})\cdot(\sin^2 t\mathbf{i} + \cos 2t\mathbf{j}) \\ &= -2\sin^2 t\sin 2t - 2\cos^2 2t \end{aligned} \tag{2}$$

Furthermore,

$$\begin{aligned} D_t\mathbf{Q}(t) &= D_t(\sin^2 t\mathbf{i} + \cos 2t\mathbf{j}) \\ &= 2\sin t\cos t\mathbf{i} - 2\sin 2t\mathbf{j} \end{aligned}$$

and thus

$$\begin{aligned} \mathbf{R}(t)\cdot[D_t\mathbf{Q}(t)] &= (\cos 2t\mathbf{i} - \sin 2t\mathbf{j})\cdot(2\sin t\cos t\mathbf{i} - 2\sin 2t\mathbf{j}) \\ &= 2\sin t\cos t\cos 2t + 2\sin^2 2t \end{aligned} \tag{3}$$

By adding the corresponding sides of Eqs. (2) and (3) we obtain

$$[D_t\mathbf{R}(t)]\cdot\mathbf{Q}(t) + \mathbf{R}(t)\cdot[D_t\mathbf{Q}(t)] = -2\sin^2 t\sin 2t - 2\cos^2 2t + 2\sin t\cos t\cos 2t + 2\sin^2 2t \tag{4}$$

and by comparing Eqs. (1) and (4) we verify Theorem 14.4.47. That is,

$$D_t[\mathbf{R}(t)\cdot\mathbf{Q}(t)] = [D_t\mathbf{R}(t)]\cdot\mathbf{Q}(t) + \mathbf{R}(t)\cdot[D_t\mathbf{Q}(t)]$$

In Exercises 17, 21 and 25, $\mathbf{R}(t) = 2\sin t\mathbf{i} + \cos t\mathbf{j} - \sin 2t\mathbf{k}$ and $\mathbf{Q}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j} + \mathbf{k}$

17.
$$\begin{aligned} D_t[\mathbf{R}(t) + \mathbf{Q}(t)] &= D_t[(2\sin t + \cos t)\mathbf{i} + (\cos t + 2\sin t)\mathbf{j} + (-\sin 2t + 1)\mathbf{k}] \\ &= (2\cos t - \sin t)\mathbf{i} + (-\sin t + 2\cos t)\mathbf{j} + (-2\cos 2t)\mathbf{k} \end{aligned} \tag{1}$$

$$D_t\mathbf{R}(t) + D_t\mathbf{Q}(t) = (2\cos t\mathbf{i} - \sin t\mathbf{j} - 2\cos 2t\mathbf{k}) + (-\sin t\mathbf{i} + 2\cos t\mathbf{j}) = (1)$$

21.
$$D_t[\mathbf{R}(t)\cdot\mathbf{Q}(t)] = D_t(2\sin t\cos t + 2\sin t\cos t - \sin 2t) = D_t\sin 2t = 2\cos 2t$$

$$\begin{aligned} \mathbf{R}'\cdot\mathbf{Q} + \mathbf{R}\cdot\mathbf{Q}' &= (2\cos t\mathbf{i} - \sin t\mathbf{j} - 2\cos 2t\mathbf{k})\cdot(\cos t\mathbf{i} + 2\sin t\mathbf{j} + \mathbf{k}) + \\ &\qquad (2\sin t\mathbf{i} + \cos t\mathbf{j} - \sin 2t\mathbf{k})\cdot(-\sin t\mathbf{i} + 2\cos t\mathbf{j}) \\ &= (2\cos^2 t - 2\sin^2 t - 2\cos 2t) + (-2\sin^2 t + 2\cos^2 t) = 2\cos 2t \end{aligned}$$

25.
$$D_t[\mathbf{R}(t)\times\mathbf{Q}(t)] = D_t\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\sin t & \cos t & -\sin 2t \\ \cos t & 2\sin t & 1 \end{vmatrix}$$

$$= D_t[(\cos t + 2\sin t\sin 2t)\mathbf{i} + (-2\sin t - \sin 2t\cos t)\mathbf{j} + (4\sin^2 t - \cos^2 t)\mathbf{k}]$$

$$= (-\sin t + 4\cos 2t\sin t + 2\sin 2t\cos t)\mathbf{i} + (-2\cos t - 2\cos 2t\cos t + \sin 2t\sin t)\mathbf{j} + 10\sin t\cos t\mathbf{k} \tag{1}$$

$$D_t\mathbf{R}\times\mathbf{Q} = (2\cos t\mathbf{i} - \sin t\mathbf{j} - 2\cos 2t\mathbf{k})\times\mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos t & -\sin t & -2\cos 2t \\ \cos t & 2\sin t & 1 \end{vmatrix}$$

$$= (-\sin t + 4\cos 2t\sin t)\mathbf{i} + (-2\cos 2t - 2\cos 2t\cos t)\mathbf{j} + 5\sin t\cos t\mathbf{k} \tag{2}$$

$$\mathbf{R}\times D_t\mathbf{Q} = (2\sin t\mathbf{i} + \cos t\mathbf{j} - \sin 2t\mathbf{k})\times(-\sin t\mathbf{i} + 2\cos t\mathbf{j}) = 2\sin 2t\cos t\mathbf{i} + \sin 2t\sin t\mathbf{j} + 5\sin t\cos t \tag{3}$$

and (2) + (3) = (1).

In Exercises 18, 22 and 26, $\mathbf{R}(t) = e^{3t}\mathbf{i} - 4e^{3t}\mathbf{j} - 2\mathbf{k}$ and $\mathbf{Q}(t) = e^{t}\mathbf{i} - e^{t}\mathbf{j} - 2e^{4t}\mathbf{k}$

18. $D_t(\mathbf{R} + \mathbf{Q}) = D_t\mathbf{R} + D_t\mathbf{Q} = (3e^{3t} + e^{t})\mathbf{i} + (-12e^{3t} - e^{t})\mathbf{j} + (-8e^{4t})$

22. $D_t(\mathbf{R} \cdot \mathbf{Q}) = \mathbf{R}' \cdot \mathbf{Q} + \mathbf{R} \cdot \mathbf{Q}' = 36e^{4t}$

26. $D_t(\mathbf{R} \times \mathbf{Q}) = \mathbf{R}' \times \mathbf{Q} + \mathbf{R} \times \mathbf{Q}' = (56e^{7t} - 2e^{t})\mathbf{i} + (14e^{7t} - 2e^{t})\mathbf{j} + 12e^{4t}\mathbf{k}$

23. $f(t) = \cos 2t$, $\mathbf{R}(t) = 5 \sin 2t\mathbf{i} - \sec 4t\mathbf{j} + 4 \cos 2t\mathbf{k}$

$D_t(f\mathbf{R}) = D_t[5 \sin 2t \cos 2t\mathbf{i} - \cos 2t \sec 4t\mathbf{j} + 4 \cos^2 2t\mathbf{k}]$

$= 10(\cos^2 2t - \sin^2 2t)\mathbf{i} + (2 \sin 2t \sec 4t - 4 \cos 2t \sec 4t \tan 4t)\mathbf{j} = f'\mathbf{R} + f\mathbf{R}'$

24. Verify Theorem 11.2.6 for the functions $f(t) = e^{t}$; $\mathbf{R}(t) = (e^{3t} + 2)\mathbf{i} + 2e^{3t}\mathbf{j} + 3 \cdot 2^{t}\mathbf{k}$

▷ $D_t[f(t)\mathbf{R}(t)] = D_t[(e^{4t} + 2e^{t})\mathbf{i} + 2e^{4t}\mathbf{j} + 3(2e)^{t}] = (4e^{4t} + 2e^{t})\mathbf{i} + 8e^{4t}\mathbf{j} + 3(1 + \ln 2)(2e)^{t}\mathbf{k}$ (1)

$f'(t)\mathbf{R}(t) + f(t)\mathbf{R}'(t) = e^{t}[(e^{3t} + 2)\mathbf{i} + 2e^{3t}\mathbf{j} + 3 \cdot 2^{t}\mathbf{k}] + e^{t}[(4e^{4t} + 2e^{t})\mathbf{i} + 6e^{3t}\mathbf{j} + 3(\ln 2)2^{t}\mathbf{k}]$

$= (4e^{4t} + 2e^{t})\mathbf{i} + 8e^{4t}\mathbf{j} + 3(1 + \ln 2)(2e)^{t}\mathbf{k}$ (2)

Comparing (1) and (2), we have verified Theorem 11.2.6.

In Exercises 27 and 28, verify Theorem 11.2.8 for the given functions.

27. $\mathbf{F}(\phi) = \phi\mathbf{i} + \phi^2\mathbf{j} + \ln \phi\mathbf{k}$ and $h(t) = e^{t}$. $D_t(\mathbf{F}(h(t)) = D_t(e^{t}\mathbf{i} + e^{2t}\mathbf{j} + t\mathbf{k}) = e^{t}\mathbf{i} + 2e^{2t}\mathbf{j} + \mathbf{k}$

$\mathbf{F}'(\phi) = \mathbf{i} + 2\phi\mathbf{j} + \phi^{-1}\mathbf{k}$. $\mathbf{F}'(h(t))h'(t) = (\mathbf{i} + 2e^{t}\mathbf{j} + e^{-t}\mathbf{k})e^{t} = e^{t}\mathbf{i} + 2e^{2t}\mathbf{j} + \mathbf{k}$

28. $\mathbf{F}(\phi) = \sin \phi\mathbf{i} + \cos \phi\mathbf{j} + \phi\mathbf{k}$ and $h(t) = \sin^{-1} t$

▷ $D_t(\mathbf{F}(h(t)) = D_t(t\mathbf{i} + \sqrt{1 - t^2}\mathbf{j} + \sin^{-1} t\mathbf{k}) = \mathbf{i} - \dfrac{t}{\sqrt{1 - t^2}}\mathbf{j} + \dfrac{1}{\sqrt{1 - t^2}}\mathbf{k}$ (1)

We have

$$\mathbf{F}'(\phi) = \cos \phi\mathbf{i} - \sin \phi\mathbf{j} + \mathbf{k} \text{ and } h'(t) = \frac{1}{\sqrt{1 - t^2}}$$

Thus,

$\mathbf{F}'(h(t))h'(t) = (\sqrt{1 - t^2}\mathbf{i} - t\mathbf{j} + \mathbf{k})\dfrac{1}{\sqrt{1 - t^2}} = \mathbf{i} - \dfrac{t}{\sqrt{1 - t^2}}\mathbf{j} + \dfrac{1}{\sqrt{1 - t^2}}\mathbf{k}$ (2)

Comparing (1) and (2), we have verified Theorem 11.2.8.

In Exercises 29-31, let $\mathbf{R}(t) = f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}$ and $\mathbf{Q}(t) = f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}$.

29. $D_t[\mathbf{R}(t) + \mathbf{Q}(t)] = D_t\{[f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}] + [f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}]\}$

$= D_t\{[f_1(t) + f_2(t)]\mathbf{i} + [g_1(t) + g_2(t)]\mathbf{j} + [h_1(t) + h_2(t)]\}$

$= [f_1'(t) + f_2'(t)]\mathbf{i} + [g_1'(t) + g_2'(t)]\mathbf{j} + [h_1'(t) + h_2'(t)]$

$= [f_1'(t)\mathbf{i} + g_1'(t)\mathbf{j} + h_1'(t)] + [f_2'(t)\mathbf{i} + g_2'(t)\mathbf{j} + h_2'(t)] = D_t\mathbf{R}(t) + D_t\mathbf{Q}(t)$

30. $D_t[f\mathbf{R}] = D_t[ff_1\mathbf{i} + fg_1\mathbf{j} + fh_1\mathbf{k}] = (f'f_1 + ff_1')\mathbf{i} + (f'g_1 + fg_1')\mathbf{j} + (f'h_1 + fh_1')\mathbf{k}$

$= f'(f_1\mathbf{i} + g_1\mathbf{j} + h_1\mathbf{k}) + f(f_1'\mathbf{i} + g_1'\mathbf{j} + h_1'\mathbf{k}) = f'\mathbf{R} + f\mathbf{R}'$

31. The derivative of $\mathbf{R} \times \mathbf{Q}$ is an immediate consequence of the following determinant theorem. When we use the rule for derivative of a product, we write the three terms vertically.

$$D_t\begin{vmatrix} f & g & h \\ f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \end{vmatrix} = D_t\left[f\begin{vmatrix} g_1 & h_1 \\ g_2 & h_2 \end{vmatrix} - g\begin{vmatrix} f_1 & h_1 \\ f_2 & h_2 \end{vmatrix} + h\begin{vmatrix} f_1 & g_1 \\ f_2 & g_2 \end{vmatrix}\right]$$

$= D_t[fg_1h_2 - fg_2h_1 - f_1gh_2 + f_1gh_2 + f_2gh_1 + f_1g_2h - f_2g_1h]$

$= [f'g_1h_2 - f'g_2h_1 - f_1'gh_2 + f_1'gh_2 + f_2'gh_1 + f_1'g_2h - f_2'g_1h]$

$+ [fg_1'h_2 - fg_2'h_1 - f_1g'h_2 + f_1g'h_2 + f_2g'h_1 + f_1g_2'h - f_2g_1'h]$

$+ [fg_1h_2' - fg_2h_1' - f_1gh_2' + f_1gh_2' + f_2gh_1' + f_1g_2h' - f_2g_1h']$

$$= \begin{vmatrix} f' & g' & h' \\ f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \end{vmatrix} + \begin{vmatrix} f & g & h \\ f_1' & g_1' & h_1' \\ f_2 & g_2 & h_2 \end{vmatrix} + \begin{vmatrix} f & g & h \\ f_1 & g_1 & h_1 \\ f_2' & g_2' & h_2' \end{vmatrix}$$

If we replace f, g, h with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the first determinant is 0 because the the derivative of a constant is 0.

32. Prove Theorem 11.2.8

▷ We have $\phi = h(t)$ and $\mathbf{F}(h(t)) = f(\phi)\mathbf{i} + g(\phi)\mathbf{j} + h(\phi)\mathbf{k}$

Applying the chain rule for real-valued function, we obtain

$D_t\mathbf{F}(h(t)) = D_t[f(\phi)\mathbf{i} + g(\phi)\mathbf{j} + h(\phi)\mathbf{k}] = [D_tf(\phi)]\mathbf{i} + [D_tg(\phi)]\mathbf{j} + [D_th(\phi)]\mathbf{k}$

$= [D_\phi f(\phi)D_t\phi]\mathbf{i} + [D_\phi g(\phi)D_t\phi]\mathbf{j} + [D_\phi h(\phi)D_t\phi]\mathbf{k} = \{[D_\phi f(\phi)]\mathbf{i} + [D_\phi g(\phi)]\mathbf{j} + [D_\phi h(\phi)]\mathbf{k}\}D_t\phi$

$= D_\phi[f(\phi)\mathbf{i} + g(\phi)\mathbf{j} + h(\phi)\mathbf{k}]D_t\phi = \mathbf{F}'(h(t))D_t\phi$

In Exercises 33–40, find the most general vector-valued function whose derivative has the given function value.

33. $\mathbf{i}\int \tan t\, dt - \mathbf{j}\int \frac{1}{t}\, dt = (\ln|\sec t| + C_1)\mathbf{i} - (\ln|t| + C_2)\mathbf{j} = \ln|\sec t|\mathbf{i} - \ln|t|\mathbf{j} + \mathbf{C}$

34. $\int [(t^2 - 9)\mathbf{i} + (2t - 5)\mathbf{j}]dt = (\frac{1}{3}t^3 - 9t)\mathbf{i} + (t^2 - 5t)\mathbf{j} + \mathbf{C}$

35. Let $\mathbf{R}'(t) = e^{3t}\mathbf{i} + \frac{1}{t-1}\mathbf{j}$. Then $\mathbf{R}(t) = \mathbf{i}\int e^{3t}dt + \mathbf{j}\int \frac{dt}{t-1} = \frac{1}{3}e^{3t}\mathbf{i} + \ln|t - 1|\mathbf{j} + \mathbf{C}$.

36. $\frac{1}{4+t^2}\mathbf{i} - \frac{4}{1-t^2}\mathbf{j}$

▸ Let $\mathbf{R}'(t) = \ln t\mathbf{i} + t^2\mathbf{j}$. Then $\mathbf{R}(t) = \mathbf{i}\int \ln t\, dt + \mathbf{j}\int t^2 dt = (t \ln t - t)\mathbf{i} + \frac{1}{3}t^3\mathbf{j} + \mathbf{C}$.

37. $\int [e^{3t}\mathbf{i} + e^{-3t}\mathbf{j} - te^{3t}\mathbf{k}]dt = \frac{1}{3}e^{3t}\mathbf{i} - \frac{1}{3}e^{-3t}\mathbf{j} + (-\frac{1}{3}te^{3t} + \frac{1}{9}e^{3t})\mathbf{k} + \mathbf{C}$

38. $\int [3^t\mathbf{i} - 2^t\mathbf{j} + e^t\mathbf{k}]dt = \frac{3^t}{\ln 3}\mathbf{i} - \frac{2^t}{\ln 2}\mathbf{j} + e^t\mathbf{k} + \mathbf{C}$

39. $\int [\tan t\mathbf{i} + \sec t\mathbf{j} + \frac{1}{t}\mathbf{k}]dt = \ln|\sec t|\mathbf{i} + \ln|\sec t + \tan t|\mathbf{j} + \ln|t|\mathbf{k} + \mathbf{C}$

40. $t \sin t\mathbf{i} - t \cos t\mathbf{j} + t\mathbf{k}$

▸ We use integration by parts. Let

$$u = t \qquad dv = \sin t\mathbf{i} + \cos t\mathbf{j}$$
$$du = dt \qquad v = -\cos t\mathbf{i} + \sin t\mathbf{j}$$

$$\int [t(\sin t\mathbf{i} + \cos t\mathbf{j}) + t\mathbf{k}]dt = t(-\cos t\mathbf{i} + \sin t\mathbf{j}) - \int [-\cos t\mathbf{i} + \sin t\mathbf{j}]dt + \tfrac{1}{2}t^2\mathbf{k}$$
$$= (-t \cos t + \sin t)\mathbf{i} + (t \sin t + \cos t)\mathbf{j} + \tfrac{1}{2}t^2\mathbf{k} + \mathbf{C}$$

41. $\mathbf{R}(3) + \int_3^t \mathbf{R}'(t)dt = (2\mathbf{i} - 5\mathbf{j}) + \int_3^t (t^2\mathbf{i} + \frac{1}{t-2}\mathbf{j})dt = (2\mathbf{i} - 5\mathbf{j}) + \left[\frac{1}{3}t^3\mathbf{i} + \ln|t - 2|\mathbf{j}\right]_3^t = (\frac{1}{3}t^3 - 7)\mathbf{i} + (\ln|t - 2| - 5)\mathbf{j}$

42. $\mathbf{R}(t) = \int \mathbf{R}'(t)dt = \mathbf{i}\int \sin^2 t\, dt + \mathbf{j}\int 2\cos^2 t\, dt = \frac{1}{4}(2t - \sin 2t)\mathbf{i} + \frac{1}{2}(2t + \sin 2t)\mathbf{j} + \mathbf{C}$

$\mathbf{R}(\pi) = \mathbf{0} \Rightarrow \mathbf{0} = \frac{1}{2}\pi\mathbf{i} + \pi\mathbf{j} + \mathbf{C}$; $\mathbf{C} = -\frac{1}{2}\pi\mathbf{i} - \pi\mathbf{j}$. Thus $\mathbf{R}(t) = \frac{1}{2}(t - \frac{1}{2}\sin 2t - \pi)\mathbf{i} + (t + \frac{1}{2}\sin 2t - \pi)\mathbf{j}$

43. $\mathbf{R}(0) + \int_0^t \mathbf{R}'(t)dt = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \int_0^t (e^t \sin t\mathbf{j} + \cos t\mathbf{j} - e^t\mathbf{k})dt = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \left[\frac{1}{2}\sqrt{2}\sin(t - \frac{1}{4}\pi)\mathbf{i} + \sin t\mathbf{j} - e^t\mathbf{k}\right]_0^t$

$= [\frac{1}{2}e^t(\sin t - \cos t) + \frac{3}{2}]\mathbf{i} + (\sin t - 1)\mathbf{j} + (2 - e^t)\mathbf{k}$

44. If $\mathbf{R}'(t) = \frac{1}{t+1}\mathbf{i} - \tan t\mathbf{j} + \frac{t}{t^2-1}\mathbf{k}$ and $\mathbf{R}(0) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, find $\mathbf{R}(t)$.

▸ $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{R}'(t)dt = \mathbf{R}(0) + \int_0^t \left(\frac{1}{t+1}\mathbf{i} - \tan t\mathbf{j} + \frac{t}{t^2-1}\right)dt$

$= (4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) + \left[\ln|t + 1| + \ln|\cos t| + \frac{1}{2}\ln|t^2 - 1|\right]_0^t = (\ln|t + 1| + 4)\mathbf{i} + (\ln|\cos t| - 3)\mathbf{j} + (\frac{1}{2}\ln|t^2 - 1| + 5)\mathbf{k}$

In Exercises 45 and 46, (a) find a cartesian equation of curve $\mathbf{R}'(t)$; (b) compute $\mathbf{R}(t) \cdot \mathbf{R}'(t)$. Interpret the result.

45. $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. $\mathbf{R}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$. Parametric equations for $\mathbf{R}'$ are $x = -\sin t$ and $y = \cos t$; a cartesian equation is $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$, the unit circle. $\mathbf{R}(t) \cdot \mathbf{R}'(t) = -\cos t \sin t + \sin t \cos t = 0$ The representation of $\mathbf{R}'(t_1)$ having its initial point at $\mathbf{R}(t_1)$ is along the tangent line to the circle at $\mathbf{R}(t_1)$; the position representation of $\mathbf{R}(t_1)$ is the radius of the circle which is perpendicular to the tangent there.

46. $\mathbf{R}(t) = \cosh t\mathbf{i} - \sinh t\mathbf{j}$. $\mathbf{R}'(t) = \sinh t\mathbf{i} - \cosh t\mathbf{j}$. Parametric equations for $\mathbf{R}$ are $x = \cosh t$, $y = -\sinh t$; a cartesian equation is $x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$, $x \geq 1$, hyperbola. $\mathbf{R}(t) \cdot \mathbf{R}'(t) = 2\cosh t \sinh t = \sinh 2t$

In Exercises 47 and 48, if α is the radian measure of the angle between $\mathbf{R}(t)$ and $\mathbf{Q}(t)$, find $D_t\alpha(t)$.

47. $\mathbf{R}(t) = 3e^{2t}\mathbf{i} - 4e^{2t}\mathbf{j} = e^{2t}(3\mathbf{i} - 4\mathbf{j})$ and $\mathbf{Q}(t) = 6e^{3t}\mathbf{j}$. $\mathbf{R}(t)$ has the same direction as $3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{Q}(t)$ has the same direction as $\mathbf{j}$. Thus $\alpha(t)$ is a constant and $D_t\alpha(t) = 0$.

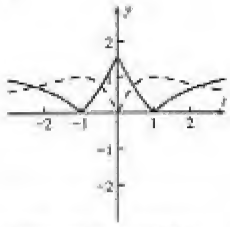
48. Given $\mathbf{R}(t) = 2t\mathbf{i} + (t^2 - 1)\mathbf{j}$ and $\mathbf{Q}(t) = 3t\mathbf{i}$.

▹ The derivative will not exist at $t = 0$ because $\mathbf{Q}$ reverses its direction there, and at $t = \pm 1$ because the undirected angle switches abruptly between decreasing and increasing there. The figure shows $\cos \alpha(t)$ dashed and $\alpha(t)$ solid.

$$\cos \alpha(t) = \frac{\mathbf{R}(t) \cdot \mathbf{Q}(t)}{\|\mathbf{R}(t)\| \|\mathbf{Q}(t)\|} = \frac{6t^2}{\sqrt{(2t)^2 + (t^2-1)^2}\sqrt{(3t)^2}} = \frac{6t^2}{\sqrt{(t^2+1)^2}|3t|} = \frac{2|t|}{t^2+1}$$

Thus,

$$\alpha(t) = \cos^{-1} \frac{2|t|}{t^2+1}$$



Because $D_t|t| = t/|t|$, then

$$\alpha'(t) = \frac{-1}{\sqrt{1 - \left(\frac{2|t|}{t^2+1}\right)^2}} \cdot \frac{(t^2+1)\frac{2t}{|t|} - 2|t|(2t)}{(t^2+1)^2} = \frac{-(t^2+1)}{\sqrt{(t^2-1)^2}} \cdot \frac{-2t(t^2-1)}{|t|(t^2+1)^2}$$

$$= \frac{2t(t^2-1)}{|t||t^2-1|(t^2+1)} = \frac{2t}{t^2+1}\operatorname{sgn}[t(t^2-1)]$$

In Exercises 49–52, find the exact length L of arc from t_1 to t_2 of the curve having the given vector equation.

49. $\mathbf{R}(t) = (t+1)\mathbf{i} - t^2\mathbf{j} + (1-2t)\mathbf{k}$. $\|\mathbf{R}'(t)\| = \|\mathbf{i} - 2t\mathbf{j} - 2\mathbf{k}\| = \sqrt{1 + 4t^2 + 4} = \sqrt{5 + 4t^2}$

$$L = \int_{-1}^{2} \|D_t\mathbf{R}(t)\| dt = \int_{-1}^{2} \sqrt{5 + 4t^2}\,dt = 2\int_{-1}^{2} \sqrt{t^2 + \tfrac{5}{4}}\,dt = 2\left[\tfrac{t}{2}\sqrt{t^2+\tfrac{5}{4}} + \tfrac{5}{2}\ln\left|t + \sqrt{t^2 + \tfrac{5}{4}}\right|\right]_{-1}^{2}$$

$$= 2\left(\sqrt{4 + \tfrac{5}{4}} + \tfrac{5}{8}\ln\left|2 + \sqrt{5 + \tfrac{5}{4}}\right|\right) - 2\left(-\tfrac{1}{2}\sqrt{1 + \tfrac{5}{4}} + \tfrac{5}{8}\ln\left|-1 + \sqrt{1 + \tfrac{5}{4}}\right|\right)$$

$$= \sqrt{21} + \tfrac{5}{4}\ln(2 + \tfrac{1}{2}\sqrt{21}) + \tfrac{3}{2} - \tfrac{5}{4}\ln\tfrac{1}{2} = \sqrt{21} + \tfrac{3}{2} + \tfrac{5}{4}\ln(4 + \sqrt{21})$$

50. $\mathbf{R}(t) = \sin 2t\mathbf{i} + \cos 2t\mathbf{j} + 2t^{3/2}\mathbf{k}$. $\|\mathbf{R}'(t)\| = \|2\cos 2t\mathbf{i} - 2\sin 2t\mathbf{j} + 3t^{1/2}\mathbf{k}\| = \sqrt{4\cos^2 2t + 4\sin^2 2t + 9t}$

$$= \sqrt{4 + 9t}.\ L = \int_0^1 (4+9t)^{1/2} dt = \tfrac{1}{9} \cdot \tfrac{2}{3}(4+9t)^{3/2}\Big]_0^1 = \tfrac{2}{27}(13^{3/2} - 8)$$

51. $\mathbf{R}(t) = 4t^{3/2}\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$. $\|\mathbf{R}'(t)\| = \|6t^{1/2}\mathbf{i} - 3\cos t\mathbf{j} - 3\sin t\mathbf{k}\| = \sqrt{36t + 9\cos^2 t + 9\sin^2 t} = 3\sqrt{4t+1}$

$$L = \int_0^2 3(4t+1)^{1/2} dt = 3 \cdot \tfrac{1}{4} \cdot \tfrac{2}{3}(4t+1)^{3/2}\Big]_0^2 = \tfrac{1}{2}(27 - 1) = 13$$

52. $\mathbf{R}(t) = t^2\mathbf{i} + (t + \tfrac{1}{3}t^3)\mathbf{j} + (t - \tfrac{1}{3}t^3)\mathbf{k}$: $t_1 = 0$; $t_2 = 1$

▹ $\mathbf{R}'(t) = 2t\mathbf{i} + (1+t^2)\mathbf{j} + (1-t^2)\mathbf{k}$

$$\|\mathbf{R}'(t)\| = \sqrt{4t^2 + (1 + 2t^2 + t^4) + (1 - 2t^2 + t^4)} = \sqrt{2 + 4t^2 + 2t^4} = \sqrt{2(1 + 2t^2 + t^4)} = \sqrt{2}(1+t^2)$$

$$L = \int_0^1 \sqrt{2}(1+t^2)dt = \sqrt{2}(t + \tfrac{1}{3}t^3)\Big]_0^1 = \tfrac{4}{3}\sqrt{2}$$

In Exercises 53–56, use NINT to approximate to 4 digits the length L of arc from t_1 to t_2 of the given curve.

53. $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. $\mathbf{R}'(t) = \mathbf{j} + 2t\mathbf{j} + 3t^2\mathbf{k}$. $L = \int_0^2 \|\mathbf{R}'(t)\| dt = \int_0^2 \sqrt{1 + 4t^2 + 9t^4}\,dt = 9.57057 \approx 9.571$

54. $\mathbf{R}(t) = e^t\mathbf{i} + e^t\mathbf{j} + \ln t\mathbf{k}$. $\mathbf{R}'(t) = e^t\mathbf{i} + e^t\mathbf{j} + t^{-1}\mathbf{k}$. $L = \int_1^2 \|\mathbf{R}'(t)\| dt = \int_1^2 \sqrt{2e^{2t} + t^{-2}}\,dt = 6.6510 \approx 6.651$

55. $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t^3\mathbf{k}$. $\mathbf{R}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 3t^2\mathbf{k}$. $L = \int_{-1}^{1} \|\mathbf{R}'\| dt = \int_{-1}^{1} \sqrt{1 + 9t^4}\,dt = 3.0957 \approx 3.096$

56. $\mathbf{R}(t) = \sin 2t\mathbf{i} + \cos 2t\mathbf{j} + t^{1/2}\mathbf{k}$; $t_1 = 0$; $t_2 = 4$

▹ $\mathbf{R}'(t) = 2\cos 2t\mathbf{i} - 2\sin 2t\mathbf{j} + \tfrac{1}{2}t^{-1/2}\mathbf{k}$

$$\|\mathbf{R}'(t)\| = \sqrt{4\cos^2 2t + 2\sin^2 2t + \frac{1}{4t}} = \sqrt{4 + \frac{1}{4t}}$$

Because $\|\mathbf{R}'(t)\|$ is unbounded at 0, the arc length integral is improper. We let $t = u^2$, $dt = 2u\,du$ to get a proper integral, then use NINT.

$$L = \int_{t=0}^{4} \sqrt{4 + \frac{1}{4t}}\,dt = \int_{u=0}^{2} \sqrt{4 + \frac{1}{4u^2}}(2u\,du) = \int_0^2 \sqrt{16u^2 + 1}\,du = 8.4093 \approx 8.409$$

The exact value of the integral is $\sqrt{65} + \tfrac{1}{16}\ln(129 + 16\sqrt{65})$.

57. $D_t[\mathbf{R}'(t) \cdot \mathbf{R}(t)] = \mathbf{R}''(t) \cdot \mathbf{R}(t) + \mathbf{R}'(t) \cdot \mathbf{R}'(t) = \mathbf{R}''(t) \cdot \mathbf{R}(t) + \|\mathbf{R}'(t)\|^2$

58. $\mathbf{R}(t)\cdot\mathbf{R}(t)=\|\mathbf{R}(t)\|^2=h^2(t)$. Differentiating, $\mathbf{R}'(t)\cdot\mathbf{R}(t)+\mathbf{R}(t)\cdot\mathbf{R}'(t)=2h(t)h'(t)\Rightarrow \mathbf{R}(t)\cdot\mathbf{R}'(t)=h(t)h'(t)$

59. From Theorem 11.2.6, $D_t\left[\dfrac{\mathbf{R}(t)}{f(t)}\right]=D_t[f(t)^{-1}\mathbf{R}(t)]=-f(t)^{-2}f'(t)\mathbf{R}(t)+f(t)^{-1}\mathbf{R}'(t)=\dfrac{f(t)\mathbf{R}'(t)-f'(t)\mathbf{R}(t)}{[f(t)]^2}$

60. Prove that if $\mathbf{A}$ and $\mathbf{B}$ are constant vectors and f and g are integrable functions, then
$\int[\mathbf{A}f(t)+\mathbf{B}g(t)]dt=\mathbf{A}\int f(t)dt+\mathbf{B}\int g(t)dt$
(*Hint:* Express $\mathbf{A}$ and $\mathbf{B}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.)

▶ Because $\mathbf{A}$ and $\mathbf{B}$ are constant vectors, there are constants such that
$\mathbf{A}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{B}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$
Therefore

$$\int[\mathbf{A}f(t)+\mathbf{B}g(t)]dt$$
$$=\int[(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})f(t)+(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})g(t)]dt$$
$$=\int[(a_1f(t)+b_1g(t))\mathbf{i}+(a_2f(t)+b_2g(t))\mathbf{j}+(a_3f(t)+b_3g(t))\mathbf{k}]dt$$
$$=\mathbf{i}\int[a_1f(t)+b_1g(t)]dt+\mathbf{j}\int[a_2f(t)+b_2g(t)]dt+\mathbf{k}\int[a_3f(t)+b_3g(t)]dt \qquad \text{(Definition 11.2.10)}$$
$$=\mathbf{i}\left[a_1\int f(t)dt+b_1\int g(t)dt\right]+\mathbf{j}\left[a_2\int f(t)dt+b_2\int g(t)dt\right]+\mathbf{k}\left[a_3\int f(t)dt+b_3\int g(t)dt\right]$$
$$=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\int f(t)dt+(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\int g(t)dt=\mathbf{A}\int f(t)dt+\mathbf{B}\int g(t)dt$$

61. Let $\mathbf{R}(t)=f_1(t)\mathbf{i}+g_1(t)\mathbf{j}+h_1(t)\mathbf{k}$ and $\mathbf{Q}(t)=f_2(t)\mathbf{i}+g_2(t)\mathbf{j}+h_2(t)\mathbf{k}$. If $\mathbf{R}'(t)\equiv\mathbf{Q}'(t)$ then $f_1{}'(t)\equiv f_2{}'(t)$, $g_1{}'(t)\equiv g_2{}'(t)$, $h_1{}'(t)\equiv h_2{}'(t)$. By Theorem 4.1.2, there are constants k, l, m such that $f_1(t)=f_2(t)+k$, $g_1(t)=g_2(t)+l$, $h_1(t)=h_2(t)+m$ and so $\mathbf{R}(t)=\mathbf{Q}(t)+\mathbf{K}$ where $\mathbf{K}=k\mathbf{i}+l\mathbf{j}+m\mathbf{k}$.

62. Let $\mathbf{G}(t)$ be any antiderivative of $\mathbf{R}$. Then $\mathbf{G}'(t)=\mathbf{F}'(t)$. By Exercise 62, $\mathbf{G}(t)=\mathbf{F}(t)+\mathbf{C}$.

63. Let $\mathbf{R}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$. Then $D_t\int_a^t\mathbf{R}(u)du=D_t\left[\int_a^t[f(u)\mathbf{i}+g(u)\mathbf{j}+h(u)\mathbf{k}]du\right]$
$=\left(D_t\int_a^t f(u)du\right)\mathbf{i}+\left(D_t\int_a^t g(u)du\right)\mathbf{j}+\left(D_t\int_a^t h(u)du\right)\mathbf{k}=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}=\mathbf{R}(t)$

64. Use the theorems of Exercises 62 and 63 to prove the following theorem that corresponds to the second fundamental theorem of the calculus (4.7.2): If the function $\mathbf{R}$ is continuous on the closed interval $[a,b]$ and if $\mathbf{F}(t)$ is any antiderivative of $\mathbf{R}$ on $[a,b]$, then $\int_a^b\mathbf{R}(u)du=\mathbf{F}(b)-\mathbf{F}(a)$.

▶ Because by hypothesis $\mathbf{F}'(t)=\mathbf{R}(t)$, and by Exercise 63, $D_t\int_a^t\mathbf{R}(u)du=\mathbf{R}(t)$, it follows from Exercise 62 that
$\mathbf{F}(t)=\int_a^t\mathbf{R}(u)du+\mathbf{C}$
where $\mathbf{C}$ is a constant vector. Because $\int_a^a\mathbf{R}(u)du=0$, it follows that
$\mathbf{F}(b)-\mathbf{F}(a)=\left(\int_a^b\mathbf{R}(u)du+\mathbf{C}\right)-\left(\int_a^a\mathbf{R}(u)du+\mathbf{C}\right)=\int_a^b\mathbf{R}(u)du$

11.3 THE UNIT TANGENT AND UNIT NORMAL VECTORS AND ARC LENGTH AS A PARAMETER

We assume that a curve has a direction (or orientation) implied by increasing values of the parameter.

11.3.1 Definition If $\mathbf{R}(t)$ is the position vector of curve C at a point P on C, then the *unit tangent vector* of C at P, denoted by $\mathbf{T}(t)$, is the unit vector in the direction of $D_t\mathbf{R}(t)$ if $D_t\mathbf{R}(t)\neq 0$.
Thus, denoting t-derivatives by dots,

$$\mathbf{T}(t)=\frac{D_t\mathbf{R}(t)}{\|D_t\mathbf{R}(t)\|}=\frac{\dot{\mathbf{R}}}{\|\dot{\mathbf{R}}\|} \qquad (1)$$

11.3.2 Definition If $\mathbf{T}(t)$ is the unit tangent vector of curve C at a point P on C, then the *unit normal vector*, denoted by $\mathbf{N}(t)$ is the unit vector in the direction of $D_t\mathbf{T}(t)$. Thus

$$\mathbf{N}(t)=\frac{D_t\mathbf{T}(t)}{\|D_t\mathbf{T}(t)\|}=\frac{\dot{\mathbf{T}}}{\|\dot{\mathbf{T}}\|} \qquad (2)$$

Moving Trihedral The *unit binormal vector* to the curve C is given by $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ The planes at P having $\mathbf{B}$, $\mathbf{N}$, and $\mathbf{T}$ as respective normal vectors are the *osculating, rectifying* and *normal plane*.

If F, f, g, and h are real-valued functions and $\mathbf{R}$ is the vector-valued function defined by
$\mathbf{R}(t) = F(t)[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]$
then the magnitude of $\mathbf{R}(t)$ is given by
$$\|\mathbf{R}(t)\| = |F(t)|\sqrt{[f(t)]^2 + [g(t)]^2 + [h(t)]^2}$$

Finding T and N We may multiply $\dot{\mathbf{R}}$ and $\dot{\mathbf{T}}$ by any positive function before using (1) and (2). See Exercise 4.
For $\mathbf{N}$, we may use the following version of the quotient rule.
$$D_t\frac{f(t)}{\sqrt{g(t)}} = D_t\{f(t)[g(t)]^{-1/2}\} = f'(t)[g(t)]^{-1/2} - \tfrac{1}{2}f(t)[g(t)]^{-3/2}g'(t) = \frac{f'(t)g(t) - \frac{1}{2}f(t)g'(t)}{[g(t)]^{3/2}}$$

11.3.3 Theorem If the vector equation of a curve C is $\mathbf{R}(s) = f(s)\mathbf{i} + g(s)\mathbf{j} + h(s)\mathbf{k}$, where s units is the length of arc measured from a particular point P_0 on C to the point P, then the unit tangent vector of C at P is given by $\mathbf{T}(s) = D_s\mathbf{R}(s)$ if it exists. Furthermore
$$\left(\frac{df}{ds}\right)^2 + \left(\frac{dg}{ds}\right)^2 + \left(\frac{dh}{ds}\right)^2 = 1 \tag{8}$$

Helix A curve which lies on a cylinder or cone and cuts the elements under constant angle. If the cylinder is circular, we have a *circular helix* with equation $\mathbf{R} = a \cos t\mathbf{i} + a \sin t\mathbf{j} + ct\mathbf{k}$, which is a circle if $c = 0$. See Exercises 8 ff. A *conical helix* has equation $\mathbf{R} = ae^t\cos t\mathbf{i} + ae^t\sin t\mathbf{j} + ce^t\mathbf{k}$ which is a logarithmic spiral if $c = 0$. See Exercises 10 ff.

Exercises 11.3 (Starred numbers refer to Exercises 11.4.)

In Exercises 1–6, find $\mathbf{T}(t)$, $\mathbf{T}(t_1)$, $\mathbf{N}(t)$, $\mathbf{N}(t_1)$, sketch a portion of the curve and representations of $\mathbf{T}(t_1)$ and $\mathbf{N}(t_1)$. In Exercises 1*–6*, use the formula $K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\|$ find the curvature K; find the radius of curvature ρ. Sketch the curve, unit tangent vector, and circle of curvature.

1. $\mathbf{R} = 3 \cos t\mathbf{i} + 3 \sin t\mathbf{j}$; $t_1 = \frac{1}{2}\pi$. $\dot{\mathbf{R}} = -3 \sin t\mathbf{i} + 3 \cos t\mathbf{j}$. $\|\dot{\mathbf{R}}\| = 3\sqrt{\sin^2 t + \cos^2 t} = 3$. $\mathbf{T} = \dfrac{\dot{\mathbf{R}}}{\|\dot{\mathbf{R}}\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$.
$\mathbf{T}(\frac{1}{2}\pi) = -\mathbf{i}$. $\dot{\mathbf{T}} = -\cos t\mathbf{i} - \sin t\mathbf{j} = \mathbf{N}$ because $\|\dot{\mathbf{T}}\| = \sqrt{\cos^2 t + \sin^2 t} = 1$. $\mathbf{N}(\frac{1}{2}\pi) = -\mathbf{j}$

1*. $K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{1}{3}$. $\rho = \frac{1}{K} = 3$. The curve is a circle of radius 3; it is its own circle of curvature.

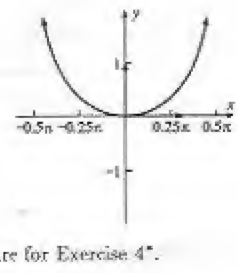
2. $\mathbf{R} = \cos 3t\mathbf{i} + \sin 3t\mathbf{j}$; $t = \frac{1}{3}\pi$. $\dot{\mathbf{R}} = -3 \sin 3t\mathbf{i} + 3 \cos 3t\mathbf{j}$. $\|\dot{\mathbf{R}}\| = 3\sqrt{\sin^2 3t + \cos^2 3t} = 3$.
$\mathbf{T} = \dot{\mathbf{R}}/\|\dot{\mathbf{R}}\| = -\sin 3t\mathbf{i} + \cos 3t\mathbf{j}$. $\mathbf{T}(\frac{1}{3}\pi) = -\mathbf{i}$. $\dot{\mathbf{T}} = -3 \cos 3t\mathbf{i} - 3 \sin 3t\mathbf{j}$. $\|\dot{\mathbf{T}}\| = 3\sqrt{\cos^2 3t + \sin^2 3t} = 3$.
$\mathbf{N} = \dot{\mathbf{T}}/\|\dot{\mathbf{T}}\| = -\cos 3t\mathbf{i} - \sin 3t\mathbf{j}$. $\mathbf{N}(\frac{1}{3}\pi) = -\mathbf{j}$.

2*. $K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{3}{3} = 1$. $\rho = \frac{1}{K} = 1$. The curve is a unit circle; it is its own circle of curvature.

3. $\mathbf{R} = \ln \sin t\mathbf{i} + t\mathbf{j}$; $0 < t < \pi$; $t_1 = \frac{1}{2}\pi$. $\dot{\mathbf{R}} = \frac{\cos t}{\sin t}\mathbf{i} + \mathbf{j}$. $(\sin t)\dot{\mathbf{R}} = \cos t\mathbf{i} + \sin t\mathbf{j} = \mathbf{T}$ because it is a unit vector.
$\mathbf{T}(\frac{1}{2}\pi) = \mathbf{j}$. $\dot{\mathbf{T}} = -\sin t\mathbf{i} + \cos t\mathbf{j} = \mathbf{N}$ because it is a unit vector. $\mathbf{N}(\frac{1}{2}\pi) = -\mathbf{i}$

3*. $\dot{\mathbf{R}}(\frac{1}{2}\pi) = \mathbf{j}$. $K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{1}{1} = 1$. $\rho = \frac{1}{K} = 1$.

4. $\mathbf{R}(t) = t\mathbf{i} - \ln \cos t\mathbf{j}$; $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$; $t_1 = 0$



▸ $D_t\mathbf{R}(t) = \mathbf{i} + \frac{\sin t}{\cos t}\mathbf{j}$
Because $\cos t > 0$ in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, then
$(\cos t)D_t\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} = \mathbf{T}(t)$
because it has the same direction and it is a unit vector.
$\mathbf{T}(0) = \mathbf{i}$
$D_t\mathbf{T}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} = \mathbf{N}(t)$
because it is a unit vector.
$\mathbf{N}(0) = \mathbf{j}$
The figure shows a portion of the curve, $\mathbf{T}(0)$ and $\mathbf{N}(0)$ and the circle of curvature for Exercise 4*.

4*. $\mathbf{R}(0) = \mathbf{0}$ $\qquad \|\dot{\mathbf{R}}(0)\| = \sqrt{1 + \tan^2 0} = 1$
$K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = 1/1 = 1$ $\qquad \rho = 1/K = 1$
$\mathbf{R}_c(0) = \mathbf{R}(0) + \rho\mathbf{N}(0) = \mathbf{j}$

5. $\mathbf{R}(t) = (\frac{1}{3}t^3 - t)\mathbf{i} + t^2\mathbf{j}$; $t_1 = 2$. $D_t\mathbf{R}(t) = (t^2-1)\mathbf{i} + 2t\mathbf{j}$.

$$\|D_t\mathbf{R}(t)\| = \sqrt{(t^2-1)^2 + 4t^2} = \sqrt{(t^2+1)^2} = t^2+1.\ \mathbf{T}(t) = \frac{D_t\mathbf{R}(t)}{\|D_t\mathbf{R}(t)\|} = \frac{t^2-1}{t^2+1}\mathbf{i} + \frac{2t}{t^2+1}\mathbf{j}.\ \mathbf{T}(2) = \tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}$$

$$D_t\mathbf{T}(t) = \frac{2t(t^2+1) - 2t(t^2-1)}{(t^2+1)^2}\mathbf{i} + \frac{2(t^2+1) - 2t(2t)}{(t^2+1)^2}\mathbf{j} = \frac{4t}{(t^2+1)^2}\mathbf{i} + \frac{2(1-t^2)}{(t^2+1)^2}\mathbf{j}$$

$$\|D_t\mathbf{T}(t)\| = \sqrt{\frac{(4t)^2}{(t^2+1)^4} + \frac{4(1-t^2)^2}{(t^2+1)^4}} = \frac{2}{(t^2+1)^2}\sqrt{4t^2 + (1 - 2t^2 + t^4)} = \frac{2}{(t^2+1)^2}\sqrt{(t^2+1)^2} = \frac{2}{t^2+1}$$

$$\mathbf{N}(t) = \frac{D_t\mathbf{T}(t)}{\|D_t\mathbf{T}(t)\|} - \left[\frac{4t}{(t^2+1)^2}\mathbf{i} + \frac{2(1-t^2)}{(t^2+1)^2}\mathbf{j}\right]\frac{t^2+1}{2} = \frac{2t}{t^2+1}\mathbf{i} + \frac{1-t^2}{t^2+1}\mathbf{j}.\ \mathbf{N}(2) = \tfrac{4}{5}\mathbf{i} - \tfrac{3}{5}\mathbf{j}$$

5*. $K = \|\dot{\mathbf{T}}(2)\|/\|\dot{\mathbf{R}}(2)\| = \frac{2}{2^2+1}/(2^2+1) = \frac{2}{25}$. $\rho = \frac{1}{K} = \frac{25}{2}$

6. $\mathbf{R} = \frac{1}{2}t^2\mathbf{i} + \frac{1}{3}t^3\mathbf{j}$, $t > 0$; $t_1 = 1$. $\dot{\mathbf{R}} = t\mathbf{i} + t^2\mathbf{j}$. $\mathbf{t} = \dot{\mathbf{R}}/t = \mathbf{i} + t\mathbf{j}$. $\|\mathbf{t}\| = \sqrt{1+t^2}$.

$$\mathbf{T} = \frac{1}{\sqrt{1+t^2}}\mathbf{i} + \frac{t}{\sqrt{1+t^2}}\mathbf{j}.\ \mathbf{T}(1) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.\ \dot{\mathbf{T}} = -\frac{t}{(1+t^2)^{3/2}}\mathbf{i} + \frac{1(1+t^2) - \frac{1}{2}t(2t)}{(1+t^2)^{3/2}}\mathbf{j} = \frac{t\mathbf{i}+\mathbf{j}}{(1+t^2)^{3/2}}.\ \mathbf{n} = -t\mathbf{i} + \mathbf{j}$$

$$\|\mathbf{n}\| = \sqrt{t^2+1}.\ \mathbf{N} = -\frac{t}{\sqrt{t^2+1}}\mathbf{i} + \frac{1}{\sqrt{t^2+1}}\mathbf{j}.\ \mathbf{N}(1) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

6*. $\|\dot{\mathbf{R}}(1)\| = \sqrt{1^2+1^2} = \sqrt{2}$. $\|\dot{\mathbf{T}}(1)\| = \frac{1}{2^{3/2}}\sqrt{1^2+1^2} = \frac{1}{2}$. $K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{1}{4}$. $\rho = \frac{1}{K} = 4$

In Exercises 7–10, find $\mathbf{T}(t)$ and $\mathbf{N}(t)$. In Exercises 11–14, find the moving trihedral at $t = t_1$. In Exercises 15 and 16, find the moving trihedral at any point. In Exercises 17–22, find equations of the osculating, rectifying, and normal planes at $t = t_1$. In Exercises 31–36, find an equation of the having with arc length s as a parameter, where s increases with t from $t = 0$. Check by using Eq. (8). In Exercises 7*–10*, use the formula $K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\|$ find the curvature K; in Exercises 11*–14*, use the formula $K(t) = \|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3$.

In Exercises 7, 11, 17, 31, 7*, and 11*, $\mathbf{R} = (\sin t - t\cos t)\mathbf{i} + (\cos t + t\sin t)\mathbf{j} + 2\mathbf{k}$.

7. $\dot{\mathbf{R}} = t\sin t\mathbf{i} + t\cos t\mathbf{j}$, $t > 0$. $\mathbf{t} = \dot{\mathbf{R}}/t = \sin t\mathbf{i} + \cos t\mathbf{j} = \mathbf{T}$. $\dot{\mathbf{T}} = \cos t\mathbf{i} - \sin t\mathbf{j} = \mathbf{N}$.

11, 17. $t_1 = \frac{1}{2}\pi$. $\mathbf{R}(\frac{1}{2}\pi) = \mathbf{i} + \frac{1}{2}\pi\mathbf{j} + 2\mathbf{k}$. $\mathbf{T}(\frac{1}{2}\pi) = \mathbf{i}$, normal: $x = 1$. $\mathbf{N}(\frac{1}{2}\pi) = -\mathbf{j}$, rectifying: $y = \frac{1}{2}\pi$

$\mathbf{B}(\frac{1}{2}\pi) = \mathbf{i} \times (-\mathbf{j}) = -\mathbf{k}$, osculating: $z = 2$

31. $\dot{s} = \|\dot{\mathbf{R}}\| = t\sqrt{\sin^2 t + \cos^2 t} = t$, $s = \int_0^t u\,du = \frac{1}{2}u^2\Big]_0^t = \frac{1}{2}t^2$; $t = \sqrt{2s}$

$\mathbf{R} = (\sin\sqrt{2s} - \sqrt{2s}\cos\sqrt{2s})\mathbf{i} + (\cos\sqrt{2s} + \sqrt{2s}\sin\sqrt{2s})\mathbf{j} + 2\mathbf{k}$

7*. $K = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = 1/t$

11*. $\ddot{\mathbf{R}} = (\sin t + t\cos t)\mathbf{i} + (\cos t - t\sin t)\mathbf{j}$. $\|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t\sin t & t\cos t & 0 \\ \sin t + t\cos t & \cos t - t\sin t & 0 \end{matrix} \right\| = |-t^2| = t^2$

$K = \|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3 = t^2/t^3 = 1/t$

In Exercises 8, 12, 20, 32, 8*, and 12*, the curve is the circular helix $\mathbf{R}(t) = \sin 3t\mathbf{i} - \cos 3t\mathbf{j} + 4t\mathbf{k}$

8. $\dot{\mathbf{R}}(t) = 3\cos 3t\mathbf{i} + 3\sin 3t\mathbf{j} + 4\mathbf{k}$

$\|\dot{\mathbf{R}}(t)\| = \sqrt{9\cos^2 3t + 9\sin^2 3t + 16} = \sqrt{9+16} = 5$

$$\mathbf{T}(t) = \frac{\dot{\mathbf{R}}(t)}{\|\dot{\mathbf{R}}(t)\|} = \tfrac{3}{5}\cos 3t\mathbf{i} + \tfrac{3}{5}\sin 3t\mathbf{j} + \tfrac{4}{5}\mathbf{k}$$

$\dot{\mathbf{T}}(t) = -\frac{9}{5}\sin 3t\mathbf{i} + \frac{9}{5}\cos 3t\mathbf{j}$

$\dot{\mathbf{T}}(t)/\frac{9}{5} = -\sin 3t\mathbf{i} + \cos 3t\mathbf{j} = \mathbf{N}(t)$

12. $t_1 = \frac{1}{3}\pi$. $\mathbf{T}(\frac{1}{3}\pi) = \frac{3}{5}\cos\pi\mathbf{i} + \frac{3}{5}\sin\pi\mathbf{j} + \frac{4}{5}\mathbf{k} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$

$\mathbf{N}(\frac{1}{3}\pi) = -\sin\pi\mathbf{i} + \cos\pi\mathbf{j} = -\mathbf{j}$

$\mathbf{B}(\frac{1}{3}\pi) = \mathbf{T}(\frac{1}{3}\pi) \times \mathbf{N}(\frac{1}{3}\pi) = (-\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}) \times (-\mathbf{j}) = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{k}$

20. $\mathbf{R}(\frac{1}{3}\pi) = \sin \pi\mathbf{i} - \cos \pi\mathbf{j} + \frac{4}{3}\pi\mathbf{k} = \mathbf{j} + \frac{4}{3}\mathbf{k}$
A normal vector to the osculating plane is $5\mathbf{B}(\frac{1}{3}\pi) = \langle 4, 0, 3\rangle$. An equation is $4x + 3(z - \frac{4}{3}) = 0$.

A normal vector to the rectifying plane is $-\mathbf{N}(\frac{1}{3}\pi) = \langle 0, 1, 0\rangle$. An equation is $y - 1 = 0$.

A normal vector to the normal plane is $5\mathbf{T}(\frac{1}{3}\pi) = \langle -3, 0, 4\rangle$. An equation is $-3x + 4(z - \frac{4}{3}) = 0$.

32. $\dot{s} = \|\dot{\mathbf{R}}(t)\| = 5$. $s = \int_0^t \|\dot{\mathbf{R}}(u)\| du = \int_0^t 5\, du = 5t$

Thus, $t = \frac{1}{5}s$. Hence,

$\mathbf{R}(s) = \sin \frac{3}{5}s\mathbf{i} - \cos \frac{3}{5}s\mathbf{j} + \frac{4}{5}s\mathbf{k}$

Because $f(s) = \sin \frac{3}{5}s$, $g(s) = -\cos \frac{3}{5}s$, and $h(s) = \frac{4}{5}s$,

$f'(s)^2 + g'(s)^2 + h'(s)^2 = (\frac{3}{5}\cos \frac{3}{5}s)^2 + (\frac{3}{5}\sin \frac{3}{5}s)^2 + (\frac{4}{5})^2 = \frac{9}{25}(\cos^2 \frac{3}{5}s + \sin^2 \frac{3}{5}s) + \frac{16}{25} = 1$

8*. $\mathrm{K} = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{9}{5}/5 = \frac{9}{25}$

12*. $\ddot{\mathbf{R}}(t) = -9 \sin 3t\mathbf{i} + 9 \cos 3t\mathbf{j}$.

$$\|\dot{\mathbf{R}}(t) \times \ddot{\mathbf{R}}(t)\| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos 3t & 3\sin 3t & 4 \\ -9\sin 3t & 9\cos 3t & 0 \end{vmatrix} = \|-36\cos 3t\mathbf{i} - 36 \sin 3t\mathbf{j} + 27\mathbf{k}\| = 45$$

$\mathrm{K}(t) = \|\dot{\mathbf{R}}(t) \times \ddot{\mathbf{R}}(t)\|/\|\dot{\mathbf{R}}(t)\|^3 = 45/5^3 = \frac{9}{25}$

In Exercises 9, 13, 19, 33, 9*, and 13*, $\mathbf{R} = \mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$, $t > 0$.

9. $\dot{\mathbf{R}} = t\mathbf{j} + t^2\mathbf{k}$. $\mathbf{t} = \dot{\mathbf{R}}/t = \mathbf{j} + t\mathbf{k}$. $\|\mathbf{t}\| = \sqrt{1+t^2}$. $\mathbf{T} = \frac{\mathbf{t}}{\|\mathbf{t}\|} = \frac{1}{\sqrt{1+t^2}}\mathbf{j} + \frac{t}{\sqrt{1+t^2}}\mathbf{k}$.

$\dot{\mathbf{T}} = -\frac{t}{(1+t^2)^{3/2}}\mathbf{j} + \frac{1}{(1+t^2)^{3/2}}\mathbf{k}$. $\mathbf{n} = (1+t^2)^{3/2}\dot{\mathbf{T}} = -t\mathbf{j} + \mathbf{k}$. $\mathbf{N} = -\frac{t}{\sqrt{t^2+1}}\mathbf{j} + \frac{1}{\sqrt{t^2+1}}\mathbf{k}$

13. $t_1 = 1$. $\mathbf{T}(1) = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$. $\mathbf{N}(1) = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$. $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \mathbf{i}$

19. $\mathbf{R}(1) = \mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{3}\mathbf{k}$. osculating: $\mathbf{B}(1) = \mathbf{i}$, $x - 1 = 0$. rectifying: $\sqrt{2}\mathbf{N}(1) = -\mathbf{j} + \mathbf{k}$, $-(y - \frac{1}{2}) + (z - \frac{1}{3}) = 0$

normal: $\sqrt{2}\mathbf{T}(1) = \mathbf{j} + \mathbf{k}$, $(y - \frac{1}{2}) + (z - \frac{1}{3}) = 0$

33. $\dot{s} = \|\dot{\mathbf{R}}\| = t\sqrt{1+t^2}$. $s = \int_0^t u\sqrt{1+u^2}du = \frac{1}{3}(1+u^2)^{3/2}\Big]_0^t = \frac{1}{3}[(1+t^2)^{3/2} - 1]$, $t = \sqrt{(3s+1)^{2/3} - 1}$

$\mathbf{R}(s) = \mathbf{i} + \frac{1}{2}[(3s+1)^{2/3} - 1]\mathbf{j} + \frac{1}{3}[(3s+1)^{2/3} - 1]^{3/2}\mathbf{k}$

9*. $\mathrm{K} = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \frac{\sqrt{t^2+1}}{(1+t^2)^{3/2}}/t\sqrt{1+t^2} = \frac{1}{t(1+t^2)^{3/2}}$

13*. $\ddot{\mathbf{R}} = \mathbf{j} + 2t\mathbf{k}$. $\|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & t & t^2 \\ 0 & 1 & 2t \end{matrix} \right\| = \|t^2\mathbf{i}\| = t^2$. $\mathrm{K} = \|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3 = t^2/(t\sqrt{1+t^2})^3 = \frac{1}{t(1+t^2)^{3/2}}$

In Exercises 10, 14, 18, 34, 10*, and 14*, the curve is the conical helix $\mathbf{R} = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j} + e^t\mathbf{k}$.

10. From §9.2, $\dot{\mathbf{R}} = \sqrt{2}e^t\cos(t + \frac{1}{4}\pi)\mathbf{i} + \sqrt{2}e^t\sin(t + \frac{1}{4}\pi)\mathbf{j} + e^t\mathbf{k}$. $\|\dot{\mathbf{R}}\| = e^t\sqrt{2\cos^2(t + \frac{1}{4}\pi) + 2\sin^2(t + \frac{1}{4}\pi) + 1} = \sqrt{3}e^t$

$\mathbf{T} = \sqrt{\frac{2}{3}}\cos(t+\frac{1}{4}\pi)\mathbf{i} + \sqrt{\frac{2}{3}}\sin(t+\frac{1}{4}\pi)\mathbf{j} + \sqrt{\frac{1}{3}}\mathbf{k}$. $\dot{\mathbf{T}} = -\sqrt{\frac{2}{3}}\sin(t+\frac{1}{4}\pi)\mathbf{i} + \sqrt{\frac{2}{3}}\cos(t+\frac{1}{4}\pi)\mathbf{j}$. $\mathbf{N} = -\sin(t+\frac{1}{4}\pi)\mathbf{i} + \cos(t+\frac{1}{4}\pi)\mathbf{j}$

14. $t_1 = 0$. $\mathbf{T}(0) = \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}}\mathbf{i} + \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}}\mathbf{j} + \sqrt{\frac{1}{3}}\mathbf{k} = \sqrt{\frac{1}{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$. $\mathbf{N}(0) = \sqrt{\frac{1}{2}}(-\mathbf{i} + \mathbf{j})$. $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \sqrt{\frac{1}{6}}(-\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

18. $\mathbf{R}(0) = \mathbf{i} + \mathbf{k}$. osculating: $-\sqrt{6}\mathbf{B}(0) = \langle 1, 1, -2\rangle$, $(x - 1) + y - 2(z - 1) = 0$. rectifying: $\sqrt{2}\mathbf{N} = \langle -1, 1, 0\rangle$,
$-(x - 1) + y = 0$. normal: $\sqrt{3}\mathbf{T} = \langle 1, 1, 1\rangle$, $(x - 1) + y + (z - 1) = 0$

34. $\dot{s} = \|\dot{\mathbf{R}}\| = \sqrt{3}e^t$, $s = \int_0^t \sqrt{3}e^u du = \sqrt{3}e^u\Big]_0^t = \sqrt{3}(e^t - 1)$. $e^t = \frac{1}{3}\sqrt{3}s + 1$, $t = \ln(\frac{1}{3}\sqrt{3}s + 1)$

$\mathbf{R} = (\frac{1}{3}\sqrt{3}s + 1)[\cos \ln(\frac{1}{3}\sqrt{3}s + 1)\mathbf{i} + \sin \ln(\frac{1}{3}\sqrt{3}s + 1)\mathbf{j} + \mathbf{k}]$

10*. $\mathrm{K} = \|\dot{\mathbf{T}}\|/\|\dot{\mathbf{R}}\| = \sqrt{\frac{2}{3}}/\sqrt{3}e^t = \frac{1}{3}\sqrt{2}e^{-t}$

14. $\dot{\mathbf{R}} = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j} + e^t\mathbf{k}$. $\ddot{\mathbf{R}} = -2e^t \sin t\mathbf{i} + 2e^t \cos t\mathbf{j} + e^t\mathbf{k}$

$$\|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t(\cos t - \sin t) & e^t(\cos t + \sin t) & e^t \\ -2e^t \sin t & 2e^t \cos t & e^t \end{matrix} \right\|$$

$= \|e^{2t}(\sin t - \cos t)\mathbf{i} - e^{2t}(\sin t + \cos t)\mathbf{j} + 2e^{2t}\mathbf{k}\| = \sqrt{6}e^{2t}$. $K = \|\dot{\mathbf{R}} \times \ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3 = \sqrt{6}e^{2t}/(\sqrt{3}e^t)^3 = \frac{1}{3}\sqrt{2}e^{-t}$

In Exercises 15, 21 and 35. $\mathbf{R} = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j} + 2\mathbf{k}$, $0 \le t \le \frac{1}{2}\pi$

15. $\dot{\mathbf{R}} = -3\cos^2 t \sin t\mathbf{i} + 3\sin^2 t \cos t\mathbf{j}$. $\dot{\mathbf{R}}/3 \sin t \cos t = -\cos t\mathbf{i} + \sin t\mathbf{j} = \mathbf{T}$. $\dot{\mathbf{T}} = \sin t\mathbf{i} + \cos t\mathbf{j} = \mathbf{N}$. $\mathbf{B} = \mathbf{T} \times \mathbf{N} = -\mathbf{k}$

21. $t_1 = \frac{1}{4}\pi$. $\mathbf{R}(\frac{1}{4}\pi) = \frac{1}{4}\sqrt{2}\mathbf{i} + \frac{1}{4}\sqrt{2}\mathbf{j} + 2\mathbf{k}$. osculating: $\mathbf{B}(\frac{1}{4}\pi) = -\mathbf{k}$, $z = 2$. rectifying: $\sqrt{2}\mathbf{N} = \langle 1, 1, 0\rangle$,

$(x - \frac{1}{4}\sqrt{2}) + (y - \frac{1}{4}\sqrt{2}) = 0$. normal: $\sqrt{2}\mathbf{T} = \langle -1, 1, 0\rangle$, $-(x - \frac{1}{4}\sqrt{2}) + (y - \frac{1}{4}\sqrt{2}) = 0$

35. $\dot{s} = \|\dot{\mathbf{R}}\| = 3 \sin t \cos t$. $s = \displaystyle\int_0^t 3 \sin u \, d(\sin u) = \frac{3}{2}\sin^2 u\Big]_0^t = \frac{3}{2}\sin^2 t$, $\sin t = \sqrt{\tfrac{2}{3}s}$, $\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - \tfrac{2}{3}s}$

$\mathbf{R} = (1 - \frac{2}{3}s)^{3/2}\mathbf{i} + (\frac{2}{3}s)^{3/2}\mathbf{j} + 2\mathbf{k}$

In Exercises 16, 22, and 36, $\mathbf{R}(t) = \cosh t\mathbf{i} + \sinh t\mathbf{j} + t\mathbf{k}$.

16. $\dot{\mathbf{R}}(t) = \sinh t\mathbf{i} + \cosh t\mathbf{j} + \mathbf{k}$

$\|\dot{\mathbf{R}}(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$

$\mathbf{T}(t) = \dot{\mathbf{R}}/\|\dot{\mathbf{R}}\| = \frac{1}{2}\sqrt{2}(\tanh t\mathbf{i} + \mathbf{j} + \operatorname{sech} t\mathbf{k})$

$\dot{\mathbf{T}}(t) = \frac{1}{2}\sqrt{2}(\operatorname{sech}^2 t\mathbf{i} - \operatorname{sech} t \tanh t\mathbf{k})$

$\sqrt{2}(\cosh t)\dot{\mathbf{T}}(t) = \operatorname{sech} t\mathbf{i} - \tanh t\mathbf{k} = \mathbf{N}(t)$

because

$\|\mathbf{N}(t)\| = \sqrt{\operatorname{sech}^2 t + \tanh^2 t} = 1$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \tanh t & 1 & \operatorname{sech} t \\ \operatorname{sech} t & 0 & -\tanh t \end{vmatrix} = -\tanh t\mathbf{i} + \mathbf{j} - \operatorname{sech} t\mathbf{k}$$

22. $t_1 = 0$. $\mathbf{R}(0) = \mathbf{i}$. osculate: $\mathbf{B}(0) = \mathbf{j} - \mathbf{k}$, $y - z = 0$. rectify: $\mathbf{N}(0) = \mathbf{i}$, $x - 1 = 0$. normal: $\mathbf{T}(0) = \mathbf{j} + \mathbf{k}$, $y + z = 0$

36. $\dot{s} = \|\dot{\mathbf{R}}(t)\| = \sqrt{2}\cosh t$

$s = \displaystyle\int_0^t \|\dot{\mathbf{R}}(u)\| du = \int_0^t \sqrt{2}\cosh u \, du = \sqrt{2}\sinh u\Big]_0^t = \sqrt{2}\sinh t$

$\sinh t = \frac{1}{2}\sqrt{2}s$

$\cosh t = \sqrt{\sinh^2 t + 1} = \sqrt{\frac{1}{2}s^2 + 1}$

$\mathbf{R}(s) = \sqrt{\frac{1}{2}s^2 + 1}\,\mathbf{i} + \frac{1}{2}\sqrt{2}s\mathbf{j} + \sinh^{-1}(\frac{1}{2}\sqrt{2}s)\mathbf{k}$

$f(s) = \sqrt{\frac{1}{2}s^2 + 1}$, $g(s) = \frac{1}{2}\sqrt{2}s$, $h(s) = \sinh^{-1}(\frac{1}{2}\sqrt{2}s)$

$$[f'(s)]^2 + [g'(s)]^2 + [h'(s)]^2 = \left[\frac{s}{2\sqrt{\frac{1}{2}s^2 + 1}}\right]^2 + [\tfrac{1}{2}\sqrt{2}]^2 + \left[\frac{\frac{1}{2}\sqrt{2}}{\sqrt{1 + \frac{1}{2}s^2}}\right]^2 = \frac{s^2}{2s^2 + 4} + \frac{1}{2} + \frac{1}{2 + s^2} = \frac{2s^2 + 4}{2s^2 + 4} = 1$$

23. $\mathbf{T}(\frac{1}{2}\pi) = \frac{1}{5}\sqrt{5}(-2\mathbf{i} + \mathbf{k})$; $\mathbf{N}(\frac{1}{2}\pi) = -\mathbf{j}$; $\mathbf{B}(\frac{1}{2}\pi) = \frac{1}{5}\sqrt{5}(\mathbf{i} + 2\mathbf{k})$; osculating: $x + 2z - 4 = 0$; rectifying: $y = 1$;

normal: $2x - z + 2 = 0$

24. Find the cosine of the angle between the vectors $\mathbf{R}(2)$ and $\mathbf{T}(2)$ for the curve $\mathbf{R}(t) = 3t^2\mathbf{i} + (t^3 - 3t)\mathbf{j}$.

▸ $D_t\mathbf{R}(t) = 6t\mathbf{i} + (3t^2 - 3)\mathbf{j}$

$\|D_t\mathbf{R}(t)\| = \sqrt{36t^2 + 9t^4 - 18t^2 + 9} = \sqrt{9t^4 + 18t^2 + 9} = 3(t^2 + 1)$

$$\mathbf{T}(t) = \frac{D_t\mathbf{R}(t)}{\|D_t\mathbf{R}(t)\|} = \frac{2t}{t^2 + 1}\mathbf{i} + \frac{t^2 - 1}{t^2 + 1}\mathbf{j}.$$

$\mathbf{R}(2) = 2\mathbf{i} + 2\mathbf{j}$. $\mathbf{T}(2) = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$

The cosine of the angle between $\mathbf{R}(2)$ and $\mathbf{T}(2)$ is given by

$$\cos\theta = \frac{\mathbf{R}(2) \cdot \mathbf{T}(2)}{\|\mathbf{R}(2)\|\|\mathbf{T}(2)\|} = \frac{12 \cdot \frac{4}{5} + 2 \cdot \frac{3}{5}}{\sqrt{144 + 4}(1)} = \frac{\frac{48}{5} + \frac{6}{5}}{2\sqrt{37}} = \frac{27}{185}\sqrt{37}$$

25. $\mathbf{R} = 2\sin t\mathbf{i} + \sin 2t\mathbf{j} + \cos 3t\mathbf{k}$. $\dot{\mathbf{R}} = 2\cos t\mathbf{i} + 2\cos t\mathbf{j} - 3\sin 3t\mathbf{k}$. $\mathbf{R}(\frac{1}{6}\pi) = \langle 1, \frac{1}{2}\sqrt{3}, 0\rangle$. $\dot{\mathbf{R}}(\frac{1}{6}\pi) = \langle\sqrt{3}, 1, -3\rangle$

$$\cos\theta = \frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{\|\mathbf{R}\|\|\dot{\mathbf{R}}\|} = \frac{\sqrt{3}+\frac{1}{2}\sqrt{3}}{(\frac{1}{2}\sqrt{7})(\sqrt{13})} = \frac{3\sqrt{3}}{\sqrt{91}} = \frac{3}{91}\sqrt{273}$$

26. $\mathbf{R} = \cos 2t\mathbf{i} - 3t\mathbf{j} + 2\sin 2t\mathbf{k}$. $\dot{\mathbf{R}} = -2\sin 2t\mathbf{i} - 3\mathbf{j} + 4\cos 2t\mathbf{k}$. $\dot{\mathbf{R}}(\pi) = -3\mathbf{j} + 4\mathbf{k}$. $\cos\theta = \mathbf{j}\cdot\dot{\mathbf{R}}(\pi)/\|\dot{\mathbf{R}}(\pi)\| = -\frac{3}{5}$

27. $\mathbf{R} = (4-3t^2)\mathbf{i} + (t^3-3t)\mathbf{j}$. At $t=1$, $\dot{\mathbf{R}} = -6t\mathbf{i} + (3t^2-3)\mathbf{j} = -6\mathbf{i}$, $\ddot{\mathbf{R}} = -6\mathbf{i} + 6t\mathbf{j} = -6\mathbf{i} + 6\mathbf{j}$.

$\cos\theta = (-\mathbf{i})\cdot\frac{1}{2}\sqrt{2}(-\mathbf{i}+\mathbf{j}) = \frac{1}{2}\sqrt{2}$, $\theta = \frac{1}{4}\pi$

In Exercises 28 and 29, express the arc length s as a function of t, where s increases with t from $t = 0$.

28. The cycloid $\mathbf{R}(t) = 2(t-\sin t)\mathbf{i} + 2(1-\cos t)\mathbf{j}$

▶ $$\frac{ds}{dt} = \|\dot{\mathbf{R}}(t)\| = \|2(1-\cos t)\mathbf{i} + 2\sin t\mathbf{j}\| = 2\sqrt{(1-\cos t)^2+\sin^2 t} = 2\sqrt{(1-\cos 2t)} = 2\sqrt{4\sin^2\tfrac{1}{2}t} = 4\sin\tfrac{1}{2}t$$

if $0 \le t \le 2\pi$. Then

$$s = \int_0^t \frac{ds}{du}du = \int_0^t 4\sin\tfrac{1}{2}u\,du = -8\cos\tfrac{1}{2}u\Big]_0^t = 8(1-\cos\tfrac{1}{2}t)$$

Because $s = 16$ when $t = 2\pi$, if $2\pi \le t \le 4\pi$, then

$$s = 16 + 8[1-\cos\tfrac{1}{2}(t-2\pi)] = 16 + 8(1+\cos\tfrac{1}{2}t)$$

In general,

$$s = 16[\![t/2\pi]\!] + 8[1-(-1)^{[\![t/2\pi]\!]}\cos\tfrac{1}{2}t]$$

29. $\mathbf{R}(t) = t\mathbf{i} + t^{3/2}\mathbf{j}$. $\dot{\mathbf{R}}(t) = \mathbf{i} + \frac{3}{2}t^{1/2}\mathbf{j}$.

$$s = \int_0^t \|\dot{\mathbf{R}}(u)\|du = \int_0^t \sqrt{1+\tfrac{9}{4}u}\,du = \tfrac{1}{2}\int_0^t (4+9u)^{1/2}du = \tfrac{1}{27}(4+9u)^{3/2}\Big]_0^t = \tfrac{1}{27}(4+9t)^{3/2} - \tfrac{8}{27}$$

30. $f(s) = \frac{1}{27}[(27s+8)^{2/3}-4]^{3/2}$, $g(s) = \frac{1}{2}[(27s+8)^{2/3}-4]$, $h(s) = 4$

$$f'(s) = \frac{1}{18}[(27s+8)^{2/3}-4]^{1/2}\Big[\frac{2}{3}(27s+8)^{-1/3}(27)\Big] = \frac{[(27s+8)^{2/3}-4]^{1/2}}{(27s+8)^{1/3}}$$

$g'(s) = \frac{2}{27}(27s+8)^{-1/3}(27) = \dfrac{2}{(27s+8)^{1/3}}$, $h'(s) = 0$. Hence $(f')^2 + (g')^2 + (h')^2 = \dfrac{(27s+8)^{2/3}-4+4}{(27s+8)^{2/3}} = 1$.

37. $\mathbf{T}\cdot\mathbf{k} = \dfrac{1}{\sqrt{a^2+1}}(-a\sin t\mathbf{i} + b\cos t\mathbf{j} + \mathbf{k})\cdot\mathbf{k} = \dfrac{1}{\sqrt{a^2+1}}$, a constant

38. If a particle moves on a line, then $\mathbf{T}$ is constant and so $\dot{\mathbf{T}} \equiv \mathbf{0}$ and has no direction. Thus $\mathbf{N}$ is undefined.

11.4 CURVATURE

Curvature If $\mathrm{T}(t)$ is the unit tangent vector to a curve C at a point P, s is the arc length measured from an arbitrarily chosen point on C to P, and s increases as t increases, then for some scalar $\mathrm{K}(t)$, called the *curvature*,

$$\mathrm{D}_s\mathbf{T}(t) = \mathrm{K}(t)\mathbf{N}(t) \tag{A}$$

Furthermore, for a scalar $\tau(t)$, called the *torsion*,

$$\mathrm{D}_s\mathbf{B}(t) = -\tau(t)\mathbf{N}(t) \tag{B}$$

and

$$\mathrm{D}_s\mathbf{N}(t) = -\mathrm{K}(t)\mathbf{T} + \tau(t)\mathbf{N} \tag{C}$$

Equations (A), (B), (C) together are known as Frenet's formulas.

Following are some formulas for the curvature.

$$\mathrm{K}(t) = \frac{\|\dot{\mathbf{T}}\|}{\|\dot{\mathbf{R}}\|} \tag{5}$$

$$\mathrm{K}(t) = \frac{\|\dot{\mathbf{R}}(t)\times\ddot{\mathbf{R}}(t)\|}{\|\dot{\mathbf{R}}(t)\|^3} \qquad \text{(Theorem 11.4.2. See Exercise 56)}$$

In the Plane $$\mathrm{K}(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} \tag{8}$$

and if $y = f(x)$ or $x = g(y)$

$$\mathrm{K}(x) = \frac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}} \text{ or } \mathrm{K}(y) = \frac{\left|d^2x/dy^2\right|}{[1+(dx/dy)^2]^{3/2}} \tag{9}$$

Polar Coordinates Let a polar equation of a curve be $r = r(\theta)$. If β is the angle from the radius vector to the tangent line, then $\tan\beta = r/(dr/d\theta)$ and the curvature is

$$\mathrm{K}(\theta) = \frac{\left|r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)\right|}{\left[r^2 + (dr/d\theta)^2\right]^{3/2}}$$

Circle of Curvature The circle on the concave side of C that is tangent to C at $\mathrm{P}(x, y)$ and whose radius is ρ is called the *circle of curvature* or the *osculating circle*. The center of this circle is called the *center of curvature* and is given by $\mathbf{R}_c = \mathbf{R} + \rho\mathbf{N} = (x_c, y_c)$ where (see Exercise 39)

$$x_c = x - \frac{dy}{dx}\cdot\frac{1+(dy/dx)^2}{d^2y/dx^2} \qquad y_c = y + \frac{1+(dy/dx)^2}{d^2y/dx^2}$$

or

$$x_c = x - \dot{y}\frac{(\dot{x})^2+(\dot{y})^2}{\dot{x}\ddot{y}-\dot{y}\ddot{x}},\ y_c = y + \dot{x}\frac{(\dot{x})^2+(\dot{y})^2}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

The locus of the center of curvature of C is called the *evolute* of C.

Exercises 11.4

For Exercises 1–14, see Exercises 1*–14* of Exercises 11.3.

In Exercises 15 and 16, compute the curvature at the indicated point using $\mathrm{K} = \|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3$.

15. $\mathbf{R} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. At $t = 0$, $\dot{\mathbf{R}} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} = \mathbf{i}$, $\ddot{\mathbf{R}} = 2\mathbf{j} + 6t\mathbf{k} = 2\mathbf{j}$. $\mathrm{K} = \|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3 = \|2\mathbf{k}\|/\|\mathbf{i}\|^3 = 2$

16. $\mathbf{R}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}$; $t = 0$

▶ $\dot{\mathbf{R}}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}$ $\qquad \dot{\mathbf{R}}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$\ddot{\mathbf{R}}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$ $\qquad \ddot{\mathbf{R}}(0) = \mathbf{i} + \mathbf{j}$

$$\left\|\dot{\mathbf{R}}(0)\times\ddot{\mathbf{R}}(0)\right\| = \left\|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 1\\ 1 & 1 & 0\end{vmatrix}\right\| = \|-\mathbf{i}+\mathbf{j}+2\mathbf{k}\| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$$

$$\left\|\dot{\mathbf{R}}(0)\right\| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\mathrm{K}(0) = \left\|\dot{\mathbf{R}}(0)\times\ddot{\mathbf{R}}(0)\right\|/\left\|\dot{\mathbf{R}}(0)\right\|^3 = \sqrt{6}/2\sqrt{2} = \tfrac{1}{2}\sqrt{3}$$

In Exercises 17–26, find the curvature K and the radius of curvature ρ at the given point. Sketch portions of the curve and tangent line, and the circle of curvature at the given point.

17. $x = \frac{1}{1+t}$, $y = \frac{1}{1-t}$, $\frac{dx}{dt} = -\frac{1}{(1+t)^2}$, $\frac{dy}{dt} = \frac{1}{(1-t)^2}$, $\frac{d^2x}{dt^2} = \frac{2}{(1+t)^3}$, $\frac{d^2y}{dt^2} = \frac{2}{(1-t)^3}$.

$$\mathrm{K}(t) = \frac{\left|\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right|}{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{3/2}} = \frac{\left|\frac{-2}{(1+t)^2(1-t)^3} - \frac{2}{(1+t)^3(1-t)^2}\right|}{\left[\frac{1}{(1+t)^4}+\frac{1}{(1-t)^4}\right]^{3/2}}$$

$$= \frac{-2(1+t)^4(1-t)^3 - 2(1+t)^3(1-t)^4}{[(1-t)^4+(1+t)^4]^{3/2}} = \frac{-2(1+t)^3(1-t)^3[(1+t)+(1-t)]}{(1-4t+6t^2-4t^3+t^4+1+4t+6t^2+4t^3+t^4)^{3/2}}$$

$$= \frac{4|(1+t)(1-t)|^3}{(2+12t^2+2t^4)^{3/2}} = \frac{\sqrt{2}|1-t^2|^3}{(1+6t^2+4)^{3/2}}.$$ Therefore $\mathrm{K}(0) = \sqrt{2}$ and $\rho(0) = \frac{1}{2}\sqrt{2}$.

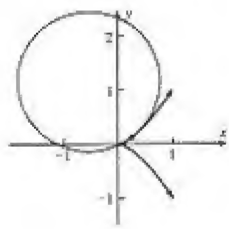
18. $x = e^t + e^{-t}$, $y = e^t - e^{-t}$. $\dot{x} = \ddot{y} = e^t - e^{-t}$, $\dot{y} = \ddot{x} = e^t + e^{-t}$. $\mathrm{K} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}$

$$= \frac{\left|(e^{2t}-2+e^{-2t})-(e^{2t}+2+e^{-2t})\right|}{[(e^{2t}-2+e^{-2t})+(e^{2t}+2+e^{-2t})]^{3/2}} = \frac{4}{[2(e^{2t}+e^{-2t})]^{3/2}} = \frac{\sqrt{2}}{(e^{2t}+e^{-2t})^{3/2}}.$$ $\mathrm{K}(0) = \frac{1}{2}$, $\rho(0) = 2$

19. $y = 2\sqrt{x}$. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$; $\dfrac{d^2y}{dx^2} = -\dfrac{1}{2x^{3/2}}$. $\mathrm{K} = \dfrac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}} = \dfrac{\dfrac{1}{2x^{3/2}}}{\left(1+\dfrac{1}{x}\right)^{3/2}} = \dfrac{1}{2(x+1)^{3/2}}$

At $(0,0)$, $\mathrm{K} = \frac{1}{2}$ and $\rho = 2$.

20. $y^2 = x^3$; $(\frac{1}{4}, \frac{1}{8})$



▸ We have the parametric equations

$$x = t^2 \qquad\qquad y = t^3$$

Calculating derivatives and evaluating at $t = \frac{1}{2}$, we have

$$\dot{x} = 2t = 2(\tfrac{1}{2}) = 1 \qquad\qquad \dot{y} = 3t^2 = 3(\tfrac{1}{2})^2 = \tfrac{3}{4}$$

$$\ddot{x} = 2 \qquad\qquad \ddot{y} = 6t = 6(\tfrac{1}{2}) = 3$$

The curvature is given by

$$\mathrm{K} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{\left|1\cdot 3 - \frac{3}{4}\cdot 2\right|}{[1^2+(\frac{3}{4})^2]^{3/2}} = \frac{\frac{3}{2}}{(\frac{25}{16})^{3/2}} = \frac{\frac{3}{2}}{\frac{125}{64}} = \frac{96}{125}$$

The radius of curvature is the reciprocal of the curvature: $\rho = \frac{1}{\mathrm{K}} = \frac{125}{96}$

To find the center of curvature (x_c, y_c), let $c = \dfrac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \dfrac{\frac{25}{16}}{\frac{3}{2}} = \dfrac{25}{24}$. Then

$$x_c = x - c\dot{y} = \tfrac{1}{4} - \tfrac{25}{24}\cdot\tfrac{3}{4} = \tfrac{7}{32} \qquad\qquad y_c = y + c\dot{x} = \tfrac{1}{8} + \tfrac{25}{24}\cdot 1 = \tfrac{7}{6}$$

The figure shows portions of the curve and tangent line, and the circle of curvature at $(\frac{1}{4}, \frac{1}{8})$.

21. $y = e^x$. $\dfrac{dy}{dx} = e^x$; $\dfrac{d^2y}{dx^2} = e^x$. $\mathrm{K} = \dfrac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}} = \dfrac{e^x}{(1+e^{2x})^{3/2}}$

At $(0,1)$, $\mathrm{K} = \dfrac{1}{(1+1)^{3/2}} = \frac{1}{2} = \frac{1}{2}\sqrt{2}$ and $\rho = 2\sqrt{2}$.

22. $y = \ln x$. $\dfrac{dy}{dx} = x^{-1}$, $\dfrac{d^2y}{dx^2} = -x^{-2}$. $\mathrm{K} = \dfrac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}} = \dfrac{x^{-2}}{(1+x^{-2})^{3/2}} = \dfrac{x}{(x^2+1)^{3/2}}$.

At $(e,1)$, $\mathrm{K} = \dfrac{e}{(e^2+1)^{3/2}}$, $\rho = \dfrac{(e^2+1)^{3/2}}{e}$

23. $x = \sin y$. $\dfrac{dx}{dy} = \cos y$; $\dfrac{d^2x}{dy^2} = -\sin y$. $\mathrm{K} = \dfrac{\left|d^2y/dx^2\right|}{[1+(dy/dx)^2]^{3/2}} = \dfrac{|-\sin y|}{(1+\cos^2 y)^{3/2}} = \dfrac{|\sin y|}{(1+\cos^2 y)^{3/2}}$.

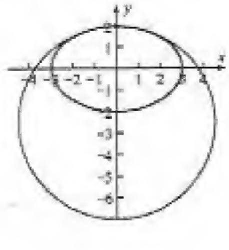
At $(\frac{1}{2}, \frac{1}{6}\pi)$, $\mathrm{K} = \dfrac{\sin\frac{1}{6}\pi}{(1+\cos^2\frac{1}{6}\pi)^{3/2}} = \dfrac{\frac{1}{2}}{(1+\frac{3}{4})^{3/2}} = \dfrac{4}{7^{3/2}} = \dfrac{4}{49}\sqrt{7}$ and $\rho = \frac{7}{4}\sqrt{7}$.

24. $4x^2 + 9y^2 = 36$; $(0,2)$

▸ We put the equation of the ellipse into standard form and write parametric equations.

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$x = 3\cos t \qquad\qquad y = 2\sin t$$



We calculate derivatives and evaluate at $t = \frac{1}{2}\pi$.

$$\dot{x} = -3\sin t = -3\sin\tfrac{1}{2}\pi = -3 \qquad \dot{y} = 2\cos t = 2\cos\tfrac{1}{2}\pi = 0$$

$$\ddot{x} = -3\cos t = -3\cos\tfrac{1}{2}\pi = 0 \qquad \ddot{y} = -2\sin t = -2\sin\tfrac{1}{2}\pi = -2$$

The curvature is given by

$$\mathrm{K} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{|(-3)(-2) - 0|}{[(-3)^2+0]^{3/2}} = \frac{6}{27} = \frac{2}{9}$$

The radius of curvature is the reciprocal of the curvature: $\rho = \frac{1}{\mathrm{K}} = \frac{9}{2}$

The center of curvature is at $(0, -\frac{5}{2})$.

The figure shows portions of the curve and tangent line, and the circle of curvature at $(0,2)$.

25. $x = \sqrt{y-1}$. $\frac{dx}{dy} = \frac{1}{2\sqrt{y-1}}$; $\frac{d^2x}{dy^2} = \frac{-1}{4(y-1)^{3/2}}$. $\mathrm{K} = \frac{\left|d^2x/dy^2\right|}{[1+(dx/dy)^2]^{3/2}} = \frac{\frac{1}{4(y-1)^{3/2}}}{\left[1+\frac{1}{4(y-1)}\right]^{3/2}} = \frac{2}{(4y-3)^{3/2}}$

At (2,5), $\mathrm{K} = \frac{2}{17^{3/2}} = \frac{2}{289}\sqrt{17}$ and $\rho = \frac{17}{2}\sqrt{17}$.

26. $x = \tan y$. At $y = \frac{1}{4}\pi$, $\frac{dx}{dy} = \sec^2 y = 2$; $\frac{d^2x}{dy^2} = 2\sec^2 y \tan y = 4$. $\mathrm{K} = \frac{\left|d^2x/dy^2\right|}{[1+(dx/dy)^2]^{3/2}} = \frac{4}{(1+2^2)^{3/2}} = \frac{4}{25}\sqrt{5}$

$\rho = \frac{5}{4}\sqrt{5}$

In Exercises 27–34, find the radius of curvature at any point.

27. $y = \sin^{-1}x$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$; $\frac{d^2y}{dx^2} = \frac{x}{(1-x^2)^{3/2}}$.

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left(1+\frac{1}{1-x^2}\right)^{3/2}}{\frac{|x|}{(1-x^2)^{3/2}}} = \frac{(1-x^2+1)^{3/2}}{|x|} = \frac{(2-x^2)^{3/2}}{|x|}$$

28. $y = \ln \sec x$

▷ $$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x,\ \sec x > 0$$

$$\frac{d^2y}{dx^2} = \sec^2 x$$

Hence,

$$\rho(x) = \frac{1}{\mathrm{K}(x)} = \frac{[1+(dy/dx)^2]^{3/2}}{\left|d^2y/dx^2\right|} = \frac{(1+\tan^2 x)^{3/2}}{\sec^2 x} = \frac{(\sec^2 x)^{3/2}}{\sec^2 x} = \frac{\left|\sec^3 x\right|}{\sec^2 x} = \sec x$$

29. The hyperbola $4x^2 - 9y^2 = 16$. $8x - 18\frac{dy}{dx} = 0$; $\frac{dy}{dx} = \frac{4x}{9y}$. $\frac{d^2y}{dx^2} = \frac{4}{9}\cdot\frac{y - x\frac{dy}{dx}}{y^2} = \frac{4}{9}\cdot\frac{y - x\frac{4x^2}{9y}}{y^2} = \frac{4(9y^2-4x^2)}{81y^3}$

$$= \frac{4(-16)}{81y^3} = -\frac{64}{81y^3}.\ \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left(1+\frac{16x^2}{81y^2}\right)^{3/2}}{\frac{64}{81\left|y^3\right|}} = \frac{(16x^2+81y^2)^{3/2}}{576}$$

30. $x = \tan^{-1}y$, $y = \tan x$, $\frac{dy}{dx} = \sec^2 x$, $\frac{d^2y}{dx^2} = 2\sec^2 x \tan x$. $\rho = \frac{[1+(\mathrm{D}_x y)^2]^{3/2}}{\left|\mathrm{D}_x{}^2 y\right|} = \frac{(1+\sec^4 x)^{3/2}}{2\sec^2 x|\tan x|}$

31. The parabola $x^{1/2} + y^{1/2} = a^{1/2}$. $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\mathrm{D}_x y = 0$; $\mathrm{D}_x y = -\frac{y^{1/2}}{2} = \frac{a^{1/2}-x^{1/2}}{2} = 1 - a^{1/2}x^{-1/2}$.

$\mathrm{D}_x{}^2 y = \frac{1}{2}a^{1/2}x^{-3/2}$. $\rho = \frac{[1+(\mathrm{D}_x y)^2]^{3/2}}{\left|\mathrm{D}_x{}^2 y\right|} = \frac{[1+(-y^{1/2}x^{-1/2})^2]^{3/2}}{\left|\frac{1}{2}a^{1/2}x^{-3/2}\right|} = \frac{2(x+y)^{3/2}}{a^{1/2}}$.

32. The logarithmic spiral $\mathbf{R}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{j}$

▷ $x = e^t\sin t$ $\quad y = e^t\cos t$

$\dot{x} = e^t(\cos t + \sin t)$ $\quad \dot{y} = e^t(\cos t - \sin t)$

$\ddot{x} = 2e^t\cos t$ $\quad \ddot{y} = -2e^t\sin t$

$$\mathrm{K}(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{\left|-2e^{2t}\sin t(\cos t+\sin t) - 2e^{2t}\cos t(\cos t - \sin t)\right|}{[e^{2t}(\cos t+\sin t)^2 + e^{2t}(\cos t - \sin t)^2]^{3/2}}$$

$$= \frac{2e^{2t}\left|\sin t\cos t + \sin^2 t + \cos^2 t - \sin t\cos t\right|}{e^{3t}[\cos^2 t + 2\cos t\sin t + \sin^2 t + \cos^2 t - 2\cos t\sin t + \sin^2 t]^{3/2}} = \frac{2}{e^t(2)^{3/2}} = \frac{1}{\sqrt{2}e^t}$$

Thus,

$$\rho(t) = \frac{1}{\mathrm{K}(t)} = \sqrt{2}e^t$$

33. The cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$. $x' = a(1 - \cos t)$, $x'' = a \sin t$; $y' = a \sin t$, $y'' = a \cos t$.

$$\rho = \frac{[\dot{x}^2 + \dot{y}^2]^{3/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|} = \frac{[a^2(1 - \cos t)^2 + a^2\sin^2 t]^{3/2}}{|a(1 - \cos t)(a \cos t) - (a \sin t)(a \sin t)|}$$

$$= \frac{(a^2 - 2a^2\cos t + a^2\cos^2 t + a^2\sin^2 t)^{3/2}}{|a^2\cos t - a^2\cos^2 t - a^2\sin^2 t|} = \frac{[2a^2(1 - \cos t)]^{3/2}}{|a^2(\cos t - 1)|} = \frac{2^{3/2}|a^3|(1 - \cos t)^{3/2}}{a^2(1 - \cos t)}$$

$$= 2^{3/2}|a|(1 - \cos t)^{1/2} = 4|a|\left(\frac{1 - \cos t}{2}\right)^{1/2} = 4\left|a \sin \tfrac{1}{2}t\right|$$

34. The tractrix. Let $t = au$. $x = t - a\tanh(t/a) = au - a \tanh u$, $y = a\,\mathrm{sech}(t/a) = a\,\mathrm{sech}\ u$. We reduce $\mathrm{sech}^2 u = 1 - \tanh^2 u$. $\dot{x} = a - a\,\mathrm{sech}^2 u = a\tanh^2 u$, $\dot{y} = -a\,\mathrm{sech}\,u \tanh u$.
$\ddot{x} = 2a\,\mathrm{sech}^2 u \tanh u = 2a(\tanh u - \tanh^3 u)$, $\ddot{y} = a(\mathrm{sech}\,u\tanh^2 u - \mathrm{sech}^3 u) = a\,\mathrm{sech}\,u(2\tanh^2 u - 1)$
$[\dot{x}^2 + \dot{y}^2]^{3/2} = [a^2(\tanh^4 u + \mathrm{sech}^2 u \tanh^2 u)]^{3/2} = [a^2\tanh^2 u]^{3/2} = |a^3\tanh^3 u|$

$|\dot{x}\ddot{y} - \dot{y}\ddot{x}| = a^2\left|\tanh^2 u\,\mathrm{sech} u(2\tanh^2 u - 1) + 2\,\mathrm{sech}\,u \tanh u(\tanh u - \tanh^3 u)\right| = a^2\tanh^2 u|\mathrm{sech}\,u|$

$$\rho = \frac{[\dot{x}^2 + \dot{y}^2]^{3/2}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|} = |a \sinh u| = |a \sinh(t/a)|$$

In Exercises 35–38, find a point on the given curve at which the curvature is an absolute maximum.

35. $y = e^x$, $y' = e^x$, $y'' = e^x$. $K = \dfrac{e^x}{(1 + e^{2x})^{3/2}}$. $K' = e^x(-\tfrac{3}{2})(1 + e^{2x})^{-5/2}(2e^{2x}) + (1 + e^{2x})^{-3/2}e^x = \dfrac{e^x(1 - 2e^{2x})}{(1 + e^{2x})^{5/2}}$

Because $K'(x) > 0$ if $e^{2x} < \frac{1}{2}$ and $K'(x) < 0$ if $e^{2x} > \frac{1}{2}$, then K has an absolute maximum value when $x = -\frac{1}{2}\ln 2$ and $y = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$.

36. $y = x^2 - 2x + 3$

▷ $\dfrac{dy}{dx} = 2x - 2$ and $\dfrac{d^2y}{dx^2} = 2$

$$K(x) = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}} = \frac{2}{[1 + (2x - 2)^2]^{3/2}} = \frac{2}{[1 + 4(x - 1)^2]^{3/2}}$$

Because $1 + 4(x - 1)^2$ has an absolute minimum at $x = 1$, then K has an absolute maximum at $x = 1$. If $x = 1$, then $y = 2$. Thus, the curvature is an absolute maximum at the point $(1, 2)$. Note that $y = (x - 1)^2 + 2$ is a parabola and that $(1, 2)$ is the vertex of the parabola.

37. $\mathbf{R}(t) = (2t - 3)\mathbf{i} + (t^2 - 1)\mathbf{j}$. $x = 2t - 3$, $y = t^2 - 1$. $\dot{x} = 2$, $x'' = 0$, $\dot{y} = 2t$, $y'' = 2$.

$$K = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[(\dot{x})^2 + (\dot{y})^2]^{3/2}} = \frac{4}{(4 + 4t^2)^{3/2}} = \frac{1}{2(1 + t^2)^{3/2}}$$

$1 + t^2$ has an absolute minimum when $t = 0$ at $(-3, -1)$, so K has an absolute maximum there.

38. $y = \sin x$. $\dfrac{dy}{dx} = \cos x$, $\dfrac{d^2y}{dx^2} = -\sin x$. $K = \dfrac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}} = \dfrac{|\sin x|}{[1 + \cos^2 x]^{3/2}} = \dfrac{|\sin x|}{(2 - \sin^2 x)^{3/2}}$

When $x = (n + \frac{1}{2})\pi$, for any integer n, the numerator has an absolute maximum and the denominator an absolute minimum. Thus K has an absolute maximum.

39. The circle of curvature is the circle $(x - x_c)^2 + (y - y_c)^2 = r^2$ such that y, y', y'' have the same values as on the curve. Differentiating twice with respect to x we get
$2(x - x_c) + 2(y - y_c)y' = 0$; $(x_c - x) + (y_c - y)y' = 0$
$-1 - (y')^2 + (y_c - y)y'' = 0$; $(y_c - y)y' = 1 + (y')^2$
Solving for $x_c - x$ and $y_c - y$ we get

$$x_c = x - \frac{y'[1 + (y')^2]}{y''} \text{ and } y_c = y + \frac{1 + (y')^2}{y''}$$

In Exercises 40–42, find the curvature K, the radius of curvature ρ, and the center of curvature at the given point. Sketch the curve and the circle of curvature.

40. $y = \ln x$; $(1, 0)$

▹ We have

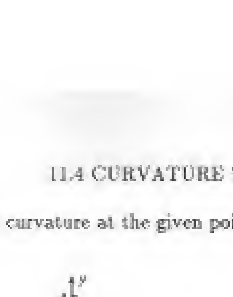
$$\frac{dy}{dx} = \frac{1}{x} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

When $x = 1$, we get

$$\frac{dy}{dx} = 1 \quad \text{and} \quad \frac{dy^2}{dx^2} = -1$$

Thus,

$$\mathrm{K} = \frac{\left|d^2y/dx^2\right|}{[1 + (dy/dx)^2]^{3/2}} = \frac{1}{2^{2/3}} \quad \text{and } \rho = \frac{1}{\mathrm{K}} = 2\sqrt{2}$$



The coordinates of the center of curvature at $(1, 0)$ are given by

$$x_c = x - \frac{(dy/dx)[1 + (dy/dx)^2]}{d^2y/dx^2} = 1 - \frac{1[1 + 1^2]}{-1} = 3 \text{ and } y_c = y + \frac{1 + (dy/dx)^2}{d^2y/dx^2} = 0 + \frac{1^2 + 1}{-1} = -2$$

The center of curvature is $(3, -2)$. The figure shows the curve and the circle of curvature at $(1, 0)$.

41. $y = x^4 - x^2$; $y' = 4x^3 - 2x$; $y'' = 12x^2 - 2$. At $(0,0)$, $y = 0$, $y' = 0$, $y'' = 2$, and we have

$$\mathrm{K} = \frac{y''}{[1 + (y')^2]^{3/2}} = \frac{|-2|}{[1 + 0]^{3/2}} = 2,\ \rho = \frac{1}{2},\ x_c = x - \frac{y'[1 + (y')^2]}{y''} = 0 - \frac{[1 + 0^2]}{-2} = 0,$$

$$y_c = y + \frac{1 + (y')^2}{y''} = 0 + \frac{1 + 0}{-2} = -\frac{1}{2}.$$ The center of curvature is at $(0, -\frac{1}{2})$.

42. $y = \cos x$, $y' = -\sin x$, $y'' = -\cos x$. At $(\frac{1}{3}\pi, \frac{1}{2})$, $y = \frac{1}{2}$, $y' = \frac{1}{2}\sqrt{3}$, $y'' = -\frac{1}{2}$, and we have

$$\mathrm{K} = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\frac{1}{2}}{[1 + \frac{3}{4}]^{3/2}} = \frac{4}{7\sqrt{7}},\ \rho = \frac{7}{4}\sqrt{7},\ x_c = x - \frac{y'[1 + (y')^2]}{y''} = \frac{1}{3}\pi - \frac{\frac{1}{2}\sqrt{3}(1 + \frac{3}{4})}{\frac{1}{2}} = \frac{1}{3}\pi - \frac{7}{4}\sqrt{3}$$

$$y_c = y + \frac{1 + (y')^2}{y''} = \frac{1}{2} + \frac{1 + \frac{3}{4}}{-\frac{1}{2}} = -3.$$ The center of curvature is at $(\frac{1}{3}\pi - \frac{7}{4}\sqrt{3}, -3)$.

In Exercises 43–46, find the coordinates of the center of curvature at any point.

43. $y^2 = 4px$, $y = 2p^{1/2}x^{1/2}$, $y' = p^{1/2}x^{-1/2}$, $y'' = -\frac{1}{2}p^{1/2}x^{-3/2}$,

$$x_c = x - \frac{y'[1 + (y')^2]}{y''} = x - \frac{p^{1/2}x^{-1/2}[1 + px^{-1}]}{-\frac{1}{2}p^{1/2}x^{-3/2}} = x + 2x + 2p = 3x + 2p$$

$$y_c = y + \frac{1 + (y')^2}{y''} = 2p^{1/2}x^{1/2} + \frac{1 + px^{-1}}{-\frac{1}{2}p^{1/2}x^{-3/2}} = 2p^{1/2}x^{1/2} - 2p^{-1/2}x^{3/2} - 2p^{1/2}x^{1/2} = -2p^{-1/2}x^{3/2}$$

Therefore, the center of curvature is $(3x + 2p, \mp 2p^{-1/2}x^{3/2}) = (3x + 2p, -\frac{1}{4}p^{-2}y^3)$.

44. $y^3 = a^2x$.

▹ Solving for y and differentiating, we have

$$y = a^{2/3}x^{1/3} \quad \frac{dy}{dx} = \frac{1}{3}a^{2/3}x^{-2/3} \quad \frac{d^2y}{dx^2} = -\frac{2}{9}a^{2/3}x^{-5/3}$$

The coordinates of the center of curvature are given by

$$x_c = x - \frac{(dy/dx)[1 + (dy/dx)^2]}{d^2y/dx^2} = x - \frac{\frac{1}{3}a^{2/3}x^{-2/3}[1 + (\frac{1}{3}a^{2/3}x^{-2/3})^2]}{-\frac{2}{9}a^{2/3}x^{-5/3}}$$

$$= x + \frac{3}{2}x(1 + \frac{1}{9}a^{4/3}x^{-4/3}) = x + \frac{3}{2}x + \frac{1}{6}a^{4/3}x^{-1/3} = \frac{5}{2}x + \frac{1}{6}a^{4/3}x^{-1/3}$$

$$y_c = y + \frac{1 + (dy/dx)^2}{d^2y/dx^2} = y + \frac{(\frac{1}{3}a^{2/3}x^{-2/3})^2 + 1}{-\frac{2}{9}a^{2/3}x^{-5/3}} = a^{2/3}x^{1/3} + \frac{\frac{1}{9}a^{4/3}x^{-4/3} + 1}{-\frac{2}{9}a^{2/3}x^{-5/3}}$$

$$= a^{2/3}x^{1/3} - \frac{1}{2}a^{2/3}x^{1/3} - \frac{9}{2}a^{-2/3}x^{5/3} = \frac{1}{2}a^{2/3}x^{1/3} - \frac{9}{2}a^{-2/3}x^{5/3}$$

45. $\mathbf{R}(t) = a \cos t\mathbf{i} + b \sin t\mathbf{j}$. Parametric equations of the ellipse are $x = a \cos t$ and $y = b \sin t$. Following the same steps as in the solution of Exercise 39, but differentiating with respect to t, we get the system

$$\begin{cases} \dot{x}(x_c - x) + \dot{y}(y_c - y) = 0 \\ \ddot{x}(x_c - x) + \ddot{y}(y_c - y) = (\dot{x})^2 + (\dot{y})^2 \end{cases}, \quad x_c = x - \frac{\dot{y}[(\dot{x})^2 + (\dot{y})^2]}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}, \; y_c = y + \frac{\dot{x}[(\dot{x})^2 + (\dot{y})^2]}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$\dot{x} = -a \sin t$, $\ddot{x} = -a \cos t$; $\dot{y} = b \cos t$, $\ddot{y} = -b \sin t$

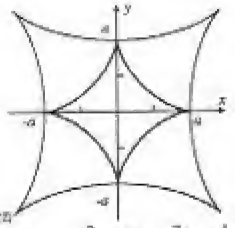
$\dot{x}y'' - \dot{y}x'' = (-a \sin t)(-b \sin t) - (b \cos t)(-a \cos t) = ab$

$$x_c = a \cos t - \frac{b \cos t(a^2\sin^2 t + b^2\cos^2 t)}{ab} = \frac{a^2\cos t(1 - \sin^2 t) - b^2\cos^3 t}{a} = \frac{a^2 - b^2}{a}\cos^3 t$$

$$y_c = b \sin t + \frac{(-a \sin t)(a^2\sin^2 t + b^2\cos^2 t)}{ab} = \frac{b^2\sin t(1 - \cos^2 t) - a^2\sin^3 t}{b} = \frac{b^2 - a^2}{b}\sin^3 t$$

The evolute is a squashed astroid.

46. Astroid $x = a \cos^3 t$, $y = a \sin^3 t$, $a > 0$.



▸ $\dot{x} = -3a \cos^2 t \sin t$, $\dot{y} = 3a \sin^2 t \cos t$. In $(0, \frac{1}{2}\pi)$,
$\dot{s} = 3a[\cos^4 t \sin^2 t + \sin^4 t \cos^2 t]^{1/2} = 3a[\sin^2 t \cos^2 t(\sin^2 t + \cos^2 t)]^{1/2}$
$= 3a \sin t \cos t > 0$. $\ddot{x} = 3a(2 \cos t \sin^2 t - \cos^3 t)$, $\ddot{y} = 3a(2 \sin t \cos^2 t - \sin^3 t)$,
$\dot{x}\ddot{y} - \dot{y}\ddot{x} = 9a^2[(-\cos^2 t \sin t)(2 \sin t \cos^2 t - \sin^3 t) - (\sin^2 t \cos t)(2 \cos t \sin^2 t - \cos^3 t)]$
$= -9a^2(\sin^2 t \cos^4 t + \sin^4 t \cos^2 t) = -9a^2\sin^2 t \cos^2 t$.
$K = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|/\dot{s}^3 = 9a^2\sin^2 t \cos^2 t/27a^3\sin^3 t \cos^3 t = 1/(3a \sin t \cos t)$
$= 2/(3a \sin 2t)$. Remark. $s'^2/D = 9a^2\sin^2 t \cos^2 t/(-9a^2\sin^2 t \cos^2 t) = -1$.
$\mathbf{C} = [a \cos^3 t - (3a \sin^2 t \cos t)(-1)]\mathbf{i} + [a \sin^3 t + (-3a \cos^2 t \sin t)(-1)]\mathbf{j}$.
To show the evolute is an astroid twice as large we rotate axes 45° to u and v. Then
$u = 2^{-1/2}(y + x) = 2^{-1/2}a(\sin^3 t + 3 \sin^2 t \cos t + 3 \sin t \cos^2 t + \cos^3 t) = 2^{-1/2}a(\sin t + \cos t)^3 = 2a \sin^3(t + \frac{1}{4}\pi)$
$v = 2^{-1/2}(y - x) = 2^{-1/2}a(\sin^3 t - 3 \sin^2 t \cos t + 3 \sin t \cos^2 t - \cos^3 t) = 2^{-1/2}a(\sin t - \cos t)^3 = 2a \cos^3(t + \frac{1}{4}\pi)$.

47. $y = \frac{1}{81}x^4$. At $x = 3$, $y = 1$, $y' = \frac{4}{81}x^3 = \frac{4}{3}$, $y'' = \frac{4}{27}x^2 = \frac{4}{3}$. $K = \dfrac{|y'|}{[1 + (y')^2]^{3/2}} = \dfrac{\frac{4}{3}}{(1 + \frac{16}{9})^{3/2}} = \dfrac{\frac{4}{3}}{\frac{125}{9}} = \dfrac{36}{125}$, $\rho = \dfrac{125}{36}$

$c = \dfrac{1 + (y')^2}{y''} = \dfrac{\frac{25}{9}}{\frac{4}{3}} = \dfrac{25}{12}$. $x_c = x - cy' = 3 - \frac{25}{12}\cdot\frac{4}{3} = \frac{2}{9}$, $y_c = y + c = 1 + \frac{25}{12} = \frac{37}{12}$

48. Show that the curvature of a line is zero at every point.

▸ An equation of a straight line is $y = mx + b$. $y' = m$, $y'' = 0$. At every point (x,y), we have

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{0}{(1 + m^2)^{3/2}} = 0$$

49. The circle of curvature is the circle for which the values of y, y', y'' at $x = 0$ equal those of the curve $y = e^x$. Let an equation of the circle be
$(x - h)^2 + (y - k)^2 = r^2$. Because $y = e^0 = 1$, we have $h^2 + (1 - k)^2 = r^2$.
$2(x - h) + 2(y - k)y' = 0$; $(x - h) + (y - k)y' = 0$. $y' = e^x = e^0 = 1$, so we have $-h + (1 - k) = 0$.
$1 + (y')^2 + (y - k)y'' = 0$. Because $y'' = e^x = e^0 = 1$, we have $1 + 1 + (1 - k) = 0$.
Therefore $k = 3$, $h = -2$, $r^2 = 8$ and the equation is $(x + 2)^2 + (y - 3)^2 = 8$.

50. Prove the formula for curvature in polar coordinates.

▸ $\tan(\beta + \theta) = \tan\varphi = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{(r \sin\theta)'}{(r \cos\theta)'} = \dfrac{r'\sin\theta + r \cos\theta}{r'\cos\theta - r \sin\theta} = \dfrac{\tan\theta + r/r'}{1 - (r/r')\tan\theta}$. Thus $\tan\varphi = r/r'$.

$\varphi = \theta + \beta = \theta + \tan^{-1}(r/r')$, $d\varphi/d\theta = 1 + [(r'^2 - rr'')/r'^2[1 + (r/r')^2] = (r^2 + 2r'^2 - rr'')/(r^2 + r'^2)$.

$K(\theta) = (d\varphi/d\theta)/(ds/d\theta) = (r^2 + 2r'^2 - rr'')/(r^2 + r'^2)^{3/2}$

In Exercises 51–54, find the curvature and the radius of curvature at the indicated point of the polar curve.

51. $r = 4 \cos 2\theta$. $\dfrac{dr}{d\theta} = -8 \sin 2\theta$, $\dfrac{d^2r}{d\theta^2} = -16 \cos 2\theta$.

At $\theta = \frac{\pi}{12}$, $r = 4 \cos\frac{\pi}{6} = 2\sqrt{3}$, $\dfrac{dr}{d\theta} = -8 \sin\frac{\pi}{6} = -4$, $\dfrac{d^2r}{d\theta^2} = -16 \cos\frac{\pi}{6} = -8\sqrt{3}$.

$$K = \frac{(2\sqrt{3})^2 + 2(-4)^2 - (2\sqrt{3})(-8\sqrt{3})}{[(2\sqrt{3})^2 + (-4)^2]^{3/2}} = \frac{12 + 32 + 48}{(12 + 16)^{3/2}} = \frac{92}{28^{3/2}} = \frac{23}{98}\sqrt{7}, \; \rho = \frac{14}{23}\sqrt{7}$$

52. $r = 1 - \sin\theta;\ \theta = 0$

▷ We have

$\dfrac{dr}{d\theta} = -\cos\theta$ and $\dfrac{d^2r}{d\theta^2} = \sin\theta$

When $\theta = 0$, we get

$r = 1\ \ \dfrac{dr}{d\theta} = -1\ \ \dfrac{d^2r}{d\theta^2} = 0$

Thus,

$$\mathrm{K} = \frac{\left|r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)\right|}{\left[r^2 + (dr/d\theta)^2\right]^{3/2}} = \frac{\left|1^2 + 2(-1)^2 - 1(0)\right|}{[1^2 + (-1)^2]^{3/2}} = \frac{3}{2^{3/2}}$$

Hence,

$$\rho = \frac{1}{\mathrm{K}} = \tfrac{2}{3}\sqrt{2}$$

53. $r = a\sec^2\frac{1}{2}\theta.\ \dfrac{dr}{d\theta} = a\sec^2\frac{1}{2}\theta\tan\frac{1}{2}\theta,\ \dfrac{d^2r}{d\theta^2} = a\sec^2\frac{1}{2}\theta\tan^2\frac{1}{2}\theta + \frac{1}{2}a\sec^4\frac{1}{2}\theta.$ At $\theta = \frac{2}{3}\pi$,

$r = a\sec^2\frac{\pi}{3} = 4a;\ \dfrac{dr}{d\theta} = a\sec^2\frac{\pi}{3}\tan\frac{\pi}{3} = 4\sqrt{3}a,\ \dfrac{d^2r}{d\theta^2} = a\sec^2\frac{\pi}{3}\tan^2\frac{\pi}{3} + \frac{1}{2}a\sec^4\frac{\pi}{3} = 12a + 8a = 20a.$

$$\mathrm{K} = \frac{(4a)^2 - 2(4\sqrt{3}a)^2 - (4a)(20a)}{[(4a)^2 + (4\sqrt{3}a)^2]^{3/2}} = \frac{16a^2 + 98a^2 - 80a^2}{(16a^2 + 48a^2)^{3/2}} = \frac{32a^2}{8^3|a|^3} = \frac{1}{16|a|},\ \rho = 16|a|$$

54. At $\theta = 1$, $r = a\theta = a$, $r' = a$, $r'' = 0$. $\mathrm{K} = \dfrac{\left|a^2 + 2a^2 - 0\right|}{(a^2 + a^2)^{3/2}} = \dfrac{3a^2}{2^{3/2}|a|^3} = \dfrac{3\sqrt{2}}{4|a|},\ \rho = \frac{2}{3}\sqrt{2}|a|$

55. $[\mathrm{K}(t)]^2 = \|D_s\mathbf{T}(t)\|^2 = D_s\mathbf{T}(t)\cdot D_s\mathbf{T}(t)$ and by Theorem 11.3.3, $\mathbf{T}(t) = D_s\mathbf{R}(t)$. Because $\mathbf{T}(t)$ is a unit vector, $0 = \mathbf{T}(t)\cdot D_s\mathbf{T}(t)$. Therefore

$0 = D_s[\mathbf{T}(t)\cdot D_s\mathbf{T}(t)] = D_s\mathbf{T}(t)\cdot D_s\mathbf{T}(t) + \mathbf{T}(t)\cdot D_s^{\,2}\mathbf{T}(t) = [\mathrm{K}(t)]^2 + D_s\mathbf{R}(t)\cdot D_s^{\,3}\mathbf{R}(t)$

Hence $D_s\mathbf{R}(t)\cdot D_s^{\,3}\mathbf{R}(t) = -[\mathrm{K}(t)]^2$.

56. Prove Theorem 11.4.2.

▷ We have by the chain rule

$\dot{\mathbf{R}} = \dfrac{d\mathbf{R}}{ds}\cdot\dfrac{ds}{dt} = \dot{s}\mathbf{T}$

$\ddot{\mathbf{R}} = \ddot{s}\mathbf{T} + \dot{s}\dot{\mathbf{T}} = \ddot{s}\mathbf{T} + \dot{s}^2\mathrm{K}\mathbf{N}$

Because $\mathbf{T}\times\mathbf{T} = \mathbf{0}$, we have

$\dot{\mathbf{R}}\times\ddot{\mathbf{R}} = \dot{s}^3\mathrm{K}(\mathbf{T}\times\mathbf{N}) = \dot{s}^3\mathrm{K}\mathbf{B}$

Because $\mathbf{T}$ and $\mathbf{B}$ are unit vectors, then

$\|\dot{\mathbf{R}}\| = |\dot{s}|$ and $\|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\| = |\dot{s}|^3\mathrm{K} = \|\dot{\mathbf{R}}\|^3\mathrm{K}$

Solving for K, we find the formula of Theorem 11.4.2

$\mathrm{K} = \dfrac{\|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\|}{\|\dot{\mathbf{R}}\|^3}$

57. $y = a\cosh\frac{x}{a},\ \dfrac{dy}{dx} = \sinh\frac{x}{a};\ \dfrac{d^2y}{dx^2} = \frac{1}{a}\cosh\frac{x}{a}.$

$$\mathrm{K} = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\left|\frac{1}{a}\cosh\frac{x}{a}\right|}{\left[1 + \sinh^2\frac{x}{a}\right]^{3/2}} = \frac{\frac{1}{a}\cosh\frac{x}{a}}{\cosh^3\frac{x}{a}} = \frac{a}{a^2\cosh^2\frac{x}{a}} = \frac{a}{y^2}$$

Because $y = a\cosh\frac{x}{a}$ has an absolute minimum at (0,a), K has an absolute maximum there.

58. $\mathbf{R} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k},\ a > 0.\ \dot{\mathbf{R}} = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b,\ \ddot{\mathbf{R}} = -a\cos t\mathbf{i} - a\sin t\mathbf{j}$

$$\|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\| = \left\|\begin{matrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -a\sin t & a\cos t & b\\ -a\cos t & -a\sin t & 0\end{matrix}\right\| = \|ab\sin t\mathbf{i} - ab\cos t\mathbf{j} - a^2\mathbf{k}\| = \sqrt{a^2b^2 + a^4} = a\sqrt{a^2 + b^2}$$

$\mathrm{K} = \|\dot{\mathbf{R}}\times\ddot{\mathbf{R}}\|/\|\dot{\mathbf{R}}\|^3 = a\sqrt{a^2 + b^2}/(\sqrt{a^2 + b^2})^3 = a/(a^2 + b^2)$

59. $\mathrm{K} = \dfrac{1}{a + b^2/a} = \dfrac{1}{(\sqrt{a} - b/\sqrt{a})^2 + 2b}$ has an absolute maximum value of $\frac{1}{2b}$ when $\sqrt{a} = b/\sqrt{a}$, $b = a$.

11.5 CURVILINEAR MOTION

11.5.1 Definition Let C be the curve having vector equation

$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

If a particle is moving along C so that its position at any time t units is is the point $P(f(t), g(t), h(t))$, and then the velocity vector $\mathbf{V}(t)$ and the acceleration vector $\mathbf{A}(t)$ at the point P are defined by

$\mathbf{V}(t) = \mathbf{R}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$

$\mathbf{A}(t) = \mathbf{V}'(t) = \mathbf{R}''(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j} + h''(t)\mathbf{k}$

Initial Value Reversing the above equations, we have

$$\mathbf{V}(t) = \mathbf{V}(a) + \int_a^t \mathbf{A}(u)du \quad \text{and} \quad \mathbf{R}(t) = \mathbf{R}(a) + \int_a^t \mathbf{V}(u)du$$

If a representation of the velocity vector $\mathbf{V}(t)$ is chosen so that its initial point is on the curve C, then the representation is tangent to C at all t. The measure of the *speed* of the particle at t is a scalar and is the magnitude of the velocity vector. Because the number of units in the speed is ds/dt, we have

$$v(t) = \frac{ds}{dt} = \|\mathbf{V}(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

The acceleration vector may be expressed as a linear combination of the unit tangent vector and the unit normal vector as follows.

$$\mathbf{A}(t) = A_T(t)\mathbf{T}(t) + A_N(t)\mathbf{N}(t)$$

where

$A_T(t) = \dfrac{d^2s}{dt^2}$ is the *tangential component* of acceleration

and

$A_N(t) = K(t)\left(\dfrac{ds}{dt}\right)^2 = \dfrac{v^2}{\rho}$ is the *normal component* of acceleration.

It follows that

$$A_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [A_T(t)]^2} \quad \text{and} \quad K(t) = A_N(t)/v(t)^2$$

Planar Motion If

$\mathbf{T} = a\mathbf{i} + b\mathbf{j}$

then

$$\mathbf{N} = \begin{cases} -b\mathbf{i} + b\mathbf{j} & \text{if } \mathbf{A} \cdot \mathbf{N} \geq 0 \\ b\mathbf{i} - a\mathbf{j} & \text{if otherwise} \end{cases}$$

If a representation of the acceleration vector $\mathbf{A}(t)$ is chosen so that its initial point in on the curve C, then the representation is directed toward the "inside" or concave side of the curve.

If a projectile is shot from a gun having angle of elevation of measure α with a muzzle velocity of v_0 ft/sec, then the position vector of the projectile at t sec is $\mathbf{R}(t)$, given by

$\mathbf{R}(t) = tv_0\cos\alpha\mathbf{i} + (tv_0\sin\alpha - \frac{1}{2}gt^2)\mathbf{j} = x\mathbf{i} + y\mathbf{j} \quad \mathbf{V}(t) = v_0\cos\alpha\mathbf{i} + (v_0\sin\alpha - gt)\mathbf{j}$

y is maximum when $t = T = v_0g^{-1}\sin\alpha$. The range is $x(2T) = v_0{}^2g^{-1}\sin 2\alpha$.

where g ft/sec^2 is the acceleration due to the force of gravity ($g = 32$).

Exercises 11.5

In Exercises 1–10, the position $\mathbf{R}$ of a particle moving in the xy plane at time t units is given. (a) Find $\mathbf{V}(t)$, $\mathbf{A}(t)$, $v(t) = \|\mathbf{V}(t)\|$, $a(t) = \|\mathbf{A}(t)\|$. (b) Find $\mathbf{V}(t_1)$ and $\mathbf{A}(t_1)$ and (c) sketch their representations on the particle's path.

1. $\mathbf{R}(t) = (t^2 + 4)\mathbf{i} + (t - 2)\mathbf{j}$; $t_1 = 3$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = 2t\mathbf{i} + \mathbf{j}$; $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = 2\mathbf{i}$;
$v(t) = \sqrt{4t^2 + 1}$; $a(t) = 2$ (b) $\mathbf{V}(3) = 6\mathbf{i} + \mathbf{j}$ $\mathbf{A}(3) = 2\mathbf{i}$

2. $\mathbf{R}(t) = (1 + t)\mathbf{i} + (t^2 - 1)\mathbf{j}$; $t_1 = 1$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \mathbf{i} + 2t\mathbf{j}$; $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = 2\mathbf{j}$;
$v(t) = \sqrt{1 + 4t^2}$; $a(t) = 2$ (b) $\mathbf{V}(1) = \mathbf{i} + 2\mathbf{j}$ $\mathbf{A}(1) = 2\mathbf{j}$

3. $\mathbf{R}(t) = 5\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j}$; $t_1 = \frac{1}{4}\pi$ (a) $\mathbf{V} = \dot{\mathbf{R}} = -10\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j}$; $\mathbf{A} = \dot{\mathbf{V}} = -20\cos 2t\mathbf{i} - 12\sin 2t\mathbf{j}$;
$v = 2\sqrt{25\sin^2 2t + 9\cos^2 2t}$; $a = 4\sqrt{25\cos^2 2t + 9\sin^2 2t}$ (b) $\mathbf{V}(\frac{1}{4}\pi) = -10\mathbf{i}$; $\mathbf{A}(\frac{1}{4}\pi) = -12\mathbf{j}$

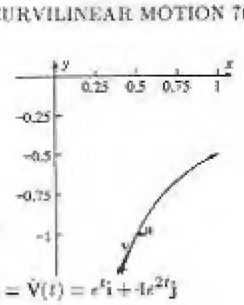
4. $\mathbf{R}(t) = \frac{2}{t}\mathbf{i} - \frac{1}{4}t\mathbf{j}$; $t_1 = 4$

▸ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = -\frac{2}{t^2}\mathbf{i} - \frac{1}{4}\mathbf{j}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = \frac{4}{t^3}\mathbf{i}$

$\|\mathbf{V}(t)\| = \sqrt{4/t^4 + \frac{1}{16}}$ $\qquad$ $\|\mathbf{A}(t)\| = \frac{4}{|t|^3}$

(b) $\mathbf{V}(4) = -\frac{1}{8}\mathbf{i} - \frac{1}{4}\mathbf{j}$ $\qquad$ $\mathbf{A}(4) = \frac{1}{16}\mathbf{i}$

(c) $\mathbf{R}(4) = \frac{1}{2}\mathbf{i} - \mathbf{j}$. The figure shows the path (a hyperbola) and representations of the velocity and acceleration vectors at $(\frac{1}{2}, -1)$.



5. $\mathbf{R}(t) = e^t\mathbf{i} + e^{2t}\mathbf{j}$, $t_1 = \ln 2$ $\qquad$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = e^t\mathbf{i} + 2e^{2t}\mathbf{j}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = e^t\mathbf{i} + 4e^{2t}\mathbf{j}$

$v(t) = e^t\sqrt{1 + 4e^{2t}}$; $a(t) = e^t\sqrt{1 + 16e^{2t}}$ (b) $\mathbf{V}(\ln 2) = 2\mathbf{i} + 8\mathbf{j}$ $\qquad$ $\mathbf{A}(\ln 2) = 2\mathbf{i} + 16\mathbf{j}$

6. $\mathbf{R}(t) = e^{2t}\mathbf{i} + e^{3t}\mathbf{j}$, $t_1 = 0$ $\qquad$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = 2e^{2t}\mathbf{i} + 3e^{3t}\mathbf{j}$ $\qquad$ $\mathbf{A}(t) = 4e^{2t}\mathbf{i} + 9e^{3t}\mathbf{j}$

$v(t) = e^{2t}\sqrt{4 + 9e^{2t}}$; $a(t) = e^{2t}\sqrt{16 + 81e^{2t}}$ (b) $\mathbf{V}(0) = 2\mathbf{i} + 3\mathbf{j}$ $\qquad$ $\mathbf{A}(0) = 4\mathbf{i} + 9\mathbf{j}$

7. $\mathbf{R}(t) = t\mathbf{i} + (\ln \sec t)\mathbf{j}$; $t_1 = \frac{1}{4}\pi$ $\qquad$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \mathbf{i} + \tan t\mathbf{j}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = \sec^2 t\mathbf{j}$

$v(t) = \sqrt{1 + \tan^2 t} = |\sec t|$; $a(t) = \sec^2 t$ (b) $\mathbf{V}(\frac{1}{4}\pi) = \mathbf{i} + \mathbf{j}$ $\qquad$ $\mathbf{A}(\frac{1}{4}\pi) = 2\mathbf{j}$

8. $\mathbf{R}(t) = 2(1 - \cos t)\mathbf{i} + 2(1 - \sin t)\mathbf{j}$; $t_1 = \frac{5}{6}\pi$

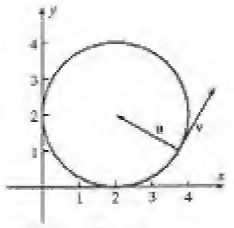
▸ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = 2\sin t\mathbf{i} - 2\cos t\mathbf{j}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$\|\mathbf{V}(t)\| = 2\sqrt{\sin^2 t + \cos^2 t} = 2$ $\qquad$ $\|\mathbf{A}(t)\| = 2\sqrt{\cos^2 t + \sin^2 t} = 2$

(b) $\mathbf{V}(\frac{5}{6}\pi) = 2(\frac{1}{2})\mathbf{i} - 2(-\frac{1}{2}\sqrt{3})\mathbf{j} = \mathbf{i} + \sqrt{3}\mathbf{j}$; $\mathbf{A}(\frac{5}{6}\pi) = 2(-\frac{1}{2}\sqrt{3})\mathbf{i} + 2(\frac{1}{2})\mathbf{j} = -\sqrt{3}\mathbf{i} + \mathbf{j}$

(c) $\mathbf{R}(\frac{5}{6}\pi) = 2(1 + \frac{1}{2}\sqrt{3})\mathbf{i} + 2(1 - \frac{1}{2})\mathbf{j} = (2 + \sqrt{3})\mathbf{i} + \mathbf{j}$

The figure shows the circular path and representations of the velocity and acceleration vectors at $(1 + \sqrt{3}, 1)$.



9. $\mathbf{R}(t) = (t^2 + 3t)\mathbf{i} + (1 - 3t^2)\mathbf{j}$; $t_1 = \frac{1}{2}$ $\qquad$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = (2t + 3)\mathbf{i} - 6t\mathbf{j}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = 2\mathbf{i} - 6\mathbf{j}$

$v = \sqrt{(2t + 3)^2 + 36t^2} = \sqrt{40t^2 + 12t + 9}$; $a = 2\sqrt{1^2 + 3^2} = 2\sqrt{10}$ (b) $\mathbf{V}(\frac{1}{2}) = 4\mathbf{i} - 3\mathbf{j}$; $\mathbf{A}(\frac{1}{2}) = 2\mathbf{i} - 6\mathbf{j}$

10. $\mathbf{R}(t) = \ln(t + 2)\mathbf{i} + \frac{1}{3}t^2\mathbf{j}$; $t_1 = 1$ $\qquad$ (a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \frac{1}{t + 2}\mathbf{i} + \frac{2}{3}t\mathbf{j}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = -\frac{1}{(t + 2)^2}\mathbf{i} + \frac{2}{3}\mathbf{j}$

$v = \sqrt{(t + 2)^{-2} + \frac{4}{9}t^2}$; $a = \sqrt{(t + 2)^{-4} + \frac{4}{9}}$ (b) $\mathbf{V}(1) = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j}$ $\qquad$ $\mathbf{A}(1) = -\frac{1}{9}\mathbf{i} + \frac{2}{3}\mathbf{j}$

In Exercises 11–16, the position $\mathbf{R}$ of a particle moving in space at time t units is given. Find $\mathbf{V}(t_1)$, $\mathbf{A}(t_1)$ and $v(t_1)$ and sketch representations of the vectors on the particle's path.

11. $\mathbf{R} = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$; $t_1 = \frac{1}{2}\pi$ $\qquad$ $\mathbf{V} = \dot{\mathbf{R}} = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k} = -2\mathbf{i} + \mathbf{k}$

$\mathbf{A} = \dot{\mathbf{V}} = -2\cos t\mathbf{i} - 2\sin t\mathbf{j} = -2\mathbf{j}$ $\qquad$ $v(\frac{1}{2}\pi) = \sqrt{2^2 + 1^2} = \sqrt{5}$

12. $\mathbf{R}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$; $t_1 = 2$

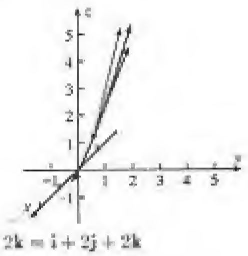
▸ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = \mathbf{j} + 2t\mathbf{k}$

$\mathbf{V}(2) = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ $\qquad$ $\mathbf{A}(2) = \mathbf{j} + 4\mathbf{k}$

$v(2) = \|\mathbf{V}(2)\| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$

$\mathbf{R}(2) = 2\mathbf{i} + 2\mathbf{j} + \frac{8}{3}\mathbf{j}$

The figure shows the path and representations of the velocity and acceleration vectors at $(2, 2, \frac{8}{3})$



13. $\mathbf{R} = t\mathbf{i} + (t^2 - 2t)\mathbf{j} + 2(t - 1)\mathbf{k}$; $t_1 = 2$ $\qquad$ $\mathbf{V} = \dot{\mathbf{R}} = \mathbf{i} + (2t - 2)\mathbf{j} + 2\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$\mathbf{A} = \dot{\mathbf{V}} = 2\mathbf{j}$ $\qquad$ $v(2) = \sqrt{1^2 + 2^2 + 2^2} = 3$

14. $\mathbf{R} = 3\cos t\mathbf{i} + 4\sin t\mathbf{j} + 2t\mathbf{k}$; $t_1 = \frac{1}{2}\pi$ $\qquad$ $\mathbf{V} = \dot{\mathbf{R}} = -3\sin t\mathbf{i} + 4\cos t\mathbf{j} + 2\mathbf{k} = -3\mathbf{i} + 2\mathbf{k}$

$\mathbf{A} = \dot{\mathbf{V}} = -3\cos t\mathbf{i} - 4\sin t\mathbf{j} = -4\mathbf{j}$ $\qquad$ $v(\frac{1}{2}\pi) = \sqrt{3^2 + 2^2} = \sqrt{13}$

15. $\mathbf{R} = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j} + e^t\mathbf{k};\ t_1 = 0$ $\qquad \mathbf{V} = \dot{\mathbf{R}} = \sqrt{2}e^t\cos(t + \frac{1}{4}\pi)\mathbf{i} + \sqrt{2}e^t\sin(t + \frac{1}{4}\pi)\mathbf{j} + e^t\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{A} = \dot{\mathbf{V}} = 2e^t\cos(t + \frac{1}{2}\pi)\mathbf{i} + 2e^t\sin(t + \frac{1}{2}\pi)\mathbf{j} + e^t\mathbf{k} = 2\mathbf{j} + \mathbf{k}$ $\qquad v(0) = \sqrt{3}$

16. $\mathbf{R}(t) = \frac{1}{2}(t^2 + 1)^{-1}\mathbf{i} + \ln(1 + t^2)\mathbf{j} + \tan^{-1}t\mathbf{k};\ t_1 = 1$

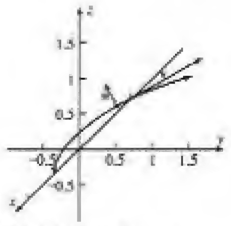
▷ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = t(t^2 + 1)^{-2}\mathbf{i} + 2t(1 + t^2)^{-1}\mathbf{j} + (1 + t^2)^{-1}\mathbf{k}$

$\mathbf{A}(t) = \dot{\mathbf{V}}(t) = (3t^2 - 1)(t^2 + 1)^{-3}\mathbf{i} + 2(1 - t^2)(1 + t^2)^{-2}\mathbf{j} - 2t(1 + t^2)^{-2}\mathbf{k}$

$\mathbf{V}(1) = -\frac{1}{4}\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$ $\qquad \mathbf{A}(1) = \frac{1}{4}\mathbf{i} + \frac{1}{2}\mathbf{k}$

$v(1) = \|\mathbf{V}(1)\| = \sqrt{\frac{1}{16} + 1 + \frac{1}{4}} = \frac{1}{4}\sqrt{21}$ $\qquad \mathbf{R}(1) = \frac{1}{4}\mathbf{i} + \ln 2\mathbf{j} + \frac{1}{4}\pi\mathbf{k}$

The figure shows the path and representations of the velocity and acceleration vectors at $(0.25, 0.69, 0.79)$.



In Exercises 17–24, find a vector equation of the path satisfying the given equation and initial conditions.

17. $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = 3\mathbf{i} + 2\mathbf{j} + \int_0^t [(u + 1)^{-2}\mathbf{i} + (u + 1)\mathbf{j}]du$

$= 3\mathbf{i} + 2\mathbf{j} + \left[-(u + 1)^{-1}\mathbf{i} + (\frac{1}{2}u^2 + u)\mathbf{j}\right]_0^t = \left(4 - \frac{1}{t + 1}\right)\mathbf{i} + (\frac{1}{2}t^2 + t + 2)\mathbf{j}$

18. $\mathbf{R}(t) = \mathbf{R}(1) + \int_1^t \mathbf{V}(u)du = 4\mathbf{i} - 3\mathbf{j} + \int_1^t [(2u - 1)\mathbf{i} + 3u^{-2}\mathbf{j}]du$

$= 4\mathbf{i} - 3\mathbf{j} + \left[(u^2 - u)\mathbf{i} - 3u^{-1}\right]_1^t = (t^2 - t + 4)\mathbf{i} - 3t^{-1}\mathbf{j}$

19. $\mathbf{V}(t) = \mathbf{V}(0) + \int_0^t \mathbf{A}(u)du = 2\mathbf{i} + \mathbf{j} + \int_0^t (e^{-u}\mathbf{i} + 2e^{3u}\mathbf{j})du = 2\mathbf{i} + \mathbf{j} + \left[-e^{-u}\mathbf{i} + \frac{2}{3}e^{3u}\mathbf{j}\right]_0^t = (3 - e^{-t})\mathbf{i} + (\frac{2}{3}e^{3t} + \frac{1}{3})\mathbf{j}$

$\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = 3\mathbf{j} + \int_0^t [(3 - e^{-u})\mathbf{i} + (\frac{2}{3}e^{3u} + \frac{1}{3})\mathbf{j}]du = 3\mathbf{j} + \left[(3u + e^{-u})\mathbf{i} + (\frac{2}{9}e^{3u} + \frac{1}{3}u)\right]_0^t$

$= (3t + e^{-t} - 1)\mathbf{i} + (\frac{2}{9}e^{3t} + \frac{1}{3}t + \frac{25}{9})\mathbf{j}$

20. $\mathbf{A}(t) = 2\cos 2t\mathbf{i} + 2\sin 2t\mathbf{j}$, $\mathbf{V}(0) = \mathbf{i} + \mathbf{j}$, and $\mathbf{R}(0) = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$

▷ $\mathbf{V}(t) = \mathbf{V}(0) + \int_0^t \mathbf{A}(u)du = \mathbf{i} + \mathbf{j} + \int_0^t (2\cos 2u\mathbf{i} + 2\sin 2u\mathbf{j})du = \mathbf{i} + \mathbf{j} + [\sin 2u\mathbf{i} - \cos 2u\mathbf{j}]_0^t$

$= (\sin 2t + 1)\mathbf{i} + (2 - \cos 2t)\mathbf{j}$

$\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \int_0^t (\sin 2u + 1)\mathbf{i} + (2 - \cos 2t)\mathbf{j}$

$= \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \left[(-\frac{1}{2}\cos 2u + u)\mathbf{i} + (2u - \frac{1}{2}\sin 2u)\mathbf{j}\right]_0^t = (-\frac{1}{2}\cos 2t + t + 1)\mathbf{i} + (2t - \frac{1}{2}\sin 2t - \frac{1}{2})\mathbf{j}$

21. $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = \mathbf{i} + 2\mathbf{j} + \int_0^t (\mathbf{i} + \mathbf{j} - 32u\mathbf{k})du = \mathbf{i} + 2\mathbf{j} + [u\mathbf{i} + 2u\mathbf{j} - 16u^2\mathbf{k}]_0^t = (t + 1)\mathbf{i} + (2t + 2)\mathbf{j} - 16t^2\mathbf{k}$

22. $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = 2\mathbf{j} + \mathbf{k} + \int_0^t [(u^2 + 2u)\mathbf{i} + 2u\mathbf{j} + 3u^2\mathbf{k}]du = 2\mathbf{j} + \mathbf{k} + \left[(\frac{1}{3}u^3 + u^2)\mathbf{i} + u^2\mathbf{j} + u^3\mathbf{k}\right]_0^t$

$= (\frac{1}{3}t^3 + t^2)\mathbf{i} + (t^2 + 2)\mathbf{j} + (t^3 + 1)\mathbf{k}$

23. $\mathbf{V}(t) = \mathbf{V}(0) + \int_0^t \mathbf{A}(u)du = 2\mathbf{i} + 3\mathbf{j} + \int_0^t (6u\mathbf{i} + 12u^2\mathbf{j} + \mathbf{k})du = 2\mathbf{i} + 3\mathbf{j} + [3u^2\mathbf{i} + 4u^3\mathbf{j} + u\mathbf{k}]_0^t$

$= (3t^2 + 2)\mathbf{i} + (4t^3 + 3)\mathbf{j} + t\mathbf{k}$. $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = 4\mathbf{k} + \int_0^t [(3u^2 + 2)\mathbf{i} + (4u^3 + 3)\mathbf{j} + u\mathbf{k}]du$

$= 4\mathbf{k} + \left[(u^3 + 2u)\mathbf{i} + (u^4 + 3u)\mathbf{j} + \frac{1}{2}u^2\mathbf{k}\right]_0^t = (t^3 + 2t)\mathbf{i} + (t^4 + 3t)\mathbf{j} + (\frac{1}{2}t^2 + 4)\mathbf{k}$

24. $\mathbf{A}(t) = -32\mathbf{k}$, $\mathbf{V}(0) = 4\mathbf{i} + 4\mathbf{j}$, $\mathbf{R}(0) = 60\mathbf{k}$

▷ $\mathbf{V}(t) = \mathbf{V}(0) + \int_0^t \mathbf{A}(u)du = 4\mathbf{i} + 4\mathbf{j} + \int_0^t (-32\mathbf{k})du = 4\mathbf{i} + 4\mathbf{j} - 32t\mathbf{k}$

$\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{V}(u)du = 60\mathbf{k} + \int_0^t (4\mathbf{i} + 4\mathbf{j} - 32u\mathbf{k})du = 60\mathbf{k} + [4u\mathbf{i} + 4u\mathbf{j} - 16u^2\mathbf{k}]_0^t = 4t\mathbf{i} + 4t\mathbf{j} + (60 - 16t^2)\mathbf{k}$

In Exercises 25–30, find $\mathbf{V}(t)$, $\mathbf{A}(t)$, $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\|\mathbf{V}(t)\|$, $A_T(t)$, $A_N(t)$ and $K(t)$; and their values when $t = t_1$. Sketch the curve and representations of $\mathbf{V}(t_1)$, $\mathbf{A}(t_1)$ and its tangential and normal components.

25. $\mathbf{R}(t) = (2t+3)\mathbf{i} + (t^2-1)\mathbf{j}$. $\mathbf{V}(t) = 2\mathbf{i} + 2t\mathbf{j}$. $\mathbf{A}(t) = 2\mathbf{j}$. $D_t s = \|\mathbf{V}(t)\| = \sqrt{4+4t^2} = 2\sqrt{1+t^2}$.

$A_T(t) = D_t{}^2 s = \dfrac{2t}{\sqrt{1+t^2}}$. $A_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [A_T(t)]^2} = \sqrt{4 - \dfrac{4t^2}{1+t^2}} = \sqrt{\dfrac{4+4t^2-4t^2}{1+t^2}} = \dfrac{2}{\sqrt{1+t^2}}$

$K(t) = \dfrac{A_N(t)}{(D_t s)^2} = \dfrac{2}{\sqrt{1+t^2}} \cdot \dfrac{1}{4(1+t^2)} = \dfrac{1}{2(1+t^2)^{3/2}}$. $\quad \mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \dfrac{1}{\sqrt{1+t^2}}\mathbf{i} + \dfrac{t}{\sqrt{1+t^2}}\mathbf{j}$.

$\mathbf{N}(t) = \dfrac{-t}{\sqrt{1+t^2}}\mathbf{i} + \dfrac{1}{\sqrt{1+t^2}}\mathbf{j}$ because $\mathbf{N}(t)\cdot\mathbf{A}(t) \geq 0$.

$\mathbf{V}(2) = 2\mathbf{i} + 4\mathbf{j}$. $\mathbf{A}(2) = 2\mathbf{j}$. $\mathbf{T}(2) = \dfrac{1}{\sqrt{5}}\mathbf{i} + \dfrac{2}{\sqrt{5}}\mathbf{j}$. $\quad \mathbf{N}(2) = \dfrac{-2}{\sqrt{5}}\mathbf{i} + \dfrac{1}{\sqrt{5}}\mathbf{j}$. $\|\mathbf{V}(2)\| = \sqrt{4+16} = 2\sqrt{5}$.

$A_T(2) = \dfrac{4}{\sqrt{5}}$. $A_N(2) = \dfrac{2}{\sqrt{5}}$. $K(2) = \dfrac{1}{2\sqrt{125}} = \dfrac{1}{10\sqrt{5}}$.

26. $\mathbf{R}(t) = (t-1)\mathbf{i} + t^2\mathbf{j}$. $\mathbf{V}(t) = \mathbf{i} + 2t\mathbf{j}$. $\mathbf{A}(t) = 2\mathbf{j}$. $\dot{s} = \|\mathbf{V}(t)\| = \sqrt{1+4t^2}$. $A_T(t) = \ddot{s} = \dfrac{4t}{\sqrt{1+4t^2}}$

$A_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [A_T(t)]^2} = \sqrt{4 - \dfrac{16t^2}{1+4t^2}} = \sqrt{\dfrac{4}{1+4t^2}} = \dfrac{2}{\sqrt{1+4t^2}}$

$K(t) = \dfrac{A_N(t)}{\dot{s}^2} = \dfrac{2}{\sqrt{1+4t^2}} \cdot \dfrac{1}{1+4t^2} = \dfrac{2}{(1+4t^2)^{3/2}}$ $\quad \mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \dfrac{\mathbf{i}+2\mathbf{j}}{\sqrt{1+4t^2}}$. $\mathbf{N}(t) = \dfrac{-2\mathbf{i}+\mathbf{j}}{\sqrt{1+4t^2}}$

$\mathbf{V}(1) = \mathbf{i} + 2\mathbf{j}$. $\mathbf{A}(1) = 2\mathbf{j}$. $\mathbf{T}(1) = \frac{1}{5}\sqrt{5}(\mathbf{i}+2\mathbf{j})$. $\quad \mathbf{N}(1) = \frac{1}{5}\sqrt{5}(-2\mathbf{i}+\mathbf{j})$. $\|\mathbf{V}(1)\| = \sqrt{5}$.

$A_T(1) = \frac{4}{5}\sqrt{5}$. $A_N(1) = \frac{2}{5}\sqrt{5}$. $K(1) = \frac{2}{25}\sqrt{5}$

27. $\mathbf{R}(t) = 5\cos 3t\mathbf{i} + 5\sin 3t\mathbf{j}$. $\mathbf{V}(t) = -15\sin 3t\mathbf{i} + 15\cos 3t\mathbf{j}$. $\mathbf{A}(t) = -45\cos 3t\mathbf{i} - 45\sin 3t\mathbf{j}$.

$D_t s = \|\mathbf{V}(t)\| = 15\sqrt{(-\sin 3t)^2 + \cos^2 3t} = 15$. $\quad \mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = -\sin 3t\mathbf{i} + \cos 3t\mathbf{j}$.

$\mathbf{N}(t) = -\cos 3t\mathbf{i} - \sin 3t\mathbf{j}$ because $\mathbf{N}(t)\cdot\mathbf{A}(t) \geq 0$. $\quad A_T(t) = D_t{}^2 s = D_t(15) = 0$.

$A_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [A_T(t)]^2} = \sqrt{\|\mathbf{A}(t)\|^2} = \|\mathbf{A}(t)\| = 45\sqrt{(-\cos 3t)^2 + \sin^2 3t} = 45$

$K(t) = \dfrac{A_N(t)}{(D_t s)^2} = \dfrac{45}{225} = \dfrac{1}{5}$. $\mathbf{V}(\frac{1}{3}\pi) = -15\sin\pi\mathbf{i} + 15\cos\pi\mathbf{j} = -15\mathbf{j}$. $\mathbf{A}(\frac{1}{3}\pi) = -45\cos\pi\mathbf{i} - 45\sin\pi\mathbf{j} = 45\mathbf{i}$.

$\mathbf{T}(\frac{1}{3}\pi) = -\sin\pi\mathbf{i} + \cos\pi\mathbf{j} = -\mathbf{j}$. $\mathbf{N}(\frac{1}{3}\pi) = -\cos\pi\mathbf{i} - \sin\pi\mathbf{j} = \mathbf{i}$. $\|\mathbf{V}(\frac{1}{3}\pi)\| = 15$.

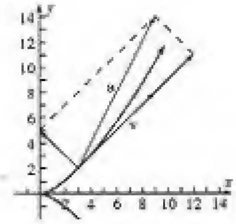
28. $\mathbf{R}(t) = 3t^2\mathbf{i} + 2t^3\mathbf{j}$; $t_1 = 1$

▸ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = 6t\mathbf{i} + 6t^2\mathbf{j}$ $\qquad \mathbf{A}(t) = \dot{\mathbf{V}}(t) = 6\mathbf{i} + 12t\mathbf{j}$

$\dot{s} = \|\mathbf{V}(t)\| = 6\sqrt{t^2+t^4} = 6|t|\sqrt{1+t^2} = 6t\sqrt{1+t^2}$

We have simplified by omitting the factor $\operatorname{sgn}(t)$ from the last expression and every subsequent expression.

$A_T = \ddot{s} = 6\sqrt{1+t^2} + 6t\dfrac{t}{\sqrt{1+t^2}} = \dfrac{6(1+2t^2)}{\sqrt{1+t^2}}$



$A_N = \sqrt{\|\mathbf{A}(t)\|^2 - A_T{}^2} = 6\sqrt{(1+4t^2) - \dfrac{1+4t^2+4t^4}{1+t^2}} = \dfrac{6t}{\sqrt{1+t^2}}$

$K(t) = \dfrac{A_N}{\dot{s}^2} = \dfrac{6t}{\sqrt{1+t^2}} \cdot \dfrac{1}{36t^2(1+t^2)} = \dfrac{1}{6t(1+t^2)^{3/2}}$

$\mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \dfrac{6t\mathbf{i}+6t^2\mathbf{j}}{6t\sqrt{1+t^2}} = \dfrac{\mathbf{i}+t\mathbf{j}}{\sqrt{1+t^2}}$

$\mathbf{N}(t) = \dfrac{-t\mathbf{i}+\mathbf{j}}{\sqrt{1+t^2}}$ because $\mathbf{N}(t)\cdot\mathbf{A}(t) = \dfrac{6t}{\sqrt{1+t^2}} \geq 0$

Substituting $t = 1$ in each formula, we obtain

$\mathbf{V}(1) = 6\mathbf{i} + 6\mathbf{j}$ $\quad \mathbf{A}(1) = 6\mathbf{i} + 12\mathbf{j}$ $\quad \mathbf{T}(1) = \frac{1}{2}\sqrt{2}(\mathbf{i}+\mathbf{j})$ $\quad \mathbf{N}(1) = \frac{1}{2}\sqrt{2}(-\mathbf{i}+\mathbf{j})$

$\|\mathbf{V}(1)\| = 6\sqrt{2}$ $\mathrm{A}_T(1) = 9\sqrt{2}$ $\mathrm{A}_N(1) = 3\sqrt{2}$ $\mathrm{K}(1) = \frac{1}{24}\sqrt{2}$

$\mathbf{R}(1) = 3\mathbf{i} + 2\mathbf{j}$. The figure shows the curve and the vectors with terminal point $(3, 2)$.

29. $\mathbf{R}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$. $\mathbf{V}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$. $\mathbf{A}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$. $D_t s = \|\mathbf{V}(t)\| = \sqrt{e^{2t} + e^{-2t}}$.

$$\mathbf{T}(t) = \frac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \frac{e^t}{\sqrt{e^{2t} + e^{-2t}}}\mathbf{i} - \frac{e^{-t}}{\sqrt{e^{2t} + e^{-2t}}}\mathbf{j}$$

$\mathbf{N}(t) = \frac{e^{-t}}{\sqrt{e^{2t} + e^{-2t}}}\mathbf{i} + \frac{e^t}{\sqrt{e^{2t} + e^{-2t}}}\mathbf{j}$ because $\mathbf{N}(t) \cdot \mathbf{A}(t) \geq 0$. $\mathrm{A}_T(t) = D_t{}^2 s = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$.

$$\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [\mathrm{A}_T(t)]^2} = \sqrt{e^{2t} + e^{-2t} - \frac{(e^{2t} - e^{-2t})^2}{e^{2t} + e^{-2t}}} = \sqrt{\frac{e^{4t} + 2 + e^{-4t} - (e^{4t} - 2 + e^{-4t})}{e^{2t} + e^{-2t}}}$$

$$= \frac{2}{\sqrt{e^{2t} + e^{-2t}}}.\ \mathrm{K}(t) = \frac{\mathrm{A}_N(t)}{(D_t s)^2} = \frac{2}{\sqrt{e^{2t} + e^{-2t}}} \cdot \frac{1}{e^{2t} + e^{-2t}} = \frac{2}{(e^{2t} + e^{-2t})^{3/2}}.$$

$\mathbf{V}(0) = \mathbf{i} - \mathbf{j}$. $\mathbf{A}(0) = \mathbf{i} + \mathbf{j}$. $\mathbf{T}(0) = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$. $\mathbf{N}(0) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$.

$\|\mathbf{V}(0)\| = \sqrt{2}$. $\mathrm{A}_T(0) = 0$. $\mathrm{A}_N(0) = \sqrt{2}$. $\mathrm{K}(0) = \frac{1}{\sqrt{2}}$

30. $\mathbf{R}(t) = \cos t^2\mathbf{i} + \sin t^2\mathbf{j}$. $\mathbf{V}(t) = -2t\sin t^2\mathbf{i} + 2t\cos t^2\mathbf{j}$. $\mathbf{A}(t) = (-2\sin t^2 - 4t^2\cos t^2)\mathbf{i} + (2\cos t^2 - 4t^2\sin t^2)\mathbf{j}$

$\dot{s} = \|\mathbf{V}(t)\| = 2t\,\mathrm{sgn}(t)$. $\mathrm{A}_T = \ddot{s} = 2\,\mathrm{sgn}(t)$. $\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - \mathrm{A}_T(t)^2} = 4t^2$. $\mathrm{K}(t) = \frac{\mathrm{A}_N}{\dot{s}^2} = 1$.

$\mathbf{T}(t) = \frac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \mathrm{sgn}(t)(-\sin t^2\mathbf{i} + \cos t^2\mathbf{j})$. $\mathbf{N}(t) = -\cos t^2\mathbf{i} - \sin t^2\mathbf{j}$ because $\mathbf{N}(t) \cdot \mathbf{A}(t) \geq 2 > 0$

$t_1 = \frac{1}{2}\sqrt{\pi}$. $\mathbf{V}(t_1) = \sqrt{\frac{1}{2}\pi}(-\mathbf{i} + \mathbf{j})$. $\mathbf{A}(t_1) \approx -3.6\mathbf{i} - 0.8\mathbf{j}$. $\mathbf{T}(t_1) = \frac{1}{2}\sqrt{2}(-\mathbf{i} + \mathbf{j})$. $\mathbf{N}(t_1) = -\frac{1}{2}\sqrt{2}(\mathbf{i} + \mathbf{j})$

$\|\mathbf{V}(t_1)\| = \sqrt{\pi}$. $\mathrm{A}_T(t_1) = 2$. $\mathrm{A}_N(t_1) = \pi$. $\mathrm{K}(t_1) = 1$

In Exercises 31–36, write $\mathbf{A}(t) = \mathrm{A}_T(t)\mathbf{T}(t) + \mathrm{A}_N(t)\mathbf{N}(t)$ without computing $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

31. $\mathbf{R} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$. $\mathbf{V} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$. $\mathbf{A} = 2\mathbf{j}$. $\dot{s} = \|\mathbf{V}\| = \sqrt{1 + 4t^2 + 1} = \sqrt{4t^2 + 2}$. $\mathrm{A}_T = \ddot{s} = \frac{4t}{\sqrt{4t^2 + 2}}$

$$\mathrm{A}_N = \sqrt{\|\mathbf{A}\|^2 - \mathrm{A}_T{}^2} = \sqrt{4 - \frac{16t^2}{4t^2 + 2}} = \sqrt{\frac{8}{4t^2 + 2}} = \frac{2\sqrt{2}}{\sqrt{4t^2 + 2}}.\ \mathbf{A} = \frac{4t\mathbf{T} + 2\sqrt{2}\mathbf{N}}{\sqrt{4t^2 + 2}}$$

32. $\mathbf{R}(t) = e^{-t}\mathbf{i} + e^t\mathbf{j} + \sqrt{2}t\mathbf{k}$

▸ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = -e^{-t}\mathbf{i} + e^t\mathbf{j} + \sqrt{2}\mathbf{k}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = e^{-t}\mathbf{i} + e^t\mathbf{j}$

$\dot{s} = \|\mathbf{V}(t)\| = \sqrt{e^{-2t} + e^{2t} + 2} = e^{-t} + e^t$ $\qquad$ $\mathrm{A}_T = \ddot{s} = -e^{-t} + e^t$

$\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - \mathrm{A}_T(t)^2} = \sqrt{4} = 2$ $\qquad$ $\mathbf{A}(t) = (-e^{-t} + e^t)\mathbf{T}(t) + 2\mathbf{N}(t)$

33. $\mathbf{R}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 2\mathbf{k}$, $t \geq 0$.

$\mathbf{V}(t) = D_t\mathbf{R}(t) = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j} = t\cos t\mathbf{i} + t\sin t\mathbf{j}$

$\mathbf{A}(t) = D_t\mathbf{V}(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}$. $\|\mathbf{A}(t)\| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$

$= \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t} = \sqrt{(\cos^2 t + \sin^2 t) + t^2(\sin^2 t + \cos^2 t)}$

$= \sqrt{1 + t^2}$. $D_t s = \|\mathbf{V}(t)\| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t$. $\mathrm{A}_T = D_t{}^2 s = 1$.

$\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - [\mathrm{A}_T(t)]^2} = \sqrt{(1 + t^2) - 1} = \sqrt{t^2} = |t| = t$. $\mathbf{A}(t) = \mathbf{T}(t) + t\mathbf{N}(t)$

34. $\mathbf{R} = 2t^2\mathbf{i} + t^2\mathbf{j} + 4t\mathbf{k}$. $\mathbf{V} = 4t\mathbf{i} + 2t\mathbf{j} + 4\mathbf{k}$. $\mathbf{A} = 4\mathbf{i} + 2\mathbf{j}$. $\dot{s} = \|\mathbf{V}\| = 2\sqrt{4t^2 + t^2 + 4} = 2\sqrt{5t^2 + 4}$.

$$\mathrm{A}_T = \ddot{s} = \frac{10}{\sqrt{5t^2 + 4}}.\ \mathrm{A}_N = \sqrt{\|\mathbf{A}\|^2 - \mathrm{A}_T{}^2} = \sqrt{20 - \frac{100}{5t^2 + 4}} = \sqrt{\frac{100t^2 - 20}{5t^2 + 4}} = 2\sqrt{\frac{25t^2 - 5}{5t^2 + 4}}$$

$$\mathbf{A} = \frac{10}{\sqrt{5t^2 + 4}}\mathbf{T} + 2\sqrt{\frac{25t^2 - 5}{5t^2 + 4}}\mathbf{N}$$

35. $\mathbf{R} = t^2\mathbf{i} + (\frac{1}{3}t^3 + t)\mathbf{j} + (\frac{1}{3}t^3 - t)\mathbf{k}$. $\mathbf{V} = 2t\mathbf{i} + (t^2+1)\mathbf{j} + (t^2-1)\mathbf{k}$. $\mathbf{A} = 2\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{j}$.
$\dot{s} = \|\mathbf{V}\| = \sqrt{4t^2 + (t^4+2t^2+1) + (t^4-2t^2+1)} = \sqrt{2t^4+4t^2+2} = \sqrt{2}(t^2+1)$. $A_T = \ddot{s} = 2\sqrt{2}t$
$A_N = \sqrt{\|\mathbf{A}\|^2 - {A_T}^2} = \sqrt{4+4t^2+4t^2-8t^2} = 2$. $\mathbf{A} = 2\sqrt{2}t\mathbf{T} + 2\mathbf{N}$

36. $\mathbf{R}(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j} + t\mathbf{k}$
▸ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \mathbf{k}$
$\mathbf{A}(t) = \dot{\mathbf{V}}(t) = (-2\sin t - t\cos t)\mathbf{i} + (2\cos t - t\sin t)\mathbf{j}$
$\dot{s} = \|\mathbf{V}(t)\| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1} = \sqrt{2+t^2}$
$A_T(t) = \ddot{s} = \dfrac{t}{\sqrt{2+t^2}}$ $\qquad A_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - {A_T}^2}$
$= \sqrt{(-2\sin t - t\cos t)^2 + (2\cos t - t\sin t)^2 - \dfrac{t^2}{2+t^2}} = \sqrt{4+t^2-\dfrac{t^2}{2+t^2}} = \sqrt{\dfrac{4t^4+5t^2+8}{2+t^2}}$
$\mathbf{A}(t) = \dfrac{t}{\sqrt{2+t^2}}\mathbf{T} + \sqrt{\dfrac{4t^4+5t^2+8}{2+t^2}}\mathbf{N}$

37. This is Theorem 11.2.9.

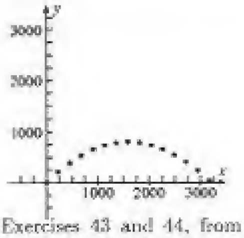
38. $\mathbf{R} = \tan t\mathbf{i} + \sinh 2t\mathbf{j} + \operatorname{sech} t\mathbf{k}$. $\mathbf{V} = \sec^2 t\mathbf{i} + 2\cosh 2t\mathbf{j} - \operatorname{sech} t\tanh t\mathbf{k}$.
$\mathbf{A} = 2\sec^2 t\tan t + 4\sinh 2t\mathbf{j} + \operatorname{sech} t(2\tanh^2 t - 1)$. $\mathbf{V}(0)\cdot\mathbf{A}(0) = (i+2\mathbf{j})\cdot(-\mathbf{k}) = 0$

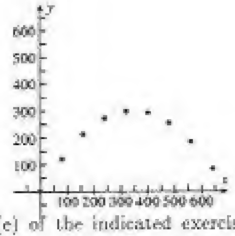
39. $\mathbf{R} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. $\dot{\mathbf{R}} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$. $\mathbf{T} = \dfrac{1}{\sqrt{1+4t^2+9t^4}}(\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})$.
$\mathbf{n} = (-4t - 18t^2)\mathbf{i} + (2 - 18t^4)\mathbf{j} + (6t + 12t^3)\mathbf{k}$. $\mathbf{R}(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$. $\mathbf{T}(1) = \dfrac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.
$\mathbf{n}(1) = -22\mathbf{i} - 16\mathbf{j} + 18\mathbf{k}$. $\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = 14(3\mathbf{i} - 3\mathbf{j} + \mathbf{k})$. Osculating: $3(x-1) - 3(y-1) + (z-1) = 0$

40. Prove that for the twisted cubic $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, if $t \neq 0$, no two of the vectors $\mathbf{R}(t)$, $\mathbf{V}(t)$, and $\mathbf{A}(t)$ are orthogonal.
▸ $\mathbf{V}(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$ $\qquad \mathbf{A}(t) = 2\mathbf{j} + 6t\mathbf{k}$
Two vectors are orthogonal if and only if their dot product is zero.
$\mathbf{R}(t)\cdot\mathbf{V}(t) = (t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k})\cdot(\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) = t + 2t^3 + 3t^5 = t(1 + 2t^2 + 3t^4)$
Because $t \neq 0$ and $3t^4 + 2t^2 + 1 = 2t^4 + (t^2+1)^2 > 0$, then $\mathbf{R}(t)\cdot\mathbf{V}(t) \neq 0$
$\mathbf{R}(t)\cdot\mathbf{A}(t) = (t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k})\cdot(2\mathbf{j} + 6t\mathbf{k}) = 2t^2 + 6t^4 = 2t^2(1 + 3t^2)$
Because $t^2 \neq 0$ and $3t^2 + 1 > 0$, then $\mathbf{R}(t)\cdot\mathbf{A}(t) \neq 0$.
$\mathbf{V}(t)\cdot\mathbf{A}(t) = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\cdot(2\mathbf{j} + 6t\mathbf{k}) = 4t + 18t^3 = t(4 + 18t^2)$
Because $t \neq 0$ and $4 + 18t^2 > 0$, then $\mathbf{V}(t)\cdot\mathbf{A}(t) \neq 0$. Thus no two of the vectors are orthogonal.

41. (a) $\mathbf{V}(0) = 320\cos\frac{1}{4}\pi\mathbf{i} + 320\sin\frac{1}{4}\pi\mathbf{j} = 160\sqrt{2}\mathbf{i} + 160\sqrt{2}\mathbf{j}$
(b) $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = 160\sqrt{2}t\mathbf{i} + (160\sqrt{2}t - 16t^2)\mathbf{j}$; (c) $2T = 2\dfrac{v_0\sin\alpha}{g} = 2\dfrac{320(\frac{1}{2}\sqrt{2})}{32} = 10\sqrt{2}$ sec
(d) range $= \dfrac{{v_0}^2}{g}\sin 2\alpha = \dfrac{(320)^2}{32}\cdot 1 = 3200$ ft (e) $y_{max} = \dfrac{{v_0}^2}{2g}\sin^2\alpha = \dfrac{(320)^2}{2(32)}\cdot\dfrac{1}{2} = 800$ ft
(f) $\mathbf{V}(2T) = 160\sqrt{2}\mathbf{i} - 160\sqrt{2}\mathbf{j}$; $v(2T) = v_0 = 320$ ft/sec (g) $\mathbf{R}(6) = 960\sqrt{2}\mathbf{i} + (960\sqrt{2} - 576)\mathbf{j}$;
$\mathbf{V}(6) = 160\sqrt{2}\mathbf{i} + (160\sqrt{2} - 192)\mathbf{j}$; $v(6) = 64\sqrt{34 - 15\sqrt{2}} \approx 229$ ft/sec; (h) $y = x - \dfrac{x^2}{3200}$

42. (a) $\mathbf{V}(0) = 160\cos\frac{1}{3}\pi\mathbf{i} + 160\sin\frac{1}{3}\pi\mathbf{j} = 80\mathbf{i} + 80\sqrt{3}\mathbf{j}$ (b) $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = 80t\mathbf{i} + (80\sqrt{3}t - 16t^2)\mathbf{j}$
(c) $2T = 2\dfrac{v_0\sin\alpha}{g} = 2\dfrac{160(\frac{1}{2}\sqrt{3})}{32} = 5\sqrt{3}$ sec (d) range $= \dfrac{{v_0}^2}{g}\sin 2\alpha = \dfrac{(160)^2}{32}(\frac{1}{2}\sqrt{3}) = 400\sqrt{3}$ ft ≈ 692.8 ft
(e) $y_{max} = \dfrac{{v_0}^2}{2g}\sin^2\alpha = \dfrac{(160)^2}{32}\cdot\dfrac{3}{4} = 600$ ft (f) $\mathbf{V}(2T) = 80\mathbf{i} - 80\sqrt{3}\mathbf{j}$; $v(2T) = v_0 = 160$ ft/sec
(g) $\mathbf{R}(4) = 320\mathbf{i} + (320\sqrt{3} - 256)\mathbf{j}$; $\mathbf{V}(4) = 80\mathbf{i} + (80\sqrt{3} - 128)\mathbf{j}$; $v(4) = \sqrt{80^2 + (80\sqrt{3} - 128)^2}$
$= 32\sqrt{41 - 20\sqrt{3}} \approx 80.7$ ft/sec (h) $t = \frac{x}{80}$, $y = 80\sqrt{3}\cdot\frac{x}{80} - 16\left(\frac{x}{80}\right)^2 = \sqrt{3}x - \frac{x^2}{400}$





In Exercises 43 and 44, from the answers in parts (b)–(c) of the indicated exercises, (a) plot the path of the projectile and simulate its motion; (b) plot the path of the projectile in dot mode and (c) sketch what you see. (d) What do the dots tell you about the speed of the projectile?

43. Exercise 41. See the figure, above left.

44. Exercise 42

▹ In Exercise 42, we found the parametric equations of the path of the projectile to be
$x = 80t$ and $y = 80\sqrt{3}t - 16t^2$
A sketch of the plot in dot mode is shown above right. The dots are closer at the vertex, indicating that the projectile's speed is least there.

45. $y_0 = 96$, $v_0 = 1600$, $\alpha = 30^\circ$. At t sec, $\mathbf{R}(t) = (t \cdot 1600 \cos 30^\circ)\mathbf{i} + (96 + t \cdot 1600 \sin 30^\circ - 16t^2)$
$= 800\sqrt{3}t\mathbf{i} + (96 + 800t - 16t^2)\mathbf{j}$ and $x = 800\sqrt{3}t$, $y = 96 + 800t - 16t^2$.
Set $y = 0$; $16t^2 - 800t - 96 = 0$. Because $t > 0$, $t = 25 + \sqrt{631} \approx 50.1$ is the time of flight.
The number of feet traveled is $x(25 + \sqrt{631}) = 20{,}000\sqrt{3} + 800\sqrt{1893} \approx 69{,}450$.

46. range $= 400 = \dfrac{v_0^2}{g}\sin 2\alpha$, $\sin 2\alpha = \dfrac{400 \cdot 32}{160^2} = \frac{1}{2}$, $2\alpha = 30^\circ$, $\alpha = 15^\circ$ or $2\alpha = 150^\circ$, $\alpha = 75^\circ$.

47. At t sec, $\mathbf{R}(t) = v_0\cos \alpha t\mathbf{i} + (v_0\sin \alpha t - 16t^2)\mathbf{j}$ so $x = v_0\cos \alpha t$ and $y = \sin \alpha t - 16t^2$.
Because the range is 2000 ft, then $x = 2000$ when $y = 0$. When $y = 0$, $t = \dfrac{v_0\sin \alpha}{16}$ and then
$2000 = \dfrac{v_0^2\sin \alpha \cos \alpha}{16}$; $v_0^2\sin \alpha \cos \alpha = 32{,}000$ (1)
Because the maximum height is 1000 ft, then $y = 1000$ when $D_t y = 0$; $v_0\sin \alpha - 32t = 0$;
$t = \dfrac{v_0\sin \alpha}{32}$. Then $\dfrac{v_0^2\sin^2\alpha}{32} - \dfrac{v_0\sin^2\alpha}{64} = 1000$; $v_0^2 = 64.000$ (2)
From (1) and (2) we obtain $\tan \alpha = 2$. Therefore $\sin \alpha = \dfrac{1}{\sqrt{5}}$ and so from (2) we get
$v_0^2 = 80{,}000$; thus $v_0 = 200\sqrt{2} \approx 283$. Hence the muzzle speed of the gun is 283 ft/sec.

48. A ball is thrown horizontally from the top of a cliff 256 ft high with an initial speed of 50 ft/sec. Find the time of flight of the ball and the distance from the base of the cliff to the point where the ball lands.

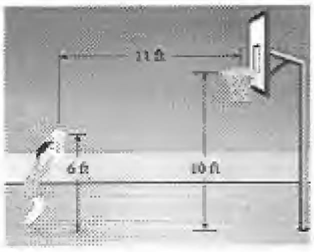
▹ We have
$256 = \frac{1}{2}gt^2 = 16t^2$; $\qquad t^2 = 16$; $\qquad t = 4$
The time of flight is 4 seconds and the horizontal distance is $4(50) = 200$ feet.

49. $y(t) = 15 \cdot \frac{1}{2}\sqrt{2}t - 16t^2 + 60 = 0$ if $t = \frac{1}{64}(15\sqrt{2} + \sqrt{15810}) \approx 2.30$ sec. $x(2.30) = 15 \cdot \frac{1}{2}\sqrt{2}(2.30) \approx 24.4 < 28$

50. $v_0 = 15$, $\alpha = 0^\circ$, $y_0 = 60$. $\mathbf{R}(t) = 15t \cos 0^\circ + (60 + 15t \sin 0^\circ - 16t^2)\mathbf{j} = 15t\mathbf{i} + (60 - 16t^2)\mathbf{j}$ at t sec and $x = 15t$,
$y = 60 - 16t^2$. Set $y = 0$: $60 - 16t = 0$; $t^2 = \frac{15}{4}$; $t = \frac{1}{2}\sqrt{15}$ sec at impact. $x(\frac{1}{2}\sqrt{15}) = 15(\frac{1}{2}\sqrt{15}) \approx 29.0 > 28$

51. $v_0 = 60$, $\alpha = 60^\circ$. $\mathbf{R}(t) = t \cdot 60 \cos 60^\circ\mathbf{i} + (5 + t \cdot 60 \sin 60^\circ - 16t^2)\mathbf{j} = 30t\mathbf{i} + (5 + 30\sqrt{3}t - 16t^2)$ at t sec and $x = 30t$, $y = 5 + 30\sqrt{3}t - 16t^2$. The ball hits the building when $x = 25$, that is, when $t = \frac{5}{6}$. When $t = \frac{5}{6}$, $y = 25\sqrt{3} - \frac{55}{9} \approx 37$. Because $y > 0$, the ball does hit the building.
$\mathbf{V}(t) = D_t\mathbf{R}(t) = 30\mathbf{i} + (30\sqrt{3} - 32t)\mathbf{j}$; $\mathbf{V}(\frac{5}{6}) = 30\mathbf{i} + (30\sqrt{3} - \frac{80}{3})\mathbf{j}$
If θ degrees is the direction of the ball when it hits, then $\tan \theta = \dfrac{30\sqrt{3} - \frac{80}{3}}{30} = \dfrac{9\sqrt{3} - 8}{9} \approx 0.8431$; $\theta = 40.1^\circ$

52. A basketball player is at a horizontal distance of 11 feet from the center of a basket 10 feet from the ground. Determine the angle α at which she should throw the ball at an initial speed of 25 ft/sec if her hands are 6 ft above the floor.



▹ The basket is 4 feet above her head The equations of motion are
$x(t) = 25(\cos\alpha)t$ and $y(t) = 25(\sin\alpha)t - 16t^2$

The ball reaches the basket when $x = 11$ and $y = 4$ Then

$$11 = 25(\cos\alpha)t;\ t = \frac{11}{25\cos\alpha}$$

and

$$4 = 25(\sin\alpha)\frac{11}{25\cos\alpha} - 16\left(\frac{11}{25\cos\alpha}\right)^2 = 11\tan\alpha - 3.0976\sec^2\alpha$$
$$= 11\tan\alpha - 3.0976\tan^2\alpha - 3.0976$$

We have the quadratic equation

$$0 = 3.0976\tan^2\alpha - 11\tan\alpha + 7.0976$$
$$\tan\alpha = \frac{11 \pm \sqrt{11^2 - 4(3.0976)(7.0976)}}{2(3.0976)} = 2.7036 \text{ or } 0.8475$$
$$\alpha = 69.7^\circ \text{ or } 40.3^\circ$$

She should throw the ball at an angle of 69.7° or 40.3°

53. $x(t) = 40\sqrt{2}t = 100$ if $t = \dfrac{100}{40\sqrt{2}} = \frac{5}{4}\sqrt{2}$. $y(t) = 40\sqrt{2}t - 16t^2$. $y(\frac{5}{4}\sqrt{2}) = 40\sqrt{2}(\frac{5}{4}\sqrt{2}) - 16(\frac{5}{4}\sqrt{2})^2 = 50 > 45$
$y(t) = 0$ when $t = \dfrac{40\sqrt{2}}{16} = \frac{5}{2}\sqrt{2}$. $x(\frac{5}{2}\sqrt{2}) = 40\sqrt{2}(\frac{5}{2}\sqrt{2}) = 200$, 25 feet from the pin.

54. Because range $= \dfrac{v_0^2}{g}\sin 2\alpha$, range is maximum when $\sin 2\alpha = 1$, $2\alpha = 90^\circ$, $\alpha = 45^\circ$.

55. Uniform circular motion. $\mathbf{R}(t) = r\cos\omega t\mathbf{i} + r\sin\omega t\mathbf{j}$ (a) $\mathbf{V}(t) = -r\omega\sin\omega t\mathbf{i} + r\omega\cos\omega t\mathbf{j}$. $v = \|\mathbf{V}\| = r\omega\sqrt{\sin^2\omega t + \cos^2\omega t} = r\omega$ (b) $\mathbf{A}(t) = -r\omega^2\cos\omega t\mathbf{i} - r\omega^2\sin\omega t\mathbf{j} = -\omega^2\mathbf{R}(t) \Rightarrow$ the direction is opposite.
(c) $\mathbf{T} = \mathbf{V}/v = -\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j}$, $\mathbf{N} = \dot{\mathbf{T}}/\omega = -\cos\omega t\mathbf{i} - \sin\omega t\mathbf{i}$. $A_T = \dot{v} = 0$, $A_N(t) = \|\mathbf{A}\| = r\omega^2$
(d) If ω is doubled, A_N is quadrupled.

Miscellaneous Exercises for Chapter 11

In Exercises 1–4, find the domain of the vector-valued function.

1. $\text{Dom}\left(\frac{1}{t-3}\mathbf{i} + \sqrt{t}\mathbf{j}\right) = \text{Dom}\left(\frac{1}{t-3}\right) \cap \text{Dom}(\sqrt{t}) = \{t \neq 3\} \cap \{t \geq 0\} = [0,3) \cup (3,+\infty)$
2. $\text{Dom}[\ln(t+1)\mathbf{i} + e^{1/t}\mathbf{j}] = \text{Dom}[\ln(t+1)] \cap \text{Dom}(e^{1/t}) = \{t > -1\} \cap \{t \neq 0\} = (-1,0) \cup (0,+\infty)$
3. $\text{Dom}(\ln|\cos t|\mathbf{i} + \sqrt{4t^2-1}\mathbf{j} + \sqrt{4-t^2}\mathbf{k}) = \{t \neq (k+\frac{1}{2})\pi\} \cap \{|t| \geq \frac{1}{2}\} \cap \{|t| \leq 2\} = [-2,-\frac{1}{2}\pi) \cup (-\frac{1}{2}\pi,-\frac{1}{2}] \cup [\frac{1}{2},\frac{1}{2}\pi) \cup (\frac{1}{2}\pi, 2]$
4. $\mathbf{R}(t) = \sqrt{9-t^2}\mathbf{i} + \tan t\mathbf{j} + \frac{t}{t-1}\mathbf{k}$

▹ $\text{Dom}(\mathbf{R}) = \text{Dom}(\sqrt{9-t^2}) \cap \text{Dom}(\tan t) \cap \text{Dom}\left(\frac{t}{t-1}\right) = \{-3 \leq t \leq 3\} \cap \{t \neq (k+\frac{1}{2}\pi)\} \cap \{t \neq 1\}$
$= [-3,-\frac{1}{2}\pi) \cup (-\frac{1}{2}\pi, 1) \cup (1, \frac{1}{2}\pi) \cup (\frac{1}{2}\pi, 3]$

In Exercises 5–8, find the indicated limit, if it exists.

5. $\lim\limits_{t\to 1}\left(\frac{1}{t+1}\mathbf{i} + \frac{\sqrt{t}-1}{t-1}\mathbf{j}\right) = \lim\limits_{t\to 1}\left(\frac{1}{t+1}\mathbf{i} + \frac{1}{\sqrt{t}+1}\mathbf{j}\right) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$
6. $\lim\limits_{t\to 0}\left(\cos t\mathbf{i} + \frac{\sin t}{t}\mathbf{j}\right) = \mathbf{i} + \mathbf{j}$
7. $\lim\limits_{t\to 0^+}\left(e^{-1/t}\mathbf{i} + \frac{1-\cos t}{t}\mathbf{j} + \frac{\sin^{-1}t}{t}\mathbf{k}\right) = \mathbf{k}$
8. $\lim\limits_{t\to 4}\left(\frac{t^2-16}{t-4}\mathbf{i} + \frac{\tan^{-1}(t-4)}{t-4}\mathbf{j} + |t-4|\mathbf{k}\right)$

▹ We apply L'Hôpital's rule to the middle term. The limit is
$\lim\limits_{t\to 4}\left((t+4)\mathbf{i} + \frac{1}{1+(t-4)^2}\mathbf{j} + |t-4|\mathbf{k}\right) = 8\mathbf{i} + \mathbf{j}$

In Exercises 9 – 12, determine the numbers at which the vector-valued function is continuous.

9. $e^{-t}\mathbf{i} + \ln t\mathbf{j} + \frac{1}{\sqrt{4-t}}\mathbf{k}$. $\{t > 0\} \cap \{t < 4\} = (0,4)$

10. $\ln|\cos t|\mathbf{i} + \frac{1}{|t|-1}\mathbf{j} + \sqrt{1-t^2}\mathbf{k}$. $\{t \neq (k+\frac{1}{2}\pi)\} \cap \{t \neq \pm 1\} \cap \{-1 \leq t \leq 1\} = (-1, 1)$

11. $\begin{cases} \frac{\sin(t-1)}{t-1}\mathbf{i} + \frac{t^2-1}{2(t-1)}\mathbf{j} + (t-1)\ln|t-1|\mathbf{k} & \text{if } t \neq 1 \\ \mathbf{i}+\mathbf{j} & \text{if } t = 1 \end{cases}$. $(-\infty, +\infty)$ because $\lim_{t\to 1}\frac{\sin(t-1)}{t-1} = 1$,

$\lim_{t\to 1}\frac{t+1}{2} = 1$ and $\lim_{t\to 1}(t-1)\ln|t-1| = 0$.

12. $\mathbf{R}(t) = \begin{cases} (1+t)^{1/t}\mathbf{i} + e^{1+t}\mathbf{j} + \frac{e^{1-t}}{1-t}\mathbf{k} & \text{if } t \neq 0 \\ e(\mathbf{i}+\mathbf{j}+\mathbf{k}) & \text{if } t \neq 0 \end{cases}$

▸ $\lim_{t\to 0}(1+t)^{1/t} = e^1 = e$

$\lim_{t\to 0}e^{1+t} = e^1 = e$

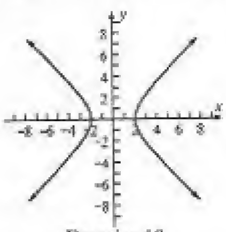
$\lim_{t\to 0}\frac{e^{1-t}}{1-t} = \frac{e^1}{1} = e$

but $\frac{e^{1-t}}{1-t}$ is not defined at $t = 1$. Thus, $\mathbf{R}(t)$ is continuous on $(-\infty, 1) \cup (1, +\infty)$.

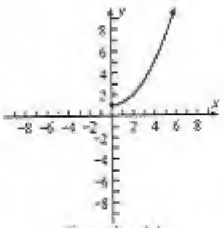
In Exercises 13–16, sketch the graph of the vector-valued function.

13. $\mathbf{R}(t) = 2\sec t\mathbf{i} + 2\tan t\mathbf{j}$

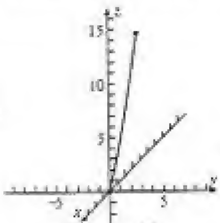
14. $\mathbf{R}(t) = 2\sqrt{t}\mathbf{i} + (t+1)\mathbf{j}$



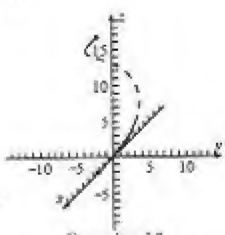
Exercise 13



Exercise 14



Exercise 15



Exercise 16

15. $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t^3\mathbf{k}$, $0 \leq t \leq 2$

16. $\mathbf{R}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 3t\mathbf{k}$, $0 \leq t \leq 2\pi$

In Exercises 17 and 18, find $\mathbf{R}'(t)$ and $\mathbf{R}''(t)$.

17. $\mathbf{R}(t) = \frac{1}{t^2+9}\mathbf{i} + \frac{1}{t}\mathbf{j} - \ln t\mathbf{k}$. $\mathbf{R}'(t) = -\frac{2t}{(t^2+9)^2}\mathbf{i} - \frac{1}{t^2}\mathbf{j} - \frac{1}{t}\mathbf{k}$. $\mathbf{R}''(t) = 6\frac{t^2-3}{(t^2+9)^2}\mathbf{i} + \frac{2}{t^3}\mathbf{j} + \frac{1}{t^2}\mathbf{k}$, $t > 0$

18. $\mathbf{R}(t) = \frac{t}{t-1}\mathbf{i} - \frac{t+1}{t}\mathbf{j} + \frac{t+1}{t-1}\mathbf{k} = \left(1+\frac{1}{t-1}\right)\mathbf{i} - \left(1+\frac{1}{t}\right)\mathbf{j} + \left(1+\frac{2}{t-1}\right)\mathbf{k}$

$\mathbf{R}'(t) = -\frac{1}{(t-1)^2}\mathbf{i} + \frac{1}{t^2}\mathbf{j} - \frac{2}{(t-1)^2}\mathbf{k}$. $\mathbf{R}''(t) = \frac{2}{(t-1)^3}\mathbf{i} - \frac{2}{t^3}\mathbf{j} + \frac{4}{(t-1)^3}\mathbf{k}$

In Exercises 19 and 20, find $D_t\|\mathbf{R}(t)\|$ and $\|D_t\mathbf{R}(t)\|$.

19. $\mathbf{R}(t) = 2(e^t-1)\mathbf{i} + 2(e^t+1)\mathbf{j} + e^t\mathbf{k}$. $D_t\sqrt{4(e^t-1)^2 + 4(e^t+1)^2 + e^{2t}} = D_t\sqrt{9e^{2t}+8} = \frac{9e^{2t}}{\sqrt{9e^{2t}+8}}$

$\|2e^t\mathbf{i} + 2e^t\mathbf{j} + e^t\mathbf{k}\| = \sqrt{4e^{2t} + 4e^{2t} + e^{2t}} = 3e^t$

20. $\mathbf{R}(t) = \cos 2t\mathbf{i} + \sin 2t\mathbf{j} + 2t^2\mathbf{k}$

▸ $D_t\|\mathbf{R}(t)\| = D_t\sqrt{\cos^2 2t + \sin^2 2t + 4t^4} = D_t\sqrt{1+4t^4} = \frac{8t^3}{\sqrt{1+4t^4}}$

$\|D_t\mathbf{R}(t)\| = \|-2\sin 2t\mathbf{i} + 2\cos 2t\mathbf{j} + 4t\mathbf{k}\| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 16t^2} = 2\sqrt{1+4t^2}$

21. $\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t\left(\frac{1}{u+2}\mathbf{i} + \frac{1}{u^2+1}\mathbf{j}\right)du = \mathbf{j} + \left[\ln(u+2)\mathbf{i} + \tan^{-1}u\mathbf{j}\right]_0^t = [\ln(t+2) - \ln 2]\mathbf{i} + (\tan^{-1}t + 1)\mathbf{j}$

22. $\mathbf{R}(t) = \mathbf{R}(1) + \int_1^t\left(\frac{1}{u}\mathbf{i} + \frac{2\ln u}{u}\mathbf{j}\right)du = 3\mathbf{i} - 2\mathbf{j} + \left[\ln u\,\mathbf{i} + (\ln u)^2\mathbf{j}\right]_1^t = (\ln t + 3)\mathbf{i} + [(\ln t)^2 - 2]\mathbf{j}$, $t > 0$

23. $\mathbf{R}(t) = \mathbf{R}(2) + \int_2^t(2e^{u/2}\mathbf{i} - 2e^{-u/2}\mathbf{j} + 2\cosh\frac{1}{2}u\mathbf{k})du = 2e\mathbf{i} - 2e^{-1}\mathbf{j} + \mathbf{k} + \left[4e^{u/2}\mathbf{i} + 4e^{-u/2}\mathbf{j} + 4\sinh\frac{1}{2}u\mathbf{k}\right]_2^t$

$= (4e^{t/2} - 2e)\mathbf{i} + (4e^{-t/2} - 6e^{-1})\mathbf{j} + (4\sinh\frac{1}{2}t - 4\sinh 1 + 1)\mathbf{k}$

24. If $\mathbf{R}'(t) = \cos^2 t\mathbf{i} + \cos 2t\mathbf{j} - 2\sin 2t\mathbf{k}$ and $\mathbf{R}(0) = \mathbf{i}$, find $\mathbf{R}(t)$.

▸ $$\mathbf{R}(t) = \mathbf{R}(0) + \int_0^t \mathbf{R}'(u)du = \mathbf{i} + \int_0^t [\tfrac{1}{2}(1+\cos 2u)\mathbf{i} + \cos 2u\mathbf{j} - 2\sin 2u\mathbf{k}]du$$
$$= \mathbf{i} + \Big[(\tfrac{1}{2}u + \tfrac{1}{4}\sin 2u)\mathbf{i} + \tfrac{1}{2}\sin 2u\mathbf{j} + \cos 2u\mathbf{k}\Big]_0^t = (\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t + 1)\mathbf{i} + \tfrac{1}{2}\sin 2t\mathbf{j} + (\cos 2t - 1)\mathbf{k}$$

In Exercises 25 and 26, find the exact length of arc L from t_1 to t_2.

25. $\|\mathbf{R}'\| = \|(\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \mathbf{k}\| = \sqrt{1+t^2+1}$. $\mathrm{L} = \int_0^{\pi/2}\sqrt{t^2+2}\,dt = \frac{1}{8}\pi\sqrt{8+\pi^2} + \sinh^{-1}\frac{1}{4}\sqrt{2}\pi$

26. $\|\mathbf{R}'(t)\| = \|-3\mathbf{i} + 4\mathbf{j} + 2t\mathbf{k}\| = \sqrt{25+4t^2}$. $\mathrm{L} = \int_0^{5/2}\sqrt{25+4t^2}\,dt = \frac{5}{4}\sqrt{34} + \frac{25}{4}\sinh^{-1}\frac{3}{5}$

In Exercises 27 and 28, use NINT find to four significant digits the length of arc L from t_1 to t_2.

27. $\|\mathbf{R}\| = \left\|\frac{1}{t}\mathbf{i} - \frac{1}{t^2}\mathbf{j} + \frac{1}{t^2}\right\| = \sqrt{\frac{1}{t^2} + \frac{2}{t^4}} = \frac{\sqrt{t^2+2}}{t^2}$. $\mathrm{L} = \int_1^2 \sqrt{t^2+2}\frac{dt}{t^2} = \left[\sinh^{-1}\frac{t}{\sqrt{2}} - \frac{\sqrt{t^2+2}}{t}\right]_1^2$
$= \sqrt{3} - \frac{1}{2}\sqrt{6} + \sinh^{-1}\sqrt{2} - \sinh^{-1}\frac{1}{\sqrt{2}} = 0.99504 \approx 0.9950$

28. $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + e^{2t}\mathbf{k}$; $t_1 = 0$; $t_2 = \frac{1}{2}\pi$

▸ $\|\mathbf{R}'(t)\| = \|-\sin t\mathbf{j} + \cos t\mathbf{j} + 2e^{2t}\| = \sqrt{\sin^2 t + \cos^2 t + 4e^{4t}} = \sqrt{1+4e^{4t}}$
$$\mathrm{L} = \int_0^{\pi/2}\sqrt{1+4e^{4t}}\,dt = 22.258 \approx 22.26$$
using NINT. To get the exact value, let $u^2 = 1 + 4e^{4t}$. Then
$$\frac{2u\,du}{u^2-1} = \frac{16e^{4t}dt}{4e^{4t}} = 4\,dt$$
and so
$$\mathrm{L} = \int_{\sqrt{5}}^{\sqrt{1+4e^{2\pi}}} u\cdot\frac{u\,du}{2(u^2-1)} = \frac{1}{2}\int_{\sqrt{5}}^{\sqrt{1+4e^{2\pi}}}\Big(1 + \frac{1}{u^2-1}\Big)du = \frac{1}{2}u + \frac{1}{4}\ln\left|\frac{u-1}{u+1}\right|\Bigg]_{\sqrt{5}}^{\sqrt{1+4e^{2\pi}}}$$
$$= \tfrac{1}{2}\sqrt{1+4e^{2\pi}} - \tfrac{1}{2}\sqrt{5} + \tfrac{1}{4}\ln\frac{\sqrt{1+4e^{2\pi}}-1}{\sqrt{1+4e^{2\pi}}+1} - \tfrac{1}{4}\ln\frac{\sqrt{5}-1}{\sqrt{5}+1}$$

In Exercises 29 and 30, find $\mathbf{T}(t)$ and $\mathbf{N}(t)$; also $\mathbf{T}(t_1)$ and $\mathbf{N}(t_1)$ and sketch their representations at $t = t_1$. In Exercises 31–34, find $\mathbf{T}(t)$, $\mathbf{N}(t)$. In Exercise 35–38, find the moving trihedral and equations of its planes at $t = t_1$. In Exercises 39 and 40, express the arc length s as a function of t measured from $t = 0$. In Exercises 41–44, find an equation of the curve having s as parameter. In Exercises 45 and 46, find the curvature K, the radius of curvature ρ and the center of curvature at $t = t_1$; in Exercises 47–50, just K.

29 and 45. $\mathbf{R}(t) = \frac{1}{2}e^{2t}\mathbf{i} + t\mathbf{j}$. $\dot{\mathbf{R}}(t) = e^{2t}\mathbf{i} + \mathbf{j}$. $\mathbf{T}(t) = \frac{1}{\sqrt{e^{4t}+1}}(e^{2t}\mathbf{i} + \mathbf{j})$. $\ddot{\mathbf{R}}(t) = 2e^{2t}\mathbf{i}$. $\mathbf{N}(t) = \frac{1}{\sqrt{e^{4t}+1}}(\mathbf{i} - e^{2t}\mathbf{j})$ because $\mathbf{N}\cdot\ddot{\mathbf{R}} = e^{2t} > 0$. At $t = \ln 2$, $e^{2t} = 4$, $\mathbf{T} = \frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$, $\mathbf{N} = \frac{1}{\sqrt{17}}\mathbf{i} - \frac{4}{\sqrt{17}}\mathbf{j}$, $x = 2$, $y = \ln 2$, $\dot{x} = 4$, $\dot{y} = 1$, $\ddot{x} = 8$, $\ddot{y} = 0$. $\mathrm{K} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{|4(0) - 1(8)|}{(4^2+1^2)^{3/2}} = \frac{8}{17^{3/2}}$. $\rho = \frac{17^{3/2}}{8}$. $c = \frac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \frac{4^2+1^2}{4(0) - 1(8)} = -\frac{17}{8}$.
$x_c = x - \dot{y}c = 2 - 1(-\frac{17}{8}) = \frac{33}{8}$. $y_c = y + \dot{x}c = \ln 2 + 4(-\frac{17}{8}) = \ln 2 - \frac{17}{2}$

30, 42, and 46. $\mathbf{R}(t) = t^2\mathbf{i} + \frac{1}{3}t^3\mathbf{j}$, $t \ge 0$. $\dot{\mathbf{R}}(t) = 2t\mathbf{i} + t^2\mathbf{j}$. $\|\dot{\mathbf{R}}(t)\| = t\sqrt{4+t^2}$. $\mathbf{T}(t) = \frac{1}{\sqrt{4+t^2}}(2\mathbf{i} + t\mathbf{j})$. $\ddot{\mathbf{R}} = 2\mathbf{i} + 2t\mathbf{j}$.
$\mathbf{N}(t) = \frac{1}{\sqrt{4+t^2}}(-t\mathbf{i} + 2\mathbf{j})$ because $\mathbf{N}\cdot\ddot{\mathbf{R}} = \frac{2t}{\sqrt{4+t^2}} \ge 0$. $\mathbf{T}(1) = \frac{1}{\sqrt{5}}(2\mathbf{i} + \mathbf{j})$. $\mathbf{N}(1) = \frac{1}{\sqrt{5}}(-\mathbf{i} + 2\mathbf{j})$
$s = \int_0^t u\sqrt{4+u^2}\,du = \frac{1}{3}(4+u^2)^{3/2}\Big]_0^t = \frac{1}{3}[(4+t^2)^{3/2} - 8]$. $3s + 8 = (4+t^2)^{3/2}$; $t = [(3s+8)^{2/3} - 4]^{1/2}$
$\mathbf{R}(s) = [(3s+8)^{2/3} - 4]\mathbf{i} + \frac{1}{3}[(3s+8)^{2/3} - 4]^{3/2}\mathbf{j}$. At $t = 1$, $x = 1$, $y = \frac{1}{3}$, $\dot{x} = 2$, $\dot{y} = 1$, $\ddot{x} = 2$, $\ddot{y} = 2$.
$\mathrm{K} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{|2(2) - 1(2)|}{(2^2+1^2)^{3/2}} = \frac{2}{5^{3/2}}$. $\rho = \frac{5^{3/2}}{2}$. $c = \frac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \frac{5}{2}$. $x_c = x - \dot{y}c = 1 - 1(\frac{5}{2}) = -\frac{3}{2}$.
$y_c = y + \dot{x}c = \frac{1}{3} + 2(\frac{5}{2}) = \frac{16}{3}$

31, 35 39, and 47. $\mathbf{R}(t) = \frac{1}{3}t^3\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$. $\dot{\mathbf{R}}(t) = t^2\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$. $\|\dot{\mathbf{R}}(t)\| = \sqrt{t^4 + 4t^2 + 4} = t^2 + 2$.

$\mathbf{T}(t) = \dfrac{t^2}{t^2+2}\mathbf{i} + \dfrac{2t}{t^2+2}\mathbf{j} + \dfrac{2}{t^2+2}\mathbf{k}$. $\dot{\mathbf{T}}(t) = \dfrac{2}{(t^2+2)^2}[2t\mathbf{i} - (t^2-2)\mathbf{j} - 2t\mathbf{k}]$. $\|\dot{\mathbf{T}}(t)\| = \dfrac{2}{t^2+2}$.

$$\mathbf{N}(t) = \frac{1}{t^2+2}[2t\mathbf{i} - (t^2-2)\mathbf{j} - 2t\mathbf{k}].\ \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{(t^2+2)^2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & 2t & 2 \\ 2t & 2-t^2 & -2t \end{vmatrix}$$

$= -\dfrac{2}{t^2+2}\mathbf{i} + \dfrac{2t}{t^2+2}\mathbf{j} - \dfrac{t^2}{t^2+2}$. $\mathbf{R}(1) = \frac{1}{3}\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. $\mathbf{B}(1) = -\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \Rightarrow$ osculating:

$-2(x - \frac{1}{3}) + 2(y-1) - (z-2) = 0$; $\mathbf{N}(1) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \Rightarrow$ rectifying: $2(x - \frac{1}{3}) + (y-1) - 2(z-2) = 0$;

$\mathbf{T}(1) = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow$ normal: $(x - \frac{1}{3}) + 2(y-1) + 2(z-2) = 0$. $s(t) = \int_0^t \|\dot{\mathbf{R}}(u)\|du = \int_0^t (u^2+2)du = \frac{1}{3}t^3 + 2t$.

$K = \|\dot{\mathbf{T}}\| / \|\dot{\mathbf{R}}\| = \dfrac{2}{(t^2+2)^2}$

32, 36, 40, and 48. $\mathbf{R}(t) = e^t\mathbf{i} + 2e^{-t}\mathbf{j} + 2t\mathbf{k}$

▸ $\dot{\mathbf{R}}(t) = e^t\mathbf{i} - 2e^{-t}\mathbf{j} + 2\mathbf{k}$ $\qquad \|\dot{\mathbf{R}}(t)\| = \sqrt{e^{2t} + 4e^{-2t} + 4} = e^t + 2e^{-t}$

$$\mathbf{T}(t) = \frac{e^t}{e^t+2e^{-t}}\mathbf{i} - \frac{2e^{-t}}{e^t+2e^{-t}}\mathbf{j} + \frac{2}{e^t+2e^{-t}}\mathbf{k}$$

$$\dot{\mathbf{T}}(t) = \frac{e^t(e^t+2e^{-t}) - e^t(e^t-2e^{-t})}{(e^t+2e^{-t})^2}\mathbf{i} - 2\frac{-e^{-t}(e^t+2e^{-t}) - e^{-t}(e^t-2e^{-t})}{(e^t+2e^{-t})^2}\mathbf{j} - 2\frac{e^t-2e^{-t}}{(e^t+2e^{-t})^2}\mathbf{k}$$

$$= \frac{2}{(e^t+2e^{-t})^2}[2\mathbf{i} + 2\mathbf{j} - (e^t - 2e^{-t})\mathbf{k}]$$

$$\|\dot{\mathbf{T}}(t)\| = \frac{2}{(e^t+2e^{-t})^2}\sqrt{4+4+(e^t-2e^{-t})^2} = \frac{2}{e^t+2e^{-t}}$$

$$\mathbf{N}(t) = \frac{1}{e^t+2e^{-t}}[2\mathbf{i} + 2\mathbf{j} - (e^t-2e^{-t})\mathbf{k}]$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{(e^t+2e^{-t})^2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & -2e^{-t} & 2 \\ 2 & 2 & 2e^{-t}-e^t \end{vmatrix}$$

$$= \frac{1}{(e^t+2e^{-t})^2}[-2(1+2e^{-2t})\mathbf{i} + (e^{2t}+2)\mathbf{j} + (2e^t+4e^{-t})\mathbf{k}] = \frac{1}{e^t+2e^{-t}}(-2e^{-t}\mathbf{i} + e^t\mathbf{j} + 2\mathbf{k})$$

$\mathbf{R}(0) = \mathbf{i} + 2\mathbf{j}$

$\mathbf{B}(0) = -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ $\qquad$ An equation of the osculating plane is $-2(x-1) + (y-2) + 2z = 0$

$\mathbf{N}(0) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ $\qquad$ An equation of the rectifying plane is $2(x-1) + 2(y-2) + z = 0$

$\mathbf{T}(0) = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ $\qquad$ An equation of the normal plane is $(x-1) - 2(y-2) + 2z = 0$

$$s(t) = \int_0^t \|\dot{\mathbf{R}}(u)\|du = \int_0^t (e^u + 2e^{-u})du = e^u - 2e^{-u}\Big]_0^t = e^t - 2e^{-t} + 1$$

$$K = \|\dot{\mathbf{T}}(t)\| / \|\dot{\mathbf{R}}(t)\| = \frac{2}{(e^t+2e^{-t})^2}$$

33, 37, 43, and 49. $\mathbf{R}(t) = 3\sin 2t\mathbf{i} + 4t\mathbf{j} + 3\cos 2t\mathbf{k}$. $\dot{\mathbf{R}}(t) = 6\cos 2t\mathbf{i} + 4\mathbf{j} - 6\sin 2t\mathbf{k}$. $\|\dot{\mathbf{R}}(t)\| = \sqrt{36+16} = 2\sqrt{13}$

$\mathbf{T}(t) = \dfrac{1}{\sqrt{13}}(3\cos 2t\mathbf{i} + 2\mathbf{j} - 3\sin 2t\mathbf{k})$. $\dot{\mathbf{T}}(t) = \dfrac{6}{\sqrt{13}}(-\sin 2t\mathbf{i} - \cos 2t\mathbf{k})$. $\|\dot{\mathbf{T}}(t)\| = \dfrac{6}{\sqrt{13}}$. $\mathbf{N}(t) = -\sin 2t\mathbf{i} - \cos 2t\mathbf{k}$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{13}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos 2t & 4t & -3\sin 2t \\ -\sin 2t & 0 & -\cos 2t \end{vmatrix} = \frac{1}{\sqrt{13}}(-2\cos 2t\mathbf{i} + 3\mathbf{j} + 2\sin 2t\mathbf{k}).\ \mathbf{R}(0) = 3\mathbf{k}.$$

$\mathbf{B}(0) = \dfrac{1}{\sqrt{13}}(-2\mathbf{i} + 3\mathbf{j}) \Rightarrow$ osculating: $-2x + 3y = 0$; $\mathbf{N}(0) = -\mathbf{k} \Rightarrow$ rectifying: $z = 3$; $\mathbf{T}(0) = \dfrac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j}) \Rightarrow$ normal: $3x + 2y = 0$. $s = \int_0^t \|\dot{\mathbf{R}}(u)\|du = \int_0^t 2\sqrt{13}du = 2\sqrt{13}t$. $t = \frac{1}{26}\sqrt{13}s$.

$\mathbf{R}(s) = 3\sin \frac{1}{13}\sqrt{13}s\mathbf{i} + \frac{2}{13}\sqrt{13}s\mathbf{j} + 3\cos\frac{1}{13}\sqrt{13}s\mathbf{k}$. $K = \|\dot{\mathbf{T}}(t)\| / \|\dot{\mathbf{R}}(t)\| = \frac{3}{13}$

34 and 50. $\mathbf{R}(t) = 3(\cos t + t\sin t)\mathbf{i} + 3(\sin t - t\cos t)\mathbf{j} + 3\mathbf{k},\ t \geq 0$. $\dot{\mathbf{R}}(t) = 3t\cos t\mathbf{i} + 3t\sin t\mathbf{j}$. $\|\dot{\mathbf{R}}(t)\| = 3t$

$\mathbf{T}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. $\dot{\mathbf{T}}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} = \mathbf{N}(t)$. $\mathrm{K} = \|\dot{\mathbf{T}}(t)\|/\|\dot{\mathbf{R}}(t)\| = 1/(3t)$

41. $\mathbf{R}(t) = 4t\mathbf{i} + \frac{1}{3}(2t+1)^{3/2}\mathbf{j}$. $\dot{\mathbf{R}}(t) = 4\mathbf{i} + (2t+1)^{1/2}\mathbf{j}$. $\|\dot{\mathbf{R}}(t)\| = \sqrt{16 + 2t + 1} = \sqrt{2t+17}$.

$s = \int_0^t \sqrt{2u+17}\,du = \frac{1}{3}(2t+17)^{3/2} - \frac{1}{3}(17)^{3/2}$; $3s + 17\sqrt{17} = (2t+17)^{3/2}$; $2t = (3s + 17\sqrt{17})^{2/3} - 17$

$\mathbf{R}(s) = [2(3s + 17\sqrt{17})^{2/3} - 34]\mathbf{i} + \frac{1}{3}[(3s + 17\sqrt{17})^{2/3} - 16]^{3/2}\mathbf{j}$

44. $\mathbf{R}(t) = 3(\cos t + t\sin t)\mathbf{i} + 3(\sin t - t\cos t)\mathbf{j} + 3\mathbf{k},\ t \geq 0$

▸ $\dot{\mathbf{R}}(t) = 3t\cos t\mathbf{i} + 3t\sin t\mathbf{j}$ $\qquad \|\dot{\mathbf{R}}(t)\| = 3t\sqrt{\cos^2 t + \sin^2 t} = 3t$

$s = \int_0^t \|\dot{\mathbf{R}}(u)\|\,du = \int_0^t 3u\,du = \frac{3}{2}u^2\Big]_0^t = \frac{3}{2}t^2$ $\qquad t = \sqrt{\frac{2}{3}s}$

$\mathbf{R}(s) = 3\left(\cos\sqrt{\tfrac{2}{3}s} + \sqrt{\tfrac{2}{3}s}\sin\sqrt{\tfrac{2}{3}s}\right)\mathbf{i} + 3\left(\sin\sqrt{\tfrac{2}{3}s} - \sqrt{\tfrac{2}{3}s}\cos\sqrt{\tfrac{2}{3}s}\right)\mathbf{j} + 3\mathbf{k}$

51. $y = \ln x$, where $x > 0$. $y' = x^{-1}$, $y'' = -x^{-2}$. $\mathrm{K} = \dfrac{y''}{[1 + (y')^2]^{3/2}} = \dfrac{|-x^{-2}|}{(1 + x^{-2})^{3/2}} = \dfrac{x}{(x^2+1)^{3/2}}$.

$$\mathrm{D}_x\mathrm{K} = \frac{(x^2+1)^{3/2} - 3x^2(x^2+1)^{1/2}}{(x^2+1)^3} = \frac{x^2+1-3x^2}{(x^2+1)^{5/2}} = \frac{1-2x^2}{(x^2+1)^{3/2}}$$

Set $\mathrm{D}_x\mathrm{K} = 0$: $x = \frac{1}{2}\sqrt{2}$. Because $\mathrm{D}_x > 0$ if $0 < x < \frac{1}{2}\sqrt{2}$ and $\mathrm{D}_x < 0$ if $x > \frac{1}{2}\sqrt{2}$, then K has an absolute maximum value when $x = \frac{1}{2}\sqrt{2}$. Then $y = \ln\frac{1}{\sqrt{2}} = -\frac{1}{2}\ln 2$ and $\mathrm{K} = \frac{1}{\sqrt{2}}/\left(\frac{1}{2}+1\right)^{3/2} = \frac{1}{\sqrt{2}}/\frac{3}{2}\cdot\frac{\sqrt{3}}{\sqrt{2}} = \frac{2}{3\sqrt{3}} = \frac{2}{9}\sqrt{3}$.

52. Find the curvature at any point of the branch of the parametric hyperbola $x = a\cosh t$ and $y = b\sinh t$.

▸ $\dot{x} = a\sinh t$ $\qquad \dot{y} = b\cosh t$

$\ddot{x} = a\cosh t$ $\qquad \ddot{y} = b\sinh t$

$$\mathrm{K}(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|(a\sinh t)(b\sinh t) - (b\cosh t)(a\cosh t)|}{(a^2\sinh^2 t + b^2\cosh^2 t)^{3/2}} = \frac{|ab|}{(a^2\sinh^2 t + b^2\cosh^2 t)^{3/2}}$$

53 and 55. $x = 3t^2$, $y = t^3 - 3t$. $\dot{x} = 6t$, $\dot{y} = 3t^2 - 3$. $\ddot{x} = 6$, $\ddot{y} = 6t$.

$$\mathrm{K}(2) = \frac{\dot{x}(2)\ddot{y}(2) - \dot{y}(2)\ddot{x}(2)}{[\dot{x}(2)^2 + \dot{y}(2)^2]^{3/2}} = \frac{12\cdot 12 - 9\cdot 6}{[12^2 + 9^2]^{3/2}} = \frac{90}{225^{3/2}} = \frac{2}{75};\ \rho(2) = \frac{75}{2},\ c = \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \frac{225}{90} = \frac{5}{2}$$

$x_c = x - c\dot{y} = 12 - \frac{5}{2}(9) = -\frac{21}{2}$, $y_c = y + c\dot{x} = 2 + \frac{5}{2}(12) = 32$

54 and 56. Find the curvature, radius of curvature and center of curvature (x_c, y_c) of the curve $y = e^{-x}$ at the point $(0, 1)$. Sketch a portion of the curve and the circle of curvature at that point.

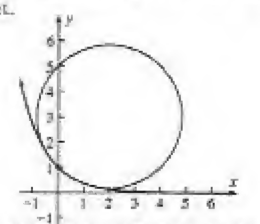
▸ $y' = -e^{-x}$ $\qquad y'(0) = -1$

$y'' = e^{-x}$ $\qquad y''(0) = 1$

$\mathrm{K} = \dfrac{|y''|}{(1 + y'^2)^{3/2}} = \dfrac{1}{2^{3/2}}$ $\qquad \rho = \dfrac{1}{\mathrm{K}} = 2^{3/2}$

$c = \dfrac{1 + y'^2}{y''} = \dfrac{2}{1} = 2$. $x_c = x - cy' = 0 - 2(-1) = 2$. $y_c = y + c = 1 + 2 = 3$

The figure shows the curve and circle of curvature at $(0, 1)$.



In Exercises 57 and 58, find $\mathbf{V}(t)$, $\mathbf{A}(t)$, $\|\mathbf{V}(t)\|$, $\|\mathbf{A}(t)\|$, $\mathbf{V}(1)$, $\mathbf{A}(1)$. In Exercises 67 and 68, find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathrm{A}_T(t)$, $\mathrm{A}_N(t)$, $\mathrm{K}(t)$. Find the particular values when $t = 1$. Sketch representations of $\mathbf{V}(1)$, $\mathbf{A}(1)$, $\mathrm{A}_T(1)\mathbf{T}(1)$, and $\mathrm{A}_N(1)\mathbf{N}(1)$ on the path.

57 and 67. $\mathbf{R}(t) = 3t\mathbf{i} + (4t - t^2)\mathbf{j}$. $\mathbf{V}(t) = 3\mathbf{i} + (4 - 2t)\mathbf{j}$. $\mathbf{A}(t) = -2\mathbf{j}$. $\|\mathbf{V}(t)\| = \sqrt{3^2 + (4 - 2t)^2} = \sqrt{4t^2 - 16t + 25}$.

$\mathbf{V}(1) = 3\mathbf{i} + 2\mathbf{j}$. $\mathbf{A}(1) = -2\mathbf{j}$. $\mathbf{T}(t) = \dfrac{1}{\sqrt{4t^2 - 16t + 25}}[3\mathbf{i} + (4 - 2t)\mathbf{j}]$. $\mathbf{N}(t) = \dfrac{1}{\sqrt{4t^2 - 16t + 25}}[(4 - 2t)\mathbf{i} - 3\mathbf{j}]$.

$$\mathrm{A}_T(t) = \dot{V}(t) = \frac{4t - 8}{\sqrt{4t^2 - 16t + 25}}.\ \mathrm{A}_N(t) = \sqrt{\|\mathbf{A}\|^2 - \mathrm{A}_T{}^2} = \sqrt{4 - \frac{16t^2 - 64t + 64}{4t^2 - 16t + 25}} = \frac{6}{\sqrt{4t^2 - 16t + 25}}.$$

$\mathrm{K}(t) = \dfrac{\mathrm{A}_N(t)}{\|\mathbf{V}(t)\|^2} = \dfrac{6}{(4t^2 - 16t + 25)^{3/2}}$. $\mathbf{T}(1) = \dfrac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$. $\mathbf{N}(1) = \dfrac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{j})$. $\|\mathbf{V}(1)\| = \sqrt{13}$. $\mathrm{A}_T(1) = \dfrac{-4}{\sqrt{13}}$

$\mathrm{A}_N(1) = \dfrac{6}{\sqrt{13}}$. $\mathrm{K}(1) = \dfrac{6}{13^{3/2}}$

58 and 68. $\mathbf{R}(t) = 2e^t\mathbf{i} + 3e^{-t}\mathbf{j}$

▷ $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = 2e^t\mathbf{i} - 3e^{-t}\mathbf{j}$ $\qquad$ $\mathbf{V}(1) = 2e\mathbf{i} - 3e^{-1}\mathbf{j} \approx 5.44\mathbf{i} - 1.10\mathbf{j}$

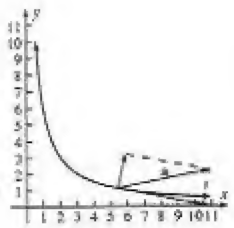
$\mathbf{A}(t) = \dot{\mathbf{V}}(t) = 2e^t\mathbf{i} + 3e^{-t}\mathbf{j}$ $\qquad$ $\mathbf{A}(1) = 2e\mathbf{i} + 3e^{-1}\mathbf{j} \approx 5.44\mathbf{i} + 1.10\mathbf{j}$

$v(t) = \|\mathbf{V}(t)\| = \sqrt{4e^{2t} + 9e^{-2t}}$ $\qquad$ $\|\mathbf{V}(1)\| = \sqrt{4e^2 + 9e^{-2}}$

$\mathbf{T}(t) = \dfrac{\mathbf{V}(t)}{\|\mathbf{V}(t)\|} = \dfrac{2e^t\mathbf{i} - 3e^{-t}\mathbf{j}}{\sqrt{4e^{2t} + 9e^{-2t}}}$ $\qquad$ $\mathbf{T}(1) = \dfrac{2e\mathbf{i} - 3e^{-1}\mathbf{j}}{\sqrt{4e^2 + 9e^{-2}}}$

$\mathbf{N}(t) = \dfrac{3e^{-t}\mathbf{i} + 2e^t\mathbf{j}}{\sqrt{4e^{2t} + 9e^{-2t}}}$ $(\mathbf{N} \cdot \mathbf{A} = 6 > 0)$ $\qquad$ $\mathbf{N}(1) = \dfrac{3e^{-1}\mathbf{i} + 2e\mathbf{j}}{\sqrt{4e^2 + 9e^{-2}}}$

$\mathrm{A}_T(t) = \dot{v}(t) = \dfrac{4e^{2t} - 9e^{-2t}}{\sqrt{4e^{2t} + 9e^{-2t}}}$ $\qquad$ $\mathrm{A}_T(1) = \dfrac{4e^2 - 9e^{-2}}{\sqrt{4e^2 + 9e^{-2}}}$



$$\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}\|^2 - \mathrm{A}_T{}^2} = \sqrt{(4e^{2t} + 9e^{-2t}) - \frac{(4e^{2t} - 9e^{-2t})^2}{4e^{2t} + 9e^{-2t}}} = \frac{12}{\sqrt{4e^{2t} + 9e^{-2t}}} \qquad \mathrm{A}_N(1) = \frac{12}{\sqrt{4e^2 + 9e^{-2}}}$$

$$\mathrm{K}(t) = \frac{\mathrm{A}_N(t)}{v(t)^2} = \frac{12}{(4e^{2t} + 9e^{-2t})^{3/2}} \qquad \mathrm{K}(1) = \frac{12}{(4e^2 + 9e^{-2})^{3/2}}$$

The figure shows the hyperbola and $\mathbf{V}(1)$, $\mathbf{A}(1)$, $\mathrm{A}_T(1)\mathbf{T}(1) \approx 5.01\mathbf{i} - 1.02\mathbf{j}$ and $\mathrm{A}_N(1)\mathbf{N}(1) \approx 0.43\mathbf{i} + 2.12\mathbf{j}$.

In Exercises 69 and 70, find $\mathrm{A}_T(t)$ and $\mathrm{A}_N(t)$ and write $\mathbf{A}(t) = \mathrm{A}_T(t)\mathbf{T}(t) + \mathrm{A}_N(t)\mathbf{N}(t)$. Don't find $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

69. $\mathbf{R} = 2t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$. $\mathbf{V} = 2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$. $\mathbf{A} = e^t\mathbf{j} + e^{-t}\mathbf{k}$. $v = \|\mathbf{V}\| = \sqrt{4 + e^{2t} + e^{-2t}}$. $\mathrm{A}_T = \dot{v} = \dfrac{e^{2t} - e^{-2t}}{\sqrt{4 + e^{2t} + e^{-2t}}}$

$$\mathrm{A}_N = \sqrt{\|\mathbf{A}\|^2 - \mathrm{A}_T{}^2} = \sqrt{e^{2t} + e^{-2t} - \frac{e^{4t} - 2 + e^{-4t}}{4 + e^{2t} + e^{-2t}}} = \sqrt{\frac{4 + 4e^{2t} + 4e^{-2t}}{4 + e^{2t} + e^{-2t}}}$$

$$\mathbf{A}(t) = \frac{e^{2t} - e^{-2t}}{\sqrt{4 + e^{2t} + e^{-2t}}}\mathbf{T}(t) + \sqrt{\frac{4 + 4e^{2t} + 4e^{-2t}}{4 + e^{2t} + e^{-2t}}}\mathbf{N}(t)$$

70. $\mathbf{R} = t^2\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$. $\mathbf{V} = 2t\mathbf{i} + 2\mathbf{j}$. $\mathbf{A} = 2\mathbf{i}$. $v = \|\mathbf{V}\| = 2\sqrt{t^2 + 1}$. $\mathrm{A}_T = \dot{v} = \dfrac{2t}{\sqrt{t^2 + 1}}$.

$$\mathrm{A}_N = \sqrt{\|\mathbf{A}\|^2 - \mathrm{A}_T{}^2} = \sqrt{4 - \frac{4t}{t^2 + 1}} = 2\sqrt{\frac{t^2 - t + 1}{t^2 + 1}}.\ \mathbf{A} = \frac{2t}{\sqrt{t^2 + 1}}\mathbf{T} + 2\sqrt{\frac{t^2 - t + 1}{t^2 + 1}}\mathbf{N}$$

In Exercises 71 and 72, find (a) velocity, acceleration, (b) speed, (c) tangential, normal components of acceleration.

71. $\mathbf{R}(t) = \cosh 2t\mathbf{i} + \sinh 2t\mathbf{j}$. $\mathbf{V}(t) = 2\sinh 2t\mathbf{i} + 2\cosh 2t\mathbf{j}$. $\mathbf{A}(t) = 4\cosh 2t\mathbf{i} + 4\sinh 2t\mathbf{j}$.

$\|\mathbf{V}(t)\| = \sqrt{4\sinh^2 2t + 4\cosh^2 2t} = 2\sqrt{\sinh^2 2t + \cosh^2 2t} = 2\sqrt{\cosh 4t} = \mathrm{D}_t s$

$\mathrm{A}_T(t) = \mathrm{D}_t{}^2 s = \dfrac{4\sinh 4t}{\sqrt{\cosh 4t}}$ $\|\mathbf{A}(t)\| = \sqrt{16\cosh^2 2t + 16\sinh^2 2t} = 4\sqrt{\cosh 4t}$

$\mathrm{A}_N(t) = \sqrt{\|\mathbf{A}(t)\|^2 - (\mathrm{A}_T(t))^2} = \sqrt{16\cosh 4t - \dfrac{16\sinh^2 4t}{\cosh 4t}} = 4\sqrt{\dfrac{\cosh^2 4t - \sin^2 4t}{\cosh 4t}} = 4\sqrt{\dfrac{1}{\cosh 4t}} = \dfrac{4}{\sqrt{\cosh 4t}}$

72. $\mathbf{R}(t) = (2\tan^{-1}t - t)\mathbf{i} + \ln(1 + t^2)\mathbf{j} + 2\mathbf{k}$

(a) $\mathbf{V}(t) = \dot{\mathbf{R}}(t) = \dfrac{1 - t^2}{1 + t^2}\mathbf{i} + \dfrac{2t}{1 + t^2}\mathbf{j}$ $\qquad$ $\mathbf{A}(t) = \dot{\mathbf{V}}(t) = \dfrac{-4t}{(1 + t^2)^2}\mathbf{i} + \dfrac{2(1 - t^2)}{(1 + t^2)^2}\mathbf{j}$

(b) $v = \|\mathbf{V}(t)\| = \dfrac{1}{1 + t^2}\sqrt{(1 - t^2)^2 + (2t)^2} = 1$

(c) $\mathrm{A}_T = \dot{v} = 0$ and so $\mathrm{A}_N = \|\mathbf{A}\| = \dfrac{2}{(1 + t^2)^2}\sqrt{(-2t)^2 + (1 - t^2)^2} = \dfrac{2}{1 + t^2}$

73 and 74. $\mathbf{V}_0 = 150\cos 30°\mathbf{i} + 150\sin 30\mathbf{j} = 75\sqrt{3}\mathbf{i} + 75\mathbf{j}$ (a) $\mathbf{R} = 75\sqrt{3}t\mathbf{i} + (75t - 16t^2)\mathbf{j}$ (b) $y = 0$ when $t = \frac{75}{16}$, range $= x(\frac{75}{16}) = 75\sqrt{3} \cdot \frac{75}{16} = \frac{5625}{16}\sqrt{3} \approx 609$ ft (c) $y(\frac{75}{32}) = \frac{5625}{64} \approx 87.9$ ft. (d) speed of impact is 150 ft/sec. The dots are closer at the vertex of the parabolic path indicating the speed is least there.

75. Range $= \frac{v_0^{\,2}}{g}\sin 2\theta = \frac{v_0^{\,2}}{32}\sin 80^\circ \geq 300$, $v_0 \geq \sqrt{\frac{9600}{\sin 80^\circ}} = 98.73$ ft/sec

76. Find the maximum height reached by a projectile of muzzle speed v_0 feet per second and angle of elevation α.

▷ We complete the square on the vertical component.

$y(t) = v_0\sin \alpha t - \frac{1}{2}gt^2 = -\frac{g}{2}\left(t^2 - \frac{2v_0}{g}\sin \alpha t + \frac{v_0^{\,2}\sin^2\alpha}{g^2}\right) + \frac{v_0^{\,2}\sin^2\alpha}{2g} = \frac{v_0^{\,2}\sin^2\alpha}{2g} - \frac{g}{2}\left(t - \frac{v_0\sin \alpha}{g}\right)^2$

Hence the maximum height is $\frac{v_0^{\,2}\sin^2\alpha}{2g}$ feet.

77. (a) $y = 20 - 24 \sin 25^\circ t - 16t^2 = 0$ when $t = \frac{1}{32}(-24 \sin 25^\circ + \sqrt{24^2\sin^2 25^\circ + 1820}) \approx 0.845$ sec
(b) $x = 20 \cos 25^\circ(0.845) = 18.38$ ft

78. (a) $y = 20 - 16t^2 = 0$ when $t = \frac{1}{2}\sqrt{5} \approx 1.12$ sec (b) $x = 20(\frac{1}{2}\sqrt{5}) = 10\sqrt{5} = 22.36$ ft

79. $\mathbf{R} \cdot \dot{\mathbf{R}} = 0 \Rightarrow D_t\|\mathbf{R}\|^2 = 0 \Rightarrow \|\mathbf{R}\|$ is constant and so the particle moves in a circle.

80. If a particle is moving along a curve, under what condition will the acceleration vector and the unit tangent vector have the same or opposite directions?

▷ For some scalar function $a(t)$ we have

$\dot{\mathbf{R}}(t) = a(t)\mathbf{T}(t)$

Differentiating with respect to t,

$\mathbf{A}(t) = \ddot{\mathbf{R}}(t) = \dot{a}(t)\mathbf{T}(t) + a(t)\dot{\mathbf{T}}(a)$

This will have the same or opposite direction as $\mathbf{T}(t)$ if and only if $\dot{\mathbf{T}}(t) = 0$. Thus $\mathbf{T}(t)$ is a constant vector and the curve is a straight line.

81. From Exercise 11.4.56, $\mathbf{T}$ and $\mathbf{B}$ have the direction of $\mathbf{V}$ and $\mathbf{V} \times \mathbf{A}$. Thus $\mathbf{N}$ has the direction of $(\mathbf{V} \times \mathbf{A}) \times \mathbf{V} = (\mathbf{V} \cdot \mathbf{V})\mathbf{A} - (\mathbf{V} \cdot \mathbf{A})\mathbf{V}$. Dividing by the magnitude gives a unit vector.

82. Because the scalar triple product is a determinant, the result is immediate from Exercise 11.2.31

T W E L V E

DIFFERENTIAL CALCULUS OF FUNCTIONS OF MORE THAN ONE VARIABLE

12.1 FUNCTIONS OF MORE THAN ONE VARIABLE

Let P be an ordered pair of real numbers (x, y), and let z be a real number. A *function of two variables* is a set of ordered pairs of the form (P, z) in which no two distinct pairs have the same first element. The set of all possible replacements of P is called the *domain* of the function, and the set of all possible values of z is called the *range* of the function. The variables x and y are the *independent variables*, and z is the *dependent variable*. If f is a function of two variables, then we denote the function value of f at P by either $f(P)$ or $f(x, y)$. A *polynomial function* f of two variables x and y is the sum of terms of the form $cx^n y^m$ where c is a real number and n and m are nonnegative integers. The *degree* of a term is the sum of the exponents; the largest degree is the degree of f. If all terms have the same degree, f is *homogeneous*.

12.1.1 Definition The set of all ordered n-tuples of real numbers is called the *n-dimensional number space* and is denoted by R^n. Each ordered n-tuple $(x_1, x_2, \ldots, x_n)$ is called a *point* in the n-dimensional number space.

12.1.2 Definition A *function of n variables* is a set of ordered pairs of the form (P, w) in which no two distinct ordered pairs have the same first element, where P is a point in n-dimensional number space and w is a real number. The set of all possible values of P is called the *domain* of the function, and the set of all possible values of w is called the *range* of the function.

12.1.3 Definition If f is a function of a single variable and g is a function of two variables, then the *composite function* $f \circ g$ is the function of two variables defined by $(f \circ g)(x, y) = f(g(x, y))$ and the domain of $f \circ g$ is the set of all points (x, y) in the domain of g such that $g(x, y)$ is in the domain of f.

12.1.4 Definition If f is a function of a single variable and g is a function of n variables, then the *composite function* $f \circ g$ is the function of n variables defined by
$(f \circ g)(x_1, x_2, \ldots, x_n) = f(g(x_1, x_2, \ldots, x_n))$
and the domain of $f \circ g$ is the set of all points $(x_1, x_2, \ldots, x_n)$ in the domain of g such that $g(x_1, x_2, \ldots, x_n)$ is in the domain of f.

12.1.5 Definition If f is a function of two variables, then the graph of f is the set of all points (x, y, z) in R^3 for which (x, y) is a point in the domain of f and $z = f(x, y)$.
Thus, the graph of a function of two variables is a surface in R^3. Because there is only one value of z for each replacement of (x, y) from the domain of the function, a line perpendicular to the xy plane intersects the surface in no more than one point.

12.1.6 Definition If f is a function of n variables, then the *graph* of f is the set of all points $(x_1, x_2, \ldots, x_n, w)$ in R^{n+1} for which $(x_1, x_2, \ldots, x_n)$ is a point in the domain of f and $w = f(x_1, x_2, \ldots, x_n)$.

Let C be the intersection of the surface $z = f(x, y)$ and the plane $z = k$. The projection of C on the xy plane, that is, the graph of $f(x, y) = k$, is called the *level curve* of the function f at k. Level curves are called *equipotential curves* if f gives electric potential. The set of all level curves of f at k if $k = k_1, k_2, \ldots, k_n$ is called a *contour map* of the function f. If the numbers $k_1, k_2, \ldots, k_n$ are equally spaced, then the resulting contour map indicates the steepness of the surface that is the graph of f. Closely spaced contour curves indicate that the surface is steep; widely space contour curves indicate that the surface is not steep; and a contour "curve" that is a region indicates that the graph is flat. Similarly, functions of three variables have *level surfaces*. The *inside* of conic section or quadric surface Q is the set of points through which there are no lines tangent to Q.

Exercises 12.1

1. $f(x,y) = \dfrac{x+y}{x-y}$

▸ (a) $f(-3,4) = \dfrac{-3+4}{-3-4} = -\frac{1}{7}$

(b) $f(\frac{1}{2},\frac{1}{3}) = \dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}-\frac{1}{3}} = \dfrac{\frac{5}{6}}{\frac{1}{6}} = 5$

(c) $f(x+1,y-1) = \dfrac{(x+1)+(y-1)}{(x+1)-(y-1)} = \dfrac{x+y}{x-y+2}$

(d) $f(-x,y) - f(x,-y) = \dfrac{-x+y}{-x-y} - \dfrac{x-y}{x+y} = \dfrac{x-y}{x+y} - \dfrac{x-y}{x+y} = 0$

2. $g(x,y) = \sqrt{x^2-y}$

▸ (a) $g(3,5) = \sqrt{3^2-5} = \sqrt{4} = 2$

(b) $g(-4,-9) = \sqrt{(-4)^2-(-9)} = \sqrt{25} = 5$

(c) $g(x+2,4x+4) = \sqrt{(x+2)^2-(4x+4)} = \sqrt{x^2} = |x|$

(d) $g\left(\frac{1}{x},\frac{-3}{x^2}\right) = \sqrt{\left(\frac{1}{x}\right)^2-\left(\frac{-3}{x^2}\right)} = \sqrt{\frac{4}{x^2}} = \frac{2}{|x|}$

3. $g(x,y,z) = \sqrt{4-x^2-y^2-z^2}$

(a) $g(1,-1,-1) = \sqrt{4-1-1-1} = 1$

(b) $g(-1,\frac{1}{2},\frac{3}{2}) = \sqrt{4-1-\frac{1}{4}-\frac{9}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$

(c) $g(\frac{1}{2}x,\frac{1}{2}y,\frac{1}{2}z) = \sqrt{4-\frac{1}{4}x^2-\frac{1}{4}y^2-\frac{1}{4}z^2} = \frac{1}{2}\sqrt{16-x^2-y^2-z^2}$

(d)
$$\begin{aligned}[g(x,y,z)]^2-[g(x+2,y+2,z)]^2 &= (4-x^2-y^2-z^2)-[4-(x+2)^2-(y+2)^2-z^2] \\ &= -x^2-y^2+x^2+4x+4+y^2+4y+4 = 4x+4y+8\end{aligned}$$

4. Let the function f of three variables x, y, and z be the set of all ordered pairs of the form (P, w) such that
$w = \dfrac{4}{x^2+y^2+z^2-9} \Leftrightarrow f(x,y,z) = \dfrac{4}{x^2+y^2+z^2-9}$

Find: (a) $f(1,2,3)$; (b) $f(2,-\frac{1}{2},\frac{3}{2})$; (c) $f\left(-\frac{2}{x},\frac{2}{x},-\frac{1}{x}\right)$; (d) $f(x+2),1,x-2)$

▸ (a) We let $x=1$, $y=2$, and $z=3$. Thus,
$$f(1,2,3) = \frac{4}{1^2+2^2+3^2-9} = \frac{4}{5}$$

(b) $f(2,-\frac{1}{2},\frac{3}{2}) = \dfrac{4}{2^2+(-\frac{1}{2})^2+(\frac{3}{2})^2-9} = \dfrac{4}{4+\frac{1}{4}+\frac{9}{4}-9} = -\frac{8}{5}$

(c) $f\left(-\frac{2}{x},\frac{2}{x},-\frac{1}{x}\right) = \dfrac{4}{\left(-\frac{2}{x}\right)^2+\left(\frac{2}{x}\right)^2+\left(-\frac{1}{x}\right)^2-9} = \dfrac{4}{\frac{9}{x^2}-9} = \dfrac{4x^2}{9(1-x^2)},\ x \neq 0$

In Exercises 5–20, determine the domain of f and sketch the domain as a region in R^2. Use dashed curves to indicate any part of the boundary not in the domain and solid curves to indicate parts of the boundary in the domain.

5. $f(x,y) = \dfrac{1}{x^2+y^2-1}$

▸ $\text{Dom}(f) = \{(x,y) \mid x^2+y^2 \neq 1\}$, the set of all points (x,y) in R^2 except those on the circle $x^2+y^2=1$.

6. $f(x,y) = \dfrac{4}{4-x^2-y^2}$

▸ $\text{Dom}(f) = \{(x,y) \mid x^2+y^2 \neq 4\}$, the set of all points (x,y) in R^2 except those on the circle $x^2+y^2=4$.

7. $f(x,y) = \sqrt{1-x^2-y^2}$

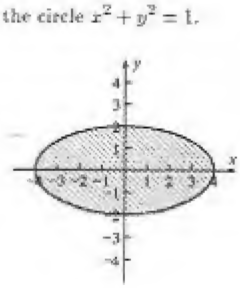
▸ $\text{Dom}(f) = \{(x,y) \mid 1-x^2-y^2 \geq 0\}$, the set of all points (x,y) in R^2 inside or on the circle $x^2+y^2=1$.

8. $f(x,y) = \sqrt{16-x^2-4y^2}$

▸ Because the square root of a negative number is not real, the value of the expression under the radical sign must be greater than or equal to zero. That is

$$16-x^2-4y^2 \geq 0$$
$$x^2+4y^2 \leq 16 \qquad (1)$$

The domain of the function f is the set of all points (x,y) that satisfy (1). The graph of this set is the region in the xy plane that is inside or on the ellipse $x^2/16+y^2/4=1$, as shown in the figure.



9. $f(x,y) = \sqrt{x^2 - y^2 - 1}$

▷ $\text{Dom}(f) = \{(x,y) \mid x^2 - y^2 \geq 1\}$, the set of all points (x,y) in $\mathbb{R}^2$ on and inside the hyperbola $x^2 - y^2 = 1$.

10. $f(x,y) = \sqrt{x^2 - 4y^2 + 16}$

▷ $\text{Dom}(f) = \{(x,y) \mid 4y^2 - x^2 \leq 16\}$, the set of all points (x,y) in $\mathbb{R}^2$ on and outside the hyperbola $4y^2 - x^2 = 16$.

11. $f(x,y) = \sqrt{x^2 + y^2 - 1}$

▷ $\text{Dom}(f) = \{(x,y) \mid x^2 + y^2 \geq 1\}$, the set of all points (x,y) in $\mathbb{R}^2$ on and outside the circle $x^2 + y^2 = 1$.

12. $f(x,y) = \sqrt{x^2 + 4y^2 - 16}$

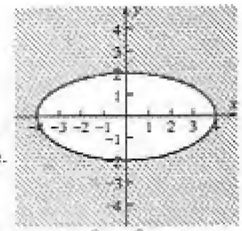
▷ If

$$x^2 + 4y^2 - 16 \geq 0$$

then

$$x^2 + 4y^2 \geq 16 \tag{1}$$

The domain of f is the set of all points that satisfy (1). This is the region in the xy plane that is on or outside the ellipse $x^2/16 + y^2/4 = 1$, as shown in the figure.



13. $f(x,y) = \dfrac{1}{\sqrt{1 - x^2 - y^2}}$

▷ The domain of f is $\{(x,y) \mid x^2 + y^2 < 1\}$, the set of all points (x,y) in $\mathbb{R}^2$ inside the circle $x^2 + y^2 = 1$.

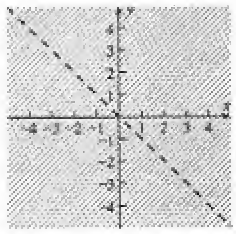
14. $f(x,y) = \dfrac{1}{\sqrt{16 - x^2 - 4y^2}}$

▷ The domain of f is $\{(x,y) \mid x^2 + 4y^2 < 16\}$, the set of all points (x,y) in $\mathbb{R}^2$ inside the ellipse $x^2 + 4y^2 = 16$.

15. $f(x,y) = \dfrac{x^4 - y^4}{x^2 - y^2}$. $\text{Dom}(f) = \{(x,y) \mid y \neq \pm x\}$, the set of all points (x,y) in $\mathbb{R}^2$ not on the lines $y = \pm x$.

16. $f(x,y) = \dfrac{x-y}{x+y}$

▷ The domain of f consists of all points in the xy plane except those on the line $x + y = 0$, as shown in the figure.



17. $f(x,y) = \cos^{-1}(x-y)$. The domain of f is $\{(x,y) \mid -1 \leq x - y \leq 1\}$, the set of all points (x,y) in $\mathbb{R}^2$ on and between the lines $y = x - 1$ and $y = x + 1$.

18. $f(x,y) = \ln(x^2 + y)$. The domain of f is $\{(x,y) \mid y > -x^2\}$, the set of all points (x,y) in $\mathbb{R}^2$ outside the parabola $y = -x^2$.

19. $f(x,y) = \ln(xy - 1)$. The domain of f is $\{(x,y) \mid xy > 1\}$, the set of all points (x,y) in $\mathbb{R}^2$ inside the hyperbola $xy = 1$.

20. $f(x,y) = \sin^{-1}(x+y)$

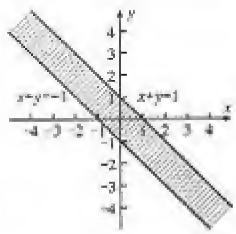
▷ Because the domain of the inverse sine function is $[-1,1]$, the domain of f is

$\{(x,y) \mid |x+y| \leq 1\}$

Thus the domain is the set of all points on or between the parallel lines

$x + y = 1$ and $x + y = -1$

See the figure.



In Exercises 21-28, determine the domain of f and describe as a region in $\mathbb{R}^3$ the set of points in the domain.

21. $f(x,y,z) = \dfrac{x+y+z}{x-y-z}$

▷ The domain of f is $\{(x,y,z) \mid x - y - z \neq 0\}$, the set of all points (x,y,z) in $\mathbb{R}^3$ not on the plane $x - y - z = 0$.

22. $f(x,y,z) = \dfrac{z}{x^2 - y}$

▷ The domain of f is $\{(x,y,z) \mid y \neq x^2\}$, the set of all points (x,y,z) in $\mathbb{R}^3$ not on the parabolic cylinder $y = x^2$.

23. $f(x,y,z) = \sqrt{16 - x^2 - 4y^2 - z^2}$. The domain of f is $\{(x,y,z) \mid x^2 + 4y^2 + z^2 \leq 16\}$, the set of all points (x,y,z) in $\mathbb{R}^3$ on and inside the ellipsoid $x^2/16 + y^2/4 + z^2/16 = 1$.

24. $f(x,y,z) = \sqrt{9 - x^2 - y^2 - z^2}$

▸ The domain consists of the points of $\mathbb{R}^3$ for which

$$9 - x^2 - y^2 - z^2 \geq 0$$

or, equivalently,

$$x^2 + y^2 + z^2 \leq 9$$

The domain is the set of points inside or on a sphere of radius 3 centered at the origin.

25. $f(x,y,z) = \sin^{-1}x + \sin^{-1}y + \sin^{-1}z$. The domain of f is $\{(x,y,z) \mid |x| \leq 1, |y| \leq 1, |z| \leq 1\}$, the set of all points (x,y,x) in $\mathbb{R}^3$ on and inside the cube $|x| \leq 1, |y| \leq 1, |z| \leq 1$.

26. $f(x,y,z) = \ln x + \ln y + \ln z$. $\text{Dom}(f) = \{(x,y,z) \mid x > 0, y > 0, z > 0\}$, the first octant in $\mathbb{R}^3$.

27. $f(x,y,z) = \ln(4 - x^2 - y^2) + |z|$

▸ $\text{Dom}(f) = \{(x,y,z) \mid x^2 + y^2 < 4\}$, the set of all points (x,y,z) in $\mathbb{R}^3$ inside the circular cylinder $x^2 + y^2 = 4$.

28. $f(x,y,z) = xz\cos^{-1}(y^2 - 1)$

▸ Because the domain of the inverse cosine function is $[-1,1]$, we must have

$$-1 \leq y^2 - 1 \leq 1$$

$$0 \leq y^2 \leq 2$$

$$-\sqrt{2} \leq y \leq \sqrt{2}$$

The domain is the set of points of $\mathbb{R}^3$ on and between the parallel planes $y = -\sqrt{2}$ and $y = \sqrt{2}$.

In Exercises 29-36, determine the domain of f and sketch the graph of f.

In Exercises 37-46, sketch a contour map of f showing the level curves at the given numbers.

29 and 37. $f(x,y) = \sqrt{16 - x^2 - y^2}$. The domain of f is the set of points (x,y) of $\mathbb{R}^2$ on and inside the circle $x^2 + y^2 = 16$. The graph of $z = \sqrt{16 - x^2 - y^2}$ is the top half of the sphere $x^2 + y^2 + z^2 = 16$. If $z = k$ we have $k^2 = 16 - x^2 - y^2$; $x^2 + y^2 = 16 - k^2$. The level curves are circles centered at the origin of radius $\sqrt{16 - k^2}$, $k = 0, 1, 2, 3, 4$ (a point).

30 and 38. $f(x,y) = 6 - 2x + 2y$. $\text{Dom}(f) = \mathbb{R}^2$. The graph of $z = 6 - 2x + 2y$ is a plane. If $z = k$ we have $k = 6 - 2x + 2y$; $y = x + \frac{1}{2}k - 3$. The level curves are the lines for $k = 10, 6, 2, 0, -2, -6$, and -10.

31 and 39. $f(x,y) = 16 - x^2 - y^2$. The domain of f is the set of all points (x,y) of $\mathbb{R}^2$. The graph of $z = 16 - x^2 - y^2$ is a hyperbolic paraboloid. If $z = k$ we have $k = 16 - x^2 - y^2$; $x^2 + y^2 = 16 - k$. The level curves are circles of radius $\sqrt{16 - k}$, $k = 16$ (a point), 12, 7, 0, -20.

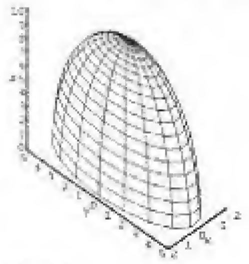
32. $f(x,y) = \sqrt{100 - 25x^2 - 4y^2}$

▸ Let $z = f(x,y)$. Then

$$z^2 = 100 - 25x^2 - 4y^2$$

$$25x^2 + 4y^2 + z^2 = 100$$

$$\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{100} = 1$$



Because $z \geq 0$, the graph of the function f is that part of an ellipsoid of semiaxes 2, 5, and 10, that is not below the xy plane, as shown in the figure. The domain of f is $\{(x,y) \mid 100 - 25x^2 - 4y^2 \geq 0\} = \{(x,y) \mid 25x^2 + 4y^2 \leq 100\}$, the set of points in the xy plane inside or on the ellipse $x^2/4 + y^2/25 = 1$. Note that the domain of f is the projection on the xy plane of the surface that is the graph of f.

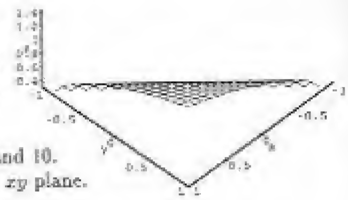
33 and 41. $f(x,y) = x^2 - y^2$. The domain of f is the set of all points (x,y) of $\mathbb{R}^2$. The graph of $z = x^2 - y^2$ is a hyperbolic paraboloid. If $z = k$, the level curves are the hyperbolas $x^2 - y^2 = k$, 16, 9, 4, 0, -4, -9, -16.

34. $f(x,y) = 144 - 9x^2 - 16y^2$. The domain of f is the set of all points (x,y) of $\mathbb{R}^2$. The graph of $z = 144 - 9x^2 - 16y^2$ is an elliptical paraboloid.

35. $f(x,y) = 4x^2 + 9y^2$. The domain of f is the set of all points (x,y) of $\mathbb{R}^2$. The graph of $z = 4x^2 + 9y^2$ is an elliptical paraboloid.

36. $f(x,y) = \sqrt{x+y}$

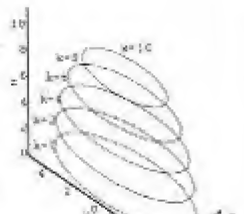
▷ The domain is $\{(x,y) \mid x+y \geq 0\}$. The graph of f is the graph of the equation $z = \sqrt{x+y}$. It is the upper half of a parabolic cylinder with rulings parallel to its trace in the xy plane, the line $x+y=0$. Its trace in the plane $z=k$ is the line $x+y=k^2$. Its trace in the perpendicular plane $x-y=0$ is $z=\sqrt{2x}$ or $z^2=4x$, $z \geq 0$. See the figure.



40. The function f for which $f(x,y) = \sqrt{100-25x^2-4y^2}$ at 0, 2, 4, 6, 8, and 10.

▷ The level curves of f at the number k is the graph of $f(x,y)=k$ in the xy plane. Then

$$\begin{aligned}
\sqrt{100-25x^2-4y^2} &= k \\
100-25x^2-4y^2 &= k^2 \\
25x^2+4y^2 &= 100-k^2 \\
\frac{x^2}{\frac{100-k^2}{25}} + \frac{y^2}{\frac{100-k^2}{4}} &= 1
\end{aligned}$$



This is an ellipse centered at the origin of semiaxes $\frac{1}{5}\sqrt{100-k^2}$ and $\frac{1}{2}\sqrt{100-k^2}$. The semiaxes for the given values of k are shown in the table. See the sketch

k	$\sqrt{100-k^2}$	$\frac{1}{5}\sqrt{100-k^2}$	$\frac{1}{2}\sqrt{100-k^2}$
0	10	2	5
2	$\sqrt{96} \approx 9.8$	1.96	4.9
4	$\sqrt{84} \approx 9.2$	1.83	4.6
6	8	1.6	4
8	6	1.2	3
10	0	0	0

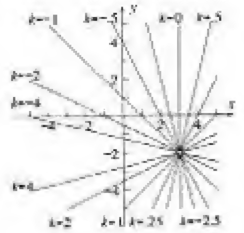
42. The level curves are the lines $x+y=k^2$, $k=10, 8, 6, 5$ and 0

43. $f(x,y) = \frac{1}{2}(x^2+y^2)$. If $f=k$ we have $k=\frac{1}{2}(x^2+y^2)$; $x^2+y^2=2k$. The level curves are circles centered at the origin of radius $\sqrt{2k}$, $k=8, 6, 4, 2, 0$.

44. $f(x,y) = (x-3)/(y+2)$ at 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, 0, $-\frac{1}{4}$, $-\frac{1}{2}$, -1, -2, and -4.

▷ If $f(x,y)=k$, then

$$\begin{aligned}
\frac{x-3}{y+2} &= k \\
x-3 &= ky+2k,\ y \neq -2 \\
x &= ky+2k+3,\ y \neq 2
\end{aligned}$$



Thus each graph is a line with the point $(3,-2)$ deleted. The figure shows the contour map obtained by letting k take on each of the given values.

45. $f(x,y)=e^{xy}$. If $f=k$ we have $k=e^{xy}$; $xy=\ln k$. The level curves are hyperbolas whose asymptotes are the axes for $k=1, 2, e, 4, \frac{1}{2}, e^{-1}, \frac{1}{4}$; $\ln k = 0$ (the axes), $\ln 2$, 1, $\ln 4$, $-\ln 2$, -1, $-\ln 4$.

46. $f(x,y)=\ln xy$. If $f=k$ we have $k=\ln xy$; $xy=e^k$. The level curves are hyperbolas in the first and third quadrants whose asymptotes are the axes, for $k=0, 1, 2, 4, -1, -1, -4$.

47. $f(x,y)=x-y$, $g(t)=\sqrt{t}$, and $h(s)=s^2$.

(a) $(g \circ f)(5,1) = g(f(5,1)) = g(5-1) = g(4) = \sqrt{4} = 2$

(b) $f(h(3), g(9)) = f(9,3) = 9-3 = 6$

(c) $f(g(x), h(y)) = f(\sqrt{x}, y^2) = \sqrt{x} - y^2$

(d) $g((h \circ f)(x,y)) = g(h(f(x,y))) = g(h(x-y)) = g((x-y)^2) = \sqrt{(x-y)^2} = |x-y|$

(e) $(g \circ h)(f(x,y)) = (g \circ h)(x-y) = g(h(x-y)) = |x-y|$

48. Given $f(x,y) = x/y^2$, $g(x) = x^2$, $h(x) = \sqrt{x}$. Find: (a) $(h \circ f)(2,1)$; (b) $f(g(2), h(4))$; (c) $f(g(\sqrt{x}), h(x^2))$; (d) $h((g \circ f)(x,y))$; (e) $(h \circ g)(f(x,y))$.

▸ (a) $(h \circ f)(2,1) = h(f(2,1)) = h\left(\frac{2}{1^2}\right) = h(2) = \sqrt{2}$

(b) $f(g(2), h(4)) = f(2^2, \sqrt{4}) = f(4,2) = \frac{4}{2^2} = 1$

(c) $f(g(\sqrt{x}), h(x^2)) = f((\sqrt{x})^2, \sqrt{x^2}) = f(x,x) = \frac{x}{x^2} = \frac{1}{x}$, $x > 0$

(d) $h((g \circ f)(x,y)) = h(g(f(x,y))) = h\left(g\left(\frac{x}{y^2}\right)\right) = h\left(\left(\frac{x}{y^2}\right)^2\right) = h\left(\frac{x^2}{y^4}\right) = \sqrt{\frac{x^2}{y^4}} = \frac{|x|}{y^2}$

(e) $(h \circ g)(f(x,y)) = h(g(f(x,y))) = \frac{|x|}{y^2}$

In Exercises 49 and 50, find $h = f \circ g$ and its domain.

49. $f(t) = \sin^{-1} t$; $g(x,y) = \sqrt{1 - x^2 - y^2}$; $h(x,y) = (f \circ g)(x,y) = f(\sqrt{1 - x^2 - y^2}) = \sin^{-1}\sqrt{1 - x^2 - y^2}$.
The domain of h is $\{(x,y) \mid x^2 + y^2 \le 1\}$. Each of these points (x,y) lies in the domain of g and $g(x,y)$ is in $[-1,1]$ which is the domain of f.

50. $f(t) = e^t$; $g(x,y) = y \ln x$. $h(x,y) = (f \circ g)(x,y) = e^{y \ln x} = xe^y$, $x > 0$.

In Exercises 51–54, match the function with one of the surfaces (a)–(d) and with one of the contour maps (i)–(iv).

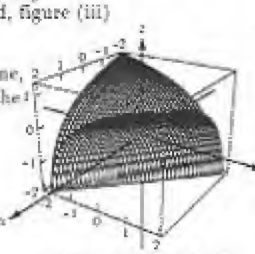
51. $f(x,y) = \dfrac{x^2}{x^2 + y^2}$. On the x axis, $y = 0$ and $z = 1$ while on the y axis $x = 0$ and $z = 1$, creating a rift at the origin. Thus figure (b). If $f = k$ we have $k = \dfrac{x^2}{x^2 + y^2}$, $kx^2 + ky^2 = x^2$; $y^2 = \frac{1-k}{k}x^2$. The level curves are lines intersecting at the origin, with the origin deleted, figure (iii)

52. $f(x,y) = \ln|x + y|$

▸ As we approach the line $x + y = 0$ in the xy plane, the graph of $z = \ln|x + y|$ approaches $-\infty$ and the graph is symmetric with respect to the plane $x + y = 0$. The graph is figure (d). If

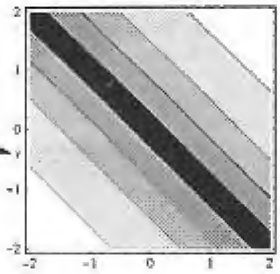
$$
\begin{aligned}
f(x,y) &= k \\
\ln|x + y| &= k \\
|x + y| &= e^k \\
x + y &= \pm e^k
\end{aligned}
$$

Thus, the level curve are lines parallel to $x + y = 0$, as shown in figure (i).



[-2, 2] by [-2, 2] by [-2, 1.5]

(d)



(i)

53. $f(x,y) = e^y \cos x$. We note the periodic behavior of the cosine curve decaying exponentially as y decreases; figure (a). If $f = k$ we have $k = e^y \cos x$; $e^y = k \sec x$; $y = \ln|k \sec x|$. This is figure (iv).

54. $f(x,y) = \sin x + \cos y$. We note the periodic behavior in the direction of both axes in figure (c). If $f = 0$, $\sin x = -\cos y$; $\cos(\frac{1}{2}\pi + x) = \cos(y + n\pi)$; $\frac{1}{2}\pi - x = y - n\pi$, $x + y = (n + \frac{1}{2}\pi)$ or $\frac{1}{2}\pi - x = n\pi - y$, $y - x = (n - \frac{1}{2})\pi$. These diagonal lines form a grid separating the groups of curves in figure (ii).

55. Let $C(x,y)$ dollars be the cost when the width is x ft and the height is y ft. Because the volume is 16 ft^3, the depth must be $16/xy$ ft. $C(x,y) = 2(\text{cost of top}) + 2(\text{cost of front}) + 2(\text{cost of other side})$

$= 2\left(.18x\frac{16}{xy} + .16xy + .12y\frac{16}{xy}\right) = .32\left(\frac{18}{y} + xy + \frac{12}{x}\right)$, $x > 0$, $y > 0$. $C(2,4) = .32(4.5 + 8 + 6) = 5.92$

56. A rectangular box without a top is to made at a cost of \$10 for the material. The material for the bottom costs \$0.15 per square foot and the material for the sides costs \$0.30 per square foot. (a) Find a mathematical model for for the volume of the box as a function of the dimensions of the bottom. State the domain of the function. (b) What is the volume of the box if the bottom is a square of side 3 ft?

▸ Let $V(x,y)$ ft^3 be the volume when the length and width are x ft and y ft, and the height is h ft.

(a) Because the cost is \$10, we have

$$
\begin{aligned}
10 &= 2(\text{cost of front}) + 2(\text{cost of side}) + (\text{cost of bottom}) \\
10 &= 2(0.30)xh + 2(0.30)yh + 0.15xy \\
10 &= 0.6h(x + y) + 0.15xy \\
h &= \frac{10 - 0.15xy}{0.6(x + y)}
\end{aligned}
$$

Then

$$V(x,y) = xyh = \frac{xy(10 - 0.15xy)}{0.6(x+y)}, \quad x > 0,\ y > 0$$

(b) $V(3,3) = \dfrac{9(10 - 0.15 \cdot 9)}{0.6(6)} = 21.625$

The volume is 21.625 ft^3.

57 and 58. $z = \frac{1}{2}(6 - x - 3y)$. $V = xyz = \frac{1}{2}xy(6 - x - 3y)$. $V(1.25, 1.25) = \frac{1}{2}(1.25)^2(1) = 0.78125\ \text{ft}^3$.
$S = 2(xy + yz + zx) = 2[xy + \frac{1}{2}(x+y)(6 - x - 3y)]$. $S(1.25, 1.25) = 2(1.25^2 + 1.25) = 5.625\ \text{ft}^2$.
The domain of each function is the triangle bounded by the coordinate axes and the line $x + 3y = 6$.

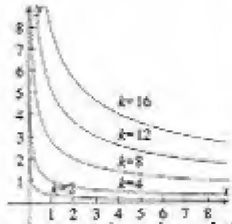
59. $V(x,y) = \dfrac{4}{\sqrt{9 - x^2 - y^2}}$. If $V(x,y) = V$ we have $9 - x^2 - y^2 = \dfrac{16}{V^2}$; $x^2 + y^2 = 9 - \dfrac{16}{V^2}$. The equipotential curves are circles centered at the origin of radius $\sqrt{9 - 16/V^2}$, $V = 16, 12, 8, 4$.

60. A production function f for a commodity has function values $f(x,y) = 4x^{1/3}y^{2/3}$, where x and y give the amounts of two inputs. Draw a contour map of f showing the constant product curves at 16, 12, 8, 4, and 2.

▶ If $f = k$, then

$$k = 4x^{1/3}y^{2/3}$$
$$k^3 = 64xy^2$$
$$x = \frac{k^3}{64y^2}$$

The figure shows the constant product curves for $k = 16, 12, 8, 4$, and 2.



61. $z = 6xy$. The constant product curves for $z = 30, 24, 18, 12$, and 6 are the first quadrant branches of the hyperbolas $xy = 5$, $xy = 4$, $xy = 3$, $xy = 2$, and $xy = 1$. The curve is restricted to the first quadrant because $x > 0$ and $y > 0$.

62. If $(x,y) = k$, then $4x^2 + 2y^2 = k$; $\dfrac{x^2}{k/4} + \dfrac{y^2}{k/2} = 1$, an ellipse centered at the origin of semiaxes $\sqrt{k/4}$ and $\sqrt{k/2}$, $k = 12, 18, 4, 1$, and 0 (point).

63. $k = 4e^{-(x^2+y^2+z^2)}$; $x^2 + y^2 + z^2 = \ln(4/k)$. The level surfaces are spheres of radius $\sqrt{\ln(4/k)}$, $k = 4, 2, 1, \frac{1}{2}$.

64. The electric potential at a point (x, y, z) in three-dimensional space is $V(x,y,z)$ volts where $V(x,y,z) = \dfrac{8}{\sqrt{16x^2 + 4y^2 + z^2}}$. The level surfaces of V are *equipotential surfaces*. Describe these surfaces at 4, 2, 1, and $\frac{1}{2}$.

▶ If $V(x, yz) = k$, then

$$k = \frac{8}{\sqrt{16x^2 + 4y^2 + z^2}}$$
$$\sqrt{16x^2 + 4y^2 + z^2} = \frac{8}{k}$$
$$16x^2 + 4y^2 + z^2 = \frac{64}{k^2}$$
$$\frac{x^2}{4/k^2} + \frac{y^2}{16/k^2} + \frac{z^2}{64/k^2} = 1$$

Thus the equipotential surfaces are ellipsoids of semiaxes $2/k$, $4/k$ and $8/k$ for $k = 4, 2, 1, \frac{1}{2}$.

12.2 LIMITS AND CONTINUITY OF FUNCTIONS OF MORE THAN ONE VARIABLE

We first consider functions of two variables. The *open disk* $B((x_0, y_0); r)$ is the set of all points in the xy plane that are inside the circle with center at the point (x_0, y_0) and radius r. Thus a point (x, y) is in $B((x_0, y_0); r)$ if and only if $\sqrt{(x - x_0)^2 + (y - y_0)^2} < r$.

12.2.5 Definition Let f be a function of two variables that is defined on some open disk $B((x_0, y_0); r)$, except possibly at the point (x_0, y_0) itself. Then the *limit of $f(x,y)$ as (x,y) approaches (x_0, y_0) is L*, written as

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

$$\text{if } 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \text{ then } |f(x,y) - L| < \epsilon$$

Let S be a set of points in the xy plane. A point (x_0, y_0) is said to be an *accumulation* point of S if and only if every open disk $B((x_0, y_0); r)$ contains infinitely many points of S.

We now extend to functions of two variables the concept of one-sided limits for functions of two variables.

12.2.8 Definition Let f be a function defined on a set of points in R^2 and let (x_0, y_0) be an accumulation point of S. Then the *limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) in S is L*, is written as

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\(P \text{ in } S)}} f(x, y) = L$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

if $0 < \|(x, y) - (x_0, y_0)\| < \delta$ and $(x, y) \in S$ then $|f(x, y) - L| < \epsilon$

If S is the domain of f, this is the definition of limit in advanced calculus. This distinction is illustrated in Exercise 24.

12.2.9 Theorem Suppose that the function f is defined for all points on an open disk having its center at (x_0, y_0), except possibly at (x_0, y_0) itself,

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

and S is any set of points in R^2 having (x_0, y_0) as an accumulation point. Then

$$\lim_{\substack{(x,y)\to(0,0)\\(P \text{ in } S)}} f(x, y) = L$$

The contrapositive of this theorem may be used to prove a limit does not exist.

12.2.10 Theorem If the function f has different limits as (x, y) approaches (x_0, y_0) through distinct sets of points having (x_0, y_0) as an accumulation point, then

$$\lim_{(x,y)\to(x_0,y_0)} \quad \text{does not exist.}$$

The limit theorems for functions of one variable, with minor modifications, apply to functions of two variables. We also have the following theorem regarding the limit of a composite function of two variables.

12.2.6 Theorem If g is a function of two variables and $\displaystyle\lim_{(x,y)\to(x_0,y_0)} = b$, and f is a function of a single variable continuous at b, then

$$\lim_{(x,y)\to(x_0,y_0)} (f \circ g)(x, y) = f(b) \Leftrightarrow$$

$$\lim_{(x,y)\to(x_0,y_0)} f(g(x, y)) = f\Big(\lim_{(x,y)\to(x_0,y_0)} g(x, y)\Big)$$

We may be able to find the limit of a function of two variables by applying the limit theorem for a function of one variable together with Theorem 12.2.6. If these theorems fail, as they do, for example, when both the numerator and denominator of a fraction have limit zero, then the following steps may sometimes be used to determine if $\displaystyle\lim_{(x,y)\to(x_0,y_0)}$ exists.

1. Let S_1 be a set of points on a curve containing (x_0, y_0) and find L_1, it it exists, where $$L_1 = \lim_{\substack{(x,y)\to(x_0,y_0)\\(P \text{ in } S_1)}} f(x, y)$$
2. If L_1 does not exist, then $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.
3. If L_1 does exist, then let S_2 be a set of points on a different curve containing (x_0, y_0), and find L_2, it it exists, where $$L_2 = \lim_{\substack{(x,y)\to(x_0,y_0)\\(P \text{ in } S_2)}} f(x, y)$$
4. If L_2 does not exist, or if $L_1 \neq L_2$, then $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (x, y)$ does not exist.
5. If $L_1 = L_2$, try to use Definition 12.2.5 with $L = L_1 = L_2$. (It is still possible that the limit does not exist.)

In most of Exercises 11–24, the numerator is homogeneous of degree m and the denominator is homogeneous of degree n. Then the limit at $(0, 0)$ exists and is zero if $m > n$ and does not exist if $m \leq n$.

12.2.12 Definition The function f of two variables x and y is said to be *continuous at the point* (x_0, y_0) if and only if the following three conditions are satisfied:

(i) $f(x_0, y_0)$ exists;

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists;

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

If a function f of two variables is discontinuous at the point (x_0, y_0) but $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, then f is said to have a *removable discontinuity* at (x_0, y_0) because if f is redefined at (x_0, y_0) so that $f(x_0, y_0) = \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ then the new function is continuous at (x_0, y_0). If the discontinuity is not removable, it is an *essential discontinuity*.

12.2.13 Theorem If f and g are two functions continuous at the point (x_0, y_0), then

(i) $f + g$ is continuous at (x_0, y_0);

(ii) $f - g$ is continuous at (x_0, y_0);

(iii) fg is continuous at (x_0, y_0);

(iv) f/g is continuous at (x_0, y_0), provided that $g(x_0, y_0) \neq 0$.

12.2.14 Theorem A polynomial function of two variables is continuous at every point in R^2.

12.2.15 Theorem A rational function of two variables is continuous at every point in its domain.

12.2.17 Theorem Suppose that f is a function of a single variable and g is a function of two variables such that g is continuous at (x_0, y_0) and f is continuous at $g(x_0, y_0)$. Then the composite function $f \circ g$ is continuous at (x_0, y_0).

The following extend the definitions and theorems to functions of more than two variables.

12.2.1 Definition If $P(x_1, x_2, \ldots, x_n)$ and $A(a_1, a_2, \ldots, a_n)$ are two points in R^n, then the distance between P and A is given by

$$\| P - A \| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2}$$

12.2.2 Definition If A is a point in R^n and r is a positive number, then the *open ball* $B(A; r)$ is the set of all points P in R^n such that $\| P - A \| < r$.

12.2.4 Definition Let f be a function of n variables that is defined on some open ball $B(A; r)$, except possibly at the point A itself. Then the *limit of $f(P)$ as P approaches A is L*, written as $\lim\limits_{P\to A} f(P) = L$ if for any $\epsilon > 0$, however small, the exists a $\delta > 0$ such that

if $\| P - A \| < \delta$ then $| f(P) - L | < \epsilon$

12.2.7 Definition A point P is said to be an *accumulation point* of a set S of points in R^n if every open ball $B(P; r)$ contains infinitely many points of S.

12.2.11 Definition Suppose that f is a function of n variables and A is a point in R^n. Then f is said to be *continuous* at the point A if and only if the following three conditions are satisfied:

(i) $f(A)$ exists;

(ii) $\lim\limits_{P\to A} f(P)$ exists;

(iii) $\lim\limits_{P\to A} f(P) = f(A)$.

12.2.16 Definition A function of n variables is said to be *continuous on an open ball* if it is continuous at every point of the open ball.

Exercises 12.2

In Exercises 1–6, evaluate the limit by use of limit theorems.

1. $\lim\limits_{(x,y)\to(2,3)} (3x^2 + xy - 2y^2) = \lim\limits_{(x,y)\to(2,3)} 3x^2 + \lim\limits_{(x,y)\to(2,3)} xy + \lim\limits_{(x,y)\to(2,3)} (-2y^2) = 12 + 6 - 18 = 0$

2. $\lim\limits_{(x,y)\to(-1,4)} (5x^2 - 2xy + y^2) = \lim\limits_{(x,y)\to(-1,4)} 5x^2 - \lim\limits_{(x,y)\to(-1,4)} 2xy + \lim\limits_{(x,y)\to(-1,4)} y^2 = 5 - (-8) + 16 = 29$

3. $\lim\limits_{(x,y)\to(2,-1)} = \dfrac{\lim\limits_{(x,y)\to(2,-1)}(3x-2y)}{\lim\limits_{(x,y)\to(2,-1)}(x+4y)} = \dfrac{6+2}{2-4} = \dfrac{8}{-2} = -4$

4. $\lim\limits_{(x,y)\to(-2,4)} y\sqrt[3]{x^3+2y}$

▸ $\lim\limits_{(x,y)\to(-2,4)} y\sqrt[3]{x^3+2y} = \left(\lim\limits_{y\to4} y\right)\cdot\sqrt[3]{\lim\limits_{(x,y)\to(-2,4)}(x^3+2y)} = 4\sqrt[3]{0} = 0$

5. $\lim\limits_{(x,y)\to(0,1)} \dfrac{x^4-(y-1)^4}{x^2-(y-1)^2} = \lim\limits_{(x,y)\to(0,1)} \dfrac{[x^2-(y-1)^2][x^2+(y-1)^2]}{x^2-(y-1)^2} = \lim\limits_{(x,y)\to(0,1)} [x^2+(y-1)^2] = 0$

6. $\lim\limits_{(x,y)\to(1,1)} \dfrac{(x-1)^{4/3}-(y-1)^{4/3}}{(x-1)^{2/3}+(y-1)^{2/3}} = \lim\limits_{(x,y)\to(1,1)} \dfrac{[(x-1)^{2/3}-(y-1)^{2/3}][(x-1)^{2/3}+(y-1)^{2/3}]}{(x-1)^{2/3}+(y+1)^{2/3}}$

$= \lim\limits_{(x,y)\to(1,1)} [(x-1)^{2/3}-(y-1)^{2/3}] = 0-0 = 0$

In Exercises 7–10, establish the limit by finding a $\delta > 0$ such that Definition 12.2.5 holds.

7. Because $3x-4y$ is defined at every point (x,y), any open disk centered at $(3,2)$ will satisfy the first requirement of Definition 12.2.5. To prove that $\lim\limits_{(x,y)\to(3,2)}(3x-4y) = 1$, we must show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$|3x-4y-1| < \epsilon$ whenever $0 < \sqrt{(x-3)^2+(y-2)^2} < \delta$ (1)

$|3x-4y-1| = |3(x-3)-4(y-2)| \le 3|x-3| + 4|y-2| \le 3\sqrt{(x-3)^2+(y-2)^2} + 4\sqrt{(x-3)^2+(y-2)^2}$

Thus $|3x-4y-1| \le 3\delta + 4\delta = 7\delta$ whenever $0 < \sqrt{(x-3)^2+(y-2)^2} < \delta$. Hence if $\delta = \frac{1}{7}\epsilon$, statement (1) holds.

8. $\lim\limits_{(x,y)\to(-2,1)}(5x+4y) = -6$

▸ Because $5x+4y$ is defined everywhere, it is defined on an open disk having $(-2,1)$ as its center. We use the Cauchy-Schwarz inequality:

$$|a_1b_1 + a_2b_2| \le \sqrt{{a_1}^2+{a_2}^2}\sqrt{{b_1}^2+{b_2}^2}$$

Thus, if

$0 < \|(x,y)-(2,-1)\| = \sqrt{(x+2)^2+(y-1)^2} < \delta$

then

$$|(5x+4y)-(-6)| = |5(x+2)+4(y-1)| \le \sqrt{5^2+4^2}\sqrt{(x+2)^2+(y-1)^2} < \sqrt{41}\,\delta$$

Therefore, if $\delta = \epsilon/\sqrt{41}$ then $|(5x+4y)-(-6)| < \epsilon$.

9. Because $3x-2y$ is defined at every point (x,y), any open disk centered at $(-1,3)$ will satisfy the first requirement of Definition 12.2.5. To prove that $\lim\limits_{(x,y)\to(0,0)}(3x-2y) = -9$, we must show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$|3x-2y+9| < \epsilon$ whenever $0 < \sqrt{(x+1)^2+(y-3)^2} < \delta$ (1)

$|3x-2y+9| = |3(x+1)-2(y-3)| \le 3|x+1| + 2|y-3| \le 3\sqrt{(x+1)^2+(y-3)^2} + 2\sqrt{(x+1)^2+(y-3)^2}$

Thus $|3x-2y+9| \le 3\delta + 2\delta = 5\delta$ whenever $0 < \sqrt{(x+2)^2+(y-3)^2} < \delta$. Thus, if $\delta = \frac{1}{5}\epsilon$, statement (1) holds.

10. Because $5x-3y$ is defined at every point (x,y), any open disk centered at $(2,4)$ will satisfy the first requirement of Definition 12.2.5. If $0 < \sqrt{(x-2)^2+(y-4)^2} < \delta$, then by the Cauchy-Schwarz inequality

$|(5x-3y)-(-2)| = |5(x-2)-3(y-4)| \le \sqrt{5^2+3^2}\sqrt{(x+2)^2+(y-4)^2} < \sqrt{34}\delta$

Therefore, if $\delta = \epsilon/\sqrt{34}$ then $|(5x-3y)-(-2)| < \epsilon$.

In Exercises 11–16, prove that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

11. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(y=0)}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0}\frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$; $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(x=0)}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0}\frac{-y^2}{y^2} = \lim_{y\to 0}(-1) = -1$

Because these limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

12. $f(x,y) = \dfrac{x^2}{x^2+y^2}$

▹ We use Theorem 12.2.10 with $S_1 = \{(x,y) \mid y = 0\}$ and $S_2 = \{(x,y) \mid x = 0\}$.

$$\lim_{\substack{(x,y)\to(0,0)\\(y=0)}} \frac{x^2}{x^2+y^2} = \lim_{x\to 0}\frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$$

$$\lim_{\substack{(x,y)\to(0,0)\\(x=0)}} \frac{x^2}{x^2+y^2} = \lim_{y\to 0}\frac{0}{y^2} = \lim_{y\to 0} 0 = 0$$

Because these limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist.

13. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(y=0)}} \frac{x^4y^4}{(x^2+y^4)^3} = \lim_{x\to 0}\frac{0}{x^6} = \lim 0 = 0$; $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(x=y^2)}} \frac{x^4y^4}{(x^2+y^4)^3} = \lim_{y\to 0}\frac{y^8y^4}{(y^4+y^4)^3} = \lim_{y\to 0}\frac{1}{8} = \frac{1}{8}$

Because these limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$ does not exist.

14. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(y=0)}} \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2} = \lim_{x\to 0}\frac{x^4}{x^4} = \lim_{x\to 0} 1 = 1$; $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(x=0)}} \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2} = \lim_{y\to 0} 0 = 0$

Because these limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}$ does not exist.

15. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(y=0)}} \frac{x^9y}{(x^6+y^2)^2} = \lim_{x\to 0}\frac{0}{x^{12}} = 0$; $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(y=x^3)}} \frac{x^9y}{(x^6+x^6)^2} = \lim_{x\to 0}\frac{x^{12}}{4x^{12}} = \lim_{x\to 0}\frac{1}{4} = \frac{1}{4}$

Because these limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^9y}{(x^6+y^2)^2}$ does not exist.

16. $f(x,y) = \dfrac{x^2y^2}{x^4+y^4}$

▹ We use Theorem 12.2.10 with $S_1 = \{(x,y) \mid y = 0\}$ and $S_2 = \{(x,y) \mid y = x\}$.

$$\lim_{\substack{(x,y)\to(0,0)\\(y=0)}} \frac{x^2y^2}{x^4+y^4} = \lim_{x\to 0}\frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\(y=x)}} \frac{x^2y^2}{x^4+y^4} = \lim_{x\to 0}\frac{x^4}{2x^4} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$$

Because these limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$ does not exist.

In Exercises 17–20, prove that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

17. To prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y+xy^2}{x^2+y^2} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{x^2y+xy^2}{x^2+y^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \qquad (1)$$

$\left|\dfrac{x^2y+xy^2}{x^2+y^2}\right| \le \dfrac{x^2|y|+|x|y^2}{x^2+y^2} \le \dfrac{(x^2+y^2)\sqrt{x^2+y^2}}{x^2+y^2} = \sqrt{x^2+y^2}$. If $\delta = \epsilon$, then statement (1) holds.

18. To prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \qquad (1)$$

$\left|\dfrac{x^3+y^3}{x^2+y^2}\right| \le \dfrac{x^2|x|+|y|y^2}{x^2+y^2} \le \dfrac{(x^2+y^2)\sqrt{x^2+y^2}}{x^2+y^2} = \sqrt{x^2+y^2}$. If $\delta = \epsilon$, then statement (1) holds.

19. To prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \qquad (1)$$

$\left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| = \dfrac{|x||y|}{\sqrt{x^2+y^2}} \le \dfrac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$. Hence, if $\delta = \epsilon$, then statement (1) holds.

20. $f(x,y) = \dfrac{x^2+2xy}{\sqrt{x^2+y^2}}$

$$\lim_{\substack{(x,y)\to(0,0)\\(x=0)}} \frac{x^2+2xy}{\sqrt{x^2+y^2}} = \lim_{y\to 0} \frac{0}{|y|} = 0$$

Therefore, if $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists, then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. We use Definition 12.2.5 to prove that the limit is 0. For any $\epsilon > 0$, we must find a $\delta > 0$ such that

$$\left|\frac{x^2+2xy}{\sqrt{x^2+y^2}}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \qquad (1)$$

Because $|x| \le \sqrt{x^2+y^2}$ and $|y| \le \sqrt{x^2+y^2}$ and by the triangle inequality $|x+2y| \le |x|+|2y|$, then

$$\left|\frac{x^2+2xy}{\sqrt{x^2+y^2}}\right| = \frac{|x||x+2y|}{\sqrt{x^2+y^2}} \le \frac{\sqrt{x^2+y^2}\cdot 3\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 3\sqrt{x^2+y^2}$$

Hence, if $\delta = \frac{1}{3}\epsilon$, then statement (1) holds.

In Exercises 21–24, determine if the limit exists.

21. To prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{x^2y^2}{x^2+y^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \qquad (1)$$

$\left|\dfrac{x^2y^2}{x^2+y^2}\right| = \dfrac{x^2y^2}{x^2+y^2} \le \dfrac{(x^2+y^2)(x^2+y^2)}{x^2+y^2} = x^2+y^2$. Therefore if $\delta = \sqrt{\epsilon}$, statement (1) holds.

22. Because $y^4 \le x^4+y^4$, then $0 \le \dfrac{x^2y^4}{x^4+y^4} \le x^2$. Because $\lim\limits_{(x,y)\to(0,0)} x^2 = 0$, then by the squeeze theorem $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^4}{x^4+y^4} = 0$.

23. $\lim\limits_{\substack{(x,y)\to(0,0)\\(x=0)}} \dfrac{x^2+y}{x^2+y^2} = \lim\limits_{y\to 0} \dfrac{y}{y^2} = \lim\limits_{y\to 0} \dfrac{1}{y}$ does not exist. Therefore $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y}{x^2+y^2}$ does not exist.

24. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3}$

▷ Let $f(x,y) = \dfrac{x^2y^2}{x^3+y^3}$. Because $x^3+y^3 = (x+y)(x^2-xy+x^2)$, the domain of consists of all points of $\mathbb{R}^2$ except those on the line $x+y=0$. Therefore f is not defined on any open disk centered at the origin and so the limit does not exist according to Definition 12.2.5. To show the limit does not exist using the advanced definition, we use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\frac{x^2y^2}{x^3+y^3} = \frac{(r^2\cos^2\theta)(r^2\sin^2\theta)}{r^3\cos^3\theta + r^3\sin^3\theta} = r\frac{\cos^2\theta\sin^2\theta}{\cos^3\theta+\sin^3\theta}$$

Because for any fixed value of r, no matter how small, the value of this expression is arbitrarily large when θ is near $\frac{3}{4}\pi$, then $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

In Exercises 25–28, show the application of Theorem 12.2.6 to find the limit.

25. Because $\tan^{-1}t$ is continuous at $t=1$ it follows by Theorem 12.2.6 that

$$\lim_{(x,y)\to(2,2)} \tan^{-1}\frac{y}{x} = \tan^{-1}\Big(\lim_{(x,y)\to(2,2)} \frac{y}{x}\Big) = \tan^{-1}\frac{2}{2} = \tan^{-1}1 = \tfrac{1}{4}\pi$$

26. Because $\displaystyle\lim_{(x,y)\to(\ln 3,\ln 2)} (x-y) = \ln 3 - \ln 2 = \ln\tfrac{3}{2}$ and e^t is continuous at $t = \ln\frac{3}{2}$, it follows by Theorem 12.2.6 that $\displaystyle\lim_{(x,y)\to(\ln 3,\ln 2)} e^{x-y} = \exp\Big(\lim_{(x,y)\to(\ln 3,\ln 2)} (x-y)\Big) = \exp(\ln\tfrac{3}{2}) = \tfrac{3}{2}$

27. Because $\sqrt{t}$ is continuous at $t = \frac{1}{4}$ it follows by Theorem 12.2.6 that

$$\lim_{(x,y)\to(4,2)} \sqrt{\frac{1}{3x-4y}} = \sqrt{\lim_{(x,y)\to(4,2)} \frac{1}{3x-4y}} = \sqrt{\frac{1}{3\cdot 4 - 4\cdot 2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

28. $\displaystyle\lim_{(x,y)\to(-2,3)} [\![5x + \tfrac{1}{2}y^2]\!]$

▷ Because

$$\lim_{(x,y)\to(-2,3)} (5x+\tfrac{1}{2}y^2) = -10 + \tfrac{9}{2} = -\tfrac{11}{2}$$

and $[\![t]\!]$ is continuous at any noninteger, and hence at $-\frac{11}{2}$, it follows by Theorem 12.2.6 that

$$\lim_{(x,y)\to(-2,3)} [\![5x+\tfrac{1}{2}y^2]\!] = [\![\lim_{(x,y)\to(-2,3)} (5x+\tfrac{1}{2}y^2)]\!] = [\![-\tfrac{11}{2}]\!] = -6$$

In Exercises 29–52, determine all points at which the function is continuous.

29. $f(x,y) = \dfrac{x^2}{y-1}$. Because f is a rational function, then by Theorem 12.2.15, f is continuous at every point in its domain, which is $\{(x,y) \mid y \neq 1\}$. Thus f is continuous at every point (x,y) in $\mathbb{R}^2$ that is not on line $y=1$.

30. $F(x,y) = \dfrac{1}{x-y}$ is continuous at every point (x,y) in $\mathbb{R}^2$ that is not on the line $y=x$.

31. $h(x,y) = \sin\frac{y}{x}$. Let $g(x,y) = \frac{y}{x}$. Because g is a rational function, then by Theorem 12.2.15 it is continuous at every point in its domain which is $\{(x,y) \mid x \neq 0\}$. Let $f(t) = \sin t$. Because the sine function is continuous everywhere, then f is continuous at all t. The function h is the composite function $f\circ g$, which by Theorem 12.2.17 is continuous at every point (x,y) in $\mathbb{R}^2$ that is not on the y axis.

32. $f(x,y) = \ln xy^2$

▷ Because $xy^2 > 0$ if $x > 0$ and $y \neq 0$, and the natural logarithmic function is continuous at every positive number, f is continuous at every point in the first and fourth quadrant.

33. $f(x,y) = \dfrac{4x^2y + 3y^2}{2x-y}$. Because f is a rational function, then by Theorem 12.2.15, f is continuous at every point in its domain, which is $\{(x,y) \mid y \neq 2x\}$. Thus, f is continuous at every point (x,y) in $\mathbb{R}^2$ not on line $y=2x$.

34. $g(x,y) = \dfrac{5xy^2+2y}{16-x^2-4y^2}$ is continuous except if $16 - x^2 - 4y^2 = 0$, that is, except on the ellipse $4x^2+y^2=16$.

35. $g(x,y) = \ln(25 - x^2 - y^2)$. Let $h(x,y) = 25 - x^2 - y^2$. Because h is a polynomial function, then by Theorem 12.2.14 it is continuous at every point in R^2. Let $f(t) = \ln t$. Because the natural logarithmic function is continuous at all positive numbers, then f is continuous at all $t > 0$. The function g is the composite function $f \circ g$, which by Theorem 12.2.17 is continuous at every point (x,y) in R^2 inside the circle $x^2 + y^2 = 25$.

36. $f(x,y) = \cos^{-1}(x+y)$

▷ The inverse cosine function is continuous at every point in the open interval $(-1,1)$. Thus, f is continuous at every point in the strip $\{(x,y) \mid -1 < x+y < 1\}$.

37. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. If $(x,y) \neq (0,0)$ f is continuous at (x,y). From Exercise 19 we have

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0)$. Thus, f is continuous at $(0,0)$. Hence, f is continuous at every point in R^2.

38. $h(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

▷ From Example 7, $\lim\limits_{(x,y)\to(0,0)} h(x,y)$ does not exist. Hence f is continuous at every point of R^2 except $(0,0)$.

39. $f(x,y) = \begin{cases} \dfrac{x+y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. If $(x,y) \neq (0,0)$ f is continuous at (x,y). Test if f is continuous at $(0,0)$.

$\lim\limits_{\substack{(x,y)\to(0,0)\\(y=x)}} f(x,y) = \lim\limits_{\substack{(x,y)\to(0,0)\\(y=x)}} \dfrac{x+y}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{2x}{2x^2} = \lim\limits_{x\to 0} \dfrac{1}{x}$. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist because this limit does not exist. Hence f is not continuous at $(0,0)$. Thus f is continuous at every point $(x,y) \neq (0,0)$ in R^2.

40. $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

▷ In Exercise 18 it was shown that

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = 0$

Because $f(0,0) = 0$, then the three conditions of Definition 12.2.12 are satisfied and hence f is continuous at $(0,0)$. If $(x_0, y_0) \neq (0,0)$, then by Theorem 12.2.15, f is continuous at (x_0, y_0). Therefore, f is continuous everywhere.

41. $g(x,y) = \begin{cases} \dfrac{xy}{|x|+|y|} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. If $(x,y) \neq (0,0)$ G is continuous at (x,y). To show that G is continuous at $(0,0)$ we must show that $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ exists and is equal to $g(0,0) = 0$ which is done by showing that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{xy}{|x|+|y|}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta \qquad (1)$$

$$\left|\frac{xy}{|x|+|y|}\right| \leq \frac{|x||y|}{|x|} = |y| \leq \sqrt{x^2+y^2}$$

Therefore, if $\delta = \epsilon$ statement (1) holds. Hence G is continuous at every point in R^2.

42. $F(x,y)=\begin{cases}\dfrac{x^2y^2}{|x^3|+|y^3|} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$. Because $x^2=|x^3|^{2/3}\le(|x^3|+|y^3|)^{2/3}$ and $y^2\le(|x^3|+|y^3|)^{2/3}$, then

$$0\le\frac{x^2y^2}{|x^3|+|y^3|}\le\frac{(|x^3|+|y^3|)^{4/3}}{|x^3|+|y^3|}=(|x^3|+|y^3|)^{1/3}\to 0 \text{ as } (x,y)\to(0,0)$$

By the squeeze theorem, F is continuous at $(0,0)$ and hence at every point in R^2. Contrast with Exercise 24.

43. $f(x,y)=\dfrac{xy}{\sqrt{16-x^2-y^2}}$. By Thms 12.2.15 and 12.2.17 the region of continuity of f is the set of all points (x,y) in R^2 for which $16-x^2-y^2>0$. This is the set of all points in R^2 that are inside the circle $x^2+y^2=16$.

44. $f(x,y)=\dfrac{y}{\sqrt{x^2-y^2-4}}$

▹ Because $g(x,y)=y$ is continuous at all points in R^2 and $h(x,y)=\sqrt{x^2-y^2-4}$ is continuous at all (x,y) for which

$x^2-y^2-4>0$

$x^2-y^2>4$

then, by Theorem 12.2.13(iv), f is continuous at all points in the interior of the hyperbola $x^2-y^2=4$.

45. $f(x,y)=\dfrac{z}{\sqrt{4x^2+9y^2-36}}$. By Thms 12.2.15 and 12.2.17 the region of continuity of f is the set of all points (x,y) in R^2 for which $4x^2+9y^2-36>0$, the set of all points in R^2 that are outside the ellipse $4x^2+9y^2=36$

46. $f(x,y)=\dfrac{x^2+y^2}{\sqrt{9-x^2-y^2}}$ is continuous if $9-x^2-y^2>0$, that is, inside the circle $x^2+y^2=9$.

47. $f(x,y)=\sec^{-1}(xy)$. If $g(t)=\sec^{-1}t$, the domain of g is $\{t\,|\,|t|\ge 1\}$, and g is continuous on its domain. Thus the region of continuity of f is the set of all points (x,y) in R^2 for which $|xy|\ge 1$.

48. $f(x,y)=\ln(x^2+y^2-9)-\ln(1-x^2-y^2)$

▹ Because the natural logarithmic function is defined for positive numbers only, $f(x,y)$ is defined only when both

$x^2+y^2-9>0$ and $1-x^2-y^2>0$

or, equivalently, only when both

$x^2+y^2>9$ and $x^2+y^2<1$

which is impossible. Thus, f is continuous at no point of R^2.

Note that the function $\ln[(x^2+y^2-9)/(1-x^2-y^2)]$ is continuous in the annulus $\{(x,y)\,|\,1<x^2+y^2<9\}$.

49. $f(x,y)=\sin^{-1}(x+y)+\ln(xy)$

▹ If $g(t)=\sin^{-1}t$, the domain of g is $\{t\,|\,|t|\le 1\}$ and g is continuous on its domain. If $h(t)=\ln t$, the domain of h is $\{t\,|\,t>0\}$, and h is continuous on its domain. Therefore f is continuous at all points (x,y) in R^2 for which $|x+y|\le 1$ and $xy>0$. These are all the points in the first and third quadrants for which $|x+y|\le 1$.

50. $f(x,y)=\sin^{-1}(xy)$ is continuous if $|xy|<1$.

51. $f(x,y)=\begin{cases}\dfrac{\sin(x+y)}{x+y} & \text{if } x+y\neq 0\\ 1 & \text{if } x+y=0\end{cases}$. Let $h(t)=\begin{cases}\dfrac{\sin t}{t} & \text{if } t\neq 0\\ 1 & \text{if } t=0\end{cases}$ and let $g(x,y)=x+y$. Then $f(x,y)=(h\circ g)(x,y)$. Observe that $\lim\limits_{t\to 0}h(t)=\lim\limits_{t\to 0}\frac{\sin t}{t}=1=h(0)$. Thus h is continuous at 0, and continuous everywhere else. The domain of g is R^2 and g is continuous on its domain. Therefore by Theorem 12.2.17 f is continuous at all points in R^2.

52. $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x - y} & \text{if } x \neq y \\ x - y & \text{if } x = y \end{cases}$

▸ By Theorem 12.2.15, f is continuous at all points not on the line $y = x$.
Let (x_0, y_0) be a point such that $x_0 = y_0$. Let S_1 be the line $y = x$ and S_2 the rest of R^2.

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ (P \text{ in } S_1)}} f(x,y) = \lim_{x\to 0}(x - x) = 0$$

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ (P \text{ in } S_2)}} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(x_0,y_0)} (x + y) = x_0 + x_0 = 2x_0$$

Thus, f is continuous if $x_0 = 0$, that is at the origin, and f is discontinuous at any other point of line $y = x$.

In Exercises 53–59, f is discontinuous at the origin because $f(0,0)$ does not exist. Determine if the discontinuity is removable or essential. If the discontinuity is removable, redefine $f(0,0)$ so that the new function is continuous.

53. $f(x,y) = \dfrac{xy}{x^2 + xy + y^2}$

▸ $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (y=x)}} \frac{xy}{x^2 + xy + y^2} = \lim_{x\to 0}\frac{x^2}{3x^2} = \lim_{x\to 0}\frac{1}{3} = \frac{1}{3}$; $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (y=-x)}} \frac{xy}{x^2 + xy + y^2} = \lim_{x\to 0}\frac{-x^2}{x^2} = \lim_{x\to 0} -1 = -1$

Because these limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + xy + y^2}$ does not exist. Therefore, the discontinuity is essential.

54. $f(x,y) = \dfrac{x}{x^2 + y^2}$. $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (y=0)}} \frac{x}{x^2 + y^2} = \lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0}\frac{1}{x}$ does not exist. Thus the discontinuity is essential.

55. $f(x,y) = (x + y)\sin\dfrac{x}{x^2 + y^2}$. Because $|\sin t| \leq 1$, then $0 \leq |f(x,y)| \leq |x + y|$. Because $\displaystyle\lim_{(x,y)\to(0,0)} (x + y) = 0$, it follows from the squeeze theorem that $\lim f(x,y) = 0$; if we define $f(0,0) = 0$ the discontinuity is removed.

56. $f(x,y) = \dfrac{x^2y^2}{x^2 + y^2}$

▸ In Exercise 21 it was shown that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Hence the discontinuity at the origin may be removed by defining $f(0,0) = 0$.

57. $f(x,y) = \dfrac{x^3y^2}{x^6 + y^4}$. $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (y=0)}} \frac{x^3y^2}{x^6 + y^4} = \lim_{x\to 0}\frac{0}{x^6} = \lim_{x\to 0} 0 = 0$; $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (y=x^{3/2})}} \frac{x^3y^2}{x^6 + y^4} = \lim_{x\to 0}\frac{x^6}{x^6 + x^6} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$

Because these two limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6 + y^4}$ does not exist. Thus, the discontinuity is essential.

58. $f(x,y) = \dfrac{2y^2 - 3xy}{\sqrt{x^2 + y^2}}$. $0 \leq |f(x,y)| = \dfrac{|2y^2 - 3xy|}{\sqrt{x^2 + y^2}} = |y|\dfrac{|2|y| + 3x|}{\sqrt{x^2 + y^2}} \leq |y|\dfrac{5\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} = |y| \to 0$.

If we define $f(0,0) = 0$ the discontinuity is removed.

59. $f(x,y) = \dfrac{x^3 - 4xy^2}{x^2 + y^2}$. $0 \leq |f(x,y)| = \left|\dfrac{x^3 - 4xy^2}{x^2 + y^2}\right| \leq |x|\dfrac{x^2 + 4y^2}{x^2 + y^2} \leq |x|\dfrac{4x^2 + 4y^2}{x^2 + y^2} = 4|x|$

From the squeeze theorem we get $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$. If we define $f(0,0) = 0$, the discontinuity is removed.

60. (a) Give a definition, similar to Definition 12.2.5, of the limit of a function of three variables as a point (x, y, z) approaches a point (x_0, y_0, z_0).
(b) Give a definition, similar to Definition 12.2.8, of the limit of a function of three variables as a point (x, y, z) approaches a point (x_0, y_0, z_0) in a specific set of points S in R^3.

▸ (a) Let f be a function of three variables that is defined on some open disk $B((x_0, y_0, z_0); r)$, except possibly at the point (x_0, y_0, z_0) itself. Then

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

if $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta$ then $|f(x,y,z)) - L| < \epsilon$

(b) Let f be a function defined on a set of points in R^3 and let (x_0, y_0, z_0) be an accumulation point of S. Then

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\(P \text{ in } S)}} f(x,y,z) = L$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that

if $0 < \|(x,y,z) - (x_0,y_0,z_0)\| < \delta$ and $(x,y,z) \in S$ then $|f(x,y,z) - L| < \epsilon$

61. (a) A theorem similar to Theorem 12.2.9 for a function of three variables is:

Suppose that the function f is defined for all points on an open ball having its center at (x_0, y_0, z_0) except possibly at (x_0, y_0, z_0) itself, and $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$.

If S is any set of points in R^3 having (x_0, y_0, z_0) as an accumulation point, then

$\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\(P \text{ in } S)}} f(x,y,z)$ exists and always has the value L.

(b) A theorem similar to Theorem 12.2.10 for a function of three variables is:

If the function f has different limits as (x,y,z) approaches (x_0, y_0, z_0) through two distinct sets of points having (x_0, y_0, z_0) as an accumulation point, then $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$ does not exist.

In Exercises 62–65, use definitions and theorems of Ex. 60–61 to prove that $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$ does not exist.

62. $f(x,y,z) = \dfrac{x^3 + yz^2}{x^4 + y^2 + z^4}$

▹ $\lim_{\substack{(x,y,z)\to(0,0,0)\\y=z=0}} f(x,y,z) = \lim_{x\to 0} \dfrac{x^3}{x^4} = \lim_{x\to 0} \dfrac{1}{x}$ does not exist. Then $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$ does not exist.

63. $\lim_{\substack{(x,y,z)\to(0,0,0)\\(y=0,\, z=0)}} \dfrac{x^2 + y^2 - z^2}{x^2 + y^2 + z^2} = \lim_{x\to 0} \dfrac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$; $\lim_{\substack{(x,y,z)\to(0,0,0)\\(x=0,\, y=0)}} \dfrac{x^2 + y^2 - z^2}{x^2 + y^2 + z^2} = \lim_{z\to 0} \dfrac{-z^2}{z^2} = \lim_{z\to 0}(-1) = -1$

Because these two limits are not equal, $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2 + y^2 - z^2}{x^2 + y^2 + z^2}$ does not exist.

64. $f(x,y,z) = \dfrac{x^4 + yx^3 + z^2x^2}{x^4 + y^4 + z^4}$

▹ $(0,0,0)$ is an accumulation point of the sets $\{(x,y,z) \mid x = 0\}$ and $\{(x,y,z) \mid x = y = z\}$. Furthermore,

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\x=0}} f(x,y,z) = \lim_{(y,z)\to(0,0)} \frac{0}{y^4 + z^4} = 0$$

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\x=y=z}} f(x,y,z) = \lim_{x\to 0} \frac{3x^4}{3x^4} = 1$$

Because these limits are not equal, then $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$ does not exist.

65. $\lim_{\substack{(x,y,z)\to(0,0,0)\\(y=0,\, z=0)}} \dfrac{x^2y^2z^2}{x^6 + y^6 + z^6} = \lim_{x\to 0} \dfrac{0}{x^6} = \lim_{x\to 0} 0 = 0$; $\lim_{\substack{(x,y,z)\to(0,0,0)\\(x=y=z)}} \dfrac{x^2y^2z^2}{x^6 + y^6 + z^6} = \lim_{x\to 0} \dfrac{x^6}{3x^6} = \lim_{x\to 0} \dfrac{1}{3} = \dfrac{1}{3}$

Because these two limits are not equal, $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2y^2z^2}{x^6 + y^6 + z^6}$ does not exist.

In Exercises 66 and 67, use the definition in Exercise 60(a) to prove that $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$ exists.

66. To prove $\lim_{(x,y,z)\to(0,0,0)} \dfrac{y^3 + xz^2}{x^2 + y^2 + z^2} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{y^3 + xz^2}{x^2 + y^2 + z^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2 + y^2 + z^2} < \delta \tag{1}$$

$$\left|\frac{y^3 + xz^2}{x^2 + y^2 + z^2}\right| \le \frac{|y^2|^{3/2} + |x|z^2}{x^2 + y^2 + z^2} \le \frac{2(x^2 + y^2 + z^2)^{3/2}}{x^2 + y^2 + z^2} = 2\sqrt{x^2 + y^2 + z^2} < 2\delta$$

Hence if $\delta = \frac{1}{2}\epsilon$, then statement (1) holds.

67. To prove $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}} = 0$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2+y^2+z^2} < \delta \qquad (1)$$

$$\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right| \le \frac{|x||y|+|x||y|+|y||z|}{\sqrt{x^2+y^2+z^2}} \le \frac{3\sqrt{(x^2+y^2+z^2)^2}}{\sqrt{x^2+y^2+z^2}} = 3\sqrt{x^2+y^2+z^2}$$

Hence if $\delta = \frac{1}{3}\epsilon$, then statement (1) holds.

68. (a) The function f of three variables x, y, z is said to be continuous at the point (x_0, y_0, z_0) if and only if the following three conditions are satisfied:
(i) $f(x_0, y_0, z_0)$ exists; (ii) $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$ exists; (iii) $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$
(b) If f and g are two functions continuous at the point (x_0, y_0, z_0), then
(i) $f+g$ is continuous at (x_0, y_0, z_0); (ii) $f-g$ is continuous at (x_0, y_0, z_0); (iii) fg is continuous at (x_0, y_0, z_0); (iv) f/g is continuous at (x_0, y_0, z_0), provided that $g(x_0, y_0, z_0) \neq 0$.

(c) Suppose that f is a function of a single variable and g is a function of three variables such that g is continuous at (x_0, y_0, z_0) and f is continuous at $g(x_0, y_0, z_0)$. Then the composite function $f \circ g$ is continuous at (x_0, y_0, z_0).

In Exercises 69–72, use the definitions and theorems of Exercise 68 to determine where the function is continuous.

69. $f(x,y,z) = \dfrac{xz}{\sqrt{x^2+y^2+z^2-1}}$ By Ex. 68(b)(iv) and (c), the region of continuity of f is the set of all points (x,y,z) in R^3 for which $x^2+y^2+z^2-1>0$, the set of all points in R^3 outside the sphere $x^2+y^2+z^2=1$.

70. $f(x,y,z) = \ln(36-4x^2-y^2-9z^2)$. Applying Ex. 68(c) to the natural logarithmic function, we see that f is continuous when $36-4x^2-y^2-9z^2<0$, that is, for points inside the ellipsoid $4x^2-y^2-9z^2=36$.

71. $f(x,y,z) = \begin{cases} \dfrac{3xyz}{x^2+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$

The function f is continuous at every point (x,y,z) in R^3 for which $(x,y,z) \neq (0,0,0)$ because it is a rational function. We show that f is also continuous at (0,0,0) by proving that $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = 0$.

$$0 \le \left|\frac{3xyz}{x^2+y^2+z^2}\right| = \frac{3|x||y||z|}{x^2+y^2+z^2} \le \frac{3(\sqrt{x^2+y^2+z^2})^3}{x^2+y^2+z^2} = 3\sqrt{x^2+y^2+z^2}$$

Because $\lim_{(x,y,z)\to(0,0,0)} (x^2+y^2+z^2) = 0$, it follows by the squeeze theorem that $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = 0$. So, f is continuous at all points in R^3.

72. $f(x,y,z) = \begin{cases} \dfrac{xz-y^2}{x^2+y^2+z^2} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$

The function f is continuous at every point (x,y,z) in R^3 for which $(x,y,z) \neq (0,0,0)$ because it is a rational function. We show that f is not continuous at $(0,0,0)$ by considering the sets $\{(x,y,z) \mid y=0,\ z=0\}$ and $\{(x,y,z) \mid y=0,\ z=x\}$, each of which has $(0,0,0)$ as an accumulation point.

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\(y=0,\,z=0)}} f(x,y,z) = \lim_{x\to 0} \frac{0}{x^2} = 0 \qquad \lim_{\substack{(x,y,z)\to(0,0,0)\\(y=0,\,z=x)}} f(x,y,z) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Because these limits are not equal, f is continuous everywhere except $(0,0,0)$.

73. $g(x,y) = \begin{cases} x^2+4y^2 & \text{if } x^2+4y^2 \le 5 \\ 3 & \text{if } x^2+4y^2 > 5 \end{cases}$

If ${x_0}^2 + 4{y_0}^2 < 5$ then $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} (x^2+4y^2) = {x_0}^2 + 4{y_0}^2 = g(x_0,y_0)$.

If ${x_0}^2 + 4{y_0}^2 > 5$ then $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} 3 = g(x_0,y_0)$.

Thus, G is continuous at all points (x_0,y_0) for which ${x_0}^2 + 4{y_0}^2 \neq 5$.

Consider the points (x_0,y_0) for which ${x_0}^2 + 4{y_0}^2 = 5$.

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\(x^2+4y^2=5)}} g(x,y) = \lim_{\substack{(x,y)\to(x_0,y_0)\\(x^2+4y^2=5)}} (x^2+4y^2) = \lim_{(x,y)\to(x_0,y_0)} 5 = 5; \quad \lim_{\substack{(x,y)\to(x_0,y_0)\\(x^2+4y^2>5)}} g(x,y) = \lim_{(x,y)\to(x_0,y_0)} 3 = 3$$

Because these two limits are not equal we conclude that $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)$ does not exist. Thus G is discontinuous at all points (x_0,y_0) for which ${x_0}^2 + 4{y_0}^2 = 5$. Hence G is continuous at all points of R^2 except those on the ellipse $x^2 + 4y^2 = 5$.

74. $F(x,y) = \begin{cases} x^2-3y^2 & \text{if } x^2-3y^2 \le 1 \\ 2 & \text{if } x^2-3y^2 > 1 \end{cases}$. If (x,y) is inside the hyperbola x^2-3y^2, then F is continuous because it is a polynomial; if (x,y) is outside, it is continuous because it is a constant. For (x_0,y_0) on the hyperbola, the limit from points inside is 1 while the limit from points outside is 2; thus the limit does not exist.

75. $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=0)}} \dfrac{f(x,y)}{g(x,y)} = \lim\limits_{x\to0} \dfrac{f(x,0)}{g(x,0)} = \lim\limits_{x\to0} \dfrac{x^n f(1,0)}{x^n g(1,0)} = \dfrac{f(1,0)}{g(1,0)}$ which exists by (ii)

$\lim\limits_{\substack{(x,y)\to(0,0)\\(y=x)}} \dfrac{f(x,y)}{g(x,y)} = \lim\limits_{x\to0} \dfrac{f(x,x)}{g(x,x)} = \lim\limits_{x\to0} \dfrac{x^n f(1,1)}{x^n g(1,1)} = \dfrac{f(1,1)}{g(1,1)}$ which also exists. But by (iii) the limits are unequal.

12.3 PARTIAL DERIVATIVES

12.3.1 Definition Let f be a function of two variables, x and y. The *partial derivative of f with respect to x* is that function, denoted by D_1f, such that its function value at any point (x,y) in the domain of f is given by

$$D_1f(x,y) = \lim_{\Delta x\to0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x} = \lim_{u\to x} \frac{f(u,y)-f(x,y)}{u-x}$$

if this limit exists. Similarly, the *partial derivative of f with respect to y* is that function, denoted by D_2f such that its function value at any point (x,y) in the domain of f is given by

$$D_2f(x,y) = \lim_{\Delta y\to0} \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y} = \lim_{v\to y} \frac{f(x,v)-f(x,y)}{v-y}$$

if this limit exists.

There are various notations for partial derivative function values. If $z = f(x,y)$, then

$$D_1f(x,y) = f_1(x,y) = f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} \quad\text{and}\quad D_2f(x,y) = f_2(x,y) = f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

We do not usually have to apply Definition 12.3.1 to find the partial derivatives of a function. To find $D_1f(x,y)$, differentiate with respect to x, with y regarded as a constant, and use the differentiation formulas for ordinary derivatives. To find $D_2f(x,y)$, differentiate with respect to y, with x regarded as a constant, and use the differentiation formulas for ordinary derivatives.

The line tangent to the curve that is the intersection of the surface $z = f(x,y)$ and the plane $x = x_0$ at the point $(x_0,y_0,f(x_0,y_0))$ has $[0,1,D_2f(x_0,y_0)]$ for a set of direction numbers. Thus, equations of the line are

$x = x_0 \quad z = z_0 + D_2f(x_0,y_0)(y-y_0)$

The line tangent to the curve that is the intersection of the surface $z = f(x,y)$ and the plane $y = y_0$ at the point $(x_0,y_0,f(x_0,y_0))$ has $[1,0,D_1f(x_0,y_0)]$ for a set of direction numbers. Thus, equations of the line are

$y = y_0 \quad z = z_0 + D_1f(x_0,y_0)(x-x_0)$

Partial derivatives for functions of more variables are defined as follows.

12.3.2 Definition Let $P(x_1, x_2, \ldots, x_3)$ be a point in R^n, and let f be a function of the n variables $x_1, x_2, \ldots, x_n$. Then the partial derivative with respect to x_k is that function, denoted by $D_k f$, such that its function value at any point P in the domain of f is given by

$$D_k f(x_1, x_2, \ldots, x_n) = \lim_{\Delta x_k \to 0} \frac{f(x_1, x_2, \ldots, x_{k-1}, x_k + \Delta x_k, x_{k+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_k}$$

if this limit exists.

We may find $D_k f(x_1, x_2, \ldots, x_n)$ by using the formulas for ordinary differentiation if we regard all variables except x_k as constants and differentiate with respect to x_k.

Higher Order If f is a function of x and y, then the second partial derivatives of f obtained by first partial-differentiating f with respect to x and then partial-differentiating the result with respect to y is represented by each of the following notations.

$$D_2(D_1 f) \quad D_{12} f \quad f_{12} \quad f_{xy} \quad \frac{\partial^2 f}{\partial y \partial x}$$

Note the order of the symbols in the last form. The second partial derivative of f obtained by partial-differentiating twice with respect to x is represented by each of the following notations.

$$D_1(D_1 f) \quad D_{11} f \quad f_{11} \quad f_{xx} \quad \frac{\partial^2 f}{\partial x^2}$$

The second partial derivative of f obtained by partial-differentiating twice with respect to y is represented by each of the following notations.

$$D_2(D_2 f) \quad D_{22} f \quad f_{22} \quad f_{yy} \quad \frac{\partial^2 f}{\partial y^2}$$

There is another possible mixed partial derivative, represented by f_{21}, f_{yx}, etc., but as the following theorem states, it is often true (but not always) that $f_{21} = f_{12}$, $f_{yx} = f_{xy}$, etc.

12.3.3 Theorem (Equality of mixed partials) Suppose that f is a function of two variables x and y defined on an open disk $B((x_0, y_0); r)$ and f_x, f_y, and f_{xy} are also defined on B. Furthermore, suppose that f_{xy} is continuous at (x_0, y_0). Then $f_{yx}(x_0, y_0)$ exists and $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

A *partial differential equation* has partial derivatives with respect to more than one variable.

Laplace's Equation in R^2 is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$; in R^3 it is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. A solution is *harmonic*. See Exercises 49–53.

Exercises 12.3

In Exercises 1–6, apply definition 12.3.1 to find the partial derivative.

1. $f(x, y) = 6x + 3y - 7$

▷ $D_1 f(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(6x + 6\Delta x + 3y - 7) - (6x + 3y - 7)}{\Delta x} = \lim_{\Delta x \to 0} \frac{6\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 6 = 6$

2. $f(x, y) = 4x^2 - 3xy$. $D_1 f(x, y) = \lim_{u \to x} \frac{f(u, y) - f(x, y)}{u - x} = \lim_{u \to x} \frac{(4u^2 - 3uy) - (4x^2 - 3xy)}{u - x}$

$= \lim_{u \to x} \left[4\frac{u^2 - x^2}{u - x} - 3y\frac{u - x}{u - x} \right] = \lim_{u \to x} [4(u + x) - 3y] = 8x - 3y$

3. $f(x, y) = 3xy + 6x - y^2$.

$D_2 f(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[3x(y + \Delta y) + 6x - (y + \Delta y)^2] - (3xy + 6x - y^2)}{\Delta y}$

$= \lim_{\Delta y \to 0} \frac{3xy + 3x\Delta y + 6x - y^2 - 2y\Delta y - (\Delta y)^2 - 3xy - 6x + y^2}{\Delta y} = \lim_{\Delta y \to 0} \frac{(3x - 2y - \Delta y)\Delta y}{\Delta y}$

$= \lim_{\Delta y \to 0} (3x - 2y - \Delta y) = 3x - 2y$. Alternatively, by formula (4),

$D_2 f(x, y) = \lim_{v \to y} \frac{f(x, v) - f(x, y)}{u - y} = \lim_{v \to y} \frac{(3xv + 6x - v^2) - (3xy + 6x - y^2)}{v - y} = \lim_{v \to y} \frac{3x(v - y) - (v^2 - y^2)}{v - y}$

$= \lim_{v \to y} [3x - (v + y)] = 3x - 2y$

4. $f(x,y) = xy^2 - 5y + 6$; $D_2 f(x,y)$

▸ $$D_2 f(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[x(y+\Delta y)^2 - 5(y+\Delta y) + 6] - [xy^2 - 5y + 6]}{\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{xy^2 + 2xy\Delta y + x(\Delta y)^2 - 5y - 5\Delta y + 6 - xy^2 + 5y - 6}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y(2xy + x\Delta y - 5)}{\Delta y}$$
$$= \lim_{\Delta y \to 0} (2xy + x\Delta y - 5) = 2xy - 5$$

5. $f(x,y) = \sqrt{x^2+y^2}$. $D_1 f(x,y) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\sqrt{(x+\Delta x)^2 + y^2} - \sqrt{x^2+y^2}}{\Delta x}$
$$= \lim_{\Delta x \to 0} \frac{(\sqrt{(x+\Delta x)^2 + y^2} - \sqrt{x^2+y^2})(\sqrt{(x+\Delta x)^2 + y^2} + \sqrt{x^2+y^2})}{\Delta x(\sqrt{(x+\Delta x)^2 + y^2} + \sqrt{x^2+y^2})}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + y^2 - x^2 - y^2}{\Delta x[\sqrt{(x+\Delta x)^2 + y^2} + \sqrt{x^2+y^2}]} = \lim_{\Delta x \to 0} \frac{2x + \Delta x}{\sqrt{(x+\Delta x)^2 + y^2} + \sqrt{x^2+y^2}} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

6. $\dfrac{\partial}{\partial y} \dfrac{x+2y}{x^2-y} = \lim\limits_{v \to y} \dfrac{1}{v-y}\left(\dfrac{x+2v}{x^2-v} - \dfrac{x+2y}{x^2-y}\right) = \lim\limits_{v \to y} \dfrac{(x+2v)(x^2-y) - (x+2y)(x^2-v)}{(v-y)(x^2-v)(x^2-y)} = \lim\limits_{v \to y} \dfrac{(2x^2+x)(v-y)}{(v-y)(x^2-v)(x^2-y)}$
$$= \lim_{v \to y} \frac{2x^2+x}{(x^2-v)(x^2-y)} = \frac{2x^2+x}{(x^2-y)^2}$$

In Exercises 7–10, apply Definition 12.3.2 to find the partial derivative.

7. $f(x,y,z) = x^2y - 3xy^2 + 2yz$
$$D_2 f(x,y,z) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{x^2(y+\Delta y) - 3x(y+\Delta y)^2 + 2(y+\Delta y)z - (x^2y - 3xy^2 + 2yz)}{\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{x^2y + x^2\Delta y - 3xy^2 - 6xy\Delta y - 3x(\Delta y)^2 + 2yz + 2z\Delta y - x^2y + 3xy^2 - 2yz}{\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{(x^2 - 6xy - 3x\Delta y + 2z)\Delta y}{\Delta y} = \lim_{\Delta y \to 0} (x^2 - 6xy - 3x\Delta y + 2z) = x^2 - 6xy + 2z$$

Alternatively, $D_2 f(x,y,z) = \lim\limits_{v \to y} \dfrac{f(x,v,z) - f(x,y,z)}{v-y} = \lim\limits_{v \to y} \dfrac{x^2v - 3xv^2 + 2vz - (x^2y - 3xy^2 + 2yz)}{v-y}$
$$= \lim_{v \to y} \frac{x^2(v-y) - 3x(v^2-y^2) + 2z(v-y)}{v-y} = \lim_{v \to y} [x^2 - 3x(v+y) + 2z] = x^2 - 6xy + 2z$$

8. $f(x,y,z) = x^2 + 4y^2 + 9z^2$; $D_1 f(x,y,z)$

▸ $$D_1 f(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 + 4y^2 + 9z^2] - [x^2 + 4y^2 + 9z^2]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

9. $f(x,y,z,r,t) = xyr + yzt + yrt + zrt$
$$D_4 f(x,y,z,r,t) = \lim_{\Delta r \to 0} \frac{f(x,y,z,r+\Delta r,t) - f(x,y,z,r,t)}{\Delta r}$$
$$= \lim_{\Delta r \to 0} \frac{xy(r+\Delta r) + yzt + y(r+\Delta r)t + z(r+\Delta r)t - (xyr + yzt + yrt + zrt)}{\Delta r}$$
$$= \lim_{\Delta r \to 0} \frac{xyr + xy\Delta r + yzt + yrt + yt\Delta r + zrt + zt\Delta r - xyr - yzt - yrt - zrt}{\Delta r}$$
$$= \lim_{\Delta r \to 0} \frac{(xy + yt + zt)\Delta r}{\Delta r} = \lim_{\Delta r \to 0} (xy + yt + zt) = xy + yt + zt$$

10. $\frac{\partial}{\partial v}(3r^2st + st^2v - 2tuv^2 - tvw + 3uw^2)$

$= \lim_{z\to v} \frac{(3r^2st + st^2z - 2tuz^2 - tzw + 3uw^2) - (3r^2st + st^2v - 2tuv^2 - tvw + 3uw^2)}{z - v}$

$= \lim_{z\to v}\left[st^2\frac{z-v}{z-v} - 2tu\frac{z^2-v^2}{z-v} - tw\frac{z-v}{z-v}\right] = \lim_{z\to v}[st^2 - 2tu(z+v) - tw] = st^2 - 4tuv - tw$

In Exercises 11 and 12, $f(x,y) = x^2 - 9y^2$.

11. (a) $D_1f(2,1) = \lim_{\Delta x\to 0} \frac{f(2+\Delta x,1) - f(2,1)}{\Delta x} = \lim_{\Delta x\to 0} \frac{(2+\Delta x)^2 - 9 - (-5)}{\Delta x} = \lim_{\Delta x\to 0} \frac{4 + 4\Delta x + (\Delta x)^2 - 4}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\Delta x(4+\Delta x)}{\Delta x} = \lim\,(4 + \Delta x) = 4$

(b) $D_1f(2,1) = \lim_{x\to 2} \frac{f(x,1) - f(2,1)}{x-2} = \lim_{x\to 2} \frac{x^2 - 9 - (2^2 - 9)}{x-2} = \lim_{x\to 2} \frac{x^2 - 2^2}{x-2} = \lim_{x\to 2}(x+2) = 4$

(c) $D_1f(x,y) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x,y) - f(x,y)}{\Delta x} = \lim_{\Delta x\to 0} \frac{(x+\Delta x)^2 - 9y^2 - (x^2 - 9y^2)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x\to 0}(2x + \Delta x) = 2x$. Thus at $(2,1)$, $D_1f(x,y) = 4$.

12. Find $D_2f(2,1)$ by: (a) applying formula (2), (b) applying formula (4), and (c) applying definition 12.3.1 and then replacing x and y by 2 and 1, respectively.

(a) $D_2f(2,1) = \lim_{\Delta y\to 0} \frac{f(2,1+\Delta y) - f(2,1)}{\Delta y} = \lim_{\Delta y\to 0} \frac{[4 - 9(1+\Delta y)^2] - [4-9]}{\Delta y}$

$= \lim_{\Delta y\to 0} \frac{4 - 9 - 18\Delta y - 9(\Delta y)^2 + 5}{\Delta y} = \lim_{\Delta y\to 0}(-18 - 9\Delta y) = -18$

(b) $D_2f(2,1) = \lim_{y\to 1}\frac{f(2,y) - f(2,1)}{y-1} = \lim_{y\to 1}\frac{(4-9y^2) - (-5)}{y-1} = \lim_{y\to 1}\frac{-9(y+1)(y-1)}{y-1} = \lim_{y\to 1}[-9(y+1)] = -18$

(c) $D_2f(x,y) = \lim_{\Delta y\to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y} = \lim_{\Delta y\to 0} \frac{[x^2 - 9(y+\Delta y)^2 - [x^2 - 9y^2]}{\Delta y}$

$= \lim_{\Delta y\to 0} \frac{x^2 - 9y^2 - 18y\Delta y - 9(\Delta y)^2 - x^2 + 9y^2}{\Delta y} = \lim_{\Delta y\to 0} \frac{\Delta y(-18y - 9\Delta y)}{\Delta y} = \lim_{\Delta y\to 0}(-18y - 9\Delta y) = -18y$

Thus, $D_2f(2,1) = -18(1) = -18$.

In Exercises 13–24, find the partial derivative by holding all but one of the variables constant and applying theorems for ordinary differentiation.

13. $D_1f(x,y) = \frac{\partial}{\partial x}[4y^3 + \sqrt{x^2+y^2}] = \frac{\partial}{\partial x}[4y^3 + (x^2+y^2)^{1/2}] = 0 + \frac{1}{2\sqrt{x^2+y^2}}(2x) = \frac{x}{\sqrt{x^2+y^2}}$

14. $\frac{\partial}{\partial y}\frac{x+y}{\sqrt{y^2-x^2}} = \frac{1(y^2-x^2) - \frac{1}{2}(x+y)(2y)}{(y^2-x^2)^{3/2}} = \frac{-x^2 - xy}{(y^2-x^2)^{3/2}}$

15. $D_2f(\theta,\phi) = \frac{\partial}{\partial\phi}[\sin 3\theta\cos 2\phi] = \sin 3\theta[-\sin 2\phi(2)] = -2\sin 3\theta\sin 2\phi$

16. $f(r,\theta) = r^2\cos\theta - 2r\tan\theta$; $f_\theta(r,\theta)$

▸ We differentiate with respect to θ with r regarded as a constant. Thus,
$f_\theta(r,\theta) = -r^2\sin\theta - 2r\sec^2\theta$

17. $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[e^{y/x}\ln\frac{x^2}{y}\right] = \left[e^{y/x}\left(\frac{1}{x}\right)\right]\ln\frac{x^2}{y} + e^{y/x}\left[\frac{y}{x^2}\left(-\frac{x^2}{y^2}\right)\right] = e^{y/x}\left(\frac{1}{x}\ln\frac{x^2}{y} - \frac{1}{y}\right) = \frac{e^{y/x}}{xy}\left(y\ln\frac{x^2}{y} - x\right)$

18. $\frac{\partial}{\partial\theta}e^{-\theta}\cos(\theta+\phi) = -e^{-\theta}\cos(\theta+\phi) - e^{-\theta}\sin(\theta+\phi)$

19. $\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z) = \frac{-z}{(x^2+y^2+z^2)}$

20. $u = \tan^{-1}(xyzw)$; $\dfrac{\partial u}{\partial w}$

▹ We differentiate with respect to w with x, y and z held constant. Thus,

$$\frac{\partial u}{\partial w} = \frac{1}{1+(xyzw)^2}\frac{\partial}{\partial w}(wxyz) = \frac{xyz}{1+x^2y^2z^2w^2}$$

21. $f_3(x,y,z) = \dfrac{\partial}{\partial z}[4xyz + \ln(2xyz)] = 4xy + \dfrac{1}{2xyz}(2xy) = 4xy + \frac{1}{z}$

22. $\dfrac{\partial}{\partial z}(e^{xy}\sinh 2z - e^{xy}\cosh 2z) = 2e^{xy}(\cosh 2z - \sinh 2z)$

23. $f_3(x,y,z) = \dfrac{\partial}{\partial z}\left[e^{xyz} + \tan^{-1}\left(\dfrac{3xy}{z^2}\right)\right] = xze^{xyz} + \dfrac{1}{1+\left(\dfrac{3xy}{z^2}\right)^2}\left(\dfrac{3x}{z^2}\right) = xze^{xyz} + \dfrac{3xz^2}{9x^2y^2+z^4}$

24. $f(r,\theta,\phi) = 4r^2\sin\theta + 5e^r\cos\theta\sin\phi - 2\cos\phi$; $f_2(r,\theta,\phi)$

▹ We differentiate with respect to θ with r and ϕ held constant. Thus,
$f_2(r,\theta,\phi) = 4r^2\cos\theta - 5e^r\sin\theta\sin\phi$

25. $f(r,\theta) = r\tan\theta - r^2\sin\theta$

▹ (a) $f_1(r,\theta) = \tan\theta - 2r\sin\theta$; $f_1(\sqrt{2}, \frac{1}{4}\pi) = \tan\frac{1}{4}\pi - 2\sqrt{2}\sin\frac{1}{4}\pi = 1 - 2\sqrt{2}(\frac{1}{2}\sqrt{2}) = -1$

(b) $f_2(r,\theta) = r\sec^2\theta - r^2\cos\theta$; $f_2(3,\pi) = 3\sec^2\pi - 9\cos\pi = 3(-1)^2 - 9(-1) = 12$

26. $f(x,y,z) = e^{xy^2} + \ln(y+z)$ (a) $f_1(x,y,z) = y^2e^{xy^2}$; $f_1(3,0,17) = 0$

(b) $f_2(x,y,z) = 2xye^{xy^2} + \dfrac{1}{y+z}$; $f_2(1,0,2) = \frac{1}{2}$ (c) $f_3(x,y,z) = \dfrac{1}{y+z}$; $f_3(0,0,1) = 1$

In Exercises 27 and 28, find $f_x(x,y)$ and $f_y(x,y)$.

27. $f(x,y) = \displaystyle\int_0^y \ln\sin t\,dt$; so $f_y(x,y) = \ln\sin y$. $f(x,y) = -\displaystyle\int_0^x \ln\sin t\,dt$; so $f_x(x,y) = -\ln\sin x$.

28. $f(x,y) = \displaystyle\int_x^y e^{\cos t}dt$

▹ The first fundamental theorem of the calculus gives the derivative with respect to the upper limit. If we hold x constant and differentiate with respect to y, we get
$f_y(x,y) = e^{\cos y}$
If we hold y constant and differentiate with respect to x, we must first interchange the limits of integration. Thus

$$f_x(x,y) = \frac{\partial}{\partial x}\int_y^x -e^{\cos t}dt = -e^{\cos x}$$

In Exercises 29–38 (a) find $D_{11}f(x,y)$, (b) find $D_{22}f(x,y)$, and (c) show that $D_{12}f(x,y) = D_{21}f(x,y)$.

29. $f(x,y) = \dfrac{x^2}{y} - \dfrac{y}{x^2}$

(a) $D_1f(x,y) = \dfrac{2x}{y} + \dfrac{2y}{x^3}$; $D_{11}f(x,y) = \dfrac{2}{y} - \dfrac{6y}{x^4}$

(b) $D_2f(x,y) = -\dfrac{x^2}{y^2} - \dfrac{1}{x^2}$; $D_{22}f(x,y) = \dfrac{2x^2}{y^3}$

(c) $D_{12}f(x,y) = -\dfrac{2x}{y^2} + \dfrac{2}{x^3}$; $D_{21}f(x,y) = -\dfrac{2x}{y^2} + \dfrac{2}{x^3}$

30. $f(x,y) = 2x^3 - 3x^2y + xy^2$

(a) $D_1f(x,y) = 6x^2 - 6xy + y^2$; $D_{11}f(x,y) = 12x - 6y$

(b) $D_2f(x,y) = -3x^2 + 2xy$; $D_{22}f(x,y) = 2x$

(c) $D_{12}f(x,y) = -6x + 2y$; $D_{21}f(x,y) = -6x + 2y$

31. $f(x,y) = e^{2x}\sin y$

(a) $D_1f(x,y) = 2e^{2x}\sin y$; $D_{11}f(x,y) = 4e^{2x}\sin y$

(b) $D_2f(x,y) = e^{2x}\cos y$; $D_{22}f(x,y) = -e^{2x}\sin y$

(c) $D_{12}f(x,y) = 2e^{2x}\cos y$; $D_{21}f(x,y) = 2e^{2x}\cos y$

32. $f(x,y) = e^{-x/y} + \ln\dfrac{y}{x}$

▹ Simplifying $f(x,y)$, we have, in quadrants 1 and 3,

$f(x,y) = e^{-x/y} + \ln|y| - \ln|x|$

Hence,

$$D_1f(x,y) = -\frac{1}{y}e^{-x/y} - \frac{1}{x} \tag{1}$$

and

$$D_2f(x,y) = \frac{x}{y^2}e^{-x/y} + \frac{1}{y} \tag{2}$$

(a) We partial-differentiate on both sides of (1) with respect to x.

$\mathrm{D}_{11}f(x,y) = \frac{1}{y^2}e^{-x/y} + \frac{1}{x^2}$

(b) We partial-differentiate on both sides of (2) with respect to y.

$\mathrm{D}_{22}f(x,y) = \frac{x}{y^2} \cdot \frac{x}{y^2}e^{-x/y} + e^{-x/y} \cdot \frac{-2x}{y^3} - \frac{1}{y^2} = \left(\frac{x^2}{y^4} - \frac{2x}{y^3}\right)e^{-x/y} - \frac{1}{y^2}$

(c) We partial differentiate on both sides of (1) with respect to y.

$\mathrm{D}_{12}f(x,y) = \left(-\frac{1}{y}\right)\left(\frac{x}{y^2}\right)e^{-x/y} + e^{-x/y}\left(\frac{1}{y^2}\right) = \left(\frac{1}{y^2} - \frac{x}{y^3}\right)e^{-x/y}$ (3)

We partial-differentiate on both sides of (2) with respect to x.

$\mathrm{D}_{21}f(x,y) = \left(\frac{x}{y^2}\right)\left(-\frac{1}{y}\right)e^{-x/y} + e^{-x/y}\left(\frac{1}{y^2}\right) = \left(\frac{1}{y^2} - \frac{x}{y^3}\right)e^{-x/y}$ (4)

From (3) and (4) we show that $\mathrm{D}_{12}f(x,y) = \mathrm{D}_{21}f(x,y)$.

33. $f(x,y) = (x^2 + y^2)\tan^{-1}\frac{y}{x}$

▶ (a) $\mathrm{D}_1f(x,y) = 2x \tan^{-1}\frac{y}{x} + (x^2+y^2)\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{-y}{x^2}\right) = 2x \tan^{-1}\frac{y}{x} - y$

$\mathrm{D}_{11}f(x,y) = 2 \tan^{-1}\frac{y}{x} + 2x\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{-y}{x^2}\right) = 2 \tan^{-1}\frac{y}{x} - \frac{2xy}{x^2+y^2}$

(b) $\mathrm{D}_2f(x,y) = 2y \tan^{-1}\frac{y}{x} + (x^2+y^2)\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{1}{x}\right) = 2y \tan^{-1}\frac{y}{x} + x$

$\mathrm{D}_{22}f(x,y) = 2 \tan^{-1}\frac{y}{x} + 2y\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{1}{x}\right) = 2 \tan^{-1}\frac{y}{x} + \frac{2xy}{x^2+y^2}$

(c) $\mathrm{D}_{12}f(x,y) = \frac{2x^3}{x^2+y^2}\left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - \frac{x^2+y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$

$\mathrm{D}_{21}f(x,y) = \frac{2x^2y}{x^2+y^2}\left(-\frac{y}{x^2}\right) + 1 = \frac{-2y^2}{x^2+y^2} + \frac{x^2+y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$

34. $f(x,y) = \sin^{-1}\frac{3y}{x^2}$. (a) $\mathrm{D}_1f(x,y) = \frac{1}{\sqrt{1-(3y/x^2)^2}}\left(-\frac{6y}{x^3}\right) = \frac{-6y}{x\sqrt{x^4-9y^2}}$;

$\mathrm{D}_{11}f(x,y) = \frac{6y}{x^2\sqrt{x^4-9y^2}} + \frac{12x^2y}{(x^4-9y^2)^{3/2}} = \frac{18y(x^4-3y^2)}{x^2(x^4-9y^2)^{3/2}}$

(b) $\mathrm{D}_2f(x,y) = \frac{1}{\sqrt{1-(3y/x^2)^2}} \cdot \frac{3}{x^2} = \frac{3}{\sqrt{x^4-9y^2}}$; $\mathrm{D}_{22}f(x,y) = \frac{27y}{(x^4-9y^2)^{3/2}}$

(c) $\mathrm{D}_{12}f(x,y) = \frac{-6}{x} \cdot \frac{1(x^4-9y^2) - \frac{1}{2}y(-18y)}{(x^4-9y^2)^{3/2}} = \frac{-6x^3}{(x^4-9y^2)^{3/2}} = \mathrm{D}_{21}f(x,y)$

35. $f(x,y) = 4x \sinh y + 3y \cosh x$

(a) $\mathrm{D}_1f(x,y) = 4 \sinh y + 3y \sinh x$; $\mathrm{D}_{11}f(x,y) = 3y \cosh x$

(b) $\mathrm{D}_2f(x,y) = 4x \cosh y + 3 \cosh x$; $\mathrm{D}_{22}f(x,y) = 4x \sinh y$

(c) $\mathrm{D}_{12}f(x,y) = 4 \cosh y + 3 \sinh x$; $\mathrm{D}_{21}f(x,y) = 4 \cosh y + 3 \sinh x$

36. $f(x,y) = x \cos y - ye^x$

▶ $\mathrm{D}_1f(x,y) = \cos y - ye^x$ (1)

$\mathrm{D}_2f(x,y) = -x \sin y - e^x$ (2)

(a) From (1) we obtain $\mathrm{D}_{11}f(x,y) = -ye^x$

(b) From (2), we obtain $\mathrm{D}_{22}f(x,y) = -x \cos y$

(c) From (1), we obtain $\mathrm{D}_{12}f(x,y) = -\sin y - e^x$

From (2), we obtain $\mathrm{D}_{21}f(x,y) = -\sin y - e^x$

Thus $\mathrm{D}_{12}f(x,y) = \mathrm{D}_{21}f(x,y)$.

37. $f(x,y) = e^x \cos y + \tan^{-1} x(\ln y)$

▸ (a) $D_1 f(x,y) = e^x \cos y + \dfrac{1}{1+x^2}(\ln y)$; $D_{11} f(x,y) = e^x \cos y - \dfrac{2x \ln y}{(1+x^2)^2}$

(b) $D_2 f(x,y) = -e^x \sin y + \tan^{-1} x\left(\dfrac{1}{y}\right)$; $D_{22} f(x,y) = -e^x \cos y - \dfrac{\tan^{-1} x}{y^2}$

(c) $D_{12} f(x,y) = -e^x \sin y + \dfrac{1}{y(1+x^2)}$; $D_{21} f(x,y) = -e^x \sin y + \dfrac{1}{y(1+x^2)}$

38. $f(x,y) = 3x \cosh y - y \sin^{-1} e^x$

▸ (a) $D_1 f(x,y) = 3 \cosh y - \dfrac{ye^x}{\sqrt{1-e^{2x}}}$; $D_{11} f(x,y) = -y \dfrac{e^x(1-e^{2x}) - \frac{1}{2}e^x(-2e^{2x})}{(1-e^{2x})^{3/2}} = \dfrac{-ye^x}{(1-e^{2x})^{3/2}}$

(b) $D_2 f(x,y) = 3x \sinh y - \sin^{-1} e^x$; $D_{22} f(x,y) = 3x \cosh y$ (c) $D_{12} f(x,y) = 3 \sinh y - \dfrac{e^x}{\sqrt{1-e^{2x}}} = D_{21} f(x,y)$

In Exercises 39–46, find the indicated partial derivatives.

39. $f(x,y) = 2x^3y + 5x^2y^2 - 3xy^2$

▸ (a) $f_1(x,y) = 6x^2y + 10xy^2 - 3y^2$; $f_{12}(x,y) = 6x^2 + 20xy - 6y$; $f_{121}(x,y) = 12x + 20y$

(b) $f_2(x,y) = 2x^3 + 10x^2y - 6xy$; $f_{21}(x,y) = 6x^2 + 20xy - 6y$; $f_{211}(x,y) = 12x + 20y$

40. $G(x,y) = 3x^3y^2 + 5x^2y^3 + 2x$; (a) $G_{yyx}(x,y)$; (b) $G_{yxy}(x,y)$

▸ (a) $G_y(x,y) = (G(x,y))_y = 6x^3y + 15x^2y^2$

$G_{yy}(x,y) = (G_y(x,y))_y = 6x^3 + 30x^2y$

$G_{yyx}(x,y) = (G_{yy}(x,y))_x = 18x^2 + 60xy$

(b) $G_{yx}(x,y) = (G_y(x,y))_x = 18x^2y + 30xy^2$

$G_{yxy}(x,y) = (G_{yx}(x,y))_y = 18x^2 + 60xy$

41. $f(x,y,z) = ye^x + ze^y + e^z$ (a) $f_x(x,y,z) = ye^x$; $f_{xz}(x,y,z) = 0$ (b) $f_y(x,y,z) = e^x + ze^y$; $f_{yz}(x,y,z) = e^y$

42. $g(x,y,z) = \sin xyz$ (a) $g_2(x,y,z) = xz \cos xyz$; $g_{23}(x,y,z) = x \cos xyz - x^2yz \sin xyz$

(b) $g_1(x,y,z) = yz \cos xyz$; $g_{12}(x,y,z) = z \cos xyz - xyz^2 \sin xyz$

43. $f(w,z) = w^2 \cos e^z$ (a) $f_1(w,z) = 2w \cos e^z$; $f_{12}(w,z) = -2we^z \sin e^z$; $f_{121}(w,z) = -2e^z \sin e^z$

(b) $f_2(w,z) = -w^2e^z \sin e^z$; $f_{21}(w,z) = -2we^z \sin e^z$; $f_{212}(w,z) = -2we^z \sin e^z - 2we^{2z} \cos e^z$

44. $f(u,v) = \ln \cos(u-v)$; (a) $f_{uuv}(u,v)$; (b) $f_{vuu}(u,v)$

▸ (a) $f_u(u,v) = -\dfrac{\sin(u-v)}{\cos(u-v)} = -\tan(u-v)$

$f_{uu}(u,v) = (f_u(u,v))_u = -\sec^2(u-v)$

$f_{uuv}(u,v) = (f_{uu}(u,v))_v = 2 \sec^2(u-v)\tan(u-v)$

(b) $f_v(u,v) = \dfrac{\sin(u-v)}{\cos(u-v)} = \tan(u-v)$

$f_{vu}(u,v) = (f_v(u,v))_u = \sec^2(u-v)$

$f_{vuu}(u,v) = (f_{vu}(u,v))_u = -2 \sec^2(u-v)\tan(u-v)$

45. $g(r,s,t) = \ln(r^2 + 4s^2 - 5t^2)$

▸ (a) $g_1(r,s,t) = \dfrac{2r}{r^2+4s^2-3t^2}$; $g_{13}(r,s,t) = \dfrac{20rt}{(r^2+4s^2-5t^2)^2}$; $g_{132}(r,s,t) = \dfrac{-2(20rt)(8s)}{(r^2+4s^2-5t^2)^3} = -\dfrac{320rst}{(r^2+4s^2-5t^2)^3}$

(b) $g_{12}(r,s,t) = \dfrac{-16rs}{(r^2+4s^2-5t^2)^2}$. Using the product rule,

$$g_{122}(r,s,t) = \frac{-16r}{(r^2+4s^2-5t^2)^2} + \frac{(-16rs)(-2)(8s)}{(r^2+4s^2-5t^2)^3} = \frac{-16r(r^2+4s^2-5t^2)+256rs^2}{(r^2+4s^2-5t^2)^3} = \frac{192rs^2-16r^3+80rt^2}{(r^2+4s^2-5t^2)^3}$$

46. $f(x,y,z) = \tan^{-1} 3xyz$. $f_1 = \dfrac{3yz}{1+9x^2y^2z^2}$ (a) $f_{11} = \dfrac{-54xy^3z^3}{(1+9x^2y^2z^2)^2}$; $f_{113}(x,y,z) = \dfrac{162y^3z^2(1+9x^2y^2z^2)+12z^2}{(1+9x^2y^2z^2)^3}$

(b) $f_{12}(x,y,z) = 3z\dfrac{1(1+9x^2y^2z^2) - y(18x^2yz^2)}{(1+9x^2y^2z^2)^2} = \dfrac{3z(1-9x^2y^2z^2)}{(1+9x^2y^2z^2)^2}$; $f_{123}(x,y,z) = 3\dfrac{1-54x^2y^2z^2+81x^4y^4z^4}{(1+9x^2y^2z^2)^3}$

47. $u = \sin\frac{r}{t} + \ln\frac{t}{r}$. $t\dfrac{\partial u}{\partial t} + r\dfrac{\partial u}{\partial r} = t\left[\cos\frac{r}{t}\left(-\frac{r}{t^2}\right)+\frac{1}{t}\right] + r\left[\cos\frac{r}{t}\left(\frac{1}{t}\right)-\frac{1}{r}\right] = -\frac{r}{t}\cos\frac{r}{t} + 1 + \frac{r}{t}\cos\frac{r}{t} - 1 = 0.$

48. Given $w = x^2y + y^2z + z^2x$. Verify $\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} + \dfrac{\partial w}{\partial z} = (x+y+z)^2$.

▸ $\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} + \dfrac{\partial w}{\partial z} = (2xy + z^2) + (x^2 + 2yz) + (y^2 + 2zx) = (x+y+z)^2$

In Exercises 49–53, show that u satisfies Laplace's equation in R^2 or R^3.

49. $u(x,y) = \ln(x^2+y^2)$

▸ $\dfrac{\partial u}{\partial x} = \dfrac{2x}{x^2+y^2}$; $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{2(x^2+y^2)-2x(2x)}{(x^2+y^2)^2} = \dfrac{2y^2-2x^2}{(x^2+y^2)^2}$

$\dfrac{\partial u}{\partial y} = \dfrac{2y}{x^2+y^2}$; $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2} = \dfrac{2x^2-2y^2}{(x^2+y^2)^2} = -\dfrac{\partial^2 u}{\partial x^2}$. Hence $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

50. $u(x,y) = \tan^{-1}\dfrac{2xy}{x^2-y^2} = 2\tan^{-1}\dfrac{y}{x}$. $\dfrac{\partial u}{\partial x} = \dfrac{2\left(-\frac{y}{x^2}\right)}{1+\frac{y^2}{x^2}} = \dfrac{-2y}{x^2+y^2}$; $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{4xy}{(x^2+y^2)^2}$. $\dfrac{\partial u}{\partial y} = \dfrac{\frac{2}{x}}{1+\frac{y^2}{x^2}} = \dfrac{2x}{x^2+y^2}$;

$\dfrac{\partial^2 u}{\partial y^2} = \dfrac{-4xy}{(x^2+y^2)^2} = -\dfrac{\partial^2 u}{\partial^2 y}$. Hence $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

51. $u(x,y) = \tan^{-1}\dfrac{y}{x} + \dfrac{x}{x^2+y^2}$

▸ $\dfrac{\partial u}{\partial x} = \dfrac{1}{1+(y/x)^2}\left(-\dfrac{y}{x^2}\right) + \dfrac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2-x^2y-y^3}{(x^2+y^2)^2}$

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{-2x-2xy}{(x^2+y^2)^2} - \dfrac{2(y^2-x^2-x^2y-y^3)(2x)}{(x^2+y^2)^3} = \dfrac{2x^3-6xy^2+2x^3y+2xy^3}{(x^2+y^2)^3}$

$\dfrac{\partial u}{\partial y} = \dfrac{1}{1+(y/x)^2}\left(\dfrac{1}{x}\right) - \dfrac{2xy}{(x^2+y^2)^2} = \dfrac{x^3+xy^2-2xy}{(x^2+y^2)^2}$

$\dfrac{\partial^2 u}{\partial y^2} = \dfrac{2xy-2x}{(x^2+y^2)^2} - \dfrac{2(x^3+xy^2-2xy)(2y)}{(x^2+y^2)^3} = \dfrac{-2x^3+6xy^2-2x^3y-2xy^3}{(x^2+y^2)^3} = -\dfrac{\partial^2 u}{\partial x^2}$. Hence $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

52. $u(x,y) = e^x\sin y + e^y\cos x$

▸ $u_x = e^x\sin y - e^y\sin x$

$u_{xx} = e^x\sin y - e^y\cos x$ (1)

and

$u_y = e^x\cos y + e^y\cos x$

$u_{yy} = -e^x\sin y + e^y\cos x$ (2)

Adding the members of (1) and (2), we obtain

$u_{xx} + u_{yy} = 0$

53. $u(x,y,z) = \dfrac{1}{(x^2+y^2+z^2)^{3/2}}$. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{-1}{(x^2+y^2+z^2)^{3/2}} - \dfrac{3}{2}\cdot\dfrac{(-x)(2x)}{(x^2+y^2+z^2)^{5/2}} = \dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$.

Similarly, $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $\dfrac{\partial^2 u}{\partial z^2} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$. Therefore

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$

54. $36x^2 - 9y^2 + 4z^2 + 36 = 0$. We hold x constant and differentiate implicitly with respect to y.

$-18y + 8z\dfrac{\partial z}{\partial y} = 0$; $\dfrac{\partial z}{\partial y} = \dfrac{9y}{4z}$. At $(1, \sqrt{12}, -3)$, $\dfrac{\partial z}{\partial y} = \dfrac{9(\sqrt{12})}{4(-3)} = -\frac{3}{2}\sqrt{3}$

55. $z = x^2 + y^2$. Therefore $\dfrac{\partial z}{\partial x} = 2x$. At $(2,1,5)$, $\dfrac{\partial z}{\partial x} = 4$.

56. Find equations of the tangent line to the curve of intersection of the surface $x^2 + y^2 + z^2 = 9$ with the plane $y = 2$ at the point $(1,2,2)$.

▹ We hold y constant and differentiate implicitly with respect to x on both sides of the equation. Thus

$$2x + 2z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}$$

At the point $(1,2,2)$, we have

$$\frac{\partial z}{\partial x} = -\frac{1}{2}$$

Thus, equations of the tangent line are

$z = 2 - \frac{1}{2}(x-1)$ and $y = 2$

57. $T = 54 - \frac{2}{3}x^2 - 4y^2$. $\dfrac{\partial T}{\partial x} = -\frac{4}{3}x$ and $\dfrac{\partial T}{\partial y} = -8y$. At the point $(3,1)$, $\dfrac{\partial T}{\partial x} = -4$ and $\dfrac{\partial T}{\partial y} = -8$. Hence, the rate of change of the temperature with respect to the distance moved in the direction of the positive x axis is -4 deg/cm and in the direction of the positive y axis is -8 deg/cm.

58. Gas law: $PV = kT$. $\dfrac{\partial}{\partial T}V(P,T)\cdot\dfrac{\partial}{\partial P}T(P,V)\cdot\dfrac{\partial}{\partial V}P(T,V) = \dfrac{\partial}{\partial T}\left(\dfrac{kT}{P}\right)\cdot\dfrac{\partial}{\partial P}\left(\dfrac{PV}{k}\right)\cdot\dfrac{\partial}{\partial V}\left(\dfrac{kT}{V}\right) = \dfrac{k}{P}\cdot\dfrac{V}{k}\cdot\dfrac{-kT}{V^2} = \dfrac{-kT}{PV} = -1$

59. $V = 100\left[\dfrac{1-(1+i)^{-t}}{i}\right]$. (a) $\dfrac{\partial V}{\partial i} = 100\left[\dfrac{it(1+i)^{-t-1} - 1 + (1+i)^{-t}}{i^2}\right] = \dfrac{100}{i^2}\left[\dfrac{ti}{(1+i)^{t+1}} + \dfrac{1}{(1+i)^t} - 1\right]$

When $t = 8$, $\dfrac{\partial V}{\partial i} = \dfrac{100}{i^2}\left[\dfrac{8i}{(1+i)^9} + \dfrac{1}{(1+i)^8} - 1\right] = \dfrac{100}{i^2}\left[\dfrac{9i+1}{(1+i)^9} - 1\right]$ (b) If $i = 0.06$, $\Delta i = 0.01$ and $t = 8$, then $\Delta V \approx \dfrac{\partial V}{\partial i}\Delta i = \dfrac{1}{0.0036}\left[\dfrac{1.54}{1.06^9} - 1\right] \approx -24.4$. Hence, the present value decreases approximately \$24.40. (c) $\dfrac{\partial V}{\partial t} = \dfrac{100}{i}(1+i)^{-t}\ln(1+i)$. When $i = 0.06$, $\dfrac{\partial V}{\partial t} = \dfrac{100}{0.06\cdot 1.06^t}\ln(1.06) = \dfrac{5000\ln(1.06)}{3\cdot 1.06^t}$. (d) If $t = 8$, $\Delta t = -1$ and $i = 0.06$, $\Delta V \approx \dfrac{\partial V}{\partial t}\Delta t = \dfrac{5000\ln(1.06)}{3\cdot 1.06^8}(-1) \approx -60.9$. Thus, the present value is decreased by about \$60.90.

60. Suppose that $10{,}000x$ dollars is the inventory carried in a store employing y clerks. P dollars is the weekly profit of the store, and

$P = 3000 + 240y + 20y(x - 2y) - 10(x-12)^2$

where $15 \le x \le 25$ and $5 \le y \le 12$. At present the inventory is \$180,000 and there are 8 clerks. (a) Find the instantaneous rate of change of P per unit change in x if y remains constant. (b) Use the result of part (a) to find the approximate change in weekly profit if the inventory changes from \$180,000 to \$200,000 and the number of clerks remains fixed at 8. (c) Find the instantaneous rate of change of P per unit change in y if x remains fixed at 18. (d) Use the result of part (c) to find the approximate change in the weekly profit if the number of clerks is increased from 8 to 10 and the inventory remains fixed at \$180,000.

▹ (a) We hold y constant and differentiate with respect to x.

$P_x(18,8) = 20y - 20(x-12)\Big]_{\substack{x=18\\y=8}} = 20(8) - 20(18-12) = 40$

The instantaneous rate of change of P is \$40 for each \$10,000 increase in inventory. (b) The approximate change in profit when the inventory is increased by $200{,}000 - 180{,}000 = 20{,}000$ is $2(\$40) = \$80$.

(c) We hold x constant and differentiate with respect to y.

$P_y(18,8) = 240 + 20x - 80y\Big]_{\substack{x=18\\y=8}} = 240 + 20(18) - 80(8) = -40$

The instantaneous rate of change is a decrease in profit of \$40 for each additional clerk. (d) Two additional clerks will cause a decrease of about $2(\$40) = \$80$.

61. $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. (a) $f_1(0,0) = \lim\limits_{x\to 0}\dfrac{f(x,0) - f(0,0)}{x - 0} = \lim\limits_{x\to 0}\dfrac{x-0}{x-0} = \lim\limits_{x\to 0} 1 = 1$

(b) $f_2(0,0) = \lim\limits_{y\to 0}\dfrac{f(0,y) - f(0,0)}{y - 0} = \lim\limits_{y\to 0}\dfrac{y-0}{y-0} = \lim\limits_{y\to 0} 1 = 1$

In Exercises 62 and 63, $f(x,y) = \begin{cases} \dfrac{x^2 - xy}{x+y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

62. (a) If $(x,y) \neq (0,0)$, $f_1(x,y) = \dfrac{(2x-y)(x+y) - (x^2 - xy)1}{(x+y)^2} = \dfrac{x^2 + 2xy - y^2}{(x+y)^2}$. If $y \neq 0$, $f_1(0,y) = \dfrac{-y^2}{y^2} = -1$

(b) $f_1(0,0) = \lim\limits_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x - 0} = \lim\limits_{x \to 0} \frac{x}{x} = 1$

63. (a) If $(x,y) \neq (0,0)$, $f_2(x,y) = \dfrac{-x(x+y) - 1(x^2 - xy)}{(x+y)^2} = \dfrac{-2x^2}{(x+y)^2}$. If $x \neq 0$, $f_2(x,0) = \dfrac{-2x^2}{x^2} = -2$

(b) $f_2(0,0) = \lim\limits_{y \to 0} \dfrac{f(0,y) - f(0,0)}{y - 0} = \lim\limits_{y \to 0} \dfrac{0-0}{y} = \lim\limits_{y \to 0} 0 = 0$

64. For the function of Example 11, show that f_{12} is discontinuous at $(0,0)$ and hence the hypothesis of Theorem 12.3.3 is not satisfied if $(x_0, y_0) = (0,0)$.

▸ The function of Example 4 is defined by

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 3 & \text{if } (x,y) = (0,0) \end{cases}$$

We show that $\lim\limits_{(x,y) \to (0,0)} f_{12}(x,y)$ does not exist. If $(x,y) \neq (0,0)$, then

$$f(x,y) = \frac{x^3 y - xy^3}{x^2 + y^2}$$

$$f_1(x,y) = \frac{(x^2+y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_{12}(x,y) = \frac{(x^2+y^2)^2(x^4 + 12x^2y^2 - 5y^4) - (x^4y + 4x^2y^3 - y^5)(4y)(x^2+y^2)}{(x^2+y^2)^4} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Along the x axis we have

$$\lim_{\substack{(x,y) \to (0,0) \\ (y=0)}} f_{12}(x,y) = \lim_{x \to 0} \frac{x^6}{x^6} = 1$$

while along the y axis we have

$$\lim_{\substack{(x,y) \to (0,0) \\ (x=0)}} f_{12}(x,y) = \lim_{y \to 0} \frac{-y^6}{y^6} = -1$$

Because these two limits are not equal, then $\lim\limits_{(x,y) \to (0,0)} f_{12}(x,y)$ does not exist. Hence f_{12} is discontinuous at $(0,0)$.

In Exercises 65–67, find $f_{12}(0,0)$ and $f_{21}(0,0)$ if they exist.

65. $f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. $f_1(0,0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - (0,0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0-0}{\Delta x} = 0$

If $y \neq 0$, $f_1(0,y) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, y) - f(0,y)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x} \cdot \dfrac{2(\Delta x)y}{(\Delta x)^2 + y^2} = \dfrac{2}{y}$.

$f_{12}(0,0) = \lim\limits_{\Delta y \to 0} \dfrac{f_1(0, \Delta y) - f_1(0,0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{1}{\Delta y}\left(\dfrac{2}{\Delta y} - 0\right)$ which doesn't exist. Similarly, $f_{21}(0,0)$ doesn't exist.

66. $f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^4+y^4} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$. $f_1(0,0)=\lim\limits_{x\to0}\dfrac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to0}\dfrac{0}{x}=0$

If $y\neq0$, $f_1(0,y)=\lim\limits_{x\to0}\dfrac{f(x,y)-f(0,y)}{x-0}=\lim\limits_{x\to0}\dfrac{1}{x}\left(\dfrac{x^2y^2}{x^4+y^4}-0\right)=0$

$f_{12}(0,0)=\lim\limits_{y\to0}\dfrac{f_1(0,y)-f_1(0,0)}{y-0}=\lim\limits_{y\to0}\dfrac{0-0}{y}=0$. Similarly, $f_{21}(0,0)=0$.

67. $f(x,y)=\begin{cases}x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y} & \text{if } x\neq0 \text{ and } y\neq0\\ 0 & \text{if either } x=0 \text{ or } y=0\end{cases}$. $f_1(0,0)=\lim\limits_{\Delta x\to0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{0-0}{\Delta x}=0$

If $y\neq0$, $f_1(0,y)=\lim\limits_{\Delta x\to0}\dfrac{f(\Delta x,y)-f(0,y)}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{(\Delta x)^2\tan^{-1}\frac{y}{\Delta x}-y^2\tan^{-1}\frac{\Delta x}{y}-0}{\Delta x}$

$=\lim\limits_{\Delta x\to0}(\Delta x)\tan^{-1}\dfrac{y}{\Delta x}-y^2\lim\limits_{\Delta x\to0}\dfrac{\tan^{-1}\frac{\Delta x}{y}}{\Delta x}=0-y^2\dfrac{\partial}{\partial x}\tan^{-1}\dfrac{x}{y}\Big|_{x=0}=-y^2\dfrac{1}{1+(x/y)^2}\cdot\dfrac{1}{y}\Big|_{x=0}=-y$

Hence $f_{12}(0,0)=\lim\limits_{\Delta y\to0}\dfrac{f_1(0,\Delta y)-f_1(0,0)}{\Delta y}=\lim\limits_{\Delta y\to0}\dfrac{-\Delta y-0}{\Delta y}=-1$.

In a similar way, we get $f_2(0,0)=0$ and if $x\neq0$, $f_2(x,0)=x$. Then

$f_{21}(0,0)=\lim\limits_{\Delta x\to0}\dfrac{f_2(\Delta x,0)-f_2(0,0)}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{\Delta x-0}{\Delta x}=1$

68. Prove that if f is a function of two variables and all the partial derivatives of the fourth order are continuous on some open disk, then $D_{1122}f=D_{2121}f$.

▸ Theorem 12.3.3 says that we may interchange adjacent subscripts. Applying the theorem three times we get $D_{1122}f=D_{1212}f=D_{2112}f=D_{2121}f$

69. $S=2W^{0.4}H^{0.7}$. $\dfrac{\partial S}{\partial W}=0.8W^{-0.6}H^{0.7}$. $\dfrac{\partial S}{\partial H}=1.4W^{0.4}H^{-0.3}$. When $W=70$ and $H=1.8$.

$\dfrac{\partial S}{\partial W}=(0.8)(70^{-0.6})(1.8^{0.7})\approx0.0943$. The surface area changes 0.0943 m^2/kg of weight.

$\dfrac{\partial S}{\partial H}=(1.4)(70^{0.4})(1.8^{-0.3})\approx6.42$. The surface area changes 6.42 m^2/m of height.

12.4 DIFFERENTIABILITY AND THE TOTAL DIFFERENTIAL

12.4.1 Definition If f is a function of two variables x and y, then the *increment of f* at the point (x_0,y_0) denoted by $\Delta f(x_0,y_0)$, is given by
$\Delta f(x_0,y_0)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$

12.4.2 Definition If f is a function of two variables x and y and the increment of f at (x_0,y_0) can be written as
$\Delta f(x_0,y_0)=D_1f(x_0,y_0)\Delta x+D_2f(x_0,y_0)\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y$
where ϵ_1 and ϵ_2 are functions of Δx and Δy such that $\epsilon_1\to0$ and $\epsilon_2\to0$ as $(\Delta x,\Delta y)\to(0,0)$, then f is said to be *differentiable* at (x_0,y_0).
In Section 12.7 we show that a function is differentiable if and only if its graph has a tangent plane and whose equation is the total differential given below.

12.4.3 Definition If f is differentiable at (x,y), the *total differential* is the function
$df(x,y,\Delta x,\Delta y)=D_1f(x,y)\Delta x+D_2f(x,y)\Delta y$
or, if $z=f(x,y)$,
$dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy$

12.4.3 Theorem If a function f of two variables is differentiable at a point, it is continuous at that point.

12.4.4 Theorem Let f be a function of two variables x and y. Suppose that D_1f and D_2f exist on an open disk $B(P_0;r)$ and D_1f and D_2f are continuous at P_0, then f is differentiable at P_0.

Note that continuity of D_1f and $D_2f\Rightarrow$ differentiability of f $\Rightarrow$ existence of D_1f and D_2f
$\Rightarrow$ continuity of f

but no other implications exist among these four properties.

The following definitions extend the preceding concepts to functions of n variables.

12.4.6 Definition If f is a function of n variables $x_1, x_2, \ldots, x_n$ and $\bar{P}$ is the point $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, then the *increment of* f at $\bar{P}$ is given by

$$\Delta f(\bar{P}) = f(\bar{x}_1 + \Delta x_1, \bar{x}_2 + \Delta x_2, \ldots, \bar{x}_n + \Delta x_n) - f(\bar{P})$$

12.4.7 Definition If f is a function of the n variables $x_1, x_2, \ldots, x_n$ and the increment of f at the point $\bar{P}$ can be written as

$$\Delta f(\bar{P}) = D_1 f(\bar{P})\Delta x_1 + D_2 f(\bar{P})\Delta x_2 + \cdots + D_n f(\bar{P})\Delta x_n + \epsilon_1 \Delta x_1 + \epsilon_2 \Delta x_2 + \cdots + \epsilon_n \Delta x_n$$

where $\epsilon_1 \to 0$, $\epsilon_2 \to 0$, ..., $\epsilon_n \to 0$ as $(\Delta x_1, \Delta x_2, \ldots, \Delta x_n) \to (0, 0, \ldots, 0)$ then f is said to be *differentiable* at $\bar{P}$.

12.4.8 Definition If f is a function of the n variables $x_1, x_2, \ldots, x_n$ and f is differentiable at P, then the *total differential* of f is the function df having function values given by

$$df(P, \Delta x_1, \Delta x_2, \ldots, \Delta x_n) = D_1 f(P)\Delta x_1 + D_2 f(P)\Delta x_2 + \cdots + D_n f(P)\Delta x_n$$

or, if $w = f(x_1, x_2, \ldots, x_n)$

$$dw = \frac{\partial w}{\partial x_1}dx_1 + \frac{\partial w}{\partial x_2}dx_2 + \cdots + \frac{\partial w}{\partial x_n}dx_n$$

Exercises 12.4

1. $f(x, y) = 3x^2 + 2xy - y^2$

▸ (a) $\Delta f(1,4) = f(1 + \Delta x, 4 + \Delta y) - f(1,4) = 3(1 + \Delta x)^2 + 2(1 + \Delta x)(4 + \Delta y) - (4 + \Delta y)^2 - (3 + 8 - 16)$
$= 3 + 6\Delta x + 3(\Delta x)^2 + 8 + 8\Delta x + 2\Delta y + 2\Delta x\,\Delta y - 16 - 8\Delta y - (\Delta y)^2 + 5$
$= 3(\Delta x)^2 + 2\Delta x\,\Delta y - (\Delta y)^2 + 14\Delta x - 6\Delta y$

(b) When $\Delta x = 0.03$ and $\Delta y = -0.02$, we have
$\Delta f(1,4) = 3(0.03)^2 + 2(0.03)(-0.02) - (0.02)^2 + 14(0.03) - 6(-0.02)$
$= 0.0027 - 0.0012 - 0.0004 + 0.42 + 0.12 = 0.5411$

(c) $df(1, 4, \Delta x, \Delta y) = D_1 f(1,4)\Delta x + D_2 f(1,4)\Delta y$. When $x = 1$ and $y = 4$,
$D_1 f(x,y) = 6x + 2y = 14$ and $D_2 f(x,y) = 2x - 2y = -6$. Thus $df(1, 4, \Delta x, \Delta y) = 14\Delta x - 6\Delta y$.

(d) $df(1, 4, 0.03, -0.02) = 14(0.03) - 6(-0.02) = 0.42 + 0.12 = 0.54$

2. $f(x, y) = 2x^2 + 5xy + 4y^2$

▸ (a) $\Delta f(2,-1) = f(2 + \Delta x, -1 + \Delta y) - f(2,-1) = 2(2 + \Delta x)^2 + 5(2 + \Delta x)(-1 + \Delta y) + 4(-1 + \Delta y)^2 - 2$
$= 2(\Delta x)^2 + 5\Delta x\,\Delta y + 4(\Delta y)^2 + 3\Delta x + 2\Delta y$

(b) When $\Delta x = -0.01$ and $\Delta y = 0.02$, we have
$\Delta f(2,-1) = 2(-0.01)^2 + 5(0.01)(0.02) + 4(0.02)^2 + 3(0.01) + 2(0.02) = 0.0108$

(c)$df(2, -1, \Delta x, \Delta y) = D_1 f(2,-1)\Delta x + D_2 f(2,-1)\Delta y$. When $x = 2$ and $y = -1$,
$D_1 f(x,y) = 4x + 5y = 3$ and $D_2 f(x,y) = 5x + 8y = 2$. Thus $df(2, -1, \Delta x, \Delta y) = 3\Delta x + 2\Delta y$

(d) $df(2, -1, -0.01, 0.02) = 3(-0.01) + 2(0.02) = 0.01$

3. $g(x, y) = xye^{xy}$ (a) $\Delta g(2,-4) = g(2 + \Delta x, -4 + \Delta y) - g(2,-4) = (2 + \Delta x)(-4 + \Delta y)e^{(2+\Delta x)(-4+\Delta y)} + 8e^{-8}$

(b) When $\Delta x = -0.1$ and $\Delta y = 0.2$, we have
$\Delta g(2,-4) = (2 - 0.1)(-4 + 0.2)e^{(2-0.1)(-4+0.2)} + 8e^{-8} = (1.9)(-3.8)e^{(1.9)(-3.8)} + 8e^{-8} = -0.0026$

(c) $dg(2, -4, \Delta x, \Delta y) = D_1 g(2,-4)\Delta x + D_2 g(2,-4)\Delta y$.
Because $D_1 g(x,y) = ye^{xy} + xy^2e^{xy}$ and $D_2 g(x,y) = xe^{xy} + x^2ye^{xy}$, we have
$dg(2, -4, \Delta x, \Delta y) = (-4e^{-8} + 32e^{-8})\Delta x + (2e^{-8} - 16e^{-8})\Delta y = 28e^{-8}\Delta x - 14e^{-8}\Delta y$

(d) $dg(2, 4, -0.1, 0.2) = 28e^{-8}(-0.1) - 14e^{-8}(0.2) = -5.6e^{-8} = -0.0019$

4. If $h(x, y) = (x + y)/(x - y)$, find: (a) $\Delta h(3,0)$, the increment of h at $(3,0)$; (b) $\Delta h(3,0)$ when $\Delta x = 0.04$ and $\Delta y = 0.03$; (c) $dh(3, 0, \Delta x, \Delta y)$, the total differential of h at $(3,0)$; (d) $dh(3, 0, 0.04, 0.03)$.

▸ (a) We apply Definition 12.4.1 to the given function h with $(x_0, y_0) = (3,0)$. Thus,

$$\Delta h(3,0) = h(3 + \Delta x, \Delta y) - h(3,0) = \frac{3 + \Delta x + \Delta y}{3 + \Delta x - \Delta y} - 1 = \frac{2\Delta y}{3 + \Delta x - \Delta y}$$

(b) We substitute $\Delta x = 0.04$ and $\Delta y = 0.03$ into the result obtained in part (a) to get

$$\frac{2(0.03)}{3 + 0 + 0.04 - 0.03} = \frac{0.06}{3.01} = 0.0199$$

(c) We apply Definition 12.4.5 to the given function h.

$dh(x, y, \Delta x, \Delta y) = \dfrac{\partial h}{\partial x}\Delta x + \dfrac{\partial h}{\partial y}\Delta y = \dfrac{(x-y)-(x+y)}{(x-y)^2}\Delta x + \dfrac{(x-y)-(x+y)(-1)}{(x-y)^2} = \dfrac{-2y\Delta x + 2x\Delta y}{(x-y)^2}$

Substituting $x = 3$ and $y = 0$, we obtain

$dh(3, 0, \Delta x, \Delta y) = \dfrac{0\Delta x + 2(3)\Delta y}{(3-0)^2} = \dfrac{2\Delta y}{3}$

(d) We substitute $\Delta x = 0.04$ and $\Delta y = 0.03$ into the result obtained in part (c).

$dh(3, 0, 0.04, 0.03) = \dfrac{2(0.03)}{3} = 0.02$

Observe that the result of part (d) is approximately equal to the result of part (b).

5. $f(x, y, z) = xy + \ln(yz)$

▹ (a) $\Delta f(4, 1, 5) = f(4 + \Delta x, 1 + \Delta y, 5 + \Delta z) - f(4, 1, 5) = (4 + \Delta x)(1 + \Delta y) + \ln[(1 + \Delta y)(5 + \Delta z)] - (4 + \ln 5)$
$= \Delta x + 4\Delta y + \Delta x\, \Delta y + \ln(1 + \Delta y) + \ln(5 + \Delta z) - \ln 5$

(b) When $\Delta x = 0.02$, $\Delta y = 0.04$, and $\Delta z = -0.03$, we have
$\Delta f(4, 1, 5) = 0.02 + 0.16 + 0.0008 + \ln(1.04) + \ln(4.97) - \ln 5 \approx 0.214003$

(c) $df(4, 1, 5, \Delta x, \Delta y, \Delta z) = D_1 f(4, 1, 5)\Delta x + D_2 f(4, 1, 5)\Delta y + D_3 f(4, 1, 5)\Delta z$. When $x = 4$, $y = 1$, $z = 5$, $D_1 f(x, y, z) = y = 1$, $D_2 f(x, y, z) = x + \frac{1}{y} = 5$, and $D_3 f(x, y, z) = \frac{1}{z} = \frac{1}{5}$. Then

$df(4, 1, 5, \Delta x, \Delta y, \Delta z) = \Delta x + 5\Delta y + \frac{1}{5}\Delta z$

(d) $df(4, 1, 5, 0.02, 0.04, -0.03) = 0.02 + 5(0.04) + \frac{1}{5}(-0.03) = 0.02 + 0.20 - 0.006 = 0.214$

6. $G(x, y, z) = x^2 y + 2xyz - z^3$. (a) $\Delta G(-3, 0, 2) = G(-3 + \Delta x, \Delta y, 2 + \Delta z) - G(-3, 0, 2)$
$= (-3 + \Delta x)^2 \Delta y + 2(-3 + \Delta x)\Delta y(2 + \Delta z) - (2 + \Delta z)^3 - 8$
$= -3\Delta y - 12\Delta z - 2\Delta x\, \Delta z - 6(\Delta z)^2 + 2\Delta x\, \Delta y\, \Delta z - (\Delta z)^3$

(b) When $\Delta x = 0.01$, $\Delta y = 0.03$, $\Delta z = -0.01$, we have
$\Delta G(-3, 0, 2) = -3(0.01)^2 - 12(-0.01) - 2(0.01)(-0.01) - 6(-0.01)^2 + 2(0.01)(0.03)(-0.01) - (-0.01)^3 = 0.031$

(c) $dG(-3, 0, 2, \Delta x, \Delta y, \Delta z) = D_1 G(-3, 0, 2)\Delta x + D_2 G(-3, 0, 2)\Delta y + D_3 G(-3, 0, 2)\Delta z$. When $x = -3$, $y = 0$, $z = 2$, $D_1 G(x, y, z) = 2xy + 2yz = 0$, $D_2 G(x, y, z) = x^2 + 2xz = -3$, $D_3 G(x, y, z) = 2xy - 3z^2 = -12$. Then
$dG(-3, 0, 2, \Delta x, \Delta y, \Delta z) = -3\Delta y - 12\Delta z$

In Exercises 7–14, find the total differential dw.

7. $w = 4x^3 - xy^2 + 3y - 7$. $dw = \dfrac{\partial w}{\partial x}\,dx + \dfrac{\partial w}{\partial y}\,dy = (12x^2 - y^2)dx + (3 - 2xy)dy$

8. $w = y \tan x^2 - 2xy$

▹ From Definition 12.4.5 we have

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy = (2xy \sec^2 x^2 - 2y)dx + (\tan x^2 - 2x)dy$$

9. $w = x\cos y - y\sin x$. $dw = \dfrac{\partial w}{\partial x}dx + \dfrac{\partial w}{\partial y}dy = (\cos y - y\cos x)dx + (-x\sin y - \sin x)dy$

10. $w = xe^{2y} + e^{-y}$. $dw = \dfrac{\partial w}{\partial x}dx + \dfrac{\partial w}{\partial y}dy = e^{2y}dx + (2xe^{2y} - e^{-y})dy$

11. $w = \ln(x^2 + y^2 + z^2)$. $dw = \dfrac{\partial w}{\partial x}\,dx + \dfrac{\partial w}{\partial y}\,dy + \dfrac{\partial w}{\partial z}\,dz$
$= \dfrac{2x}{x^2 + y^2 + z^2}\,dx + \dfrac{2y}{x^2 + y^2 + z^2}\,dy + \dfrac{2z}{x^2 + y^2 + z^2}\,dz = \dfrac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}$

12. $w = \dfrac{xyz}{x + y + z}$

▹ By Definition 12.4.8,

$dw = \dfrac{\partial w}{\partial x}dx + \dfrac{\partial w}{\partial y}dy + \dfrac{\partial w}{\partial z}dz = \dfrac{(x+y+z)(yz) - xyz}{(x+y+z)^2}dx + \dfrac{(x+y+z)(xz) - xyz}{(x+y+z)^2}dy + \dfrac{(x+y+z)(xy) - xyz}{(x+y+z)^2}dz$
$= \dfrac{yz(y+z)dx + xz(x+z)dy + xy(x+y)dz}{(x+y+z)^2}$

13. $w = x\tan^{-1} z - \dfrac{y^2}{z}$. $dw = \dfrac{\partial w}{\partial x}dx + \dfrac{\partial w}{\partial y}dy + \dfrac{\partial w}{\partial z}dz = \tan^{-1} z\,dx - \dfrac{2y}{z}\,dy + \left(\dfrac{x}{1+z^2} + \dfrac{y^2}{z^2}\right)dz$

14. $w = e^{yz} - \cos xz$. $dw = \dfrac{\partial w}{\partial x}dx + \dfrac{\partial w}{\partial y}dy + \dfrac{\partial w}{\partial z}dz = z\sin xz\,dx + ze^{yz}dy + (ye^{yz} + x\cos xz)dz$

In Exercises 15–18, prove that f is differentiable at all points in its domain by doing each of the following: (a) Find $\Delta f(x_0, y_0)$; (b) find an ϵ_1 and an ϵ_2 so that the equation of Definition 12.4.2 holds; (c) show that the ϵ_1 and the ϵ_2 found in part (b) both approach zero as $(\Delta x, \Delta y) \to (0, 0)$.

15. $f(x, y) = x^2 y - 2xy$

▷ (a) $\Delta f(x, y) = f(x + \Delta x, y + \Delta y) - f(x, y) = (x + \Delta x)^2(y + \Delta y) - 2(x + \Delta x)(y + \Delta y)$
$= 2(xy - y)\Delta x + (x^2 - 2x)\Delta y + (y\Delta x + \Delta x\Delta y)\Delta x + 2(x\Delta x - \Delta x)\Delta y$

(b) $D_1 f(x, y) = 2xy - 2y$ and $D_2 f(x, y) = x^2 - 2x$. Hence $\epsilon_1 = y\Delta x + \Delta x\Delta y$ and $\epsilon_2 = 2x\Delta x - 2\Delta x$

(c) $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} \epsilon_1 = \lim\limits_{(\Delta x, \Delta y) \to (0,0)} (y\Delta x + \Delta x\Delta y) = 0$; $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} \epsilon_2 = \lim\limits_{(\Delta x, \Delta y) \to (0,0)} (2x\Delta x - 2\Delta x) = 0$

16. $f(x, y) = 2x^2 + 3y^2$

▷ (a)
$$\begin{aligned}\Delta f(x_0, y_0) &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)\\ &= [2(x_0 + \Delta x)^2 + 3(y_0 + \Delta y)^2] - [2x_0{}^2 + 3y_0{}^2]\\ &= 2x_0{}^2 + 4x_0\Delta x + 2(\Delta x)^2 + 3y_0{}^2 + 6y_0\Delta y + 3(\Delta y)^2 - 2x_0{}^2 - 3y_0{}^2\\ &= 4x_0\Delta x + 6y_0\Delta y + 2(\Delta x)^2 + 3(\Delta y)^2 \qquad (1)\end{aligned}$$

(b) The equation of Definition 12.4.2 states that

$\Delta f(x_0, y_0) = D_1 f(x_0, y_0)\Delta x + D_2 f(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$ (2)

Partial-differentiating the given function, we have

$D_1 f(x_0, y_0) = 4x_0$ (3)

$D_2 f(x_0, y_0) = 6y_0$ (4)

Substituting from Eqs. (1), (3), and (4) into (2), we get

$4x_0\Delta x + 6y_0\Delta y + 2(\Delta x)^2 + 3(\Delta y)^2 = 4x_0\Delta x + 6y_0\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$

$$2(\Delta x)^2 + 3(\Delta y)^2 = \epsilon_1\Delta x + \epsilon_2\Delta y \qquad (5)$$

Equation (5) will be an identity if

$\epsilon_1 = 2\Delta x$ and $\epsilon_2 = 3\Delta y$

(c) $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} \epsilon_1 = \lim\limits_{\Delta x \to 0} 2\Delta x = 0$ $\qquad \lim\limits_{(\Delta x, \Delta y) \to (0,0)} \epsilon_2 = \lim\limits_{\Delta y \to 0} 3\Delta y = 0$

17. $f(x, y) = \dfrac{x^2}{y}$

▷ (a) $\Delta f(x, y) = f(x + \Delta x, y + \Delta y) - f(x, y) = \dfrac{(x + \Delta x)^2}{y + \Delta y} - \dfrac{x^2}{y} = \dfrac{x^2 y + 2xy\Delta x + y(\Delta x)^2 - x^2 y - x^2\Delta y}{y(y + \Delta y)}$

$= \dfrac{2xy\Delta x + y(\Delta x)^2 - x^2\Delta y}{y(y + \Delta y)} = \dfrac{2xy + y\Delta x}{y(y + \Delta y)}\Delta x - \dfrac{x^2}{y(y + \Delta y)}\Delta y$

(b) $D_1 f(x, y) = \dfrac{2x}{y}$ and $D_2 f(x, y) = -\dfrac{x^2}{y^2}$. Thus,

$\Delta f(x, y) - D_1 f(x, y)\Delta x - D_2 f(x, y)\Delta y = \left[\dfrac{2xy + y\Delta x}{y(y + \Delta y)} - \dfrac{2x}{y}\right]\Delta x + \left[-\dfrac{x^2}{y(y + \Delta y)} + \dfrac{x^2}{y^2(y + \Delta y)}\right]\Delta y$

$= \dfrac{2xy + y\Delta x - 2xy - 2x\Delta y}{y(y + \Delta y)}\Delta x + \dfrac{-x^2 y + x^2 y + x^2\Delta y}{y^2(y + \Delta y)}\Delta y = \dfrac{y\Delta x - 2x\Delta y}{y(y + \Delta y)}\Delta x + \dfrac{x^2\Delta y}{y^2(y + \Delta y)}$

Thus, $\epsilon_1 = \dfrac{y\Delta x - 2x\Delta y}{y(y + \Delta y)}$ and $\epsilon_2 = \dfrac{x^2\Delta y}{y^2(y + \Delta y)}$.

(c) $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} \epsilon_1 = \lim\limits_{(\Delta x, \Delta y) \to (0,0)} \dfrac{y\Delta x - 2x\Delta y}{y(y + \Delta y)} = 0$; $\lim\limits_{(\Delta x, \Delta y) \to (0,0)} \epsilon_2 = \lim\limits_{(\Delta x, \Delta y) \to (0,0)} \dfrac{x^2\Delta y}{y^2(y + \Delta y)} = 0$

18. $f(x, y) = \dfrac{y}{x}$ (a) $\Delta f(x, y) = f(x + \Delta x, y + \Delta y) - f(x, y) = \dfrac{y + \Delta y}{x + \Delta x} - \dfrac{y}{x} = \dfrac{x\Delta y - y\Delta x}{x(x + \Delta x)}$

(b) $D_1 f(x, y) = -\dfrac{y}{x^2}$ and $D_2 f(x, y) = \dfrac{1}{x}$. Thus, $\Delta f(x, y) - D_1 f(x, y)\Delta x - D_2 f(x, y)\Delta y$

$= \left(-\dfrac{y}{x(x + \Delta x)} + \dfrac{y}{x^2}\right)\Delta x + \left(\dfrac{1}{x + \Delta x} - \dfrac{1}{x}\right)\Delta y = \dfrac{y\Delta x}{x^2(x + \Delta x)}\Delta x - \dfrac{\Delta x}{x(x + \Delta x)}\Delta y$

Thus, $\epsilon_1 = \dfrac{y\Delta x}{x^2(x + \Delta x)}$ and $\epsilon_2 = \dfrac{\Delta x}{x(x + \Delta x)}$ and (c) $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

In Exercises 19–26, Use Theorem 12.4.4 to prove that the function is differentiable at all points in its domain.

19. $g(x,y) = 2x^4 - 3x^2y^2 + x^{-2}y^{-2}$. The domain of g is all points in R^2 except (0,0).
$D_1g(x,y) = 8x^3 - 6xy^2 - 2x^{-3}y^{-2}$; $D_2g(x,y) = -6x^2y - 2x^{-2}y^{-3}$
At all points in the domain of g, D_1g and D_2g exist and are continuous; so g is differentiable by Thm 12.4.4.

20. $f(x,y) = \dfrac{3x-4y}{x^2+8y}$

▸ The domain of f is the set of all points (x,y) not on the parabola $y = -\frac{1}{8}x^2$. The partial derivatives of f are

$$D_1f(x,y) = \frac{3(x^2+8y)-(3x-4y)(2x)}{(x^2+8y)^2} = \frac{-3x^2+24y+8xy}{(x^2+8y)^2}$$

and

$$D_2f(x,y) = \frac{(-4)(x^2+8y)-(3x-4y)8}{(x^2+8y)^2} = \frac{-4x^2-24x}{(x^2+8y)^2}$$

Both D_1f and D_2f are continuous at every point in the domain of f. Therefore, by Theorem 12.4.4, g is differentiable at all points in its domain.

21. $f(x,y) = 3\ln xy + 5\sin x$. The domain of f is the first and third quadrants in R^2.
$D_1f(x,y) = 3\cdot\frac{1}{xy}\cdot y + 5\cos x = \frac{3}{x} + 5\cos x$; $D_2f(x,y) = 3\cdot\frac{1}{xy}\cdot x = \frac{3}{y}$
At all points in the domain of f, D_1f and D_2f exist and are continuous; so f is differentiable by Thm. 12.4.4.

22. $f(x,y) = \sin\frac{y}{x} + \cos\frac{x}{y}$. The domain of f is all points not on either axis.
$D_1f(x,y) = -\frac{y}{x^2}\cos\frac{y}{x} - \frac{1}{y}\sin\frac{x}{y}$ and $D_2f(x,y) = \frac{1}{x}\cos\frac{y}{x} + \frac{x}{y^2}\sin\frac{x}{y}$
At all points in the domain of f, D_1f and D_2f exist and are continuous; so f is differentiable by Thm. 12.4.4.

23. $h(x,y) = \tan^{-1}(x+y) + \frac{1}{x-y}$. The domain of h is $\{(x,y) \mid x \neq y\}$.
$D_1h(x,y) = \dfrac{1}{1+(x+y)^2} - \dfrac{1}{(x-y)^2}$; $D_2h(x,y) = \dfrac{1}{1+(x+y)^2} + \dfrac{1}{(x-y)^2}$
At all points in the domain of h, D_1h and D_2h exist and are continuous; so h is differentiable by Thm. 12.4.4.

24. $g(x,y) = y\ln x - \frac{x}{y}$

▸ The domain of g is $\{(x,y) \mid x > 0 \text{ and } y \neq 0\}$. The partial derivatives of g are
$D_1g(x,y) = g_x(x,y) = \frac{y}{x} - \frac{1}{y}$ and $D_2g(x,y) = g_y(x,y) = \ln x + \frac{x}{y^2}$
Both D_1g and D_2g are continuous at every point in the domain of g. Therefore, by Theorem 12.4.4, g is differentiable at all points in its domain.

25. $f(x,y) = ye^{3x} - xe^{-3y}$. The domain of f is all of R^2.
$D_1f(x,y) = 3ye^{3x} - e^{-3y}$; $D_2f(x,y) = e^{3x} + 3xe^{-3y}$
At all points in the domain of f, D_1f and D_2f exist and are continuous; so f is differentiable by Thm. 12.4.4.

26. $f(x,y) = e^{2x}\sin y + e^{-2x}\cos y$. The domain of f is all of R^2.
$D_1f(x,y) = 2e^{2x}\sin y - 2e^{-2x}\cos y$ and $D_2f(x,y) = e^{2x}\cos y - e^{-2x}\sin y$
At all points in the domain of f, D_1f and D_2f exist and are continuous; so f is differentiable by Thm. 12.4.4.

27. $f(x,y) = \begin{cases} x+y-2 & \text{if } x = 1 \text{ or } y = 1 \\ 2 & \text{if } x \neq 1 \text{ and } y \neq 1 \end{cases}$

$$D_1f(1,1) = \lim_{\Delta x \to 0}\frac{f(1+\Delta x,1)-f(1,1)}{\Delta x} = \lim_{\Delta x \to 0}\frac{(1+\Delta x+1-2)-0}{\Delta x} = 1$$

$$D_2f(1,1) = \lim_{\Delta y \to 0}\frac{f(1,1+\Delta y)-f(1,1)}{\Delta y} = \lim_{\Delta y \to 0}\frac{(1+1+\Delta y-2)-0}{\Delta y} = 1$$

But $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(y=x)}} f(x,y) = \lim_{x\to 1} 2 = 2 \neq 0 = f(1,1)$ so f is discontinuous at (1,1) and so not differentiable there.

28. Given $f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Prove that $D_1f(0,0)$ and $D_2f(0,0)$ exist but D_1f and D_2f are not continuous at $(0,0)$.

▸ (a) $D_1f(0,0) = \lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0}\dfrac{0-0}{x} = 0$

If $(x,y) \neq (0,0)$, by the quotient rule,

$$D_1f(x,y) = \frac{(x^2+y^2)(6xy)-(3x^2y)(2x)}{(x^2+y^2)^2} = \frac{6xy^3}{(x^2+y^2)^2}$$

Because

$$\lim_{\substack{(x,y)\to(0,0)\\(y=x)}} D_1(x,y) = \lim_{x\to 0}\frac{6x^4}{4x^4} = \frac{3}{2} \neq D_1(0,0)$$

then D_1f is not continuous at $(0,0)$.

(b) $D_2f(0,0) = \lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y-0} = \lim\limits_{y\to 0}\dfrac{0-0}{y} = 0$

If $(x,y) \neq (0,0)$, by the quotient rule,

$$D_2f(x,y) = \frac{(x^2+y^2)(3x^2)-(3x^2y)(2y)}{(x^2+y^2)^2} = \frac{3x^4-3x^2y^2}{(x^2+y^2)^2}$$

Because

$$\lim_{\substack{(x,y)\to(0,0)\\(y=0)}} D_2f(x,y) = \lim_{x\to 0}\frac{3x^4}{x^4} = 3 \neq D_2(0,0)$$

then D_2f is not continuous at $(0,0)$.

In Exercises 29 and 30, prove that $D_1f(0,0)$ and $D_2f(0,0)$ exist but f is not differentiable at $(0,0)$.

29. $f(x,y) = \begin{cases} \dfrac{3x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$D_1f(0,0) = \lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0}\dfrac{0-0}{x} = 0 \qquad D_2f(0,0) = \lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y-0} = \lim\limits_{y\to 0}\dfrac{0-0}{y} = 0$

But $\lim\limits_{\substack{(x,y)\to(0,0)\\(y=x)}} f(x,y) = \lim\limits_{x\to 0}\dfrac{3x^4}{2x^4} = \dfrac{3}{2} \neq f(0,0)$; f is not continuous at $(0,0)$ and so not differentiable there.

30. $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$D_1f(0,0) = \lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0}\dfrac{0-0}{x} = 0 \qquad D_2f(0,0) = \lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y-0} = \lim\limits_{y\to 0}\dfrac{0-0}{y} = 0$

But $\lim\limits_{\substack{(x,y)\to(0,0)\\(x=y^2)}} f(x,y) = \lim\limits_{y\to 0}\dfrac{y^4}{2y^4} = \dfrac{1}{2} \neq f(0,0)$; f is not continuous at $(0,0)$ and so not differentiable there.

In Exercises 31 and 32, prove that f is differentiable at all points in R^3 by doing each of the following: (a) Find $\Delta f(x,y,z)$; (b) find an ϵ_1, ϵ_2 and ϵ_3 so that the equation of Definition 12.4.7 holds; (c) show that the ϵ_1, ϵ_2, and ϵ_3 found in part (b) all approach zero as $(\Delta x, \Delta y, \Delta z)\to(0,0,0)$.

31. $f(x,y,z) = xy - xz + z^2$

(a) $\Delta f(x,y,z) = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$

$= (x+\Delta x)(y+\Delta y) - (x+\Delta x)(z+\Delta z) + (z+\Delta z)^2 - (xy - xz + z^2)$

$= xy + x\Delta y + y\Delta x + (\Delta x)(\Delta y) - xz - x\Delta z - z\Delta x - (\Delta x)(\Delta z) + z^2 + 2z\Delta z + (\Delta z)^2 - xy + xz - z^2$

$= x\Delta y + y\Delta x - x\Delta z - z\Delta x + 2z\Delta z + (\Delta x)(\Delta y) - (\Delta x)(\Delta z) + (\Delta z)^2$

$= (y-z)\Delta x + x\Delta y + (2z-x)\Delta z - \Delta z\Delta x + \Delta x\Delta y + \Delta z\Delta z$

(b) $D_1 f(x,y) = y - z$, $D_2 f(x,y,z) = x$, $D_3 f(x,y,z) = 2z - x$. Thus, $\epsilon_1 = -\Delta z$, $\epsilon_2 = \Delta x$, $\epsilon_3 = \Delta z$.

(c) $\displaystyle\lim_{(\Delta x, \Delta y, \Delta z)\to(0,0,0)} \epsilon_1 = \lim_{\Delta z \to 0}(-\Delta z) = 0$; $\displaystyle\lim_{(\Delta x, \Delta y, \Delta z)\to(0,0,0)} \epsilon_2 = \lim_{\Delta x\to 0} \Delta x = 0$;

$\displaystyle\lim_{(\Delta x, \Delta y, \Delta z)\to(0,0,0)} \;=\; \lim_{\Delta z\to 0} \Delta z = 0$

It follows from Definition 12.4.8 that f is differentiable at all points in R^3.

32. $f(x,y,z) = 2x^2 z - 3yz^2$

(a) $\Delta f(x,y,z)$

$= f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$

$= [2(x+\Delta x)^2(z+\Delta z) - 3(y+\Delta y)(z+\Delta z)^2 - [2x^2 z - 3yz^2]$

$= 2x^2\Delta z + 4xz\Delta x + 4x\Delta x\Delta z + 2z(\Delta x)^2 + 2(\Delta x)^2\Delta z - 6yz\Delta z - 3y(\Delta z)^2 - 3z^2\Delta y - 6z\Delta y\Delta z - 3\Delta y(\Delta z)^2 \quad (1)$

(b) From the equation of Definition 12.4.7 with $n = 3$

$\epsilon_1\Delta x + \epsilon_2\Delta y + \epsilon_3\Delta z = \Delta f(x,y,z) - D_1 f(x,y,z)\Delta x - D_2 f(x,y,z)\Delta y - D_3 f(x,y,z)\Delta z \quad (2)$

Taking partial derivatives of f, we obtain

$D_1 f(x,y,z)\Delta x = 4xz\Delta x \qquad D_2 f(x,y,z)\Delta y = -3z^2\Delta y \qquad D_3 f(x,y,z)\Delta z = (2x^2 - 6yz)\Delta z \quad (3)$

Substituting from (1) and (3) into (2), we get

$$\begin{aligned}\epsilon_1\Delta x + \epsilon_2\Delta y + \epsilon_3\Delta z &= 4x\Delta x\Delta z + 2z(\Delta x)^2 + 2(\Delta x)^2\Delta z - 3y(\Delta z)^2 - 6z\Delta y\Delta z - 3\Delta y(\Delta z)^2\\ &= (4x\Delta z + 2z\Delta x + 2\Delta x\Delta z)\Delta x + (-6z\Delta z - 3(\Delta z)^2)\Delta y + (-3y\Delta z)\Delta z\end{aligned}$$

Equating corresponding coefficients of Δx, Δy, and Δz in the above, we take

$\epsilon_1 = 4x\Delta z + 2z\Delta x + 2\Delta x\Delta z \qquad \epsilon_2 = -6z\Delta z - 3(\Delta z)^2 \qquad \epsilon_3 = -3y\Delta z$

(c) Since each term contains at least one factor of Δx, Δy, or Δz, all approach zero as $(\Delta x, \Delta y, \Delta z)\to(0,0,0)$.

In Ex. 33 and 34, prove that $D_1 f(0,0,0)$, $D_2 f(0,0,0)$ and $D_3 f(0,0,0)$ exist but f is not differentiable at $(0,0)$.

33. $f(x,y,z) = \begin{cases} \dfrac{xy^2 z}{x^4 + y^4 + z^4} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$

(a) $D_1 f(0,0,0) = \displaystyle\lim_{x\to 0}\frac{f(x,0,0) - f(0,0,0)}{x - 0} = \lim_{x\to 0}\frac{0-0}{x} = \lim_{x\to 0} 0 = 0$

$D_2 f(0,0,0) = \displaystyle\lim_{y\to 0}\frac{f(0,y,0) - f(0,0,0)}{y - 0} = \lim_{y\to 0}\frac{0-0}{y} = 0$; $D_3 f(0,0,0) = \displaystyle\lim_{z\to 0}\frac{f(0,0,z) - f(0,0,0)}{z - 0} = \lim_{z\to 0}\frac{0-0}{z} = 0$

(b) $\displaystyle\lim_{\substack{(x,y,z)\to(0,0,0)\\(x=y=z)}} f(x,y,z) = \lim_{x\to 0}\frac{x^4}{3x^4} = \lim_{x\to 0}\frac{1}{3} = \frac{1}{3} \neq 0 = f(0,0,0)$.

Therefore f is discontinuous at $(0,0,0)$, and so f is not differentiable at $(0,0,0)$.

34. $f(x,y,z) = \begin{cases} \dfrac{3yz}{x^4 + y^2 + z^2} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$

(a) $D_1 f(0,0,0) = \displaystyle\lim_{x\to 0}\frac{f(x,0,0) - f(0,0,0)}{x - 0} = \lim_{x\to 0}\frac{0-0}{x} = \lim_{x\to 0} 0 = 0$

$D_2 f(0,0,0) = \displaystyle\lim_{y\to 0}\frac{f(0,y,0) - f(0,0,0)}{y - 0} = \lim_{y\to 0}\frac{0-0}{y} = 0$; $D_3 f(0,0,0) = \displaystyle\lim_{z\to 0}\frac{f(0,0,z) - f(0,0,0)}{z - 0} = \lim_{z\to 0}\frac{0-0}{z} = 0$

(b) $\displaystyle\lim_{\substack{(x,y,z)\to(0,0,0)\\(y=z=x^2)}} f(x,y,z) = \lim_{x\to 0}\frac{3x^4}{3x^4} = \lim_{x\to 0} 1 = 1 \neq 0 = f(0,0,0)$.

Therefore f is discontinuous at $(0,0,0)$, and so f is not differentiable at $(0,0,0)$.

35. ℓ meters is the length, w m the width, and h m the height of the solid. If V m^3 is the volume of the solid, then

$V = \ell wh$; $dV = \dfrac{\partial V}{\partial \ell}\,d\ell + \dfrac{\partial V}{\partial w}\,dw + \dfrac{\partial V}{\partial h}\,dh = wh\,d\ell + \ell h\,dw + \ell w\,dh$

With $\ell = 8$, $d\ell = 0.08$, $w = 5$, $dw = 0.08$, $h = 4$, and $dh = 0.08$, we have

$dV = (5)(4)(0.08) + (8)(4)(0.08) + (8)(5)(0.08) = 1.60 + 2.56 + 3.20 = 7.36$. $\Delta V \approx 7.36$.

In fact, $\Delta V = (8.08)(5.08)(4.08) - (8)(5)(4) = 7.47$.

36. Use the total differential to find approximately the greatest error in calculating the area of a right triangle from the lengths of the legs if they are measured to be 6 cm and 8 cm respectively, with a possible error of 0.1 cm for each measurement. Also find the approximate percent error.

▶ Let x cm and y cm be the lengths of the legs of a right triangle, and let A cm^2 be the area. Then
$A = \frac{1}{2}xy$
We take the total differential of A. Thus,

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = \tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy$$

We are given that $x = 6$, $y = 8$, $|dx| \leq 0.1$, and $|dy| \leq 0.1$. Using the triangle inequality, we have

$$|dA| \leq \left|\tfrac{1}{2}y\right||dx| + \left|\tfrac{1}{2}x\right||dy| \leq 4(0.1) + 3(0.1) = 0.7$$

Thus, the greatest error in calculating the area is approximately 0.7 cm^2. Because $A = \frac{1}{2}(6)8 = 24$, the percent error is found by

$$\frac{|dA|}{A} \leq \frac{0.7}{24} = 0.029$$

Thus, the error is approximately 2.9% of the calculated error.

37. Let ℓ cm be the length of the hypotenuse and x cm and y cm be the lengths of the legs.
$\ell = \sqrt{x^2 + y^2}$; $d\ell = \dfrac{x}{\sqrt{x^2+y^2}}\,dx + \dfrac{y}{\sqrt{x^2+y^2}}\,dy$. Let $x = 6$, $y = 8$, $dx = 0.1$, and $dy = 0.1$. Then
$d\ell = \frac{6}{10}(\pm 0.1) + \frac{8}{10}(\pm 0.1) = \pm 0.14$; $\Delta\ell \approx 0.14$. Hence the greatest error in calculating the length of the hypotenuse is approximately 0.14 cm. $\frac{d\ell}{\ell} = \frac{\pm 0.14}{10} = \pm 0.014$, and so the approximate percent error is 1.4%.

38. $PV = kT$; $P = kTV^{-1}$, $\left|\frac{dP}{P}\right| = \left|\frac{kV^{-1}dT - kTV^{-2}dV}{kTV^{-1}}\right| = \left|\frac{dT}{T} - \frac{dV}{V}\right| \leq \left|\frac{dT}{T}\right| + \left|\frac{dV}{V}\right| = 0.3\% + 0.8\% = 1.1\%$

39. $s = \dfrac{A}{A - W}$. $ds = \dfrac{-W}{(A-W)^2}\,dA + \dfrac{A}{(A-W)^2}\,dW$. Let $A = 20$, $W = 12$, $dA = \pm 0.01$ and $dW = \pm 0.02$. Then
$s = \dfrac{20}{20 - 12} = 2.5$; $|ds| \leq \left|\frac{-12}{64}\cdot 0.01\right| + \left|\frac{20}{64}\cdot 0.02\right| = \frac{0.12}{64} + \frac{0.40}{64} = 0.008125$
The approximate relative error is $\frac{ds}{s} = \pm\frac{0.00875}{2.5} = \pm 0.00325 = \pm 0.325\%$.

40. A wooden box is to be made of lumber that is $\frac{2}{3}$ in. thick. The inside length is to be 6 ft; the inside width is to be 3 ft; the inside depth is to be 4 ft, and the box is to have no top. Use the total differential to find the approximate amount of lumber to be used in the box.

▶ Let x ft be the length, y ft the width, z ft be the depth, and V ft^3 be the volume of a rectangular box. Then
$V = xyz$
We have $x = 6$, $y = 3$, and $z = 4$. The thickness of the box is $\frac{2}{3}$ in $= \frac{1}{18}$ ft. Thus, take $dx = 2\cdot\frac{1}{18} = \frac{1}{9}$ and $dy = \frac{1}{9}$. Because there is no top for the box, we take $dz = \frac{1}{18}$. Hence,

$$\begin{aligned} dV &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \\ &= yz\,dx + xz\,dy + xy\,dz = 3\cdot 4\cdot\tfrac{1}{9} + 6\cdot 4\cdot\tfrac{1}{9} + 6\cdot 3\cdot\tfrac{1}{18} = 5 \end{aligned}$$

Therefore, approximately 5 ft^3 of lumber is needed.

41. Let the dimensions of one crate be x m, y m, z m. If A m^2 is the total surface area,
$A = 2(xy + xz + yz)$; $dA = 2[(y + z)dx + (x + z)dy + (x + y)dz]$
Let $x = 3$, $dx = \pm 0.005$, $y = 4$, $dy = \pm 0.005$, $z = 5$, and $dz = \pm 0.005$. Then
$\Delta A \approx dA = 2[9(\pm 0.005) + 8(\pm 0.005) + 7(\pm 0.005)] = \pm 0.240$
The greatest possible error in the amount of wood used for one crate is approximately 0.24 m^2. Thus, the greatest possible error in the estimate of the cost of wood for manufacturing 10,000 crates is 10,000(0.24)($3) = $7200.

In Exercises 42–45, we show that a function f may be differentiable at a point even though it is not continuously differentiable there, where

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

42. Find $\Delta f(0,0)$. See (2) below.

43. $D_1 f(0,0)=0$ and $D_2 f(0,0)=0$; see (3) and (4) below. If $(x,y)\neq(0,0)$, the formulas for differentiation give

$$D_1 f(x,y)=\begin{cases}2x\sin\dfrac{1}{\sqrt{x^2+y^2}}-\dfrac{x}{\sqrt{x^2+y^2}}\cos\dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

$$D_2 f(x,y)=\begin{cases}2y\sin\dfrac{1}{\sqrt{x^2+y^2}}-\dfrac{y}{\sqrt{x^2+y^2}}\cos\dfrac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

44. Prove that f is differentiable at $(0,0)$ by using Definition 12.4.2 and the results of Exercises 42 and 43.

▸ By Definition 12.4.2 we must find an ϵ_1 and an ϵ_2 where ϵ_1 and ϵ_2 are functions of Δx and Δy such that $\epsilon_1\to 0$ and $\epsilon_2\to 0$ as $(\Delta x,\Delta y)\to(0,0)$ and where ϵ_1 and ϵ_2 satisfy the equation

$$\Delta f(0,0)=D_1 f(0,0)\Delta x+D_2 f(0,0)\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y \tag{1}$$

By Definition 12.4.1,

$$\Delta f(0,0)=f(\Delta x,\Delta y)-f(0,0)=[(\Delta x)^2+(\Delta y)^2]\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \tag{2}$$

and by Definition 16.4.1,

$$D_1 f(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(\Delta x)^2\sin\dfrac{1}{|\Delta x|}}{\Delta x}=\lim_{\Delta x\to 0}\Delta x\sin\frac{1}{|\Delta x|}$$

Because

$$-1\le\sin\frac{1}{|\Delta x|}<1$$

then

$$-|\Delta x|\le|\Delta x|\sin\frac{1}{|\Delta x|}\le|\Delta x|$$

Therefore, by the squeeze theorem,

$$\lim_{\Delta x\to 0}|\Delta x|\sin\frac{1}{|\Delta x|}=0$$

Thus,

$$D_1 f(0,0)=0 \tag{3}$$

Interchanging x and y, we find also

$$D_2 f(0,0)=0 \tag{4}$$

Substituting from Eqs. (2), (3), and (4) into (1), we obtain

$$[(\Delta x)^2+(\Delta y)^2]\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\epsilon_1\Delta x+\epsilon_2\Delta y \tag{5}$$

Equation (5) is an identity if

$$\epsilon_1=\Delta x\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\quad\text{and}\quad\epsilon_2=\Delta y\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

Because

$$-1\le\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\le 1$$

then

$$-|\Delta x|\le\epsilon_1\le|\Delta x|$$

and

$$-|\Delta y|\le\epsilon_2\le|\Delta y|$$

so, by the squeeze theorem,

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\epsilon_1=0\quad\text{and}\quad\lim_{(\Delta x,\Delta y)\to(0,0)}\epsilon_2=0$$

Therefore, by Definition 12.4.2, f is differentiable at $(0,0)$.

45. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(y=0,\ x>0)}} D_1 f(x,y) = \lim_{x\to 0^+}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$. $\displaystyle\lim_{x\to 0^+} 2x\sin\frac{1}{x} = 0$ by the squeeze theorem but $\displaystyle\lim_{x\to 0^+}\cos\frac{1}{x}$ does not exist. Therefore, $\lim D_1 f(x,y)$ does not exist, and so $D_1 f$ is discontinuous at $(0,0)$. In a similar way we prove that $\displaystyle\lim_{(x,y)\to(0,0)} D_2 f(x,y)$ does not exist by showing that $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(x=0,\ y>0)}} D_2 f(x,y)$ does not exist.

46. $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

If $(x,y)\neq(0,0)$, $D_1 f(x,y) = \dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$ and $D_1 f(0,0) = \displaystyle\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{0-0}{\Delta x} = 0.$

If $(x,y)\neq(0,0)$, $D_2 f(x,y) = \dfrac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$ and $D_2 f(0,0) = \displaystyle\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{0-0}{\Delta y} = 0.$

Both $D_1 f$ and $D_2 f$ exist on every open disk centered at $(0,0)$. To show $D_1 f$ is continuous at $(0,0)$ we will show that $\displaystyle\lim_{(x,y)\to(0,0)} D_1 f(x,y) = 0$.

$$\left|\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}\right| \le \frac{|x^4||y|+4|x^2||y^3|+|y^5|}{(x^2+y^2)^2} \le \frac{6(\sqrt{x^2+y^2})^5}{(\sqrt{x^2+y^2})^4} = 6\sqrt{x^2+y^2}$$

Therefore, if $\delta = \frac{1}{6}\epsilon$, $|D_1 f(x,y)| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$ and so $\displaystyle\lim_{(x,y)\to(0,0)} D_1 f(x,y) = 0$. Hence $D_1 f$ is continuous at $(0,0)$. Similarly, $D_2 f$ is continuous at $(0,0)$. Hence, by Theorem 12.4.4 f is differentiable at $(0,0)$.

47. $f(x,y,z) = \begin{cases} \dfrac{xyz^2}{x^2+y^2+z^2} & \text{if } (x,y,z)\neq(0,0,0) \\ 0 & \text{if } (x,y,z)=(0,0,0) \end{cases}$. Let $r = \|(x,y,z)-(0,0,0)\| = \sqrt{x^2+y^2+z^2}$.

If $(x,y,z)\neq(0,0,0)$, then $D_1 f(x,y,z) = \dfrac{yz^2(x^2+y^2+z^2)-xyz^2(2x)}{r^4} = \dfrac{-x^2yz^2+y^3z^2+yz^4}{r^4}$

$|D_1 f(x,y,z)| \le \dfrac{3r^5}{r^4} = 3r \to 0$ as $r\to 0$ and $D_1 f(0,0,0) = \displaystyle\lim_{x\to 0}\frac{f(x,0,0)-f(0,0,0)}{x-0} = \lim_{x\to 0}\frac{0}{x} = 0.$

Similarly, $D_2 f(x,y,z)$ and $D_3 f(x,y,z)$ are continuous at $(0,0,0)$. By Thm 12.2.4, f is differentiable at $(0,0,0)$.

12.5 THE CHAIN RULE FOR FUNCTIONS OF MORE THAN ONE VARIABLE

12.5.1 Theorem (The Chain Rule). If u is a differentiable function of x and y, defined by $u = f(x,y)$, and $x = F(r,s)$, $y = G(r,s)$ and $\partial x/\partial r$, $\partial x/\partial s$, $\partial y/\partial r$, and $\partial y/\partial s$ all exist, then u is a function of r and s and

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

If in Theorem 12.5.1 both x and y are differentiable functions of the single variable t, then u is a function of t and the total derivative of u with respect to t is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \qquad (1)$$

The chain rule may be extended to functions of three or more variables as follows.

12.5.2 Theorem (The General Chain Rule) Suppose that u is a differentiable function of the n variables x_1, x_2, …, x_n and each of these variables is in turn a function of the m variables $y_1, y_2, \ldots, y_m$. Suppose further that each of the mn partial derivatives $\partial x_i/\partial y_j$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$) exists. Then u is a function of $y_1, y_2, \ldots, y_m$, and

$$\frac{\partial u}{\partial y_1} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_1} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_1} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_1}$$
$$\frac{\partial u}{\partial y_2} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_2} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_2} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_2}$$
$$\vdots$$
$$\frac{\partial u}{\partial y_m} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_m} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_m} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_m}$$

If in Theorem 12.5.2 each x_i is a differentiable function of t, then u is a function of t, and the total derivative of u with respect to t is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dt} + \cdots + \frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dt} \tag{2}$$

If t also appears explicitly in u, then we must add the term $\partial u/\partial t$ to Eqs. (1) and (2). See Exercises 18, 21, and 22.

The next two theorems tell us how to calculate derivatives of functions defined implicitly.

12.5.3 Theorem Let $F(x,y)$ be a function defined in an open disk $B((x_0,y_0);r)$ such that F has continuous first partial derivatives in B and that $F(x_0,y_0)=0$, $F_y(x_0,y_0)\neq 0$. Then inside some smaller disk $B((x_0,y_0);s)$ there is a unique function $y=f(x)$ such that $F(x,y)\equiv 0$. y has a continuous derivative given by

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$

12.5.4 Theorem Let $F(x,y,z)$ be a function defined in an open ball $B((x_0,y_0,z_0);r)$ such that F has continuous first partial derivatives in B and that $F(x_0,y_0,z_0)=0$, $F_z(x_0,y_0,z_0)\neq 0$. Then inside some smaller disk $B((x_0,y_0,z_0);s)$ there is a unique function $z=f(x,y)$ such that $F(x,y,z)\equiv 0$. z has continuous partial derivatives given by

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \quad\text{and}\quad \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

Cauchy-Riemann If $u=f(x,y)$ and $v=g(x,y)$, the Cauchy-Riemann equations are $u_x=v_y$ and $v_x=-u_y$. See Exercises 41, 42, and 52.

Exercises 12.5

In Exercises 1–6, find the indicated partial derivative by two methods: (a) use the chain rule and (b) make the substitutions for x and y before differentiating.

1. $u=x^2-y^2$; $x=3r-s$; $y=r+2s$

▸ (a) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = 2x(3)-2y(1) = 6x-2y$; $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = 2x(-1)-2y(2) = -2x-4y$

(b) $u=(3r-s)^2-(r+2s)^2 = 9r^2-6rs+s^2-r^2-4rs-4s^2 = 8r^2-10rs-3s^2$. $\frac{\partial u}{\partial r} = 16r-10s$; $\frac{\partial u}{\partial s} = -10r-6s$

2. $u=3x-4y^2$; $x=5pq$; $y=3p^2-2q$

(a) $\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial p} = 3(5q)-8y(6p) = 15q-48py$; $\frac{\partial u}{\partial q} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial q} = 3(5p)-8y(-2) = 15p+16y$

(b) $u=3(5pq)-4(3p^2-2q)^2 = 15pq-36p^4+48p^2q-16q^2$. $\frac{\partial u}{\partial p} = 15q-144p^3+96pq$. $\frac{\partial u}{\partial q} = 15p+48p^2$

3. $u=3x^2+xy-2y^2+3x-y$; $x=2r-3s$; $y=r+s$

▸ (a) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = (6x+y+3)(2)+(x-4y-1)(1) = 13x-2y+5$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = (6x+y+3)(-3)+(x-4y-1)(1) = -17x-7y-10$

(b) $u=3(2r-3s)^2+(2r-3s)(r+s)-2(r+s)^2+3(2r-3s)-(r+s)$
$=12r^2-36rs+27s^2+2r^2-rs-3s^2-2r^2-4rs-2s^2+6r-9s-r-s = 12r^2-41rs+22s^2+5r-10s$

$\frac{\partial u}{\partial r} = 24r-41s+5$; $\frac{\partial u}{\partial s} = -41r+44s-10$.

4. $u = x^2 + y^2$; $x = \cosh r \cos t$; $y = \sinh r \sin t$; $\partial u/\partial r$; $\partial u/\partial t$

▷ (a) $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial r} = 2x \sinh r \cos t + 2y \cosh r \sin t$

$= 2(\cosh r \cos t)\sinh r \cos t + 2(\sinh r \sin t)\cosh r \sin t$

$\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial t} = -2x \cosh r \sin t + 2y \sinh r \cos t$

$= -2(\cosh r \cos t)\cosh r \sin t + 2(\sinh r \sin t)\sinh r \cos t$

(b) Substituting before differentiating, we have

$u = \cosh^2 r \cos^2 t + \sinh^2 r \sin^2 t$

$\partial u/\partial r = 2 \cosh r \sinh r \cos^2 t + 2 \sinh r \cosh r \sin^2 t$ $\qquad \partial u/\partial t = -2 \cosh^2 r \cos t \sin t + 2 \sinh^2 r \sin t \cos t$

The results of (a) and (b) agree.

5. $u = e^{y/x}$; $x = 2r \cos t$; $y = 4r \sin t$

▷ (a) $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} = e^{y/x}\left(-\dfrac{y}{x^2}\right)(2 \cos t) + e^{y/x}\left(\dfrac{1}{x}\right)(4 \sin t) = \dfrac{2e^{y/x}}{x^2}(2x \sin t - y \cos t)$

$\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t} = e^{y/x}\left(-\dfrac{y}{x^2}\right)(-2r \sin t) + e^{y/x}\left(\dfrac{1}{x}\right)(4r \cos t) = \dfrac{2re^{y/x}}{x^2}(y \sin t + 2x \cos t)$

(b) $u = e^{4r \sin t/2r \cos t} = e^{2 \tan t}$. $\dfrac{\partial u}{\partial r} = 0$; $\dfrac{\partial u}{\partial t} = 2e^{2 \tan t} \sec^2 t$.

6. $V = \pi x^2 y$; $x = \cos z \sin t$; $y = z^2 e^t$

▷ (a) $\dfrac{\partial V}{\partial z} = \dfrac{\partial V}{\partial x}\dfrac{\partial x}{\partial z} + \dfrac{\partial V}{\partial y}\dfrac{\partial y}{\partial z} = 2\pi xy(-\sin z \sin t) + \pi x^2(2ze^t)$; $\dfrac{\partial V}{\partial t} = \dfrac{\partial V}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial V}{\partial y}\dfrac{\partial y}{\partial t} = 2\pi xy(\cos z \cos t) + \pi x^2(z^2e^t)$

(b) $V = \pi \cos^2 z \sin^2 t\, z^2 e^t$

$\dfrac{\partial V}{\partial z} = \pi \sin^2 t\, e^t(-2 \cos z \sin z\, z^2 + 2 \cos^2 z\, z)$; $\dfrac{\partial V}{\partial t} = \pi \cos^2 z\, z^2(2 \sin t \cos t\, e^t + \sin^2 t\, e^t)$

In Exercises 7–14, find the indicated partial derivatives by using the chain rule.

7. $u = x^2 + xy$; $x = r^2 + s^2$; $y = 3r - 2s$

▷ $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} = (2x + y)(2r) + x(3) = 2r(2x + y) + 3x$

$\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} = (2x + y)(2s) + x(-2) = 2s(2x + y) - 2x$

8. $u = xy + xz + yz$; $x = rs$; $y = r^2 - s^2$; $z = (r - s)^2$; $\partial u/\partial r$; $\partial u/\partial s$

▷ $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\cdot\dfrac{\partial z}{\partial r} = (y + z)s + (x + z)(2r) + (x + y)2(r - s)$

$\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial s} + \dfrac{\partial u}{\partial z}\cdot\dfrac{\partial z}{\partial s} = (y + z)r + (x + z)(-2s) + (x + y)(-2)(r - s)$

9. $u = \sin^{-1}(3x + y)$; $x = r^2 e^s$; $y = \sin rs$

▷ $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} = \dfrac{3}{\sqrt{1 - (3x + y)^2}}(2re^s) + \dfrac{1}{\sqrt{1 - (3x + y)^2}}(s \cos rs) = \dfrac{6re^s + s \cos rs}{\sqrt{1 - (3x + y)^2}}$

$\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} = \dfrac{3}{\sqrt{1 - (3x + y)^2}}(r^2e^s) + \dfrac{1}{\sqrt{1 - (3x + y)^2}}(r \cos rs) = \dfrac{3r^2e^s + r \cos rs}{\sqrt{1 - (3x + y)^2}}$

10. $u = \sin xy$; $x = 2ze^t$; $y = t^2 e^{-z}$

▷ $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t} = y(\cos xy)2ze^t + x(\cos xy)2te^{-z}$; $\dfrac{\partial u}{\partial z} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial z} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial z} = y(\cos xy)2e^t + x(\cos xy)(-t^2e^{-z})$

11. $u = \cosh \dfrac{y}{x}$; $x = 3r^2 s$; $y = 6se^r$

▷ $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} = \sinh \dfrac{y}{x}\left(-\dfrac{y}{x^2}\right)(6rs) + \sinh \dfrac{y}{x}\left(\dfrac{1}{x}\right)(6se^r) = \dfrac{6s}{x^2} \sinh \dfrac{y}{x}(xe^r - ry)$

$\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} = \sinh \dfrac{y}{x}\left(-\dfrac{y}{x^2}\right)(3r^2) + \sinh \dfrac{y}{x}\left(\dfrac{1}{x}\right)(6e^r) = \dfrac{3}{x^2} \sinh \dfrac{y}{x}(2xe^r - yr^2) = 0$

12. $u = e^{-y}$; $x = \tan^{-1}(rst)$; $y = \ln(3rs + 5st)$; $\partial u/\partial r$; $\partial u/\partial s$; $\partial u/\partial t$

▷ We have $y = \ln s + \ln(3r + 5t)$. Then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = e^{-y}\frac{st}{1 + r^2s^2t^2} - xe^{-y}\frac{3}{3r + 5t}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} = e^{-y}\frac{rt}{1 + r^2s^2t^2} - xe^{-y}\frac{1}{s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} = e^{-y}\frac{rs}{1 + r^2s^2t^2} - xe^{-y}\frac{5}{3r + 5t}$$

13. $u = x^2 + y^2 + z^2$; $x = r\sin\phi\cos\theta$; $y = r\sin\phi\sin\theta$; $z = r\cos\phi$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = 2x\sin\phi\cos\theta + 2y\sin\phi\sin\theta + 2z\cos\phi$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \phi} = 2xr\cos\phi\cos\theta + 2yr\cos\phi\sin\theta - 2zr\sin\phi$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = -2xr\sin\phi\sin\theta + 2yr\sin\phi\cos\theta + 0$$

14. $u = x^2yz$; $x = rs^{-1}$; $y = re^s$; $z = re^{-s}$. $u_r = u_xx_r + u_yy_r + u_zz_r = 2xyzs^{-1} + x^2ze^s + x^2ye^{-s}$;

$$u_s = u_xx_s + u_yy_s + u_zz_s = 2xyz(-rs^{-2}) + x^2zre^s + x^2y(-re^{-s})$$

In Exercises 15–18, find the total derivative du/dt by two methods: (a) Use the chain rule; (b) make the substitutions for x and y (and z) before differentiating.

15. $u = ye^x + xe^y$; $x = \cos t$; $y = \sin t$

▷ (a) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} = (ye^x + e^y)(-\sin t) + (e^x + xe^y)(\cos t) = e^x(\cos t - y\sin t) + e^y(x\cos t - \sin t)$

(b) $u = \sin t\, e^{\cos t} + \cos t\, e^{\sin t}$

$$\frac{du}{dt} = \cos t\, e^{\cos t} + \sin t\, e^{\cos t}(-\sin t) - \sin t\, e^{\sin t} + \cos t\, e^{\sin t}(\cos t) = e^{\cos t}(\cos t - \sin^2 t) + e^{\sin t}(\cos^2 t - \sin t)$$

16. $u = \ln xy + y^2$; $x = e^t$; $y = e^{-t}$

▷ We have $u = \ln x + \ln y + y^2$.

(a) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dt} = \dfrac{1}{x}e^t + \left(\dfrac{1}{y} + 2y\right)(-e^{-t}) = e^{-t}e^t + (e^t + 2e^{-t})(-e^{-t}) = -2e^{-2t}$

(b) Substituting for x and y, we have

$$u = \ln e^t + \ln e^{-t} + (e^{-t})^2 = t - t + e^{-2t} = e^{-2t}$$

$$\frac{du}{dt} = -2e^{-2t}$$

17. $u = \sqrt{x^2 + y^2 + z^2}$; $x = \tan t$; $y = \cos t$; $z = \sin t$; $0 < t < \frac{1}{2}\pi$. (a) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dt}$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}}(\sec^2 t) + \frac{y}{\sqrt{x^2 + y^2 + z^2}}(-\sin t) + \frac{z}{\sqrt{x^2 + y^2 + z^2}}(\cos t) = \frac{x\sec^2 t - y\sin t + z\cos t}{\sqrt{x^2 + y^2 + z^2}}$$

(b) $u = \sqrt{\tan^2 t + \cos^2 t + \sin^2 t} = \sqrt{\tan^2 t + 1} = \sqrt{\sec^2 t} = |\sec t| = \sec t$. $\dfrac{du}{dt} = \sec t\tan t$.

18. $u = \dfrac{t + e^x}{y - e^t}$; $x = 3\sin t$; $y = \ln t$.

▷ (a) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial u}{\partial t} = \dfrac{e^x}{y - e^t}(3\cos t) - \dfrac{t + e^x}{(y - e^t)^2}\cdot\dfrac{1}{t} + \dfrac{1(y - e^t) + (t + e^x)e^t}{(y - e^t)^2}$

(b) $u = \dfrac{t + e^{3\sin t}}{\ln t - e^t}$. $\dfrac{du}{dt} = \dfrac{(1 + 3\cos t\, e^{3\sin t})(\ln t - e^t) - (t + e^{3\sin t})(1/t - e^t)}{(\ln t - e^t)^2}$

In Exercises 19–22, find the total derivative du/dt by using the chain rule.

19. $u = \tan^{-1}\left(\dfrac{y}{x}\right)$; $x = \ln t$; $y = e^t$

▷ $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} = \dfrac{1}{1 + \dfrac{y^2}{x^2}}\left(-\dfrac{y}{x^2}\right)\left(\dfrac{1}{t}\right) + \dfrac{1}{1 + \dfrac{y^2}{x^2}}\left(\dfrac{1}{x}\right)e^t = \dfrac{-y}{t(x^2 + y^2)} + \dfrac{xe^t}{x^2 + y^2} = \dfrac{txe^t - y}{t(x^2 + y^2)}$

20. $u = xy + xz + yz$; $x = t\cos t$; $y = t\sin t$; $z = t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} = (y + z)(\cos t - t\sin t) + (x + z)(\sin t + t\cos t) + (x + y)$$

21. $u = \frac{x+t}{y+t}$; $x = \ln t$; $y = \ln \frac{1}{t} = -\ln t$. $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial t} = \frac{1}{y+t}\left(\frac{1}{t}\right) + \left[-\frac{x+t}{(y+t)^2}\right]\left(-\frac{1}{t}\right) + \frac{(y+t)-(x+t)}{(y+t)^2}$

$= \frac{y+t+x+t}{t(y+t)^2} + \frac{y-x}{(y+t)^2} = \frac{x+y+2t+ty-tx}{t(y+t)^2}$

22. $u = \ln(x^2+y^2+t^2)$; $x = t\sin t$; $y = \cos t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial t} = \frac{2x}{x^2+y^2+t^2}(\sin t + t\cos t) + \frac{2y}{x^2+y^2+t^2}(-\sin t) + \frac{2t}{x^2+y^2+t^2}$$

In Exercises 23–26, use Theorem 12.5.3: $\frac{dy}{dx} = -\frac{\mathrm{F}_x(x,y)}{\mathrm{F}_y(x,y)}$. Compare with the indicated exercise in Exercises 2.9.

23. $x^3 + y^3 - 8xy = 0$. $\frac{dy}{dx} = -\frac{\mathrm{F}_x(x,y)}{\mathrm{F}_y(x,y)} = -\frac{3x^2-8y}{3y^2-8x}$

24. $2x^3y + 3xy^3 = 5$; Exercise 24

▸ We have $\mathrm{F}(x,y) = 2x^3y + 3xy^3 - 5 = 0$.

$\frac{dy}{dx} = -\frac{\mathrm{F}_x(x,y)}{\mathrm{F}_y(x,y)} = -\frac{6x^2y+3y^3}{2x^3+9xy^2}$

We get the same result as Exercise 2.9.24 with less work.

25. $x\sin y + y\cos x - 1 = 0$. $\frac{dy}{dx} = \frac{\sin y - y\sin x}{x\cos y + \cos x}$

26. $\cos(x+y) - y\sin x = 0$. $\frac{dy}{dx} = -\frac{-\sin(x+y) - y\cos x}{-\sin(x+y) - \sin x}$

In Exercises 27–30, assume that the equation defines z as a function of x and y. Find $\partial z/\partial x$ and $\partial z/\partial y$ by two methods: (a) use Theorem 12.5.4; (b) differentiate implicitly.

27. $3x^2 + y^2 + z^2 - 3xy + 4xz - 15 = 0$ (a) $\frac{\partial z}{\partial x} = -\frac{\mathrm{F}_x(x,y,z)}{\mathrm{F}_z(x,y,z)} = -\frac{6x-3y+4z}{2z+4x}$; $\frac{\partial z}{\partial y} = -\frac{\mathrm{F}_y(x,y,z)}{\mathrm{F}_z(x,y,z)} = -\frac{2y-3x}{2z+4x}$

(b) $6x + 2z\frac{\partial z}{\partial x} - 3y + 4z + 4x\frac{\partial z}{\partial x} = 0$; $\frac{\partial z}{\partial x} = \frac{3y-6x-4z}{4x+2z}$. $2y + 2z\frac{\partial z}{\partial y} - 3x + 4x\frac{\partial z}{\partial y} = 0$; $\frac{\partial z}{\partial y} = \frac{3x-2y}{4x+2z}$.

28. $z = (x^2+y^2)\sin xz$

▸ (a) We have $\mathrm{F}(x,y,z) = (x^2+y^2)\sin xz - z = 0$. By Theorem 12.5.4,

$\frac{\partial z}{\partial x} = -\frac{\mathrm{F}_x(x,y,z)}{\mathrm{F}_z(x,y,z)} = -\frac{2x\sin xz + (x^2+y^2)z\cos xz}{(x^2+y^2)x\cos xz - 1}$

$\frac{\partial z}{\partial y} = -\frac{\mathrm{F}_y(x,y,z)}{\mathrm{F}_z(x,y,z)} = -\frac{2y\sin xz}{(x^2+y^2)x\cos xz - 1}$

(b) For $\partial z/\partial x$, we treat y as a constant and differentiate on both sides with respect to x. By the product rule,

$$\frac{\partial z}{\partial x} = (x^2+y^2)\frac{\partial(\sin xz)}{\partial x} + \sin xz\frac{\partial(x^2+y^2)}{\partial x} = (x^2+y^2)(\cos xz)\frac{\partial(xz)}{\partial x} + (\sin xz)2x$$

$$= (x^2+y^2)(\cos xz)\left(x\frac{\partial z}{\partial x} + z\right) + 2x\sin xz$$

Solving for $\partial z/\partial x$, we have

$$\frac{\partial z}{\partial x} = [x(x^2+y^2)\cos xz]\frac{\partial z}{\partial x} + z(x^2+y^2)\cos xz + 2x\sin xz$$

$$[1 - x(x^2+y^2)\cos xz]\frac{\partial z}{\partial x} = z(x^2+y^2)\cos xz + 2x\sin xz$$

$$\frac{\partial z}{\partial x} = \frac{z(x^2+y^2)\cos xz + 2x\sin xz}{1 - x(x^2+y^2)\cos xz}$$

For $\partial z/\partial y$, we regard x as a constant and differentiate on both sides with respect to y. Thus,

$$\frac{\partial z}{\partial y} = (x^2+y^2)(\cos xz)(x)\frac{\partial z}{\partial x} + (\sin xz)2y$$

$$[1 - x(x^2+y^2)\cos xz]\frac{\partial z}{\partial y} = 2y\sin xz$$

$$\frac{\partial z}{\partial y} = \frac{2y\sin xz}{1 - x(x^2+y^2)\cos xz}$$

29. $ye^{xyz}\cos 3xz - 5 = 0$. (a) $\dfrac{\partial z}{\partial x} = -\dfrac{F_x(x,y,z)}{F_z(x,y,z)} = -\dfrac{y^2ze^{xyz}\cos 3xz - 3yz\sin 3xz}{xy^2e^{xyz}\cos 3xz - 3xy\sin 3xz} = -\dfrac{z}{x}$;

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = -\frac{(e^{xyz} + xyze^{xyz}) + \cos 3xz}{xy^2e^{xyz}\cos 3xz - 3xy\sin 3xz}$$

(b) $ye^{xyz}\left(yz + xy\dfrac{\partial z}{\partial x}\right)\cos 3xz + ye^{xyz}(-\sin 3xz)\left(3z + 3x\dfrac{\partial z}{\partial x}\right) = 0$

$(y^2e^{xyz}\cos 3xz - 3ye^{xyz}\sin 3xz)\left(z + x\dfrac{\partial z}{\partial x}\right) = 0$; $\dfrac{\partial z}{\partial x} = -\dfrac{z}{x}$

$e^{xyz}\cos 3xz + ye^{xyz}\left(xz + xy\dfrac{\partial z}{\partial y}\right)\cos 3xz + ye^{xyz}(-\sin 3xz)\left(3x\dfrac{\partial z}{\partial y}\right) = 0$

$(xy^2\cos 3xz - 3xy\sin 3xz)\dfrac{\partial z}{\partial y} = -(xyz+1)\cos 3xz$; $\dfrac{\partial z}{\partial y} = \dfrac{-(xyz+1)\cos 3xz}{xy^2\cos 3xz - 3xy\sin 3xz} = \dfrac{xyz+1}{3xy\tan 3xz - xy^2}$

30. $ze^{yz} + 2xe^{xz} - 4e^{xy} - 3 = 0$. $\dfrac{\partial z}{\partial x} = -\dfrac{F_x(x,y,z)}{F_z(x,y,z)} = -\dfrac{2e^{xz} + 2xze^{xz} - 4ye^{xy}}{e^{yz} + yze^{yz}}$; $\dfrac{\partial z}{\partial y} = -\dfrac{F_y(x,y,z)}{F_z(x,y,z)} = -\dfrac{z^2e^{yz} - 4xe^{xy}}{e^{yz}(1+yz)}$

31. $z = f(bx - ay)$. Let $u = bx - ay$. Then $\dfrac{\partial z}{\partial x} = f_u\dfrac{\partial u}{\partial x} = bf_u$ and $\dfrac{\partial z}{\partial y} = f_u\dfrac{\partial u}{\partial y} = -af_u$.

Therefore $a\dfrac{\partial z}{\partial x} + b\dfrac{\partial z}{\partial y} = a(bf_u) + b(-af_u) = 0$

32. If f is a differentiable function of two variables u and v, let $u = x - y$ and $v = y - x$ and prove that $f(x-y, y-x)$ satisfies the equation $\partial z/\partial x + \partial z/\partial y = 0$.

▸ We have $z = f(u,v)$. By the chain rule,

$$\frac{\partial z}{\partial x} = f_1(u,v)\frac{\partial u}{\partial x} + f_2(u,v)\frac{\partial v}{\partial x} = f_1(u,v) - f_2(u,v) \tag{1}$$

and

$$\frac{\partial z}{\partial y} = f_1(u,v)\frac{\partial u}{\partial y} + f_2(u,v)\frac{\partial v}{\partial y} = -f_1(u,v) + f_2(u,v) \tag{2}$$

Adding the members of (1) and (2), we obtain the desired result

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

33. $u = f(x,y)$; $x = \cosh v\cos w$; $y = \sinh v\sin w$

▸ $\dfrac{\partial u}{\partial v} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial v} = \sinh v\cos w\dfrac{\partial u}{\partial x} + \cosh v\sin w\dfrac{\partial u}{\partial y}$

$\dfrac{\partial u}{\partial w} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial w} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial w} = -\cosh v\sin w\dfrac{\partial u}{\partial x} + \sinh v\cos w\dfrac{\partial u}{\partial y}$

In Exercises 34–39, find the second partial derivative two ways: (a) first substitute; (b) use the chain rule.

34. $u = e^y\cos x$, $x = 2t$, $y = t^2$

▸ (a) $u = e^{t^2}\cos 2t$. $u_t = 2e^{t^2}(t\cos 2t - \sin 2t)$. $u_{tt} = 2e^{t^2}(2t^2\cos 2t - 4t\sin 2t - \cos 2t)$

(b) $\dfrac{du}{dt} = \dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dt} = -e^y\sin x(2) + e^y\cos x(2t) = 2e^y(t\cos x - \sin x)$

$\dfrac{d^2u}{dt^2} = \dfrac{\partial}{\partial x}\left(\dfrac{du}{dt}\right)\dfrac{dx}{dt} + \dfrac{\partial}{\partial y}\left(\dfrac{du}{dt}\right)\dfrac{dy}{dt} + \dfrac{\partial}{\partial t}\left(\dfrac{du}{dt}\right) = 2e^y(-t\sin x - \cos x)(2) + 2e^y(t\cos x - \sin x)(2) + 2e^y\cos x$

35. $u = 3xy - 4y^2$, $x = 2se^r$, and $y = re^{-s}$

(a) $u = 3(2se^r)(re^{-s}) - 4(re^{-s})^2 = 6rse^{r-s} - 4r^2e^{-2s}$, $\dfrac{\partial u}{\partial r} = 6se^{r-s} + 6rse^{r-s} - 8re^{-2s}$

$\dfrac{\partial^2 u}{\partial r^2} = 6se^{r-s} + 6se^{r-s} + 6rse^{r-s} - 8e^{-2s} = 6se^{r-s}(2+r) - 8e^{-2s} = 12se^{r-s} + 6rse^{r-s} - 8e^{-2s}$

(c) $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} = 3y(2se^r) + (3x - 8y)e^{-s}$. $\dfrac{\partial^2 u}{\partial r^2} = 3e^{-s}(2se^r) + 3y(2se^r) + (3\cdot 2se^r - 8\cdot e^{-s})e^{-s}$

$= 6se^{r-s} + 3re^{-s}(2se^r) + 6se^{r-s} - 8e^{-2s} = 12se^{r-s} + 6rse^{r-s} - 8e^{-2s}$

36. $u = 3xy - 4y^2$, $x = 2se^r$, $y = re^{-s}$; $\partial^2 u/\partial s\partial r$

▷ (a) $u = 3(2se^r)(re^{-s}) - 4(re^{-s})^2 = 6rse^{r-s} - 4r^2e^{-2rs}$

Thus,

$$\frac{\partial u}{\partial r} = -6rse^{r-s} + 6se^{r-s} - 8re^{-2s}$$

Hence

$$\frac{\partial^2 u}{\partial s\partial r} = -6rse^{r-s} + 6re^{r-s} - 6se^{r-s} + 6e^{r-s} + 16re^{-2s}$$

(b) By using the chain rule, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = 3y\cdot 2se^r + (3x - 8y)e^{-s} = 6yse^r + (3x - 8y)e^{-s}$$

Partial-differentiating with respect to s and using the formula for the derivative of a product, we obtain

$$\begin{aligned}\frac{\partial^2 u}{\partial s\partial r} &= 6e^r\left(\frac{\partial y}{\partial s}s + 1\cdot y\right) + \frac{\partial(3x - 8y)}{\partial s}e^{-s} + (3x - 8y)(-e^{-s})\\ &= 6e^r(-rse^{-s} + y) + (6e^r + 8re^{-s})e^{-s} + (3x - 8y)(-e^{-s})\\ &= -6rse^{r-s} + 6ye^r + 6e^{r-s} + 8re^{-2s} - 3xe^{-s} + 8ye^{-s}\end{aligned}$$

In Exercises 37–39, $u = 9x^2 + 4y^2$; $x = r\cos\theta$; $y = r\sin\theta$

37. (a) $u = 9r^2\cos^2\theta + 4r^2\sin^2\theta$; $\frac{\partial u}{\partial r} = 18r\cos^2\theta + 8r\sin^2\theta$; $\frac{\partial^2 u}{\partial r^2} = 18\cos^2\theta + 8\sin^2\theta = 10\cos^2\theta + 8$

(b) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = 18x\cos\theta + 8y\sin\theta = 18r\cos^2\theta + 8r\sin^2\theta$; $\frac{\partial^2 u}{\partial r^2} = 18\cos^2\theta + 8\sin^2\theta = 10\cos^2\theta + 8$

38. (a) $u = 9r^2\cos^2\theta + 4r^2\sin^2\theta = r^2[\frac{9}{2}(1 + \cos 2\theta) + 2(1 - \cos 2\theta)] = r^2(\frac{13}{2} + \frac{5}{2}\cos 2\theta)$;

$\frac{\partial u}{\partial \theta} = -5r^2\sin 2\theta$; $\frac{\partial^2 u}{\partial \theta^2} = -10r^2\cos 2\theta$

(b) $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = 18x(-r\sin\theta) + 8y(r\cos\theta) = r(-18x\sin\theta + 8y\cos\theta)$;

$\frac{\partial^2 u}{\partial \theta^2} = r[-18(-r\sin\theta)\sin\theta - 18x\cos\theta + 8(r\cos\theta)\cos\theta + 8y(-\sin\theta)]$

39. (a) $u = 9r^2\cos^2\theta + 4r^2\sin^2\theta$; $\frac{\partial u}{\partial \theta} = -18r^2\cos\theta\sin\theta + 8r^2\sin\theta\cos\theta = -10r^2\sin\theta\cos\theta$

$\frac{\partial^2 u}{\partial r\partial \theta} = -20r\sin\theta\cos\theta = -10r\sin 2\theta$

(b) $\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = 18x(-r\sin\theta) + 8y(r\cos\theta) = -18r\sin\theta(r\cos\theta) + 8r\cos\theta(r\sin\theta)$

$= -10r^2\sin\theta\cos\theta$. $\frac{\partial^2 u}{\partial r\partial \theta} = 20r\sin\theta\cos\theta = -10r\sin 2\theta$.

40. Suppose f is a differentiable function of x, y, and z and $u = f(x, y, z)$. If $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, and $z = r\cos\phi$, express $\partial u/\partial r$, $\partial u/\partial\phi$, and $\partial u/\partial\theta$ in terms of $\partial u/\partial x$, $\partial u/\partial y$, *and* $\partial u/\partial z$.

▷
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r} = \frac{\partial u}{\partial x}\sin\phi\cos\theta + \frac{\partial u}{\partial y}\sin\phi\sin\theta + \frac{\partial u}{\partial z}\cos\phi$$

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \phi} = \frac{\partial u}{\partial x}r\cos\phi\cos\theta + \frac{\partial u}{\partial y}r\cos\phi\sin\theta - \frac{\partial u}{\partial z}r\sin\phi$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin\phi\sin\theta + \frac{\partial u}{\partial y}r\sin\phi\cos\theta$$

41. $u = \frac{1}{2}\ln(x^2 + y^2)$; $v = \tan^{-1}(\frac{y}{x})$

▷ $\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$; $\frac{\partial v}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)}\cdot\frac{1}{x} = \frac{x}{x^2 + y^2}$. Therefore $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

$\frac{\partial u}{\partial y} = \frac{1}{2}\cdot\frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$; $\frac{\partial v}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)}\cdot\frac{-y}{x^2} = \frac{-y}{x^2 + y^2}$. Therefore $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

42. (Cauchy-Riemann equation in polar coordinates) $x = r\cos\theta$, $y = r\sin\theta$.

▷ $u_r - r^{-1}v_\theta = (u_x x_r + u_y y_r) - r^{-1}(v_x x_\theta + v_y y_\theta) = (u_x\cos\theta + u_y\sin\theta) - r^{-1}(-v_x r\sin\theta + v_y r\cos\theta)$

$= (u_x - v_y)\cos\theta + (u_y + v_x)\sin\theta = 0 + 0 = 0$

$v_r + r^{-1}u_\theta = (v_x x_r + v_y y_r) + r^{-1}(u_x x_\theta + u_y y_\theta) = (v_x\cos\theta + v_y\sin\theta) + r^{-1}(-u_x r\sin\theta + u_y r\cos\theta)$

$= (v_x + u_y)\cos\theta + (v_y - u_x)\sin\theta = 0 + 0 = 0$

43. At t min, let x cm be the length of the first leg, y cm the length of the second leg, and θ radians the measure of the angle opposite the leg of length y cm. Then

$\theta = \tan^{-1}\frac{y}{x}; \frac{d\theta}{dt} = \frac{\partial\theta}{\partial x}\frac{dx}{dt} + \frac{\partial\theta}{\partial y}\frac{dy}{dt} = \frac{-y}{x^2+y^2}\frac{dx}{dt} + \frac{x}{x^2+y^2}$

At the given instant $x = 10$, $\frac{dx}{dt} = 1$, $y = 12$, $\frac{dy}{dt} = -2$, so $\frac{d\theta}{dt} = \frac{-12}{244}(1) + \frac{10}{244}(-2) = -\frac{8}{61}$. Hence the measure of the angle is decreasing at a rate of $\frac{8}{61}$ rad/min.

44. Water is flowing into a tank in the form of a right circular cylinder at the rate of $\frac{4}{3}\pi$ m^3/min. The tank is stretching in such a way that even though it remains cylindrical, it radius is increasing at the rate of 0.2 cm/min. How fast is the surface of the water rising when the radius is 2 m and the volume of water in the tank is 20π m^3?

▷ After t minutes, the radius of the tank is r meters, the depth of the water is h meters, and the volume of the water is V m^3. We solve the volume formula for h:

$V = \pi r^2 h$

$h = \frac{1}{\pi}r^{-2}V$

Because 0.2 cm = 0.002 m, we are given

$\frac{dV}{dt} = \frac{4}{5}\pi$ and $\frac{dr}{dt} = 0.002$

We calculate dh/dt by the chain rule and substitute $r = 2$ and $V = 20\pi$. Thus,

$$\frac{dh}{dt} = \frac{\partial h}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial h}{\partial V}\cdot\frac{dV}{dt} = \frac{1}{\pi}\left(-2r^{-3}V\frac{dr}{dt} + r^{-2}\frac{dV}{dt}\right) = \frac{1}{\pi}\left[-2(\tfrac{1}{8})20\pi(0.002) + \tfrac{1}{4}(\tfrac{4}{5}\pi)\right] = 0.19$$

Thus, the surface of the water is rising at the rate of 0.19 meters per minute.

45. At t min, let h cm be the height of the cylinder, r cm its radius and V cm^3 its volume.

$V = \pi r^2 h; \frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$

Because $h = 50$, $\frac{dh}{dh} = -10$, $r = 16$, $\frac{dr}{dt} = 4$, then $\frac{dV}{dt} = 2\pi(16)(50)(4) + \pi(16)^2(-10) = 3840\pi$.

Thus at the given instant the volume is increasing at a rate of 3840π cm^3/min.

46. At t min, h cm and r cm are the height and radius of the cone. $\frac{dh}{dt} = 40$, $\frac{dr}{dt} = -5$, $h = 200$, $r = 60$. $V = \frac{1}{3}\pi r^2 h$.

$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \frac{1}{3}\pi[2(60)(200)(-15) + 60^2(40)] = -72{,}000\pi$ cm^3/min (decreasing)

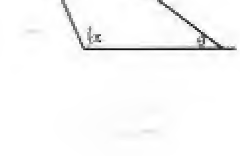
47. $PV = kT$; $V = \frac{kT}{P}$; $\frac{dV}{dt} = \frac{\partial V}{\partial T}\frac{dT}{dt} + \frac{\partial V}{\partial P}\frac{dP}{dt} = \frac{k}{P}\frac{dT}{dt} - \frac{kT}{P^2}\frac{dP}{dt}$.

Because $k = 1.2$, $T = 300$, $\frac{dT}{dt} = 3$, $P = 6$, $\frac{dP}{dt} = -0.1$, then $\frac{dV}{dt} = \frac{1.2}{6}(3) - \frac{1.2(300)}{6^2}(-0.1) = 1.6$.

Hence at the given instant the volume is increasing at a rate of 1.6 liters/min.

48. A wall makes an angle of radian measure $\frac{2}{3}\pi$ with the ground. A ladder of length 20 ft is leaning against the wall and its top is sliding down the wall at the rate of 3 ft/sec. How fast is the area of the triangle formed by the ladder, the wall and the ground changing when the ladder makes an angle of $\frac{1}{6}\pi$ radians with the ground?

▷ After t minutes, the top of the ladder is x feet from the ground and the bottom of the ladder makes an of θ with the ground and the area is A ft^2. See the figure. From the law of sines,



$\frac{x}{\sin\theta} = \frac{20}{\sin\frac{2}{3}\pi} = \frac{20}{\frac{1}{2}\sqrt{3}}$ and so $x = \frac{40}{\sqrt{3}}\sin\theta$

Because the angle between the wall and the ladder is $\pi - (\theta + \frac{2}{3}\pi) = \frac{1}{3}\pi - \theta$, then

$A = \frac{1}{2}(20)x\sin(\frac{1}{3}\pi - \theta) = \frac{400}{\sqrt{3}}\sin\theta\sin(\frac{1}{3}\pi - \theta)$

$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{400}{\sqrt{3}}[\cos\theta\sin(\frac{1}{3}\pi + \theta) - \sin\theta\cos(\frac{1}{3}\pi - \theta)]\frac{d\theta}{dt}$

When $\theta = \frac{1}{6}\pi$, we have

$\frac{dA}{dt} = \frac{400}{\sqrt{3}}(\cos\frac{1}{6}\pi\sin\frac{1}{6}\pi - \sin\frac{1}{6}\pi\cos\frac{1}{6}\pi)\frac{d\theta}{dt} = 0$

Thus, the triangle is isosceles and its area is not changing.

49. $\dfrac{\partial\beta}{\partial P}+\dfrac{\partial\kappa}{\partial T}=\dfrac{\partial}{\partial P}\left(V^{-1}\dfrac{\partial V}{\partial T}\right)-\dfrac{\partial}{\partial T}\left(V^{-1}\dfrac{\partial V}{\partial P}\right)=-V^{-2}\dfrac{\partial V}{\partial P}\dfrac{\partial V}{\partial T}+V^{-1}\dfrac{\partial^2 V}{\partial P\partial T}+V^{-2}\dfrac{\partial V}{\partial T}\dfrac{\partial V}{\partial P}-V^{-1}\dfrac{\partial^2 V}{\partial T\partial P}=0$

50. $(P+aV^{-2})(V-b)=RT$, $F(P,V,T)=P+aV^{-2}-RT(V-b)^{-1}=0$

$\beta=\dfrac{1}{V}\dfrac{\partial V}{\partial T}=-\dfrac{1}{V}\dfrac{\partial F/\partial T}{\partial F/\partial V}=-\dfrac{1}{V}\cdot\dfrac{-R(V-b)^{-1}}{-2aV^{-3}+RT(V-b)^{-2}}\cdot\dfrac{V^2(V-b)^2}{V^2(V-b)^2}=\dfrac{RV^2(V-b)}{RTV^3-2a(V-b)^2}$

$\kappa=-\dfrac{1}{V}\dfrac{\partial V}{\partial P}=\dfrac{1}{V}\dfrac{\partial F/\partial P}{\partial F/\partial V}=\dfrac{1}{V}\cdot\dfrac{1}{-2aV^{-3}+RT(v-b)^{-2}}\cdot\dfrac{V^2(V-b)^2}{V^2(V-b)^2}=\dfrac{V^2(V-b)^2}{RTV^3-2a(V-b)^2}$

If $a=b=0$, $\beta=\dfrac{RV^3}{RTV^3}=\dfrac{1}{T}$, $k=\dfrac{V^4}{RTV^3}=\dfrac{V}{RT}$

51. (Changing to polar coordinates) $u=f(x,y)$, $x=r\cos\theta$, and $y=r\sin\theta$. Then

$\dfrac{\partial u}{\partial r}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r}$; $\dfrac{\partial u}{\partial r}=\dfrac{\partial u}{\partial x}\cos\theta+\dfrac{\partial u}{\partial y}\sin\theta$ (1)

$\dfrac{\partial u}{\partial \theta}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \theta}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \theta}$; $\dfrac{\partial u}{\partial \theta}=\dfrac{\partial u}{\partial x}(-r\sin\theta)+\dfrac{\partial u}{\partial y}(r\cos\theta)$ (2)

Multiplying (1) by $r\cos\theta$ and (2) by $(-\sin\theta)$ and adding, we obtain

$r\cos\theta\,\dfrac{\partial u}{\partial r}-\sin\theta\,\dfrac{\partial u}{\partial \theta}=\dfrac{\partial u}{\partial x}(r\cos^2\theta+r\sin^2\theta)$; $\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial r}\cos\theta-\dfrac{\partial u}{\partial \theta}\dfrac{\sin\theta}{r}$

Multiplying (1) by $r\sin\theta$ and (2) by $\cos\theta$ and adding, we obtain

$r\sin\theta\,\dfrac{\partial u}{\partial r}+\cos\theta\,\dfrac{\partial u}{\partial \theta}=\dfrac{\partial u}{\partial y}(r\sin^2\theta+r\cos^2\theta)$; $\dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial r}\sin\theta+\dfrac{\partial u}{\partial \theta}\dfrac{\cos\theta}{r}$

52. Suppose that $u=f(x,y)$, $v=g(x,y)$, and f and g and their first and second partial derivatives are continuous. Prove that if u and v satisfy the Cauchy Riemann equations, they also satisfy Laplace's equation (§12.3).

▸ The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ \text{ and }\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \qquad (1)$$

Partial-differentiating with respect to x on both sides of the first equation in (1), we obtain

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y} \qquad (2)$$

Partial-differentiating with respect to y on both sides of the second equation in (1), we obtain

$$\frac{\partial^2 v}{\partial y\partial x}=-\frac{\partial^2 u}{\partial y^2} \qquad (3)$$

Because g and its first and second partial derivatives are continuous, then $\partial^2 v/\partial y\partial x=\partial^2 v/\partial x\partial y$. Thus, Eq.(3) is equivalent to

$$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial x\partial y} \qquad (4)$$

Adding the members of (2) and (4), we obtain

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

which is Laplace's equation. Similarly, we may partial-differentiate with respect to y on both sides of the fist equation in (1) and partial-differentiase with respect to x on both sides of the second equation in (1). We obtain

$$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 v}{\partial y^2}\ \text{ and }\ \frac{\partial^2 v}{\partial x^2}=-\frac{\partial^2 u}{\partial x\partial y} \qquad (5)$$

Because f and its first and second partial derivatives are continuous, then $\partial^2 u/\partial y\partial x=\partial^2 u/\partial x\partial y$. Thus, if we add the members of Eq.(5), the result is Laplace's equation for the function v:

$$\frac{\partial^2 v}{\partial x^2}+\frac{\theta^2 v}{\partial y^2}=0$$

12.6 DIRECTIONAL DERIVATIVES AND GRADIENTS

12.6.1 Definition Let f be a function of two variables x and y. If $\mathbf{U}$ is the unit vector $\cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, then the *directional derivative* of f in the direction of $\mathbf{U}$, denoted by $D_U f$, is given by

$$D_U f(x,y) = \lim_{h\to 0}\frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h}$$

if this limit exists.

12.6.2 Theorem If f is a differentiable function of x and y, and $\mathbf{U} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, then

$$D_U(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

12.6.3 Definition If f is a function of two variables x and y and f_x and f_y exist, then the *gradient* of f, denoted by ∇f (read: "del f") is defined by

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

The gradient of f can be used to find the directional derivative.

If f is a differentiable function of x and y and $\mathbf{U}$ is a unit vector, then

$$D_U f(x,y) = \mathbf{U}\cdot\nabla f(x,y)$$

12.6.4 Theorem Let f be a function of two variables and differentiable at (x_0, y_0) where $\nabla f(x_0, y_0) \neq 0$. Let $\mathbf{U}$ be any unit vector, so that $D_U f(x_0, y_0)$ is a function of $\mathbf{U}$.

(i) The maximum value of $D_U f(x_0, y_0)$ is $\|\nabla f(x_0, y_0)\|$, attained when $\mathbf{U}$ has the same direction as $\nabla f(x_0, y_0)$.

(ii) The minimum value of $D_U f(x_0, y_0)$ is $-\|\nabla f(x_0, y_0)\|$, attained when $\mathbf{U}$ has the opposite direction of $\nabla f(x_0, y_0)$.

We extend the definition of a directional derivative to a function of three variables.

12.6.5 Definition Let f be a function of three variables x, y, and z. If $\mathbf{U}$ is the unit vector $\cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$ then the *directional derivative* of f in the direction of $\mathbf{U}$, denoted by $D_U f$, is given by

$$D_U f(x,y,z) = \lim_{h\to 0}\frac{f(x + h\cos\alpha, y + h\cos\beta, z + h\cos\gamma) - f(x,y,z))}{h}$$

if this limit exists.

12.6.6 Theorem If f is a differentiable function of x, y, and z and $\mathbf{U} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$, then

$$D_U(x,y,z) = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_z(x,y,z)\cos\gamma$$

12.6.7 Definition If f is a function of three variables x, y, and z and f_x, f_y, and f_z exist, then the *gradient* of f, denoted by ∇f is defined by

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

If $\mathbf{U}$ is a unit vector in R^3, then

$$D_U f(x,y,z) = \mathbf{U}\cdot\nabla f(x,y,z)$$

Exercises 12.6

In Exercises 1–6, find the directional derivative of the function in the direction of the unit vector $\mathbf{U}$ by using the appropriate Definition (12.6.1 or 12.6.5). Verify your result by applying Theorem 12.6.2 or Theorem 12.6.6.

1. $f(x,y) = 2x^2 + 5y^2$; $\mathbf{U} = \cos\frac{1}{4}\pi\mathbf{i} + \sin\frac{1}{4}\pi\mathbf{j}$

▸ $D_U f(x,y) = \lim_{h\to 0}\dfrac{f(x + h\cos\frac{1}{4}\pi, y + h\sin\frac{1}{4}\pi) - f(x,y)}{h} = \lim_{h\to 0}\dfrac{2(x + \frac{1}{2}\sqrt{2}h)^2 + 5(y + \frac{1}{2}\sqrt{2}h)^2 - (2x^2 + 5y^2)}{h}$

$= \lim_{h\to 0}\dfrac{2x^2 + 2\sqrt{2}hx + h^2 + 5y^2 + 5\sqrt{2}hy + \frac{5}{2}h^2 - 2x^2 - 5y^2}{h} = \lim_{h\to 0}(2\sqrt{2}x + h + 5\sqrt{2}y + \frac{5}{2}h)$

$= 2\sqrt{2}x + 5\sqrt{2}y$

$D_U f(x,y) = f_x(x,y)\cos\frac{1}{4}\pi + f_y(x,y)\sin\frac{1}{4}\pi = 4x(\frac{1}{2}\sqrt{2}) + 10y(\frac{1}{2}\sqrt{2}) = 2\sqrt{2}x + 5\sqrt{2}y$

2. $g(x,y) = 3x^2 - 4y^2$; $\mathbf{U} = \cos\frac{1}{3}\pi\mathbf{i} + \sin\frac{1}{3}\pi\mathbf{j}$

▸ $D_U g(x,y) = \lim_{h\to 0}\dfrac{g(x + h\cos\frac{1}{3}\pi, y + h\sin\frac{1}{3}\pi) - g(x,y)}{h} = \lim_{h\to 0}\dfrac{3(x + \frac{1}{2}h)^2 - 4(y + \frac{1}{2}\sqrt{3}h)^2 - (3x^2 - 4y^2)}{h}$

$= \lim_{h\to 0}\dfrac{3x^2 + 3xh + \frac{3}{4}h^2 - 4y^2 - 4\sqrt{3}hy - 3h^2 - 3x^2 + 4y^2}{h} = \lim_{h\to 0}(3x + \frac{3}{4}h - 4\sqrt{3}y - 3h) = 3x - 4\sqrt{3}y$

$D_U g(x,y) = g_x(x,y)\cos\frac{1}{3}\pi + g_y(x,y)\sin\frac{1}{3}\pi = 6x(\frac{1}{2}) - 8y(\frac{1}{2}\sqrt{3}) = 3x - 4\sqrt{3}y$

3. $h(x,y,z) = 3x^2 + y^2 - 4z^2$; $\mathbf{U} = \cos \frac{1}{3}\pi\mathbf{i} + \cos \frac{1}{4}\pi\mathbf{j} + \cos \frac{2}{3}\pi\mathbf{k}$

▸ $$D_U h(x,y,z) = \lim_{t\to 0} \frac{h(x + t\cos\frac{1}{3}\pi, y + t\cos\frac{1}{4}\pi, z + t\cos\frac{2}{3}\pi) - h(x,y,z)}{t}$$

$$= \lim_{t\to 0} \frac{3tx + \frac{3}{4}t^2 + \sqrt{2}ty + \frac{1}{2}t^2 + 4tz - t^2}{t} = \lim_{t\to 0}(3x + \frac{3}{4}t + \sqrt{2}y + \frac{1}{2}t + 4z - t) = 3x + \sqrt{2}y + 4z$$

$$D_U f(x,y,z) = h_x(x,y,z)\cos\tfrac{1}{3}\pi + h_y(x,y,z)\cos\tfrac{1}{4}\pi + h_z(x,y,z)\cos\tfrac{2}{3}\pi = 6x(\tfrac{1}{2}) + 2y(\tfrac{1}{2}\sqrt{2}) - 8z(-\tfrac{1}{2})$$

$$= 3x + \sqrt{2}y + 4z$$

4. $f(x,y,z) = 6x^2 - 2xy + yz$; $\mathbf{U} = \frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$

▸ We use Definition 12.6.4 with

$\cos\alpha = \frac{3}{7}$ $\quad\cos\beta = \frac{2}{7}$ $\quad\cos\gamma = \frac{6}{7}$

Thus,

$$\begin{aligned}
D_U f(x,y,z) &= \lim_{h\to 0} \frac{f(x + \frac{3}{7}h, y + \frac{2}{7}h, z + \frac{6}{7}h) - f(x,y,z)}{h} \\
&= \lim_{h\to 0} \frac{[6(x + \frac{3}{7}h)^2 - 2(x + \frac{3}{7}h)(y + \frac{2}{7}h) + (y + \frac{2}{7}h)(z + \frac{6}{7}h)] - [6x^2 - 2xy + yz]}{h} \\
&= \lim_{h\to 0} \frac{6x^2 + \frac{36}{7}xh + \frac{54}{49}h^2 - 2xy - \frac{4}{7}xh - \frac{6}{7}yh - \frac{12}{49}h^2 + yz + \frac{6}{7}yh + \frac{2}{7}hz + \frac{12}{49}h^2 - 6x^2 + 2xy - yz}{h} \\
&= \lim_{h\to 0} \frac{\frac{32}{7}xh + \frac{54}{49}h^2 + \frac{2}{7}hz}{h} = \lim_{h\to 0}(\tfrac{32}{7}x + \tfrac{54}{49}h + \tfrac{2}{7}z) = \tfrac{32}{7}x + \tfrac{2}{7}z
\end{aligned}$$

Next, we apply Theorem 12.6.6. We have

$f_x(x,y,z) = 12x - 2y \quad f_y(x,y,z) = -2x + z \quad f_z(x,y,z) = y$

Substituting into the formula of the theorem, we get

$D_U f(x,y,z) = (12x - 2y)\frac{3}{7} + (-2x + z)\frac{2}{7} + \frac{6}{7}y = \frac{32}{7}x + \frac{2}{7}z$

Therefore, the result of using Definition 12.6.5 agrees with the result of applying Theorem 12.6.6.

5. $g(x,y) = \dfrac{1}{x-y}$; $\mathbf{U} = -\dfrac{12}{13}\,\mathbf{i} + \dfrac{5}{13}\,\mathbf{j}$

▸ $$D_U g(x,y) = \lim_{h\to 0} \frac{g(x - \frac{12}{13}h, y + \frac{5}{13}h) - g(x,y)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x - \frac{12}{13}h) - (y + \frac{5}{13}h)} - g(x,y)\right]$$

$$= \lim_{h\to 0} \frac{x - y - x + \frac{12}{13}h + y + \frac{5}{13}h}{h(x-y)(x - y - \frac{12}{13}h - \frac{5}{13}h)} = \lim_{h\to 0} \frac{\frac{17}{13}h}{h(x-y)(x - y - \frac{17}{13}h)} = \lim_{h\to 0} \frac{\frac{17}{13}}{(x-y)(x - y - \frac{17}{13}h)} = \frac{17}{13(x-y)^2}$$

$$D_U g(x,y) = g_x(x,y)\left(-\frac{12}{13}\right) + g_y(x,y)\left(\frac{5}{13}\right) = -\frac{1}{(x-y)^2}\left(-\frac{12}{13}\right) + \frac{1}{(x-y)^2}\left(\frac{5}{13}\right) = \frac{17}{13(x-y)^2}$$

6. $f(x,y) = \dfrac{1}{x^2 + y^2}$; $\mathbf{U} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$. $D_U f(x,y) = \lim\limits_{h\to 0} \dfrac{f(x + \frac{3}{5}h, y - \frac{4}{5}h) - f(x,y)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x + \frac{3}{5}h)^2 + (y - \frac{4}{5}h)^2} - \frac{1}{x^2 + y^2}\right] = \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x^2 + y^2) - (x^2 + \frac{6}{5}hx + \frac{9}{25}h^2 + y^2 - \frac{8}{5}yh + \frac{16}{25}h^2)}{[(x + \frac{3}{5}h)^2 + (y - \frac{4}{5}h)^2](x^2 + y^2)}$$

$$= \lim_{h\to 0} \frac{-\frac{6}{5}x - \frac{9}{25}h + \frac{8}{5}y - \frac{16}{25}h}{[(x + \frac{3}{5}h)^2 + (y - \frac{4}{5}h)^2](x^2 + y^2)} = \frac{-\frac{6}{5}x + \frac{8}{5}y}{(x^2 + y^2)^2}$$

$$D_U f(x,y) = f_x(x,y)\tfrac{3}{5} + f_y(x,y)(-\tfrac{4}{5}) = -\frac{2x}{(x^2 + y^2)^2} \cdot \frac{3}{5} - \frac{2y}{(x^2 + y^2)^2}(-\tfrac{4}{5}) = \frac{-\frac{6}{5}x + \frac{8}{5}y}{(x^2 + y^2)^2}$$

In Exercises 7–14, find the gradient of the function.

7. $f(x,y) = 4x^2 - 3xy + y^2$. $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y,z)\mathbf{j} = (8x - 3y)\mathbf{i} + (2y - 3x)\mathbf{j}$

8. $g(x,y) = \dfrac{xy}{x^2 + y^2}$

▸ We apply Definition 17.1.3. Thus,

$$\nabla g(x,y) = g_x(x,y)\mathbf{i} + g_y(x,y)\mathbf{j}$$

$$= \frac{(x^2 + y^2)(y) - (xy)(2x)}{(x^2 + y^2)^2}\mathbf{i} + \frac{(x^2 + y^2)(x) - (xy)(2y)}{(x^2 + y^2)^2}\mathbf{j} = \frac{-x^2y + y^3}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^3 - xy^2}{(x^2 + y^2)^2}\mathbf{j}$$

9. $g(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$. $\nabla g(x,y) = g_x(x,y)\mathbf{i} + g_y(x,y)\mathbf{j} = \dfrac{x}{x^2+y^2}\mathbf{i} + \dfrac{y}{x^2+y^2}\mathbf{j}$

10. $f(x,y) = e^y \tan 2x$. $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = 2e^y \sec^2 2y\mathbf{i} + e^y \tan 2x\mathbf{j}$

11. $f(x,y,z) = \dfrac{x-y}{x+z}$. $\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k} = \dfrac{y+z}{(x+z)^2}\mathbf{i} - \dfrac{1}{x+z}\mathbf{j} - \dfrac{x-y}{(x+z)^2}\mathbf{k}$

12. $f(x,y,z) = 3z \ln(x+y)$

▷ We apply Definition 12.6.7. Thus,

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k} = \frac{3z}{x+y}\mathbf{i} + \frac{3z}{x+y}\mathbf{j} + 3\ln(x+y)\mathbf{k}$$

13. $g(x,y,z) = xe^{-2y}\sec z$.

▷ $\nabla g(x,y,z) = g_x(x,y,z)\mathbf{i} + g_y(x,y,z)\mathbf{j} + g_z(x,y,z)\mathbf{k} = e^{-2y}\sec z\mathbf{i} - 2xe^{-2y}\sec z\mathbf{j} + xe^{-2y}\sec z \tan z\mathbf{k}$

14. $g(x,y,z) = e^{2z}(\sin x - \cos y)$. $\nabla g(x,y,z) = g_x\mathbf{i} + g_y\mathbf{j} + g_z\mathbf{k} = e^{2z}\cos x\mathbf{i} + e^{2z}\sin y\mathbf{j} + 2e^{2z}(\sin x - \cos y)\mathbf{k}$

In Exercises 15–22, find the value of the directional derivative at point P_0 for the function in the direction of **U**.

15. $f(x,y) = x^2 - 2xy^2$; $\mathbf{U} = \cos\pi\mathbf{i} + \sin\pi\mathbf{j} = -\mathbf{i}$

▷ $\nabla f(x,y) = (2x - 2y^2)\mathbf{i} - 4xy\mathbf{j}$; $\nabla f(1,-2) = -6\mathbf{i} + 8\mathbf{j}$. $D_U f(1,-2) = \mathbf{U}\cdot\nabla f(1,-2) = -\mathbf{i}\cdot(-6\mathbf{i} + 8\mathbf{j}) = 6$

16. $g(x,y) = 3x^3y + 4y^2 - xy$; $\mathbf{U} = \cos\frac{1}{4}\pi\,\mathbf{i} + \sin\frac{1}{4}\pi\,\mathbf{j}$; $P_0 = (0,3)$

▷ We apply Definition 12.6.2. We have

$g_x(x,y) = 9x^2y - y \qquad\qquad g_y(x,y) = 3x^3 + 8y - x$

And because we are given that

$\mathbf{U} = \cos\frac{1}{4}\pi\mathbf{i} + \sin\frac{1}{4}\pi\mathbf{j}$

we take $\theta = \frac{1}{4}\pi$. Therefore, by Definition 12.6.2,

$$\begin{aligned} D_U g(x,y) &= g_x(x,y)\cos\theta + g_y(x,y)\sin\theta \\ &= (9x^2y - y)\cos\tfrac{1}{4}\pi + (3x^3 + 8y - x)\sin\tfrac{1}{4}\pi = (9x^2y - y)(\tfrac{1}{2}\sqrt{2}) + (3x^3 + 8y - x)(\tfrac{1}{2}\sqrt{2}) \end{aligned}$$

Substituting the coordinates of the given point $(0,3)$, we have

$D_U g(0,3) = (-3)(\frac{1}{2}\sqrt{2}) + (24)(\frac{1}{2}\sqrt{2}) = \frac{21}{2}\sqrt{2}$

17. $g(x,y) = y^2\tan^2 x$; $\mathbf{U} = -\frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j}$. $\nabla g(x,y) = 2y^2\tan x \sec^2 x\mathbf{i} + 2y\tan^2 x\mathbf{j}$.

$\nabla g(\frac{1}{3}\pi, 2) = 8\tan\frac{1}{3}\pi\sec^2\frac{1}{3}\pi\mathbf{i} + 4\tan^2\frac{1}{3}\pi\mathbf{j} = 8(\sqrt{3})(2)^2\mathbf{i} + 4(\sqrt{3})^2\mathbf{j} = 32\sqrt{3}\mathbf{i} + 12\mathbf{j}$

$D_U g(\frac{1}{3}\pi, 2) = \mathbf{U}\cdot\nabla g(\frac{1}{3}\pi, 2) = (-\frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j})\cdot(32\sqrt{3}\mathbf{i} + 2\mathbf{j}) = -48 + 6 = -42$

18. $f(x,y) = xe^{2y}$; $\mathbf{U} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$. $\nabla f = e^{2y}\mathbf{i} + 2xe^{2y}\mathbf{j}$. $D_U f(2,0) = \nabla f(2,0)\cdot\mathbf{U} = (\mathbf{i} + 4\mathbf{j})\cdot(\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}) = \frac{1}{2} + 2\sqrt{3}$

19. $h(x,y,z) = \cos(xy) + \sin(xy)$; $\mathbf{U} = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

▷ $\nabla h(x,y,z) = h_x(x,y,z)\mathbf{i} + h_y(x,y,z)\mathbf{j} + h_z(x,y,z)\mathbf{k} = -y\sin(xy)\mathbf{i} + [-x\sin(xy) + z\cos(yz)]\mathbf{j} + y\cos(yz)\mathbf{k}$

$\nabla h(2,0,-3) = 0\mathbf{i} + [-2\cdot 0 - 3\cdot 1]\mathbf{j} + 0\mathbf{k} = -3\mathbf{j}$. $D_U h(2,0,-3) = \mathbf{U}\cdot\nabla h(2,0,-3) = (-\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k})\cdot(-3\mathbf{j}) = -2$

20. $f(x,y,z) = \ln(x^2 + y^2 + z^2)$ $\quad \mathbf{U} = \dfrac{1}{\sqrt{3}}\mathbf{i} - \dfrac{1}{\sqrt{3}}\mathbf{j} - \dfrac{1}{\sqrt{3}}\mathbf{k}$ $\quad P_0 = (1,3,2)$

▷ We have

$f_x(x,y,z) = \dfrac{2x}{x^2+y^2+z^2} \qquad f_y(x,y,z) = \dfrac{2y}{x^2+y^2+z^2} \qquad f_z(x,y,z) = \dfrac{2z}{x^2+y^2+z^2}$

Thus,

$f_x(1,3,2) = \frac{1}{7} \qquad f_y(1,3,2) = \frac{3}{7} \qquad f_z(1,3,2) = \frac{2}{7}$

Applying Theorem 12.6.6, we have

$$D_U(1,3,2) = f_x(1,3,2)\cos\alpha + f_y(1,3,2)\cos\beta + f_z(1,3,2)\cos\gamma = \frac{1}{7}\cdot\frac{1}{\sqrt{3}} + \frac{3}{7}\left(-\frac{1}{\sqrt{3}}\right) + \frac{2}{7}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{4}{7\sqrt{3}}$$

21. $f(x,y) = e^{-3x}\cos 3y$; $\mathbf{U} = \cos(-\frac{1}{12}\pi)\mathbf{i} + \sin(-\frac{1}{12}\pi)\mathbf{j}$

$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = -3e^{-3x}\cos 3y\mathbf{i} - 3e^{-3x}\sin 3y\mathbf{j}$. $\nabla f(-\frac{1}{12}\pi, 0) = -3e^{\pi/4}\mathbf{i}$

$D_U f(-\frac{1}{12}\pi, 0) = [\cos(-\frac{1}{12}\pi)\mathbf{i} + \sin(-\frac{1}{12}\pi)\mathbf{j}]\cdot(-3e^{\pi/4}\mathbf{i}) = -3e^{\pi/4}\cos\frac{1}{12}\pi$

22. $g(x,y,z) = \cos 2x\cos 3y\sinh 4z$; $\mathbf{U} = \frac{1}{3}\sqrt{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

▷ $\nabla g(x,y,z) = -2\sin 2x\cos 3y\sinh 4z\mathbf{i} - 3\cos 2x\sin 3y\sinh 4z\mathbf{j} + 4\cos 2x\cos 3y\cosh 4z\mathbf{k}$

$D_U g(\frac{1}{2}\pi, 0, 0) = \nabla g(\frac{1}{2}\pi, 0, 0)\cdot\mathbf{U} = -4\mathbf{k}\cdot\frac{1}{3}\sqrt{3}(\mathbf{i} - \mathbf{j} + \mathbf{k}) = -\frac{4}{3}\sqrt{3}$

In Exercises 23–26, find (a) the gradient of f at P and (b) the rate of change of f in the direction of U at P.

23. $f(x,y) = x^2 - 4y$; $\mathbf{U} = \cos\frac{1}{3}\pi\mathbf{i} + \sin\frac{1}{3}\pi\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$

(a) $\nabla f(x,y) = 2x\mathbf{i} - 4\mathbf{j}$; $\nabla f(-2,2) = -4\mathbf{i} - 4\mathbf{j}$

(b) $D_U f(-2,2) = (\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}) \cdot (-4\mathbf{i} - 4\mathbf{j}) = -2 - 2\sqrt{3}$

24. $f(x,y) = e^{2xy}$; $P = (2,1)$; $\mathbf{U} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

▹ (a) $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = 2ye^{2xy}\mathbf{i} + 2xe^{2xy}\mathbf{j}$

Thus,

$\nabla f(2,1) = 2e^4\mathbf{i} + 4e^4\mathbf{j}$

(b) The rate of change of the function in the direction of U at P is given by

$\mathbf{U} \cdot \nabla f(2,1) = (\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}) \cdot (2e^4\mathbf{i} + 4e^4\mathbf{j}) = \frac{8}{5}e^4 - \frac{12}{5}e^4 = -\frac{4}{5}e^4$

25. $f(x,y,z) = y^2 + z^2 - 4xz$; $\mathbf{U} = \frac{2}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$

▹ (a) $\nabla f(x,y,z) = -4z\mathbf{i} + 2y\mathbf{j} + (2z - 4x)\mathbf{k}$; $\nabla f(-2,1,3) = -12\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}$

(b) $D_U f(-2,1,3) = \left(\frac{2}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}\right) \cdot (-12\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}) = -\frac{24}{7} - \frac{12}{7} + \frac{42}{7} = \frac{6}{7}$

26. $f(x,y,z) = 2x^3 + xy^2 + xz^2$; $P = (1,1,1)$; $\mathbf{U} = \frac{1}{7}\sqrt{21}\mathbf{j} - \frac{2}{7}\sqrt{7}\mathbf{k}$

(a) $\nabla f(x,y,z) = (6x^2 + y^2 + z^2)\mathbf{i} + 2xy\mathbf{j} + 2xz\mathbf{k}$; $\nabla f(1,1,1) = 8\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

(b) $D_U f(1,1,1) = (\frac{1}{7}\sqrt{21}\mathbf{j} - \frac{2}{7}\sqrt{7}\mathbf{k}) \cdot (8\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \frac{2}{7}\sqrt{21} - \frac{4}{7}\sqrt{7}$

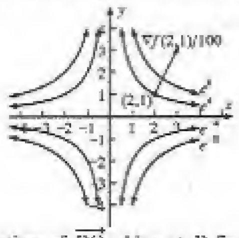
27. $f(x,y) = x^2 - 4y$. The level curves of f at 8, 4, 0, −4, and −8 are, respectively, the parabolas $x^2 - 4y = 8$, $x^2 - 4y = 4$, $x^2 - 4y = 0$, $x^2 - 4y = -4$, and $x^2 - 4y = -8$.

28. Draw a contour map showing the level curves of the function of Exercise 24 at e^8, e^4, 1, e^{-4}, e^{-8}. Also show the representation of $\nabla f(2,1)$ having its initial point at (2,1).

▹ The function of Exercises 24 is defined by $f(x,y) = e^{2xy}$. If $f(x,y) = e^k$, then $e^{2xy} = e^k$ or equivalently, $2xy = k$, $k = 8, 4, 0, -4, -8$. If $k \neq 0$, these are equilateral hyperbolas passing through the point $(1, \frac{1}{2}k)$. If $k = 0$, then $xy = 0$ and the level curve consists of the lines $x = 0$ and $y = 0$, that is, the axes. Because

$\nabla f(2,1) \approx 109.2\mathbf{i} + 218.4\mathbf{j}$

the figure shows the level curves and a representation of $\frac{1}{100}\nabla f(2,1)$.



In Exercises 29–32, find $D_U f$ at the point P for which U is a unit vector on the direction of $\overrightarrow{PQ}$. Also at P find the maximum of $D_U f$.

29. $f(x,y) = e^x\tan^{-1}y$; $P = (0,1)$; $Q = (3,5)$. $\mathbf{V}(\overrightarrow{PQ}) = \langle 3-0, 5-1\rangle = \langle 3,4\rangle$; $\left\|\mathbf{V}(\overrightarrow{PQ})\right\| = \sqrt{9+16} = 5$

Therefore, a unit vector in the direction of $\overrightarrow{PQ}$ is $\mathbf{U} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.

$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = e^x\tan^{-1}y\mathbf{i} + \dfrac{e^x}{1+y^2}\mathbf{j}$; $\nabla f(0,1) = \frac{\pi}{4}\mathbf{i} + \frac{1}{2}\mathbf{j}$.

$D_U f(0,1) = \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) \cdot \left(\frac{1}{4}\pi\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{3}{20}\pi + \frac{2}{5} \approx 0.871$

The maximum of $D_U f(0,1)$ is $\|\nabla f(0,1)\| = \sqrt{\dfrac{\pi^2}{16} + \dfrac{1}{4}} = \frac{1}{4}\sqrt{\pi^2 + 4} \approx 0.931$.

30. $f(x,y) = e^x\cos y + e^y\sin x$; $P(1,0)$, $Q(-3,3)$. $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \langle -3,3\rangle - \langle 1,0\rangle = -4\mathbf{i} + 3\mathbf{j}$. $\left\|\overrightarrow{PQ}\right\| = \sqrt{4^2 + 3^2} = 5$.
$\mathbf{U} = \frac{1}{5}(-4\mathbf{i} + 3\mathbf{j})$. $\nabla f(x,y) = f_x\mathbf{i} + f_y\mathbf{j} = (e^x\cos y + e^y\cos x)\mathbf{i} + (-e^x\sin y + e^y\sin x)$;
$\nabla f(1,0) = (e + \cos 1)\mathbf{i} + \sin 1\mathbf{j}$. $D_U f(0,1) = \frac{1}{5}(-4\mathbf{i} + 3\mathbf{j}) \cdot [(e + \cos 1)\mathbf{i} + \sin 1\mathbf{j}] = \frac{1}{5}(-4e - 4\cos 1 + 3\sin 1)$

The maximum of $D_U f(0,1)$ is $\|\nabla f(1,0)\| = \sqrt{(e + \cos 1)^2 + \sin^2 1} = \sqrt{e^2 + 2e\cos 1 + 1}$.

31. $f(x,y,z) = x - 2y + z^2$; $P = (3,1,-2)$; $Q = (10,7,4)$

$\mathbf{V}(\overrightarrow{PQ}) = \langle 10-3, 7-1, 4+2\rangle = \langle 7,6,6\rangle$; $\left\|\mathbf{V}(\overrightarrow{PQ})\right\| = \sqrt{49+36+36} = \sqrt{121} = 11$

Therefore, a unit vector in the direction of $\overrightarrow{PQ}$ is $\mathbf{U} = \frac{7}{11}\mathbf{i} + \frac{6}{11}\mathbf{j} + \frac{6}{11}\mathbf{k}$.
$\nabla f(x,y,z) = \mathbf{i} - 2\mathbf{j} + 2z\mathbf{k}$. Hence, $\nabla f(3,1,-2) = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$. Therefore

$D_U f(3,1,-2) = \left(\frac{7}{11}\mathbf{i} + \frac{6}{11}\mathbf{j} + \frac{6}{11}\mathbf{k}\right) \cdot (\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}) = \frac{7}{11} - \frac{12}{11} - \frac{24}{11} = -\frac{29}{11} \approx -2.64$

The maximum of $D_U f(3,1,-2)$ is $\|\nabla f(3,1,-2)\| = \sqrt{1+4+16} = \sqrt{21} \approx 4.58$.

32. $f(x,y,z) = x^2 + y^2 - 4xz$; $P(3,1,-2)$, $Q(-6,3,4)$

▷ We find the gradient vector. Thus,

$$\nabla f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k} = (2x-4z)\mathbf{i} + 2y\mathbf{j} - 4x\mathbf{k}$$

Hence,

$$\nabla f(3,1,-2) = 14\mathbf{i} + 2\mathbf{j} - 12\mathbf{k} \tag{1}$$

Next, we find $\mathbf{U}$, a unit vector in the direction of $\overrightarrow{PQ}$. We have

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (-6,3,4) - (3,1,-2) = -9\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

$$\left\|\overrightarrow{PQ}\right\| = \sqrt{81+4+36} = \sqrt{121} = 11$$

Thus,

$$\mathbf{U} = \overrightarrow{PQ}/\left\|\overrightarrow{PQ}\right\| = \tfrac{1}{11}(-9\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) \tag{2}$$

From (1) and (2) we obtain

$$D_U f(3,1,-2) = \mathbf{U}\cdot\nabla f(3,1,-2) = \tfrac{1}{11}(-9\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})\cdot(14\mathbf{i} + 2\mathbf{j} - 12\mathbf{k}) = -\tfrac{194}{11}$$

The maximum value of D_U, attained when $\mathbf{U}$ has the direction of $\nabla f(3,1,-2)$, is

$$\|\nabla f(3,1,-2)\| = \|14\mathbf{i} + 2\mathbf{j} - 12\mathbf{k}\| = \sqrt{196+4+144} = 2\sqrt{86}$$

33. $f(x,y) = e^{2y}\tan^{-1}\frac{y}{3x}$. $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = -\dfrac{3ye^{2y}}{9x^2+y^2}\mathbf{i} + \left(2e^{2y}\tan^{-1}\dfrac{y}{3x} + \dfrac{3xe^{2y}}{9x^2+y^2}\right)\mathbf{j}$

$\nabla f(1,3) = -\frac{1}{2}e^6\mathbf{i} + (\frac{1}{2}\pi e^6 + \frac{1}{6}e^6)\mathbf{j}$. Let $\mathbf{U} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. We seek the values of θ for which

$\nabla f(1,3)\cdot\mathbf{U} = 0$; $-\frac{1}{2}e^6\cos\theta + (\frac{1}{2}\pi e^6 + \frac{1}{6}e^6)\sin\theta = 0$; $-\cos\theta + (\pi + \frac{1}{3})\sin\theta = 0$; $\tan\theta = \dfrac{3}{3\pi+1}$

Therefore $\theta = \tan^{-1}\dfrac{3}{3\pi+1}$ and $\theta = \tan^{-1}\dfrac{3}{3\pi+1} - \pi$.

34. $\rho(x,y) = (x^2+y^2+3)^{-1/2}$. $\nabla\rho(x,y) = -(x^2+y^2+3)^{-3/2}(x\mathbf{i} + y\mathbf{j})$. $\nabla\rho(3,2) = -\frac{1}{64}(3\mathbf{i} + 2\mathbf{j})$.

(a) $\mathbf{U} = \cos\frac{2}{3}\pi\mathbf{i} + \sin\frac{2}{3}\pi\mathbf{j}$. $D_U\rho(3,2) = -\frac{1}{64}(3\mathbf{i} + 2\mathbf{j})\cdot(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}) = -\frac{1}{64}(-\frac{3}{2} + \sqrt{3})$ slugs/ft^2/ft

(b) The greatest magnitude is $\frac{1}{64}\|3\mathbf{i} + 2\mathbf{j}\| = \frac{1}{64}\sqrt{9+4} = \frac{1}{64}\sqrt{13}$ in the direction of $-(3\mathbf{i} + 2\mathbf{j})$.

35. $T = 3x^2 + 2xy$. (a) $\nabla T(x,y) = T_x(x,y)\mathbf{i} + T_y(x,y)\mathbf{j} = (6x+2y)\mathbf{i} + 2x\mathbf{j}$; $\nabla T(3,-6) = 6\mathbf{i} + 6\mathbf{j}$. The number of degrees per meter in the maximum rate of change of T at $(3,-6)$ is $\|\nabla T(3,-6)\| = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$
(b) The maximum rate of change of T occurs in the direction of $\nabla T(3,-6) = 6\mathbf{i} + 6\mathbf{j}$. A unit vector in this direction is $\frac{1}{2}\sqrt{2}\mathbf{i} + \frac{1}{2}\sqrt{2}\mathbf{j}$.

36. The temperature is $T(x,y,z)$ degrees at any point (x,y,z) in three-dimensional space and $T(x,y,z) = 60/(x^2+y^2+z^2+3)$. Distance is measured in inches. Find: (a) the rate of change of the temperature at the point $(3,-2,2)$ in the direction of the vector $-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$, and (b) the direction and magnitude of the greatest rate of change of T at $(3,-1,2)$

▷ Because

$$T(x,y,z) = \frac{60}{x^2+y^2+z^2+3}$$

then

$$\begin{aligned}\nabla T(x,y,z) &= f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}\\ &= \frac{-120x}{(x^2+y^2+z^2+3)^2}\mathbf{i} + \frac{-120y}{(x^2+y^2+z^2+3)^2}\mathbf{j} + \frac{-120z}{(x^2+y^2+z^2+3)^2}\mathbf{k}\end{aligned}$$

Hence,

$$\nabla T(3,-2,2) = \frac{-120}{(9+4+4+3)^2}(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = -\tfrac{3}{10}(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

(a) Because $\|-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}\| = \sqrt{4+9+36} = 7$, a unit vector in the direction of $-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ is given by

$$\mathbf{U} = \tfrac{1}{7}(-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) \tag{2}$$

Hence, from (2) and (1), we obtain

$$D_U T(3,-2,2) = \mathbf{U}\cdot\nabla T(3,-2,2) = \tfrac{1}{7}(-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})\cdot(-\tfrac{3}{10})(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = \tfrac{36}{35}$$

Therefore, the temperature is increasing at the rate of $\frac{36}{35}$ degrees per inch.
(b) The greatest rate of change of T is in the direction of $\nabla T(3,-2,2)$, that is, in the direction of $-3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Its magnitude is $\frac{3}{10}\|3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}\| = \frac{3}{10}\sqrt{17}$. Thus, the greatest rate of change in the temperature is about 1.24 degrees per inch.

37. $V = e^{-2x}\cos 2y$.

▸ (a) $U = \cos\frac{1}{6}\pi\mathbf{i} + \sin\frac{1}{6}\pi\mathbf{j} = \frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j}$. $\nabla V(x,y) = V_x(x,y)\mathbf{i} + V_y(x,y)\mathbf{j} = -2e^{-2x}\cos 2y\mathbf{i} - 2e^{-2x}\sin 2y\mathbf{j}$

$\nabla V(0,\frac{1}{4}\pi) = -2e^0\cos\frac{1}{2}\pi\mathbf{i} - 2e^0\sin\frac{1}{2}\pi\mathbf{j} = -2\mathbf{j}$. The rate of change of V at the point $(0,\frac{1}{4}\pi)$ in the direction of **U** is $D_U V(0,\frac{1}{4}\pi) = \mathbf{U}\cdot\nabla V(0,\frac{1}{4}\pi) = (\frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j})\cdot(-2\mathbf{j}) = -1$. (b) The greatest rate of change of V at $(0,\frac{1}{4}\pi)$ occurs in the direction of $\nabla V(0,\frac{1}{4}\pi)$. The direction is that of the unit vector $-\mathbf{j}$ and its magnitude is 2.

38. $z = 1200 - 3x^2 - 2y^2$. $\nabla z = -6x\mathbf{i} - 4y\mathbf{j}$. $\nabla z(-10,5) = 60\mathbf{i} - 20\mathbf{j}$ (a) Steepest ascent in direction $(3\mathbf{i}-\mathbf{j})/\sqrt{10}$
(b) $\nabla z\cdot\mathbf{i} = 60$: ascending 60 meters for each meter east (c) $\nabla z\cdot -(\mathbf{i}+\mathbf{j})/\sqrt{2} = -40/\sqrt{2} = -20\sqrt{2}$ m/m
(c) level is orthogonal: $\pm(\mathbf{i}+3\mathbf{j})/\sqrt{10}$

12.7 TANGENT PLANES AND NORMALS TO SURFACES

Surfaces The graph of a continuous function $z = f(x,y)$ is a surface. f is differentiable at (x_0,y_0) if and only if the graph of $z = f(x,y)$ has a tangent plane at (x_0,y_0). The equation of the tangent plane is

$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

The graph of a function is a special case of the parametric surface $x = f(u,v)$, $y = g(u,v)$, $z = g(u,v)$ where f, g, and h are continuous and the representation is one-to-one, that is, distinct values of (u,v) give distinct values of (x,y,z). Thus, the cylinder $x^2 + y^2 = a^2$, which cannot be represented in the form $z = f(x,y)$, can be represented as $x = a\cos u$, $y = a\sin u$, $z = v$, $0 \le u < 2\pi$ and the sphere $x^2 + y^2 + z^2 = a^2$ can be represented as $x = a\cos u\sin v$, $y = a\sin u\sin v$, $z = a\cos v$, $0 \le u < 2\pi$, $0 \le v \le \pi$. (Compare with cylindrical and spherical coordinates in §13.1.) Representations such as these were used to prepare the computer graphics in the text. The grids seen in some figures represent equally spaced values of each parameter.

Suppose f, g, and h have partial derivatives with respect to u and v. If v is held constant, then $\mathbf{R} = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$ is a curve and $\mathbf{R}_u = f_u(u,v)\mathbf{i} + g_u(u,v)\mathbf{j} + h_u(u,v)\mathbf{k}$ is a vector tangent to the curve, and hence to the surface. Similarly, the vector $\mathbf{R}_v = f_v(u,v)\mathbf{i} + g_v(u,v)\mathbf{j} + h_v(u,v)\mathbf{k}$ is tangent to the surface. If, in addition, f, g, and h are differentiable, then all the tangents to the surface lie in a plane whose normal vector is $\mathbf{n} = \mathbf{R}_u \times \mathbf{R}_v$. If, in addition, $\mathbf{n}$ is continuous, then the representation is one-to-one if $\mathbf{n} \ne \mathbf{0}$.

Often we can eliminate the parameters u and v and express the surface in the form $F(x,y,z) = 0$.

12.7.1 Definition A vector that is orthogonal to a tangent vector of every curve C through a point P_0 on a surface S is called a *normal vector* to S at P_0.

12.7.2 Theorem If an equation of a surface S is $F(x,y,z) = 0$, where F is differentiable and F_x, F_y, and F_z are not all zero at the point $P_0(x_0,y_0,z_0)$ on S, then a normal vector to S at P_0 is $\nabla F(x_0,y_0,z_0)$.

12.7.6 Theorem If an equation of a curve C is $F(x,y) = 0$, where F is differentiable and F_x and F_y, are not both zero at the point $P_0(x_0,y_0)$ on C, then a normal vector to C at P_0 is $\nabla F(x_0,y_0)$.

12.7.3 Definition If an equation of a surface S is $F(x,y,z) = 0$, then the *tangent plane* of S at a point $P_0(x_0,y_0,z_0)$ is the plane through P_0 having as normal vector $\nabla F(x_0,y_0,z_0)$.

An equation of the tangent plane of Definition 12.7.3 is

$\nabla F(x_0,y_0,z_0)\cdot[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$

or, equivalently,

$F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0$

Two surfaces are *tangent* at a point P if they have a common tangent plane at P or, equivalently, their normal vectors at P are parallel.

12.7.4 Definition The *normal line* to a surface S at a point P_0 on S is the line through P_0 parallel to a normal vector to S at P_0.

If an equation of the surface S is $F(x,y,z) = 0$, then parametric equations of the normal line to S at $P_0(x_0,y_0,z_0)$ are

$x = x_0 + tF_x(x_0,y_0,z_0)$, $y = y_0 + tF_y(x_0,y_0,z_0)$, $z = z_0 + tF_z(x_0,y_0,z_0)$

12.7.5 Definition The *tangent line* to a curve C at a point P_0 is the line through P_0 parallel to a tangent vector to C at P_0.

If the curve C is the intersection of two surfaces with normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$, then a tangent to the curve is parallel to $\mathbf{N}_1 \times \mathbf{N}_2$.

Exercises 12.7

In Exercises 1–12, find an equation of the tangent plane and equations of the normal line to the surface at the point.

1. The given surface is the sphere $x^2 + y^2 + z^2 = 17$. Let $F(x,y,z) = x^2 + y^2 + z^2 - 17$.
   $\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$; $\nabla F(2,-2,3) = 4\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$. A normal vector is $2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. At $(2,-2,3)$, an equation of the tangent plane is $2(x-2) - 2(y+2) + 3(z-3) = 0$; $2x - 2y + 3z = 17$;
   symmetric equations of the normal line are $\frac{x-2}{2} = \frac{y+2}{-2} = \frac{z-3}{3}$.
2. The given surface is ellipsoid $4x^2 + y^2 + 2z^2 = 26$. Let $F(x,y,z) = 4x^2 + y^2 + 2z^2 - 26$.
   $\nabla F(x,y,z) = 8x\mathbf{i} + 2y\mathbf{j} + 4z\mathbf{k}$; $\nabla F(1,-2,3) = 8\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$. A normal vector is $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. At $(1,-2,3)$, an equation of the tangent plane is $2(x-1) - (y+2) + 3(z-3) = 0$; $2x - y + 3z - 13 = 0$;
   symmetric equations of the normal line are $\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z-3}{3}$.
3. The given surface is the paraboloid of revolution $x^2 + y^2 - 3z = 2$. Let $F(x,y,z) = x^2 + y^2 - 3z - 2$.
   $\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$; $\nabla F(-2,-4,6) = -4\mathbf{i} - 8\mathbf{j} - 3\mathbf{k}$. At $(-2,-4,6)$, an equation of the tangent plane is $-4(x+2) - 8(y+4) - 3(z-6) = 0$; $4x + 8y + 3z + 22 = 0$;
   symmetric equations of the normal line are $\frac{x+2}{4} = \frac{y+4}{8} = \frac{z-6}{3}$.
4. $x^2 + y^2 - z^2 = 6$; $(3,-1,2)$

▸ Let F be the function defined by
$F(x,y,z) = x^2 + y^2 - z^2 - 6$
Then
$\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$
A normal to the hyperboloid of one sheet $F(x,y,z) = 0$ at $(3,-1,2)$ is given by
$\nabla F(3,-1,2) = 6\mathbf{i} - 2\mathbf{j} - 4z\mathbf{k}$
or by
$3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$
Thus, at $(3,-1,2)$ an equation of the tangent plane is
$3(x-3) - (y+1) - 2(z-2) = 0$
$3x - y - 2z = 0$
and symmetric equations of the normal line are
$\frac{x-3}{3} = \frac{y+1}{-1} = \frac{z-2}{-2}$

5. The given surface is $y = e^x \cos z$. Let $F(x,y,z) = e^x \cos z - y$.
   $\nabla F(x,y,z) = e^x \cos z\mathbf{i} - \mathbf{j} - e^x \sin z\mathbf{k}$; $\nabla F(1,e,0) = e\mathbf{i} - \mathbf{j} + 0\mathbf{k}$. At $(1,e,0)$, an equation of the tangent plane is $e(x-1) - (y-e) + 0(z-0) = 0$; $ex - y = 0$;
   symmetric equations of the normal line are $\frac{x-1}{e} = \frac{y-e}{-1}$, $z = 0$.
6. The given surface is $z = e^{3x} \sin 3y$. Let $F(x,y,z) = e^{3x} \sin 3y - z$.
   $\nabla F(x,y,z) = 3e^{3x} \sin 3y\mathbf{i} + 3e^{3x} \cos 3y\mathbf{j} - \mathbf{k}$. $\nabla F(0,\frac{1}{6}\pi,1) = \langle 3,0,-1\rangle$. At $(0,\frac{1}{6}\pi,1)$, an equation of the tangent plane is $3x - (z-1) = 0$; $3x - z + 1 = 0$;
   symmetric equations of the normal line are $\frac{x}{3} = \frac{z-1}{-1}$, $y = \frac{1}{6}\pi$
7. The given surface is the parabolic cylinder $x^2 = 12y$. Let $F(x,y,z) = x^2 - 12y$.
   $\nabla(x,y,z) = 2x\mathbf{i} - 12\mathbf{j} + 0\mathbf{k}$; $\nabla F(6,3,3) = 12\mathbf{i} - 12\mathbf{j} + 0\mathbf{k}$. At $(6,3,3)$, an equation of the tangent plane is $12(x-3) - 12(y-3) + 0(z-3) = 0$; $x - y - 3 = 0$;
   symmetric equations of the normal line are $\frac{x-6}{1} = \frac{y-3}{-1}$, $z = 3$.

8. $z = x^{1/2} + y^{1/2}$; $(1, 1, 2)$

▶ Let F be the function defined by

$F(x, y, z) = x^{1/2} + y^{1/2} - z$

$\nabla F(x, y, z) = \frac{1}{2}x^{-1/2}\mathbf{i} + \frac{1}{2}y^{-1/2}\mathbf{j} - \mathbf{k}$

A normal to the graph of $F(x, y, z) = 0$ at $(1, 1, 2)$ is given by

$\nabla F(1, 1, 2) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$

or

$\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Thus, at $(1, 1, 2)$ an equation of the tangent plane is

$(x - 1) + (y - 1) - 2(z - 2) = 0$

$x + y - 2z + 2 = 0$

and symmetric equations of the normal line are

$\frac{x - 1}{1} = \frac{y - 1}{1} = \frac{z - 2}{-2}$

9. The given surface is $x^{1/2} + y^{1/2} + z^{1/2} = 4$. Let $F(x, y, z) = x^{1/2} + y^{1/2} + z^{1/2} - 4$.
$\nabla F(x, y, z) = \frac{1}{2}x^{-1/2}\mathbf{i} + \frac{1}{2}y^{-1/2}\mathbf{j} + \frac{1}{2}z^{-1/2}\mathbf{k}$; $\nabla F(4, 1, 1) = \frac{1}{4}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$. A normal vector is $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. At $(4, 1, 1)$, an equation of the tangent plane is $(x - 4) + 2(y - 1) + 2(z - 1) = 0$; $x + 2y + 2z - 8 = 0$; symmetric equations of the normal line are $\frac{x - 4}{1} = \frac{y - 1}{2} = \frac{z - 1}{2}$.

10. The given surface is $zx^2 - xy^2 - yz^2 = 18$. Let $F(x, y, z) = zx^2 - xy^2 - yz^2 - 18$.
$\nabla F(x, y, z) = (2xz - y^2)\mathbf{i} - (2xy + z^2)\mathbf{j} + (x^2 - 2zy)\mathbf{k}$; $\nabla F(0, -2, 3) = -4\mathbf{i} - 9\mathbf{j} + 12\mathbf{j}$. At $(0, -2, 3)$, an equation of the tangent plane is $-4x - 9(y + 2) + 12(z - 3) = 0$; $-4x - 9y + 12z - 54 = 0$; symmetric equations of the normal line are $\frac{x}{-4} = \frac{y + 2}{-9} = \frac{z - 3}{13}$.

11. The given surface is $x^{2/3} + y^{2/3} + z^{2/3} = 14$. Let $F(x, y, z) = x^{2/3} + y^{2/3} + z^{2/3} - 14$.
$\nabla F(x, y, z) = \frac{2}{3}x^{-1/3}\mathbf{i} + \frac{2}{3}y^{-1/3}\mathbf{j} + \frac{2}{3}z^{-1/3}\mathbf{k}$; $\nabla F(-8, 27, 1) = -\frac{1}{3}\mathbf{i} + \frac{2}{9}\mathbf{j} + \frac{2}{3}\mathbf{k}$. Normal: $-3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$. At $(-8, 27, 1)$, an equation of the tangent plane is $-3(x + 8) + 2(y - 27) + 6(z - 1) = 0$; $3x - 2y - 6z + 84 = 0$; symmetric equations of the normal line are $\frac{x + 8}{-3} = \frac{y - 27}{2} = \frac{z - 1}{6}$.

12. $x^{1/2} + z^{1/2} = 8$; $(25, 2, 9)$

▶ Let F be the function defined by

$F(x, y, z) = x^{1/2} + z^{1/2} - 8$

$\nabla F(x, y, z) = \frac{1}{2}x^{-1/2}\mathbf{i} + \frac{1}{2}z^{-1/2}\mathbf{k}$

A normal to the parabolic cylinder $F(x, y, z) = 0$ at $(25, 2, 9)$ is given by

$\nabla F(25, 2, 9) = \frac{1}{10}\mathbf{i} + \frac{1}{6}\mathbf{k}$

or

$3\mathbf{i} + 5\mathbf{k}$

Thus, at $(25, 2, 9)$ an equation of the tangent plane is

$3(x - 25) + 5(z - 9) = 0$

$3x + 5z - 120 = 0$

and symmetric equations of the normal line are

$\frac{x - 25}{3} = \frac{z - 9}{5} \quad y = 2$

In Exercises 13–20, if the two surfaces intersect in a curve, find equations of the tangent line to the curve of intersection at the given point; if the two surfaces are tangent at the given point, prove it.

13. $x^2 + y^2 - z = 8$, $x - y^2 + z^2 = -2$; $(2, -2, 0)$. Let $F(x, y, z) = x^2 + y^2 - 8$ and $G(x, y, z) = x - y^2 + z^2 + 2$. Then $\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ and $G(x, y, z) = \mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$.
$\mathbf{n}_1 = \nabla F(2, -2, 0) = 4\mathbf{i} - 4\mathbf{j} - \mathbf{k}$; $\mathbf{n}_2 = \nabla G(2, -2, 0) = \mathbf{i} + 4\mathbf{j} + 0\mathbf{k}$. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & -1 \\ 1 & 4 & 0 \end{vmatrix} = 4\mathbf{i} - \mathbf{j} + 20\mathbf{k}$.

Equations of the tangent line of the curve of intersection are $\frac{x - 2}{4} = \frac{y + 2}{-1} = \frac{z}{20}$.

14. The given surfaces are $F(x,y,z) = x^2 + y^2 - 2z + 1 = 0$ and $G(x,y,z) = x^2 + y^2 - z^2 = 0$.
Then $\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} - 2\mathbf{k}$ and $\nabla G(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$.
$\mathbf{n}_1 = \nabla F(0,1,1) = 2\mathbf{j} - 2\mathbf{k}$; $\mathbf{n}_2 = \nabla G(0,1,1) = 2\mathbf{j} - 2\mathbf{k}$. Because $\mathbf{n}_1 = \mathbf{n}_2$, the surfaces are tangent at $(0,1,1)$.

15. $y = x^2$ and $y = x^2$, $y = 16 - z^2$; $(4,16,0)$. Let $F(x,y,z) = x^2 - y$ and
$G(x,y,z) = y^2 + z^2 - 16$. Then $\nabla F(x,y,z) = 2x\mathbf{i} - \mathbf{j}$ and $\nabla G(x,y,z) = \mathbf{j} + 2z\mathbf{k}$.

$\mathbf{n}_1 = \nabla F(4,16,0) = 8\mathbf{i} - \mathbf{j}$; $\mathbf{n}_2 = \nabla G(4,16,0) = \mathbf{j}$. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 8\mathbf{k}$.

Equations of the tangent line of the curve of intersection are $x = 4$, $y = 16$.

16. $x = 2 + \cos \pi yz$, $y = 1 + \sin \pi xz$; $(3,1,2)$

▸ Let F and G be the functions defined by
$F(x,y,z) = 2 + \cos \pi yz - x$ $\qquad$ $G(x,y,z) = 1 + \sin \pi xz - y$
We have
$\nabla F(x,y,z) = -\mathbf{i} - \pi z \sin \pi yz\mathbf{j} - \pi y \sin \pi yz\mathbf{k}$ $\qquad$ $\nabla G(x,y,z) = \pi z \cos \pi xz\mathbf{i} - \mathbf{j} + \pi x \cos \pi xz\mathbf{k}$
We take
$\mathbf{N}_1 = \nabla F(3,1,2) = -\mathbf{i}$ $\qquad$ $\mathbf{N}_2 = \nabla G(3,1,2) = 2\pi\mathbf{i} - \mathbf{j} + 3\pi\mathbf{k}$
Because the normal $\mathbf{N}_1$ to the surface $F(x,y,z) = 0$ and the normal $\mathbf{N}_2$ to the surface $G(x,y,z) = 0$ are not parallel, the surfaces intersect in a curve. A tangent to this curve is given by

$$\mathbf{N}_1 \times \mathbf{N}_2 = (-\mathbf{i}) \times (2\pi\mathbf{i} - \mathbf{j} + 3\pi\mathbf{k}) = 3\pi\mathbf{j} + \mathbf{k}$$

Symmetric equations of the tangent line at $(3,1,2)$ are

$$x = 3 \quad \frac{y-1}{3\pi} = \frac{z-2}{1}$$

17. $y = e^x \sin 2\pi z + 2$, $z = y^2 - \ln(x+1) - 3$; $(0,2,1)$

▸ Let $F(x,y,z) = e^x \sin 2\pi z - y + 2$ and $G(x,y,z) = y^2 - \ln(x+1) - z - 3$.

$\nabla F(x,y,z) = e^x \sin 2\pi z\mathbf{i} - \mathbf{j} + 2\pi e^x \cos 2\pi z\mathbf{k}$ and $\nabla G(x,y,z) = -\dfrac{1}{x+1}\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$.

$\mathbf{n}_1 = \nabla F(0,2,1) = -\mathbf{j} + 2\pi\mathbf{k}$; $\mathbf{n}_2 = \nabla G(0,2,1) = -\mathbf{i} + 4\mathbf{j} - \mathbf{k}$. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 2\pi \\ -1 & 4 & -1 \end{vmatrix} = (1 - 8\pi)\mathbf{i} - 2\pi\mathbf{j} - \mathbf{k}$.

Equations of the tangent line of the curve of intersection are $\dfrac{x}{1-8\pi} = \dfrac{y-2}{-2\pi} = \dfrac{z-1}{-1}$.

18. The given surfaces are $F(x,y,z) = x^2 - 3xy + y^2 - z = 0$ and $G(x,y,z) = 2x^2 + y^2 - 3z + 27 = 0$.
$\nabla F(x,y,z) = (2x - 3y)\mathbf{i} + (-3x + 2y)\mathbf{j} - \mathbf{k}$ and $\nabla G(x,y,z) = 4x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$

$\mathbf{n}_1 = \nabla F(1,-2,11) = 8\mathbf{i} - 7\mathbf{j} - \mathbf{k}$; $\mathbf{n}_2 = \nabla G(1,-2,11) = 4\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -7 & -1 \\ 4 & -4 & -3 \end{vmatrix} = 17\mathbf{i} + 20\mathbf{j} - 4\mathbf{k}$.

Equations of the tangent line of the curve of intersection are $\dfrac{x-1}{17} = \dfrac{y+2}{20} = \dfrac{z-11}{-4}$.

19. $x^2 + z^2 + 4y = 0$, $x^2 + y^2 + z^2 - 6z + 7 = 0$; $(0,-1,2)$

▸ Let $F(x,y,z) = x^2 + z^2 + 4y$ and $G(x,y,z) = x^2 + y^2 + z^2 - 6z + 7$.
$\nabla F(x,y,z) = 2x\mathbf{i} + 4\mathbf{j} + 2z\mathbf{k}$ and $\nabla G(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + (2z - 6)\mathbf{k}$.
$\mathbf{n}_1 = \nabla F(0,-1,2) = 0\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n}_2 = \nabla G(0,-1,2) = 0\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = -\frac{1}{2}\mathbf{n}_1$.
Hence $\mathbf{n}_1$ is parallel to $\mathbf{n}_2$, and so the surfaces are tangent at $(0,-1,2)$. Tangent surfaces may have a curve of intersection, but adding the first equation to twice the second, we get $3x^2 + 2y^2 + 4y + 3z^2 - 2z + 14 = 0$; $3x^2 + 2(y+1)^2 + 3(z-2)^2 = 0$, and so the surfaces intersect only at $(0,-1,2)$.

20. $x^2 + y^2 + z^2 = 8$, $yz = 4$; $(0,2,2)$

▸ Let F and G be the functions defined by
$F(x,y,z) = x^2 + y^2 + z^2 - 8$ and $\qquad$ $G(x,y,z) = yz - 4$
We have
$\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $\qquad$ $\nabla G(x,y,z) = z\mathbf{j} + y\mathbf{k}$
We take
$\mathbf{N}_1 = \nabla F(0,2,2) = 4\mathbf{j} + 4\mathbf{k}$ $\qquad$ $\mathbf{N}_2 = \nabla G(0,2,2) = 2\mathbf{j} + 2\mathbf{k}$
Because $\mathbf{N}_1 = 2\mathbf{N}_2$, the normal vectors to the surfaces are parallel and so the surfaces are tangent at $(0,2,2)$.

In Exercises 21–24, use the gradient to find an equation of the tangent line to the curve at the indicated point.

21. $F(x,y) = 9x^3 - y^3 - 1 = 0$. $\nabla F(x,y) = 27x^2\mathbf{i} - 3y^2\mathbf{j}$. $\nabla F(1,2) = 27\mathbf{i} - 12\mathbf{j}$. $27(x-1) - 12(y-2) = 0$; $9x - 4y = 1$

22. $F(x,y) = 16x^4 + y^4 - 32 = 0$. $\nabla F(x,y) = 64x^3\mathbf{i} + 4y^3\mathbf{j}$. $\nabla F(1,2) = 64\mathbf{i} + 32\mathbf{j}$. $2(x-1) + (y-2) = 0$; $2x + y = 4$

23. $F(x,y) = 2x^3 + 2y^3 - 9xy = 0$. $\nabla F(x,y) = (6x^2 - 9y)\mathbf{i} + (6y^2 - 9x)\mathbf{j}$. $\nabla F(1,2) = -12\mathbf{i} + 15\mathbf{j}$.
$-4(x-1) + 5(y-2) = 0$; $-4x + 5y = 6$

24. $x^4 + 2xy - y^2 = 4$; $(2,-2)$

▸ Let F be the function defined by
$F(x,y) = x^4 + 2xy - y^2 - 4$
We have
$\nabla F(x,y) = (4x^3 + 2y)\mathbf{i} + (2x - 2y)\mathbf{j}$
A normal vector is
$\nabla F(2,-2) = 28\mathbf{i} + 8\mathbf{j}$
An equation of the tangent line is
$7(x-2) + 2(y+2) = 0$
$7x + 2y - 10 = 0$

25. The given spheres are $x^2 + y^2 + z^2 = a^2$ and $(x-b)^2 + y^2 + z^2 = (b-a)^2$.
Let $F(x,y,z) = x^2 + y^2 + z^2 - a^2$ and $G(x,y,z) = (x-b)^2 + y^2 + z^2 - (b-a)^2$.
$\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla G(x,y,z) = 2(x-b)\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. $\mathbf{n}_1 = \nabla F(a,0,0) = 2a\mathbf{i}$ and $\mathbf{n}_2 = \nabla G(a,0,0) = 2(a-b)\mathbf{i} = \frac{a-b}{a}\mathbf{n}_1$. Hence $\mathbf{n}_1$ is parallel to $\mathbf{n}_2$, and so the spheres are tangent at $(a,0,0)$. Tangent spheres cannot have a curve of intersection.

26. $F(x,y,z) = xyz - 36 = 0$; $G(x,y,z) = 4x^2 + y^2 + 9z^2 - 108 = 0$. $\nabla F(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$.
$\nabla G(x,y,z) = 8x\mathbf{i} + 2y\mathbf{j} + 18z\mathbf{k}$. $\mathbf{n}_1 = \nabla F(3,6,2) = 12\mathbf{i} + 6\mathbf{j} + 18\mathbf{k}$; $\mathbf{n}_2 = \nabla G(3,6,2) = 24\mathbf{i} + 12\mathbf{j} + 36\mathbf{k} = 2\mathbf{n}_1$

27. The surface $x^2 - 2yz + y^3 = 4$ is perpendicular to every member of the family of surfaces $x^2 + (4c-2)y^2 - cz^2 + 1 = 0$ at the point $(1,-1,2)$ if their normal vectors are orthogonal.
Let $F(x,y,z) = x^2 - 2yz + y^3 - 4$ and $G(x,y,z) = x^2 + (4c-2)y^2 - cz^2 + 1$.
$\nabla F(x,y,z) = 2x\mathbf{i} + (3y^2 - 2z)\mathbf{j} - 2y\mathbf{k}$ and $\nabla G(x,y,z) = 2x\mathbf{i} + 4(2c-1)y\mathbf{j} - 2cz\mathbf{k}$.
Let $\mathbf{n}_1 = \nabla F(1,-1,2) = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{n}_2 = \nabla G(1,-1,2) = 2\mathbf{i} - 4(2c-1)\mathbf{j} - 4c\mathbf{k}$. Then
$\mathbf{n}_1 \cdot \mathbf{n}_2 = 4 + 4(2c-1) - 8c = 0$ so $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal.

28. Prove that every normal line to the sphere $x^2 + y^2 + z^2 = a^2$ passes through the center of the sphere.

▸ Let $F(x,y,z) = x^2 + y^2 + z^2 - a^2$. Then $\nabla F(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and so a normal line to the sphere at the point $P(x,y,z)$ has the same direction as the position vector $V(\overrightarrow{OP}) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Hence the normal line passes through O, the center of the sphere.

12.8 EXTREMA OF FUNCTIONS OF TWO VARIABLES

12.8.1 Definition The function f of two variables is said to have an *absolute maximum value* on its domain D in the xy plane if there is some point (x_0, y_0) in D such that $f(x_0,y_0) \geq f(x,y)$ for all points (x,y) in D. f is said to have an *absolute minimum value* on its domain D in the xy plane if there is some point (x_0,y_0) in D such that $f(x_0,y_0) \leq f(x,y)$ for all points (x,y) in D.

12.8.2 Definition The function f of two variables is said to have a *relative maximum value* at the point (x_0,y_0) if there exists an open disk $B((x_0,y_0);r)$ such that $f(x_0,y_0) \geq f(x,y)$ for all (x,y) in the open disk that are in the domain of f. f is said to have a *relative minimum value* at the point (x_0,y_0) if there exists an open disk $B((x_0,y_0);r)$ such that $f(x_0,y_0) \leq f(x,y)$ for all (x,y) in the open disk that are in the domain of f.

12.8.3 Theorem If $f(x,y)$ exists at all points in some open disk $B((x_0,y_0);r)$, f has a relative extremum at (x_0,y_0), and $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist, then
$f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$

12.8.4 Definition If $f(x,y)$ exists at all points in some open disk $B((x_0,y_0);r)$, the point (x_0,y_0) is a *critical point* of f if one of the following conditions holds:

(i) $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$;

(ii) $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ does not exist.

It is not possible to test a critical point of $f(x,y)$ for relative extrema by using only the first-order partial derivatives of f.

12.8.5 Theorem (Second Derivative Test) Let f be a differentiable function of two variables x and y such that f_x and f_2 are differentiable in some open disk $B((a,b);r)$. Suppose further that $f_x(a,b)=0$ and $f_y(a,b)=0$. Let the *Hessian* of f be

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}{}^2(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

Then

(i) f has a relative minimum value at (a,b) if
$D(a,b) > 0$ and $f_{xx}(a,b) > 0$ or $f_{yy} > 0$

(ii) f has a relative maximum value at (a,b) if
$D(a,b) > 0$ and $f_{xx}(a,b) < 0$ or $f_{yy} < 0$

(iii) $f(a,b)$ is not a relative extremum if $D(a,b) < 0$.

(iv) No conclusion regarding relative extrema can be made if $D(a,b) = 0$.

The level curves of f near (a,b) are given approximately by
$z = \frac{1}{2}[f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2]$ where $h = x - a$ and $k = y - b$.
If $D(a,b) < 0$, then the graph of $f(x,y)$ lies on both sides of its tangent plane at (a,b). We call such a point a *hyperbolic point* of the surface. Every point of a hyperbolic paraboloid is a hyperbolic point. If $D(a,b) < 0$ at a critical point, as in (iii) above, the hyperbolic point is called a *saddle point*.

P is a *boundary point* of a set R if every open disk centered at P contains at least one point of R and one point not of R. A set is *closed* if it contains all of its boundary points. A set is *bounded* if it is contained in some open disk. The interior of an ellipse is a region that is bounded but not closed; a parabola with its interior is a region that is closed but not bounded. The continuous function $f(x,y) = x^2 + y^2$ does not have an absolute maximum value on either of these two sets. Thus both hypotheses on R are essential for the following theorem.

12.8.6 Theorem (Extreme Value Theorem for functions of Two Variables) Let R be a closed and bounded region in the xy plane, and let f be a function of two variables that is continuous on R. Then there is at least one point in R where f has an absolute maximum value and at least one point in R where f has an absolute minimum value.

Because an absolute extremum is a relative extremum, it follows from Theorems 12.8.3 and 12.8.8 that a continuous function f on a closed and bounded domain D has a maximum and a minimum which are attained either

(i) at a point of the boundary of D;

(ii) at an interior point of D which is a critical point.

Suppose $f(x,y)$ can be expressed as $g(x) + h(y)$ or $g(x)h(y)$ where g and h are positive. If $g(x_0) \geq g(x)$ and $h(y_0) \geq h(y)$ then $g(x_0) + h(x_0) \geq g(x) + h(y)$ and $g(x_0)h(y_0) \geq f(x)h(y)$, that is, the absolute maximum of f is the sum or product of the maxima of g and h. See Exercises 24 and 32. In Exercise 26 we prove the following theorems stated in §1.3.

Maximum Product of a set of positive numbers of constant sum is when factors are equal.

Minimum Sum of a set of positive numbers of constant product is when terms are equal.

Regression Line The regression line of y on x for the set of data points $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ obtained from the method of least squares is $y = mx + b$, where

$$m = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i{}^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \tag{I}$$

and

$$b = \frac{1}{n}\left[\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i\right] \tag{II}$$

The regression for x on y for the same set of data points is not the same line.

Exercises 12.8

In Exercises 1–6, find any relative extrema by finding the critical points and applying Definition 12.8.2.

1. $f(x,y) = \sqrt{16 - x^2 - y^2}$. $f_x = \dfrac{-x}{\sqrt{16 - x^2 - y^2}} = 0$, $x = 0$. $f_y = \dfrac{-y}{\sqrt{16 - x^2 - y^2}} = 0$, $y = 0$

   $f(0,0) = \sqrt{16} \geq \sqrt{16 - x^2 - y^2}$ for any (x,y). f has an absolute maximum at $(0,0)$.

2. $f(x,y) = \sqrt{x^2 + y^2 + 9}$. $f_x = \dfrac{x}{\sqrt{x^2 + y^2 + 9}} = 0$, $x = 0$. $f_y = \dfrac{y}{\sqrt{x^2 + y^2 + 9}} = 0$, $y = 0$

   $f(0,0) = \sqrt{9} \leq \sqrt{x^2 + y^2 + 9}$ for any (x,y). f has an absolute minimum at $(0,0)$.

3. $f(x,y) = x^2 + y^2 - 4x - 8y + 16$. $f_x = 2x - 4 = 0$, $x = 2$. $f_y = 2y - 8 = 0$, $y = 4$.
   $f(x,y) = (x-2)^2 + (y-4)^2 - 4 \geq -4 = f(2,4)$. f has an absolute minimum at $(2,4)$.

4. $f(x,y) = 2 + 2x + 6y - x^2 - y^2$

▷ To locate the critical point, we set the partial derivatives to 0.
   $f_x(x,y) = 2 - 2x = 0$
   $f_y(x,y) = 6 - 2y = 0$
   Solving for x and y, we find
   $x = 1 \quad y = 3$
   To apply Definitions 12.8.1 or 12.8.2, we calculate

$$\begin{aligned} f(3,1) - f(x,y) &= 12 - (2 + 2x + 6y - x^2 - y^2) \\ &= x^2 - 2x + y^2 - 6y + 10 \\ &= (x-1)^2 + (y-3)^2 \geq 0 \end{aligned}$$

   Because $f(3,1) \geq f(x,y)$ for any (x,y), we conclude that f has an absolute maximum value of 12 at $(3,1)$.

5. $f(x,y) = 9 - \sqrt{x^2 + y^2 - 2x + 1}$. $f_x = -\dfrac{x-1}{\sqrt{x^2 + y^2 - 2x + 1}} = 0$, $x = 1$. $f_y = -\dfrac{y}{\sqrt{x^2 + y^2 - 2x + 1}} = 0$, $y = 0$

   $f(x,y) = 9 - \sqrt{(x-1)^2 + y^2} \leq 9 = f(1,0)$ for any (x,y). f has an absolute minimum at $(1,0)$.

6. $f(x,y) = \sqrt{x^2 + y^2} + 1$. $f_x = \dfrac{x}{\sqrt{x^2 + y^2}} = 0$, $x = 0$. $f_y = \dfrac{y}{\sqrt{x^2 + y^2}} = 0$, $y = 0$

   $f(x,y) = \sqrt{x^2 + y^2} + 1 \geq 1 = f(0,0)$ for any (x,y). f has an absolute minimum value at $(0,0)$

In Exercises 7–18, determine the relative extrema of f, if there are any, and locate any saddle points.

7. $f(x,y) = x^3 + y^2 - 6x^2 + y - 1$. $\quad f_x(x,y) = 3x^2 - 12x \quad f_y(x,y) = 2y + 1$
   $f_{xx}(x,y) = 6x - 12 \quad f_{yy}(x,y) = 2 \quad f_{xy}(x,y) = 0$
   $f_x(x,y) = 0$: $3x^2 - 12x = 0$; $3x(x-4) = 0$; $x = 0$, $x = 4$. $f_y(x,y) = 0$: $2y + 1 = 0$; $y = -\frac{1}{2}$
   The critical points of f are $(0, -\frac{1}{2})$ and $(4, -\frac{1}{2})$. The results of applying the second-derivative test at these points are summarized in the following table.

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}{}^2$	f	Conclusion
$(0, -\frac{1}{2})$	-12	2	0	-24	$-\frac{3}{4}$	saddle point
$(4, -\frac{1}{2})$	12	2	0	24	$-\frac{133}{4}$	relative minimum value

8. $f(x,y) = 18x^2 - 32y^2 - 36x - 128y - 110$

▷ First we apply Definition 12.8.4 to find each critical point.
   $f_x(x,y) = 36x - 36 = 36(x-1)$
   $f_y(x,y) = -64y - 128 = -64(y+2)$
   If $f_x(x,y) = 0$, then $x = 1$.
   If $f_y(x,y) = 0$, then $y = -2$.
   Therefore, $(1,-2)$ is the only critical point. By Theorem 12.8.3, if f has a relative extremum, it must occur at the critical point $(1,-2)$. We apply the second-derivative test to determine if f has a relative extremum at $(1,-2)$.
   $f_{xx}(x,y) = 36 \quad f_{yy}(x,y) = -64 \quad f_{xy}(x,y) = 0$
   Because

$$f_{xx}(x,y)f_{yy}(x,y) - f_{xy}{}^2(x,y) = 36(-64) - 0^2 = -2048 < 0$$

   by Theorem 12.8.5(iii) we conclude that f does not have a relative extremum at $(1,-2)$. Therefore, the function f does not have a relative extremum at any point.

9. $f(x,y) = y^2 - x^2 + 2x - 4y + 3$ $\quad f_x(x,y) = -2x + 2 = 0,\ x = 1$ $\quad f_y(x,y) = 2y - 4 = 0,\ y = 2$
$f_{xx}(x,y) = -2$ $\quad f_{yy}(x,y) = 2$ $\quad f_{xy}(x,y) = 0$
$D(x,y) = (-2)2 - 0^2 - 4 < 0$. The critical point $(1,2)$ is a saddle point.

10. $f(x,y) = x^2 - y^2 + 6x - 8y + 25$ $\quad f_x(x,y) = 2x + 6 = 0,\ x = -3$ $\quad f_y(x,y) = -2y - 8 = 0,\ y = -4$
$f_{xx}(x,y) = 2$ $\quad f_{yy}(x,y) = -2$ $\quad f_{xy}(x,y) = 0$
$D(x,y) = 2(-2) - 0^2 = -4 < 0$. The critical point $(-3,-4)$ is a saddle point.

11. $f(x,y) = \frac{1}{x} - \frac{64}{y} + xy.$ $\quad f_x(x,y) = -\frac{1}{x^2} + y$ $\quad f_y(x,y) = \frac{64}{y^2} + x$

$f_{xx}(x,y) = \frac{2}{x^3}$ $\quad f_{yy}(x,y) = -\frac{128}{y^3}$ $\quad f_{xy}(x,y) = 1$

$f_x(x,y) = 0$: $y = \frac{1}{x^2}$. $f_y = 0$: $64 + xy^2 = 0$; $64 + \frac{1}{x^3} = 0$; $x = -\frac{1}{4}$, $y = 16$. Critical point: $(-\frac{1}{4}, 16)$

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	f	Conclusion
$(-\frac{1}{4}, 16)$	-128	$\frac{1}{32}$	1	3	-12	relative maximum value

12. $f(x,y) = x^2 - 4xy + y^3 + 4y$

▹ We calculate the partial derivatives and set them to 0.
$f_x(x,y) = 2x - 4y = 0$ (1)
$f_y(x,y) = -4x + 3y^2 + 4 = 0$ (2)
Solving (1) for x and substituting in (2), we get

$$x = 2y$$
$$3y^2 - 8y + 4 = 0$$
$$(3y-2)(y-2) = 0$$

$y = \frac{2}{3},\ x = \frac{4}{3}$
$y = 2,\ x = 4$
$f_{xx}(x,y) = 2$ $\quad f_{yy}(x,y) = 6y$ $\quad f_{xy}(x,y) = -4$

The critical points of f are $(\frac{4}{3}, \frac{2}{3})$ and $(4,2)$. The results of applying the second-derivative test at these points are summarized in the following table.

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	f	Conclusion
$(\frac{4}{3}, \frac{2}{3})$	2	4	-4	$2(4) - (-4)^2 = -8$	$\frac{32}{27}$	saddle point
$(4,2)$	2	12	-4	$2(12) - (-4)^2 = 8$	0	relative minimum

13. $f(x,y) = 4xy^2 - 2x^2y - x.$ $\quad f_x(x,y) = 4y^2 - 4xy - 1$ $\quad f_y = 8xy - 2x^2$
$f_{xx}(x,y) = -4y$ $\quad f_{yy}(x,y) = 8x$ $\quad f_{xy}(x,y) = 8y - 4x$
$f_x(x,y) = 0$: $8xy - 2x^2 = 0$; $2x(4y - x) = 0$; $x = 0$ or $x = 4y$
$f_y(x,y) = 0$: $4y^2 - 4xy - 1 = 0$. If $x = 0$, $4y^2 - 1 = 0$; $y = \pm\frac{1}{2}$.
If $x = 4y$, $4y^2 - 16y^2 - 1 = 0$; $12y^2 = -1$; no solution. Critical points: $(0, \frac{1}{2})$ and $(0, -\frac{1}{2})$.

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	Conclusion
$(0, \frac{1}{2})$	-2	0	4	-16	saddle point
$(0, -\frac{1}{2})$	2	0	-4	-16	saddle point

14. $f(x,y) = y^4 - 4y^3 + 2x^2 + 8xy$ $\quad f_x(x,y) = 4x + 8y = 0,\ x = -2y$ $\quad f_y(x,y) = 4y^3 - 12y^2 + 8x = 0$
$0 = 4y^3 - 12y^2 - 16y = 4y(y^2 - 3y - 4) = 4y(y-4)(y+1)$
$f_{xx}(x,y) = 4$ $\quad f_{yy}(x,y) = 12y^2 - 24y = 12y(y-2)$ $\quad f_{xy}(x,y) = 8$

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	f	Conclusion
$(0,0)$	4	0	8	$0 - 8^2 = -64$	0	saddle point
$(-8,4)$	4	96	8	320	-128	absolute minimu
$(2,-1)$	4	20	8	16	-3	relative minimum

15. $f(x,y) = x^3 + y^3 + 3y^2 - 3x - 9y + 2$ $f_x(x,y) = 3x^2 - 3$ $f_y(x,y) = 3y^2 + 6y - 9$
$f_{xx}(x,y) = 6x$ $f_{yy}(x,y) = 6y + 6$ $f_{xy}(x,y) = 0$
$f_{xx}(x,y) = 0$: $x^2 - 1 = 0$; $x = \pm 1$. $f_{yy}(x,y) = 0$; $y^2 + 2y - 3 = 0$; $(y-1)(y+3) = 0$; $y = 1, -3$

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - {f_{xy}}^2$	f	Conclusion
$(1,1)$	6	12	0	72	-5	relative minimum value
$(1,-3)$	6	-12	0	-72	27	saddle point
$(-1,1)$	-6	12	0	-72	-1	saddle point
$(-1,-3)$	-6	-12	0	72	31	relative maximum value

16. $f(x,y) = e^x \sin y$
▷ $f_x(x,y) = e^x \sin y$ $f_y(x,y) = e^x \cos y$
Because
$f_x(x,y)^2 + f_y(x,y)^2 = e^{2x}(\sin^2 y + \cos^2) = e^{2x} > 0$
then f_x and f_y are never both 0. We conclude that there are no critical points for the function f. Because f is continuous at every point in R^2, by Theorem 12.8.3 we conclude that f does not have a relative extremum. See Figure 12.1.14b in the text.

17. $f(x,y) = e^{xy}$. $f_x(x,y) = ye^{xy}$ $f_y(x,y) = xe^{xy}$
$f_{xx}(x,y) = y^2 e^{xy}$ $f_{yy}(x,y) = x^2 e^{xy}$ $f_{xy}(x,y) = e^{xy} + xye^{xy}$
$f_x = 0$: $ye^{xy} = 0$; $y = 0$. $f_y = 0$: $xe^{xy} = 0$; $x = 0$. Critical point: $(0,0)$
At $(0,0)$, $f_{xx}f_{yy} - {f_{xy}}^2 = -1$; hence $(0,0)$ is a saddle point.

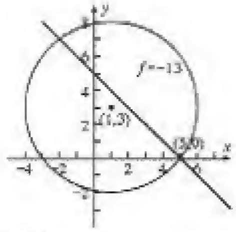
18. $f(x,y) = x^3 + y^3 - 18xy$ $f_x(x,y) = 3x^2 - 18y = 0$, $y = \frac{1}{6}x^2$ $f_y(x,y) = 3y^2 - 18x = 0$
$0 = 3(\frac{1}{6}x^2)^2 - 18x = \frac{1}{12}x(x^3 - 216)$, $x = 0, 6$
$f_{xx}(x,y) = 6x$ $f_{yy}(x,y) = 6y$ $f_{xy}(x,y) = -18$

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - {f_{xy}}^2$	f	Conclusion
$(0,0)$	0	0	-18	-324	0	saddle point
$(6,6)$	36	36	-18	972	-216	relative minimum value

In Exercises 19–24, find the absolute extrema of the function on the closed bounded region R.

19. The function of Ex. 3; R is the triangular region with vertices A(0,0), B(4,0), C(0,8).
▷ The absolute minimum is -4 at $(2,4)$, the only interior critical point. We look for the absolute maximum on the boundary. $f(x,y) = (x-2)^2 + (y-4)^2 - 4$ is greatest at the point furthest from $(2,4)$; it has the value 16 at each of the vertices.

20. The function of Exercise 4; R is the triangular region whose sides are the x axis, the y axis, and the line $x + y = 5$.
▷ From Exercise 4, the absolute maximum is 12 at $(1,3)$, the only interior critical point. We look for the absolute minimum on the boundary. Because
$f(x,y) = 12 - [(x-1)^2 + (y-3)^2]$
f is least at the point furthest from $(1,3)$. This is the vertex $(5,0)$. See the figure. The minimum value is
$12 - [4^2 + 3^2] = -13$.



21. $f(x,y) = 3x^2 + xy$; R is the region bounded by the parabola $y = x^2$ and the line $y = 4$.
▷ $f_x = 6x + y = 0$, $f_y = x = 0$. The only critical point is $(0,0)$. $f_{xx} = 6$, $f_{yy} = 0$, $f_{xy} = 1$, $\mathrm{D} = 6(0) - 1^2 < 0$, a saddle point. Thus the extrema are on the boundary.
On $y = 4$, $f = 3x^2 + 4x$, $-2 \le x \le 2$. $f' = 6x + 4$. $f(-2) = 4$, $f(-\frac{2}{3}) = -\frac{4}{3}$, $f(2) = 20$.
On $y = x^2$, $f = 3x^2 + x^3$, $-2 \le x \le 2$. $f' = 6x + 3x^2 = 3x(2+x)$. $f(-2) = 4$, $f(0) = 0$, $f(2) = 20$.
Thus the absolute maximum is 20 at $(2,4)$ and the absolute minimum is $-\frac{4}{3}$ at $(-\frac{2}{3}, 4)$.

22. $f(x,y) = x^2 - 2xy + 2y$; R is the region bounded by the parabola $y = 4 - x^2$ and the x axis.
▷ $f_x = 2x - 2y = 0$, $f_y = -2x + 2$. The critical point is $(1,1)$. $f_{xx} = 2$, $f_{yy} = 0$, $f_{xy} = -2$, $\mathrm{D} = 2(0) - (-2)^2 < 0$ a saddle point. Thus the extrema are on the boundary.
On $y = 0$, $f = x^2$, $-2 \le x \le 2$. f has a minimum of 0 at 0 and a maximum of 4 at ± 2.
On $y = 4 - x^2$, $f = 2x^3 - x^2 - 8x + 8$, $-2 \le x \le -2$. $f' = 6x^2 - 2x - 8 = 2(x+1)(3x-4)$. $f(-2) = f(2) = 4$, $f(-1) = 13$, $f(\frac{4}{3}) = \frac{8}{27}$. Thus, the absolute maximum is 13 at $(-1,3)$ and the absolute minimum is 0 at $(0,0)$.

23. $f(x,y) = y^3 - x^2 - 3y$; R is the region bounded by the circle $x^2 + (y-1)^2 = 1$.

▸ $f_x = -2x = 0$, $f_y = 3y^2 - 3 = 3(y-1)(y+1)$. The critical points are $(0,1)$ and $(0,-1)$. $f_{xx} = 0$, $f_{yy} = 6y$, $f_{xy} = 0$, D = 0 at both points. $f(0,1) = -2$. $f(0,-1) = 2$. If $x^2 = 1 - (y-1)^2$, $f = y^3 - y^2 - y$, $0 \le y \le 2$
$f' = 3y^2 - 2y - y = (y+1)(3y-1)$. $f(0) = 0$, $f(\frac{1}{3}) = -\frac{11}{27}$, $f(2) = 2$.
The absolute maximum is 2 at $(0,-1)$ and the absolute minimum is -2 at $(0,-1)$.

24. $f(x,y) = \sin x + \sin y$; R is the region bounded by the square having vertices at $(0,0)$, $(\pi,0)$, $(0,\pi)$ and (π,π).

▸ Because on R
$0 \le \sin x \le 1$ and $0 \le \sin y \le 1$
then the absolute maximum value of f is $1+1=2$ at $(\frac{1}{2}\pi, \frac{1}{2}\pi)$ and the absolute minimum value of f is $0+0=0$ at $(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π).

25. Let x, y, and z be the three positive numbers whose sum is 24. Let P be their product.
$x + y + z = 24$; $z = 24 - x - y$. Then $P(x,y) = xyz = xy(24 - x - y) = 24xy - x^2y - xy^2$.
If $x \ge 24$ or $y \ge 24$ or $x = 0$ or $y = 0$, then $P \le 0$. Hence the maximum of P occurs at a critical point inside the square $0 \le x \le 24$, $0 \le y \le 24$.
$P_x(x,y) = 24y - 2xy - y^2$ $\qquad$ $P_y(x,y) = 24x - x^2 - 2xy$
$P_x(x,y) = 0$: $y(24 - 2x - y) = 0$. $\qquad$ $P_y(x,y) = 0$: $x(24 - x - 2y) = 0$.
Because $x > 0$, $y > 0$, the only critical point is $(8,8)$ which gives the absolute maximum.
Then $z = 8$, and so the three positive numbers are 8, 8, and 8. Alternatively, see Exercise 26.

26. Find the n positive numbers whose product is a^n such that their sum is as small as possible.

▸ Suppose the numbers are not all equal. We replace the smallest, say x, and the largest, say y by two numbers a and xy/a with the same product. We show that the sum is decreased. In fact
$(x+y) - \left(a + \frac{xy}{a}\right) = \frac{ax + ay - a^2 - xy}{a} = \frac{(a-x)(y-a)}{a}$. Because x is the smallest and y is the largest, then $a - x > 0$ and $y - a > 0$. Since each step introduces a new term of a, in at most n steps of decreasing sum we arrive at the case of all terms equal. In particular, if the product of three positive numbers is 24, the sum is least when each number is $\sqrt[3]{24}$. To prove the other theorem, we replace x and y by two numbers a and $x + y - a$ with the same sum and show the product is increased. In fact, $a(x + y - a) - xy = (a-x)(y-a) > 0$.

27. Let w units be the distance from the point $(1,-2,3)$ to a point (x,y,z) in the plane
$3x + 2y - z - 5$; $z = 3x + 2y - 5$. Then
$w^2 = (x-1)^2 + (y+2)^2 + (z-3)^2$; $w^2 = (x-1)^2 + (y+2)^2 + (3x+2y-8)^2$
Because w will be a minimum when w^2 is a minimum, we seek the absolute minimum value of
$f(x,y) = (x-1)^2 + (y+2)^2 + (3x+2y-8)^2$. Because $f(x,y) \ge 100$ when $(x-1)^2 + (y+2)^2 \ge 100$
the minimum must occur at a critical point inside the circle $(x-1)^2 + (y+2)^2 = 100$.
$f_x(x,y) = 2(x-1) + 2(3x+2y-8)(3) = 20x + 12y - 50$. $f_x(x,y) = 0$: $10x + 6y = 25$
$f_y(x,y) = 2(y+2) + 2(3x+2y-8)(2) = 12x + 10y - 28$. $f_y(x,y) = 0$: $6x + 5y = 14$
The only critical point is $(\frac{41}{14}, -\frac{5}{7})$ which must be the absolute minimum. Then $z = \frac{33}{14}$.
Therefore, the point in the plane closest to $(1,-2,3)$ is $(\frac{41}{14}, -\frac{5}{7}, \frac{33}{14})$ and the minimum distance is
$\sqrt{(\frac{41}{14}-1)^2 + (-\frac{5}{7}+2)^2 + (\frac{33}{14}-3)^2} = \frac{9}{14}\sqrt{14}$

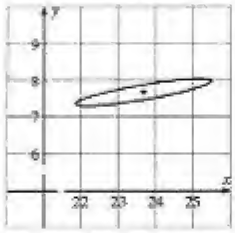
28. Find the points on the surface $y^2 - xz = 4$ that are closest to the origin and find the minimum distance.

▸ Let F be the square of the number of units between the origin and any point (x,y,z) on the hyperboloid of one sheet $y^2 - xy = 4$. The distance is a minimum when F is a minimum. We have
$F = x^2 + y^2 + z^2 = x^2 + z^2 + xz + 4$
$F_x = 2x + z$ $\qquad\qquad$ $F_y = 2z + x$
Therefore, $x = 0$ and $z = 0$ is the only critical point of the function, and when $x = z = 0$, we have $F = 4$. We show that 2 is the absolute minimum value of F. Consider a sphere of radius 3 which intersects the xz plane in a circle of radius 3. This circle with its interior is a closed and bounded set R and so, by Theorem 12.8.8, F has an absolute minimum value on R. Because F has the value 3 on the boundary of R and 3 is greater than 2, then the absolute minimum of F on R cannot occur on the boundary. Hence, the absolute minimum of F on R must occur at the critical point. Moreover, $F > 3$ at all points outside R. Therefore, the critical point gives the absolute minimum value of F for all points. Furthermore, if $x = 0$ and $z = 0$, we have $y^2 = 4$. Thus the minimum distance of 2 occurs at the points. $(0,2,0)$ and $(0,-2,0)$.
ALTERNATE SOLUTION: Completing the square on x, we have
$F = (x + \frac{1}{2}z)^2 + \frac{3}{4}z^2 + 4 \ge 4$

and equality holds if and only if $x + \frac{1}{2}z = 0$ and $z = 0$, that is, if and only if $x = 0$ and $z = 0$. Therefore F has an absolute minimum value of 4 when $x = 0$ and $z = 0$.

29. Let w units be the distance from the point $P(x, y, z)$ of the ellipsoid $x^2 + 4y^2 + 4z^2 = 4$ to the origin. Then $y^2 + z^2 = 1 - \frac{1}{4}x^2$ so that $w^2 = x^2 + y^2 + z^2 = x^2 + (1 - \frac{1}{4}x^2) = \frac{3}{4}x^2 + 1$ which has an absolute minimum value of 1 when $x = 0$. Because P lies on the ellipsoid and on the plane $x - 4y - z = 0$, we find the y and z coordinates of P by solving the system $y^2 + z^2 = 1$, $4y + z = 0$. Then $z = -4y$; $y^2 + 16y^2 = 1$; $y = \pm\frac{1}{\sqrt{17}}$, $z = \mp\frac{4}{\sqrt{17}}$. The points are $\left(0, \frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}}\right)$ and $\left(0, -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$.

30. $C = 14x + 13y + y^3 + x^2 - 8xy + 600$, $x \geq 3$, $y \geq 3$, x and y integers. $C_x = 14 + 2x - 8y = 0$, $x = 4y - 7$
$C_y = 13 + 3y^2 - 8x = 0$, $0 = 3y^2 - 8(4y - 7) + 13 = 3y^2 - 32y + 69 = (3y - 23)(y - 3)$. $y = 3$, $x = 7$ or $y = \frac{23}{3}$, $x = \frac{71}{3}$, $C_{xx} = 2$, $C_{yy} = 6y$, $C_{xy} = -8$, $D = 12y - 64$.
$C_{yy}(7, 3) = 18$, $D(7, 3) = -24$, saddle point.



$C_{yy}(\frac{71}{3}, \frac{23}{3}) = 46$, $D(\frac{71}{3}, \frac{23}{3}) = 28$, relative minimum. At $(\frac{71}{3}, \frac{23}{3})$, $C = 590.2$ and the level curves are approximately $z = \frac{1}{2}(2h^2 - 16hk + 46k^2) = h^2 - 8hk + 23k^2$.
We choose $z = 1$ and complete the square on h to get a parametric representation. Thus, $1 = (h - 4k)^2 + 7k^2$, $k = \sin\theta/\sqrt{7}$, $h = 4k + \cos\theta$
$x = \frac{71}{3} + 4\sin\theta/\sqrt{7} + \cos\theta$, $y = \frac{23}{3} + \sin\theta/\sqrt{7}$. We see that the nearest integer point $(24, 8)$ lies outside the curve, but $(25, 8)$ lies inside. In fact, $C(24, 8) = 592$ but $C(25, 8) = 591$ is the minimum.

31. $R = x^2y^3(c - x - y) = cx^2y^3 - x^3y^3 - x^2y^4$, $x \geq 0$, $y \geq 0$. If $x \geq c$, $y \geq c$, $x = 0$, or $y = 0$, then $R \leq 0$. Hence the maximum we seek must occur at a critical point inside the square $0 \leq x \leq c$, $0 \leq y \leq c$.
$R_x = 2cxy^3 - 3x^2y^3 - 2xy^4$ and $R_y = 3cx^2y^2 - 3x^3y^2 - 4x^2y^3$.
$R_x = 0$: $xy^3(2c - 3x - 2y) = 0$; $3x + 2y = c$. $R_y = 0$: $x^2y^2(3c - 3x - 4y) = 0$; $3x + 4y = 3c$
The only critical point is when $x = \frac{1}{3}c$, $y = \frac{1}{2}c$. Hence the maximum response is when $\frac{1}{3}c$ mg of drug A and $\frac{1}{2}c$ mg of drug B is injected.

32. Suppose that t hr after the injection of x mg of adrenalin the response is R units, and $R = te^{-t}(c - x)x$ where c is a positive constant. What values of x and t will cause the maximum response?

▸ Let
$R(t, x) = te^{-t}(c - x)x$
Partial-differentiating, we obtain
$R_t(t, x) = (1 - t)e^{-t}(c - x)x$ and $\qquad\qquad R_x(t, x) = te^{-t}(c - 2x)$
If $R_t(t, x) = 0$, then either $t = 1$, $x = 0$ or $x = c$. If $R_x(t, x) = 0$, then either $t = 0$ or $x = \frac{1}{2}c$. The critical points are $(0, 0)$, $(0, c)$, and $(1, \frac{1}{2}c)$.
The domain of R is the closed but unbounded set $\{(t, x): t \geq 0, 0 \leq x \leq c\}$. To show that

$$R(1, \tfrac{1}{2}c) = \tfrac{1}{4}e^{-1}c^2 \approx 0.09c^2$$

is an absolute maximum value, consider the closed and bounded set

$$D = \{(t, x): 0 \leq t \leq 2, 0 \leq x \leq c\}$$

On the three sides of the boundary, $t = 0$, $x = 0$, and $x = c$, F has the value 0. Because

$$(c - x)x = \tfrac{1}{4}c^2 - (x - \tfrac{1}{2}c)^2$$

then on the side $t = 2$, we have

$$R(2, x) = 2e^{-2}(c - x)x \leq \tfrac{1}{2}e^{-2}c^2 \approx 0.07c^2 < R(1, \tfrac{1}{2}c)$$

Thus R has an absolute maximum value on D which must occur at the interior critical point. Furthermore, because $D_t(te^{-t}) = (1 - t)e^{-t} < 0$ for $x > 1$, then te^{-t} is decreasing for $t > 1$. Therefore, for any point (t, x) outside of D we have

$$R(t, x) = te^{-t}(c - x)x < 2e^{-2}(c - x)x < R(2, x)$$

Thus the absolute maximum on D is an absolute maximum for the entire domain of R. Therefore, the maximum response occurs when $t = 1$ and $x = \frac{1}{2}c$.
Alternatively, $R(t, x) = g(t)h(x)$, where $g(t) = te^{-t}$, $t \geq 0$ and $h(x) = (c - x)x$, $0 \leq x \leq c$. Because g and h are positive, we maximize each function separately.
$g'(t) = (1 - t)e^{-t}$
Because $g'(t) > 0$ when $0 \leq t < 1$ and $g'(t) < 0$ if $t > 1$, then g has an absolute maximum value when $t = 1$.
$h'(x) = c - 2x$
Because $h'(x) > 0$ if $0 \leq x < \frac{1}{2}c$ and $h'(x) < 0$ if $\frac{1}{2}c < x \leq c$, then h has an absolute maximum value when $x = \frac{1}{2}c$. Thus, R has an absolute maximum value when $t = 1$ and $x = \frac{1}{2}c$.

33. Let 2ℓ, $2w$ and $2h$ be the number of units in the length, width, and height, respectively, of the parallelepiped P. Then in the first octant the vertex of P is at (ℓ, w, h). Since this point is on the ellipsoid, $36\ell^2 + 9w^2 + 4h^2 = 36$; $h^2 = \frac{9}{4}(4 - 4\ell^2 - w^2)$. If V cubic units is the the volume of P, then $V = (2\ell)(2w)(2h) = 8\ell wh$. Let
$f(\ell, w) = V^2 = 64\ell^2w^2h^2 = 144\ell^2w^2(4 - 4\ell^2 - w^2) = 144(4\ell^2w^2 - 4\ell^4w^2 - \ell^2w^4)$
The domain of f is the closed region bounded by the ellipse $4\ell^2 + w^2 = 4$, and $f(\ell, w) = 0$ on the boundary. Hence the maximum we seek must occur at an interior critical point.
$f_\ell(\ell, w) = 144(8\ell w^2 - 16\ell^3w^2 - 2\ell w^4)$, $f_w(\ell, w) = 144(8\ell^2w - 8\ell^4w - 4\ell^2w^3)$
$f_\ell(\ell, w) = 0$: $288\ell w^2(4 - 8\ell^2 - w^2) = 0$; $8\ell^2 + w^2 = 4$ ($\ell = 0$ or $w = 0$ gives a boundary point)
$f_w(\ell, w) = 0$: $576\ell^2w(2 - 2\ell^2 - w^2) = 0$; $2\ell^2 + w^2 = 2$
Hence $\ell^2 = \frac{1}{3}$ and $w^2 = \frac{4}{3}$. The only critical point, and thus the absolute maximum value, is when $\ell = \dfrac{1}{\sqrt{3}}$ and $w = \dfrac{2}{\sqrt{3}}$. Then $h^2 = \frac{9}{4}(4 - \frac{4}{3} - \frac{4}{3}) = 3$; $h = \sqrt{3}$ and the maximum volume is $8\left(\dfrac{1}{\sqrt{3}}\right)\left(\dfrac{2}{\sqrt{3}}\right)(\sqrt{3}) = \dfrac{16}{\sqrt{3}} = \frac{16}{3}\sqrt{3}$

Alternatively, because the sum $36\ell^2 + 9w^2 + 4h^2$ is a constant, the product $\frac{81}{4}V^2 = (36\ell^2)(9w^2)(4h^2)$ is greatest when the terms are equal, and so $36\ell^2 = 9w^2 = 4h^2 = 12$, $\ell = 1/\sqrt{3}$, $w = 2/\sqrt{3}$, $h = \sqrt{3}$.

34. Let ℓ ft, w ft, and h ft be the length, width, and height, of the box, and V ft^3 the volume. Then $.15\ell w + .30(2\ell h + 2wh) = 10$, $3\ell w + 12\ell h + 12wh = 200$. Because the sum is constant, the product $432V^2 = (3\ell w)(12\ell h)(12wh)$ is greatest when $3\ell w = 12\ell h = 12wh = \frac{200}{3}$, $432\ell^2w^2h^2 = (\frac{200}{3})^3$, $\ell wh = \frac{500}{27}\sqrt{2}$.
$\ell = \ell wh/\ell w = \frac{10}{3}\sqrt{2} = w$, $h = \ell wh/wh = \frac{5}{6}\sqrt{2}$.

35. Let ℓ ft, w ft, and h ft be the length, width, and height, respectively, of the box. Let C cents be the cost of the materials. Then
$\ell wh = 16$; $h = \dfrac{16}{\ell w}$ and $C = 18\ell w + 16wh + 12\ell h = 18\ell w + \frac{256}{\ell} + \frac{192}{w}$
If $\ell \geq 1000$ and $w \geq 0.1$, or $\ell \geq 0.1$ and $w \geq 1000$, then $C \geq 18(1000)(0.1) = 1800$. If $\ell \leq 0.1$, then $C \geq 256/(0.1) = 2560$; if $w \leq 1$, then $C \geq 192/(0.1) = 1920$. Hence the absolute minimum value of C must occur at a critical point in the interior of the square $0.1 \leq \ell \leq 1000$, $0.1 \leq w \leq 1000$.
$\dfrac{\partial C}{\partial \ell} = 18w - \dfrac{256}{\ell^2} = 0$; $w = \dfrac{64}{9\ell^2}$. $\dfrac{\partial C}{\partial w} = 18\ell - \dfrac{192}{w^2} = 0$; $18\ell - 192\left(\dfrac{9\ell^2}{64}\right)^2$; $\ell^3 = \dfrac{512}{27}$
The only critical point, and hence the absolute minimum value, is when $\ell = \frac{8}{3}$ and $w = 2$. Then $h = 3$ and $C = 288$.

36. Suppose that T degrees is the temperature at any point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 4$, and $T = 100xy^2z$. Find the greatest and least temperatures and the points on the sphere where they occur.

▶ Because $y^2 = 4 - x^2 - z^2$, T is a function of x and z and
$$T(x, z) = 100xz(4 - x^2 - z^2);\quad x^2 + z^2 \leq 4 \tag{1}$$
Thus,
$$T_x(x, z) = 100[xz(-2x) + z(4 - x^2 - z^2)] = -100z(3x^2 + z^2 - 4)$$
and
$$T_z(x, z) = 100[xz(-2z) + x(4 - x^2 - z^2)] = -100x(3z^2 + x^2 - 4)$$
If $T_x(x, z) = 0$ and $T_z(x, z) = 0$, we have
$$z(3x^2 + z^2 - 4) = 0 \tag{2}$$
$$x(3z^2 + x^2 - 4) = 0 \tag{3}$$
If $z = 0$, then from Eq. (3) we get $x = 0$ or $x = \pm 2$.
If $x = 0$, then from Eq. (2) we get $z = 0$ or $z = \pm 2$
If $x \neq 0$ and $z \neq 0$, then we have
$$3x^2 + z^2 - 4 = 0 \quad \text{and} \quad 3z^2 + x^2 - 4 = 0 \tag{4}$$
The solutions of Eqs. (4) are $x = \pm 1$ and $z = \pm 1$. We use (1) to find the value of T at each critical point. Thus,
$T(0, 0) = 0 \quad T(0, 2) = 0 \quad T(0, -2) = 0 \quad T(2, 0) = 0 \quad T(-2, 0) = 0$
and
$T(1, 1) = T(-1, -1) = 200$
$T(1, -1) = T(-1, 1) = -200$
Because $x^2 + z^2 \leq 4$ is a closed and bounded set and T has the value 0 on the boundary, the absolute maximum and minimum values of T are found at the critical points. Furthermore, $y^2 = 2$ when $x = \pm 1$ and $z = \pm 1$. Therefore, 200 degrees is the greatest temperature, and this temperature occurs at the points

$(1, \pm\sqrt{2}, 1)$ and $(-1, \pm\sqrt{2}, -1)$. The least temperature is -200 degrees, which occurs at the points $(1, \pm\sqrt{2}, -1)$ and $(-1, \pm\sqrt{2}, 1)$.
Alternatively, the extrema of T occur at the maximum of T^2.
$T^2 = 2500(2x^2)y^2y^2(2z^2)$
Because the sum of the factors
$2x^2 + y^2 + y^2 + 2z^2 = 8$
is a constant, the product is greatest when the factors are equal, that is when
$2x^2 = y^2 = 2z^2 = \frac{1}{4}(8) = 2$
$x = \pm 1, y = \pm\sqrt{2}, z = \pm 1$
which leads to the same conclusion.

37. When the production of the commodity requires x machine hours and y person-hours, the cost of production is given by $f(x,y) = 2x^3 - 6xy + y^2 + 500$, $x \geq 0$, $y \geq 0$.
$f_x(x,y) = 6x^2 - 6y = 0$; $y = x^2$ and $f_y(x,y) = -6x + 2y = 0$; $-6x + 2x^2 = 0$; $x = 0, 3$
The critical points are $(0,0)$ and $(3,9)$. $f(0,0) = 500$ and $f(3,9) = 473$. Now $f(x,y) = x^2(2x-9) + (3x-y)^2 + 500$. If $x \geq 10$, $f(x,y) \geq 100(11) + 500 = 1600 > 473$; if $0 \leq x \leq 10$ and $y \geq 60$ then $f(x,y) \geq -9(100) + (30)^2 + 500 = 500 > 473$. Hence the absolute minimum must occur at a critical point inside the rectangle $0 \leq x \leq 10$, $0 \leq y \leq 60$, that is, at $(3,9)$.

38. The unit profits are $x - 40$ and $y - 50$.
$P = (x-40)(3200 - 50x + 25y) + (y-50)(25x - 25y) = 3950x + 250y - 50x^2 + 50xy - 25y^2 - 128{,}000$
$P_x = 3950 - 100x + 50y = 0$. $P_y = 250 + 50x - 50y = 0$.
$P_x + P_y = 4200 - 50x = 0$, $x = 84$. $P_x + 2P_y = 4450 - 50y = 0$, $y = 89$
$P_{xx} = -100$, $P_{yy} = -50$, $P_{xy} = 50$, $D = 2500 > 0$. The critical point is a relative maximum. Because P is a quadric, it is also an absolute maximum. Sell the first at \$84, the second at \$89 for a profit of \$49,025.

In Exercises 39–46, only parentheses actually needed on a calculator will be used. Every multiplication will be explicitly indicated. Negative numbers are entered by following the number with the $\boxed{+/-}$ key. Answers are given to 3 digits.

39. Let x be the number of 20-year periods since 1921, and let y be the value of the painting $20x$ years after 1921 in hundreds of dollars.

x	0	1	2	3
y	1	46	110	200

$\sum_{i=1}^{4} x_iy_i = 866$, $\sum_{i=1}^{4} x_i = 6$, $\sum_{i=1}^{4} y_i = 357$, $\sum_{i=1}^{4} x_i^2 = 14$, $m = \dfrac{(4 \times 866 - 6 \times 357)}{(4 \times 14 - 6^2)} = 66.1$,
$b = \dfrac{(357 - 66.1 \times 6)}{4} = -9.9$. The regression line has the equation $y = 66.1x - 9.9$.
When $x = 4$, $y = 66.1 \times 4 - 9.9 = 254.8$, so in 2001 the painting will be worth \$25,450.

40. A 1991 model car was sold as a used car in 1992 for \$6,800. Its value was \$6,200 in 1993, \$5,700 in 1984 and \$4,800 in 1996. Use the method of least squares to estimate what its value was in 1995.

► Let y dollars be the value of the car x years after 1991. We have $n = 4$ and
$x_1 = 1 \qquad x_2 = 2 \qquad x_3 = 3 \qquad x_4 = 5$
$y_1 = 6{,}800 \qquad y_2 = 6{,}200 \qquad y_3 = 5{,}700 \qquad y_4 = 4{,}800$
$\sum_{i=1}^{n} x_i = 11 \qquad \sum_{i=1}^{n} y_i = 23{,}500 \qquad \sum_{i=1}^{n} x_i^2 = 39 \qquad \sum_{i=1}^{n} x_iy_i = 60{,}300$
Substituting into equations (I) and (II), we obtain
$m = \dfrac{(4 \times 60300 - 11 \times 23500)}{4 \times 39 - 11^2} = -494$ and $b = \frac{1}{4}[23500 - (-494) \times 11] = 7233$
Thus the regression line for the data points is
$y = -494x + 7234$
Because $x = 4$ in 1989, then
$y = -494 \times 4 + 7234 = 5258$
Our estimate of the value of the car in 1995 is \$5300.

41.

x (Week number)	1	2	3	4	5
y (Attendance/100)	50	45	41	39	35

$\sum_{i=1}^{5} x_i y_i = 594$, $\sum_{i=1}^{5} x_i = 15$, $\sum_{i=1}^{5} y_i = 210$, $\sum_{i=1}^{5} x_i^2 = 55$.

(a) $m = \dfrac{(5 \times 594 - 15 \times 210)}{(5 \times 55 - 15^2)} = -3.6$, $b = \dfrac{(210 + 3.6 \times 15)}{5} = 52.8$.

The regression line has the equation $y = -3.6x + 52.8$. When $x = 6$, $y = -3.6 \times 6 + 52.8 = 31.2$. The expected attendance for the sixth week is $100y = 3120$.

(b) We find the regression line for predicting x. $\sum_{i=1}^{5} y_i^2 = 8952$. $m = \dfrac{(5 \times 594 - 15 \times 210)}{(5 \times 8952 - 210^2)} = -0.273$,

$b = (15 + 0.273 \times 210)/5 = 14.5$. The regression line has the equation $x = -0.273y + 14.5$. This is not the same line as in part (a). When $y = 22.5$, $x = -0.273 \times 22.5 + 14.5 = 8.35$. The film will play out the ninth week.

42.

x (μg hormone)	28	57	38	75	82
y (leaves detached)	206	350	300	620	719

$n = 5$. $\sum_{i=1}^{n} x_i y_i = 142{,}632$, $\sum_{i=1}^{n} x_i = 280$, $\sum_{i=1}^{n} y_i = 2197$,

$\sum_{i=1}^{n} x_i^2 = 17{,}826$. (a) $m = \dfrac{(5 \times 142632 - 280 \times 2197)}{(5 \times 17826 - 280^2)} = 9.133$, $b = \dfrac{(2197 - 9.133 \times 280)}{5} = -72.06$

The regression line has the equation $y = 9.133x - 72.1$. (b) When $x = 100$, $y = 9.133 \times 100 - 72.06 = 841$.

43.

x (sec)	300.5	350.6	407.3	326.2	512.8
y (mi/min/kg)	418.5	375.6	350.2	400.2	325.8

$\sum_{i=1}^{5} x_i y_i = 697{,}696.55$, $\sum_{i=1}^{5} x_i = 1897.4$, $\sum_{i=1}^{n} y_i = 1870.3$,

$\sum_{i=1}^{5} x_i^2 = 748{,}484.18$. $m = \dfrac{(5 \times 697696.55 - 1897.4 \times 1870.3)}{(5 \times 748484.18 - 1897.4^2)} = -0.423$, $b = \dfrac{(1870.3 + 0.423 \times 1897.4)}{5} = 534.7$.

The regression line has the equation $y = -0.423x + 534.7$. If $x = 340.4$, $y = -0.423 \times 340.4 + 534.7 = 390.6$.

44. The score on a student's entrance examination was used to predict the student's grade-point average at the end of the freshman year. The following table gives the data for six students, where x is the test score and y is the grade point average. (a) Find an equation of the regression line for the data. (b) Predict the grade-point average average of a student whose scores 88.

	Student 1	Student 2	Student 3	Student 4	Student 5	Student 6
x	92	81	73	98	79	85
y	3.4	2.7	3.1	3.8	2.2	3.0

▷ We have $n = 6$ and

$\sum_{i=1}^{n} x_i = 508$ $\quad \sum_{i=1}^{n} y_i = 18.1$ $\quad \sum_{i=1}^{n} x_i^2 = 43{,}424$ $\quad \sum_{i=1}^{n} x_i y_i = 1559.0$

Substituting into equations (I) and (II), we obtain

$m = \dfrac{6 \times 1559 - 508 \times 18.1}{6 \times 43424 - 508^2} = 0.0642$ and $b = (18.1 - 0.0642 \times 508)/6 = -2.419$

Therefore, the regression line is

$y = 0.0642x - 2.419$

(b) Substituting $x = 88$, we get

$y = 0.0642 \times 88 - 2.419 = 3.23$

We estimate the student's grade-point average to be 3.2.

45.

x (thousand units)	65	72	82	90	100
y (thousand dollars)	30	35	42	48	60

$n = 5$. $\sum_{i=1}^{n} x_i y_i = 18234$, $\sum_{i=1}^{n} x_i = 409$, $\sum_{i=1}^{n} y_i = 215$,

$\sum_{i=1}^{n} x_i^2 = 34233$. $m = \dfrac{(5 \times 18234 - 409 \times 215)}{(5 \times 34233 - 409^2)} = 0.833$, $b = \dfrac{(215 - 0.833 \times 409)}{5} = -25.13$.

The regression line has the equation $y = 0.833x - 25.13$. $y(105) = 62.32$. Profit will be about \$62,300.

46.

w kg	20	30	35	40	50
$x = \ln w$	2.996	3.401	3.555	3.689	3.912
y mm hg	70	85	90	96	100

$n = 5$. $\sum_{i=1}^{n} x_i y_i = 1564.12$, $\sum_{i=1}^{n} x_i = 17.55$, $\sum_{i=1}^{n} y_i = 441$,

$\sum_{i=1}^{n} x_i^2 = 62.09$. $m = \dfrac{(5 \times 1564.12 - 17.55 \times 441)}{(5 \times 62.09 - 17.55^2)} = 33.75$, $b = \dfrac{(441 - 33.75 \times 17.55)}{5} = -30.28$

The regression curve has equation $y = 33.75(\ln w) - 30.28$. $y(45) = 33.75 \times \ln 45 - 30.28 = 98.2$

47. $P = R - C = x(100 - 2x) + y(120 - 3y) - (12x + 12y + 4xy) = -2x^2 - 4xy - 3y^2 + 88x + 108y$
$P_x = -4x - 4y + 88 = 0$, $P_y = -4x - 6y + 108 = 0$. $3P_x - 2P_y = -4x + 48 = 0$, $x = 12$. $P_x - P_y = 2y - 20 = 0$, $y = 10$. $P_{xx} = -4$, $P_{yy} = -6$, $P_{xy} = -4$, $D = 8$. $(12, 10)$ is a relative maximum; because P is a quadric, it is an absolute maximum. $P(12, 10) = 1064$. Produce 12 of type 1, 10 of type 2, with profit of \$1064.

48. Prove that the box having the largest volume that can be placed inside a sphere is in the shape of a cube.
▸ Let a units be the diameter of the sphere, where a is a constant. We assume that the box is a rectangular solid and that the box with greatest volume is inscribed in the sphere. If the sides of the box have lengths x, y, and z units, and the volume is V cubic units, then $v = xyz$. Because the box is inscribed in the sphere, a diagonal of the box is a diameter of the sphere. Thus, $x^2 + y^2 + z^2 = a^2$. We eliminate the variable z. Then

$$V^2 = x^2y^2z^2 = x^2y^2(a^2 - x^2 - y^2)$$

Let f be the function defined by

$$f(x, y) = x^2y^2(a^2 - x^2 - y^2) = a^2x^2y^2 - x^4y^2 - x^2y^4$$

and the maximum value of V occurs at the point where f has a maximum value. To find the critical points of f we set the partial derivatives equal to 0. Thus,

$$f_x(x, y) = 2a^2xy^2 - 4x^3y^2 - 2xy^4 = 0 \qquad (1)$$

$$f_y(x, y) = 2a^2x^2y - 2x^4y - 4x^2y^3 = 0 \qquad (2)$$

We eliminate a from the system of Eqs. (1) and (2). Because $x \neq 0$ and $y \neq 0$, we divide Eq. (1) by y and Eq. (2) by x. This results in

$$2a^2xy - 4x^3y - 2xy^3 = 0 \qquad (3)$$

$$2a^2xy - 2x^3y - 4xy^3 = 0 \qquad (4)$$

Subtracting Eq. (4) from Eq. (3), we have

$$-2x^3y + 2xy^3 = 0$$

Dividing on both sides by $2xy$, we get

$$-x^2 + y^2 = 0$$
$$y^2 = x^2$$
$$y = x$$

Substituting $y = x$ into Eq. (1), we have

$$2a^2x^3 - 4x^5 - 2x^5 = 0$$
$$2a^2 = 6x^2$$
$$x = \tfrac{1}{3}\sqrt{3}a$$
$$y = \tfrac{1}{3}\sqrt{3}a$$

We show that $f(\frac{1}{3}\sqrt{3}a, \frac{1}{3}\sqrt{3}a) = \frac{1}{27}a^6$ is an absolute maximum value. Because the domain of f is the closed and bounded set $x^2 + y^2 \leq a^2$ and f has the value 0 on the boundary, then f has an maximum value which must occur at the interior critical point.

Because $x^2 + y^2 + z^2 = a^2$, then by substituting the values found for x and y, we obtain

$$z^2 = a^2 - x^2 - y^2 = a^2 - \tfrac{1}{3}a^2 - \tfrac{1}{3}a^2 = \tfrac{1}{3}a^2$$

Hence,

$$z = \tfrac{1}{3}\sqrt{3}a$$

Therefore, $x = y = z$ and we conclude that V has a maximum volume if the box is in the shape of a cube.

Alternatively, because the sum of the factors of

$$V^2 = x^2y^2z^2$$

is the constant

$$x^2 + y^2 + z^2 = a^2$$

the product is greatest when the factors are equal. Therefore, $x = y = z$ and we conclude that V has a maximum volume if the box is in the shape of a cube.

49. Let ℓ, w and h be the number of units in the length, width and height of the box. Let S square units be its surface area (S is a constant) and V cubic units its volume.

$S = \ell w + 2\ell h + 2wh$; $h = \dfrac{S - \ell w}{2(\ell + w)}$. $V = \ell wh = \dfrac{S\ell w - \ell^2 w^2}{2(\ell + w)}$, $\ell > 0$, $w > 0$.

$\dfrac{\partial V}{\partial \ell} = \dfrac{Sw^2 - \ell^2 w^2 - 2\ell w^3}{2(\ell + w)} = 0$; $S - \ell^2 - 2\ell w = 0$ and $\dfrac{\partial V}{\partial w} = \dfrac{S\ell^2 - \ell^2 w^2 - 2\ell^3 w}{2(\ell + w)^2} = 0$; $S - w^2 - 2\ell w = 0$

Subtracting, we get $\ell^2 = w^2$; $\ell = w$ and so $(\sqrt{S/3}, \sqrt{S/3})$ is the only critical point. Then $h = \frac{1}{2}\sqrt{S/3}$ and $V = \frac{1}{2}(S/3)^{3/2}$. If $\ell \geq 10\sqrt{S}$ and $w \geq 0.1\sqrt{S}$ or $\ell \geq 0.1\sqrt{S}$ and $w \geq 10\sqrt{S}$, then

$V = \dfrac{\ell w(S - \ell w)}{2(\ell + w)} \le 0$. If $w \le 0.1\sqrt{S}$, $V \le \dfrac{\ell(0.1\sqrt{S})S}{2\ell} = \dfrac{S^{3/2}}{20} < \frac{1}{2}\left(\frac{S}{3}\right)^3$ and similarly if $\ell \le 0.1\sqrt{S}$. Hence the absolute maximum volume occurs at a critical point inside the square $0.1\sqrt{S} \le \ell \le 10\sqrt{S}$, $0.1\sqrt{S} \le w \le 10\sqrt{S}$, that is when $\ell{:}w{:}h = 1{:}1{:}\frac{1}{2}$.
Alternatively, because the sum $\ell w + 2\ell h + 2wh$ is a constant, the product $4V^2 = (\ell w)(2\ell h)(2wh)$ is greatest when the terms are equal. Thus, $\ell w = 2\ell h = 2wh$, $1/h = 2/w = 2/\ell$, $\ell{:}w{:}h = 2{:}2{:}1$.

50. Unit profit: $p - 2$, $q - 1$, $P = (p-2)(11 - 2p - 2q) + (q-1)(19 - 2p - 3q) = 17p + 26q - 2p^2 - 4pq - 3q^2 - 41$
$P_p = 17 - 4p - 4q = 0$, $P_q = 26 - 4p - 6q = 0$. $3P_p - 2P_q = -1 - 4p = 0$, $p = -\frac{1}{4}$. $P_p - P_q = -9 + 2q = 0$, $q = \frac{9}{2}$
Because the only critical point is not in the domain, the maximum is on the boundary. If $p = 0$, $P = 26q - 3q^2 - 41$, $P' = 26 - 6q = 0$, $q = \frac{13}{3}$, $P = \frac{46}{3}$. If $q = 0$, $17p - 2p^2 - 41 \le 0$. Thus, the maximum profit occurs if the staplers are free and a box of staples sells for \$4.33.

51. If $\nabla f = 0\mathbf{i} + 0\mathbf{j}$, then a normal vector to the plane is $\langle 0, 0, 1\rangle$, that is the plane is horizontal.

52. Obtain Equation (1) from $\left(\sum_{i=1}^{n} x_i^2\right)m + \left(\sum_{i=1}^{n} x_i\right)b = \sum_{i=1}^{n} x_i y_i$ and $\left(\sum_{i=1}^{n} x_i\right)m + nb = \sum_{i=1}^{n} y_i$.

▸ We omit the limits. Multiply the first equation by n and the second by $\sum x_i$ to get
$n(\sum x_i^2)m + n(\sum x_i)b = n\sum x_i y_i$
$(\sum x_i)^2 m + n(\sum x_i)b = \sum x_i \sum y_i$
Subtract to get
$[n(\sum x_i^2) - (\sum x_i)^2]m = [n\sum x_i y_i - \sum x_i \sum y_i]$
Divide to get equation (4).

53. We give a complete proof that (3) and (4) give an *absolute* minimum value of f.
$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$, $X = \sum_{i=1}^{n} x_i^2$, $Y = \sum_{i=1}^{n} y_i^2$, $Z = \sum_{i=1}^{n} x_i y_i$, $\sigma_x^2 = \frac{1}{n}X - \bar{x}^2$, $\sigma_y^2 = \frac{1}{n}Y - \bar{y}^2$.
Then $f = \sum_{i=1}^{n}[m(x_i - \bar{x}) + b - y_i]^2 = n\sigma_x^2\left[m - \dfrac{Z - n\bar{x}\bar{y}}{n\sigma_x^2}\right]^2 + n(b - \bar{y})^2 + n\sigma_y^2 - \dfrac{(Z - n\bar{x}\bar{y})^2}{n\sigma_x^2}$.

Hence f is least when $b = \bar{y}$ and $m = \dfrac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} = \dfrac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$

12.8 SUPPLEMENT

1. Prove part (ii) of Theorem 12.8.5.

▸ We are given $D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$ and $f_{xx}(a,b) < 0$. Let $g(x,y) = -f(x,y)$.
Then $g_{xx}(a,b)g_{yy}(a,b) - g_{xy}^2 = [-f_{xx}(a,b)][-f_{yy}(a,b)] - [-f_{xy}(a,b)]^2 = D > 0$ and $g_{xx}(a,b) > 0$. By part (i) of the theorem, g has a relative minimum value at (a,b). Hence f has a relative maximum value at (a,b).

2. If $D(a,b) < 0$ then the discriminant of the right side of Eq.(10), $(2f_{xy})^2 - 4f_{xx}f_{yy} = -4D(a,b)$, is positive, and so the right side factors into two distinct linear factors, which determine two lines through (a,b). In two of the sectors, the expression is positive and in two it is negative. Thus, (a,b) is neither a relative maximum nor a relative minimum.

12.9 LAGRANGE MULTIPLIERS

If f and g are differentiable functions of x, y, and z, then the critical numbers of f subject to the constraint $g(x,y,z) = 0$, are among the critical points of the function F defined as follows.
$F(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z)$
Note that the condition $F_\lambda(x,y,z,\lambda) = 0$ is equivalent to the constraint.
Suppose that at the critical point $g_z \ne 0$. Then we can find $\partial x/\partial z$ and $\partial y/\partial z$ either by differentiating g implicitly or by applying Theorem 12.5.3 to g. We can now evaluate f_x, f_y, f_{xx}, f_{yy}, and f_{xy} and apply the second derivative test (Theorem 12.8.5). A direct test for relative extrema using determinants of 3rd and 4th order can be found in books on mathematical economics. It may be difficult to prove that absolute extrema exist. See Exercise 16.
If we use F to find the critical numbers of f, we are using the method of *Lagrange multipliers*. The method may be extended to include more than one constraint. Thus, the critical numbers of f subject to the constraints $g(x,y,z) = 0$ and $h(x,y,z) = 0$ are among the critical numbers of G defined as follows.
$G(x,y,z,\lambda,\mu) = f(x,y,z) + \lambda g(x,y,z) + \mu h(x,y,z)$

Exercises 12.9

In Exercises 1–4, use Lagrange multipliers to find the critical points of the function subject to the constraint.

1. $f(x,y) = 25 - x^2 - y^2$, and $x^2 + y^2 - 4y = 0$. $F(x,y,\lambda) = 25 - x^2 - y^2 + \lambda(x^2 + y^2 - 4y)$.
   $F_x(x,y,\lambda) = 2x + 2\lambda x$; $F_y(x,y,\lambda) = -2y + \lambda(2y - 4)$; $F_\lambda(x,y,\lambda) = x^2 + y^2 - 4y$
   Setting $F_x(x,y,\lambda) = 0$, we get $\lambda = 1$ or $x = 0$. Setting $F_y(x,y,\lambda) = 0$ we see that $\lambda = 1$ is impossible. Setting $F_\lambda(x,y,\lambda) = 0$ and with $x = 0$ we get $y = 0$ and $y = 4$. Hence the critical points are $(0,0)$ and $(0,4)$.

2. $f(x,y) = 4x^2 + 2y^2 + 5$ and $g(x,y) = x^2 + y^2 - 2y = 0$. $F(x,y,\lambda) = 4x^2 + 2y^2 + 5 + \lambda(x^2 + y^2 - 2y)$.
   $F_x(x,y,\lambda) = 8x + 2\lambda x = 2x(4 + \lambda)$; $F_y(x,y,\lambda) = 4y + 2\lambda y = 2y(2 + \lambda) = 0$. From F_x, either $x = 0$ and from g, $y = 0$ or $y = 2$ which satisfy F_y with $\lambda = -2$; or $\lambda = -4$, and from F_y, $y = 0$ so $x = 0$ from g. Thus $(0,0)$ and $(0,2)$ are the critical points.

3. $f(x,y,z) = x^2 + y^2 + z^2$, and $3x - 2y + z - 4 = 0$. $F(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(3x - 2y + z - 4)$.
   $F_x = 2x + 3\lambda = 0$; $F_y = 2y - 2\lambda = 0$; $F_z = 2z + \lambda = 0$; $F_\lambda = 3x - 2y + z - 4 = 0$
   Solving simultaneously, we get $\lambda = -\frac{4}{7}$, $x = \frac{6}{7}$, $y = -\frac{4}{7}$, and $z = \frac{2}{7}$. Hence the critical point is $(\frac{6}{7}, -\frac{4}{7}, \frac{2}{7})$.

4. $f(x,y,z) = x^2 + y^2 + z^2$ with constraint $y^2 - x^2 = 1$
   ▶ Let $g(x,y,z) = x^2 - y^2 + 1$. Let F be the function defined by
   $$\begin{aligned} F(x,y,z,\lambda) &= f(x,y,z) + \lambda g(x,y,z) \\ &= x^2 + y^2 + z^2 + \lambda x^2 - \lambda y^2 + \lambda \end{aligned}$$
   We find the critical points of the function F. Let
   $$F_x(x,y,z,\lambda) = 2x + 2\lambda x = 2x(1 + \lambda) = 0 \quad (1)$$
   $$F_y(x,y,z,\lambda) = 2y - 2\lambda y = 2y(1 - \lambda) = 0 \quad (2)$$
   $$F_z(x,y,z,\lambda) = 2z = 0 \quad (3)$$
   $$F_\lambda(x,y,z,\lambda) = x^2 - y^2 + 1 = 0 \quad (4)$$
   From (3) we have $z = 0$. From (2) we have either $\lambda = 1$ or $y = 0$. We reject $y = 0$ because then Eq. (4) would require that $x^2 + 1 = 0$, which is impossible. If $\lambda = 1$ in Eq. (1), we get $x = 0$. Substituting $x = 0$ in Eq. (4), we have $-y^2 + 1 = 0$, $y = \pm 1$. Therefore, $(0,1,0)$ and $(0,-1,0)$ are the critical points.

In Exercise 5–8, use Lagrange multipliers to find the absolute extrema of f subject to the constraint. Also find the points at which the extrema occur. In Exercises 9–12, find the absolute maximum and minimum in the region.

5. $f(x,y) = x^2 + y$, and $x^2 + y^2 = 9$. $F(x,y,\lambda) = x^2 + y + \lambda(x^2 + y^2 - 9)$.
   $F_x(x,y,\lambda) = 2x + 2\lambda x = 0$; $F_y(x,y,\lambda) = 1 + 2\lambda y = 0$; $F_\lambda(x,y,\lambda) = x^2 + y^2 - 9 = 0$
   Solving these equations simultaneously, we get the solutions:
   $x = 0,\ y = 3,\ \lambda = -\frac{1}{6}$; $x = 0,\ y = -3,\ \lambda = \frac{1}{6}$; $x = \frac{1}{2}\sqrt{35},\ y = \frac{1}{2},\ \lambda = -1$; $x = -\frac{1}{2}\sqrt{35},\ y = \frac{1}{2},\ \lambda = -1$
   $f(0,3) = 3$; $f(0,-3) = -3$, absolute minimum; $f(\pm\frac{1}{2}\sqrt{35}, \frac{1}{2}) = \frac{37}{4}$, absolute maximum.

9. $f(x,y) = x^2 + y$, $x^2 + y^2 \le 9$. $f_x = 2x = 0$, $f_y = 1 = 0$. There are no critical points. The absolute extrema are those on the boundary given in Ex. 5.

6. $f(x,y) = x^2 y$ and $g(x,y) = x^2 + 8y^2 - 24$. $F(x,y,\lambda) = x^2 y + \lambda(x^2 + 8y^2 - 24)$.
   $F_x(x,y,\lambda) = 2xy + 2\lambda x = 2x(y + \lambda) = 0$; $F_y(x,y,\lambda) = x^2 + 16\lambda y = 0$. If $x = 0$, then from g, $y = \pm\sqrt{3}$.
   If $\lambda = -y$, then from F_y, $x^2 = 16y^2$, and from g, $y = \pm 1$, $x = \pm 4$. $f(0, \pm\sqrt{3}) = 0$; $f(\pm 4, 1) = 4$, absolute maximum; $f(\pm 4, -1) = -4$, absolute minimum.

10. $f(x,y) = x^2 y$ and $x^2 + 8y^2 \le 24$. $f_x = 2xy = 0$; $f_y = x^2 = 0$. The only critical point is $(0,0)$. $f(0,0) = 0$. The absolute extrema are those on the boundary given in Ex. 6.

7. $f(x,y,z) = xyz$, and $x^2 + 2y^2 + 4z^2 = 4$ $f(x,y,z,\lambda) = xyz + \lambda(x^2 + 2y^2 + 4z^2 - 4)$.
   $F_x = yz + 2x\lambda = 0$; $xyz = -2x^2\lambda$. $F_y = xz + 4y\lambda = 0$; $xyz = -4y^2\lambda$. $F_z = xy + 8z\lambda = 0$; $xyz = -8z^2\lambda$.
   Hence $x^2 = 4z^2$, $y^2 = 2z^2$, and $4z^2 + 2(2z^2) + 4z^2 = 4$; $z^2 = \frac{1}{3}$, $z = \pm\sqrt{\frac{1}{3}}$. Then $x = \pm\sqrt{\frac{4}{3}}$, $y = \pm\sqrt{\frac{2}{3}}$.
   $$f(-\sqrt{\tfrac{4}{3}}, -\sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{1}{3}}) = f(-\sqrt{\tfrac{4}{3}}, \sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{3}}) = f(\sqrt{\tfrac{4}{3}}, -\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{3}}) = f(\sqrt{\tfrac{4}{3}}, \sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{1}{3}}) = -\tfrac{2}{9}\sqrt{6}$$
   $$f(\sqrt{\tfrac{4}{3}}, \sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{3}}) = f(\sqrt{\tfrac{4}{3}}, -\sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{1}{3}}) = f(-\sqrt{\tfrac{4}{3}}, \sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{1}{3}}) = f(-\sqrt{\tfrac{4}{3}}, -\sqrt{\tfrac{2}{3}}, \sqrt{\tfrac{1}{3}}) = \tfrac{2}{9}\sqrt{6}$$
   If $\lambda = 0$, we have 6 additional critical points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ at which $f = 0$.
   Because the constraint set is closed and bounded, f has extrema, taken at the critical points. Hence f has a maximum value of $\frac{2}{9}\sqrt{6}$ and a minimum value of $-\frac{2}{9}\sqrt{6}$.

11. $f(x,y,z) = xyz$, $x^2 + 2y^2 + 4z^2 \le 4$. $f_x = yz = 0$; $f_y = xz = 0$; $f_z = xy = 0$. Each point of each axis is a critical point and $f = 0$. The absolute extrema are those on the boundary given in Ex. 7.

8. $f(x,y,z) = y^3 + xz^2$ with constraint $x^2 + y^2 + z^2 = 1$.

▹ Let $g(x,y,z) = x^2 + y^2 + z^2 - 1$. Let F be the function defined by

$$\begin{aligned} F(x,y,z,\lambda) &= f(x,y,z) + \lambda g(x,y,z) \\ &= y^3 + xz^2 + \lambda x^2 + \lambda y^2 + \lambda z^2 - \lambda \end{aligned}$$

We find the critical points of the function F. Let

$F_x(x,y,z,\lambda) = z^2 + 2\lambda x = 0$ (1)

$F_y(x,y,z,\lambda) = 3y^2 + 2\lambda y = y(3y + 2\lambda)$ (2)

$F_z(x,y,z,\lambda) = 2xz + 2\lambda z = 2z(x + \lambda)$ (3)

$F_\lambda(x,y,z,\lambda) = x^2 + y^2 + z^2 - 1 = 0$ (4)

From Eq. (3), either $z = 0$ or $x = -\lambda$.

If $z = 0$, then from Eq. (1), either

$x = 0$: then from Eq. (4), $y = \pm 1$, or

$\lambda = 0$: then from (2), $y = 0$ and from (4) $x = \pm 1$

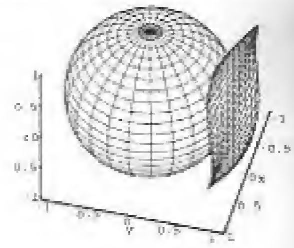
If $x = -\lambda$, then from Eq. (1) $z^2 = 2\lambda^2$ and from Eq. (2) either

$y = 0$: then from (4), $3\lambda^3 - 1 = 0$. $\lambda = \pm\frac{1}{3}\sqrt{3}$ so $x = \mp\frac{1}{3}\sqrt{3}$ and $z^2 = \frac{2}{3}$ or

$y = -\frac{2}{3}\lambda$: then from (4), $\lambda^2 + \frac{4}{9}\lambda^2 + 2\lambda^2 - 1 = 0$. $\lambda^2 = \frac{9}{31}$

We list the critical points, the values of f and our conclusion. The figure shows the constraint and the level surface for $f = 1$ tangent at $(0,1,0)$.

x	y	z	λ	$f(x,y,z)$	
0	1	0	$-\frac{3}{2}$	1	absolute maximum
0	-1	0	$\frac{3}{2}$	-1	absolute minimum
1	0	0	0	0	
-1	0	0	0	0	
$\frac{1}{3}\sqrt{3}$	0	$\frac{1}{3}\sqrt{6}$	$-\frac{1}{3}\sqrt{3}$	$\frac{2}{9}\sqrt{3}$	
$\frac{1}{3}\sqrt{3}$	0	$-\frac{1}{3}\sqrt{6}$	$-\frac{1}{3}\sqrt{3}$	$\frac{2}{9}\sqrt{3}$	
$-\frac{1}{3}\sqrt{3}$	0	$\frac{1}{3}\sqrt{6}$	$\frac{1}{3}\sqrt{3}$	$-\frac{2}{9}\sqrt{3}$	
$-\frac{1}{3}\sqrt{3}$	0	$-\frac{1}{3}\sqrt{6}$	$\frac{1}{3}\sqrt{3}$	$-\frac{2}{9}\sqrt{3}$	
$\frac{3}{\sqrt{31}}$	$\frac{2}{\sqrt{31}}$	$\frac{3\sqrt{2}}{\sqrt{31}}$	$\frac{-3}{\sqrt{31}}$	$\frac{2}{\sqrt{31}}$	
$\frac{3}{\sqrt{31}}$	$\frac{2}{\sqrt{31}}$	$\frac{-3\sqrt{2}}{\sqrt{31}}$	$\frac{-3}{\sqrt{31}}$	$\frac{2}{\sqrt{31}}$	
$\frac{-3}{\sqrt{31}}$	$\frac{-2}{\sqrt{31}}$	$\frac{3\sqrt{2}}{\sqrt{31}}$	$\frac{3}{\sqrt{31}}$	$\frac{-2}{\sqrt{31}}$	
$\frac{-3}{\sqrt{31}}$	$\frac{-2}{\sqrt{31}}$	$\frac{3\sqrt{2}}{\sqrt{31}}$	$\frac{-3}{\sqrt{31}}$	$\frac{-2}{\sqrt{31}}$	



12. $f(x,y,z) = y^3 + xz^2$, $x^2 + y^2 + z^2 \le 1$.

▹ We find the critical points of f.

$f_x(x,y,z) = z^2 = 0$

$f_y(x,y,z) = 3y^2 = 0$

$f_z(x,y,z) = xz = 0$

Thus, any point of the x axis is a critical point. Because $f(x,y,z) = 0$ if $z = z = 0$, the absolute extrema are those on the boundary given in Exercise 8.

In Exercises 13 and 14, find the absolute minimum value of f, and in Ex. 15 and 16, find the absolute maximum.

13. $f(x,y,z) = x^2 + y^2 + z^2$, and $xyz = 1$. $f(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(xyz - 1)$.

$F_x(x,y,z,\lambda) = 2x + \lambda yz = 0$; $2x^2 = -\lambda xyz = -\lambda$. $F_y(x,y,z,\lambda) = 2y + \lambda xz = 0$; $2y^2 = -\lambda xyz = -\lambda$.

$F_z(x,y,z,\lambda) = 2z + \lambda xy = 0$; $2z^2 = -\lambda xyz = -\lambda$.

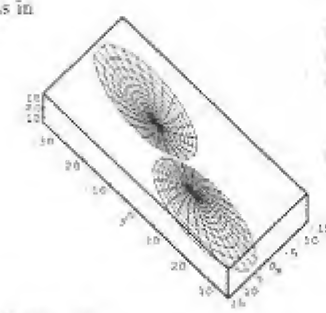
Hence $x^2 = y^2 = z^2$. From $xyz = 1$ we have the four critical points $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$. At each of these points $f(x,y,z) = 3$ which is the minimum of f.

14. $f(x,y,z) = xyz$, and $x^2 + y^2 + z^2 = 4$ $f(x,y,z,\lambda) = xyz + \lambda(x^2 + y^2 + z^2 - 1)$.
$F_x = yz + 2x\lambda = 0$; $xyz = -2x^2\lambda$. $F_y = xz + 2y\lambda = 0$; $xyz = -2y^2\lambda$. $F_z = xy + 2z\lambda = 0$; $xyz = -2z^2\lambda$
Hence $x^2 = y^2 = z^2 = \frac{4}{3}$, $x = \pm\frac{2}{3}\sqrt{3}$. At $+++$, $+--$, $-+-$, $+--$, $f = \frac{8}{9}\sqrt{3}$ and at $---$, $-++$, $+-+$, $++-$, $f = -\frac{8}{9}\sqrt{3}$. If $\lambda = 0$, we have $(\pm 2, 0, 0)$, $(0, \pm 2, 0)$, $(0, 0, \pm 2)$ at which $f = 0$. The minimum is $-\frac{8}{9}\sqrt{3}$.

15. $f(x,y,z) = x + y + z$, and $x^2 + y^2 + z^2 = 9$. $f(x,y,z,\lambda) = x + \; + z + \lambda(x^2 + y^2 + z^2 - 9)$.
$F_x = 1 + 2\lambda x = 0$; $x = -1/2\lambda$. $F_y = 1 + 2\lambda y = 0$; $y = -1/2\lambda$. $F_z = 1 + 2\lambda z = 0$; $z = -1/2\lambda$. $x = y = z$.
$F_\lambda = x^2 + y^2 + z^2 - 9 = 0$; $x^2 + y^2 + x^2 - 9 = 0$; $x = \pm\sqrt{3}$
Critical points are $(\sqrt{3}, \sqrt{3}, \sqrt{3})$, $(-\sqrt{3}, -\sqrt{3}, -\sqrt{3})$. $f(\sqrt{3}, \sqrt{3}, \sqrt{3}) = 3\sqrt{3}$, $f(-\sqrt{3}, -\sqrt{3}, -\sqrt{3}) = -3\sqrt{3}$.
Because the constraint set is closed and bounded, f has extreme values, assumed at the critical points. Therefore, the maximum value of f is $3\sqrt{3}$.

16. $f(x,y,z) = xyz$ with constraint $2xy + 3xz + yz = 72$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

▷ The constraint set is a hyperboloid of two sheets (see below) and meets the coordinate planes in the hyperbolas $zy = 72$, $xz = 24$, and $xy = 36$. Because it is not bounded, we must prove there is an absolute maximum. If $x = y = z$, then $6x^2 = 72$, $x = 2\sqrt{3}$ and $f = x^3 = 24\sqrt{3} > 41$. Now we show that f has smaller values outside a certain closed and bounded set. We solve the constraint for z and substitute in f. Then $z = \dfrac{72 - 2xy}{3x + y}$ and $f = xy\dfrac{72 - 2xy}{3x + y}$. Because the maximum is positive, we have $xy \leq 36$ and $f \leq 36 \cdot \dfrac{72}{3x + y}$. If (x, y) is on or outside the square $S = [0, 20] \times [0, 20]$, then $f \leq \dfrac{36 \cdot 72}{60 + 20} = 32.4$. Therefore, f has a maximum at an interior point of S, that is, at a critical point.
Let F be the function defined by
$F(x,y,z,\lambda) = xyz + \lambda(2xy + 3xz + yz - 72)$
We find the critical points of the function F. Let
$F_x(x,y,z,\lambda) = yz + \lambda(2y + 3z) = 0$ (1)
$F_y(x,y,z,\lambda) = xz + \lambda(2x + z) = 0$ (2)
$F_z(x,y,z,\lambda) = xy + \lambda(3x + y) = 0$ (3)
$F_\lambda(x,y,z,\lambda) = 2xy + 3xz + yz - 72 = 0$ (4)
We multiply of Eq. (1) by x, Eq. (2) by y, and Eq. (3) by z. This results in



$xyz + \lambda(2xy + 3xz) = 0$ (5)
$xyz + \lambda(2xy + yz) = 0$ (6)
$xyz + \lambda(3xz + yz) = 0$ (7)
From Eqs. (5) and (6) we conclude that
$3xz = yz$
$x = \frac{1}{3}y$ (8)
From Eqs. (6) and (7) we conclude that
$2xy = 3xz$
$z = \frac{2}{3}y$ (9)
Substituting from (8) and (9) into (4), we have
$2(\frac{1}{3}y)y + 3(\frac{1}{3}y)(\frac{2}{3}y) + y(\frac{2}{3}y) - 72 = 0$
$2y^2 = 72$
Then $y = 6$, $x = 2$, and $z = 4$ and the maximum value is $f(2, 6, 4) = (2)(6)(4) = 48$.
Alternatively, consider $6f^2 = 6x^2y^2z^2 = (2xy)(3xz)(yz)$. Because the sum of the factors is constant (72) the product is maximum when they are equal, leading immediately to Eqs. (8) and (9).
We show how to graph the constraint. Start with the simpler surface
$xy + yz + zx = 1$
Substitute $x = u + v$, $y = u - v$, and complete the square to show that we have a hyperboloid of two sheets.
$u^2 + 2uz - v^2 = 1$
$(u + z)^2 - z^2 - v^2 = 1$
Now let $(u + z)^2 = \sec^2 r$, $z^2 + v^2 = \tan^2 r$ which leads to
$u + z = \sec r$, $z = \tan r \sin s$, $v = \tan r \cos s$
The parametric equations are
$x = (u + z) + v - z = \sec r + \tan r \cos s - \tan r \sin s$
$y = (u + z) - v - z = \sec r - \tan r \cos s - \tan r \sin s$
$z = \tan r \sin s$
For surface of our problem, multiply the formulas for x, y, z by $2\sqrt{3}$, $6\sqrt{3}$, $4\sqrt{3}$.

17. $f(x,y,z) = x^2 + 4y^2 + 16z^2$, and (a) $xyz = 1$; (b) $xy = 1$; (c) $x = 1$.
Because z, and hence $f(x,y,z)$, can be be arbitrarily large, f has no maximum so we expect the critical points to give minima.
(a) $xyz = 1$. $f(x,y,z,\lambda) = x^2 + 4y^2 + 16z^2 + \lambda(xyz - 1)$.
$F_x = 2x + yz\lambda = 0$; $2x^2 = -xyz\lambda$. $F_y = 8y + xz\lambda = 0$; $8y^2 = -xyz\lambda$. $F_z = 32z + xy\lambda = 0$; $32z^2 = -xyz\lambda$.
Hence $x^2 = 16z^2$; $x = \pm 4z$ and $y^2 = 4z^2$; $y = \pm 2z$. From $xyz = 1$ we have $(\pm 4z)(\pm 2z)z = 1$; $z = \pm\frac{1}{2}$.
Because $xyz = 1$, the 4 critical points are $(2,1,\frac{1}{2})$, $(2,-1,-\frac{1}{2})$, $(-2,1,-\frac{1}{2})$ and $(-2,-1,\frac{1}{2})$.
At each of these, $f(x,y,z) = 4+4+4 = 12$. Therefore, the minimum value of f is 12.
(b) $xy = 1$. $f(x,y,z,\lambda) = x^2 + 4y^2 + 16z^2 + \lambda(xy - 1)$.
$F_x = 2x + \lambda y = 0$; $2x^2 = -\lambda xy$. $F_y = 8y + \lambda x = 0$; $8y^2 = -\lambda xy$. $x^2 = 4y^2$; $x = \pm 2y$. $F_z = 32z = 0$; $z = 0$.
From $xy = 1$, if $x = 2y$, $2y^2 = 1$, $y = \pm\frac{1}{2}\sqrt{2}$; $x = -2y$ is impossible. The 2 critical points are $(\sqrt{2}, \frac{1}{2}\sqrt{2}, 0)$ and $(-\sqrt{2}, -\frac{1}{2}\sqrt{2}, 0)$. At each, $f(x,y,z) = 4$. Hence, the minimum value of f is 4.
(c) $x = 1$. $f(x,y,z,\lambda) = x^2 + 4y^2 + 16z^2 + \lambda(x - 1)$
$F_x = 2x + \lambda = 0$. $F_y = 8y = 0$; $y = 0$. $F_z = 32z = 0$; $z = 0$. Thus the critical point is $(1,0,0)$.
Because $f(1,0,0) = 1$, the minimum value of f is 1.

18. Let w units be the distance from the point $(1,3,0)$ to the point (x,y,z) in the plane $4x + 2y - z = 5$. w has a minimum when w^2 does, and has no maximum. Let $f(x,y,z) = w^2 = (x-1)^2 + (y-3)^2 + z^2$.
$F(x,y,z,\lambda) = (x-1)^2 + (y-3)^2 + z^2 + \lambda(4x + 2y - z - 5)$
$F_x = 2(x-1) + 4\lambda = 0$, $x = 1 - 2\lambda$. $F_y = 2(y-3) + 2\lambda = 0$, $y = 3 - \lambda$. $F_z = 2z - \lambda = 0$, $z = \frac{1}{2}\lambda$.
$4(1-2\lambda) + 2(3-\lambda) - \frac{1}{2}\lambda = 5$, $10 - \frac{21}{2}\lambda = 5$, $\lambda = \frac{10}{21}$, $x = \frac{1}{21}$, $y = \frac{53}{21}$, $z = \frac{5}{21}$. Therefore, f has the critical point $(\frac{1}{21}, \frac{53}{21}, \frac{5}{21})$ and at this point $w = \sqrt{(\frac{1}{21}-1)^2 + (\frac{53}{21}-3)^2 + (\frac{5}{21})^2} = \frac{5}{21}\sqrt{21}$.

19. Let w units be the distance from the point $(1,-1,-1)$ to the point (x,y,z) in the plane $x + 4y + 3z = 2$. w has a minimum when w^2 does, and has no maximum. Let $f(x,y,z) = w^2 = (x-1)^2 + (y+1)^2 + (z+1)^2$.
$F(x,y,z,\lambda) = (x-1)^2 + (y+1)^2 + (z+1)^2 + \lambda(x + 4y + 3z - 2)$
$F_x = 2(x-1) + \lambda = 0$; $x = 1 - \frac{1}{2}\lambda$. $F_y = 2(y+1) + 4\lambda = 0$; $y = -1 - 2\lambda$. $F_z = 2(z+1) + 3\lambda = 0$; $z = -1 - \frac{3}{2}\lambda$
$(1 - \frac{1}{2}\lambda) + 4(-1 - 2\lambda) + 3(-1 - \frac{3}{2}\lambda) = 2$; $\lambda = -\frac{8}{13}$, $x = \frac{17}{13}$, $y = \frac{3}{13}$, $z = -\frac{1}{13}$. Therefore, f has the critical point $(\frac{17}{13}, \frac{3}{13}, -\frac{1}{13})$ and at this point $w = \sqrt{(\frac{17}{13}-1)^2 + (\frac{3}{13}+1)^2 + (-\frac{1}{13}+1)^2} = \sqrt{(\frac{4}{13})^2 + (\frac{16}{13})^2 + (\frac{12}{13})^2} = \frac{4}{13}\sqrt{26}$

20. Find the least and greatest distance from the origin to a point on the ellipse $x^2 + 4y^2 = 16$.
▸ Because the ellipse is a closed and bounded set, the distance has a maximum and a minimum which occurs at a critical point. The distance from the origin to any point (x,y) is $\sqrt{x^2+y^2}$ units. Because the least and greatest distances occur at the points where the square of the distance has its extrema, we let
$f(x,y) = x^2 + y^2$
We must find the points where the function f has its extrema, subject to the constraint
$$x^2 + 4y^2 = 16 \tag{1}$$
Let F be the function defined by
$F(x,y,\lambda) = x^2 + y^2 + \lambda(x^2 + 4y^2 - 16)$
We find the critical points of the function F. Let
$$F_x(x,y,\lambda) = 2x + 2\lambda x = 2x(1+\lambda) = 0 \tag{2}$$
$$F_y(x,y,\lambda) = 2y + 8\lambda y = 2y(1+4\lambda) = 0 \tag{3}$$
From Eq.(2) either $x = 0$ or $\lambda = -1$.
If $x = 0$, then from Eq. (1) we have $y = \pm 2$ and from Eq. (3) we have $\lambda = -\frac{1}{4}$.
If $\lambda = -1$, then from Eq. (3) we have $y = 0$ and from Eq. (1) we have $x = \pm 4$.
We list each critical point of F, the value of $\sqrt{x^2+y^2}$ and our conclusion. Note that the extrema occur at the ends of the axes.

x	y	λ	$\sqrt{x^2+y^2}$	
0	2	$-\frac{1}{4}$	2	absolute minimum
0	-2	$-\frac{1}{4}$	2	absolute minimum
4	0	-1	4	absolute maximum
-4	0	-1	4	absolute maximum

21. Let w units be the distance from the origin to a point (x, y, z) on the ellipsoid $9x^2 + 4y^2 + z^2 = 36$. The ellipsoid is a closed and bounded set so there are extrema for w attained at the extrema of w^2. Let
$f(x, y, z) = w^2 = x^2 + y^2 + z^2$. $F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(9x^2 + 4y^2 + z^2 - 36)$
$F_x(x, y, z, \lambda) = 2x + 18\lambda x = 0$; $x = 0$ or $\lambda = -\frac{1}{9}$. $F_y(x, y, z, \lambda) = 2y + 8\lambda y = 0$; $y = 0$ or $\lambda = -\frac{1}{4}$.
$F_z(x, y, z, \lambda) = 2z + 2\lambda z = 0$; $z = 0$ or $\lambda = -1$. $F_\lambda(x, y, z, \lambda) = 9x^2 + 4y^2 + z^2 - 36 = 0$.
The three possible solutions are:
$\lambda = -\frac{1}{9}$, $y = 0$, $z = 0$; $9x^2 - 36 = 0$; $x = \pm 2$. Critical points $(2, 0, 0)$ and $(-2, 0, 0)$. $w = 2$.
$x = 0$, $\lambda = -\frac{1}{4}$, $z = 0$; $4y^2 - 36 = 0$; $y = \pm 3$. Critical points $(0, 3, 0)$ and $(0, -3, 0)$. $w = 3$.
$x = 0$, $y = 0$, $\lambda = -1$. $z^2 - 36 = 0$; $z = \pm 6$. Critical points $(0, 0, 6)$ and $(0, 0, -6)$. $w = 6$.
Therefore, the least distance from the origin to a point on the ellipsoid is 2 and the greatest distance is 6. The points $(0, \pm 3, 0)$ are saddle points of f.

22. $f = 2x^2 + 3y^2 + z^2$, $g = x + y + z = 5$. $F(x, y, z, \lambda) = 2x^2 + 3y^2 + z^2 + \lambda(x + y + z - 5)$
$F_x = 4x + \lambda = 0$, $x = -\frac{1}{4}\lambda$; $F_y = 6y + \lambda = 0$, $y = -\frac{1}{6}\lambda$; $F_z = 2z + \lambda = 0$, $z = -\frac{1}{2}\lambda$
$-\frac{1}{4}\lambda - \frac{1}{6}\lambda - \frac{1}{2}\lambda = 5$, $\lambda = -\frac{60}{11}$, $x = \frac{15}{11}$, $y = \frac{10}{11}$, $z = \frac{30}{11}$. Although the constraint set is a plane which is unbounded, the level surfaces are ellipsoids and it is clear that some smallest ellipsoid touches the plane.

23. $f(x, y, z) = x^2 + y^2 + z^2$, and $x + 2y + 3z = 6$, and $x - y - z = -1$.
$f(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 6) + \mu(x - y - z + 1)$
The constraint set is a line, closed but unbounded, so f has a minimum but no maximum.
$F_x = 2x + \lambda + \mu = 0$; $x = -\frac{1}{2}\lambda - \frac{1}{2}\mu$. $F_y = 2y + 2\lambda - \mu = 0$; $y = -\lambda + \frac{1}{2}\mu$. $F_z = 2z + 3\lambda - \mu = 0$; $z = -\frac{3}{2}\lambda + \frac{1}{2}\mu$.
$F_\lambda = x + 2y + 3z - 6 = 0$; $-7\lambda + 2\mu = 6$. $F_\mu = x - y - z + 1 = 0$; $2\lambda - \frac{3}{2}\lambda = -1$.
Therefore $\lambda = -\frac{14}{13}$, $\mu = -\frac{10}{13}$, $x = \frac{12}{13}$, $y = \frac{9}{13}$, $z = \frac{16}{13}$. The critical point is $(\frac{12}{13}, \frac{9}{13}, \frac{16}{13})$.
$f(\frac{12}{13}, \frac{9}{13}, \frac{16}{13}) = (\frac{12}{13})^2 + (\frac{9}{13})^2 + (\frac{16}{13})^2 = \frac{481}{169} = \frac{37}{13}$, which is the minimum value of f.

24. Use Lagrange multipliers to find the absolute minimum function value of f if $f(x, y, z) = x^2 + y^2 + z^2$, with the two constraints

$$x + y + z = 1 \tag{1}$$
$$3x - 2y + z = -4 \tag{2}$$

▷ The constraint is the line formed by the intersection of two planes; its distance from the origin attains an absolute minimum at a critical point. Let F be the function defined by
$F(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(x + y + 2z - 1) + \mu(3x - 2y + z + 4)$
Then

$$F_x(x, y, z, \lambda, \mu) = 2x + \lambda + 3\mu = 0 \tag{3}$$
$$F_y(x, y, z, \lambda, \mu) = 2y + \lambda - 2\mu = 0 \tag{4}$$
$$F_z(x, y, z, \lambda, \mu) = 2z + 2\lambda + \mu = 0 \tag{5}$$

We solve Eq. (3) for x, Eq. (4) for y, and Eq. (5) for z.

$$x = -\tfrac{1}{2}(\lambda + 3\mu) \tag{6}$$
$$y = -\tfrac{1}{2}(\lambda - 2\mu) \tag{7}$$
$$z = -\tfrac{1}{2}(2\lambda + \mu) \tag{8}$$

Substituting from Eqs. (6), (7), and (8) into (1), we get

$$-\tfrac{1}{2}(\lambda + 3\mu) - \tfrac{1}{2}(\lambda - 2\mu) - (2\lambda + \mu) - 1 = 0$$
$$-\lambda - 3\mu - \lambda + 2\mu - 4\lambda - 2\mu - 2 = 0$$
$$6\lambda + 3\mu = -2 \tag{9}$$

Substituting from Eqs. (6), (7), and (8) into (2), we get

$$-\tfrac{3}{2}(\lambda + 3\mu) + (\lambda - 2\mu) - \tfrac{1}{2}(2\lambda + \mu) + 4 = 0$$
$$-3\lambda - 9\mu + 2\lambda - 4\mu - 2\lambda - \mu + 8 = 0$$
$$3\lambda + 14\mu = 8 \tag{10}$$

Solving,
14(Eq. 9) − 3(Eq. 10): $75\lambda + 52 = 0$, $\lambda = -\frac{52}{75}$
(Eq. 9) − 2(Eq. 10): $-25\mu + 18 = 0$, $\mu = \frac{18}{25}$

Substituting into (6), (7), and (8), we obtain

$x = -\frac{1}{2}(-\frac{52}{75} + \frac{54}{25}) = -\frac{11}{15}$ $\qquad$ $y = -\frac{1}{2}(-\frac{52}{75} - \frac{36}{25}) = \frac{16}{15}$ $\qquad$ $z = -\frac{1}{2}(-\frac{104}{75} + \frac{18}{25}) = \frac{1}{3}$

Evaluating f at the critical point gives the absolute minimum value $f(-\frac{11}{15}, \frac{16}{15}, \frac{1}{3}) = (-\frac{11}{15})^2 + (\frac{16}{15})^2 + (\frac{1}{3})^2 = \frac{134}{75}$

25. $f(x,y,z) = xyz$, $x+y+z=4$, $x-y-z=3$. $F(x,y,z,\lambda,\mu) = xyz + \lambda(x+y+z-4) + \mu(x-y-z-3)$.
$F_x = yz + \lambda + \mu = 0$ (1). $F_y = xz + \lambda - \mu = 0$ (2). $F_z = xy + \lambda - \mu = 0$ (3). $F_\lambda = x+y+z-4 = 0$ (4)
$F_\mu = x-y-z-3 = 0$ (5)
From (4) and (5) we have $2x-7=0$; $x = \frac{7}{2}$. From (2) and (3) we get $y = z$. From (4) we get
$y = \frac{1}{4}$, $z = \frac{1}{4}$. The critical point is $P(\frac{7}{2}, \frac{1}{4}, \frac{1}{4})$. $f(\frac{7}{2}, \frac{1}{4}, \frac{1}{4}) = (\frac{7}{2})(\frac{1}{4})(\frac{1}{4}) = \frac{7}{32}$.
Consider x and y as functions of z and denote $D_z x$ by x' and $D_z y$ by y'. From the constraints, $x'+y'+1=0$, $x'-y'-1=0$. Hence $x'=0$, $y'=-1$. $f' = x'yz + xy'z + xy = -xz + xy$.
$f'' = -x'z - x + x'y + xy' = -2x = -7 < 0$ at P. Hence f has a relative maximum value at P.

26. $f(x,y,z) = x^3+y^3+z^3$, $x+y+z=1$ (1), $x+y-z=0$ (2). $F = x^3+y^3+z^3+\lambda(x+y+z-1)+\mu(x+y-z)$
$F_x = 3x^2+\lambda+\mu=0$ (3). $F_y = 3y^2+\lambda+\mu=0$ (4). $F_z = 3z^2+\lambda-\mu=0$ (5) . From (1) and (2) we have $2z=1$, $z=\frac{1}{2}$. From (3) and (4) we get $x^2=y^2$. From (1), if $y=x$: $x+x+\frac{1}{2}=1$, $x=\frac{1}{4}$; if $y=-x$, $x-x+\frac{1}{2}=1$, impossible. At the critical point P, $f(\frac{1}{4},\frac{1}{4},\frac{1}{2}) = \frac{1}{64}+\frac{1}{64}+\frac{1}{8} = \frac{5}{32}$. Let $'$ be the derivative with respect to x. Because $z=\frac{1}{2}$, we have from (1) or (2), $1+y'=0$, $y'=-1$. $f' = 3x^2+3y^2y' = 3x^2-3y^2$.
$f'' = 6x - 6yy' = 6x+6y = 3 > 0$ at P. Thus, P is an absolute minimum value.

In Exercises 27–36, use Lagrange multipliers to solve the indicated exercise of Exercises 12.8.

27. Let x, y, and z be the numbers. Then $f(x,y,z) = xyz$ and $x+y+z=24$, $x\geq 0$, $y \geq 0$, $z\geq 0$. The constraint set is a triangle, which is closed and bounded so f has a maximum and a minimum. The minimum is 0 attained at all points of the boundary so any critical point is a maximum. Let $f(x,y,z,\lambda) = xyz + \lambda(x+y+z-24)$.
$F_x = yz+\lambda=0$; $xyz=-\lambda x$. $F_y = xz+\lambda=0$; $xyz=-\lambda y$. $F_z = xy+\lambda=0$; $xyz=-\lambda z$. $x=y=z$.
$F_\lambda = x+y+z-24=0$; $3x=24$. Hence $x=8$, $y=8$, $z=8$ gives the maximum $xyz=512$ of f.

28. Find three positive numbers whose product is 24 such that their sum is as small as possible.

▸ Let $x\geq 0$, $y\geq 0$, $z \geq 0$ be the numbers. We want to determine x, y, and z such that the function f defined by
$f(x,y,z) = x+y+z$
has a minimum value, subject to the constraint
$xyz = 24$ (1)
If $x>3$, $y>3$, $z>3$, then $f > 3+3+3 = 9$. Therefore, if the relative extremum on the closed and bounded cube $0\leq x\leq 3$, $0 \leq y \leq 3$, $0 \leq z \leq 3$ is less than 9, it must be the absolute minimum. Let F be the function defined by
$F(x,y,z,\lambda) = x+y+z+\lambda(xyz-24)$
Then
$F_x(x,y,z,\lambda) = 1+\lambda yz = 0$ (2)
$F_y(x,y,z,\lambda) = 1+\lambda xz = 0$ (3)
$F_z(x,y,z,\lambda) = 1+\lambda xy = 0$ (4)
We multiply on both sides of Eq. (2) by x, on both sides of Eq. (2) by y, and on both sides of Eq. (4) by z. This results in
$x+\lambda xyz = 0$ (5)
$y+\lambda xyz = 0$ (6)
$z+\lambda xyz = 0$ (7)
From Eqs. (5), (6), (7) we conclude that
$x=y=z$ (8)
Substituting from Eqs. (8) into (1) we get
$x^3 = 24$
$x = \sqrt[3]{24} < \sqrt[3]{27} = 3$
Therefore each number is $\sqrt[3]{24}$. Because the sum is less than 9, we have the absolute minimum.

29. Let w units be the distance from the point $(1,-2,3)$ to a point (x,y,z) in the plane $3x+2y-z-5=0$. w will be a minimum when w^2 is a minimum.
$f(x,y,z) = w^2 = (x-1)^2+(y+2)^2+(z-3)^2$
$F(x,y,z,\lambda) = (x-1)^2+(y+2)^2+(z-3)^2+\lambda(3x+2y-z-5)$
$F_x = 2(x-1)+3\lambda=0$; $x = 1-\frac{3}{2}\lambda$. $F_y = 2(y+2)+2\lambda=0$; $y=-2-\lambda$. $F_z = 2(z-3)-\lambda=0$; $z = 3+\frac{1}{2}\lambda$.
$F_\lambda = 3x+2y-z-5=0$; $3(1-\frac{3}{2}\lambda)+2(-2-\lambda)-(3+\frac{1}{2}\lambda)-5=0$; $7\lambda=-9$.
Hence $\lambda = -\frac{9}{7}$, $x=\frac{41}{14}$, $y=-\frac{5}{7}$, $z=\frac{33}{14}$. At the critical point $P(\frac{41}{14}, -\frac{5}{7}, \frac{33}{14})$,
$w = \sqrt{(\frac{41}{14}-1)^2+(-\frac{5}{7}+2)^2+(\frac{33}{14}-3)^2} = \frac{9}{14}\sqrt{14}$
Since x, and hence $f(x,y,z)$, can be arbitrarily large, w has a minimum at P.

30. Find the points on the surface $g = y^2 - xz = 4$ that are closest to the origin and find the minimum distance.

▸ Because $g = y^2 - \frac{1}{4}[(x+z)^2 - (x-z)^2] = y^2 - \frac{1}{4}(x+z)^2 + \frac{1}{4}(x-z)^2$, the surface is a hyperboloid of one sheet. $F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(y^2 - xz - 4)$. $F_x = 2x + \lambda z = 0$, $z = -(2/\lambda)x$. $F_y = 2y - 2\lambda y = 2y(1-\lambda) = 0$. $F_z = 2z + \lambda x = 0$, $z = -(\lambda/2)x$. If $\lambda \neq \pm 2$, then $x = z = 0$, $y^2 = 4$, $y = \pm 2$, $d = 2$. If $\lambda = 2$, $z = -x$, $y = 0$, $x^2 = 4$, $x = \pm 2$, $z = \mp 2$, $d = 2\sqrt{2}$. If $\lambda = 2$, $z = x$, $y = 0$, $-x^2 = 4$, impossible. The minimum distance is 2 at $(0, \pm 2, 2)$.

31. Let w units be the distance to the origin of a point (x, y, z) on $x^2 + 4y^2 + 4z^2 = 4$ and $x - 4y - z = 0$. w will be a minimum when w^2 is a minimum. Let $f(x, y, z) = w^2 = x^2 + y^2 + z^2$.
$F(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(x^2 + 4y^2 + 4z^2 - 4) + \mu(x - 4y - z)$
$F_x = 2x + 2\lambda x + \mu = 0$. $F_y = 2y + 8\lambda y - 4\mu = 0$. $F_z = 2z + 8\lambda z - \mu = 0$.
$F_y - 4F_z$: $(2 + 8\lambda)(y - 4z) = 0$; $y = 4z$ or $\lambda = -\frac{1}{4}$. If $\lambda = -\frac{1}{4}$, from F_y we get $\mu = 0$ and from F_x we get $x = 0$.
$F_\lambda = x^2 + 4y^2 + 4z^2 - 4 = 0$. $F_\mu = x - 4y - z = 0$.
If $y = 4z$, then $x - 16z - \ = 0$; $x = 17z$ and $289z^2 + 64z^2 + 4z^2 - 4 = 0$; $357z^2 = 4$. Therefore
$z = \pm\frac{2}{\sqrt{357}}$, the critical points are $\left(\frac{34}{\sqrt{357}}, \frac{8}{\sqrt{357}}, \frac{2}{\sqrt{357}}\right)$ and $\left(-\frac{34}{\sqrt{357}}, -\frac{8}{\sqrt{357}}, -\frac{2}{\sqrt{357}}\right)$ so $f = \frac{1124}{357}$.
If $x = 0$, then $0 - 4y - z = 0$; $z = -4y$ and $0 + 4y^2 + 64y^2 - 4 = 0$; $17y^2 = 1$. Therefore
$y = \pm\frac{1}{\sqrt{17}}$, the critical points are $P\left(0, \frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}}\right)$ and $Q\left(0, -\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\right)$ so $f = 1$.
Because the constraint set is closed and bounded, f attains a minimum at a critical point. Hence P and Q are closest to the origin and the minimum distance is 1.

32. Exercise 34. A rectangular box without a top is to be made at a cost of \$10 for the material. If the material for the bottom of the box costs \$0.15 per square foot and material for the sides costs \$0.30 per square foot, find the dimensions of the box of greatest volume that can be made.

▸ Let x ft, y ft, and z ft be the length, width, and height of the box, $x \geq 0$, $y \geq 0$, $z \geq 0$. We must find x, y, z so that the volume function defined by
$V(x, y, z) = xyz$
is a maximum, subject to the constraint
$0.15xy + 0.30(2xz + 2yz) = 10.00$ (1)
As in Exercise 16, the constraint set is ubounded but there is an absolute maximum occuring at a critical point. Let F be the function defined by
$F(x, y, z, \lambda) = xyz + \lambda(0.15xy + 0.60xz + 0.60yz - 10)$
Then
$F_x(x, y, z, \lambda) = yz + 0.15\lambda y + 0.60\lambda z = 0$ (2)
$F_y(x, y, z, \lambda) = xz + 0.15\lambda x + 0.60\lambda z = 0$ (3)
$F_z(x, y, z, \lambda) = xy + 0.60\lambda x + 0.60\lambda y = 0$ (4)
We multiply on both sides of Eq. (2) by x, on both sides of Eq. (2) by y, and on both sides of Eq. (4) by z. This results in
$xyz + \lambda(0.15xy + 0.60xz) = 0$ (5)
$xyz + \lambda(0.15xy + 0.60yz) = 0$ (6)
$xyz + \lambda(0.60xz + 0.60yz) = 0$ (7)
From Eqs (5) and (6), we conclude that
$x = y$
From Eqs. (6) and (7), we have
$0.15y = 0.60z$
$y = 4z$
Replacing x and y by $4z$ in Eq. (1), we obtain

$$\begin{aligned} 0.15(4z)(4z) + 0.60(4z)z + 0.60(4z)z &= 10 \\ 7.2z^2 &= 10 \\ z &= \tfrac{5}{6}\sqrt{2} \\ x = y &= \tfrac{10}{3}\sqrt{2} \end{aligned}$$

Thus, the largest box has a square base of side $\frac{10}{3}\sqrt{2}$ ft and a height of $\frac{5}{6}\sqrt{2}$ ft.

33. Let ℓ ft, w ft, h ft be the length, width and height of the box, C cents is the cost.
$C = 18\ell w + 16wh + 12\ell h$ and $\ell wh = 16$. $F(\ell, w, h, \lambda) = 18\ell w + 16wh + 12\ell h + \lambda(\ell wh - 16)$.
$\frac{\partial F}{\partial \ell} = 18w + 12h + \lambda wh = 0$; $\frac{18}{h} + \frac{12}{w} = -\lambda$
$\frac{\partial F}{\partial w} = 18\ell + 16h + \lambda \ell h = 0$; $\frac{18}{h} + \frac{16}{\ell} = -\lambda$; $\frac{12}{w} = \frac{16}{\ell}$; $w = \frac{3}{4}\ell$
$\frac{\partial F}{\partial h} = 16w + 12\ell + \lambda \ell w = 0$; $\frac{16}{\ell} + \frac{12}{w} = -\lambda$; $\frac{18}{h} = \frac{16}{\ell}$; $h = \frac{9}{8}\ell$
$\frac{\partial F}{\partial \lambda} = \ell wh - 16 = 0$; $\ell(\frac{3}{4}\ell)(\frac{9}{8}\ell) = 16$; $\ell^3 = \frac{512}{27}$; $\ell = \frac{8}{3}$. The critical point is $P(\frac{8}{3}, 2, 3)$.
Because wh, and hence C, can be arbitrarily large, C has a minimum at P.

34. $T = 100xy^2z$, $g = x^2 + y^2 + z^2 = 4$. $F = 100xy^2z - \lambda(x^2 + y^2 + z^2 - 4)$. $F_x = 100y^2z - 2\lambda x = 0$.
$F_y = 200xyz - 2\lambda y = 2y(100xz - \lambda) = 0$. $F_z = 100xy^2 - 2\lambda z = 0$. If $x = 0$, $y = 0$, or $z = 0$, then $T = 0$.
Otherwise, F_y: $\lambda = 100xz$. F_x: $100y^2z - 200x^2z = 0$, $x^2 = \frac{1}{2}y^2$. F_z: $100xy^2 - 200xz^2 = 0$, $z^2 = \frac{1}{2}y^2$.
$\frac{1}{2}y^2 + y^2 + \frac{1}{2}y^2 = 4$, $y = \pm\sqrt{2}$, $x = \pm 1$, $z = \pm 1$ (8 points). Max: $T = 200$ if $x = z$, min: -200 if $x = -z$.

35. Let ℓ, w, and h be the number of units in the length, width and height of the box. Let V cubic units be its volume and S square units its surface area (S is a constant).
$V = \ell wh$ and $S = \ell w + 2\ell h + 2wh$, $\ell > 0$, $w > 0$, $h > 0$. $f(\ell, w, h, \lambda) = \ell wh + \lambda(\ell w + 2\ell h + 2wh - S)$.
$wh < \frac{1}{2}S$, so $V < \frac{1}{2}\ell S$, and V can be arbitrarily small. Any critical point is a maximum.
$\frac{\partial F}{\partial \ell} = wh + \lambda(w + 2h) = 0$; $\frac{1}{h} + \frac{2}{w} = -\frac{1}{\lambda}$. $\frac{\partial F}{\partial w} = \ell h + \lambda(\ell + 2h) = 0$; $\frac{1}{h} + \frac{2}{\ell} = -\frac{1}{\lambda}$.
$\frac{\partial F}{\partial h} = \ell w + \lambda(2\ell + 2w) = 0$; $\frac{2}{w} + \frac{2}{\ell} = -\frac{1}{\lambda}$. Hence $\frac{1}{\ell} = -\frac{1}{4\lambda}$, $\frac{1}{w} = -\frac{1}{4\lambda}$, $\frac{1}{h} = -\frac{1}{2\lambda}$.
Therefore V has a maximum when $\ell{:}w{:}h = 1{:}1{:}\frac{1}{2}$.

36. Prove that the box having the largest volume that can be placed inside a sphere is in the shape of a cube.
▹ Let x, y, z be the number of units in the dimensions of the box. If a units is the diameter of the sphere, because the largest box is inscribed in the sphere, the diagonal of the box is a units. Thus we we want to show that the volume function defined by
$V(x, y, z) = xyz$
subject to the constraint
$x^2 + y^2 + z^2 = a^2$ (1)
has an absolute maximum value if $x = y = z$.
Let F be the function defined by
$F(x, y, z, \lambda) = xyz + \lambda(0.15xy + 0.60xz + 0.60yz - 10)$
Then
$F_x(x, y, z, \lambda) = yz + 2\lambda x = 0$ (2)
$F_y(x, y, z, \lambda) = xz + 2\lambda y = 0$ (3)
$F_z(x, y, z, \lambda) = xy + 2\lambda z = 0$ (4)
We multiply on both sides of Eq. (2) by x, on both sides of Eq. (2) by y, and on both sides of Eq. (4) by z. This results in
$xyz + 2\lambda x^2 = 0$
$xyz + 2\lambda y^2 = 0$
$xyz + 2\lambda z^2 = 0$
and we conclude that $x^2 = y^2 = z^2$, and thus, $x = y = z$. Because the constraint set is closed and bounded, V has an absolute maximum value which must occur at the critical point. Therefore, the box having the largest volume that can be placed in a sphere is a cube.

In Exercises 37 and 38 we wish to maximize U subject to $g = 2x + 3y + 4z = 90$.

37. $U = \exp(x^2yz)$. $12x^2yz = xx(3y)(4z)$. Because $x + x + 3y + 4z$ is constant, the product is maximum if $x = 3y = 4z = \frac{90}{4} = \frac{45}{2}$, $y = \frac{15}{2}$, $z = \frac{45}{8}$. This point maximizes U.

38. $U = x^2y^3z$. $4x^2y^3z = xxyyy(4z)$. Because $x + x + y + y + y + 4z$ is constant, the product is maximum if $x = y = 4z = \frac{90}{6} = 15$. $z = \frac{15}{4}$. This point maximizes U.

39. $T = 2x^2 + y^2 - y$. Because the disk $x^2 + y^2 \le 1$ is closed and bounded, extrema are attained at a critical point or a boundary point. At a critical point $T_x = 4x = 0$; $x = 0$ and $T_y = 2y - 1 = 0$; $y = \frac{1}{2}$. $T(0, \frac{1}{2}) = -\frac{1}{4}$.
On the boundary, $x^2 = 1 - y^2$, $-1 \le y \le 1$, $T = 2(1 - y^2) + y^2 - y = 2 - y^2 - y$. $T_y = -2y - 1$; $y = \frac{1}{2}$.
When $y = -1$, $x = 0$ and $T = 2$. When $y = -\frac{1}{2}$, $x = \pm\frac{1}{2}\sqrt{3}$ and $T = \frac{9}{4}$. When $y = 1$, $x = 0$ and $T = 0$.

Thus the temperature is hottest at the points $(\pm\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ and coldest at the point $(0, \frac{1}{2})$.

40. If $T(x, y)$ degrees is the temperature at any point (x, y) of the top half of a circular disk, defined by $x^2 + y^2 \le 1$ and $y \ge 0$ and $T(x, y) = 2x^2 - 3xy + 5y^2$, find the hottest and coldest points in the region and the temperature at those points.

▸ Because the region is closed and bounded, extrema are attained at a critical point or a boundary point.
$T_x = 4x - 3y = 0$
$T_y = -3x + 10y = 0$
Thus the only critical point is $(0, 0)$ and $T(0, 0) = 0$, which turns out to be the absolute minimum.
If $y = 0$, $-1 \le x \le 1$, then $T = 2x^2$, which has a minimum of 0 at 0 and a maximum of 1 at ± 1.
If $x^2 + y^2 = 1$, $y \ge 0$, let
$F(x, y, \lambda) = 2x^2 - 3xy + 5y^2 + \lambda(x^2 + y^2 - 1)$
$F_x(x, y, \lambda) = 4x - 3y + 2\lambda x = (4 + 2\lambda)x - 3y = 0$ (1)
$F_y(x, y, \lambda) = -3x + 10y + 2\lambda y = -3x + (10 + 2\lambda)y = 0$ (2)
This homogeneous linear system has the solution $(0, 0)$. Otherwise, solving Eqs (1) and (2) for y, we obtain
$y = \frac{4 + 2\lambda}{3}x$ and $y = \frac{3}{10 + 2\lambda}$
and so

$$\frac{4 + 2\lambda}{3} = \frac{3}{10 + 2\lambda}$$

$$4\lambda^2 + 28\lambda + 40 = 9$$

$$4\lambda^2 + 28\lambda + 31 = 0 \qquad (3)$$

$$\lambda = \tfrac{1}{2}(-7 \pm 3\sqrt{2})$$

$$\frac{4 + 2\lambda}{3} = -1 \pm \sqrt{2} \qquad (4)$$

Substituting into the constraint, we get

$$1 = x^2 + y^2 = x^2 + (-1 \pm \sqrt{2})x^2 = (4 \pm 2\sqrt{2})x^2$$

and so

$$x^2 = \frac{1}{4 \pm 2\sqrt{2}} = \tfrac{1}{4}(2 \pm \sqrt{2})$$

$$x = \pm\sqrt{2 \pm \sqrt{2}}$$

If we select the positive sign in (4), we choose the positive root. Then
$x = \frac{1}{2}\sqrt{2 + \sqrt{2}}$, $y = \frac{1}{2}\sqrt{2 - \sqrt{2}}$ and $T = \frac{7}{2} - \frac{3}{2}\sqrt{2}$
(after much work). If we choose the negative sign in (3), we choose the negative root. Then
$x = -\frac{1}{2}\sqrt{2 - \sqrt{2}}$, $y = \frac{1}{2}\sqrt{2 + \sqrt{2}}$, $T = \frac{7}{2} + \frac{3}{2}\sqrt{2}$
which is the absolute maximum value.
The critical points are the *eigenvectors* of the system $4x - 3y = 0$, $-3x + 10y = 0$, and Eq. (3) is the *characteristic equation*. The TI-85 can find these by using the eigVc function in the MATRX MATH menu; other advanced graphics calculators can too.
Alternatively, let $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le \pi$.

$$T = 2\cos^2\theta - 3\sin\theta\cos\theta + 5\cos^2\theta = (1 + \cos 2\theta) - \tfrac{3}{2}\sin 2\theta + \tfrac{5}{2}(1 - \cos 2\theta) = \tfrac{7}{2} - \tfrac{3}{2}\sin 2\theta - \tfrac{3}{2}\cos 2\theta$$

$$= \tfrac{7}{2} - \tfrac{3}{2}\sqrt{2}(\tfrac{1}{2}\sqrt{2}\sin 2\theta + \tfrac{1}{2}\sqrt{2}\cos 2\theta) = \tfrac{7}{2} - \tfrac{3}{2}\sqrt{2}\sin(2\theta + \tfrac{1}{4}\pi)$$

The maximum value of T on the arc, and the absolute maximum value, is $\frac{7}{2} + \frac{3}{2}\sqrt{2}$ if

$$\sin(2\theta + \tfrac{1}{4}\pi) = -1$$

$$2\theta + \tfrac{1}{4}\pi = \tfrac{3}{2}\pi$$

$$2\theta = \tfrac{5}{4}\pi$$

$x = \cos\theta = -\sqrt{\frac{1}{2}(1 + \cos\frac{5}{4}\pi)} = -\frac{1}{2}\sqrt{2 - \sqrt{2}}$ $\qquad$ $y = \sin\theta = \sqrt{\frac{1}{2}(1 - \cos\frac{5}{4}\pi)} = \frac{1}{2}\sqrt{2 + \sqrt{2}}$

The minimum on the arc is $\frac{7}{2} - \frac{3}{2}\sqrt{2} > 0$ when $x = \frac{1}{2}\sqrt{2+\sqrt{2}}$ and $y = \frac{1}{2}\sqrt{2-\sqrt{2}}$.

A third alternative is to simplify T without changing the constraint by using the rotation of axes formulas A.10.3.

41. Given $2x + 3y + 4z + s + t = 150$, we maximize $24U = (2x)(3y)(4z)st$ when $2x = 3y = 4z = s = t = \frac{150}{5} = 30$, $x = 15$, $y = 10$, $z = 7.5$.

42. Minimize $C = (3x^2 + 200) + (y^2 + 400) + (2z^2 + 300) = 3x^2 + y^2 + 2z^2 + 1100$ subject to $x + y + z = 1100$.
$F = 3x^2 + y^2 + 2z^2 + 1100 - \lambda(x + y + z - 100)$. $F_x = 6x - \lambda = 0$, $x = \frac{1}{6}\lambda$. $F_y = 2y - \lambda = 0$, $y = \frac{1}{2}\lambda$.
$F_z = 4z - \lambda = 0$, $z = \frac{1}{4}\lambda$. $\frac{1}{6}\lambda + \frac{1}{2}\lambda + \frac{1}{4}\lambda = \frac{11}{12}\lambda = 1100$, $\lambda = 1200$. $x = 200$, $y = 600$, $z = 300$.

43. See Exercise 12.8.26.

Miscellaneous Exercises for Chapter 12

In Exercises 1–4, determine the domain of f and sketch as a region in R^2 the set of points in the domain.

1. $f(x,y) = \sqrt{x^2 + 4y^2 - 16}$. The domain of f is $\{(x,y) \mid x^2 + 4y^2 \geq 16\}$, which is the set of all points in R^2 outside and on the ellipse $x^2/16 + y^2/4 = 1$.

2. $f(x,y) = 6/\sqrt{16 - x^2 - y^2}$. $\text{dom}(f) = \{(x,y) \mid x^2 + y^2 < 16\}$, the points in R^2 inside the circle $x^2 + y^2 = 16$.

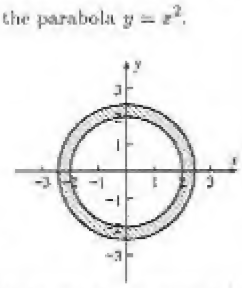
3. $f(x,y) = \ln(y - x^2)$. $\text{dom}(f) = \{(x,y) \mid y > x^2\}$, the set of all points in R^2 inside the parabola $y = x^2$.

4. $f(x,y) = \sin^{-1}(5 - x^2 - y^2)$

▸ Because the domain of $\sin^{-1} x$ is $[-1, 1]$, then the domain of f is the set of all points (x,y) such that

$$-1 \leq 5 - x^2 - y^2 \leq 1$$
$$-6 \leq -x^2 - y^2 \leq -4$$
$$6 \geq x^2 + y^2 \geq 4$$

Thus the domain is the set of all points that lie either on or between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 6$ of radii 2 and $\sqrt{6}$, as shown in the figure.



5. $f(x,y,z) = \frac{x}{|y| - |z|}$. $f(x,y,z)$ is defined if $|y| - |z| \neq 0$. Thus the domain of f is the set of all points (x,y,z) in R^3 except those on the planes $y = \pm z$. The range of f is $(-\infty, +\infty)$.

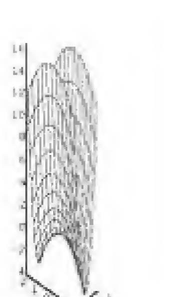
6. $f(x,y,z) = \ln(x^2 + y^2 + z^2 - 4)$. $\text{dom}(f) = \{(x,y,z) \mid x^2 + y^2 + z^2 > 4\}$, the points in R^3 outside the sphere $x^2 + y^2 + z^2 = 4$.

In Exercises 7 and 8, determine the domain of f and sketch the graph of f.

7. $f(x,y) = \sqrt{36 - 4x^2 - 9y^2}$. The domain of f is $\{(x,y) \mid 4x^2 + 9y^2 \leq 36\}$, the set of all points (x,y) in R^2 inside and on the ellipse $x^2/9 + y^2/4 = 1$. The graph of $z = \sqrt{36 - 4x^2 - 9y^2}$ is the upper half of the ellipsoid $x^2/9 + y^2/4 + z^2/36 = 1$.

8. $f(x,y) = 16x^2 - y^2$

▸ The domain of f is all points of R^2. The graph of f is the graph of $z = 16x^2 - y^2$, a hyperbolic paraboloid, as shown in the figure.



9. $f(x,y) = 4x^{1/2}y$. If $f = k$ we have $k = 4x^{1/2}y$. The constant product curves are $y = k/4x^{1/2}$, $k = 16, 8, 4, 2$.

10. $t(x,y) = x^2 + 2y$. If $t = k$ then $k = x^2 + 2y$. The isothermals are the parabolas $y = \frac{1}{2}(k - x^2)$, $k = 0, 2, 4, 6, 8$.

In Exercises 11–24, find the indicated partial derivatives.

11. $f(x,y) = 2x^2y - 3xy^2 + 4x - 2y$

(a) $D_1 f(x,y) = 4xy - 3y^2 + 4$ (b) $D_2 f(x,y) = 2x^2 - 6xy - 2$

(c) $D_{11} f(x,y) = \frac{\partial}{\partial x}(4xy - 3y^2 + 4) = 4y$ (d) $D_{22} f(x,y) = \frac{\partial}{\partial y}(2x^2 - 6xy - 2) = -6x$

(e) $D_{12} f(x,y) = \frac{\partial}{\partial y}(4xy - 3y^2 + 4) = 4x - 6y$ (f) $D_{21} f(x,y) = \frac{\partial}{\partial x}(2x^2 - 6xy - 2) = 4x - 6y$

12. $f(x,y) = (4x^2 - 2y)^3$; (a) $f_1(x,y)$; (b) $f_2(x,y)$; (c) $f_{11}(x,y)$; (d) $f_{22}(x,y)$; (e) $f_{12}(x,y)$; (f) $f_{21}(x,y)$

▸ (a) $f_1(x,y) = D_x(4x^2 - 2y)^3 = 3(4x^2 - 2y)^2 D_x(4x^2 - 2y) = 3(4x^2 - 2y)^2(8x) = 96x(2x^2 - y)^2$

(b) $f_2(x,y) = D_y(4x^2 - 2y)^3 = 3(4x^2 - 2y)^2 D_y(4x^2 - 2y) = 3(4x^2 - 2y)^2(-2) = -24(2x^2 - y)^2$

(c) Applying the result of part (a), we obtain

$$f_{11}(x,y) = D_x f_1(x,y) = D_x[96x(2x^2 - y)^2] = 96[(2x^2 - y)^2 + 2x(2x^2 - y)(4x)]$$
$$= 96(2x^2 - y)[2x^2 - y + 8x^2] = 96(2x^2 - y)(10x^2 - y)$$

(d) Applying the result of part (b), we obtain

$$f_{22}(x,y) = D_y f_2(x,y) = D_y[-24(2x^2 - y)^2] = (-24)(2)(2x^2 - y)D_y(2x^2 - y) = 48(2x^2 - y)$$

(e) Applying the result of part (a), we obtain

$$f_{12}(x,y) = D_y[96x(2x^2 - y)^2] = (96x)(2)(2x^2 - y)(-1) = -192x(2x^2 - y)$$

(f) Applying the result of part (b), we obtain

$$f_{12}(x,y) = D_x f_2(x,y) = D_x[-24(2x^2 - y)^2] = (-24)(2)(2x^2 - y)(4x) = -192x(2x^2 - y)$$

13. $f(x,y) = \dfrac{x^2 - y}{3y^2}$ (a) $f_x(x,y) = \dfrac{2x}{3y^2}$ (b) $f_y(x,y) = \dfrac{1}{3} \cdot \dfrac{(-1)(y^2) - (x^2 - y)(2y)}{(y^2)^2} = \dfrac{1}{3} \cdot \dfrac{y^2 - 2x^2 y}{y^4} = \dfrac{y - 2x^2}{3y^3}$

(c) $f_{xy}(x,y) = \dfrac{(-2)2x}{3y^3} = -\dfrac{4x}{3y^3}$ (d) $f_{yx}(x,y) = \dfrac{2(-2x)}{3y^3} = -\dfrac{4x}{3y^3}$

14. $f(r,s) = re^{2rs}$ (a) $D_r f = e^{2rs} + 2rse^{2rs} = (1 + 2rs)e^{2rs}$ (b) $D_s f = 2r^2 e^{2rs}$
(c) $D_{rs} f = 2re^{2rs} + (1 + 2rs)2re^{2rs} = (4r + 4r^2 s)e^{2rs}$ (d) $D_{sr} f = D_{rs}$.

15. $g(s,t) = \sin(st^2) + te^s$. (a) $D_s g(s,t) = t^2 \cos(st^2) + te^s$; (b) $D_t g(s,t) = 2st \cos(st^2) + e^s$

(c) $D_{st} g(s,t) = 2t \cos(st^2) + t^2[-2st \sin(st^2)] + e^s = 2t \cos(st^2) - 2st^3 \sin(st^2) + e^s$

(d) $D_{ts} g(s,t) = 2t \cos(st^2) + 2st[-t^2 \sin(st^2)] + e^s = 2t \cos(st^2) - 2st^3 \sin(st^2) + e^s$

16. $h(x,y) = \tan^{-1} \dfrac{x^3}{y^2}$; (a) $D_1 h(x,y)$; (b) $D_2 h(x,y)$; (c) $D_{11} h(x,y)$; (d) $D_{22} h(x,y)$

▸ (a) $D_1 h(x,y) = \dfrac{1}{1 + \left(\dfrac{x^3}{y^2}\right)^2} \cdot D_x\left(\dfrac{x^3}{y^2}\right) = \dfrac{y^4}{x^6 + y^4} \cdot \dfrac{3x^2}{y^2} = \dfrac{3x^2 y^2}{x^6 + y^4}$

(b) $D_2 h(x,y) = \dfrac{1}{1 + \left(\dfrac{x^3}{y^2}\right)^2} \cdot D_x\left(\dfrac{x^3}{y^2}\right) = \dfrac{y^4}{x^6 + y^4} \cdot \dfrac{-2x^3}{y^3} = \dfrac{-2x^3 y}{x^6 + y^4}$

(c) $D_{11} h(x,y) = D_x\left(\dfrac{3x^2 y^2}{x^6 + y^4}\right) = \dfrac{6xy^2(x^6 + y^4) - 3x^2 y^2 \cdot 6x^5}{(x^6 + y^4)^2} = \dfrac{-6xy^2(2x^6 - y^4)}{(x^6 + y^4)^2}$

(d) $D_{22} h(x,y) = D_y\left(\dfrac{-2x^3 y}{x^6 + y^4}\right) = \dfrac{-2x^3(x^6 + y^4) - (-2x^3 y)(4y^3)}{(x^6 + y^4)^2} = \dfrac{-2x^3(x^6 - 3y^4)}{(x^6 + y^4)^2}$

17. $f(x,y) = e^{x/y} + \ln \dfrac{x}{y} = e^{x/y} + \ln|x| - \ln|y|$

(a) $f_x(x,y) = \dfrac{1}{y}e^{x/y} + \dfrac{1}{x}$ (b) $f_y(x,y) = -\dfrac{x}{y^2}e^{x/y} - \dfrac{1}{y}$ (c) $f_{xx}(x,y) = \dfrac{1}{y^2}e^{x/y} - \dfrac{1}{x^2}$

(d) $f_{yy}(x,y) = \dfrac{2x}{y}e^{x/y} - \dfrac{x}{y^2}e^{x/y}\left(-\dfrac{x}{y^2}\right) + \dfrac{1}{y^2} = \left(\dfrac{2x}{y^3} + \dfrac{x^2}{y^4}\right)e^{x/y} + \dfrac{1}{y^2}$

18. $f(x,y) = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln(x^2 + y^2)$ (a) $f_1 = \dfrac{x}{x^2 + y^2}$ (b) $f_{11} = \dfrac{1(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$

(c) $f_{12} = \dfrac{-2xy}{(x^2 + y^2)^2}$ (d) $f_{121} = \dfrac{-2y(x^2 + y^2)^2 + 2xy \cdot 4x(x^2 + y^2)}{(x^2 + y^2)^4} = \dfrac{-2y(x^2 + y^2) + 8x^2 y}{(x^2 + y^2)^3} = \dfrac{2y(3x^2 - y^2)}{(x^2 + y^2)^3}$

19. $f(x,y,z) = \dfrac{x}{x^2 + y^2 + z^2}$ (a) $D_1 f(x,y,z) = \dfrac{x^2 + y^2 + z^2 - x(2x)}{(x^2 + y^2 + z^2)^2} = \dfrac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2}$

(b) $D_2 f(x,y,z) = \dfrac{-x(2y)}{(x^2 + y^2 + z^2)^2} = \dfrac{-2xy}{(x^2 + y^2 + z^2)^2}$ (c) $D_3 f(x,y,z) = \dfrac{-x(2z)}{(x^2 + y^2 + z^2)^2} = \dfrac{-2xz}{(x^2 + y^2 + z^2)^2}$

20. $f(x,y,z) = \sqrt{x^2+3yz-z^2}$; (a) $f_x(x,y,z)$; (b) $f_y(x,y,z)$; (c) $f_z(x,y,z)$

▹ (a) $f_x(x,y,z) = \frac{1}{2}(x^2+3yz-z^2)^{-1/2}\mathrm{D}_x(x^2+3yz-z^2) = \dfrac{x}{\sqrt{x^2+3yz-z^2}}$

(b) $f_y(x,y,z) = \frac{1}{2}(x^2+3yz-z^2)^{-1/2}\mathrm{D}_y(x^2+3yz-z^2) = \dfrac{3z}{2\sqrt{x^2+3yz-z^2}}$

(c) $f_z(x,y,z) = \frac{1}{2}(x^2+3yz-z^2)^{-1/2}\mathrm{D}_z(x^2+3yz-z^2) = \dfrac{3y-2z}{2\sqrt{x^2+3yz-z^2}}$

21. $f(u,v,w) = \ln(u^2+4v^2-5w^2)$. $f_u(u,v,w) = \dfrac{2u}{u^2+4v^2-5w^2}$.

(a) $f_{uw}(u,v,w) = \dfrac{-(2u)(-10w)}{(u^2+4v^2-5w^2)^2} = \dfrac{20uw}{(u^2+4v^2-5w^2)^2}$. $f_{uwv}(u,v,w) = \dfrac{-2(20uw)(8v)}{(u^2+4v^2-5w^2)^3} = \dfrac{-320uvw}{(u^2+4v^2-5w^2)^3}$

(b) $f_{uv} = \dfrac{-(2u)(8v)}{(u^2+4v^2-5w^2)^2} = \dfrac{-16uv}{(u^2+4v^2-5w^2)^2}$

$f_{uvv} = -16u\left[\dfrac{1}{(u^2+4v^2-5w^2)^2} - \dfrac{2v(8v)}{(u^2+4v^2-5w^2)^3}\right] = -16u\dfrac{u^2+4v^2-5w^2-16v^2}{(u^2+4v^2-5w^2)^3} = \dfrac{16u(12v^2+5w^2-u^2)}{(u^2+4v^2-5w^2)^3}$

22. $f(r,s,t) = t^2e^{4rst}$ (a) $f_r = 4st^3e^{4rst}$ (b) $f_{rt} = 4s(3t^2e^{4rst} + t^3\cdot 4rse^{4rst}) = 4(3st^2+rs^2t^3)e^{4rst}$

$f_{rts} = 4[(3t^2+2rst^3)e^{4rst} + (3st^2+rs^2t^3)\cdot 4rte^{4rst}] = 4(3+t^2+20rst^3+16r^2s^2t^4)e^{4rst}$

23. $f(r,s,t) = \dfrac{\ln 4rs}{t^2} = \dfrac{\ln r + \ln 4s}{t^2}$ (a) $\mathrm{D}_1f(r,s,t) = \dfrac{1}{rt^2}$ (b) $\mathrm{D}_{13}f(r,s,t) = \dfrac{-2}{rt^3}$ (c) $\mathrm{D}_{131}f(r,s,t) = \dfrac{2}{r^2t^3}$

24. $f(u,v,w) = w\cos 2v + 3v\sin u - 2uv\tan w$;
(a) $\mathrm{D}_2f(u,v,w)$; (b) $\mathrm{D}_1f(u,v,w)$; (c) $\mathrm{D}_{131}f(u,v,w)$

▹ (a) $\mathrm{D}_2f(u,v,w) = \mathrm{D}_v(w\cos 2v + 3v\sin u - 2uv\tan w) = -2w\sin 2v + 3\sin u - 2u\tan w$
(b) $\mathrm{D}_1f(u,v,w) = \mathrm{D}_u(w\cos 2v + 3v\sin u - 2uv\tan w) = 3v\cos u - 2v\tan w$
(c) $\mathrm{D}_{13}f(u,v,w) = \mathrm{D}_w\mathrm{D}_1f(u,v,w) = \mathrm{D}_w(3v\cos u - 2v\tan w) = -2v\sec^2 w$ and
$\mathrm{D}_{131}f(u,v,w) = \mathrm{D}_u(-2v\sec^2 w) = 0$

25. $w = x^2y - xy^2 + y^2z - z^2y + z^2x - x^2z$

$\dfrac{\partial w}{\partial x} = 2xy - y^2 + z^2 - 2xz$; $\dfrac{\partial w}{\partial y} = x^2 - 2yx + 2yz - z^2$; $\dfrac{\partial w}{\partial z} = y^2 - 2zy + 2zx - x^2$. Thus $\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} + \dfrac{\partial w}{\partial z} = 0$.

26. Show that $u = (x^2+y^2+z^2)^{-1/2}$ satisfies Laplace's equation.

▹ $u_x = -x(x^2+y^2+z^2)^{-3/2}$. $u_{xx} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$
$= \dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$. Similarly, $u_{yy} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$; $u_{zz} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$ and so $u_{xx}+u_{yy}+u_{zz} = 0$

In Exercises 27 and 28, find $\partial u/\partial t$ and $\partial u/\partial s$ by two methods.

27. $u = y\ln(x^2+y^2)$, $x = 2s+3t$, and $y = 3t-2s$.

(a) $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t} = \dfrac{2xy}{x^2+y^2}(3) + \left[\ln(x^2+y^2) + \dfrac{2y^2}{x^2+y^2}\right](3) = \dfrac{6y(x+y)}{x^2+y^2} + 3\ln(x^2+y^2)$

$\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} = \dfrac{2xy}{x^2+y^2}(2) + \left[\ln(x^2+y^2) + \dfrac{2y^2}{x^2+y^2}\right](-2) = \dfrac{4y(x-y)}{x^2+y^2} - 2\ln(x^2+y^2)$

(b) $u = (3t-2s)\ln(8s^2+18t^2)$

$\dfrac{\partial u}{\partial t} = (3t-2s)\dfrac{18t}{4s^2+9t^2} + 3\ln(8s^2+18t^2)$; $\dfrac{\partial u}{\partial s} = (3t-2s)\dfrac{8s}{4s^2+9t^2} - 2\ln(8s^2+18t^2)$

28. $u = e^{2x+y}\cos(2y - x)$, $x = 2s^2 - t^2$, $y = s^2 + 2t^2$

▶ We use the chain rule. Thus,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \\ &= [e^{2x+y}\sin(2y - x) + 2\cos(2y - x)e^{2x+y}](-2t) + [-2e^{2x+y}\sin(2y - x) + e^{2x+y}\cos(2y - x)](4t) \\ &= -10te^{2x+y}\sin(2y - x)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \\ &= [e^{2x+y}\sin(2y - x) + 2\cos(2y - x)e^{2x+y}]4s + [-2e^{2x+y}\sin(2y - x) + e^{2x+y}\cos(2y - x)]2s \\ &= 10se^{2x+y}\cos(2y - x)\end{aligned}$$

For the second method, we eliminate x and y before partial-differentiating. Because

$2x + y = 5s^2$ and $2y - x = 5t^2$

we have

$u = e^{5s^2}\cos 5t^2$

Thus,

$u_t = -10te^{5s^2}\sin 5t^2$

and

$u_s = 10se^{5s^2}\cos 5t^2$

29. $u = 3x^2y + 2xy - 3yz - 2z^2$; $x = e^{3rs}$; $y = r^3s^2$; $z = \ln 4$

(a) $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r} = (6xy + 2y)(3se^{3rs}) + (3x^2 + 2x - 3z)(3r^2s^2) + (-3y - 4z)(0)$

$= 18xyse^{3rs} + 6yse^{3rs} + 9x^2r^2s^2 + 6xr^2s^2 - 9zr^2s^2$

(b) $u = 3r^3s^2e^{6rs} + 2r^3s^2e^{3rs} - 3(\ln 4)r^3s^2 - 2\ln^2 4$

$\dfrac{\partial u}{\partial r} = 9r^2s^2e^{6rs} + 18r^3s^3e^{6rs} + 6r^2s^2e^{3rs} + 6r^3s^3e^{3rs} - 9(\ln 4)r^2s^2 = [9(1 + 2rs)e^{6rs} + 6(1 + rs)e^{3rs} - 9\ln 4]r^2s^2$

30. $u = e^{x^2+y^2} - \dfrac{3x}{y} + 3z$, $x = \sin\theta$, $y = \cos\theta$, $z = \tan\theta$

▶ (a) $\dfrac{du}{d\theta} = \dfrac{\partial u}{\partial x}\dfrac{dx}{d\theta} + \dfrac{\partial u}{\partial y}\dfrac{dy}{d\theta} + \dfrac{\partial u}{\partial z}\dfrac{dz}{d\theta} = \left(2xe^{x^2+y^2} - \dfrac{3}{y}\right)\cos\theta - \left(2ye^{x^2+y^2} + \dfrac{3x}{y^2}\right)\sin\theta + 3\sec^2\theta$

(b) $u = e - \tan\theta + 3\tan\theta = e + 2\tan\theta$, $\dfrac{du}{d\theta} = 2\sec^2\theta$

31. $u = xy + x^2$; $x = 4\cos t$; $y = 3\sin t$

(a) $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} = (y + 2x)(-4\sin t) + x(3\cos t) = 3x\cos t - 4(y + 2x)\sin t$

At $t = \frac{1}{4}\pi$, $x = 2\sqrt{2}$, $y = \frac{3}{2}\sqrt{2}$ and $\dfrac{du}{dt} = 3(2\sqrt{2})(\frac{1}{2}\sqrt{2}) - 4(\frac{3}{2}\sqrt{2} + 4\sqrt{2})(\frac{1}{2}\sqrt{2}) = -16.$

(b) $u = 12\sin t\cos t + 16\cos^2 t = 6\sin 2t + 8 + 8\cos 2t$

$\dfrac{du}{dt} = 12\cos 2t - 16\sin 2t$. At $t = \frac{1}{4}\pi$, $\dfrac{du}{dt} = 12\cos\frac{1}{2}\pi - 16\sin\frac{1}{2}\pi = -16$

32. If $f(x, y) = x^2 + ye^x$, find: (a) $\Delta f(0, 2)$, the increment of f at $(0, 2)$; (b) $\Delta f(0, 2)$ when $\Delta x = -0.1$ and $\Delta y = 0.2$; (c) $df(0, 2, \Delta x, \Delta y)$, the total differential of f at $(0, 2)$; (d) $df(0, 2, -0.1, 0.2)$.

▶ (a) $\Delta f(0, 2) = f(\Delta x, 2 + \Delta y) - f(0, 2) = (\Delta x)^2 + (2 + \Delta y)e^{\Delta x} - 2$

(b) If $\Delta x = -0.1$ and $\Delta y = 0.2$, then from the result given part (a) we have

$\Delta f(0, 2) = (-0.1)^2 + (2 + 0.2)^2e^{-0.1} - 2 = 0.01 + 2.2e^{-0.1} - 2 \approx 0.000642$

(c) $df(x, y, \Delta x, \Delta y) = D_1 f(x, y)\Delta x + D_2 f(x, y)\Delta y = (2x + ye^x)\Delta x + e^x\Delta y$

Thus, $df(0, 2, \Delta x, \Delta y) = 2\Delta x + \Delta y$

(d) From the result given in part (c), we obtain

$df(0, 2, -0.1, 0.2) = 2(-0.1) + 0.2 = 0$

33. $f(x, y, z) = 3xy^2 - 5xz^2 - 2xyz$. (a) $\Delta f(-1,3,2) = f(-1 + \Delta x, 3 + \Delta y, 2 + \Delta z) - f(-1,3,2)$

$= 3(-1 + \Delta x)(3 + \Delta y)^2 - 5(-1 + \Delta x)(2 + \Delta z)^2 - 2(-1 + \Delta x)(3 + \Delta y)(2 + \Delta z) - 5$

(b) When $\Delta x = 0.02$, $\Delta y = -0.01$, and $\Delta z = -0.02$, then

$\Delta f(-1,3,2) = 3(-0.98)(2.99)^2 - 5(-0.98)(1.98)^2 - 2(-0.98)(2.99)(1.98) - 5 = -0.470$

(c) At $(-1,3,2)$, $D_1 f(x, y, z) = 3y^2 - 5z^2 - 2yz = -5$, $D_2 f(x, y, z) = 6xy - 2xz = -14$.

$D_3 f(x, y, z) = -10xz - 2xy = 26$. Therefore

$df(-1,3,2,\Delta x,\Delta y,\Delta z) = D_1 f(-1,3,2)\Delta x + D_2 f(-1,3,2)\Delta y + D_3 f(x,y,z)\Delta z = -5\Delta x - 14\Delta y + 26\Delta z.$

(d) $df(-1,3,2,0.02,-0.01,-0.02) = -5(0.02) - 14(0.01) + 26(-0.02) = -0.48$

34. $f(x) = x^2 + 1$, $g(x, y) = \frac{2x}{3y}$, $h(x) = \frac{1}{x}$ (a) $(h \circ g)(-3, 4) = \frac{3y}{2x} = \frac{12}{-6} = -2$ (b) $g(f(3), h(\frac{1}{4})) = g(10, 4) = \frac{20}{12} = \frac{5}{3}$

(c) $g(f(x), h(y)) = g\left(x^2 + 1, \frac{1}{y}\right) = \frac{2(x^2+1)}{3(1/y)} = \frac{2}{3}y(x^2+1)$, $y \neq 0$ (d) $f(h \circ g) = f\left(\frac{3y}{2x}\right) = \frac{9y^2}{4x^2} + 1$

In Exercises 35–37, evaluate the limit by the use of limit theorems.

35. $\lim\limits_{(x,y)\to(e,0)} \ln\left(\frac{x^2}{y+1}\right) = \ln\left(\lim\limits_{(x,y)\to(e,0)} \frac{x^2}{y+1}\right) = \ln e^2 = 2$

36. $\lim\limits_{(x,y)\to(0,\pi/2)} \frac{xy^2 + e^x}{\cos x + \sin y}$

▷ Because the limit of the denominator is not zero, we may apply the limit of a quotient theorem after obtaining the limits of the numerator and denominator.

$$\lim_{(x,y)\to(0,\pi/2)} \frac{xy^2 + e^x}{\cos x + \sin y} = \frac{0(\frac{1}{2}\pi)^2 + e^0}{\cos 0 + \sin \frac{1}{2}\pi} = \frac{1}{2}$$

37. $\lim\limits_{(x,y)\to(1,3)} \sin^{-1}\left(\frac{3x}{2y}\right) = \sin^{-1}\left(\lim\limits_{(x,y)\to(1,3)} \frac{3x}{2y}\right) = \sin^{-1}\frac{1}{2} = \frac{1}{6}\pi$

In Exercises 38–40, establish the limit by finding a $\delta > 0$ for any $\epsilon > 0$ such that Definition 12.2.5 holds.

38. $\lim\limits_{(x,y)\to(4,-1)} (4x - 5y) = 21$

▷ $|4x - 5y - 21| = |4(x-4) - 5(y+1)| \leq \sqrt{4^2 + 5^2}\sqrt{(x-4)^2 + (y+1)^2} < \sqrt{41}\delta \leq \epsilon$ whenever $\delta < \epsilon/\sqrt{41}$

39. To prove $\lim\limits_{(x,y)\to(2,-2)} (3x^2 - 4y^2) = -4$, we show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|(3x^2 - 4y^2) + 4\right| < \epsilon \text{ whenever } 0 < \sqrt{(x-2)^2 + (y+2)} < \delta \qquad (1)$$

$$\left|3x^2 - 4y^2 + 4\right| = \left|3(x^2 - 4) - 4(y^2 - 4)\right| \leq 3|x-2||x+2| + 4|y+2||y-2| \qquad (2)$$

To get an upper bound for $|x+2|$ and $|y-2|$, we restrict $\delta \leq 1$. Then whenever

$|x - 2| \leq \sqrt{(x-2)^2 + (y+2)^2} < \delta \leq 1$, we have $-1 < x - 2 < 1$, and so $3 < x + 2 < 5$. Also whenever

$|y + 2| \leq \sqrt{(x-2)^2 + (y+2)^2} < \delta \leq 1$, we have $-1 < y + 2 < 1$; $-5 < y - 2 < -3$; $3 < |y - 2| < 5$.

Therefore, whenever $0 < \sqrt{(x-2)^2 + (y+2)^2} < \delta \leq 1$, we have

$$3|x-2||x+2| + 4|y+2||y-2| < 3 \cdot \delta \cdot 5 + 4 \cdot \delta \cdot 5 = 35\delta \qquad (3)$$

Hence, if $\delta = \min(1, \frac{1}{35}\epsilon)$ then from (2) and (3) we have statement (1).

40. $\lim\limits_{(x,y)\to(3,1)} (x^2 - y^2 + 2x - 4y) = 10$

▷ For any $\epsilon > 0$, we must find a $\delta > 0$ such that
$\left|x^2 - y^2 + 2x - 4y - 10\right| < \epsilon$ whenever $0 < \sqrt{(x-3)^2 + (y-1)^2} < \delta$.
We use the triangle inequality and then the Cauchy-Schwarz inequality.

If $0 < \sqrt{(x-3)^2 + (y-1)^2} < \delta$ then

$\left|x^2 - y^2 + 2x - 4y - 10\right|$

$= \left|(x^2 - 6x + 9) - (y^2 - 2y + 1) + (8x - 24) - (6y - 6)\right| = \left|(x-3)^2 - (y-1)^2 + 8(x-3) - 6(y-1)\right|$

$\leq (x-3)^2 + (y-1)^2 + \left|8(x-3) - 6(y-1)\right| < \delta^2 + \sqrt{8^2 + 6^2}\sqrt{(x-3)^2 + (y-1)^2} < \delta^2 + 10\delta \leq 11\delta$ if $\delta \leq 1$

$= \epsilon$

if $\delta = \min(1, \frac{1}{11}\epsilon)$

In Exercises 41–44, determine if the limit exists.

41. $0 < \left|\frac{x^3y^3}{x^2+y^2}\right| = \frac{|x|^3|y|^3}{x^2+y^2} \leq \frac{(x^2+y^2)^{3/2}(x^2+y^2)^{3/2}}{x^2+y^2} = (x^2+y^2)^2$. By the squeeze theorem $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2} = 0$.

42. See Exercise 48.

43. $\lim\limits_{\substack{(x,y)\to(0,0)\\(x=0)}} \frac{x^9y}{(x^6+y^2)^2} = \lim\limits_{y\to0} 0 = 0$; $\lim\limits_{\substack{(x,y)\to(0,0)\\y=x^3}} \frac{x^9y}{(x^6+y^2)^2} = \lim\limits_{x\to0} \frac{x^9x^3}{(x^6+x^6)^2} = \lim\limits_{x\to0} \frac{x^{12}}{4x^{12}} = \lim\limits_{x\to0} \frac{1}{4} = \frac{1}{4}$

Because these two limits are not equal, then $\lim\limits_{(x,y)\to(0,0)} \frac{x^9y}{(x^6+y^2)^2}$ does not exist.

44. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^3+4x^2y}{x^2+y^2}$

▸ $0 \le \left|\dfrac{2x^3+4x^2y}{x^2+y^2}\right| = \dfrac{x^2}{x^2+y^2}|2x+4y| \le |2x+4y|$

Because $\displaystyle\lim_{(x,y)\to(0,0)} |2x+4y| = 0$, then by the squeeze theorem $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^3+4x^2y}{x^2+y^2} = 0$.

In Exercises 45–48, determine all points at which f is continuous.

45. $f(x,y) = \dfrac{x^2+4y^2}{x^2-4y^2}$. Because f is a rational function it is continuous at all points in its domain. Thus, f is continuous at all points (x,y) in R^2 not on the lines $x = \pm 2y$.

46. $f(x,y) = 1/(\cos^2 \frac{1}{2}\pi x + \cos^2 \frac{1}{2}\pi y)$. f is continuous unless both cosines are 0, ie. unless x and y are odd integers.

47. $f(x,y) = \begin{cases} \dfrac{x^3y^3}{x^2+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. If $(x,y) \ne (0,0)$, f is a rational function and hence is continuous.

Furthermore, $f(0,0) = 0$ and from Exercise 41 $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$, and so f is continuous at $(0,0)$. Therefore, f is continuous at all points (x,y) in R^2.

48. $f(x,y) = \begin{cases} \dfrac{x^4-y^4}{x^4+y^4} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

▸ f is continuous at all points $(x,y) \ne (0,0)$ because then $x^4+y^4 \ne 0$. We wish to determine whether $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim_{x\to 0} \frac{x^4}{x^4} = 1$$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = \lim_{y\to 0} \frac{-y^4}{y^4} = -1$$

Because the restricted limits are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, so f is discontinuous at $(0,0)$.

In Exercises 49–53, find the value of the directional derivative at P_0 for the function in the direction of U.

49. $f(x,y) = 3x^2 - 2xy + 1$; $\mathbf{U} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$.

▸ $D_U f(x,y) = f_x(x,y)\left(\frac{3}{5}\right) + f_y(x,y)\left(-\frac{4}{5}\right) = \frac{3}{5}(6x-2y) - \frac{4}{5}(-2x) = \frac{26}{5}x - \frac{6}{5}y$. $D_U f(5,10) = \frac{26}{5}(5) - \frac{6}{5}(10) = 14$

51. $h(x,y) = e^x + y^2\cos x$; $\mathbf{U} = \frac{1}{2}\sqrt{2}\mathbf{i} - \frac{1}{2}\sqrt{2}\mathbf{j}$

$D_U h(x,y) = h_x(x,y)(\frac{1}{2}\sqrt{2}) + h_y(x,y)(-\frac{1}{2}\sqrt{2}) = (e^x - y^2\sin x)(\frac{1}{2}\sqrt{2}) + 2y\cos x(-\frac{1}{2}\sqrt{2})$

$D_U h(0,3) = \frac{1}{2}\sqrt{2}(1 - 9\cdot 0) - \frac{1}{2}\sqrt{2}(6\cdot 1) = \frac{1}{2}\sqrt{2} - 3\sqrt{2} = -\frac{5}{2}\sqrt{2}$

52. $f(x,y) = x^2 - 2x^2y + \ln x$; $\mathbf{U} = \cos\pi\mathbf{i} + \sin\pi\mathbf{j}$; $P_0 = (1,-2)$

▸ First, we find the value of the gradient of f at the point P_0.

$\nabla f(x,y) = \langle 2x - 4xy + x^{-1}, -2x^2\rangle$

$\nabla f(1,-2) = \langle 2+8+1, -2\rangle = \langle 11, -2\rangle$

We are given that

$\mathbf{U} = \langle \cos\pi, \sin\pi\rangle = \langle -1, 0\rangle$

Thus, the value of the required directional derivative is given by

$D_U f(1,-2) = \nabla f(1,-2)\cdot \mathbf{U} = \langle 11,-2\rangle \cdot \langle -1,0\rangle = -11$

53. $f(x,y,z) = xy^2z - 3xyz + 2xz^2$; $\mathbf{U} = -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

$D_U f(x,y,z) = [f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}]\cdot \mathbf{U}$

$= [(y^2z - 3yz + 2z^2)\mathbf{i} + (2xyz - 3xz)\mathbf{j} + (xy^2 - 3xy + xz)\mathbf{k}]\cdot \mathbf{U}$

$D_U f(2,1,1) = [(1-3+2)\mathbf{i} + (4-6)\mathbf{j} + (2-6+8)\mathbf{k}]\cdot \mathbf{U} = (-2\mathbf{j} + 4\mathbf{k})(-\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}) = -2\cdot\frac{2}{3} - 4\cdot\frac{1}{3} = -\frac{8}{3}$

In Exercises 54–57, find (a) the gradient of f at P_0; (b) the rate of change of f in the direction of $\mathbf{U}$ at P_0.

54. $f(x,y) = 3x^2 - 2xy^3$; $\mathbf{U} = \cos\frac{1}{6}\pi\mathbf{i} + \sin\frac{1}{6}\pi\mathbf{j} = \frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j}$ (a) $\nabla f(x,y) = (6x - 2y^3)\mathbf{i} - 6xy^2\mathbf{j}$;
$\nabla f(-3,1) = -20\mathbf{i} + 18\mathbf{j}$ (b) $D_U f(-3,1) = \nabla f(-3,1)\cdot\mathbf{U} = (-20\mathbf{i} + 18\mathbf{j})\cdot(\frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j}) = -10\sqrt{3} + 9$

55. $f(x,y) = \frac{1}{2}\ln(x^2 + y^2)$; $\mathbf{U} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}$ (a) $\nabla f(x,y) = \dfrac{x}{x^2 + y^2}\mathbf{i} + \dfrac{y}{x^2 + y^2}\mathbf{j}$; $\nabla f(1,1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$
(b) The rate of change of $f(x,y)$ in the direction of $\mathbf{U}$ is
$D_U f(1,1) = \nabla f(1,1)\cdot\mathbf{U} = (\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j})\cdot(\frac{1}{2}\mathbf{i} + \frac{1}{2}\sqrt{3}\mathbf{j}) = \frac{1}{4} + \frac{1}{4}\sqrt{3}$

56. $f(x,y,z) = yz - y^2 - xz$; $\mathbf{U} = \frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$; $P_0 = (1,2,3)$
▹ (a) $\nabla f(x,y,z) = -z\mathbf{i} + (z - 2y)\mathbf{j} + (y - x)\mathbf{k}$
$\nabla f(1,2,3) = -3\mathbf{i} - \mathbf{j} + \mathbf{k}$
(b) We have
$D_U f(1,2,3) = \nabla f(1,2,3)\cdot\mathbf{U} = (-3\mathbf{i} - \mathbf{j} + \mathbf{k})\cdot(\frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}) = -\frac{19}{7}$
The function value is decreasing at the rate of $\frac{19}{7}$ units in the direction of $\mathbf{U}$ at P_0.

57. $f(x,y,z) = x^3 + y^3 + 2xyz$; $\mathbf{U} = \dfrac{3}{\sqrt{14}}\mathbf{i} - \dfrac{2}{\sqrt{14}}\mathbf{j} + \dfrac{1}{\sqrt{14}}\mathbf{k}$
(a) $\nabla f(x,y,z) = (3x^2 + 2yz)\mathbf{i} + (3y^2 + 2xz)\mathbf{j} + 2xy\mathbf{k}$; $\nabla f(2,-1,0) = 12\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$
(b) The rate of change of $f(x,y,z)$ in the direction of $\mathbf{U}$ at $(2,-1,0)$ is
$D_U f(2,-1,0) = \nabla f(2,-1,0)\cdot\mathbf{U} = (12\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})\cdot\left(\dfrac{3}{\sqrt{14}}\mathbf{i} - \dfrac{2}{\sqrt{14}}\mathbf{j} + \dfrac{1}{\sqrt{14}}\mathbf{k}\right) = \dfrac{36}{\sqrt{14}} - \dfrac{6}{\sqrt{14}} - \dfrac{4}{\sqrt{14}} = \frac{13}{7}\sqrt{14}$

In Exercises 58 and 59, determine the relative extrema of f, if there are any.

58. $f(x,y) = 2x^2 - 3xy + 2y^2 + 10x - 11y$. $f_x = 4x - 3y + 10 = 0$; $f_y = -3x + 4y - 11 = 0$. $4f_x + 3f_y = 7x + 7 = 0$, $x = -1$. $3f_x + 4f_y = 7y - 14 = 0$, $y = 2$. $f_{xx} = 4 > 0$. $f_{yy} = 4$. $f_{xy} = -3$. $D = 4\cdot 4 - (-3)^2 = 7 > 0$. rel. min.

59. $f(x,y) = x^3 + y^3 + 3xy$. $f_x(x,y) = 3x^2 + 3y$; $f_y(x,y) = 3y^2 + 3x$.
$f_{xx}(x,y) = 6x \quad f_{yy}(x,y) = 6y \quad f_{xy}(x,y) = 3$
$f_x(x,y) = 0$: $x^2 + y = 0$; $y = -x^2$. $f_y(x,y) = 0$: $y^2 + x = 0$; $x^4 + x = 0$; $x(x^3 + 1) = 0$; $x = 0, -1$.
The critical points are $(0,0)$ and $(-1,-1)$. We apply the second-derivative test.
$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0) = 0\cdot 0 - 3^2 < 0$. Hence $f(0,0)$ is not a relative extremum.
$f_{xx}(-1,-1) = -6 < 0$ and $f_{xx}(-1,-1)f_{yy}(-1,-1) - f_{xy}{}^2(-1,-1) = (-6)(-6) - 3^2 = 27 > 0$
Therefore $f(-1,-1) = 1$ is a relative maximum value.

In Exercises 60 and 61, prove that f is differentiable at all points in its domain by showing Definition 12.4.2. holds.

60. $f(x,y) = 3xy^2 - 4x^2 + y^2$
▹ By Definition 16.5.2, we must find ϵ_1 and ϵ_2 such that
$$\epsilon_1\Delta x + \epsilon_2\Delta y = \Delta f(x_0, y_0) - D_1 f(x_0, y_0)\Delta x - D_2 f(x_0, y_0)\Delta y \tag{1}$$
where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. We have
$$\begin{aligned}\Delta f(x_0, y_0) &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)\\ &= [3(x_0 + \Delta x)(y_0 + \Delta y)^2 - 4(x_0 + \Delta x)^2 + (y_0 + \Delta y)^2] - [3x_0 y_0{}^2 - 4x_0{}^2 + y_0{}^2]\\ &= 6x_0y_0\Delta y + 3x_0(\Delta y)^2 + 3y_0{}^2\Delta x + 6y_0\Delta x\Delta y + 3\Delta x(\Delta y)^2 - 8x_0\Delta x - 4(\Delta x)^2 + 2y_0\Delta y + (\Delta y)^2 \end{aligned}\tag{2}$$
Next, we find the following.
$$D_1 f(x_0, y_0)\Delta x = 3y_0{}^2\Delta x - 8x_0\Delta x \tag{3}$$
$$D_2 f(x_0, y_0)\Delta y = 6x_0y_0\Delta y + 2y_0\Delta y \tag{4}$$
Substituting from (2), (3), and (4) into (1), we obtain
$\epsilon_1\Delta x + \epsilon_2\Delta y = [6y_0\Delta y - 4\Delta x + 3(\Delta y)^2]\Delta x + (3x_0\Delta y + \Delta y)\Delta y$
Therefore, we take $\epsilon_1 = 6y_0\Delta y - 4\Delta x + 3(\Delta y)^2$ and $\epsilon_2 = 3x_0\Delta y + \Delta y$ and $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.
Thus, f is differentiable at all points in R^2.

61. $f(x,y) = \dfrac{2x + y}{y^2}$. $D_1 f(x,y) = \dfrac{2}{y^2}$ and $D_2 f(x,y) = \dfrac{-4x - y}{y^3}$. Therefore
$$\Delta f(x,y) - D_1 f(x,y)\Delta x - D_2 f(x,y)\Delta y = \frac{2(x + \Delta x) + (y + \Delta y)}{(y + \Delta y)^2} - \frac{2x + y}{y^2} - \frac{2}{y^2}\Delta x + \frac{4x + y}{y^3}\Delta y$$
$$= \frac{-4y^2\Delta x - 2y(\Delta y)^2}{y^3(y + \Delta y)^2}\Delta x + \frac{y^2\Delta y + 6xy\Delta y + 4x(\Delta y)^2 + y(\Delta y)^2}{y^3(y + \Delta y)^2}\Delta y$$
In Definition 12.4.2 let $\epsilon_1 = \dfrac{-4y^2\Delta x - 2y(\Delta y)^2}{y^3(y + \Delta y)^2}$ and $\epsilon_2 = \dfrac{y^2\Delta y + 6xy\Delta y + 4x(\Delta y)^2 + y(\Delta y)^2}{y^3(y + \Delta y)^2}$.
As $(\Delta x, \Delta y) \to (0,0)$, $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ so f is differentiable at all points in its domain.

62. $\sin\alpha = ac^{-1}$. $|d\sin\alpha| = \left|c^{-1}da - ac^{-2}dc\right| \le c^{-1}da + ac^{-2}dc = (7.14)^{-1}(0.01) + 3.52(7.14)^{-2}(0.01) = 0.00209$

63. Let x meters be the length of the ceiling, y meters the width of the ceiling, and z meters the height of the room. If C dollars is the cost of the job, then
$C = 2(2xz + 2yz + xy) = 4xz + 4yz + 2xy$; $dC = (4z + 2y)dx + (4z + 2x)dy + (4x + 4y)dz$
Let $x = 5$, $y = 4$, $z = 3$, $dx = dy = dz = \pm 0.005$. Then
$\Delta C \approx dC = 20(\pm 0.005) + 22(\pm 0.005) + 36(\pm 0.005) = \pm 0.39$
Therefore, the greatest error in estimating the cost of the job is 39 cents.

64. At a given instant, the length of one side of a rectangle is 6 cm and it is increasing at the rate of 1 cm/s; the length of another side of the rectangle is 10 cm and it is decreasing at the rate of 2 cm/s. fins the rate of change of the area of the rectangle at the given instant.

▸ At t seconds, let x cm and y cm be the lengths of the sides of the rectangle and A cm^2 be its area. We are given that $dx/dt = 1$ and $dy/dt = -2$ when $x = 6$ and $y = 10$. To find dA/dt at this moment, we calculate the total differential and evaluate it. Thus,

$$A = xy$$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = 10 \cdot 1 + 6(-2) = -2$$

Therefore, the area is decreasing at the rate of 2 cm^2/s at the given moment.

65. Let r cm be the radius of a right-circular cylinder and let h cm be its height. If V cm^3 is the volume, then
$V = \pi r^2 h$; $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = 2\pi rh\dfrac{dr}{dt} + \pi r^2\dfrac{dh}{dt}$
When $r = 20$, $\dfrac{dr}{dt} = -5$, $h = 40$, $\dfrac{dh}{dt} = 12$, we have $\dfrac{dv}{dt} = 2\pi(20)(40)(-5) + \pi(20)^2(12) = -3200\pi$.
Thus, at the given instant the volume is decreasing at the rate of 3200π cm^3/min.

66. If $x = 4$ we have $f(y,z) = 396 - 16y^2 + 9z^2 = 0$. $\partial z/\partial y = -f_y/f_z = 32y/18z$. $y = 9$ and $z = 10$ gives $8/5$

67. The ideal gas law: if P atm is the pressure, V liters is the volume, and T degrees is the temperature, then
$P = \dfrac{kT}{V}$; $\dfrac{dP}{dt} = \dfrac{\partial P}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial P}{\partial V}\dfrac{dV}{dt} = \dfrac{k}{V}\dfrac{dT}{dt} - \dfrac{kT}{V^2}\dfrac{dV}{dt}$
If $k = 1.4$, $V = 20$, $T = 75$, $\dfrac{dV}{dt} = 0.3$ and $\dfrac{dT}{dt} = 0.5$, then $\dfrac{dP}{dt} = \dfrac{1.4}{20}(0.5) - \dfrac{1.4(75)}{20^2}(0.3) = -0.044$.
Thus at the given instant the pressure is decreasing at the rate of 0.044 atm/min.

In Ex. 68–70, find an equation of the tangent plane and equations of the normal line to the surface at the point.

68. $z = x^2 + 2xy$; $(1,3,7)$

▸ Let F be the function defined by
$F(x,y,z) = x^2 + 2xy - z$
Thus, the hyperbolic paraboloid is the graph of $F(x,y,z) = 0$. We have

$$\nabla F(x,y,z) = (2x + 2y)\mathbf{i} + 2x\mathbf{j} - \mathbf{k}$$
$$\nabla F(1,3,7) = 8\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

Thus, an equation of the tangent plane to the surface at $(1,3,7)$ is

$$8(x-1) + 2(y-3) - (z-7) = 0$$
$$8x + 2y - z - 7 = 0$$

and equations of the normal line to the surface at $(1,3,7)$ are

$$\frac{x-1}{8} = \frac{y-3}{2} = \frac{z-7}{-1}$$

69. Let $f(x,y,z) = x^2 + 2y + z - 8$. Then $\nabla f(x,y,z) = 2x\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Hence $\nabla f(2,1,2) = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.
An equation of the tangent plane at $(2,1,2)$ is
$4(x-2) + 2(y-1) + (z-2) = 0$; $4x + 2y + z - 12 = 0$
Equations of the normal line at $(2,1,2)$ are $\dfrac{x-2}{4} = \dfrac{y-1}{2} = \dfrac{z-2}{1}$.

70. Let $f(x,y,z) = 3x^2 + 2xy - y^2 - 15$. Then $\nabla f = (6x + 2y)\mathbf{i} + (2x - 2y)\mathbf{j}$. $\nabla f(2,3,4) = 18\mathbf{i} - 2\mathbf{j}$.
Tangent plane: $9(x-2) - (y-3) = 0$; $9x - y = 15$. Normal line: $(x-2)/9 = -(y-3)$, $z = 4$

71. Let $f(x,y,z) = x^2 - 3xy + y^2 - z$ and $G(x,y,z) = 2x^2 + y^2 - 3z + 27$.
$\nabla f(x,y,z) = (2x-3y)\mathbf{i} + (-3x+2y)\mathbf{j} - \mathbf{k}$ and $\nabla G(x,y,z) = 4x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$.
Let $\mathbf{n}_1 = \nabla f(1,-2,11) = 8\mathbf{i} - 7\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \nabla G(1,-2,11) = 4\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$.

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -7 & -1 \\ 4 & -4 & -3 \end{vmatrix} = 17\mathbf{i} + 20\mathbf{j} - 4\mathbf{k}$$

Equations of the tangent line to the curve of intersection at $(1,-2,11)$ are $\frac{x-1}{17} = \frac{y+2}{20} = \frac{z-11}{-4}$.

72. Find equations of the tangent line to the curve of intersection of the surface $z = 3x^2 + y^2 + 1$ with the plane $x = 2$ at the point $(2,-1,14)$.

▸ Because $x = 2$, we have
$z = y^2 + 13$
$\left.\frac{dz}{dy}\right|_{y=-1} = \left.2y\right|_{y=-1} = -2$
Thus the equations of the tangent line at $(2,-1,14)$ are
$x = 2$ and $z - 14 = -2(y+1)$ or, equivalently, $z + 2y = 12$

73. $z = 900 - 3xy$. Let $f(x,y) = 900 - 3xy$. $\nabla f(x,y) = -3y\mathbf{i} - 3x\mathbf{j}$.
(a) The direction of steepest ascent at $(50,4,300)$ is $\nabla f(50,4) = -12\mathbf{i} - 150\mathbf{j}$.
(b) The y axis points to the south so the direction of north is $-\mathbf{j}$. If $\mathbf{U} = -\mathbf{j}$, then
$D_U f(x,y) = (-12\mathbf{i} - 150\mathbf{j}) \cdot (-\mathbf{j}) = 150 > 0$. Therefore the climber is ascending.
(c) Let $\mathbf{U} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. The climber is traveling a level path when his direction is that for which $D_U f(x,y) = 0$, that is,
$(-12\mathbf{i} - 150\mathbf{j}) \cdot (\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = 0$; $-12\cos\theta - 150\sin\theta = 0$; $\tan\theta = -\frac{2}{25}$
Then either $\cos\theta = \frac{25}{\sqrt{629}}$ and $\sin\theta = -\frac{2}{\sqrt{629}}$ or $\cos\theta = -\frac{25}{\sqrt{629}}$ and $\sin\theta = \frac{2}{\sqrt{629}}$.
A level path lies in either of the directions $\frac{25}{\sqrt{629}}\mathbf{i} - \frac{2}{\sqrt{629}}\mathbf{j}$ or $-\frac{25}{\sqrt{629}}\mathbf{i} + \frac{2}{\sqrt{629}}\mathbf{j}$.

74. $f(x,y) = x^2 + 6xy + 3y^2$ (a) $f(15,8) = 15^2 + 6(15)8 + 3(8)^2 = 1137$ (b) $f_x(x,y) = 2x + 6y$, $f_x(15,8) = 2(15) + 6(8) = 78$ (c) $f_y(x,y) = 6x + 6y$, $f_y(15,8) = 6(15) + 6(8)138$

In Exercises 75–78, use Lagrange multipliers to find the critical point(s) of the function subject to the constraint and determine their nature.

75. $f(x,y) = 5 + x^2 - y^2$, and $x^2 - 2y^2 = 5$. $f(x,y,\lambda) = 5 + x^2 - y^2 + \lambda(x^2 - 2y^2 - 5)$.
$F_x(x,y,\lambda) = 2x + 2\lambda x = 0$; $x = 0$ or $\lambda = -1$. $F_y(x,y,\lambda) = -2y - 4\lambda y = 0$; $y = 0$ or $\lambda = -\frac{1}{2}$.
$F_\lambda(x,y,\lambda) = x^2 - 2y^2 - 5 = 0$. If $x = 0$, $-2y^2 - 5 = 0$ is impossible. If $y = 0$, $x^2 - 5 = 0$; $x = \pm\sqrt{5}$.
The critical points are $(\sqrt{5},0)$ and $(-\sqrt{5},0)$. Consider y as a function of x.
From the constraint, $2x - 4y\frac{dy}{dx} = 0$; $\frac{dy}{dx} = \frac{x}{2y}$. $\frac{df}{dx} = 2x - 2y\frac{dy}{dx} = 2x - 2y \cdot \frac{x}{2y} = x$. $\frac{d^2f}{dx^2} = 1 > 0$.
Therefore f has a relative minimum at each critical point and $f(\pm\sqrt{5},0) = 10$.

76. $f(x,y,z) = x^2 + y^2 + z^2$ with constraint $x^2 - y^2 = 1$

▸ Let F be the function defined by
$F(x,y,z) = x^2 + y^2 + z^2 + \lambda(x^2 - y^2 - 1)$
Then
$F_x(x,y,z,\lambda) = 2x + 2\lambda x = 2x(1+\lambda) = 0$ (1)
$F_y(x,y,z,\lambda) = 2y - 2\lambda y = 2y(1-\lambda) = 0$ (2)
$F_z(x,y,z,\lambda) = 2z = 0$ (3)
$F_\lambda(x,y,z,\lambda) = g(x,y) = x^2 - y^2 - 1 = 0$ (4)
If $\lambda \neq \pm 1$, then from (1), $x = 0$; from (2), $y = 0$; from (4), $-1 = 0$, impossible.
If $\lambda = 1$, then from (1), $x = 0$; from (4) $y^2 = -1$, impossible.
If $\lambda = -1$, then from (2), $y = 0$; from (4) $x^2 = 1$, $x = \pm 1$.
Thus the critical points are $(1,0,0)$ and $(-1,0,0)$.
Because $g(x,y) = 0$ defines x as a function of y, then
$\frac{dx}{dy} = -\frac{\partial g/\partial y}{\partial g/\partial x} = \frac{y}{x}$
Then

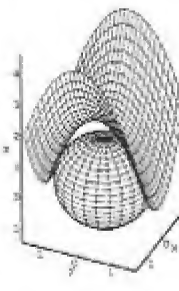
$f_y = 2x \cdot \frac{y}{x} + 2y = 4y$
$f_z = 2z$
$f_{yy} = 4$
$f_{zz} = 2$
$f_{yz} = 0$
$D = f_{yy}f_{zz} - f_{yz} = 4 \cdot 2 - 0^2 = 8$
and so each critical point is a relative minimum; the function value is 1. In fact, each is an absolute minimum.

77. $f(x, y, z) = y + xz - 2x^2 - y^2 - z^2$, and $z = 35 - x - y$.
$f(x, y, z, \lambda) = y + xz - 2x^2 - y^2 - z^2 + \lambda(35 - x - y - z)$.
$F_x(x, y, z, \lambda) = z - 4x - \lambda = 0$; $z = 4x + \lambda$. $F_y(x, y, z, \lambda) = 1 - 2y - \lambda = 0$; $y = \frac{1}{2}(1 - \lambda)$.
$F_z(x, y, z, \lambda) = x - 2z - \lambda = 0$; $x - 8x - 2\lambda - \lambda = 0$; $x = -\frac{3}{7}\lambda$, $z = -\frac{5}{7}\lambda$.
$F_\lambda(x, y, z, \lambda) = 35 - x - y - z = 0$; $-\frac{3}{7}\lambda + \frac{1}{2}(1 - \lambda) - \frac{5}{7}\lambda = 35$; $-6\lambda + 7 - 7\lambda - 0\lambda = 490$; $-23\lambda = 463$.
$\lambda = -21$. $x = -\frac{3}{7}\lambda = -\frac{3}{7}(-21) = 9$. $y = \frac{1}{2}(1 - \lambda) = \frac{1}{2}(1 + 21) = 11$, $z = -\frac{5}{7}\lambda = -\frac{5}{7}(-21) = 15$.
The critical point is P(9, 11, 15) and $f(9, 11, 15) = -362$.
Consider z as a function of x and y. From the constraint, $\frac{\partial z}{\partial x} = -1$ and $\frac{\partial z}{\partial y} = -1$.
$f_x = z + x\frac{\partial z}{\partial x} - 4x - 2z\frac{\partial z}{\partial x} = z - x - 4x + 2z = 3z - 5x$; $f_y = 1 + x\frac{\partial z}{\partial y} - 2y - 2z\frac{\partial z}{\partial y} = 1 - x - 2y + 2z$.
$f_{xx} = 3\frac{\partial z}{\partial x} - 5 = -3 - 5 = -8 < 0$; $f_{yy} = -2 + 2\frac{\partial z}{\partial y} = -2 - 2 = -4$; $f_{xy} = 3\frac{\partial z}{\partial y} = -3$.
$f_{xx}f_{yy} - f_{xy}{}^2 = (-8)(-4) - (-3)^2 = 23$. Hence f has a relative maximum at P.

78. See Exercise 12.9.9

79. Let w units be the distance from the point (4, 1, 2) to a point (x, y, z) in the plane $x - y + 2z = 0$. w will be a minimum when w^2 is a minimum. Let
$f(x, y, z) = w^2 = (x - 4)^2 + (y - 1)^2 + (z - 2)^2$
$f(x, y, z, \lambda) = (x - 4)^2 + (y - 1)^2 + (z - 2)^2 + \lambda(x - y + 2z)$
$F_x = 2x - 8 + \lambda = 0$; $x = \frac{1}{2}(8 - \lambda)$. $F_y = 2y - 2 - \lambda = 0$; $y = \frac{1}{2}(2 + \lambda)$. $F_z = 2z - 4 + 2\lambda = 0$; $z = 2 - \lambda$.
$F_\lambda = x - y + 2z = 0$; $\frac{1}{2}(8 - \lambda) - \frac{1}{2}(2 + \lambda) + 2(2 - \lambda) = 0$. $\lambda = \frac{7}{3}$, $x = \frac{17}{6}$, $y = \frac{13}{6}$, $z = -\frac{1}{3}$. The critical point is
$P(\frac{17}{6}, \frac{13}{6}, -\frac{1}{3})$ and at P, $w = \sqrt{(\frac{17}{6} - 4)^2 + (\frac{13}{6} - 1)^2 + (-\frac{1}{3} - 2)^2} = \sqrt{\frac{49}{36} + \frac{49}{36} + \frac{49}{9}} = \frac{7}{6}\sqrt{6}$.
Because $f(x, y, z) \geq 10$ on or outside the sphere $(x - 4)^2 + (y - 1)^2 + (z - 2)^2 = 10$, f has a minimum at a critical point inside the closed set bounded by the intersection of this sphere and the given plane, that is at P.

80. Use Lagrange multipliers to find the point on the surface $z = x^2 - y^2 + 2$ that is closest to the origin.
▸ Because $\sqrt{x^2 + y^2 + z^2}$ is the distance from the origin to any point (x, y, z), we wish to find the absolute minimum value of the function f defined by
$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
subject to the constraint



$z = x^2 - y^2 + 2$ (1)
Because f has a minimum value whenever f^2 has a minimum value, we let F be the function defined by
$F(x, y, z) = x^2 + y^2 + z^2 + \lambda(x^2 - y^2 - z + 2)$
The critical points of f are among those of the auxiliary function F. Then
$F_x(x, y, z, \lambda) = 2x + 2\lambda x = 2x(1 + \lambda) = 0$ (2)
$F_y(x, y, z, \lambda) = 2y - 2\lambda y = 2y(1 - \lambda) = 0$ (3)
$F_z(x, y, z, \lambda) = 2z - \lambda = 0$
so that
$z = \frac{1}{2}\lambda$ (4)
If $\lambda \neq \pm 1$, then from (2), $x = 0$; from (3), $y = 0$; and from (1), $z = 2$. Then $f(0, 0, 2) = 2$.
If $\lambda = 1$, then from (2), $x = 0$; from (4), $z = \frac{1}{2}$; and from (1), $y = \pm\frac{1}{2}\sqrt{6}$. Then $f(0, \pm\frac{1}{2}\sqrt{6}, \frac{1}{2}) = \frac{1}{2}\sqrt{7} \approx 1.32$.
If $\lambda = -1$, then from (3), $y = 0$; from (4), $z = -\frac{1}{2}$; and from (1), $x^2 = -\frac{5}{2}$, which is impossible.
Thus, the points on the hyperbolic paraboloid closest to the origin are $(0, \frac{1}{2}\sqrt{6}, \frac{1}{2})$ and $(0, -\frac{1}{2}\sqrt{6}, \frac{1}{2})$. The figure shows the surface and a sphere centered at the origin of radius $\frac{1}{2}\sqrt{7}$ tangent to the surface at the extrema.

81. Let the three numbers be x, y, and $100 - x - y$. Let $f(x,y) = x^2 + y^2 + (100 - x - y)^2$.
$f_x(x,y) = 4x + 2y - 200 = 0$; $f_y(x,y) = 4y + 2x - 200 = 0$. Then $x = \frac{100}{3}$, $y = \frac{100}{3}$, $100 - x - y = \frac{100}{3}$.
$f(\frac{100}{3}, \frac{100}{3}) = \frac{10{,}000}{3}$. If $|x| \geq 100$ or $|y| \geq 100$, then $f(x,y) \geq 10{,}000$, so f has an absolute minimum at a critical point inside the square $|x| \leq 100$, $|y| \leq 100$, i.e. at $(\frac{100}{3}, \frac{100}{3})$. Thus the three numbers are $\frac{100}{3}$, $\frac{100}{3}$, and $\frac{100}{3}$.

82. $P(x,y) = 33x + 66y - x^2 + xy - 3y^2$. $P_x = 33 - 2x + y = 0$. $P_y = 66 + x - 6y = 0$. $6P_x + P_y = 264 - 11x = 0$, $x = 24$. $P_x + 2P_y = 165 - 11y = 0$, $y = 15$, $P_{xx} = -2$, $P_{yy} = -6$, $P_{xy} = 1$, $D = (-2)(-6) - 1^2 = 11$, maximum

83. Let 2r, 2s and 2t be the measures of the edges of the rectangular parallelpiped P parallel to the x, y, and z axes. If V cubic units is the volume of P, then V = 8rst. Let $f = V^2/64 = r^2s^2t^2$, $r \geq 0$, $s \geq 0$, $t \geq 0$. Because the vertex (r,s,t) is on the ellipsoid $x^2 + 9y^2 + z^2 = 9$, then $r^2 + 9s^2 + t^2 = 9$; $r^2 = 9 - 9s^2 - t^2$. Therefore
$f = (9 - 9s^2 - t^2)s^2t^2 = 9s^2t^2 - 9s^4t^2 - s^2t^4$
If $s \geq 1$, $s = 0$, $t \geq 3$, or $t = 0$, then $f \leq 0$. Hence f has an absolute maximum value at a critical point inside the rectangle $0 \leq s \leq 1$, $0 \leq t \leq 3$.
$f_s = 18st^2 - 36s^3t^2 - 2st^4 = 2st^2(9 - 18s^2 - 2t^2) = 0$; $18s^2 + 2t^2 = 9$
$f_t = 18s^2t - 18s^4t - 4s^2t^3 = 2s^2t(9 - 9s^2 - 4t^2) = 0$; $9s^2 + 4t^2 = 9$
Solving simultaneously for s^2 and t^2 we get $s^2 = \frac{1}{3}$, $t^2 = 3$, so the only critical point in R is $(\frac{1}{3}\sqrt{3}, \sqrt{3})$ which must give the maximum. Then $r^2 = 3$; $r = \sqrt{3}$ so the dimensions of the rectangular parallepiped of greatest volume are $2\sqrt{3}$ by $\frac{2}{3}\sqrt{3}$ by $2\sqrt{3}$.

84. The temperature is T degrees at any point (x,y) of the curve $4x^2 + 12y^2 = 1$ and $T(x,y) = 4x^2 + 24y^2 - 2x$. Find the points on the curve where the temperature is the greatest and where it is least. Also find the temperature at these points.

▸ Let F be the function defined by
$F(x, y, \lambda) = 4x^2 + 24y^2 - 2x + \lambda(4x^2 + 12y^2 - 1)$
Then

$$F_x(x,y,\lambda) = 8x - 2 + 8x\lambda = 8x(1 + \lambda) - 2 = 0 \quad (1)$$
$$F_y(x,y,\lambda) = 48y + 24y\lambda = 24y(2 + \lambda) = 0 \quad (2)$$
$$F_\lambda(x,y,\lambda) = 4x^2 + 12y^2 - 1 = 0 \quad (3)$$

From (2) we have either $y = 0$ or $\lambda = -2$. If $y = 0$, from (3) we get $x = \pm\frac{1}{2}$. If $\lambda = -2$, from (1) we get $x = -\frac{1}{4}$, and from (3) we get $y = \pm\frac{1}{4}$. Thus, the critical points are $(\pm\frac{1}{2}, 0)$ and $(-\frac{1}{4}, \pm\frac{1}{4})$. Then

$T(\frac{1}{2}, 0) = 0$ $T(-\frac{1}{2}, 0) = 2$ $T(-\frac{1}{4}, \frac{1}{4}) = \frac{9}{4}$ $T(-\frac{1}{4}, -\frac{1}{4}) = \frac{9}{4}$
Because the curve is an ellipse which a closed and bounded set, the extrema are among the critical points. Therefore the maximum temperature is $\frac{9}{4}°$, which occurs at the points $(-\frac{1}{4}, \pm\frac{1}{4})$, and the minimum temperature is 0°, which occurs at the points $(\pm\frac{1}{2}, 0)$.

85. The temperature is T degrees at any point (x,y) and $T(x,y) = \dfrac{44}{x^2 + y^2 + 9}$, where distance is in centimeters from the origin. (a) We wish to find $D_UT(3,2)$ where $U = \cos\frac{1}{6}\pi\mathbf{i} + \sin\frac{1}{6}\pi\mathbf{j}$.
$T_x(x,y) = \dfrac{-88x}{(x^2 + y^2 + 9)^2}$; $T_x(3,2) = \dfrac{-88 \cdot 3}{22^2} = -\frac{6}{11}$. $T_y(x,y) = \dfrac{-88y}{(x^2 + y^2 + 9)^2}$; $T_y(3,2) = \dfrac{-88 \cdot 2}{22^2} = -\dfrac{4}{11}$
$\nabla T(3,2) = -\frac{6}{11}\mathbf{i} - \frac{4}{11}\mathbf{j}$. Hence $D_UT(3,2) = U \cdot \nabla T(3,2) = (\frac{1}{2}\sqrt{3}\mathbf{i} + \frac{1}{2}\mathbf{j}) \cdot (-\frac{6}{11}\mathbf{i} - \frac{4}{11}\mathbf{j}) = -\frac{3}{11}\sqrt{3} - \frac{2}{11}$.
The rate of change of the temperature at the point (3,2) in the direction of U is $-\frac{3}{11}\sqrt{3} - \frac{2}{11}$ degrees per cm.
(b) $D_UT(3,2)$ is maximum when U has the direction of $\nabla T(3,2)$. $|\nabla T(3,2)| = \frac{1}{11}\sqrt{36 + 16} = \frac{1}{11}\sqrt{52} = \frac{2}{11}\sqrt{13}$
The greatest rate of change of T at (3,2) is $\frac{2}{11}\sqrt{13}$ degrees per cm in the direction of $-\frac{3}{13}\sqrt{13}\mathbf{i} - \frac{2}{13}\sqrt{13}\mathbf{j}$.

86. The base is x ft by y ft, the height is z ft. Then $xy + 2xz + 2yz = 216$ (1). WE wish maximize the volume $V = xyz$. Let $F = xyz + \lambda(xy + 2xz + 2yz - 216)$. $F_x = yz + \lambda(y + 2z) = 0$, $1 + \lambda(1/z + 2/y) = 0$ (2).
$F_y = xz + \lambda(x + 2z) = 0$, $1 + \lambda(1/z + 2/x) = 0$ (3). $F_z = xy + \lambda(2x + 2y) = 0$, $1 + \lambda(2/y + 2/x) = 0$ (4).
From (2) and (3), $y = x$. From (2) and (4), $z = \frac{1}{2}x$. From (1), $x^2 + x^2 + x^2 = 216$, $x = 6\sqrt{2} = y$, $z = 3\sqrt{2}$

87. Let x ft, y ft, z ft be the length, width, and height of the crate. Then
$4x + 4y + 4z = 216$; $x + y + z = 54$; $z = 54 - x - y$
If V ft^3 is the volume of the box, then $V = xyz = xy(54 - x - y) = 54xy - x^2y - xy^2$.
If $x \geq 54$, $x = 0$, $y \geq 54$, or $y = 0$, then $V \leq 0$. Hence V has an absolute maximum value at a critical point inside the square $0 \leq x \leq 54$, $0 \leq y \leq 54$.
$V_x = 54y - 2xy - y^2 = y(54 - 2x - y) = 0$; $2x + y = 54$
$V_y = 54x - x^2 - 2xy = x(54 - x - 2y) = 0$; $x + 2y = 54$
The only critical point is (18,18), which gives the desired maximum. The dimensions of the crate of greatest volume are 18 ft by 18 ft by 18 ft.

88. A piece of wire L feet long is cut into three pieces. One piece is bent into the shape of a circle; a second piece is bent into the shape of a square; and the third piece is bent into the shape of an equilateral triangle. How should the wire be cut so that the combined area of the three figures is (a) as small as possible, and (b) as large as possible.

▹ Let r ft be the radius of the circle, x ft the side of the square, y ft the side of the triangle, and A ft^2 the combined area of the three figures. Then

$$A = \pi r^2 + x^2 + \tfrac{1}{4}\sqrt{3}y^2 \tag{1}$$

Because the circumference of the circle is $2\pi r$ ft, the perimeter of the square is $4x$ ft, and the perimeter of the triangle is $3y$ ft, we also have

$$L = 2\pi r + 4x + 4y,\ r \geq 0,\ x \geq 0,\ y \geq 0 \tag{2}$$

We want to find the extrema of A subject to the constraint (2). We use the method of Lagrange multipliers to find the critical points. Let F be the function defined by

$$F(r, x, y, \lambda) = \pi r^2 + x^2 + \tfrac{1}{4}\sqrt{3} + \lambda(2\pi r + 4x + 3y - L)$$

Then

$$F_r(r, x, y, \lambda) = 2\pi r + 2\pi\lambda = 0,\ r = -\lambda$$
$$F_x(r, x, y, \lambda) = 2x + 4\lambda = 0,\ x = -2\lambda$$
$$F_y(r, x, y, \lambda) = \tfrac{1}{2}\sqrt{3}y + 3\lambda = 0,\ y = -2\sqrt{3}\lambda$$

Substituting the values for r, x, y in (2), we obtain

$$L = -2\pi\lambda - 8\lambda - 6\sqrt{3}\lambda$$

Solving for λ gives

$$\lambda = \frac{-L}{2\pi + 8 + 6\sqrt{3}} \approx -0.04L$$

Thus,

$$r = -\lambda \approx 0.04L$$
$$x = -2\lambda \approx 0.08L$$
$$y = -2\sqrt{3}\lambda \approx 0.14L$$

Substituting these values into (1), we have

$$A = \pi(-\lambda)^2 + (-2\lambda)^2 + \tfrac{1}{4}\sqrt{3}(-2\sqrt{3}\lambda)^2 = (-\lambda)^2(\pi + 4 + 3\sqrt{3}) = \left(\frac{L}{2\pi + 8 + 6\sqrt{3}}\right)^2(\pi + 4 + 3\sqrt{3})$$
$$= \frac{L^2}{4(\pi + 4 + 3\sqrt{3})} \approx 0.02L^2 \tag{3}$$

Because the level surfaces of A are ellipsoids, we need only compare the value of A at the critical point with the value of A at the corners of the triangle which is the graph of (2).
If $x = 0$ and $y = 0$, then from (2) we have $r = L/2\pi$ and from (1) we obtain

$$A = \pi\left(\frac{L}{2\pi}\right)^2 \approx 0.08L^2 \tag{4}$$

If $r = 0$ and $y = 0$, then $x = \frac{1}{4}L$ and

$$A = (\tfrac{1}{4}L)^2 \approx 0.06L^2 \tag{5}$$

If $r = 0$ and $x = 0$, then $y = \frac{1}{3}L$ and

$$A = \tfrac{1}{4}\sqrt{3}(\tfrac{1}{3}L)^2 \approx 0.05L^2 \tag{6}$$

Comparing the values of A given in Eqs. (3), (4), (5) and (6), we conclude that the minimum combined area is $0.02L^2$ ft^2 and that it occurs when the radius is 0.04L ft, the side of the square is 0.08L ft long, a side of the triangle is 0.14L ft long. The maximum combined area is $0.08L^2$ ft^2, and it occurs when all the wire is used to make the circle.

89. See Exercises 12.8.49 and 12.9.35 for two different solutions.

90. Find the greatest and least distances from the origin to the curve of intersection of the ellipsoid $x^2 + 3y^2 + 2z^2 = 30$ and the elliptical cone $x^2 = 2yz$.

▸ We seek the extrema of $d^2 = x^2 + y^2 + z^2 = y^2 + 2yz + z^2 = (y+z)^2$ subject to $3y^2 + 2yz + 2z^2 = 30$, $yz \geq 0$. Let $F = y + z + \lambda(3y^2 + 2yz + 2z^2)$. $F_y = 1 + \lambda(6y + 2z) = 0$. $F_z = 1 + \lambda(2y + 4z) = 0$. Thus $6y + 2z = 2y + 4z$, $z = 2y$, and so $3y^2 + 4y^2 + 8y^2 = 30$, $y^2 = 2$, $y = \pm\sqrt{2}$, $z = \pm 2\sqrt{2}$, $x = \pm 2\sqrt{2}$, $d = \sqrt{8+2+8} = 3\sqrt{2}$, the absolute maximum. If $y = 0$, then $x = 0$ and $d = z = \sqrt{15}$. If $z = 0$ then $x = 0$ and $d = y = \sqrt{10}$, the min.

91.

x (years)	54	46	40	36	30
y (days)	15	12	9	10	8

$\sum_{i=1}^{5} x_i y_i = 2322$, $\sum_{i=1}^{5} x_i = 206$, $\sum_{i=1}^{5} y_i = 54$, $\sum_{i=1}^{5} x_i^2 = 8828$.

$m = \dfrac{(5 \times 2322 - 206 \times 54)}{(5 \cdot 8828 - 206^2)} = 0.285$, $b = \dfrac{(54 - 0.285 \times 206)}{5} = -0.951$. The regression line has the equation $y = 0.285x - 0.951$. When $x = 42$, $y = 0.285 \times 42 - .951 = 11.0$. The stay is 11 days.

92. In the following table, a patients systolic blood pressure and corresponding heart rate are given, where x millimeters of mercury is the systolic blood pressure and y beats per minute is the heart rate.

	Patient 1	Patient 2	Patient 3	Patient 4	Patient 5	Patient 6
x mm hg	110	117	133	146	115	127
y beats/min	70	74	80	85	60	77

(a) Find an equation of the regression line for the data in the table. (b) Use the regression line to estimate a patients heart rate if the systolic blood pressure is 85 mm of mercury.

▸ (a) The sums required to find the regression line of y on x are

$$\sum_{i=1}^{6} x_i = 748 \quad \sum_{i=1}^{6} y_i = 446 \quad \sum_{i=1}^{6} x_i^2 = 94{,}148 \quad \sum_{i=1}^{6} x_i y_i = 56{,}087$$

Then, with $n = 6$,

$$m = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{(6 \times 56087 - 748 \times 446)}{(6 \times 94148\text{-}748^2)} = 0.5412$$

$$b = \frac{1}{n}\left[\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i\right] = \frac{1}{6}(446 - 0.5412 \times 748) = 6.863$$

Therefore, the regression line is

$$y = 0.5412x + 6.863$$

(b) If $x = 85$, then

$$y = 0.5412 \times 85 + 6.863 = 52.87$$

The estimate of the heart rate is 53 beats per minute when the blood pressure is 85 mm of mercury.

93.

x (mm/yr/100)	1	2	4	5	6	6.5
y (kg/ha/100)	10	19	32	44	58	64

$\sum_{i=1}^{6} x_i y_i = 1160$, $\sum_{i=1}^{6} x_i = 24.5$, $\sum_{i=1}^{6} y_i = 227$,

$\sum_{i=1}^{6} x_i^2 = 124.25$. $m = \dfrac{(6 \times 1160 - 24.5 \times 227)}{(6 \cdot 124.25 - 24.5^2)} = 9.628$, $b = \dfrac{(227 - 9.628 \times 24.5)}{6} = -1.482$.
The regression line has the equation $y = 9.628x - 1.482$. When $x = 3$, $100y = 100 \times (9.628 \times 3 - 1.482) = 2740$.

94.

x (cents)	130	140	150	160
y (M cases)	100	85	75	63

$n = 4$. $\sum_{i=1}^{n} x_i y_i = 46230$, $\sum_{i=1}^{n} x_i = 580$, $\sum_{i=1}^{n} y_i = 323$, $\sum_{i=1}^{n} x_i^2 = 84600$

$m = \dfrac{(4 \times 46230 - 580 \times 323)}{(4 \times 84600 - 580^2)} = -1.21$, $m = \dfrac{(323 + 1.21 \times 580)}{4} = 256.2$. Regression line: $y = -1.21x + 256.2$
$y(120) = 256.2 - 1.21 \times 120 = 111$ thousand cases. $y(170) = 256.2 - 1.21 \times 170 = 50.5$ thousand cases.

95. $f(x,y) = \begin{cases} \dfrac{12x^2y - 3y^2}{x^2 + y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

(a) $f_2(x,0) = \lim\limits_{\Delta y \to 0} \dfrac{f(x,\Delta y) - f(x,0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{\dfrac{12x^2\Delta y - 3(\Delta y)^2}{x^2 + \Delta y} - 0}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{12x^2 - 3\Delta y}{x^2 + \Delta y} = \dfrac{12x^2}{x^2} = 12$

(b) $f_2(0,0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{\dfrac{0 - 3(\Delta y)^2}{0 + \Delta y} - 0}{\Delta y} = \lim\limits_{\Delta y \to 0} (-3) = -3$

96. Verify that $u(x,y) = \sinh x \sin y$ satisfies Laplace's equation in $\mathbb{R}^2$: $u_{xx} + u_{yy} = 0$

▷ $u_x = \cosh y \sin y$

$u_{xx} = \sinh x \sin y$ (1)

Furthermore

$u_y = \sinh x \cos y$

$u_{yy} = -\sinh x \sin y$ (2)

Adding Eqs. (1) and (2), we obtain

$u_{xx} + u_{yy} = \sinh x \sin y + (-\sinh x \sin y) = 0$

97. $z = xy + f(x^2 + y^2)$ and $u = x^2 + y^2$. $\dfrac{\partial z}{\partial x} = y + \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} = y + \dfrac{\partial f}{\partial u}(2x)$; $\dfrac{\partial z}{\partial y} = x + \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} = x + \dfrac{\partial f}{\partial u}(2y)$.

$y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = y\left[y + \dfrac{\partial f}{\partial u}(2x)\right] - x\left[x + \dfrac{\partial f}{\partial u}(2y)\right] = y^2 + 2xy\dfrac{\partial f}{\partial u} - x^2 - 2xy\dfrac{\partial f}{\partial u} = y^2 - x^2$

98. $u(r,\theta) = r^n \sin n\theta$. $u_r = nr^{n-1}\sin n\theta$. $u_\theta = nr^n \cos n\theta$. $u_{rr} = n(n-1)r^{n-2}\sin n\theta$. $u_{\theta\theta} = -n^2 r^n \sin n\theta$

$r^2 u_{rr} + r u_r + u_{\theta\theta} = n(n-1)r^n \sin n\theta + nr^n \sin n\theta - n^2 r^n \sin n\theta = 0$, Laplace's equation in polar coordinates.

99. $u(x,y,z) = e^{3x+4y}\sin 5z$. $\dfrac{\partial u}{\partial x} = 3e^{3x+4y}\sin 5z$; $\dfrac{\partial u}{\partial y} = 4e^{3x+4y}\sin 5z$; $\dfrac{\partial u}{\partial z} = 5e^{3x+4y}\cos 5z$

$\dfrac{\partial^2 u}{\partial x^2} = 9e^{3x+4y}\sin 5z$; $\dfrac{\partial^2 u}{\partial y^2} = 16e^{3x+4y}\sin 5z$; $\dfrac{\partial^2 u}{\partial z^2} = -25e^{3x+4y}\sin 5z$

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = e^{3x+4y}\sin 5z(9 + 16 - 25) = 0$

100. Verify that $u(x,t) = A\cos(kat)\sin(kx)$, where A and k are arbitrary constants, satisfies the partial differential equation for a vibrating string: $u_{tt} = a^2 u_{xx}$.

▷ Partial-differentiating u with respect to t, we obtain

$u_t = -kaA\sin(kat)\sin(kx)$

$u_{tt} = -k^2a^2A\cos(kat)\sin(kx)$ (1)

Partial-differentiating u with respect to x, we obtain

$u_x = kA\cos(kat)\cos(kx)$

$u_{xx} = -k^2A\cos(kat)\sin(kx)$

Thus,

$a^2u_{xx} = -k^2a^2A\cos(kat)\sin(kx)$ (2)

Comparing Eqs. (1) and (2), we obtain the desired result

$u_{tt} = a^2u_{xx}$

101. $u(x,t) = \sin\dfrac{n\pi x}{L}\exp\left(-\dfrac{n^2\pi^2k^2}{L^2}t\right)$. $\dfrac{\partial u}{\partial x} = \dfrac{n\pi}{L}\cos\dfrac{n\pi x}{L}\exp\left(-\dfrac{n^2\pi^2k^2}{L^2}t\right)$;

$\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{n^2\pi^2}{L^2}\sin\dfrac{n\pi x}{L}\exp\left(-\dfrac{n^2\pi^2k^2}{L^2}t\right)$. Then $\dfrac{\partial u}{\partial t} = -\dfrac{n^2\pi^2k^2}{L^2}\sin\dfrac{n\pi x}{L}\exp\left(-\dfrac{n^2\pi^2k^2}{L^2}t\right) = k^2\dfrac{\partial^2 u}{\partial x^2}$.

102. Interchange x and y in Exercises 12.4.30.

103. $f(x,y,z)=\begin{cases}\dfrac{x^2y^2z^2}{(x^2+y^2+z^2)^2} & \text{if } (x,y,z)\neq(0,0,0)\\ 0 & \text{if } (x,y,z)=(0,0,0)\end{cases}$. $D_1f(0,0,0)=\lim\limits_{\Delta x\to 0}\dfrac{0-0}{\Delta x}=0=D_2f(0,0,0)=D_3f(0,0,0)$.

Therefore $\Delta f(0,0,0)-D_1f(0,0,0)-D_2f(0,0,0)-D_3f(0,0,0)$

$$=\frac{(\Delta x)^2(\Delta y)^2(\Delta z)^2}{[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]^2}=\frac{(\Delta x)(\Delta y)^2(\Delta z)^2}{[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]^2}\Delta x+0\Delta y+0\Delta z$$

In Definition 12.4.7 let $\epsilon_1=\dfrac{(\Delta x)(\Delta y)^2(\Delta z)^2}{[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]^2}$, $\epsilon_2=0$, $\epsilon_3=0$. Then

$$0\le|\epsilon_1|<\frac{\Delta x[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2][(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]}{[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]^2}=|\Delta x|$$

Hence as $(\Delta x,\Delta y,\Delta z)\to(0,0,0)$, $\epsilon_1\to 0$. Also $\epsilon_2\to 0$ and $\epsilon_3\to 0$. Thus f is differentiable

In Exercises 104 and 105, f is the function defined by $f(x,y)=\begin{cases}\dfrac{e^{-1/x^2}y}{e^{-2/x^2}+y^2} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$.

104. Prove that f is discontinuous at the origin.

▷ We wish to determine whether $\lim\limits_{(x,y)\to(0,0)}f(x,y)=f(0,0)=0$. Because $\lim\limits_{x\to 0}e^{-1/x^2}=0$, then $(0,0)$ is an accumulation point of the curve $y=me^{-1/x^2}$. Furthermore,

$$\lim_{\substack{(x,y)\to(0,0)\\ y=me^{-x^2}}}f(x,y)=\lim_{x\to 0}\frac{e^{-1/x^2}(me^{-1/x^2})}{e^{-1/x^2}+m^2e^{-1/x^2}}=\frac{m}{1+m^2}$$

Thus the restricted limit depends on m. $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ does not exist and f is discontinuous at the origin.

105. $D_1f(0,0)=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{0-0}{\Delta x}=0$; $D_2f(0,0)=\lim\limits_{\Delta y\to 0}\dfrac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\dfrac{0-0}{\Delta y}=0$

106. $u=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$

▷ $u_r=u_xx_r+u_yy_r=u_x\cos\theta+u_y\sin\theta$, $u_\theta=u_xx_\theta+u_yy_\theta=u_x(-r\sin\theta)+u_y(r\cos\theta)$

$(u_r)^2+r^{-2}(u_\theta)^2=(u_x{}^2\cos^2\theta+2u_xu_y\sin\theta\cos\theta+u_y{}^2\sin^2\theta)+(u_x{}^2\sin^2\theta-2u_xu_y\sin\theta\cos\theta+u_y{}^2\cos^2\theta)$

$=(\cos^2\theta+\sin^2\theta)u_x{}^2+(\sin^2\theta+\cos^2\theta)u_y{}^2=u_x{}^2+u_y{}^2$

107. We are given $\dfrac{d^2f}{dx^2}=-\lambda^2f(x)$, $\dfrac{dg}{dt}=-k^2\lambda^2g(t)$ and $u=f(x)g(t)$. Then $\dfrac{\partial u}{\partial x}=g(t)\dfrac{df}{dx}$, and

$\dfrac{\partial^2u}{\partial x^2}=g(t)\dfrac{d^2f}{dx^2}=g(t)[-\lambda^2f(x)]$. Thus $\dfrac{\partial u}{\partial t}=f(x)\dfrac{dg}{dt}=f(x)[-k^2\lambda^2g(t)]=k^2g(t)[-\lambda^2f(t)]=k^2\dfrac{\partial^2u}{\partial x^2}$.

108. The partial differential equation for a vibrating string is $u_{tt}=a^2u_{xx}$. Show that if f is a function of x satisfying the equation $f_{xx}+\lambda^2f=0$ and g is a function of t satisfying the equation $g_{tt}+a^2\lambda^2g=0$, where a and λ are constants, then $u=f(x)g(t)$ satisfies the differential equation.

▷ We have

$u_t=f(x)g_t(t)$

$u_{tt}=f(x)g_{tt}(t)$

and

$u_x=f_x(x)g(t)$

$u_{xx}=f_{xx}(x)g(t)$

Furthermore,

$f_{xx}=-\lambda^2f$ and $g_{tt}=-a^2\lambda^2g$

Thus,

$u_{tt}-a^2u_{xx}=f(x)g_{tt}(t)-a^2f_{xx}g(t)=f(x)[-a^2\lambda^2g(t)]-a^2[-\lambda^2f(x)]g(t)=0$

Therefore, $u_{tt}=a^2u_{xx}$ and the differential equation is satisfied.

109. $u = f(x+at) + g(x-at)$

$$\frac{\partial u}{\partial x} = f'(x+at)\frac{\partial}{\partial x}(x+at) + g'(x-at)\frac{\partial}{\partial x}(x-at) = f'(x+at) + g'(x-at)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+at)\frac{\partial}{\partial x}(x+at) + g''(x-at)\frac{\partial}{\partial x}(x-at) = f''(x+at) + g''(x-at)$$

$$\frac{\partial u}{\partial t} = f'(x+at)\frac{\partial}{\partial t}(x+at) + g'(x-at)\frac{\partial}{\partial t}(x-at) = af'(x+at) - ag'(x-at)$$

$$\frac{\partial^2 u}{\partial t^2} = af''(x+at)\frac{\partial}{\partial t}(x+at) - ag''(x-at)\frac{\partial}{\partial t}(x-at) = a^2f''(x+at) + a^2g''(x-at) = a^2\frac{\partial^2 u}{dx^2}$$

110. $\mathrm{V} = r^{-1}\phi(r,t)$ (a) $r^2\mathrm{V}_r = r^2[-r^{-2}\phi(r,t) + r^{-1}\phi_r(r,t)] = r\phi_r(r,t) - \phi(r,t)$. Let $k^2 = \mu\epsilon$.

$0 = r^{-2}(r^2\mathrm{V}_r)_r - k^2\mathrm{V}_{tt} = r^{-2}[\phi_r(r,t) + r\phi_{rr} - \phi_r(r,t)] - k^2r^{-1}\phi_{tt}(r,t) = r^{-1}\phi_{rr} - k^2\phi_{tt}(r,t) \rightarrow \phi_{rr} - k^2\phi_{tt} = 0$

(b) $\phi = f(t-kr)$. $\phi_t = f'(t-kr)$; $\phi_{tt} = f''(t-kr)$. $\phi_r = f'(t-kr)(-k)$; $\phi_{rr} = f''(t-kr)k^2 = k^2\phi_{tt}$

111. $h = \mathrm{H}\cos(m\pi a^{-1}x)\cos(n\pi b^{-1}y)$. $h_x = -\mathrm{H}m\pi a^{-1}\sin(m\pi a^{-1}x)\cos(n\pi b^{-1}y)$. $h_y = -\mathrm{H}n\pi b^{-1}\cos(m\pi a^{-1}x)\sin(n\pi b^{-1}y)$

$$h_{xx} + h_{yy} + \mathrm{K}^2h = -\mathrm{H}m^2\pi^2a^{-2}\cos(m\pi a^{-1}x)\cos(n\pi b^{-1}y) - \mathrm{H}n^2\pi^2b^{-2}\cos(m\pi a^{-1}x)\cos(n\pi b^{-1}y) + (m^2\pi^2a^{-2} + n^2\pi^2b^{-2})\cos(m\pi a^{-1}x)\cos(n\pi b^{-1}y) = 0$$

T H I R T E E N

MULTIPLE INTEGRATION

13.1 CYLINDRICAL AND SPHERICAL COORDINATES

Cylindrical We take the polar axis as the positive x axis with the polar plane as the xy plane. If cylindrical coordinates (r, θ, z) of a point P are given, we use the equations

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z \tag{I}$$

to uniquely determine the cartesian coordinates (x, y, z) of P. If the Cartesian coordinates of P are given, then cylindrical coordinates satisfy equations (II) where the rule for determining θ is given by (VI)

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \quad z = z \tag{II}$$

Spherical If spherical coordinates (ρ, θ, ϕ) of a point P are given, we use the equations

$$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$$

to uniquely determine the cartesian coordinates of P. Furthermore, a set of cylindrical coordinates of P is given by

$$r = \rho\sin\phi \quad \theta = \theta \quad z = \rho\cos\phi \tag{IV}$$

If the Cartesian coordinates of P are given and P is not the origin, then the spherical coordinates satisfy

$$\rho = \sqrt{x^2 + y^2 + z^2} \quad \tan\theta = \frac{y}{x} \quad \phi = \cos^{-1}\frac{z}{\rho} \tag{V}$$

In spherical and cylindrical coordinates we have

$$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \frac{1}{2}\pi \operatorname{sgn}(y) & \text{if } x = 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases} \tag{VI}$$

Arc Length In cylindrical coordinates, $L = \int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$. See Exercise 33.

In spherical coordinates, $L = \int_a^b \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2}\, dt$. See Exercise 34.

Exercises 13.1

The cartesian, cylindrical and spherical coordinates of a point P will be denoted by $P_{\text{cartesian}}$, $P_{\text{cylindrical}}$, and $P_{\text{spherical}}$.

1. (a) $P_{\text{cylindrical}} = (3, \frac{1}{2}\pi, 5)$. Then $x = 3\cos\frac{1}{2}\pi = 0$, $y = 3\sin\frac{1}{2}\pi = 3$, $z = 5$. $P_{\text{cartesian}} = (0, 3, 5)$.

   (b) $P_{\text{cylindrical}} = (7, \frac{2}{3}\pi, -4)$. $x = 7\cos\frac{2}{3}\pi = -\frac{7}{2}$, $y = 7\sin\frac{2}{3}\pi = \frac{7}{2}\sqrt{3}$, $z = -4$. $P_{\text{cartesian}} = (-\frac{7}{2}, \frac{7}{2}\sqrt{3}, -4)$.

   (c) $P_{\text{cylindrical}} = (1, 1, 1)$. $x = 1\cos 1 = \cos 1$, $y = 1\sin 1 = \sin 1$, $z = 1$. $P_{\text{cartesian}} = (\cos 1, \sin 1, 1)$.

2. (a) $P_{\text{cartesian}} = (4, 4, -2)$. $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$, $\theta = \tan^{-1}\frac{4}{4} = \frac{1}{4}\pi$, $z = -2$. $P_{\text{cylindrical}} = (4\sqrt{2}, \frac{1}{4}\pi, -2)$.

   (b) $P_{\text{cartesian}} = (-3\sqrt{3}, 3, 6)$. $r = \sqrt{27 + 9} = 6$, $\theta = \frac{3}{-3\sqrt{3}} + \pi = -\frac{1}{6}\pi + \pi = \frac{5}{6}\pi$, $z = 6$. $P_{\text{cylindrical}} = (6, \frac{5}{6}\pi, 6)$

   (c) $P_{\text{cartesian}} = (1, 1, 1)$. $r = \sqrt{1 + 1} = \sqrt{2}$, $\theta = \tan^{-1}\frac{1}{1} = \frac{1}{4}\pi$, $z = 1$. $P_{\text{cylindrical}} = (\sqrt{2}, \frac{1}{4}\pi, 1)$

3. (a) $P_{\text{spherical}} = (4, \frac{1}{6}\pi, \frac{1}{4}\pi)$. Then $x = 4\sin\frac{1}{4}\pi\cos\frac{1}{6}\pi = 4(\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{3}) = \sqrt{6}$,
   $y = 4\sin\frac{1}{4}\pi\sin\frac{1}{6}\pi = 4(\sqrt{2})(\frac{1}{2}) = \sqrt{2}$, $z = 4\cos\frac{1}{4}\pi = 4(\frac{1}{2}\sqrt{2}) = 2\sqrt{2}$. $P_{\text{cartesian}} = (\sqrt{6}, \sqrt{2}, 2\sqrt{2})$.

   (b) $P_{\text{spherical}} = (4, \frac{1}{2}\pi, \frac{1}{3}\pi)$. Then $x = 4\sin\frac{1}{3}\pi\cos\frac{1}{2}\pi = 4(\frac{1}{2}\sqrt{3})(0) = 0$,
   $y = 4\sin\frac{1}{3}\pi\sin\frac{1}{2}\pi = 4(\frac{1}{2}\sqrt{3})(1) = 2\sqrt{3}$, $z = 4\cos\frac{1}{3}\pi = 4(\frac{1}{2}) = 2$. $P_{\text{cartesian}} = (0, 2\sqrt{3}, 2)$.

   (c) $P_{\text{spherical}} = (\sqrt{6}, \frac{1}{3}\pi, \frac{3}{4}\pi)$. Then $x = \sqrt{6}\sin\frac{3}{4}\pi\cos\frac{1}{3}\pi = \sqrt{6}(\frac{1}{2}\sqrt{2})(\frac{1}{2}) = \frac{1}{2}\sqrt{3}$,
   $y = \sqrt{6}\sin\frac{3}{4}\pi\sin\frac{1}{3}\pi = \sqrt{6}(\frac{1}{2}\sqrt{2})(\frac{1}{2}\sqrt{3}) = \frac{3}{2}$, $z = \sqrt{6}\cos\frac{3}{4}\pi = \sqrt{6}(-\frac{1}{2}\sqrt{2}) = -\sqrt{3}$. $P_{\text{cartesian}} = (\frac{1}{2}\sqrt{3}, \frac{3}{2}, -\sqrt{3})$

4. Find a set of spherical coordinates of the point having the given Cartesian coordinates: (a) $(1,-1,-\sqrt{2})$; (b) $(-1,\sqrt{3},2)$; (c) $(2,2,2)$

▹ (a) We use formulas (V) and (VI) where $x>0$. Thus,

$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{1^2+(-1)^2+(-\sqrt{2})^2}=\sqrt{4}=2$

$\theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\left(\frac{-1}{1}\right)=-\frac{1}{4}\pi$

$\phi=\cos^{-1}\frac{z}{\rho}=\cos^{-1}\left(\frac{-\sqrt{2}}{2}\right)=\frac{3}{4}\pi$

Spherical coordinates are $(2,-\frac{1}{4}\pi,\frac{3}{4}\pi)$.

(b) We use formulas (V) and (VI) where $x<0$. Thus,

$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{(-1)^2+(\sqrt{3})^2+2^2}=\sqrt{8}=2\sqrt{2}$

$\theta=\tan^{-1}\frac{y}{x}+\pi=\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)+\pi=-\frac{1}{3}\pi+\pi=\frac{2}{3}\pi$

$\phi=\cos^{-1}\frac{z}{\rho}=\cos^{-1}\frac{2}{2\sqrt{2}}=\frac{1}{4}\pi$

Spherical coordinates are $(2\sqrt{2},\frac{2}{3}\pi,\frac{1}{4}\pi)$.

(c) We use formulas (V) and (VI) where $x>0$. Thus,

$\rho=\sqrt{2^2+2^2+2^2}=\sqrt{12}=2\sqrt{3}$

$\theta=\tan^{-1}\frac{2}{2}=\frac{1}{4}\pi$

$\phi=\cos^{-1}\frac{2}{2\sqrt{3}}=\cos^{-1}\frac{1}{3}\sqrt{3}$

Spherical coordinates are $(2\sqrt{3},\frac{1}{4}\pi,\cos^{-1}\frac{1}{3}\sqrt{3})$.

5. From (IV), $r=\rho\sin\phi$. (a) $P_{\text{spherical}}=(4,\frac{2}{3}\pi,\frac{5}{6}\pi)$.

$r=4\sin\frac{5}{6}\pi=4(\frac{1}{2})=2$, $\theta=\frac{2}{3}\pi$, $z=4\cos\frac{5}{6}\pi=4(-\frac{1}{2}\sqrt{3})=-2\sqrt{3}$. $P_{\text{cylindrical}}=(2,\frac{2}{3}\pi,-2\sqrt{3})$.

(b) $P_{\text{spherical}}=(\sqrt{2},\frac{3}{4}\pi,\pi)$. Then $r=\sqrt{2}\sin\pi=\sqrt{2}(0)=0$, $\theta=\frac{3}{4}\pi$, $z=\sqrt{2}\cos\pi=\sqrt{2}(-1)=-\sqrt{2}$.

$P_{\text{cylindrical}}=(0,\frac{3}{4}\pi,-\sqrt{2})$.

(c) $P_{\text{spherical}}=(2\sqrt{3},\frac{1}{3}\pi,\frac{1}{4}\pi)$. Then

$r=2\sqrt{3}\sin\frac{1}{4}\pi=2\sqrt{3}(\frac{1}{2}\sqrt{2})=\sqrt{6}$, $\theta=\frac{1}{3}\pi$, $z=2\sqrt{3}\cos\frac{1}{4}\pi=2\sqrt{3}(\frac{1}{2}\sqrt{2})=\sqrt{6}$. $P_{\text{cylindrical}}=(\sqrt{6},\frac{1}{3}\pi,\sqrt{6})$.

6. (a) $P_{\text{cylindrical}}=(3,\frac{1}{6}\pi,3)$. $\rho=\sqrt{3^2+3^2}=3\sqrt{2}$, $\theta=\frac{1}{6}\pi$, $\phi=\cos^{-1}\frac{3}{3\sqrt{2}}=\frac{1}{4}\pi$. $P_{\text{spherical}}=(3\sqrt{2},\frac{1}{6}\pi,\frac{1}{4}\pi)$

(b) $P_{\text{cylindrical}}=(3,\frac{1}{2}\pi,2)$. $\rho=\sqrt{3^2+2^2}=\sqrt{13}$, $\theta=\frac{1}{2}\pi$, $\phi=\cos^{-1}\frac{2}{\sqrt{13}}$. $P_{\text{spherical}}=(\sqrt{13},\frac{1}{2}\pi,\cos^{-1}\frac{2}{\sqrt{13}})$

(c) $P_{\text{cylindrical}}=(2,\frac{5}{6}\pi,-4)$. $\rho=\sqrt{2^2+4^2}=2\sqrt{5}$, $\theta=\frac{5}{6}\pi$, $\phi=\cos^{-1}\frac{-4}{2\sqrt{5}}$. $P_{\text{spherical}}=(2\sqrt{5},\frac{5}{6}\pi,\cos^{-1}\frac{2}{5}\sqrt{5})$

In Exercises 7–12, find an equation in cylindrical coordinates of the surface and identify the surface. The cartesian, cylindrical and spherical equation of a surface S will be denoted by $S_{\text{cartesian}}$, $S_{\text{cylindrical}}$, and $S_{\text{spherical}}$.

7. $S_{\text{cartesian}}$ is $x^2+y^2+4z^2=16$. S is an ellipsoid. Because $x^2+y^2=r^2$, $S_{\text{cylindrical}}$ is $r^2+4z^2=16$

8. $x^2-y^2=9$

▹ The surface is a hyperbolic cylinder with rulings parallel to the z axis. The vertices of the hyperbola–given by the xy trace of the cylinder–are at the points $(\pm 3,0)$, and the lines $y=\pm x$ are the asymptotes of the hyperbola. To find an equation in cylindrical coordinates for the cylinder, we replace x by $r\cos\theta$ and y by $r\sin\theta$ in the given equation. Thus,

$$\begin{aligned}(r\cos\theta)^2-(r\sin\theta)^2&=9\\ r^2(\cos^2\theta-\sin^2\theta)&=9\\ r^2\cos 2\theta&=9\\ r^2&=9\sec 2\theta\\ r&=\pm 3\sqrt{\sec 2\theta}\end{aligned}$$

9. $S_{\text{cartesian}}$ is $x^2+y^2=3z$. S is a paraboloid of revolution. Because $x^2+y^2=r^2$, $S_{\text{cylindrical}}$ is $r^2=3z$

10. $S_{\text{cartesian}}$ is $9x^2 + 4y^2 = 36$. S is an elliptic cylinder. Because $x = r\cos\theta$ and $y = r\sin\theta$,
$S_{\text{cylindrical}}$ is $36 = 9(r\cos\theta)^2 + 4(r\sin\theta)^2 = r^2(9\cos^2\theta + 4\sin^2\theta) = r^2(5\cos^2\theta + 4)$

11. $S_{\text{cartesian}}$ is $x^2 - y^2 = 3z^2$. S is an elliptic cone, Because $x = r\cos\theta$ and $y = r\sin\theta$,
$S_{\text{cylindrical}}$ is $r^2\cos^2\theta - r^2\sin^2\theta = 3z^2$; $r^2(\cos^2\theta - \sin^2\theta) = 3z^2$; $r^2\cos 2\theta = 3z^2$.

12. $x^2 + y^2 = z^2$

▹ Because the given equation is equivalent to $x^2 + y^2 - z^2 = 0$, the equation is of type III in Section 15.7. Thus, the graph is a right circular cone with axis the z axis. Because $x^2 + y^2 = r^2$, a cylindrical-coordinate equation is $r^2 = z^2$. Because the point (r, θ, z) is the same as the point $(-r, \theta \pm \pi, z)$, the graph of the equation $r = z$ is the same as the graph of $r = -z$. Thus, the graph of $r^2 = z^2$ is the same as the graph of $r = z$, which is a cylindrical-coordinate equation of the cone $x^2 + y^2 = z^2$.

In Exercises 13–17, find an equation in spherical coordinates of the surface, and identify the surface.

13. $S_{\text{cartesian}}$ is $x^2 + y^2 + z^2 - 9z = 0$. S is a sphere. Because $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho\cos\phi$,
$S_{\text{spherical}}$ is $\rho^2 - 9\rho\cos\phi = 0$; $\rho = 9\cos\phi$. The origin is not lost by discarding the factor ρ.

14. $S_{\text{cartesian}}$ is $x^2 + y^2 = z^2$ is a circular cone; $x^2 + y^2 + z^2 = 2z^2$. $S_{\text{spherical}}$ is $\rho^2 = 2\rho^2\cos^2\phi$; $\cos\phi = \pm\frac{1}{2}\sqrt{2}$
$\phi = \frac{1}{4}\pi$ and $\phi = \frac{3}{4}\pi$, two nappes of the same cone.

15. $S_{\text{cartesian}}$ is $x^2 + y^2 = 9$. S is a right circular cylinder. Because $x^2 + y^2 = r^2 = (\rho\sin\phi)^2$,
$S_{\text{spherical}}$ is $(\rho\sin\phi)^2 = 9$; $\rho\sin\phi = 3$.

16. $x^2 + y^2 = 2z$

▹ The graph of the equation is a paraboloid of revolution whose axis is the z axis. The given equation is equivalent to
$r^2 = 2z$
Substituting from (IV), we obtain
$\rho^2\sin^2\phi = 2\rho\cos\phi$
Thus, either $\rho = 0$ or
$\rho\sin^2\phi = 2\cos\phi$ (1)
Because Eq. (1) contains the origin $(0, 0, \frac{1}{2}\pi)$, we do not need $\rho = 0$. Thus, (1) is spherical-coordinate equation of the paraboloid of revolution.

17. $S_{\text{cartesian}}$ is $x^2 + y^2 + z^2 - 8x = 0$. S is a sphere. Because $x^2 + y^2 + z^2 = \rho^2$ and $x = \rho\sin\phi\cos\theta$,
$S_{\text{spherical}}$ is $\rho^2 - 8\rho\sin\phi\cos\theta$; $\rho = 8\sin\phi\cos\theta$. The origin is not lost by discarding the factor ρ.

In Exercises 18–22, find an equation in Cartesian coordinates for the surface whose equation is given in cylindrical coordinates. In Exercises 18–20, identify the surface.

18. $S_{\text{cylindrical}}$ is $r = 3\cos\theta$. S is a circular cylinder containing the origin. $r^2 = 3r\cos\theta$, $x^2 + y^2 = 3x$

19. (a) $S_{\text{cylindrical}}$ is $r = 4$. S is a right circular cylinder. Because $x^2 + y^2 = r^2$, $S_{\text{cartesian}}$ is $x^2 + y^2 = 16$
(b) $S_{\text{cylindrical}}$ is $\theta = \frac{1}{4}\pi$. S is a plane through the z axis. Because $\tan\theta = y/x$, and $\tan\theta = \tan\frac{1}{4}\pi = 1$, $S_{\text{cartesian}}$ is $x = y$.

20. $r = 3 + 2\cos\theta$

▹ The surface is a cylinder with rulings parallel to the z axis, and whose generatrix is a limaçon with a dent. Because $\cos\theta \geq -1$, then $r \geq 3 - 1 = 2$. Hence, multiplying by r adds the origin as an isolated point. Thus,

$$r^2 - 2r\cos\theta = 3r, \quad r \neq 0$$
$$(r^2 - 2r\cos\theta)^2 = 9r^2, \quad r \neq 0$$

Substituting $r^2 = x^2 + y^2$, $r\cos\theta = x$, we obtain

$$(x^2 + y^2 - 2x)^2 = 9(x^2 + y^2), \quad x^2 + y^2 \neq 0$$

which is an equation in Cartesian coordinates for the cylinder.

21. $S_{\text{cylindrical}}$ is $r^2\cos 2\theta = z^3$; $r^2(\cos^2\theta - \sin^2\theta) = z^3$. Because $r\cos\theta = x$, $r\sin\theta = y$, $S_{\text{cartesian}}$ is $x^2 - y^2 = z^3$.

In Exercises 23–28, find an equation in Cartesian coordinates for the surface whose equation is given in spherical coordinates. In Exercises 23–25, identify the surface.

22. $S_{\text{cylindrical}}$ is $z^2\sin^3\theta = r^3$; $z^2(r^3\sin^3\theta) = (r^2)^3$. $S_{\text{cartesian}}$ is $z^2y^3 = (x^2 + y^2)^3$

23. (a) $S_{\text{spherical}}$ is $\rho = 9$. S is a sphere. Because $\rho^2 = x^2 + y^2 + z^2$, $S_{\text{cartesian}}$ is $x^2 + y^2 + z^2 = 81$.
(b) $S_{\text{spherical}}$ is $\theta = \frac{1}{4}\pi$. S is a plane through the z axis. Because $\tan\theta = y/x$ and $\tan\theta = \frac{1}{4}\pi = 1$, $S_{\text{cartesian}}$ is $x = y$.
(c) $S_{\text{spherical}}$ is $\phi = \frac{1}{4}\pi$. S is a cone with vertex at the origin. $r = z\tan\phi = z\tan\frac{1}{4}\pi = 1$. Because $r = \sqrt{x^2+y^2}$, $S_{\text{cartesian}}$ is $z = \sqrt{x^2+y^2}$.

24. $\rho = 9\sec\phi$
▷ The given equation is equivalent to
$\rho\cos\theta = 9$
Because $\rho\cos\phi = z$, a Cartesian equation of the surface is
$z = 9$
The graph is a plane perpendicular to the z axis at the point where $z = 9$.

25. $S_{\text{spherical}}$ is $\rho = 6\csc\phi$; $\rho\sin\phi = 6$. Because $\rho\sin\theta = r$ and $r^2 = x^2 + y^2$, $S_{\text{cartesian}}$ is $x^2 + y^2 = 36$. S is a right circular cylinder whose axis is the z axis.

26. $S_{\text{spherical}}$ is $\rho = 3\cos\phi$; $\rho^2 = 3\rho\cos\phi$. $S_{\text{cartesian}}$ is $x^2 + y^2 + z^2 = 3z$; $x^2 + y^2 + (z - \frac{3}{2})^2 = \frac{9}{4}$, a sphere.

27. $S_{\text{spherical}}$ is $\rho = 2\tan\theta$. Because $\rho = \sqrt{x^2+y^2+z^2}$ and $\tan\theta = y/x$, $S_{\text{cartesian}}$ is
$\sqrt{x^2+y^2+z^2} = \dfrac{2y}{x}$; $x\sqrt{x^2+y^2+z^2} = 2y$

28. $\rho = 6\sin\phi\sin\theta + 3\cos\phi$
▷ Because $(0, 0, \frac{1}{2}\pi)$ satisfies the given equation, the graph contains the origin. Thus, we may multiply on both sides by ρ. We have
$\rho^2 = 6\rho\sin\phi\sin\theta + 3\rho\cos\phi$
Substituting from (III) and (V) into the above, we get

$$x^2 + y^2 + z^2 = 6y + 3z$$
$$x^2 + (y^2 - 6y + 9) + (z^2 - 3z + \tfrac{9}{4}) = \tfrac{45}{4}$$
$$x^2 + (y-3)^2 + (z - \tfrac{3}{2})^2 = \tfrac{45}{4}$$

which is the cartesian equation of a sphere.

In Exercises 29–32, match the equation, which is in either cylindrical or spherical coordinates, with one of the surfaces shown in (i)–(x).

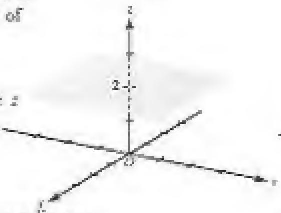
29. (a) $r = 4$. A circular cylinder of radius 4. Fig. (iii). (b) $\rho = 4$. A sphere of radius 4. Fig (vi). (c) $r = 2\sin\theta$; $r^2 = 2r\sin\theta$; $x^2 + y^2 = 2y$; $x^2 + (y-1)^2 = 1$, a circular cylinder of radius 1, centered at $(0, 1)$. Fig. (viii).

30. (a) $\theta = \frac{1}{3}\pi$. A plane perpendicular to the xy plane; Fig. (ii) (b) $\phi = \frac{1}{3}\pi$, a half-cone; Fig (iv). (c) $r = 2\cos\theta$; $r^2 = 2r\cos\theta$; $x^2 + y^2 = 2x$; $(x-1)^2 + y^2 = 1$, a circular cylinder of radius 1, centered at $(1, 0)$. Fig. (x).

31. (a) $\rho\sin\phi = 2$; $r = 2$. Circular cylinder of radius 2; Fig. (ix) . (b) $r^2 = 4z$; $x^2 + y^2 = 4z$; paraboloid of revolution; Fig. (v).

32. (a) $\rho\cos\phi = 2$; (b) $z^2 + r^2 = 4$
▷ (a) We have $z = 2$, a plane perpendicular to the z axis at the point $z = 2$ as shown in Figure (vii).
(b) We have $z^2 + x^2 + y^2 = 4$, a sphere of radius 2 as shown in Figure (i).

(vii)



(i)

33. Prove the formula for arc length in cylindrical coordinates.
▷ Because $x = r\cos\theta$, $D_t x = \cos\theta\, D_t r - r\sin\theta\, D_t\theta$,
$(D_t x)^2 = \cos^2\theta(D_t r)^2 - 2r\sin\theta\cos\theta(D_t r)(D_t\theta) = r^2\sin^2\theta(D_t\theta)^2$
Because $y = r\sin\theta$, $D_t y = \sin\theta\, D_t r + r\cos\theta\, D_t\theta$,
$(D_t y)^2 = \sin^2\theta(D_t r)^2 + 2r\sin\theta\cos\theta(D_t r)(D_t\theta) + r^2\cos^2\theta(D_t\theta)^2$
Thus $(D_t x)^2 + (D_t y)^2 = (D_t r)^2(\cos^2\theta + \sin^2\theta) + r^2(D_t\theta)^2(\sin^2\theta + \cos^2\theta) = (D_t r)^2 + r^2(D_t\theta)^2$.
If L units is the length of arc of C from $t = a$ to $t = b$, then from the formula of Theorem 11.2.1,
$\mathrm{L} = \int_a^b \sqrt{(D_t x)^2 + (D_t y)^2 + (D_t z)^2}\,dt = \int_a^b \sqrt{(D_t r)^2 + r^2(D_t\theta)^2 + (D_t z)^2}$

34. Prove the formula for arc length in spherical coordinates.

▸ $x'^2 + y'^2 + z'^2 = (\rho \sin\phi \cos\theta)'^2 + (\rho \sin\phi \sin\theta)'^2 + (\rho \cos\phi)'^2$
$= (\rho' \sin\phi \cos\theta + \rho \cos\phi \cos\theta\, \phi' - \rho \sin\phi \sin\theta\, \theta')^2 + (\rho' \sin\phi \sin\theta + \rho \cos\phi \sin\theta\, \phi' + \rho \sin\phi \cos\theta\, \theta')^2$
$+ (\rho' \cos\phi - \rho \sin\phi\, \phi')^2$
$= \rho'^2(\sin^2\phi \cos^2\theta + \sin^2\phi \sin^2\theta + \cos^2\phi) + \rho^2\phi'^2(\cos^2\phi \cos^2\theta + \cos^2\phi \sin^2\theta + \sin^2\phi)$
$+ \rho^2\theta'^2(\sin^2\phi \sin^2\theta + \sin^2\phi \cos^2\theta) + 2\rho\rho'\phi'(\sin\phi \cos\phi \cos^2\theta + \sin\phi \cos\phi \sin^2\theta - \sin\phi \cos\phi)$
$+ 2\rho\rho'\theta'(-\sin^2\phi \sin\theta \cos\theta + \sin^2\phi \sin\theta \cos\theta) + \rho^2\phi'\theta'(-\sin\phi \cos\phi \sin\theta \cos\theta + \sin\phi \cos\phi \sin\theta \cos\theta)$
$= \rho'^2 + \rho^2\phi'^2 + \rho^2\theta'^2 \sin^2\phi + 0 + 0 + 0$

35. (a) A set of parametric equations of the circular helix H is $x = a\cos t$, $y = a\sin t$, $z = t$.
If the cylindrical-coordinate representation of a point is (r, θ, z) then $r^2 = x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$ and $\tan\theta = y/x = \tan t$. Hence a set of parametric equations of H is $r = a$, $\theta = t$, $z = t$.
(b) Using the parametric equations from part (a), $D_t r = 0$, $D_t\theta = 1$, and $D_t z = 1$. Hence if L units is the length of arc of H from $t = 0$ to $t = 2\pi$, we have from Exercise 23

$$L = \int_0^{2\pi} \sqrt{0 + r^2(1)}\,dt = \int_0^{2\pi} \sqrt{a^2+1}\,dt = \sqrt{a^2+1}\,t\Big]_0^{2\pi} = 2\pi\sqrt{a^2+1}$$

36. Use the formula for arc length in spherical coordinates to find the length of arc from $t = 0$ to $t = 2\pi$ of the helix having parametric equation $\rho = t$, $\theta = t$, $\phi = \frac{1}{4}\pi$.

▸ We have

$$\frac{d\rho}{dt} = 1 \quad \frac{d\theta}{dt} = 1 \quad \frac{d\phi}{dt} = 0 \quad \sin\phi = \tfrac{1}{2}\sqrt{2}$$

Therefore,

$$L = \int_a^b \sqrt{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2}\,dt = \int_0^{2\pi} \sqrt{1 + \tfrac{1}{2}t^2}\,dt$$

Let $u = \tan^{-1}\frac{1}{2}\sqrt{2}t$; $c = \tan^{-1}\sqrt{2}\pi$. Then $\sqrt{1 + \frac{1}{2}t^2} = \sec u$; $dt = \sqrt{2}\sec^2 u\,du$ and so

$$L = \sqrt{2}\int_0^c \sec^3 u\,du = \tfrac{1}{2}\sqrt{2}\Big[\sec u \tan u + \ln|\sec u + \tan u|\Big]_0^c$$

We have $\tan c = \sqrt{2}\pi$ and $\sec c = \sqrt{1 + \tan^2 c} = \sqrt{1 + 2\pi^2}$. Thus,

$$L = \tfrac{1}{2}\sqrt{2}\left[\sqrt{1+2\pi^2}\sqrt{2}\pi + \ln\left(\sqrt{1+2\pi^2} + \sqrt{2}\pi\right)\right]$$

13.2 DOUBLE INTEGRALS

13.2.1 Definition We partition a rectangle with lines parallel to its sides. The *norm* of a partition Δ, denoted by $\|\Delta\|$, is the length of the longest diagonal of a rectangular subregion of Δ.
Let f be a function defined on a closed rectangular region R. The number L is said to be the *limit* of sums of the form $\sum_{i=1}^{n} f(u_i, v_i)\Delta_i A$ if L satisfies the property that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for every partition Δ for which $\|\Delta\| < \delta$ and for all possible selections of the point (u_i, v_i) in the ith rectangle, $i = 1, 2, \ldots, n$,

$$\left|\sum_{i=1}^{n} f(u_i, v_i)\Delta_i A - L\right| < \epsilon$$

If such a number exists, we write

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(u_i, v_i)\Delta_i A = L$$

13.2.2 Definition A function f of two variables is said to be *integrable* on a rectangular region R if f is defined on R and the number L of Definition 13.2.1 exists. This number L is called the *double integral* of f on R, and we write

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(u_i, v_i)\Delta_i A = \iint_R f(x, y)\,dA$$

13.2.3 Theorem If a function f of two variables is continuous on a closed rectangular region R, then f is integrable on R.
A *closed region* is one whose boundary consists of a finite number of arcs of smooth curves that are joined together to form a simple closed curve.

13.2.3′ Theorem If a function f of two variables is bounded and continuous on a closed region R except for a set of points which, for any $\epsilon > 0$, is contained in a union of rectangles the sum of the measures of whose area is less than ϵ, then f is integrable on R.

13.2.4 Theorem Let f be a function of two variables that is continuous on a closed region R in the xy plane and $f(x,y) \geq 0$ for all (x,y) in R. If V(S) is the measure of the solid S having the region R as its base and having an altitude of measure $f(x,y)$ at the point (x,y) in R, then

$$\mathrm{V(S)} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(u_i, v_i)\Delta_i \mathrm{A} = \iint_R f(x,y)d\mathrm{A}$$

In particular, if $f(x,y) \equiv 1$, then the measure A of the area of R is given by

$$\mathrm{A} = \iint_R d\mathrm{A}$$

13.2.5 Theorem If c is a constant and the function f is integrable on a closed region R, then cf is integrable on R and

$$\iint_R cf(x,y)d\mathrm{A} = c\iint_R f(x,y)d\mathrm{A}$$

13.2.6 Theorem If the functions f and g are integrable on a closed region R, then the function $f+g$ is integrable on R and

$$\iint_R [f(x,y) + g(x,y)]d\mathrm{A} = \iint_R f(x,y)d\mathrm{A} + \iint_R g(x,y)d\mathrm{A}$$

13.2.7 Theorem If the functions f and g are integrable on the closed region R and $f(x,y) \geq g(x,y)$ for all (x,y) in R, then

$$\iint_R f(x,y)d\mathrm{A} \geq \iint_R g(x,y)d\mathrm{A}$$

13.2.8 Theorem Let the function f be integrable on a closed region R, and suppose that m and M are two numbers such that $m \leq f(x,y) \leq \mathrm{M}$ for all (x,y) in R. If A is the measure of the area of region R, then

$$m\mathrm{A} \leq \iint_R f(x,y)d\mathrm{A} \leq \mathrm{MA}$$

13.2.9 Theorem Suppose that the function f is continuous on the closed region R and that region R is composed of the two subregions R_1 and R_2 which have no points in common except for points on parts of their boundaries. Then

$$\iint_R f(x,y)d\mathrm{A} = \iint_{R_1} f(x,y)d\mathrm{A} + \iint_{R_2} f(x,y)d\mathrm{A}$$

Evaluation Let f be a continuous function of two variables x and y. Let R be a region in the xy plane that is bounded on the left by the line $x = x_1$, bounded on the right by the line $x = x_2$, bounded below by the smooth curve $y = g_1(x)$ and bounded above by the smooth curve $y = g_2(x)$. Then the double integral of f on R is equivalent to an iterated integral, and

$$\iint_R f(x,y)d\mathrm{A} = \int_{x_1}^{x_2}\int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$$

Let R be a region in the xy plane that is bounded below by the line $y = y_1$, bounded above by the line $y = y_2$, bounded on the left by the smooth curve $x = g_1(y)$ and bounded on the right by the smooth curve $x = g_2(y)$. Then

$$\iint_R f(x,y)d\mathrm{A} = \int_{y_1}^{y_2}\int_{g_1(y)}^{g_2(y)} f(x,y)dx\,dy$$

If R can be described both ways, we can reverse the order of integration. See Ex. 55 and 56.

If the limits of integration are constants, so that R is a rectangle, and f can be expressed as the product $f(x,y) = g(x)h(y)$, then the iterated integral can be expressed as a product

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dy\,dx = \int_{x_1}^{x_2} g(x)dx \int_{y_1}^{y_2} h(y)dy$$

Improper Suppose f or R is unbounded. Let $\mathrm{R}_1 \subset \mathrm{R}_2 \subset \cdots \subset \mathrm{R}_n \subset \cdots \subset \mathrm{R}$ be such that each point of R, except possibly those in a set of arbitrarily small area, is in R_n for some n. Let $f(x,y) \geq 0$ be continuous in R_n for each n. Then $\iint_R f(x,y)d\mathrm{A} = \lim_{n\to\infty}\iint_{R_n} f(x,y)d\mathrm{A}$ if this limit exists.

Exercises 13.2

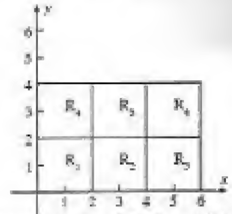
1. Let $f(x,y) = 3x - 2y + 1$. $\iint_R (3x - 2y + 1)d\mathrm{A}$
$\approx f(\frac{1}{2}, -\frac{1}{2})\cdot 1 + f(\frac{3}{2}, -\frac{1}{2})\cdot 1 + f(\frac{5}{2}, -\frac{1}{2})\cdot 1 + f(\frac{1}{2}, -\frac{3}{2})\cdot 1 + f(\frac{3}{2}, -\frac{3}{2})\cdot 1 + f(\frac{5}{2}, -\frac{3}{2})\cdot 1$
$= \frac{7}{2} + \frac{13}{2} + \frac{19}{2} + \frac{11}{2} + \frac{17}{2} + \frac{23}{2} = 45$

2. Let $f(x,y) = y^2 - 4x$. $\Delta A = 1$. $u_i = -0.5, 0.5$. $v_j = 0.5, 1.5, 2.5$. $\displaystyle\iint_R (y^2 - 4x)dA$
$\approx f(-0.5,0.5) + f(-0.5,1.5) + f(-0.5,2.5) + f(0.5,0.5) + f(0.5,1.5) + f(0.5,2.5) = 17.5$ using
sum seq$(y^2 + 2, y, .5, 2.5, 1)$ + sum seq$(y^2 - 2, y, .5, 2.5, 1)$ $\boxed{\text{ENTER}}$ on the TI-85.

In Exercises 3–8, find an approximate value of the given double integral where R is the rectangular region having the vertices P and Q. Δ is a partition of R, and (u_i, v_j) is the midpoint of each subregion.

3. Let $f(x,y) = x^2 + y$. $\Delta A = 1$. $\displaystyle\iint_R (x^2 + y)dA \approx$
$f(\frac{1}{2},\frac{1}{2}) + f(\frac{3}{2},\frac{1}{2}) + f(\frac{5}{2},\frac{1}{2}) + f(\frac{7}{2},\frac{1}{2}) + f(\frac{1}{2},\frac{3}{2}) + f(\frac{3}{2},\frac{3}{2}) + f(\frac{5}{2},\frac{3}{2}) + f(\frac{7}{2},\frac{3}{2}) = \frac{3}{4} + \frac{11}{4} + \frac{27}{4} + \frac{51}{4} + \frac{7}{4} + \frac{15}{4} + \frac{31}{4} + \frac{55}{4} = 50$

4. $\iint_R (2 - x - y)dA$; P(0,0); Q(6,4); Δ: $x_1 = 2$, $x_2 = 4$, $y_1 = 2$



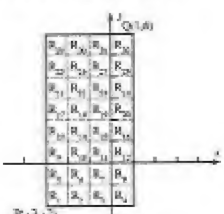
▷ The figure shows the region R which is partitioned into 6 subregions of equal area $\Delta A = 2 \times 2 = 4$ square units and the midpoint of each region. If f is the function defined by $f(x,y) = 2 - x - y$, then we have, approximately
$\displaystyle\iint_R (2 - x - y)dA$
$\approx 4[f(1,1) + f(3,1) + f(5,1) + f(1,3) + f(3,3) + f(5,3)$
$= 4[0 + (-2) + (-4) + (-2) + (-4) + (-6)]$
$= 4(-18) = -72$
You can use the sum and sequence capability of some graphics calculators (for example the TI-82 and TI-85) to automatically calculate and add the first three terms where $f(x,1) = 1 - x$ and the last three terms where $f(x,3) = -x - 1$:
$4 \times$ (sum seq$(1 - x, x, 1, 5, 2)$ + sum seq$(-1 - x, x, 1, 5, 2)$) $\boxed{\text{ENTER}}$ -72

5. Let $f(x,y) = xy + 3y^2$. $\displaystyle\iint_R (xy + 3y^2)dA$
$\approx f(-1,1)\cdot 4 + f(1,1)\cdot 4 + f(3,1)\cdot 4 + f(3,3)\cdot 4 + f(1,3)\cdot 4 + f(-1,3)\cdot 4 + f(-1,5)\cdot 4 + f(1,5)\cdot 4 + f(3,5)\cdot 4$
$= 2\cdot 4 + 4\cdot 4 + 6\cdot 4 + 36\cdot 4 + 30\cdot 4 + 24\cdot 4 + 70\cdot 4 + 80\cdot 4 + 90\cdot 4 = 1368$

6. Let $f(x,y) = xy + 3y^2$. $\Delta A = 2 \times 2 = 4$. $u_i = 1, 3, 5$. $v_j = -1, 1, 3$. $\displaystyle\iint_R (xy + 3y^2)dA \approx$
$4[f(1,-1) + f(1,1) + f(1,3) + f(3,-1) + f(3,1) + f(3,3) + f(5,-1) + f(5,1) + f(5,3)] = 504$ using
$4 \times$ (sum seq$(3 - x, x, 1, 5, 2)$ + sum seq$(x + 3, x, 1, 5, 2)$ + sum seq$(3x + 27, x, 1, 5, 2)$) $\boxed{\text{ENTER}}$

7. Let $f(x,y) = x^2y - 2xy^2$. $\displaystyle\iint_R (x^2y - 2xy^2)dA$
$\approx f(-2,-1)\cdot 4 + f(0,-1)\cdot 4 + f(-2,1)\cdot 4 + f(0,1)\cdot 4 + f(-2,3)\cdot 4 + f(0,3)\cdot 4 + f(-2,5)\cdot 4 + f(0,5)\cdot 4$
$= 0\cdot 4 + 0\cdot 4 + 8\cdot 4 + 0\cdot 4 + 48\cdot 4 + 0\cdot 4 + 120\cdot 4 + 0\cdot 4 = 704$

8. $\iint_R (x^2y - 2xy^2)dA$; P(−3,−2); Q(1,6); Δ: $x_1 = -2$, $x_2 = -1$, $x_3 = 0$, $y_1 = -1$, $y_2 = 0$, $y_3 = 1$, $y_4 = 2$, $y_5 = 3$, $y_6 = 4$, $y_7 = 5$.



▷ The figure shows the region R which is partitioned into 32 subregions of equal area $\Delta A = 1 \times 1 = 1$ square unit, with midpoints (u_i, v_j). The u_i are -2.5, -1.5, -0.5, and 0.5 and the v_j are -1.5, -0.5, 0.5, 1.5, 2.5, 3.5, 4.5, 5.5. If f is the function defined by $f(x,y) = x^2y - 2xy^2$, we have
$\displaystyle\iint_R (x^2y - 2xy^2)dA$
$\approx f(-2.5,-1.5) + f(-2.5,-0.5) + f(-2.5,0.5) + f(-2.5,1.5) + f(-2.5,2.5) + f(-2.5,3.5) + f(-2.5,4.5) + f(-2.5,5.5)$
$+ f(-1.5,-1.5) + f(-1.5,-0.5) + f(-1.5,0.5) + f(-1.5,1.5) + f(-1.5,2.5) + f(-1.5,3.5) + f(-1.5,4.5) + f(-1.5,5.5)$
$+ f(-0.5,-1.5) + f(-0.5,-0.5) + f(-0.5,0.5) + f(-0.5,1.5) + f(-0.5,2.5) + f(-0.5,3.5) + f(-0.5,4.5) + f(-0.5,5.5)$
$+ f(0.5,-1.5) + f(0.5,-0.5) + f(0.5,0.5) + f(0.6,1.5) + f(0.5,2.5) + f(0.5,3.5) + f(0.5,4.5) + f(0.5,5.5) = 736.000$
which is calculated as
sum seq$(6.25y + 5y^2, y, -1.5, 5.5, 1)$
+ sum seq$(2.25y + 3y^2, y, -1.5, 5.5, 1)$
+ sum seq$(.25y + y^2, y, -1.5, 5.5, 1)$
+ sum seq$(.25y - y^2, y, -1.5, 5.5, 1)$ $\boxed{\text{ENTER}}$

In Exercises 9–12, find an approximate value of the double integral, where R is the rectangular region having the vertices P and Q, Δ is a partition of R and (u_i, v_i) is an arbitrary point in each subregion.

9. Let $f(x,y) = x^2 + y$. Because $\Delta_i A = 1$, $\int\int_R (x^2+y)dA$

$\approx [f(.25,.5) + f(1.75,0) + f(2.5,.25) + f(4,1) + f(.75,1.75) + f(1.25,1.5) + f(2.5,2) + f(3,1)]1$
$= (.25^2 + .5) + 1.75^2 + (2.45^2 + .25) + (4^2+1) + (.75^2 + 1.75) + (1.25^2 + 1.5) + (2.5^2 + 2) + (3^2+1) = 50.75$

10. Let $f(x,y) = 2 - x - y$. Because $\Delta_i A = 4$ and $(2,2)$ belongs to 4 rectangles, $\int\int_R (2-x-y)dA$
$\approx [f(0.5,1.5) + f(3,1) + f(5.5,0.5) + f(2,2) + f(2,2) + f(5,3)]4$
$= [0-2-4-2-2-6]4 = -64$

11. Let $f(x,y) = xy + 3y^2$. Because $\Delta_i A = 4$, $\int\int_R (xy+3y^2)dA$
$\approx [f(-.5,.5) + f(1,1.5) + f(2.5,2) + f(-1.5,3.5) + f(0,3) + f(4,4) + f(-1,4.5) + f(1,4.5) + f(3,4.5)]4$
$= [0.5 + 8.75 + 17 + 31.5 + 27 + 64 + 56.25 + 165.25 + 74.75]4 = 1376$

12. $\int\int_R (xy+3y^2)dA$; $P(-2,0)$; $Q(4,6)$; Δ: $x_1 = 0$; $x_2 = 2$; $y_1 = 2$; $y_2 = 4$; $(u_1,v_1) = (-2,0)$; $(u_2,v_2) = (0,0)$; $(u_3,v_3) = (2,0)$; $(u_4,v_4) = (-2,2)$; $(u_5,v_5) = (0,2)$; $(u_6,v_6) = (2,2)$; $(u_7,v_7) = (-2,4)$; $(u_8,v_8) = (0,4)$; $(u_9,v_9) = (2,4)$

▷ If ΔA is the number of square units in each region, then $\Delta A = 2\times 2 = 4$. If $f(x,y) = xy + 3y^2$, then

$$\int\int_R (xy+3y^2)dA \approx 4[f(-2,0) + f(0,0) + f(2,0) + f(-2,2) + f(0,2) + f(2,2) + f(-2,4) + f(0,4) + f(2,4)]$$
$$= 4[0+0+0+8+12+16+40+48+56] = 720$$

which is calculated as
$4 \times (\text{sum seq}(0, x, -2, 2, 2) + \text{sum seq}(2x+12, x, -2, 2, 2) + \text{sum seq}(4x+48, x, -2, 2, 2))$ $\boxed{\text{ENTER}}$

13. Let V cubic units be the volume of the solid. $\Delta_i A = 1$. Let $f(x,y) = \sqrt{64 - x^2 - y^2}$. Then

$$V = \int\int_R \sqrt{64-x^2-y^2}\, dA \approx f(\tfrac12,\tfrac12) + f(\tfrac32,\tfrac12) + f(\tfrac52,\tfrac12) + f(\tfrac12,\tfrac32) + f(\tfrac32,\tfrac32) + f(\tfrac52,\tfrac32) + f(\tfrac12,\tfrac52) + f(\tfrac32,\tfrac52) + f(\tfrac52,\tfrac52)$$
$$= \sqrt{64-\tfrac14-\tfrac14} + \sqrt{64-\tfrac94-\tfrac14} + \sqrt{64-\tfrac{25}{4}-\tfrac14} + \sqrt{64-\tfrac14-\tfrac94} + \sqrt{64-\tfrac94-\tfrac94} + \sqrt{64-\tfrac{25}{4}-\tfrac94} + \sqrt{64-\tfrac14-\tfrac{25}{4}}$$
$$+ \sqrt{64-\tfrac94-\tfrac{25}{4}} + \sqrt{64-\tfrac{25}{4}-\tfrac{25}{4}}$$
$$= \tfrac12(\sqrt{254} + \sqrt{246} + \sqrt{230} + \sqrt{246} + \sqrt{238} + \sqrt{222} + \sqrt{230} + \sqrt{222} + \sqrt{206}) \approx 68.6$$

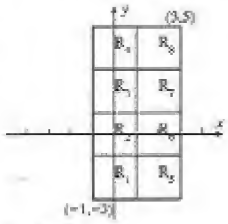
14. $\Delta_i A = 1$. Let $f = 2x + y + 4$. $V = \int\int_R (2x+y+4)dA$
$\approx f(.5,.5) + f(.5,1.5) + f(.5,2.5) + f(1.5,.5) + f(1.5,1.5) + f(1.5,2.5) = 5.5 + 6.5 + 7.5 + 7.5 + 8.5 + 9.5 = 45$

15. Let V cubic units be the volume of the solid. $\Delta_i A = 1$. Let $f(x,y) = 10 - \frac14 x^2 - \frac19 y^2$. Then

$$V = \int\int_R (10 - \tfrac14 x^2 - \tfrac19 y^2)dA \approx f(\tfrac12,\tfrac12) + f(\tfrac32,\tfrac12) + f(\tfrac12,\tfrac32) + f(\tfrac32,\tfrac32)$$
$$= (10 - \tfrac{1}{16} - \tfrac{1}{36}) + (10 - \tfrac{9}{16} - \tfrac{1}{36}) + (10 - \tfrac{1}{16} - \tfrac{9}{36}) + (10 - \tfrac{9}{16} - \tfrac{9}{36}) \approx 38.2$$

16. Approximate the volume of the solid bounded by the surface $100z = 300 - 25x^2 - 4y^2$, the planes $x = -1$, $x = 3$, $y = 1$ and $y = 3$, and the xy plane. To find an approximate value of the double integral, take a partition of the region in the xy plane formed by the lines $x = 1$, $y = -1$, and $y = 3$, and take (u_i, v_i) at the center of the ith region.

▷ The figure shows a sketch of the region R in the xy plane which is the base of the solid. R is divided into 8 subrectangles each of area $\Delta A = 2\times 2 = 4$ square units. If the midpoints are (u_i, v_i), then the u_i are 0 and 2, and the v_i are -2, 0, 2 and 4. Solving the equation of the surface for z and setting $z = f(x,y)$, we get
$f(x,y) = 3 - \frac14 x^2 - \frac{1}{25} y^2$
The number of cubic units in the volume of the solid is



$$\int\int_R (3 - \tfrac14 x^2 - \tfrac{1}{25} y^2)dA$$
$\approx 4[f(0,-2) + f(0,0) + f(0,2) + f(0,4) + f(2,-2) + f(2,0) + f(2,2) + f(2,4)] = 72.32$
which is calculated as
$4 \times (\text{sum seq}(3 - y^2/25, y, -2, 4, 2) + \text{sum seq}(2 - y^2/25, y, -2, 4, 2))$ $\boxed{\text{ENTER}}$
We conclude that the volume is approximately 72.3 cubic units. The exact value is about 69.2 cubic units.

In Exercises 17–20, apply Theorem 13.2.8 to find a closed interval containing the value of the double integral.

17. Let $f(x,y) = 2x + 5y$. Because f is linear, its absolute extrema occur at vertices of the rectangular region R.
$f(0,0) = 0 \quad f(1,0) = 2 \quad f(1,2) = 12 \quad f(0,2) = 10$
Therefore $0 \le f(x,y) \le 12$ for all (x,y) in R. Because $A = 2$, we have from Theorem 13.2.8
$0 \cdot 2 \le \iint_R (2x+5y)dA \le 12 \cdot 2;\ \ 0 \le \iint_R (2x+5y)dA \le 24$

18. $f(x,y) = x^2 + y^2$ is the square of the distance from $(0,0)$. Its least value is 0 at $(0,0)$; its greatest value is 2 at $(1,1)$. $A = 1$. $0 = 0 \cdot 1 \le \iint_R (x^2+y^2)dA \le 2 \cdot 1 = 2$

19. Let $f(x,y) = e^{xy}$. Because e^t is increasing, $e^{xy} \ge e^{0 \cdot 0} = 1$ and $e^{xy} \le e^{1 \cdot 1} = e$ for all (x,y) in R. Because $A = 1$, we have from Theorem 13.2.8 $1 \cdot 1 \le \iint_R e^{xy}dA \le e \cdot 1;\ \ 1 \le \iint_R e^{xy}dA \le e$

20. $\iint_R (\sin x + \sin y)dA$ where R is the rectangular region having vertices $(0,0)$, $(\pi,0)$, (π,π), and $(0,\pi)$.
▸ Let $f(x,y) = \sin x + \sin y$. Because f is the sum of two sine functions each of which having a minimum value of 0 and maximum value of 1 on R, we take $m = 0 + 0 = 0$ and $M = 1 + 1 = 2$ in Theorem 13.2.8. Furthermore, because the area of the region is π^2 square units, we have $A = \pi^2$ in the theorem. Thus,
$mA \le \iint_R f(x,y)dA \le MA$
becomes
$0 \le \iint_R (\sin x + \sin y)dA \le 2\pi^2$

In Exercises 21–30, evaluate the iterated integral.

21. $\int_0^2 \int_0^{2x} xy^3 dy\, dx = \int_0^2 \left[\int_0^{2x} xy^3 dy\right] dx = \int_0^2 \left[\frac{1}{4} xy^4\right]_0^{2x} dx = \int_0^2 4x^5 dx = \frac{2}{3} x^6\Big]_0^2 = \frac{2}{3}(64-1) = 42$

22. $\int_0^4 \int_0^y dx\, dy = \int_0^4 \left[\int_0^y dx\right] dy = \int_0^4 y\, dy = \frac{1}{2}y^2\Big]_0^4 = 8$

23. $\int_0^4 \int_0^y \sqrt{9+y^2}\, dx\, dy = \int_0^4 \left[\int_0^y \sqrt{9+y^2}\, dx\right] dy = \int_0^4 \left[x\sqrt{9+y^2}\right]_0^y = \int_0^4 y(9+y^2)^{1/2} dy = \frac{1}{3}(9+y^2)^{3/2}\Big]_0^4$
$= \frac{1}{3}(125-27) = \frac{98}{3}$

24. $\int_{-1}^1 \int_1^{e^x} \frac{x}{y} dy\, dx$
▸ $\int_{-1}^1 \int_1^{e^x} \frac{x}{y} dy\, dx = \int_{-1}^1 \left[x \ln y\right]_1^{e^x} dx = \int_{-1}^1 x(\ln e^x - \ln 1) dx = \int_{-1}^1 x^2 dx = \frac{1}{3}x^3\Big]_{-1}^1 = \frac{2}{3}$

25. $\int_1^4 \int_{y^2}^y \sqrt{\frac{y}{x}}\, dx\, dy = \int_0^4 \left[\int_{y^2}^y y^{1/2}x^{-1/2} dx\right] dy = \int_1^4 \left[2y^{1/2}x^{1/2}\right]_{y^2}^y dy = 2\int_1^4 (y - y^{3/2}) dy = 2\left[\frac{1}{2}y^2 - \frac{2}{5}y^{5/2}\right]_1^4$
$= 2\left[\left(8 - \frac{64}{5}\right) - \left(\frac{1}{2} - \frac{2}{5}\right)\right] = -\frac{49}{5}$

26. $\int_1^4 \int_{x^2}^x \sqrt{\frac{y}{x}} dy\, dx = \int_1^4 x^{-1/2}\left[\int_{x^2}^x y^{1/2} dy\right] dx = \int_1^4 x^{-1/2}\left[\frac{2}{3}y^{3/2}\right]_{x^2}^x dx = \frac{2}{3}\int_1^4 x^{-1/2}(x^{3/2} - x^3) dx$
$= \frac{2}{3}\int_1^4 (x - x^{5/2}) dx = \frac{2}{3}\left[\frac{1}{2}x^2 - \frac{2}{7}x^{7/2}\right]_1^4 = -\frac{403}{21}$

27. Because $|x-y|$ is symmetric with respect to the line $y = x$,
$\int_0^1 \int_0^1 |x-y| dy\, dx = 2\int_0^1 \left[\int_0^x |x-y| dy\right] dx = 2\int_0^1 \left[\int_0^x (x-y) dy\right] dx = 2\int_0^1 \left[xy - \frac{1}{2}y^2\right]_0^x dx = \int_0^1 x^2 dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}$

28. $\int_0^3 \int_0^x x^2 e^{xy} dy\, dx$
▸ $\int_0^3 \int_0^x x^2 e^{xy} dy\, dx = \int_0^3 \left[xe^{xy}\right]_0^x dx = \int_0^3 (xe^{x^2} - x) dx = \frac{1}{2}e^{x^2} - \frac{1}{2}x^2\Big]_0^3 = (\frac{1}{2}e^9 - \frac{9}{2}) - \frac{1}{2}e^0 = \frac{1}{2}e^9 - 5$

29. $\int_{\pi/2}^{\pi} \int_0^x \sin(4x-y) dy\, dx = \int_{\pi/2}^{\pi} \left[\int_0^x \sin(4x-y) dy\right] dx = \int_{\pi/2}^{\pi} \left[\cos(4x-y)\right]_0^x dx$
$= \int_{\pi/2}^{\pi} (\cos 3x - \cos 4x) dx = \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x\Big]_{\pi/2}^{\pi} = 0 - \left(\frac{1}{3}\sin\frac{3}{2}\pi - \frac{1}{4}\sin 2\pi\right) = \frac{1}{3}$

30. $\int_{\pi/2}^{\pi} \int_0^{y^2} \sin\frac{x}{y}\, dx\, dy = \int_{\pi/2}^{\pi} \left[-y\cos\frac{x}{y}\right]_{x=0}^{y^2} dy = \int_{\pi/2}^{\pi} (y - y\cos y) dy = \left[\frac{1}{2}y^2 - y\sin y - \cos y\right]_{\pi/2}^{\pi} = \frac{3}{8}\pi^2 + \frac{1}{2}\pi + 1$

In Exercises 31–38, find the exact value of the double integral.

31. $\displaystyle\iint_R (3x-2y+1)\,dA = \int_0^3\int_{-2}^{0}(3x-2y+1)\,dy\,dx = \int_0^3 \Big[3xy-y^2+y\Big]_{-2}^{0} dx = \int_0^3 (6x+6)\,dx = 3x^2+6x\Big]_0^3 = 45$

32. $\iint_R (y^2-4x)\,dA$, where R is the rectangular region having opposite vertices $(-1,0)$ and $(1,3)$.

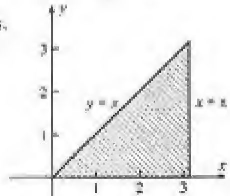
▸ $\displaystyle\iint_R (y^2-4x)\,dA = \int_{-1}^{1}\int_0^3 (y^2-4x)\,dy\,dx = \int_{-1}^{1}\Big[\tfrac{1}{3}y^3-4xy\Big]_0^3 dx = \int_{-1}^{1}(9-12x)\,dx = 9x-6x^2\Big]_{-1}^{1} = 18$

33. $\displaystyle\iint_R (x^2+y)\,dA = \int_0^4\int_0^2 (x^2+y)\,dy\,dx = \int_0^4\Big[x^2y+\tfrac{1}{2}y^2\Big]_0^2 dx = \int_0^4 (2x^2+2)\,dx = \tfrac{2}{3}x^3+2x\Big]_0^4 = \tfrac{128}{3}+8 = \tfrac{152}{3}$

34. $\displaystyle\iint_R (xy+3y^2)\,dA = \int_0^6\int_{-2}^{4}(xy+3y^2)\,dy\,dx = \int_0^6\Big[\tfrac{1}{2}xy^2+y^3\Big]_{-2}^{4} dx = \int_0^6 (6x+72)\,dx = 3x^2+72x\Big]_0^6 = 540$

35. $\displaystyle\iint_R \sin x\,dA = \int_0^{\pi}\int_{x/2}^{2x}\sin x\,dy\,dx = \int_0^{\pi}\Big[y\sin x\Big]_{x/2}^{2x} dx = \int_0^{\pi}\tfrac{3}{2}x\sin x\,dx = \tfrac{3}{2}\Big[\sin x - x\cos x\Big]_0^{\pi} = \tfrac{3}{2}\pi$

36. $\iint_R \cos(x+y)\,dA$, where R is bounded by the lines $y=x$ and $x=\pi$, and the x axis.



▸ The figure shows a sketch of the region R, which is bounded by below by the line $y=0$, bounded above by the line $y=x$, bounded on the left by the line $x=0$, and bounded on the right by the line $x=\pi$. Therefoe, we have

$$\iint_R \cos(x+y)\,dA = \int_0^{\pi}\int_0^{x}\cos(x+y)\,dy\,dx = \int_0^{\pi}\Big[\sin(x+y)\Big]_{y=0}^{x}$$
$$= \int_0^{\pi}(\sin 2x - \sin x)\,dx = -\tfrac{1}{2}\cos 2x + \cos x\Big]_0^{\pi} = -2$$

37. Because the integrand is symmetric with respect to the x and y axes,

$$\iint_R x^2\sqrt{9-y^2}\,dA = 4\int_0^3\int_0^{\sqrt{9-y^2}} x^2\sqrt{9-y^2}\,dx\,dy = 4\int_0^3\Big[\tfrac{1}{3}x^3\sqrt{9-y^2}\Big]_0^{\sqrt{9-y^2}} dy = \frac{4}{3}\int_0^3 (9-y^2)^2 dy$$
$$= \frac{4}{3}\int_0^3 (81-18y^2+y^4)\,dy = \frac{4}{3}\Big[81y-6y^3+\tfrac{1}{5}y^5\Big]_0^3 = \frac{4}{3}(81\cdot 3 - 6\cdot 3^3 + \tfrac{1}{5}\cdot 3^5) = \frac{864}{5}$$

38. $\displaystyle\iint_R \frac{y^2}{x^2}\,dA = \int_1^2\int_{1/y}^{y} y^2x^{-2}\,dx\,dy = \int_1^2\Big[-y^2x^{-1}\Big]_{1/y}^{y} dy = \int_1^2 (y^3-y)\,dy = \Big[\tfrac{1}{4}y^4-\tfrac{1}{2}y^2\Big]_1^2 = \tfrac{9}{4}$

39. $\displaystyle V = 4\iint_R x\,dA = 8\int_0^4\int_0^{\sqrt{16-x^2}} x\,dy\,dx = 8\int_0^4\Big[xy\Big]_0^{\sqrt{16-x^2}} dx = 8\int_0^4 x(16-x^2)^{1/2}dx = -\tfrac{8}{3}(16-x^2)^{3/2}\Big]_0^4 = \tfrac{512}{3}$

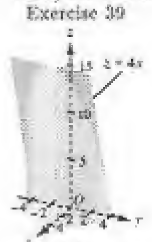
40. Find the volume of the solid bounded by the planes $x=y+2z+1$, $x=0$, $y=0$, and $3y+z=3$.

▸ Refer to the figure below. The projection of the solid onto the xy plane is not contained in the base in the xy plane, and the projection onto the xz plane is not contained in the base in the xz plane, while the projection onto the yz plane is contained in the base in the yz plane. Hence, we let R be the region in the yz plane bounded by the lines $y=0$, $y=1$, $z=0$, and $z=3-3y$ and let $f(y,z)=y+2z+1$. If V cubic units is the volume of the solid, then

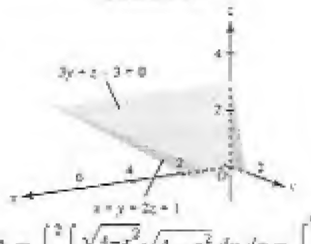
$$V = \iint_R f(y,z)\,dA = \int_0^1\int_0^{3-3y}(y+2z+1)\,dz\,dy = \int_0^1\Big[yz+z^2+z\Big]_{z=0}^{3-3y}$$
$$= \int_0^1 [y(3-3y)+(3-3y)^2+(3-3y)]\,dy = \int_0^1 (6y^2-18y+12)\,dy = 2y^3-9y^2+12y\Big]_0^1 = 5$$

Thus, the volume is 5 cubic units.

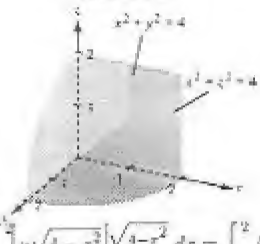
Exercise 39



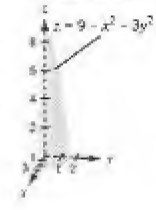
Exercise 40



Exercise 41



Exercise 42



41. $\displaystyle V = \iint_R \sqrt{4-x^2}\,dA = \int_0^2\int_0^{\sqrt{4-x^2}}\sqrt{4-x^2}\,dy\,dx = \int_0^2\Big[y\sqrt{4-x^2}\Big]_0^{\sqrt{4-x^2}} dx = \int_0^2 (4-x^2)\,dx$

$\displaystyle = 4x-\tfrac{1}{3}x^3\Big]_0^2 = 8-\tfrac{8}{3} = \tfrac{16}{3}$

42. $V = \int_0^{\sqrt{3}} \int_0^{\sqrt{9-3y^2}} (9 - 3y^2 - x^2)dx\,dy = \int_0^{\sqrt{3}} \left[(9-3y^2)x - \frac{1}{3}x^3\right]_0^{\sqrt{9-3y^2}} dy = \frac{2}{3}\int_0^{\sqrt{3}} (9-3y)^{3/2} dy$

Let $y = \sqrt{3}\sin\theta$. $V = \frac{2}{3}\int_0^{\pi/2} (3\cos\theta)^3 \sqrt{3}\cos\theta\,d\theta = 18\sqrt{3}\int_0^{\pi/2} \cos^4\theta\,d\theta$

$= 18\sqrt{3}\left[\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta)\right]_0^{\pi/2} = \frac{27}{8}\pi$

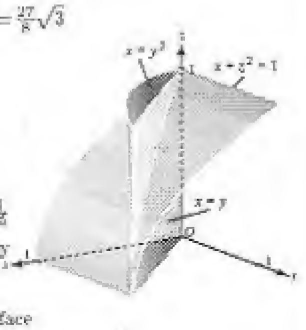
Alternatively, a z-slice is a quarter of the ellipse $x^2 + 3y^2 = 9 - z$, $\frac{x^2}{9-z} + \frac{y^2}{(9-z)/3} = 1$ of area

$\frac{1}{4}\sqrt{9-z}\sqrt{\frac{1}{3}(9-z)} = \frac{1}{4}\sqrt{3}(9-z)$. $V = \frac{1}{12}\sqrt{3}\int_0^9 (9-z)dz = -\frac{1}{24}\sqrt{3}(9-z)^2\Big]_0^9 = \frac{27}{8}\sqrt{3}$

43. $V = \iint_R \sqrt{1-x}\,dA = \int_0^1 \int_{y^2}^{y} (1-x)^{1/2} dx\,dy = \int_0^1 \left[-\frac{2}{3}(1-x)^{3/2}\right]_{y^2}^{y} dy$

$= -\frac{2}{3}\int_0^1 [(1-y)^{3/2} - (1-y^2)^{3/2}]dy$. Let $y = \sin\theta$, $dy = \cos\theta\,d\theta$ in the second term. Then $V = \left[\frac{4}{15}(1-y)^{5/2}\right]_0^1 + \frac{2}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta$

$= -\frac{4}{15} + \frac{2}{3}\left[\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\left(\frac{1}{2}\cos\theta\sin\theta - \frac{1}{2}\theta\right)\right]_0^{\pi/2} = -\frac{4}{15} + \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{1}{8}\pi - \frac{4}{15}$



44. Find by double integration the volume of the portion of the solid bounded by the sphere $x^2 + y^2 + z^2 = 16$ and lying in the first octant.

▸ Let f be the function defined by $f(x,y) = \sqrt{16 - x^2 - y^2}$. Then the solid is bounded below by a region R in the xy plane and bounded above by the surface $z = f(x,y)$. R is bounded by the lines $x = 0$, $x = 4$, and the curves $y = 0$ and $y = \sqrt{16 - x^2}$. Then

$$V = \iint_R f(x,y)dA = \int_0^4 \int_0^{\sqrt{16-x^2}} \sqrt{16 - x^2 - y^2}\,dy\,dx$$

To evaluate the inner integral, we regard x as a constant. Let $y = \sqrt{16-x^2}\sin\theta$. Then $dy = \sqrt{16-x^2}\cos\theta\,d\theta$, and $\sqrt{16-x^2-y^2} = \sqrt{16-x^2}\cos\theta$. When $y = 0$, then $\theta = 0$, and when $y = \sqrt{16-x^2}$, then $\theta = \frac{1}{2}\pi$, and the inner integral becomes

$$\int_0^{\pi/2} (16-x^2)\cos^2\theta\,d\theta = \frac{1}{2}(16-x^2)\int_0^{\pi/2} (1+\cos 2\theta)d\theta = \frac{1}{2}(16-x^2)\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{1}{4}\pi(16-x^2)$$

Alternatively, for each value of x the inner integral represents a quarter of the area of a circle of radius $\sqrt{16-x^2}$ which is $\frac{1}{4}\pi(16-x^2)$, just as if we had used slices parallel to the xy plane. Hence,

$$V = \frac{1}{4}\pi\int_0^4 (16-x^2)dx = \frac{1}{4}\pi\left[16x - \frac{1}{3}x^3\right]_0^4 = \frac{32}{3}\pi$$

Thus, the volume of the solid is $\frac{32}{3}\pi$ cubic units.

In Exercises 45–48, use double integrals to find the area of the region bounded by the curves in the xy plane. Sketch the region.

45. $A = \iint_R dA = \int_0^1 \int_{x^3}^{x^2} dy\,dx = \int_0^1 \left[y\right]_{x^3}^{x^2} dx = \int_0^1 (x^2 - x^3)dx = \frac{1}{3}x^3 - \frac{1}{4}x^4\Big]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

46. $y^2 = 4x$, $x^2 = 4y$, $x^4 = 16y^2 = 16\cdot 4x$, $x = 0, 4$. $A = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy\,dx = \int_0^4 (2x^{1/2} - \frac{1}{4}x^2)dx = \left[\frac{4}{3}x^{3/2} - \frac{1}{12}x^3\right]_0^4 = \frac{16}{3}$

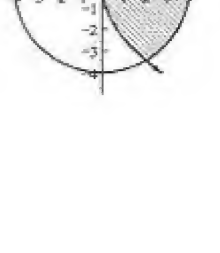
47. Because the region is symmetric with respect to the x and y axes,

$A = \iint_R dA = 4\int_0^3 \int_0^{9-x^2} dy\,dx = 4\int_0^3 \left[y\right]_0^{9-x^2} dx = 4\int_0^3 (9-x^2)dx = 4\left[9x - \frac{1}{3}x^3\right]_0^3 = 4(27-9) = 72$

48. $x^2 + y^2 = 16$ and $y^2 = 6x$

▸ The figure shows the region, which is bounded by the lines $y = \pm 2\sqrt{3}$, and by the curves $x = \frac{1}{6}y^2$ and $x = \sqrt{16-y^2}$. Let R be the upper half of the region. Thus,

$$A = 2\iint_R dx\,dy = 2\int_0^{2\sqrt{3}} \int_{y^2/6}^{\sqrt{16-y^2}} dx\,dy = 2\int_0^{2\sqrt{3}} \left[x\right]_{y^2/16}^{\sqrt{16-y^2}} dy$$
$$= 2\int_0^{2\sqrt{3}} [\sqrt{16-y^2} - \frac{1}{6}y^2]dy = 2\int_0^{2\sqrt{3}} \sqrt{16-y^2}\,dy - \left[\frac{1}{9}y^3\right]_0^{2\sqrt{3}}$$
$$= 2\int_0^{2\sqrt{3}} \sqrt{16-y^2}\,dy - \frac{8}{3}\sqrt{3} \qquad (1)$$



For the remaining integral in (1), we take $y = 4\sin\theta$. Thus,

$$A = 2\int_0^{\pi/3} 16\cos^2\theta\, d\theta - \tfrac{8}{3}\sqrt{3} = 16\int_0^{\pi/3}(1+\cos 2\theta)d\theta - \tfrac{8}{3}\sqrt{3} = 16\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/3} - \tfrac{8}{3}\sqrt{3} = \tfrac{16}{3}\pi + \tfrac{4}{3}\sqrt{3}$$

Therefore, the area is $\frac{4}{3}(4\pi + \sqrt{3})$ square units.

49. Taking 8 times the volume in the first octant,

$$V = 8c\iint_R \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dA = 8c\int_0^a \int_0^{b\sqrt{1-(x/a)^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx$$

To evaluate this integral, let $x = au$, $dx = a\, du$, $y = bv$, $dy = b\, dv$. Then

$$V = abc\cdot 8\int_0^1 \int_0^{\sqrt{1-u^2}} \sqrt{1 - u^2 - v^2}\, dv\, du = abc\cdot(\text{volume of unit sphere}) = \tfrac{4}{3}\pi abc$$

50. $4x = y^2 = 5 - x^2$, $0 = x^2 + 4x - 5 = (x+5)(x-1)$, $x = 1$ (reject -5), $y = \pm 2$.

(a) $A = \int_0^2 \int_{y^2/4}^{\sqrt{5-y^2}} dx\, dy = \int_0^2 (\sqrt{5-y^2} - \frac{1}{4}y^2)dy$. Let $y = \sqrt{5}\sin\theta$, $\sin b = \frac{2}{5}\sqrt{5}$, $\cos b = \frac{1}{5}\sqrt{5}$.

$A = \int_0^b \sqrt{5}\cos\theta\cdot\sqrt{5}\cos\theta\, d\theta - \frac{1}{12}y^3\Big]_0^2 = \frac{5}{2}\int_0^b (1+\cos 2\theta)d\theta - \frac{2}{3} = \frac{5}{2}\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_0^b - \frac{2}{3} = (\frac{5}{2}\sin^{-1}\frac{2}{5}\sqrt{5} + 1) - \frac{2}{3}$

$= \frac{5}{2}\sin^{-1}\frac{2}{5}\sqrt{5} + \frac{1}{3}$ (b) $A = \int_0^1 2x^{1/2}dx + \int_1^{\sqrt{5}} \sqrt{5 - x^2}\, dx$. Let $x = \sqrt{5}\cos\theta$.

$A = \frac{4}{3}x^{3/2}\Big]_0^1 + \int_b^0 \sqrt{5}\sin\theta(-\sqrt{5}\sin\theta\, d\theta) = \frac{4}{3} + \frac{5}{2}\int_0^b (1 - \cos 2\theta)d\theta = \frac{4}{3} + (\frac{5}{2}\sin^{-1}\frac{2}{5}\sqrt{5} - 1) = \frac{5}{2}\sin^{-1}\frac{2}{5}\sqrt{5} + \frac{1}{3}$

51. The parabola $y^2 = 2x$ and the line $2x + 3y = 10$ intersect in the first quadrant at $(2,2)$, the parabola intersects the x axis at $(0,0)$, and the line intersects the x axis at $(5,0)$.

$V = \iint_R \left(4 - \frac{1}{2}x - \frac{4}{3}y\right)dA$ (a) $V = \int_0^2 \int_{y^2/2}^{(10-3y)/2} \left(4 - \frac{1}{2}x - \frac{4}{3}y\right)dx\, dy = \int_0^2 \Big[4x - \frac{1}{2}x^2 - \frac{4}{2}xy\Big]_{y^2/2}^{(10-3y)/2} dy$

$= \int_0^2 (\frac{55}{4} - \frac{107}{12}y - \frac{9}{16}y^2 + \frac{2}{3}y^3 + \frac{1}{16}y^4)dy = \frac{55}{4}y - \frac{107}{24}y^2 - \frac{3}{16}y^3 + \frac{1}{6}y^4 + \frac{1}{80}y^5\Big]_0^2 = \frac{55}{2} - \frac{107}{6} - \frac{3}{2} + \frac{8}{3} + \frac{2}{5} = \frac{337}{30}$

(b) $V = \int_0^2 \int_0^{\sqrt{2x}} \left(4 - \frac{1}{2}x - \frac{4}{3}y\right)dy\, dx + \int_2^5 \int_0^{(10-2x)/3} \left(4 - \frac{1}{2}x - \frac{4}{3}y\right)dy\, dx$

$= \int_0^2 \Big[4y - \frac{1}{2}xy - \frac{2}{3}y^2\Big]_0^{\sqrt{2x}} dx + \int_0^5 \Big[4y - \frac{1}{2}xy - \frac{2}{2}y^2\Big]_0^{(10-2x)/3} dx$

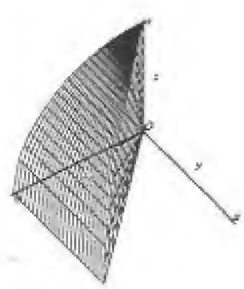
$= \int_0^2 (4\sqrt{2}x^{1/2} - \frac{1}{2}\sqrt{2}x^{3/2} - \frac{4}{3}x)dx + \int_2^5 \left(\frac{160}{27} - \frac{37}{27}x + \frac{1}{27}x^2\right)dx$

$= \Big[\frac{8}{3}\sqrt{2}x^{3/2} - \frac{1}{5}\sqrt{2}x^{5/2} - \frac{2}{3}x^2\Big]_0^2 + \Big[\frac{160}{27}x - \frac{37}{54}x^2 + \frac{1}{81}x^3\Big]_2^5 = \frac{32}{5} + \frac{87}{18} = \frac{337}{30}$

In Exercises 52 and 53, (a) sketch the solid, the measure of whose volume is represented by the given iterated integral; (b) evaluate the iterated integral; (c) write the iterated integral the gives the measure of the volume of the same solid with the order of integration reversed.

52. $\int_0^a \int_0^x \sqrt{a^2 - x^2}\, dy\, dx$

▷ (a) The base of the solid is the triangle bounded by the lines $x = 0$ and $x = a$, and by the lines $y = 0$ and $y = x$. And the solid is bounded by the cylindrical surface $z = \sqrt{a^2 - x^2}$. The figure shows a sketch of the solid.



(b) $\int_0^a \int_0^x \sqrt{a^2 - x^2}\, dy\, dx = \int_0^a \Big[y\sqrt{a^2 - x^2}\Big]_{y=0}^x dx$

$= \int_0^a x\sqrt{a^2 - x^2}\, dx = -\frac{1}{3}(a^2 - x^2)^{3/2}\Big]_0^a = \frac{1}{3}a^3$

(c) We may regard the triangular region in the xy plane as being bound by the lines $y = 0$, $y = a$, and the lines $x = y$ and $x = a$. Therefore,

$\int_0^a \int_0^x \sqrt{a^2 - x^2}\, dy\, dx = \int_0^a \int_y^a \sqrt{a^2 - x^2}\, dx\, dy$

53. (b) $\frac{2}{3}\int_0^a \int_0^{\sqrt{a^2-x^2}} (2x + y)dy\, dx = \frac{2}{3}\int_0^a \Big[2xy + \frac{1}{2}y^2\Big]_0^{\sqrt{a^2-x^2}} dx = \frac{2}{3}\int_0^a \Big[2x\sqrt{a^2 - x^2} + \frac{1}{2}(a^2 - x^2)\Big]dx$

$= \frac{2}{3}\Big[-\frac{2}{3}(a^2 - x^2)^{3/2} + \frac{1}{2}(a^2x - \frac{1}{3}x^3)\Big] = \frac{2}{3}(\frac{2}{3}a^3 + \frac{1}{3}a^3) = \frac{2}{3}a^3$ (c) $\frac{2}{3}\iint_R (2x + y)dA = \frac{2}{3}\int_0^a \int_0^{\sqrt{a^2-y^2}} (2x + y)dx\, dy$

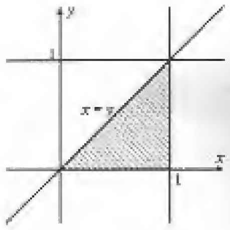
54. Let the axes of the cylinders be the x- and y-axes. The volume is 16 times the region above the part of the first octant where $y \leq x$. There $z = \sqrt{r^2 - x^2}$ (the part *not* visible) and so

$$V = 16\int_0^r \int_0^x \sqrt{r^2 - x^2}\,dy\,dx = 16\int_0^r x\sqrt{r^2 - x^2}\,dx = 16\Big[-\tfrac{1}{3}(r^2 - x^2)^{3/2}\Big]_0^r = \tfrac{16}{3}r^3$$

In Exercises 55 and 56, the iterated integral cannot be evaluated exactly in terms of elementary functions by the given order of integration. Reverse the order of integration and perform the computation.

55. $$\int_0^4 \int_{\sqrt{x}}^2 \sin \pi y^3\,dy\,dx = \int_0^2 \int_0^{y^2} \sin \pi y^3\,dx\,dy = \int_0^2 \Big[x \sin \pi y^3\Big]_0^{y^2} dy = \int_0^2 y^2 \sin \pi y^3\,dy = -\frac{1}{3\pi}\cos \pi y^3\Big]_0^2$$
$$= -\frac{1}{3\pi}(\cos 8\pi - \cos 0) = -\frac{1}{3\pi}(1 - 1) = 0$$

56. $$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$$



- The region R over which the double integral is being taken is bounded by the lines $y = 0$, $y = 1$, and by the "curves" $x = y$ and $x = 1$, as illustrated in the figure. We may also regard R as being bounded by the lines $x = 0$, $x = 1$, and by the "curves" $y = 0$ and $y = x$. Therefore,

$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 \Big[ye^{x^2}\Big]_{y=0}^{x} dx = \int_0^1 xe^{x^2}\,dx = \tfrac{1}{2}e^{x^2}\Big]_0^1$$
$$= \tfrac{1}{2}(e - 1)$$

13.3 APPLICATIONS OF DOUBLE INTEGRALS

If a lamina has the shape of a closed region R in the xy plane and $\rho(x,y)$ is the measure of the area density of the lamina at any point (x,y) of R, where ρ is continuous on R, then the measure of mass of the lamina is given by

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \rho(u_i, v_i)\Delta_i A = \iint_R \rho(x,y)\,dA$$

The measure M_x of the moment of mass with respect to the x axis of the lamina is given by

$$M_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i \rho(u_i, v_i)\Delta_i A = \iint_R y\rho(x,y)\,dA$$

Analogously, the measure M_y of its moment of mass with respect to the y axis is given by

$$M_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i \rho(u_i, v_i)\Delta_i A = \iint_R x\rho(x,y)\,dA$$

The center of mass of the lamina is denoted by the point $(\bar{x}, \bar{y})$ and

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

If the region R and the density function are symmetric in x and y, then $\bar{x} = \bar{y}$. See Exercise 12.

The *moment of inertia* of a particle, whose mass is m kg, about an axis is mr^2 kg-m^2, where r meters is the perpendicular distance from the particle to the axis.

Suppose that we are given a lamina occupying a region R in the xy plane such that the area density at the point (x,y) has measure $\rho(x,y)$, where ρ is continuous on R. Then the measure of the moment of inertia of the lamina about the x axis, denoted by I_x, is defined by

$$I_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i^2 \rho(u_i, v_i)\Delta_i A = \iint_R y^2 \rho(x,y)\,dA$$

Similarly, the measure of the moment of inertia of the lamina about the y axis, denoted by I_y, is defined by

$$I_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i^2 \rho(u_i, v_i)\Delta_i A = \iint_R x^2 \rho(x,y)\,dA$$

Finally, the measure of the *polar moment of inertia*, denoted by I_0, is defined by

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2)\rho(x,y)\,dA$$

If I is the measure of the moment of inertia about an axis L of a distribution of mass in a plane and M is the measure of the total mass of the distribution, then the *radius of gyration* of the distribution about L has measure r, where

$$r^2 = \frac{I}{M}$$

Surface Area If a surface is defined by the parametric equations over a region R
$x = f(u,v) \quad y = g(u,v) \quad z = h(u,v)$
then the surface area is given by

$$\sigma = \iint_R \|(x_u, y_u, z_u) \times (x_v, y_v, z_v)\| du\, dv = \iint_R \sqrt{\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}^2 + \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix}^2 + \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}^2}\, du\, dv$$

In the special case $x = x \quad y = y \quad z = f(x,y)$
we have (Theorem 13.3.4)

$$\sigma = \iint_R \sqrt{f_x^{\,2}(x,y) + f_y^{\,2}(x,y) + 1}\; dx\, dy$$

If a surface is given by $g(x,y,z) = 0$ and its projection on the xy plane is R, then

$\sigma = \iint_R \|\nabla g(x,y,z)\| / |g_z(x,y,z)|\; dx\, dy$. See Exercise 44.

Cone For the cone $x^2 + y^2 = z^2$, $\sigma = \iint_R \sqrt{2}\; dA = \sqrt{2}\;\text{area(R)}$. See Exercise 32.

Revolution Suppose that the function f is positive on $[a,b]$ and f' is continuous on $[a,b]$. Then the measure of the area of the surface of revolution obtained by revolving the curve $y = f(x)$, with $a \le x \le b$, about the x axis is given by $\sigma = 2\pi \int_a^b f(x)\sqrt{[f'(x)]^2 + 1}\,dx$

Pappus Theorem If an arc of length L units is revolved about the x axis, the surface area is $2\pi \bar{y} L$ square units.

Sphere The surface area of a sphere of radius r units is $4\pi r^2$ square units. The area of a *zone*, the part of the surface cut off by two parallel planes h units apart, is $2\pi r h$ square units. See Ex. 38.

Exercises 13.3

In Exercises 1–12, find the mass and the center of mass of a lamina in the given shape if the area density is as indicated. Mass is measured in kilograms and distance is measured in meters.

- In Exercises 1–12, M kg is the mass of the lamina R, M_x kg-m and M_y kg-m are the moments of mass with respect to the x and y axes, and the center of mass is at $(\bar{x}, \bar{y})$. $\rho(x,y)$ kg/m^2 is the area density at the point (x,y) and k is a constant of proportionality.

1. R is bounded by $x = 3$, $y = 2$, and the coordinate axes. $\rho(x,y) = xy^2$.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i v_i^{\,2} \Delta_i A = \iint_R xy^2 dA = \int_0^3 \int_0^2 xy^2 dy\, dx = \int_0^3 \tfrac{1}{3} xy^3 \Big]_0^2 dx = \int_0^3 \tfrac{8}{3} x\, dx = \tfrac{4}{3} x^2 \Big]_0^3 = 12$$

$$M_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i (u_i v_i^{\,2}) \Delta_i A = \iint_R xy^3 dA = \int_0^3 \int_0^2 xy^3 dy\, dx = \int_0^3 \tfrac{1}{4} xy^4 \Big]_0^2 dx = \int_0^3 4x\, dx = 2x^2 \Big]_0^3 = 18$$

$$M_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i (u_i v_i^{\,2}) \Delta_i A = \iint_R x^2 y^2 dA = \int_0^3 \int_0^2 x^2 y^2 dy\, dx = \int_0^3 \tfrac{1}{3} x^2 y^3 \Big]_0^2 dx = \int \tfrac{8}{3} x^2 dx = \tfrac{8}{9} x^3 \Big]_0^3 = 24$$

$$\bar{x} = \frac{M_y}{M} = \frac{24}{12} = 2 \text{ and } \bar{y} = \frac{M_x}{M} = \frac{18}{12} = \frac{3}{2}$$

2. R is bounded by the coordinates axes, $x = 4$ and $y = 5$. $\rho(x,y) = x^2 + y$.

$$M = \iint_R (x^2 + y) dA = \int_0^4 \int_0^5 (x^2 + y) dy\, dx = \int_0^4 \Big[x^2 y + \tfrac{1}{2} y^2 \Big]_0^5 dx = \int_0^4 (5x^2 + \tfrac{25}{2}) dx = \Big[\tfrac{5}{3} x^3 + \tfrac{25}{2} x \Big]_0^4 = \tfrac{470}{3}$$

$$M_y = \iint_R (x^2 + y) x\, dA = \int_0^4 \int_0^5 (x^2 + y) x\, dy\, dx = \int_0^4 (5x^3 + \tfrac{25}{2} x) dx = \Big[\tfrac{5}{4} x^4 + \tfrac{25}{4} x^2 \Big]_0^4 = 45.\ \bar{x} = \tfrac{3}{470} \cdot 45 = \tfrac{81}{94}$$

$$M_x = \iint_R (x^2 + y) y\, dA = \int_0^4 \int_0^5 (x^2 y + y^2) dy\, dx = \int_0^4 \Big[\tfrac{1}{2} x^2 y^2 + \tfrac{1}{3} y^3 \Big]_0^5 dx = \int_0^4 (\tfrac{25}{2} x^2 + \tfrac{125}{3}) dx = \tfrac{25}{6} x^3 + \tfrac{125}{3} x \Big]_0^4$$

$$= \tfrac{1300}{3}.\ \bar{y} = \tfrac{3}{470} \cdot \tfrac{1300}{3} = \tfrac{130}{47}$$

3. R is bounded by the coordinate axes and $x + 2y = 6$. $\rho(x,y) = y^2$.

$$M = \iint_R y^2 dA = \int_0^3 \int_0^{6-2y} y^2 dx\, dy = \int_0^3 xy^2 \Big]_0^{6-2y} dy = \int_0^3 (6y^2 - 2y^3) dy = 2y^3 - \tfrac{1}{2} y^4 \Big]_0^3 = 54 - \frac{81}{2} = \frac{27}{2}$$

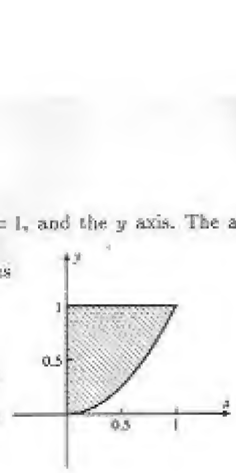
$$M_x = \iint_R y \cdot y^2 dA = \int_0^3 \int_0^{6-2y} y^3 dx\, dy = \int_0^3 xy^3 \Big]_0^{6-2y} dy = \int_0^3 (6y^3 - 2y^4) dy = \tfrac{3}{2} y^4 - \tfrac{2}{5} y^5 \Big]_0^3$$

$$= \tfrac{1}{2}(3^5) - \tfrac{2}{5}(3^5) = \frac{1}{10}(3^5) = \frac{243}{10}.\ \bar{y} = \frac{1}{M} M_x = \frac{2}{27} \cdot \frac{243}{10} = \frac{9}{5}$$

$$M_y = \iint_R xy^2 dA = \int_0^3 \int_0^{6-2y} xy^2 dx\, dy = \int_0^3 \frac{1}{2} x^2 y^2 \Big]_0^{6-2y} dy = 2 \int_0^3 (3-y)^2 y^2 dy$$

$$= 2 \int_0^3 (9y^2 - 6y^3 + y^4) dy = 2\Big[3y^3 - \frac{3}{2} y^4 + \frac{1}{5} y^5 \Big]_0^3 = 2\Big[3^4 - \tfrac{3}{2}(3^4) + \tfrac{3}{5}(3^4)\Big] = \frac{81}{5}.\ \bar{x} = \frac{1}{M} M_y = \frac{2}{27} \cdot \frac{81}{5} = \frac{6}{5}.$$

4. The region in the first quadrant bounded by the parabola $y = x^2$, the line $y = 1$, and the y axis. The area density at any point is $(x + y)\text{kg/m}^2$.



▷ The figure shows the region R. Let $\rho(x, y) = x + y$. Then the number of kilograms in the mass of the lamina is given by

$$\mathrm{M} = \iint_R \rho(x,y)d\mathrm{A} = \int_0^1 \int_{x^2}^1 (x+y)dy\,dx = \int_0^1 \Big[xy + \tfrac{1}{2}y^2\Big]_{y=x^2}^{1} dx$$
$$= \int_0^1 (\tfrac{1}{2} + x - x^3 - \tfrac{1}{2}x^4)dx = \tfrac{1}{2}x + \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{10}x^5\Big]_0^1 = \tfrac{13}{20} \qquad (1)$$

Thus, the mass is $\frac{13}{20}$ kilograms. Moreover, if $(\bar{x}, \bar{y})$ is the center of mass, we have

$$\mathrm{M}_y = \iint_R \rho(x,y)x\,dA = \int_0^1 \int_{x^2}^1 (x+y)x\,dy\,dx = \int_0^1 x\Big[xy + \tfrac{1}{2}y^2\Big]_{y=x^2}^{1} dx$$
$$= \int_0^1 (\tfrac{1}{2}x + x^2 - x^4 - \tfrac{1}{2}x^5)dx = \tfrac{1}{4}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{12}x^6\Big]_0^1 = \tfrac{3}{10} \qquad (2)$$

and

$$\mathrm{M}_x = \iint_R \rho(x,y)y\,dA = \int_0^1 \int_{x^2}^1 (x+y)y\,dy\,dx = \int_0^1 \Big[\tfrac{1}{2}xy^2 + \tfrac{1}{3}y^3\Big]_{y=x^2}^{1} dx$$
$$= \int_0^1 (\tfrac{1}{3} + \tfrac{1}{2}x - \tfrac{1}{2}x^5 - \tfrac{1}{3}x^6)dx = \tfrac{1}{3}x + \tfrac{1}{4}x^2 - \tfrac{1}{12}x^6 - \tfrac{1}{21}x^7\Big]_0^1 = \tfrac{19}{42} \qquad (3)$$

From (1), (2) and (3), we have

$\bar{x} = \frac{1}{\mathrm{M}} \cdot \mathrm{M}_y = \frac{20}{13} \cdot \frac{3}{10} = \frac{6}{13}$ and $\bar{y} = \frac{1}{\mathrm{M}} \cdot \mathrm{M}_x = \frac{20}{13} \cdot \frac{19}{42} = \frac{190}{273}$

Therefore, the center of mass is at $(\frac{6}{13}, \frac{190}{273})$.

5. R is in the first quadrant bounded by $x^2 = 8y$, $y = 2$ and the y axis. $\rho(x, y) = k(y + 1)$.

$$\mathrm{M} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k(v_i + 1)\Delta_i \mathrm{A} = k\iint_R (y+1)d\mathrm{A} = k\int_0^2 \int_0^{\sqrt{8y}} (y+1)dx\,dy = k\int_0^2 x(y+1)\Big]_0^{\sqrt{8y}} dy$$
$$= k\sqrt{8}\int_0^2 (y^{3/2} + y^{1/2})dy = k\sqrt{8}\Big[\tfrac{2}{5}y^{5/2} + \tfrac{2}{3}y^{3/2}\Big]_0^2 = k\sqrt{8}\Big(\tfrac{8}{5}\sqrt{2} + \tfrac{4}{3}\sqrt{2}\Big) = k\sqrt{8}\Big(\tfrac{44}{15}\sqrt{2}\Big) = \tfrac{176}{15}k$$

$$\mathrm{M}_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i[k(v_i + 1)]\Delta_i \mathrm{A} = k\iint_R (y^2+y)d\mathrm{A} = k\int_0^2 \int_0^{\sqrt{8y}} (y^2+y)dx\,dy = k\int_0^2 x(y^2+y)\Big]_0^{\sqrt{8y}}$$
$$= k\sqrt{8}\int_0^2 (y^{5/2} + y^{3/2})dy = k\sqrt{8}\Big[\tfrac{2}{7}y^{7/2} + \tfrac{2}{5}y^{5/2}\Big]_0^2 = k\sqrt{8}\Big(\tfrac{16}{7}\sqrt{2} + \tfrac{8}{5}\sqrt{2}\Big) = k\sqrt{8}\Big(\tfrac{136}{35}\sqrt{2}\Big) = \tfrac{544}{35}k$$

$$\mathrm{M}_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} u_i[k(v_i + 1)]\Delta_i \mathrm{A} = k\iint_R x(y+1)d\mathrm{A} = k\int_0^2 \int_0^{\sqrt{8y}} x(y+1)dx\,dy = k\int_0^2 \frac{1}{2}x^2(y+1)\Big]_0^{\sqrt{8y}} dy$$
$$= k\int_0^2 4y(y+1)dy = 4k\Big[\frac{1}{3}y^3 + \frac{1}{2}y^2\Big]_0^2 = 4k\Big(\frac{8}{3} + 2\Big) = \frac{56}{3}k$$

$$\bar{x} = \frac{1}{\mathrm{M}}\mathrm{M}_y = \frac{15}{176k} \cdot \frac{56k}{3} = \frac{35}{22} \text{ and } \bar{y} = \frac{1}{\mathrm{M}}\mathrm{M}_x = \frac{15}{176k} \cdot \frac{544k}{35} = \frac{102}{77}$$

6. R is bounded by the coordinate axes, $y = e^x$, and $x = 1$. $\rho(x, y) = ky$.

$$\mathrm{M} = \int_0^1 \int_0^{e^x} ky\,dy\,dx = \int_0^1 \Big[\tfrac{1}{2}ky^2\Big]_0^{e^x} dx = \int_0^1 \tfrac{1}{2}ke^{2x}dx = \tfrac{1}{4}ke^{2x}\Big]_0^1 = \tfrac{1}{4}k(e^2 - 1)$$

$$\mathrm{M}_y = \int_0^1 \int_0^{e^x} kxy\,dy\,dx = \int_0^1 \tfrac{1}{2}kxe^{2x}dx = \Big[\tfrac{1}{2}k(\tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x})\Big]_0^1 = \tfrac{1}{2}k(\tfrac{1}{4}e^2 + \tfrac{1}{4}).\ \bar{x} = \frac{\frac{1}{8}k(e^2+1)}{\frac{1}{4}k(e^2-1)} = \frac{e^2+1}{2(e^2-1)}$$

$$\mathrm{M}_x = \int_0^1 \int_0^{e^x} ky^2dy\,dx = \int_0^1 \Big[\tfrac{1}{3}ky^3\Big]_0^{e^x} dx = \int_0^1 \tfrac{1}{3}kxe^{3x}dx = \tfrac{1}{9}ke^{3x}\Big]_0^1 = \tfrac{1}{9}k(e^3 - 1).\ \bar{y} = \frac{\frac{1}{9}k(e^3-1)}{\frac{1}{4}k(e^2-1)} = \frac{4(e^3-1)}{9(e^2-1)}$$

7. R is in the first quadrant bounded by $x^2 + y^2 = a^2$. $\rho(x, y) = k(x + y)$.

$$\mathrm{M} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k(u_i + v_i)\Delta_i \mathrm{A} = \iint_R k(x+y)d\mathrm{A} = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x+y)dy\,x = k\int_0^a \Big[xy + \frac{1}{2}y^2\Big]_0^{\sqrt{a^2-x^2}} dx$$
$$= k\int_0^a \Big[x\sqrt{a^2-x^2} + \frac{1}{2}(a^2-x^2)\Big]dx = k\Big[-\frac{1}{3}(a^2-x^2)^{3/2} + \frac{1}{2}a^2x - \frac{1}{6}x^3\Big]_0^a = k\Big[\frac{1}{2}a^3 - \frac{1}{6}a^3 + \frac{1}{3}a^3\Big] = \frac{2}{3}ka^3$$

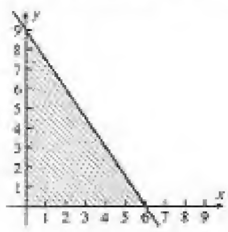
$$\mathrm{M}_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} v_i[k(u_i + v_i)]\Delta_i \mathrm{A} = \iint_R k(xy+y^2)d\mathrm{A} = k\int_0^a \int_0^{\sqrt{a^2-x^2}} (xy+y^2)dy\,dx$$
$$= k\int_0^a \frac{1}{2}xy^2 + \frac{1}{3}y^3\Big]_0^{\sqrt{a^2-x^2}} = k\int_0^a \Big[\frac{1}{2}x(a^2-x^2) + \frac{1}{3}(a^2-x^2)^{3/2}\Big]dx$$

Let $x = a\sin\theta$, $dx = a\cos\theta$ in the second term. From Exercise 13.2.43.

$\mathrm{M}_x = k\left[\frac{1}{4}a^2x^2 - \frac{1}{8}x^4\right]_0^a + \frac{ka^4}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta = k\left(\frac{1}{4}a^4 - \frac{1}{8}a^4\right) + \frac{ka^4}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{1}{16}ka^4(2+\pi)$

$\bar{y} = \frac{1}{\mathrm{M}}\mathrm{M}_x = \frac{3}{2ka^3}\cdot\frac{ka^4(2+\pi)}{16} = \frac{3}{32}a(2+\pi)$. $\bar{x} = \bar{y}$ because of symmetry about the line $y = x$.

8. The region bounded by the triangle whose sides are segments of the coordinate axes and the line $3x + 2y = 18$. The area density varies as the product of the distances from the coordinate axes.



- The figure shows the region R. Let $\rho(x,y) = kxy$. The measure of the mass is

$$\mathrm{M} = \iint_R \rho(x,y)d\mathrm{A} = \int_0^6\int_0^{(-3/2)x+9} kxy\, dy\, dx = \tfrac{1}{2}k\int_0^6 xy^2\Big]_{y=0}^{(-3/2)x+9}dx$$
$$= \tfrac{1}{2}k\int_0^6\left(\tfrac{9}{4}x^3 - 27x^2 + 81x\right)dx = \tfrac{1}{2}k\left[\tfrac{9}{16}x^4 - 9x^3 + \tfrac{81}{2}x^2\right]_0^6 = \tfrac{243}{2}k \qquad (1)$$

Thus, the mass is $\frac{243}{2}k$ kilograms. If $(\bar{x},\bar{y})$ is the center of mass, we have

$$\mathrm{M}_y = \iint_R \rho(x,y)x\, d\mathrm{A} = \int_0^6\int_0^{(-3/2)x+9} kx^2y\, dy\, dx$$
$$= \tfrac{1}{2}k\int_0^6 x^2y^2\Big]_{y=0}^{(-3/2)x+9}dx = \tfrac{1}{2}k\int_0^6\left(\tfrac{9}{4}x^4 - 27x^3 + 81x^2\right)dx$$
$$= \tfrac{1}{2}k\left[\tfrac{9}{20}x^5 - \tfrac{27}{4}x^4 + 27x^3\right]_0^6 = \tfrac{1458}{5}k \qquad (2)$$

and, reversing the order of integration to expand a lower power,

$$\mathrm{M}_x = \iint_R kxy^2d\mathrm{A} = \tfrac{1}{2}k\int_0^9\int_0^{(-2/3)y+6} kxy^2dx\, dy = \tfrac{1}{2}k\int_0^9 y^2x^2\Big]_{x=0}^{(-2/3)y+6}dy$$
$$= \tfrac{1}{2}k\int_0^9\left(\tfrac{4}{9}y^4 - 8y^3 + 36y^2\right)dy = \tfrac{1}{2}k\left[\tfrac{4}{45}y^5 - 2y^4 + 12y^3\right]_0^9 = \tfrac{2187}{5}k \qquad (3)$$

Substituting from (1), (2) and (3), we have

$\bar{x} = \frac{1}{\mathrm{M}}\cdot\mathrm{M}_y = \frac{2}{243k}\cdot\frac{1458k}{5} = \frac{12}{5}$ and $\bar{y} = \frac{1}{\mathrm{M}}\cdot\mathrm{M}_x = \frac{2}{243k}\cdot\frac{2187k}{5} = \frac{18}{5}$

Therefore, the center of mass is $(\frac{12}{5},\frac{18}{5})$.

9. R is bounded by $y = \sin x$ and $[0,\pi]$ on the x axis. $\rho(x,y) = ky$.

$\mathrm{M} = \iint_R ky\, d\mathrm{A} = k\int_0^\pi\int_0^{\sin x} y\, dy\, dx = k\int_0^\pi \frac{1}{2}y^2\Big]_0^{\sin x} dx = \frac{1}{2}k\int_0^\pi \sin^2x\, dx = \frac{1}{2}k\left[-\frac{1}{2}\sin x\cos x + \frac{1}{2}x\right]_0^\pi = \frac{1}{4}k\pi$

$\mathrm{M}_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n v_i(kv_i)\Delta_i\mathrm{A} = \iint_R ky^2d\mathrm{A} = k\int_0^\pi\int_0^{\sin x} y^2dy\, dx = k\int_0^\pi \frac{1}{3}y^3\Big]_0^{\sin x} dx = \frac{1}{3}k\int_0^\pi \sin^3x\, dx$

$= \frac{1}{3}k\int_0^\pi (1-\cos^2x)\sin x\, dx = \frac{1}{3}k\left[-\cos + \frac{1}{3}\cos^3x\right]_0^\pi = \frac{1}{3}k\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right] = \frac{4}{9}k$. $\bar{y} = \frac{1}{\mathrm{M}}\mathrm{M}_x = \frac{4}{k\pi}\cdot\frac{4k}{9} = \frac{16}{9\pi}$.

$\bar{x} = \frac{1}{2}\pi$ because $y = \sin x$ is symmetric with respect to $x = \frac{1}{2}\pi$ and ρ is independent of x.

10. R is bounded by $y = \sqrt{x}$ and $y = x$. $\rho(x,y) = kx$.

$\mathrm{M} = \int_0^1\int_{y^2}^y kx\, dx\, dy = \int_0^1 \frac{1}{2}kx^2\Big]_{y^2}^y dy = \int_0^1 \frac{1}{2}k(y^4 - y^2)dy = \frac{1}{2}k\left[\frac{1}{5}y^5 - \frac{1}{3}y^3\right]_0^1 = \frac{1}{15}k$

$\mathrm{M}_y = \int_0^1\int_{y^2}^y kx^2dx\, dy = \int_0^1 \frac{1}{3}kkx^3\Big]_{y^2}^y dy = \int_0^1 \frac{1}{3}k(y^6 - y^3)dy = \frac{1}{3}k\left[\frac{1}{7}y^7 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{28}k$. $\bar{x} = \frac{15}{28}$

$\mathrm{M}_x = \int_0^1\int_{y^2}^y kxy\, dx\, dy = \int_0^1 \frac{1}{2}k(y^5 - y^3)dy = \frac{1}{2}k\left[\frac{1}{6}y^6 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{24}k$. $\bar{y} = \frac{15}{24} = \frac{5}{8}$

11. R is in the first quadrant bounded by $x^2 + y^2 = 4$ and $x + y = 2$. $\rho(x,y) = xy$.

$\mathrm{M} = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n u_iv_i\Delta_i\mathrm{A} = \iint_R xy\, d\mathrm{A} = \int_0^2\int_{2-x}^{\sqrt{4-x^2}} xy\, dy\, dx = \int_0^2 \frac{1}{2}xy^2\Big]_{2-x}^{\sqrt{4-x^2}}$

$= \frac{1}{2}\int_0^2 [x(4-x^2) - x(2-x)^2]dx = \frac{1}{2}\int_0^2 (4x^2 - 2x^3)dx = \frac{1}{2}\left[\frac{4}{3}x^3 - \frac{1}{2}x^4\right]_0^2 = \frac{1}{2}\left(\frac{32}{3} - 8\right) = \frac{4}{3}$

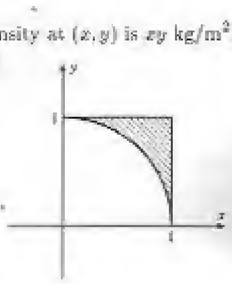
$\mathrm{M}_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n v_i(u_iv_i)\Delta_i\mathrm{A} = \iint_R xy^2d\mathrm{A} = \int_0^2\int_{2-x}^{\sqrt{4-x^2}} xy^2dy\, dx = \int_0^2 \frac{1}{3}xy^3\Big]_{2-x}^{\sqrt{4-x^2}} dx$

$= \frac{1}{3}\int_0^2 [x(4-x^2)^{3/2} - x(2-x)^3]dx = \frac{1}{3}\left[-\frac{1}{5}(4-x^2)^{5/2} + \frac{1}{4}x(2-x)^4 + \frac{1}{20}(2-x)^5\right]_0^2 = \frac{1}{3}\left[\frac{32}{5} + 0 + \frac{32}{20}\right] = \frac{8}{5}$

$\bar{y} = \frac{1}{\mathrm{M}}\mathrm{M}_x = \frac{3}{4}\cdot\frac{8}{5} = \frac{6}{5}$. $\bar{x} = \bar{y}$ because of symmetry with respect to the line $y = x$.

12. The region bounded by the circle $x^2 + y^2 = 1$ and lines $x = 1$, $y = 1$. The area density at (x, y) is xy kg/m^2.

▹ The figure shows the region R. The number of kilograms in the mass is given by



$$M = \iint_R xy\, dA = \int_0^1 \int_{\sqrt{1-x^2}}^1 xy\, dy\, dx = \tfrac{1}{2}\int_0^1 xy^2\Big]_{y=\sqrt{1-x^2}}^1 dx$$
$$= \tfrac{1}{2}\int_0^1 x[1-(1-x^2)]dx = \tfrac{1}{2}\int_0^1 x^3 dx = \tfrac{1}{8}$$

Thus, the mass is $\frac{1}{8}$ kilogram. If $(\bar{x}, \bar{y})$ is the center of mass, then $\bar{y} = \bar{x}$ because both the region and the density function are symmetric in x and y. Thus,

$$M_y = \iint_R x \cdot xy\, dA = \int_0^1 \int_{\sqrt{1-x^2}}^1 x^2 y\, dy\, dx = \tfrac{1}{2}\int_0^1 x^2y^2\Big]_{y=\sqrt{1-x^2}}^1 dx$$
$$= \tfrac{1}{2}\int_0^1 x^4 dx = \tfrac{1}{10}$$

and

$$\bar{x} = \frac{1}{M}\cdot M_y = 8\cdot\tfrac{1}{10} = \tfrac{4}{5}$$

Therefore, the center of mass is $(\frac{4}{5}, \frac{4}{5})$.

In Exercises 13–18, find the moment of inertia of the homogeneous lamina of the given shape about the indicated axis if the area density is k kilograms per square meter and distance is measured in meters.

▹ In Exercises 13–22, I_x kg-m^2 and I_y kg-m^2 are the moments of inertia of the lamina R about the x and y axes, r m is the radius of gyration about the x axis, and I_0 kg-m^2 is the polar moment of inertia.

13. R is bounded by $4y = 3x$, $x = 4$, and the x axis; about the x axis.

$$I_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n v_i^2 \Delta_i A = \iint_R ky^2 dA = k\int_0^4\int_0^{3x/4} y^2 dy\, dx = k\int_0^4 \tfrac{1}{3}y^3\Big]_0^{3x/4} dx = \frac{9k}{64}\int_0^4 x^3 dx = \frac{9k}{64}\Big[\tfrac{1}{4}x^4\Big]_0^4 = 9k$$

14. R is bounded by $4y = 3x$, $x = 4$, and the x axis; about the line $x = 4$.

$$I_{x=4} = k\int_0^4\int_0^{3x/4}(4-x)^2 dy\, dx = \tfrac{3}{4}k\int_0^4 x(4-x)^2 dx = \tfrac{3}{4}k\int_0^4(16x - 8x^2 + x^3)dx = \tfrac{3}{4}k\Big[8x^2 - \tfrac{8}{3}x^3 + \tfrac{1}{4}x^4\Big]_0^4 = 16k$$

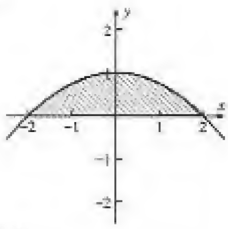
15. R is bounded by $x^2 + y^2 = a^2$. $I_0 = \iint_R k(x^2 + y^2)dA = \iint_R kx^2 dA + \iint_R ky^2 dA = 2\iint_R ky^2 dA$

$$= 8k\int_0^a\int_0^{\sqrt{a^2-x^2}} y^2 dy\, dx = 8k\int_0^a \frac{1}{3}y^3\Big]_0^{\sqrt{a^2-x^2}} dx = \frac{8k}{3}\int_0^a (a^2-x^2)^{3/2}dx.$$ Let $x = a\sin\theta$, $dx = a\cos\theta$.

From Exercise 13.2.43, $I_0 = \dfrac{8ka^4}{3}\displaystyle\int_0^{\pi/2}\cos^4\theta\, d\theta = \frac{8ka^4}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{k\pi a^4}{2}$.

16. The region bounded by the parabola $x^2 = 4 - 4y$ and the x axis; about the x axis.

▹ The figure shows the region R. The number of kg-m^2 in the moment of inertia about the x axis is given by



$$I_x = \iint_R ky^2 dA = k\int_{-2}^2\int_0^{1-x^2/4} y^2 dy\, dx = \tfrac{1}{3}k\int_{-2}^2 y^3\Big]_0^{1-x^2/4} dx$$
$$= \tfrac{1}{3}k\int_{-2}^2\left(1 - \tfrac{3}{4}x^2 + \tfrac{3}{16}x^4 - \tfrac{1}{64}x^6\right)dx = \tfrac{1}{3}k\Big[x - \tfrac{1}{4}x^3 + \tfrac{3}{80}x^5 - \tfrac{1}{448}x^7\Big]_{-2}^2 = \tfrac{64}{105}k$$

Thus, the moment of inertia about the x axis is $\frac{64}{105}k$ kg-m^2.

17. R is bounded by $x^2 = 4 - 4y$ and the x axis.

$$I_0 = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n k(u_i^2 + v_i^2)\Delta_i A = \iint_R k(x^2 + y^2)dA = k\int_{-2}^2\int_0^{(4-x^2)/4}(x^2 + y^2)dy\, dx$$
$$= k\int_{-2}^2\Big[x^2 y + \tfrac{1}{3}y^3\Big]_0^{(4-x^2)/4} dx = k\int_{-2}^2\left[\frac{4x^2 - x^4}{4} + \frac{(4-x^2)^3}{192}\right]dx = 2k\int_0^2\left(x^2 - \frac{x^4}{4} + \frac{1}{3} - \frac{x^2}{4} + \frac{x^4}{16} - \frac{x^6}{192}\right)dx$$
$$= 2k\int_0^2\left(\frac{1}{3} + \frac{3}{4}x^2 - \frac{3}{16}x^4 - \frac{1}{192}x^6\right)dx = 2k\Big[\frac{1}{3}x + \frac{1}{4}x^3 - \frac{3}{5\cdot 16}x^5 - \frac{1}{7\cdot 192}x^7\Big]_0^2 = 2k\left(\frac{2}{3} + 2 - \frac{6}{5} - \frac{2}{21}\right) = \frac{96}{35}k$$

18. A lamina in the shape of a triangle of sides a m, b m, c m, about side a.

▹ Let h m be the altitude. Place side a on the x axis. A y-section has length $\frac{a}{h}(h - y)$. The area of the triangle is $\frac{1}{2}ah = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a + b + c)$. Then

$$I_a = \int_0^h\int ky^2 dx\, dy = \frac{ak}{h}\int_0^h y^2(h-y)dy = \frac{ak}{h}\Big[\tfrac{1}{3}y^3 h - \tfrac{1}{4}y^4\Big]_0^h = \frac{kah^3}{12} = \frac{k}{12a^2}\cdot 8(\tfrac{1}{2}ah)^3 = \frac{2k}{3a^2}[s(s-a)(s-b)(s-c)]^{3/2}$$

In Exercises 19–22, find for the lamina each of the following: (a) the moment of inertia about the x axis; (b) the moment of inertia about the y axis; (c) the radius of gyration about the x axis; (d) the polar moment of inertia.

19. R is bounded by $x = 3$, $y = 2$, and the coordinate axes. $\rho(x, y) = xy^2$.

(a) $I_x = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} v_i^2(u_i v_i^2)\Delta_i A = \iint_R xy^4 dA = \int_0^3 \int_0^2 xy^4 dy\, dx = \int_0^3 \frac{1}{5} xy^5 \Big]_0^2 dx = \frac{32}{5} \int_0^3 x\, dx = \frac{16}{5} x^2 \Big]_0^3 = \frac{144}{5}$

(b) $I_y = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} u_i^2(u_i v_i^2)\Delta_i A = \iint_R x^3 y^2 dy\, dx = \int_0^3 \int_0^2 x^3 y^2 dy\, dx = \int_0^3 \frac{1}{3} x^3 y^3 \Big]_0^2 = \frac{8}{3} \int_0^3 x^3\, dx = \frac{2}{3} x^4 \Big]_0^3 = 54$

(c) From Exercise 1. M = 12. Thus $r = \sqrt{\frac{1}{m} \cdot I_x} = \sqrt{\frac{1}{12} \cdot \frac{144}{5}} = \sqrt{\frac{12}{5}} = \frac{2}{5}\sqrt{15}$.

(d) $I_0 = I_x + I_y = \frac{144}{5} + 54 = \frac{414}{5}$

20. The lamina of Exercise 4.

▹ The figure for Exercise 4 shows the region R. In Exercise 4 we are given that $\rho(x, y) = x + y$.

(a) $I_x = \iint_R (x + y)y^2 dA = \int_0^1 \int_0^{\sqrt{y}} (x + y)y^2 dx\, dy = \int_0^1 y^2 \Big[\frac{1}{2}x^2 + xy\Big]_{x=0}^{\sqrt{y}} dy$
$= \int_0^1 y^2[\frac{1}{2}y + y^{3/2}]dy = \int_0^1 (\frac{1}{2}y^3 + y^{7/2})dy = \frac{1}{8}y^4 + \frac{2}{9}y^{9/2}\Big]_0^1 = \frac{25}{72}$

Thus, the moment of inertia about the x axis is $\frac{25}{72}$ kg-m^2.

(b) $I_y = \iint_R (x + y)x^2 dA = \int_0^1 \int_{x^2}^1 (x + y)x^2 dy\, dx = \int_0^1 x^2 \Big[xy + \frac{1}{2}y^2\Big]_{y=x^2}^1 dx = \int_0^1 x^2[x + \frac{1}{2} - x^3 - \frac{1}{2}x^4]dx$
$= \int_0^1 (\frac{1}{2}x^2 + x^3 - x^5 - \frac{1}{2}x^6)dx = \frac{1}{6}x^3 + \frac{1}{4}x^4 - \frac{1}{6}x^6 - \frac{1}{14}x^7\Big]_0^1 = \frac{5}{28}$

Thus, the moment of inertia about the y axis is $\frac{5}{28}$kg-m^2.

(c) Substituting from parts (a) and (b), we have

$r^2 = \frac{1}{M} I_x = \frac{20}{13} \cdot \frac{25}{72} = \frac{125}{234}$

$r = \frac{5}{78}\sqrt{130}$

Thus, the radius of gyration about the x axis is $\frac{5}{78}\sqrt{130}$ meters.

(d) $I_0 = I_x + I_y = \frac{25}{72} + \frac{5}{28} = \frac{265}{504}$

Thus, the polar moment of inertia is $\frac{265}{504}$ kg-m^2.

21. R is bounded by $y = \sin x$ and $[0, \pi]$ on the x axis. $\rho(x, y) = ky$.

(a) $I_x = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} v_i^2(kv_i)\Delta_i A = \iint_R ky^3 dA = k \int_0^\pi \int_0^{\sin x} y^3 dy\, dx = k \int_0^\pi \frac{1}{4} y^4 \Big]_0^{\sin x} dx = \frac{1}{2} \int_0^\pi \sin^4 x\, dx$
$= \frac{1}{4}k\Big[-\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\Big(-\frac{1}{2}\sin x \cos x + \frac{1}{2}x\Big)\Big]_0^\pi = \frac{1}{4}k \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \pi = \frac{3}{32}\pi k$

(b) $I_y = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} u_i^2(kv_i)\Delta_i A = \iint_R kx^2 y dA = k \int_0^\pi \int_0^{\sin x} x^2 y\, dy\, dx = k \int_0^\pi \frac{1}{2}x^2 y^2 \Big]_0^{\sin x} dx = \frac{1}{2}k \int_0^\pi x^2 \sin^2 x\, dx$
$= \frac{1}{4}k \int_0^\pi (x^2 - x^2 \cos 2x)dx = \frac{1}{4}k\Big[\frac{1}{3}x^3 - \frac{1}{2}x^2 \sin 2x - \frac{1}{2}x \cos 2x + \frac{1}{4}\sin 2x\Big]_0^\pi = \frac{1}{4}k\Big(\frac{1}{3}\pi^3 - \frac{1}{2}\pi\Big)$

(c) From Exercise 9, $M = \frac{1}{4}\pi k$. Therefore $r = \sqrt{\frac{1}{M} \cdot I_x} = \sqrt{\frac{4}{\pi k} \cdot \frac{3\pi k}{32}} = \frac{1}{4}\sqrt{6}$.

(d) $I_0 = I_x + I_y = \frac{3}{32}\pi k + \frac{1}{12}\pi^3 k - \frac{1}{8}\pi k = (\frac{1}{12}\pi^3 - \frac{1}{32}\pi)k$

22. R is bounded by $y = \sqrt{x}$ and $y = x$. $\rho(x, y) = kx$.

(a) $I_x = \int_0^1 \int_{y^2}^{y} kxy^2 dx\, dy = \frac{1}{2}k \int_0^1 x^2 y^2 \Big]_{x=y^2}^{y} dy = \frac{1}{2}k \int_0^1 y^2(y^2 - y^4)dy = \frac{1}{2}k\Big[\frac{1}{5}y^5 - \frac{1}{7}y^7\Big]_0^1 = \frac{1}{35}k$

(b) $I_y = \int_0^1 \int_{y^2}^{y} kx^3 dx\, dy = \frac{1}{4}k \int_0^1 x^4 \Big]_{y^2}^{y} dy = \frac{1}{4}k \int_0^1 (y^4 - y^8)dy = \frac{1}{4}k\Big[\frac{1}{5}y^5 - \frac{1}{9}y^9\Big]_0^1 = \frac{1}{45}k$

(c) From Exercise 10, $M = \frac{1}{15}k$. Therefore $r = \sqrt{\frac{1}{M} \cdot I_x} = \sqrt{\frac{15}{35}} = \frac{1}{7}\sqrt{21}$

(d) $I_0 = I_x + I_y = \frac{1}{35}k + \frac{1}{45}k = \frac{16}{315}k$

23. R is bounded by $y = 2x - x^2$ and the x axis. $\rho(x, y) = k(4 - y)$. If I kg-m^2 is the moment of inertia of the lamina about the line $y = 4$.

$I = \lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} (4 - v_i)^2[k(4 - y)]\Delta_i A = k \iint_R (4 - y)^3 dA = k \int_0^2 \int_0^{2x - x^2} (4 - y)^3 dy\, dx$

$$= k\int_0^2 -\tfrac{1}{4}(4-y)^4\Big]_0^{2x-x^2} dx = -\tfrac{1}{4}k\int_0^2 [(4-2x+x^2)^4 - 256]dx = -\tfrac{1}{4}k\int_0^2 \{[3+(x-1)^2]^4 - 256\}dx$$

Let $w = x - 1$, then $dw = dx$. Thus

$$\mathrm{I} = -\tfrac{1}{4}k\int_0^1 [(w^2+3)^4 - 256]dw = -\tfrac{1}{2}k\int_0^1 (w^8 + 12w^6 + 54w^4 + 108w^2 + 81 - 256)dw$$

$$= -\tfrac{1}{2}k\Big[\tfrac{1}{9}w^9 + \tfrac{12}{7}w^7 + \tfrac{54}{5}w^5 + 36w^3 - 175w\Big]_0^1 = -\tfrac{1}{2}k\Big(\tfrac{1}{9} + \tfrac{12}{7} + \tfrac{54}{5} + 36 - 175\Big) = \frac{19{,}904}{315}k$$

24. A homogeneous lamina of area density k slugs/ft^2 is in the shape of the region bounded by the curve $x = \sqrt{y}$, the x axis, and the line $x = a$, where $a > 0$. Find the moment of inertia of the lamina about the line $x = a$.

▹ The figure shows the region R. $r = a - x$ is the distance from any point (x, y) to the axis $x = a$. The measure of the moment of inertia of the lamina about the line $x = a$ is

$$\mathrm{I} = \iint_R k(a-x)^2 dA = \int_0^a\int_0^{x^2} k(a-x)^2 dy\, dx$$

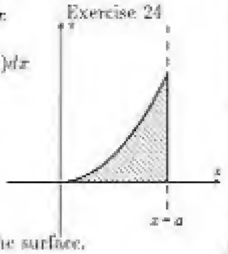
$$= \int_0^a k(a-x)^2 x^2 dx = k\int_0^a (a^2x^2 - 2ax^3 + x^4)dx$$

$$= k\Big[\tfrac{1}{3}a^2x^3 - \tfrac{1}{2}ax^4 + \tfrac{1}{5}x^5\Big]_0^a$$

$$= k(\tfrac{1}{3}a^5 - \tfrac{1}{2}a^5 + \tfrac{1}{5}a^5) = \tfrac{1}{30}ka^5$$

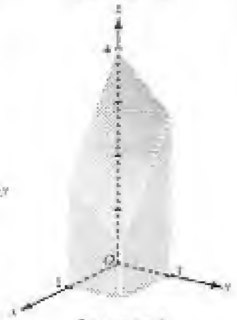
The moment of inertia is $\frac{1}{30}ka^5$ slug-ft^2.

Exercise 24



Exercise 26

Exercise 25



In Exercises 25–42, σ square units is the area of the surface.

25. $z = f(x,y) = 4 - 2x - y$; $f_x(x,y) = -2$; $f_y(x,y) = -1$.

$$\sigma = \iint_R \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\, dx\, dy$$

$$= \int_0^1\int_0^1 \sqrt{4+1+1}\, dx\, dy = \sqrt{6}\int_0^1 x\Big]_0^1 dy$$

$$= \sqrt{6}\int_0^1 dy = \sqrt{6}$$

26. $z = f(x,y) = 5 + 2x + y$; $f_x(x,y) = 2$; $f_y(x,y) = 1$.

$$\sigma = \iint_R \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\, dx\, dy = \int_0^4\int_0^2 \sqrt{4+1+1}\, dx\, dy = 8\sqrt{6}$$

27. The region R in the xy plane is bounded by the triangle with sides $x = 0$, $y = 0$, and $y = -\frac{9}{4}x + 9$.

$z = f(x,y) = 16 - 4x - \frac{16}{9}y$; $f_x(x,y) = -4$; $f_y(x,y) = -\frac{16}{9}$.

$$\sigma = \iint_R \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\, dy\, dx = \int_0^4\int_0^{-9x/4+9} \sqrt{16 + \tfrac{256}{81} + 1}\, dy\, dx = \tfrac{1}{9}\sqrt{1633}\int_0^4 (-\tfrac{9}{4}x + 9)dx$$

$$= \tfrac{1}{9}\sqrt{1633}\Big[-\tfrac{9}{8}x^2 + 9x\Big]_0^4 = \sqrt{1633}(-2+4) = 2\sqrt{1633}$$

28. Find the area of the surface that is cut from the plane $z = ax + by$ by the planes $x = 0$, $x = a$, $y = 0$, and $y = b$, where $a > 0$ and $b > 0$.

▹ Let $f(x,y) = ax + by$. R is the region in the xy plane bounded by lines $x = 0$, $x = a$, $y = 0$, and $y = b$. Then $f_x(x,y) = a$ and $f_y(x,y) = b$

Therefore the number of square units in the area of the surface is

$$\sigma = \iint_R \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\, dx\, dy = \int_0^b\int_0^a \sqrt{a^2+b^2+1}\, dx\, dy = \int_0^b a\sqrt{a^2+b^2+1}\, dy$$

$$= ab\sqrt{a^2+b^2+1}$$

The area of the surface is $ab\sqrt{a^2+b^2+1}$ square units.

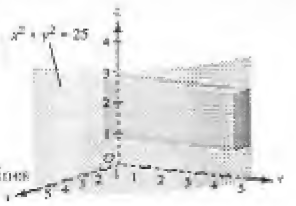
▹ In Exercises 29 and 30, the cylinder is not the graph of a function of x and y.

29. In the first octant $y \geq 0$; $x^2 + y^2 = 9$ gives $y = \sqrt{9 - x^2}$; $\dfrac{\partial y}{\partial x} = -\dfrac{x}{\sqrt{9-x^2}}$;

$$\frac{\partial z}{\partial x} = 0.\ \sigma = \iint_R \sqrt{y_x^2 + y_z^2 + 1}\, dz\, dx = \int_0^3\int_0^x \sqrt{\frac{x^2}{9-x^2} + 0 + 1}\, dz\, dx$$

$$= \int_0^3\int_0^x 3(9-x^2)^{-1/2} dz\, dx = 3\lim_{b\to 3^-}\int_0^b x(9-x^2)^{-1/2} dx$$

$$= 3\lim_{b\to 3^-}\Big[-(9-x^2)^{1/2}\Big]_0^b = 3\lim_{b\to 3^-}[-(9-b^2)^{1/2} + 3] = 9$$



30. In the first octant $y = f(x,z) = \sqrt{25-x^2}$. $y_x = -\dfrac{x}{\sqrt{25-x^2}}$; $y_z = 0$

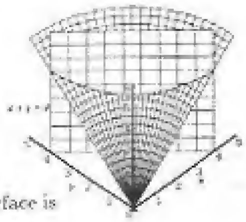
$$\sigma = \iint_R \sqrt{{y_x}^2 + {y_z}^2 + 1}\,dA = \int_0^1 \int_1^3 \sqrt{\frac{x^2}{25-x^2} + 0 + 1}\,dz\,dx$$
$$= 2\int_0^1 \frac{5dx}{\sqrt{25-x^2}} = 10\sin^{-1}\frac{x}{5}\Big]_0^1 = 10\sin^{-1}\tfrac{1}{5}$$

31. The region R in the xy plane is bounded by the triangle whose sides are the lines $y = 2x$, $x = 0$, and $y = 4$. $z = f(x,y) = 2 + 5x + y^2$; $f_x = 5$; $f_y(x,y) = 2y$.

$$\sigma = \iint_R \sqrt{{f_x}^2(x,y) + {f_y}^2(x,y) + 1}\,dx\,dy = \int_0^4 \int_0^{y/2} \sqrt{25 + 4y^2 + 1}\,dx\,dy = \int_0^4 x\sqrt{4y^2+26}\Big]_0^{y/2} dy$$
$$= \frac{1}{2}\int_0^4 y(4y^2+26)^{1/2}dy = \frac{1}{24}(4y^2+26)^{3/2}\Big]_0^4 = \frac{1}{24}(90\sqrt{90} - 26\sqrt{26}) = \frac{1}{12}(135\sqrt{10} - 13\sqrt{26})$$

32. Find the area of the surface in the first octant which is cut from the cone $x^2 + y^2 = z^2$ by the plane $x + y = 4$.

▸ The figure shows the part of the surface that lies over the triangle R in the xy plane bounded by the x axis, the y axis, and the line $x + y = 4$, viewed from behind the plane. We take



$f(x,y) = \sqrt{x^2+y^2}$

Thus,

$f_x(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$ and $f_y(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$

${f_x}^2(x,y) + {f_y}^2(x,y) + 1 = \dfrac{x^2}{x^2+y^2} + \dfrac{y^2}{x^2+y^2} + 1 = 2$

The area of the triangle is $\frac{1}{2}(4)(4) = 8$ square units. Hence, the measure of the surface is

$$\sigma = \iint_R \sqrt{2}\,dA = 8\sqrt{2}$$

Thus, the area of the surface is $8\sqrt{2}$ square units.

33. The required area is 8 times the part above the first quadrant.

$z = f(x,y) = \sqrt{4-x^2}$; $f_x(x,y) = \dfrac{-x}{\sqrt{4-x^2}}$; $f_y(x,y) = 0$. $\sigma = 8\iint_R \sqrt{{f_x}^2(x,y) + {f_y}^2(x,y) + 1}\,dA$

$$= 8\iint_R \sqrt{\frac{x^2}{4-x^2} + 0 + 1}\,dA = 16\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{\sqrt{4-x^2}}\,dy\,dx = 16\int_0^2 dx = 32$$

34. $x^2 + y^2 = 2x$ is $(x-1)^2 + y^2 = 1$. The required area is 4 times the part above the first quadrant.

$f(x,y) = \sqrt{x^2+y^2}$ so ${f_x}^2 + {f_y}^2 + 1 = 2$. Area of the semicircle is $\frac{1}{2}\pi$. $\sigma = 4\iint_R \sqrt{2}\,dA = 4\sqrt{2}\cdot\frac{1}{2}\pi = 2\sqrt{2}\pi$

35. $z = f(x,y) = \sqrt{x^2+y^2}$ so ${f_x}^2 + {f_y}^2 + 1 = 2$. $\sigma = 2\iint_R \sqrt{2}\,dA = 2\int_{-1}^2 \int_{y^2}^{2+y} \sqrt{2}\,dx\,dy$

$$= 2\sqrt{2}\int_{-1}^2 (2 + y - y^2)dy = 2\sqrt{2}\Big[2y + \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\Big]_{-1}^2 = 2\sqrt{2}\Big[\Big(4 + 2 - \tfrac{8}{3}\Big) - \Big(-2 + \tfrac{1}{2} + \tfrac{1}{3}\Big)\Big] = 9\sqrt{2}$$

36. Find the area of the portion of the plane $x = z$ that lies between the planes $y = 0$ and $y = 6$ and within the hyperboloid $9x^2 - 4y^2 + 16z^2 = 144$.

▸ The projection of the intersection of $x = z$ and $9x^2 - 4y^2 + 16z^2 = 144$ on the xy plane is the hyperbola $25x^2 - 4y^2 = 144$. We take twice the part over the first quadrant where $x = \frac{2}{5}\sqrt{y^2+36}$. Let $z = f(x,y) = x$. Then

$f_x(x,y) = 1 \quad f_y(x,y) = 0$

Therefore,

$$\sigma = \iint_R \sqrt{{f_x}^2(x,y) + {f_y}^2(x,y) + 1}\,dA = 2\int_0^6 \int_0^{2\sqrt{y^2+36}/5} \sqrt{1^2 + 0 + 1}\,dx\,dy = 2\sqrt{2}\int_0^6 x\Big]_0^{2\sqrt{y^2+36}/5} dy$$
$$= \tfrac{4}{5}\sqrt{2}\int_0^6 \sqrt{y^2+36}\,dy = \tfrac{2}{5}\sqrt{2}\Big[y\sqrt{36+y^2} + 36\ln(y + \sqrt{36+y^2})\Big]_0^6 = \tfrac{2}{5}\sqrt{2}[36\sqrt{2} + 36\ln(6 + 6\sqrt{2}) - 36\ln 6]$$
$$= \tfrac{72}{5}[2 + \sqrt{2}\ln(1 + \sqrt{2})]$$

Thus, the area of the surface is $\frac{72}{5}[2 + \sqrt{2}\ln(1 + \sqrt{2})]$ square units.

37. The line from (0,0) to (a,b) has equation $y = f(x) = \frac{b}{a}x$; $f'(x) = \frac{b}{a}$. From Theorem 13.3.4,

$$\sigma = 2\pi \int_0^a f(x)\sqrt{[f'(x)]^2 + 1}\, dx = 2\pi \int_0^a \frac{b}{a}x\sqrt{\frac{b^2}{a^2} + 1}\, dx = \frac{2\pi b\sqrt{a^2+b^2}}{a^2}\int_0^a x\, dx = \frac{2\pi b\sqrt{a^2+b^2}}{a^2}\cdot\frac{x^2}{2}\Big]_0^a = \pi b\sqrt{a^2+b^2}$$

Or, by Pappus, segment has length $\sqrt{a^2+b^2}$ and the midpoint is $\frac{1}{2}b$ from the x axis, so $\sigma = 2\pi\cdot\frac{1}{2}b\cdot\sqrt{a^2+b^2}$.

38. We revolve the circle $x^2+y^2=r^2$ about the x axis. $y = f(x) = \sqrt{r^2-x^2}$, $f'(x) = -x/\sqrt{r^2-x^2}$.

$$\sigma = 2\pi\int_a^b f(x)\sqrt{[f'(x)]^2+1}\,dx = 2\pi\int_a^b \sqrt{r^2-x^2}\sqrt{\frac{x^2}{r^2-x^2}+1}\,dx = 2\pi\int_a^b r\,dx = 2\pi r(b-a) = 2\pi rh$$

If $h = 2r$, we find the area of the entire sphere is $4\pi r^2$.

39. Let σ_x and σ_y denote the measure of the surface when a curve is revolved about the x and y axes. Reversing x and y, Theorem 13.3.2 gives $\sigma_x = 2\pi\int_a^b y\text{ ds}$. Also, $\sigma_y = \int_c^d x\text{ ds}$. If $y = a\cosh\frac{x}{a}$,

$$\sigma_y = 2\pi\int_0^a x\sqrt{1+y'^2}\,dx = 2\pi\int_0^a x\sqrt{1+\sinh^2\frac{x}{a}}\,dx = 2\pi\int_0^a x\cosh\frac{x}{a}\,dx = 2\pi\Big[ax\sinh\frac{x}{a} - a^2\cosh\frac{x}{a}\Big]_0^a$$
$$= 2\pi(a^2\sinh 1 - a^2\cos 1 + a^2) = 2\pi a^2(1-e^{-1})$$

40. Find the area of the surface of revolution obtained by revolving the catenary $y = a\cosh(x/a)$ from $x=0$ to $x = a$ about the x axis.

▷ We take

$f(x) = a\cosh(x/a)$

$f'(x) = \sinh(x/a)$

$\sqrt{[f'(x)]^2+1} = \sqrt{\sinh^2(x/a)+1} = \sqrt{\cosh^2(x/a)} = \cosh(x/a)$

Therefore, the measure of the surface area is

$$\sigma = 2\pi\int_0^a a\cosh(x/a)\cdot\cosh(x/a)\,dx = \pi a\int_0^a[1+\cosh(2x/a)]dx$$
$$= \pi a\Big[x + \tfrac{1}{2}a\sinh(2x/a)\Big]_0^a = \pi a(1+\tfrac{1}{2}a\sinh 2)$$

Hence the surface area is $\pi a(1+\frac{1}{2}a\sinh 2)$ square units.

41. Taking $y \geq 0$. $18y^2 = x(6-x)^2$ gives $y = f(x) = \frac{1}{6}\sqrt{2}(6x^{1/2}-x^{3/2})$; $f'(x) = \frac{1}{6}\sqrt{2}(3x^{-1/2}-\frac{3}{2}x^{1/2})$.

$$\sigma = 2\pi\int_0^6 f(x)\sqrt{[f'(x)]^2+1}\,dx = 2\pi\int_0^6 \tfrac{1}{6}\sqrt{2}(6x^{1/2}-x^{3/2})\sqrt{1+\tfrac{1}{18}(3x^{-1/2}-\tfrac{3}{2}x^{1/2})^2}\,dx$$
$$= \tfrac{\pi}{3}\sqrt{2}\int_0^6 x^{1/2}(6-x)\sqrt{1+\tfrac{1}{18}(9x^{-1}-9+\tfrac{9}{4}x)}\,dx = \tfrac{\pi}{3}\sqrt{2}\pi\int_0^6 x^{1/2}(6-x)\sqrt{\tfrac{1}{2}(x^{-1}+1+\tfrac{1}{4}x)}dx$$
$$= \tfrac{\pi}{3}\sqrt{2}\int_0^6 x^{1/2}(6-x)\tfrac{1}{2}\sqrt{2}\sqrt{(x^{-1/2}+\tfrac{1}{2}x^{1/2})^2}dx = \frac{\pi}{3}\int_0^6 x^{1/2}(x^{-1/2}+\tfrac{1}{2}x^{1/2})dx = \frac{\pi}{3}\int_0^6(6+2x-\tfrac{1}{2}x^2)dx$$
$$= \tfrac{\pi}{3}\Big[6x+x^2-\tfrac{1}{6}x^3\Big]_0^6 = \tfrac{\pi}{3}(36+36-36) = 12\pi$$

42. $y = \ln x$, $x = e^y$. $\sigma = 2\pi\int_0^{\ln 2} x\sqrt{1+(dx/dy)^2}dy = 2\pi\int_0^{\ln 2}\sqrt{1+e^{2y}}e^y dy$. Let $e^y = \tan u$, $e^y dy = \sec^2 u\, du$.

$\tan b = 2$, $\sec b = \sqrt{5}$. $\sigma = 2\pi\int_{\pi/4}^b \sec u(\sec^2 u\, du) = \pi(\sec u\tan u + \ln|\sec u + \tan u|)\Big]_{\pi/4}^b$

$= \pi[2\sqrt{5} - \sqrt{2} + \ln(\sqrt{5}+2) - \ln(\sqrt{2}+1)]$

43. Choose the coordinate axes so that the vertices of the triangle are at $(h, -\frac{1}{2}b)$, $(h, \frac{1}{2}b)$ and (0,0). An equation of the side of the triangle in the first quadrant is $y = bx/2h$.

$$I_x = k\iint_R y^2 dA = 2k\int_0^h\int_0^{bx/2h} y^2 dy\, dx = 2k\int_0^h \frac{1}{3}y^3\Big]_0^{bx/2h}dx = \frac{kb^3}{12h^3}\int x^3 dx = \frac{kb^3}{12h^3}\Big[\frac{1}{4}h^4\Big]_0^h = \frac{1}{48}khb^3.$$

$$M = k\iint_R dA = 2k\int_0^h\int_0^{bx/2h} dy\, dx = 2k\int_0^h \frac{bx}{2h}dx = 2k\Big[\frac{bx^2}{4h}\Big]_0^h = \frac{1}{2}khb.$$

The x axis is the axis of symmetry and $r = \sqrt{\frac{1}{M}\cdot I_x} = \sqrt{\frac{2}{kbh}\cdot\frac{khb^3}{48}} = \sqrt{\frac{6b^2}{144}} = \frac{1}{12}b\sqrt{6}$.

44. Suppose that f and its first partial derivatives are continuous on the closed region R in the xy plane. Show that if σ square units is the area of the portion of of the surface $z = f(x, y)$ that lies over R, then $\sigma = \iint_R \|\nabla g(x,y,z)\| dx\, dy$ where $g(x,y,z) = z - f(x,y)$.

▸ We apply the definition of gradient and norm to the right side of the required equation. Thus,

$$\iint_R \|\nabla g(x,y,z)\| dx\, dy = \iint_R \|(f_x(x,y), f_y(x,y), -1)\| dx\, dy = \iint_R \sqrt{f_x^{\,2}(x,y) + f_y^{\,2}(x,y) + 1}\, dx\, dy = \sigma$$

More generally, if a surface is given by $g(x,y,z) = 0$ and its projection on the xy plane is R, then

$$\sigma = \iint_R \sqrt{z_x^{\,2} + z_y^{\,2} + 1}\, dx\, dy = \iint_R \sqrt{\left(-\frac{g_x}{g_z}\right)^2 + \left(-\frac{g_y}{g_z}\right)^2 + 1}\, dx\, dy = \iint_R \frac{\sqrt{g_x^{\,2} + g_y^{\,2} + g_z^{\,2}}}{|g_z|} dx\, dy$$

$$= \iint_R \frac{\|\nabla g\|}{|g_z|} dx\, dy$$

13.4 THE DOUBLE INTEGRAL IN POLAR COORDINATES

Let f be a function of two variables r and θ. Let R be a region in the polar plane bounded by the rays $\theta = \theta_1$, $\theta = \theta_2$, and by the curves $r = g_1(\theta)$ and $r = g_2(\theta)$ where $0 \le g_1(\theta) \le g_2(\theta)$. Then the double integral of f on R is equivalent to an iterated integral, and

$$\iint_R f(r,\theta) dA = \iint_R f(r,\theta) r\, dr\, d\theta = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta) r\, dr\, d\theta \tag{I}$$

Double integrals in polar coordinates are used to calculate areas, mass, volume, moments, and centers of mass, just as were double integrals in Cartesian coordinates. For example, if S is a solid that is bounded below by some region R in the polar plane, bounded above by the surface $z = f(r,\theta)$, where $f(r,\theta) \ge 0$, and bounded on the sides by a cylindrical surface whose rulings are perpendicular to the polar plane, then the measure of the volume of S is given by (I) and the surface area is given by

$$\sigma = \iint_R \sqrt{f_r^{\,2}(r,\theta) + r^{-2} f_\theta^{\,2}(r,\theta) + 1}\; r\, dr\, d\theta \tag{II}$$

Proof Substituting from Exercise 12.5.51 into Theorem 13.3.4, we obtain

$$\sigma = \iint_R \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\, dx\, dy$$

$$= \iint_R \sqrt{(\cos\theta\, f_r - r^{-1}\sin\theta\, f_\theta)^2 + (\sin\theta\, f_r + r^{-1}\cos\theta\, f_\theta)^2}\; r\, dr\, d\theta$$

which is equivalent to (II).

Exercises 13.4

In Exercises 1–6, use double integrals to find the area of the region.

▸ A square units is the required area and V cubic units is the required volume. M kg is the mass of the given lamina and M_x kg-m and M_y kg-m are its moments about the x and y axes. I_0 kg-m^2 is the polar moment of inertia, r m is the required radius of gyration. $\rho(r,\theta)$ kg/m^2 is the area density at the point (r,θ), and k is a constant of proportionality.

1. The region is inside the cardioid $r = 2(1 + \sin\theta)$.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \bar{r}_i \Delta_i r\, \Delta_i \theta = \iint_R r\, dr\, d\theta = 2\int_{-\pi/2}^{\pi/2} \int_0^{2(1+\sin\theta)} r\, dr\, d\theta = 2\int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 \Big]_0^{2(1+\sin\theta)} d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} (1 + 2\sin\theta + \sin^2\theta) d\theta = 4\Big[\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\Big]_{-\pi/2}^{\pi/2} = 4 \cdot \frac{3}{2} \cdot \Big[\frac{\pi}{2} - \Big(-\frac{\pi}{2}\Big)\Big] = 6\pi$$

2. One leaf of the rose $r = a\cos 2\theta$

$$A = 2\int_0^{\pi/4} \int_0^{a\cos 2\theta} r\, dr\, d\theta = \int_0^{\pi/4} a^2\cos^2 2\theta\, d\theta = \tfrac{1}{2}a^2 \int_0^{\pi/4} (1 + \cos 4\theta) d\theta = \tfrac{1}{2}a^2\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4} = \tfrac{1}{8}a^2\pi$$

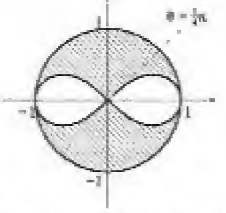
3. The region is inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} r_i \Delta_i r\, \Delta_i \theta = \iint_R r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{a(1+\cos\theta)} r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2 \Big]_a^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (a^2 + 2a^2\cos\theta + a^2\cos^2\theta - a^2) d\theta = \frac{1}{2}a^2 \int_{-\pi/2}^{\pi/2} (2\cos\theta + \cos^2\theta) d\theta$$

$$= \frac{1}{2}a^2\Big[2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\Big]_{-\pi/2}^{\pi/2} = \frac{1}{2}a^2\Big[\Big(2 + 0 + \frac{1}{2}\cdot\frac{\pi}{2}\Big) - \Big(-2 + 0 - \frac{1}{2}\cdot\frac{\pi}{2}\Big)\Big] = \frac{1}{2}a^2\Big(4 + \frac{1}{2}\pi\Big) = \frac{1}{4}a^2(8 + \pi)$$

4. The region inside the circle $r = 1$ and outside the lemniscate $r^2 = \cos 2\theta$.



▷ The figure shows the region. Let R be the first-quadrant part of the region inside the lemniscate. We take the area of the circle minus four times the area of R. Because $\cos 2\theta = 0$ when $\theta = \frac{1}{4}\pi$, the region R is bounded by the rays $\theta = 0$ and $\theta = \frac{1}{4}\pi$. Furthermore, R is bounded by the curves $r = 0$ and $r = \sqrt{\cos 2\theta}$ for $0 \le \theta \le \frac{1}{4}\pi$. Thus, the measure of the area of R is given by

$$A = \iint_R r\,dr\,d\theta = \int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/4} r^2\Big]_0^{\sqrt{\cos 2\theta}} d\theta$$
$$= \frac{1}{2}\int_0^{\pi/4}\cos 2\theta\,d\theta = \frac{1}{4}\sin 2\theta\Big]_0^{\pi/4} = \frac{1}{4}$$

Hence, $4A = 1$. Because the area of the circle is π square units, then the area of the region outside the lemniscate and inside the circle is $(\pi - 1)$ square units.

5. A_1 and A_2 square units are the areas of the large and small loops of $r = 2 - 4\sin\theta$.

$$A_1 = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} r_i\Delta_i r\,\Delta_i\theta = \iint_R r\,dr\,d\theta = 2\int_{-\pi/2}^{\pi/6}\int_0^{2-4\sin\theta} r\,dr\,d\theta = 2\int_{-\pi/2}^{\pi/6}\frac{1}{2}r^2\Big]_0^{2-4\sin\theta}$$
$$= \int_{-\pi/2}^{\pi/6}(2 - 4\sin\theta)^2 d\theta = 4\int_{-\pi/2}^{\pi/6}(1 - 4\sin\theta + 4\sin^2\theta)d\theta = 4\Big[\theta + 4\cos\theta + 2\theta - \sin 2\theta\Big]_{-\pi/2}^{\pi/6}$$
$$= 4\Big[\Big(\frac{\pi}{6} + 2\sqrt{3} + \frac{\pi}{2} - \frac{1}{2}\sqrt{3}\Big) - \Big(-\frac{\pi}{2} + 0 - \pi + 0\Big)\Big] = 4\Big(2\pi + \frac{3}{2}\sqrt{3}\Big) = 8\pi + 6\sqrt{3}$$
$$A_2 = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} r_i\Delta_i r\,\Delta_i\theta = \iint_R r\,dr\,d\theta = 2\int_{\pi/6}^{\pi/2}\int_0^{4\sin\theta - 2} r\,dr\,d\theta = 2\int_{\pi/6}^{\pi/2}\frac{1}{2}r^2\Big]_0^{4\sin\theta - 2}$$
$$= \int_{\pi/6}^{\pi/2}(2 - 4\sin\theta)^2 d\theta = 4\Big[\theta + 4\cos\theta + 2\theta - \sin 2\theta\Big]_{\pi/6}^{\pi/2} = 4\Big[(\tfrac{1}{2}\pi + 0 + \pi - 0) - (\tfrac{1}{6}\pi + 2\sqrt{3} + \tfrac{1}{3}\pi - \tfrac{1}{2}\sqrt{3})\Big]$$
$$= 4(\pi - \tfrac{3}{2}\sqrt{3}) = 4\pi - 6\sqrt{3}.$$ Therefore $A = A_1 - A_2 = (8\pi + 6\sqrt{3}) - (4\pi - 6\sqrt{3}) = 4\pi + 12\sqrt{3}$.

6. The limaçon $r = 3 - \cos\theta$ meets the circle $r = 5\cos\theta$ when $3 - \cos\theta = 5\cos\theta$, $\cos\theta = \frac{1}{2}$, $\theta = \pm\frac{1}{3}\pi$.

$$A = 2\Big[\int_{\pi/3}^{\pi}\int_0^{3-\cos\theta} r\,dr\,d\theta - \text{segment of circle}\Big] = 2\Big[\frac{1}{2}\int_{\pi/3}^{\pi}(3 - \cos\theta)^2 d\theta - [\tfrac{1}{3}\pi(\tfrac{5}{2})^2 - \tfrac{1}{4}\sqrt{3}(\tfrac{5}{2})^2]\Big]$$
$$= \int_{\pi/3}^{\pi}[9 - 6\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)]d\theta - \tfrac{25}{24}\pi + \tfrac{25}{16}\sqrt{3} = \Big[\tfrac{19}{2}\theta - 6\sin\theta + \tfrac{1}{4}\sin 2\theta\Big]_{\pi/3}^{\pi} - \tfrac{25}{12}\pi + \tfrac{25}{8}\sqrt{3}$$
$$= \tfrac{19}{2}\cdot\tfrac{2}{3}\pi + 6\cdot\tfrac{1}{2}\sqrt{3} - \tfrac{1}{4}\cdot\tfrac{1}{2}\sqrt{3} - \tfrac{25}{12}\pi + \tfrac{25}{8}\sqrt{3} = \tfrac{17}{4}\pi + 6\sqrt{3}$$

In Exercises 7–12, find the volume of the solid.

7. The solid is bounded by the ellipsoid $z^2 + 9r^2 = 9$.

$$V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 3(1 - r_i^2)^{1/2} r_i\Delta_i r\,\Delta_i\theta = 3\iint_R r(1 - r^2)^{1/2} dr\,d\theta = 24\int_0^{\pi/2}\int_0^1 r(1 - r^2)^{1/2} dr\,d\theta$$
$$= 24\int_0^{\pi/2} -\tfrac{1}{3}(1 - r^2)^{3/2}\Big]_0^1 d\theta = 8\int_0^{\pi/2} d\theta = 8\theta\Big]_0^{\pi/2} = 4\pi$$

8. The solid cut out of the sphere $z^2 + r^2 = 4$ by the cylinder $r = 1$.

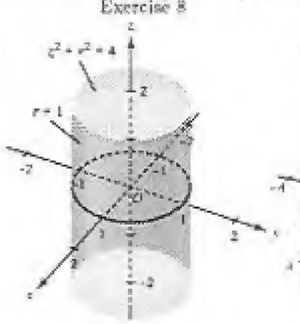
▷ The required volume is twice that of the part above the polar plane. See the figure. R is bounded by the circle $r = 1$. Solving the equation of the sphere for the positive value of z, we obtain $z = f(r, \theta) = \sqrt{4 - r^2}$. If V is the measure of the volume of the entire solid, we have V =

$$2\iint_R f(r,\theta)r\,dr\,d\theta = 2\int_0^{2\pi}\int_0^1\sqrt{4 - r^2}\,r\,dr\,d\theta = -\tfrac{2}{3}\int_0^{2\pi}\Big[(4 - r^2)^{3/2}\Big]_0^1 d\theta = -\tfrac{2}{3}\int_0^{2\pi}(3^{3/2} - 8)d\theta = \tfrac{4}{3}\pi(8 - 3\sqrt{3})$$

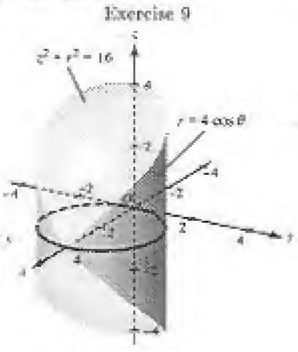
Therefore, the volume of the solid is $\frac{4}{3}\pi(8 - 3\sqrt{3})$ cubic units.

Exercise 7

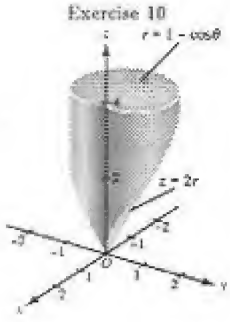
Exercise 8



Exercise 9



Exercise 10



9. The solid is cut out of the sphere $z^2 + r^2 = 16$ by the cylinder $r = 4 \cos \theta$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (16 - r_i^2)^{1/2} r_i \Delta_i r\, \Delta_i \theta = \iint_R r(16 - r^2)^{1/2} dr\, d\theta = 4\int_0^{\pi/2}\int_0^{4\cos\theta} r(16-r^2)^{1/2} dr\, d\theta$$

$$= 4\int \left[-\tfrac{1}{3}(16 - r^2)^{3/2}\right]_0^{4\cos\theta} d\theta = -\frac{4}{3}\int_0^{\pi/2}[(16 - 16\cos^2\theta)^{3/2} - 64]d\theta =$$

$$= -\frac{256}{3}\int_0^{\pi/2} \sin\theta(1 - \cos^2\theta)d\theta + \frac{256}{3}\theta\Big]_0^{\pi/2} = -\frac{256}{3}\Big[-\cos\theta + \frac{1}{3}\cos^3\theta\Big]_0^{\pi/2} + \frac{128}{2}\pi$$

$$= -\tfrac{256}{3}\Big[(-0+0) - (-1 + \tfrac{1}{3})\Big] + \tfrac{128}{3}\pi = \tfrac{128}{3}\pi - \tfrac{512}{9} = \tfrac{128}{9}(3\pi - 4)$$

10. $$V = 2\int_0^{\pi}\int_0^{1-\cos\theta} 2r \cdot r\, dr\, d\theta = \tfrac{4}{3}\int_0^{\pi}(1 - \cos\theta)^3 d\theta = \tfrac{4}{3}\int_0^{\pi}(1 - 3\cos\theta + 3\cos^2\theta - 3\cos^3\theta)d\theta$$

$$= \tfrac{4}{3}\int_0^{\pi}[1 + \tfrac{3}{2}(1 + \cos 2\theta)]d\theta + \tfrac{4}{3}\int_0^{\pi}(-3\cos\theta - 3\cos^3\theta)d\theta = \tfrac{4}{3}\Big[\tfrac{5}{2}\theta + \tfrac{3}{4}\sin 2\theta\Big]_0^{\pi} + 0 = \tfrac{10}{3}\pi$$

11. The solid is bounded by the paraboloid $z = 4 - r^2$, the cylinder $r = 1$, and the polar plane.

$$V = \iint_R (4 - r^2)r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 (4r - r^3)dr\, d\theta = \int_0^{2\pi}\Big[2r^2 - \frac{1}{4}r^4\Big]_0^1 d\theta = \frac{7}{4}\int_0^{2\pi} d\theta = \frac{7}{4}\theta\Big]_0^{2\pi} = \tfrac{7}{2}\pi$$

12. The solid above the paraboloid $z = r^2$ and below the plane $z = 2r\sin\theta$.

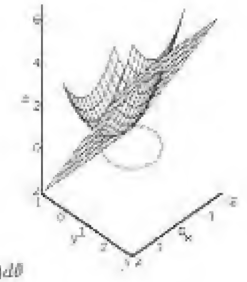
▹ We find the intersection of the paraboloid and the plane. Eliminating z from the two given equations, we have

$$r^2 - 2r\sin\theta = 0$$
$$r(r - 2\sin\theta) = 0$$

Hence

$r = 0$ (the pole) and $r = 2\sin\theta$ (which contains the pole)

The given surfaces intersect in a curve whose projection onto the polar plane is the circle $r = 2\sin\theta$. See the figure. The volume of the solid is given by



$$V = \iint_R (2r\sin\theta - r^2)dA = \int_0^{\pi}\int_0^{2\sin\theta}(2r\sin\theta - r^2)r\, dr\, d\theta$$

$$= \int_0^{\pi}\Big[\tfrac{2}{3}r^3\sin\theta - \tfrac{1}{4}r^4\Big]_0^{2\sin\theta} d\theta = \tfrac{4}{3}\int_0^{\pi}\sin^4\theta\, d\theta = \tfrac{1}{3}\int_0^{\pi}(1 - 2\cos 2\theta + \cos^2 2\theta)d\theta$$

$$= \tfrac{1}{3}\int_0^{\pi}(\tfrac{3}{2} - 2\cos 2\theta + \tfrac{1}{2}\cos 4\theta)d\theta = \tfrac{1}{3}\Big[\tfrac{3}{2}\theta - \sin 2\theta + \tfrac{1}{8}\sin 4\theta\Big]_0^{\pi} = \tfrac{1}{2}\pi$$

Thus, the volume is $\frac{1}{2}\pi$ cubic units.

In Exercises 13–19, find the mass and center of mass (in Cartesian coordinates) of a lamina having the shape of the given region if the area density is as indicated. Mass is measured in kilograms and distance is measured in meters.

13. The region is inside the cardioid $r = 2(1 + \sin\theta)$. $\rho(r, \theta) = kr$.

$$M = \lim_{\|\Delta\| \to 0}\sum_{i=1}^{n}(k\bar{r}_i)\bar{r}_i\Delta_i r\, \Delta_i\theta = k\iint_R r^2 dr\, d\theta = 2k\int_{-\pi/2}^{\pi/2}\int_0^{2(1+\sin\theta)} r^2 dr\, d\theta = 2k\int_{-\pi/2}^{\pi/2}\frac{1}{3}r^3\Big]_0^{2(1+\sin\theta)} d\theta$$

$$= \tfrac{2}{3}k\int_{-\pi/2}^{\pi/2} 8(1 + 3\sin\theta + 3\sin^2\theta + \sin^3\theta)d\theta$$

Because $\sin\theta$ and $\sin^3\theta$ are odd and $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ is even,

$$M = \frac{16}{3}k\int_0^{\pi/2}[2 + 3(1 - \cos 2\theta)]d\theta = \frac{16}{3}k\Big[5 - \frac{3}{2}\sin 2\theta\Big|_0^{\pi/2} = \frac{16}{3}\cdot 5\cdot\frac{\pi}{2} = \frac{40}{3}k\pi$$

$$M_x = \lim_{\|\Delta\| \to 0}\sum_{i=1}^{n}\bar{y}_i(k\bar{r}_i)\bar{r}_i\Delta_i r\, \Delta_i\theta = \lim_{\|\Delta\| \to 0}\sum_{i=1}^{n}(\bar{r}_i\sin\theta_i)(k\bar{r}_i)\bar{r}_i\Delta_i r\, \Delta_i\theta = \iint_R kr^3\sin\theta\, dr\, d\theta$$

$$= 2k\int_{-\pi/2}^{\pi/2}\int_0^{2(1+\sin\theta)} r^3\sin\theta\, dr\, d\theta = 2k\int_{-\pi/2}^{\pi/2}\frac{1}{4}r^4\Big]_0^{2(1+\sin\theta)}\sin\theta\, d\theta = \frac{k}{2}\int_{-\pi/2}^{\pi/2}16(1 + \sin\theta)^4\sin\theta\, d\theta$$

$$= 8k\int_{-\pi/2}^{\pi/2}(\sin\theta + 4\sin^2\theta + 6\sin^3\theta + 4\sin^4\theta + \sin^5\theta)d\theta = 64k\int_0^{\pi/2}(\sin^4\theta + \sin^2\theta)d\theta$$

$$= 64k\Big[-\frac{1}{4}\sin^3\theta\cos\theta + \frac{7}{4}\int\sin^2\theta\, d\theta\Big]_0^{\pi/2} = 64k\Big[0 + \frac{7}{4}\Big(-\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\Big)\Big]_0^{\pi/2} = 64k\cdot\frac{7}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 28k\pi$$

Because R is symmetric with respect to the y axis, the center of mass is at $(0, \bar{y})$ and

$$\bar{y} = \frac{1}{M}M_x = \frac{3}{40k\pi}\cdot 28k\pi = \frac{21}{10}$$

14. The region is inside the rose $r = a\cos 2\theta$. $\rho(r,\theta) = kr$. $M = 2\int_0^{\pi/4}\int_0^{a\cos 2\theta} kr\cdot r\,dr\,d\theta = \frac{2}{3}ka^3\int_0^{\pi/4}\cos^3 2\theta\,d\theta$

$= \frac{2}{3}ka^3\int_0^{\pi/4}(1-\sin^2 2\theta)\cos 2\theta\,d\theta = \frac{1}{3}ka^3\left[\sin 2\theta - \frac{1}{3}\sin^3 2\theta\right]_0^{\pi/4} = \frac{1}{3}ka^3\cdot\frac{2}{3} = \frac{2}{9}ka^3$. $\bar{y} = 0$ by symmetry.

$M_y = 2\int_0^{\pi/4}\int_0^{a\cos\theta} kr(r\cos\theta)r\,dr\,d\theta = \frac{1}{2}ka^4\int_0^{\pi/4}\cos\theta\cos^3 2\theta\,d\theta = \frac{1}{2}ka^4\int_0^{\pi/4}(1-2\sin^2\theta)^3\cos\theta\,d\theta$

$= \frac{1}{2}ka^4\int_0^{\pi/4}(1-6\sin^2\theta+12\sin^4\theta-8\sin^6\theta)d(\sin\theta) = \frac{1}{2}ka^4\left[\sin\theta - 2\sin^3\theta + \frac{12}{5}\sin^5\theta - \frac{8}{7}\sin^7\theta\right]_0^{\pi/4}$

$= \frac{1}{2}ka^4\left(\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2} + \frac{3}{5}\sqrt{2} - \frac{1}{7}\sqrt{2}\right) = \frac{4}{35}ka^4\sqrt{2}$. $\bar{x} = \frac{9}{2}\cdot\frac{4}{35}\sqrt{2}a = \frac{18}{35}\sqrt{2}a$

15. The region is inside the limaçon $r = 2 - \cos\theta$. $\rho(r,\theta) = kr$.

$M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(kr_i)r_i\Delta_i r\,\Delta_i\theta = k\iint_R r^2 dr\,d\theta = k\int_0^{2\pi}\int_0^{2-\cos\theta} r^2 dr\,d\theta = k\int_0^{2\pi}\frac{1}{3}r^3\Big]_0^{2-\cos\theta}d\theta$

$= \frac{k}{3}\int_0^{2\pi}(2-\cos\theta)^3 d\theta = \frac{k}{3}\int_0^{2\pi}(8-12\cos\theta+6\cos^2\theta-\cos^3\theta)d\theta = \frac{k}{3}\int_0^{2\pi}(8+6\cos^2\theta)d\theta$

$= \frac{k}{3}\int_0^{2\pi}(11+3\cos 2\theta)d\theta = \frac{k}{3}\int_0^{2\pi}11\,d\theta = \frac{22}{3}k\pi$

$M_y = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\bar{x}_i(k\bar{r}_i)\bar{r}_i\Delta_i r\,\Delta_i\theta = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\bar{r}_i\cos\bar{\theta}_i)(k\bar{r}_i)\bar{r}_i\Delta_i r\,\Delta_i\theta = \iint_R kr^3\cos\theta\,dr\,d\theta$

$= k\int_0^{2\pi}\int_0^{2-\cos\theta} r^3\cos\theta\,dr\,d\theta = k\int_0^{2\pi}\frac{1}{4}r^4\Big]_0^{2-\cos\theta}\cos\theta\,d\theta = \frac{k}{4}\int_0^{2\pi}(2-\cos\theta)^4\cos\theta\,d\theta$

$= \frac{k}{4}\int_0^{2\pi}(16\cos\theta - 32\cos^2\theta + 24\cos^3\theta - 8\cos^4\theta + \cos^5\theta)d\theta = -2k\int_0^{2\pi}(\cos^4\theta + 4\cos^2\theta)d\theta$

$= -2k\left[\frac{1}{4}\cos^3\theta\sin\theta + \left(\frac{3}{4}+4\right)\int\cos^2\theta\,d\theta\right]_0^{2\pi} = -2k\cdot\frac{19}{4}\left[\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right]_0^{2\pi} = -2k\cdot\frac{19}{4}\cdot\pi = -\frac{19}{2}k\pi$

Because R is symmetric with respect to the polar axis, the centroid is at $(\bar{x},0)$ and

$\bar{x} = \frac{1}{M}M_y = \frac{3}{22k\pi}\cdot\frac{-19k\pi}{2} = -\frac{57}{44}$

16. The region bounded by the convex limaçon $r = 2+\cos\theta$, $0 \le \theta \le \pi$, and the polar axis. The area density at any point is $k\sin\theta$ kg/m^2.

▹ The measure of the mass is given by

$$M = \iint_R k\sin\theta\,dA = \int_0^{\pi}\int_0^{2+\cos\theta} k\sin\theta\,r\,dr\,d\theta = \frac{1}{2}k\int_0^{\pi}\sin\theta\,r^2\Big]_0^{2+\cos\theta}d\theta$$

$$= \frac{1}{2}k\int_0^{\pi}(2+\cos\theta)^2\sin\theta\,d\theta = -\frac{1}{6}k(2+\cos\theta)^3\Big]_0^{\pi} = \frac{13}{3}k \qquad (1)$$

Thus, the mass is $\frac{13}{3}k$ kilograms.

$$M_y = \iint_R (k\sin\theta)x\,dA = \int_0^{\pi}\int_0^{2+\cos\theta}(k\sin\theta)(r\cos\theta)r\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+\cos\theta}\sin\theta\cos\theta\,r^2 dr\,d\theta$$

$$= \frac{1}{3}k\int_0^{\pi}\sin\theta\cos\theta(2+\cos\theta)^3 d\theta = \frac{1}{3}k\int_0^{\pi}[8\cos\theta + 12\cos^2\theta + 6\cos^3\theta + \cos^4\theta](-d\cos\theta)$$

$$= -\frac{1}{3}k\left[4\cos^2\theta + 4\cos^3\theta + \frac{3}{2}\cos^4\theta + \frac{1}{5}\cos^5\theta\right]_0^{\pi} = \frac{14}{5}k \qquad (2)$$

and

$$M_x = \iint_R (k\sin\theta)y\,dA = \int_0^{\pi}\int_0^{2+\cos\theta}(k\sin\theta)(r\sin\theta)r\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+\cos\theta} r^2\sin^2\theta\,dr\,d\theta$$

$$= \frac{1}{3}k\int_0^{\pi}\sin^2\theta(2+\cos\theta)^3 d\theta = \frac{1}{3}k\int_0^{\pi}\sin^2\theta(8+12\cos\theta+6\cos^2\theta+\cos^3\theta)d\theta$$

$$= \frac{1}{3}k\left[8\int_0^{\pi}\sin^2\theta\,d\theta + 12\int_0^{\pi}\sin^2\theta\cos\theta\,d\theta + 6\int_0^{\pi}\sin^2\theta\cos^2\theta\,d\theta + \int_0^{\pi}\sin^2\theta\cos^3\theta\,d\theta\right]$$

$$= \frac{1}{3}k\left[4\int_0^{\pi}(1-\cos 2\theta)d\theta + 12\cdot 0 + \frac{3}{4}\int_0^{\pi}(1-\cos 4\theta)d\theta + 0\right]$$

$$= \frac{1}{3}k\left[4\left(\theta - \frac{1}{2}\sin 2\theta\right) + \frac{3}{4}\left(\theta - \frac{1}{4}\sin\theta\right)\right]_0^{\pi} = \frac{19}{12}\pi k \qquad (3)$$

The integrals involving odd powers of cosine over $[0,\pi]$ vanish. Substituting from (1), (2) and (3), we have

$\bar{x} = \frac{3}{13}\cdot\frac{14}{5} = \frac{42}{65}$ and $\bar{y} = \frac{3}{13}\cdot\frac{19}{12}\pi = \frac{19}{52}\pi$

Therefore, the center of mass is $(\frac{42}{65}, \frac{19}{52}\pi)$.

17. The region is the top half of the limaçon $r = 2r\cos\theta$. $\rho(r,\theta) = kr\sin\theta$.

$$M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(k\bar{r}_i\sin\theta_i)\bar{r}_i\Delta_i r\,\Delta_i\theta = \iint_R kr^2\sin\theta\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+\cos\theta} r^2\sin\theta\,dr\,d\theta$$

$$= k\int_0^{\pi}\frac{1}{3}r^3\Big]_0^{2+\cos\theta}\sin\theta\,d\theta = \frac{k}{3}\int_0^{\pi}(2+\cos\theta)^3\sin\theta\,d\theta = \frac{k}{3}\Big[-\frac{1}{4}(2+\cos\theta)^4\Big]_0^{\pi} = -\frac{k}{12}(1^4-3^4) = \frac{20}{3}k$$

$$M_y = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\bar{x}_i(k\bar{r}_i\sin\bar{\theta}_i)\bar{r}_i\Delta_i r\,\Delta_i\theta = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(r_i\cos\theta_i)(kr_i\sin\theta_i)r_i\Delta_i r\,\Delta_i\theta$$

$$= k\iint_R r^3\sin\theta\cos\theta\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+\cos\theta} r^3\sin\theta\cos\theta\,dr\,d\theta = k\int_0^{\pi}\frac{1}{4}r^4\Big]_0^{2+\cos\theta}\sin\theta\cos\theta\,d\theta$$

$$= \frac{k}{4}\int_0^{\pi}(2+\cos\theta)^4\sin\theta\cos\theta\,d\theta = \frac{k}{4}\int_0^{\pi}(16\cos\theta + 32\cos^2\theta + 24\cos^3\theta + 8\cos^4\theta + \cos^5\theta)\sin\theta\,d\theta$$

$$= \frac{k}{4}\Big[-8\sin^2\theta - \frac{32}{3}\cos^3\theta - 6\cos^4\theta - \frac{8}{5}\cos^5\theta - \frac{1}{6}\cos^6\theta\Big]_0^{\pi} = \frac{k}{4}\Big(\frac{32}{3}+\frac{8}{5}+\frac{32}{3}+\frac{8}{5}\Big) = \frac{92}{15}k$$

$$M_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\bar{y}_i(k\bar{r}_i\sin\bar{\theta}_i)r_i\Delta_i r\,\Delta_i\theta = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(r_i\sin\theta_i)(kr_i\sin\theta_i)r_i\Delta_i r\,\Delta_i\theta$$

$$= k\iint_R r^3\sin^2\theta\,dr\,d\theta = k\int_0^{\pi}\int_0^{2+\cos\theta} r^3\sin^2\theta\,dr\,d\theta = k\int_0^{\pi}\frac{1}{4}r^4\Big]_0^{2+\cos\theta}\sin^2\theta\,d\theta$$

$$= \frac{k}{4}\int_0^{\pi}(2+\cos\theta)^4\sin^2\theta\,d\theta = \frac{k}{4}\int_0^{\pi}(16 + 32\cos\theta + 24\cos^2\theta + 8\cos^3\theta + \cos^4\theta)(1-\cos^2\theta)d\theta$$

$$= \frac{k}{4}\int_0^{\pi}(-\cos^6\theta - 8\cos^5\theta - 23\cos^4\theta - 24\cos^3\theta + 8\cos^2\theta + 32\cos\theta + 16)d\theta$$

$$= \frac{k}{4}\int_0^{\pi}(-\cos^6\theta - 23\cos^4\theta + 8\cos^2\theta + 16)d\theta = \frac{k}{4}\Big[-\frac{1}{6}\cos^5\theta\sin\theta + \int\Big(\Big(-\frac{5}{6}-23\Big)\cos^4\theta + 8\cos^2\theta + 16\Big)d\theta\Big]_0^{\pi}$$

$$= \frac{k}{4}\Big[0 - \frac{143}{6}\cdot\frac{1}{4}\cos^3\theta\sin\theta + \int\Big(\Big(-\frac{143}{6}\cdot\frac{3}{4}+8\Big)\cos^2\theta + 16\Big)d\theta\Big]_0^{\pi}$$

$$= \frac{k}{4}\Big[0 - \frac{79}{8}\cdot\frac{1}{2}\cos\theta\sin\theta + \Big(-\frac{1}{2}\cdot\frac{79}{8}+16\Big)\theta\Big]_0^{\pi} = \frac{k}{4}\Big(0 + \frac{177}{16}\pi\Big) = \frac{177}{64}k\pi$$

The center of mass is at $(\bar{x},\bar{y})$ and $\bar{x} = \frac{1}{M}M_y = \frac{3}{20k}\cdot\frac{92k}{15} = \frac{23}{25}$, $\bar{y} = \frac{1}{M}\cdot M_x = \frac{3}{20k}\cdot\frac{177k\pi}{64} = \frac{531}{1280}\pi$.

18. $M = 2k\Big[\int_{\pi/3}^{\pi/2}\int_{5\cos\theta}^{3-\cos\theta} r\cdot r\,dr\,d\theta + \int_{\pi/2}^{\pi} r\cdot r\,dr\,d\theta\Big] = \frac{2}{3}k\Big[\int_{\pi/3}^{\pi/2}[(3-\cos\theta)^3 - (5\cos\theta)^3]d\theta + \int_{\pi/2}^{\pi}(3-\cos\theta)^3 d\theta\Big]$

$= k(14\pi - \frac{500}{9} + \frac{159}{4}\sqrt{3})$. $\bar{y} = 0$ by symmetry. $M_y = 2k\Big[\int_{\pi/3}^{\pi/2}\int_{5\cos\theta}^{3-\cos\theta} r(r\cos\theta)r\,dr\,d\theta + \int_{\pi/2}^{\pi} r(r\cos\theta)r\,dr\,d\theta\Big]$

$= \frac{1}{2}k\Big[\int_{\pi/3}^{\pi/2}[(3-\cos\theta)^4 - (5\cos\theta)^4]d\theta + \int_{\pi/2}^{\pi}(3-\cos\theta)^4 d\theta\Big] = k(\frac{11613}{160}\sqrt{3} - \frac{500}{3} - \frac{39}{2}\pi)$. $\bar{x} = M_y/M$

19. The region is the small loop of the limaçon $r = 2 - 4\sin\theta$. $\rho(r,\theta) = kr$.

$$M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(kr_i)r_i\Delta_i r\,\Delta_i\theta = k\iint_R r^2 dr\,d\theta = k\int_{\pi/6}^{\pi/2}\int_0^{4\sin\theta-2} r^2 dr\,d\theta = 2k\int_{\pi/4}^{\pi/2}\frac{1}{3}r^3\Big]_0^{4\sin\theta-2}d\theta$$

$$= \frac{2}{3}k\int_{\pi/6}^{\pi/2}(4\sin\theta-2)^3 d\theta = \frac{16}{3}\int_{\pi/6}^{\pi/2}(2\sin\theta-1)^3 d\theta = \frac{16}{3}k\int_{\pi/6}^{\pi/2}(8\sin^3\theta - 12\sin^2\theta + 6\sin\theta - 1)d\theta$$

$$= \frac{16}{3}k\Big\{-\frac{1}{3}\cdot 8\sin^2\theta\cos\theta\Big]_{\pi/6}^{\pi/2} + \int_{\pi/6}^{\pi/2}\Big[-12\sin^2\theta + \Big(\frac{2}{3}\cdot 8 + 6\Big)\sin\theta - 1\Big]d\theta\Big\}$$

$$= \frac{16}{3}k\Big\{\frac{8}{3}\cdot\frac{1}{4}\cdot\frac{1}{2}\sqrt{3} + 6\sin\theta\cos\theta\Big]_{\pi/6}^{\pi/2} + \int_{\pi/6}^{\pi/2}\Big[\frac{34}{3}\sin\theta - (6+1)\Big]d\theta\Big\}$$

$$= \frac{16}{3}k\Big\{\frac{1}{3}\sqrt{3} - 6\cdot\frac{1}{2}\cdot\frac{1}{2}\sqrt{3} + \Big[-\frac{34}{3}\cos\theta - 7\theta\Big]_{\pi/6}^{\pi/2} = \frac{16}{3}k\Big(\frac{1}{3}\sqrt{3} - \frac{3}{2}\sqrt{3} + \frac{34}{3}\cdot\frac{1}{2}\sqrt{3} - 7\cdot\frac{1}{3}\pi\Big) = \frac{8}{9}(27\sqrt{3} - 14\pi)k$$

$$M_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\bar{y}_i(k\bar{r}_i)\bar{r}_i\Delta_i r\,\Delta_i\theta = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\bar{r}_i\sin\theta_i)(k\bar{r}_i)\bar{r}_i\Delta_i r\,\Delta_i\theta = \iint_R kr^3\sin\theta\,dr\,d\theta$$

$$= 2k\int_{\pi/6}^{\pi/2}\int_0^{4\sin\theta-2} r^3\sin\theta\,dr\,d\theta = 2k\int_{\pi/6}^{\pi/2}\frac{1}{4}r^4\Big]_0^{4\sin\theta-2}\sin\theta\,d\theta = \frac{k}{2}\int_{\pi/6}^{\pi/2}(2-4\sin\theta)^4\sin\theta\,d\theta$$

$= 8k\int_{\pi/6}^{\pi/2}(2\sin\theta-1)^4\sin\theta\,d\theta = 8kI$. Using recursion for powers of sine gives

$$\int(2\sin\theta-1)^4\sin\theta\,d\theta = \int(16\sin^5\theta - 32\sin^4\theta + 24\sin^3\theta - 8\sin^2\theta + \sin\theta)d\theta$$

$$= -16\cdot\frac{1}{5}\sin^4\theta\cos\theta + 32\cdot\frac{1}{4}\sin^3\theta\cos\theta + \int\Big[\Big(16\cdot\frac{4}{5}+24\Big)\sin^3\theta - \Big(32\cdot\frac{3}{4}+8\Big)\sin^2\theta + \sin\theta\Big]d\theta$$

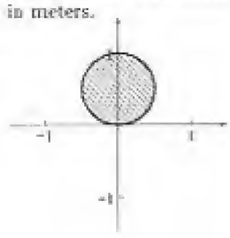
$$= -\tfrac{16}{5}\sin^4\theta\cos\theta + 8\sin^3\theta\cos\theta - \tfrac{184}{5}\cdot\tfrac{1}{3}\sin^2\theta\cos\theta + 32\cdot\frac{1}{3}\sin\theta\cos\theta + \int\Big[(\tfrac{184}{5}\cdot\tfrac{2}{3}+1)\sin\theta - 32\cdot\tfrac{1}{2}\Big]d\theta$$
$$= -\tfrac{16}{5}\sin^4\theta\cos\theta + 8\sin^3\theta\cos\theta - \tfrac{184}{15}\sin^2\theta\cos\theta + 16\sin\theta\cos\theta - \tfrac{383}{15}\cos\theta - 16\theta$$
$$8kI = 8k\Big[\tfrac{16}{5}(\tfrac{1}{2})^4\tfrac{1}{2}\sqrt{3} - 8(\tfrac{1}{2})^3\tfrac{1}{2}\sqrt{3} + \tfrac{184}{15}(\tfrac{1}{2})^2\tfrac{1}{2}\sqrt{3} - 16(\tfrac{1}{2})\sqrt{3} + \tfrac{383}{15}\cdot\tfrac{1}{2}\sqrt{3} - 16\cdot\tfrac{1}{3}\pi\Big] = \tfrac{4}{15}(297\sqrt{3} - 160\pi)$$

Because R is symmetric with respect to the y axis the center of mass is at $(0,\bar{y})$ and

$$\bar{y} = \frac{1}{M}M_x = \frac{3(297\sqrt{3} - 160\pi)}{10(27\sqrt{3} - 14\pi)} \approx 1.268$$

In Exercises 20–24, find the moment of inertia of a lamina in the shape of the given region and area density k kg/m^2 about the indicated axis or point. Mass is measured in kilograms and distance in meters.

20. The region enclosed by the circle $r = \sin\theta$, about the $\frac{1}{2}\pi$ axis.



▸ The figure shows the region. The distance from any point (r,θ) to the $\frac{1}{2}\pi$ axis is $x = r\cos\theta$ meters. Thus, the measure of the moment of inertia about the $\frac{1}{2}\pi$ axis is given by

$$I = \iint_R k(r\cos\theta)^2 dA = k\int_0^\pi\int_0^{\sin\theta} r^3\cos^2\theta\, dr\, d\theta = \tfrac{1}{4}k\int_0^\pi \sin^4\theta\cos^2\theta\, d\theta$$
$$= \tfrac{1}{4}k\int_0^\pi \sin^2\theta(\sin\theta\cos\theta)^2 d\theta = \tfrac{1}{32}k\int_0^\pi (1-\cos 2\theta)\sin^2 2\theta\, d\theta$$
$$= \tfrac{1}{32}k\int_0^\pi (\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta - \sin^2 2\theta\cos 2\theta)d\theta = \tfrac{1}{32}k\Big[\tfrac{1}{2}\theta\Big]_0^\pi - 0 - 0 = \tfrac{1}{64}k\pi$$

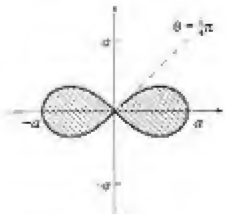
■ The moment of inertia is $\frac{1}{64}\pi k$ kg·m^2.

21. $$I_x = \lim_{\|\Delta\|\to 0}\sum \bar{y}_i^{\,2}(k)\bar{r}_i\Delta_i r\,\Delta_i\theta = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n (r_i\sin\theta_i)(k)r_i\Delta_i r\Delta_i\theta = k\iint_R r^3\sin^2\theta\, dr\, d\theta$$
$$= k\int_0^\pi\int_0^{\sin\theta} r^3\sin^2\theta\, dr\, d\theta = k\int_0^\pi \frac{1}{4}r^4\Big]_0^{\sin\theta}\sin^2\theta = \frac{k}{4}\int_0^\pi \sin^6\theta\, d\theta = \frac{k}{4}\Big[-\frac{1}{6}\sin^5\theta\cos\theta + \frac{5}{6}\int\sin^4\theta\, d\theta\Big]_0^\pi$$
$$= \frac{k}{4}\Big[0 - \frac{5}{6}\cdot\frac{1}{4}\sin^4\theta\cos\theta + \frac{5}{6}\cdot\frac{3}{4}\int\sin^2\theta\, d\theta\Big]_0^\pi = \frac{k}{4}\Big[0 - \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\sin\theta\cos\theta + \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\theta\Big]_0^\pi = \frac{k}{4}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\pi = \frac{5}{64}k\pi$$

22. $I_0 = 2k\int_0^\pi\int_0^{a(1-\cos\theta)} r^2\, r\, dr\, d\theta = \frac{1}{2}ka^4\int_0^\pi (1-\cos\theta)^4 d\theta$. Let $u = \frac{1}{2}\theta$. $I_0 = 16ka^4\int_0^{\pi/2}\sin^8 u\, du$
$$= 16ka^4\Big[-\tfrac{1}{8}\sin^7 u\cos u + \tfrac{7}{8}\int_0^{\pi/2}\sin^6 u\, du\Big] = 16ka^4\Big[0 - \tfrac{7}{8}\cdot\tfrac{1}{6}\sin^5 u\cos u + \tfrac{7}{8}\cdot\tfrac{5}{6}\int_0^{\pi/2}\sin^4 u\, du\Big]$$
$$= 16ka^4\Big[0 - \tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{1}{4}\sin^3 u\cos u + \tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{3}{4}\int_0^{\pi/2}\sin^2 u\, du\Big] = 16ka^4\Big[0 - \tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\sin u\cos u + \tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\int_0^{\pi/2} du\Big]$$
$$= 16ka^4\cdot\tfrac{7}{8}\cdot\tfrac{5}{6}\cdot\tfrac{3}{4}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\pi = \tfrac{35}{16}ka^4\pi$$

23. The region is bounded by a cardioid $r = a(1+\cos\theta)$ and a circle $r = 2a\cos\theta$. $\rho(r,\theta) = k$.
$$I_0 = \lim_{\|\Delta\|\to 0}\sum_{i=1}^n \bar{r}_i^{\,2}(k)\bar{r}_i\Delta_i r\Delta_i\theta = k\iint_R r^3 dr\, d\theta = 2k\int_0^\pi\int_0^{a(1+\cos\theta)} r^3 dr\, d\theta - 2k\int_0^{\pi/2}\int_0^{2a\cos\theta} r^3 dr\, d\theta$$
$$= 2k\int_0^\pi \frac{1}{4}r^4\Big]_0^{a(1+\cos\theta)}d\theta - 2k\int_0^{\pi/2}\frac{1}{4}r^4\Big]_0^{2a\cos\theta}d\theta = \frac{k}{2}a^4\int_0^\pi[(1+\cos\theta)^4 - 8\cos^4\theta]d\theta$$
$$= \frac{k}{2}a^4\int_0^\pi(-7\cos^4\theta + 4\cos^3\theta + 6\cos^2\theta + 4\cos\theta + 1)d\theta = \frac{k}{2}a^4\int_0^\pi(-\cos^4\theta + 6\cos^2\theta + 1)d\theta$$
$$= \frac{k}{2}a^4\Big[-\frac{1}{4}\cos^3\theta\sin\theta + \int\Big(\Big(-\frac{21}{4}+6\Big)\cos^2\theta + 1\Big)d\theta\Big]_0^\pi = \frac{k}{2}a^4\Big[0 + \frac{3}{4}\cdot\frac{1}{2}\cos\theta\sin\theta + \Big(\frac{3}{4}\cdot\frac{1}{2}+1\Big)\theta\Big]_0^\pi = \frac{11}{2}k\pi a^4$$

24. The region enclosed by the lemniscate $r^2 = a^2\cos 2\theta$; about the polar axis.



▸ The figure shows the region. By symmetry, the moment is 4 times the moment of the part in the first quadrant. The distance from a point (r,θ) in the first quadrant to the polar axis is $y = r\sin\theta$. Thus the moment of inertia is

$$I = 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} k(r\sin\theta)^2 r\, dr\, d\theta = 4k\int_0^{\pi/4}\Big[\tfrac{1}{4}r^4\sin^2\theta\Big]_0^{a\sqrt{\cos 2\theta}}$$
$$= ka^4\int_0^{\pi/4}\cos^2 2\theta\sin^2\theta\, d\theta = \tfrac{1}{2}ka^4\int_0^{\pi/4}\cos^2 2\theta(1-\cos 2\theta)d\theta$$
$$= \tfrac{1}{2}ka^4\int_0^{\pi/4}[\tfrac{1}{2}(1+\cos 4\theta) - (1-\sin^2 2\theta)\cos 2\theta]d\theta$$
$$= \tfrac{1}{2}ka^4\Big[\tfrac{1}{2}\theta + \tfrac{1}{8}\sin 4\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{6}\sin^3 2\theta\Big]_0^{\pi/4} = \tfrac{1}{2}ka^4(\tfrac{1}{8}\pi + \tfrac{1}{8}\sin\pi - \tfrac{1}{2}\sin\tfrac{1}{2}\pi + \tfrac{1}{6}\sin^3\tfrac{1}{2}\pi$$

$= \frac{1}{2}ka^4(\frac{1}{8}\pi - \frac{1}{3}) = \frac{1}{48}ka^4(3\pi - 8)$

- The moment of inertia is $\frac{1}{48}ka^4(3\pi - 8)$ kg m^2.

25. The region is bounded by one loop of the lemniscate $r^2 = \cos 2\theta$. $\rho(r,\theta) = k$.

$\mathrm{M} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k r_i \Delta_i r \, \Delta_i \theta = k \iint_R r \, dr \, d\theta = 2k \int_0^{\pi} \int_0^{\sqrt{\cos 2\theta}} r \, dr \, d\theta = 2k \int_0^{\pi} \frac{1}{2} r^2 \Big]_0^{\sqrt{\cos 2\theta}} d\theta$

$= k \int_0^{\pi/4} \cos 2\theta \, d\theta = \frac{k}{2} \sin 2\theta \Big]_0^{\pi/4} = \frac{k}{2}$

$\mathrm{I}_0 = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} r_i^2(k) r_i \Delta_i r \, \Delta_i \theta = k \iint_R r^3 dr \, d\theta = 2k \int_0^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r^3 dr \, d\theta = 2k \int_0^{\pi/4} \frac{1}{4} r^4 \Big]_0^{\sqrt{\cos 2\theta}} d\theta$

$= \frac{k}{2} \int_0^{\pi/4} \cos^2 2\theta \, d\theta = \frac{k}{4} \int_0^{\pi/4} (1 + \cos \theta) d\theta = \frac{k}{4}\theta + \frac{1}{4} \sin 4\theta \Big]_0^{\pi/4} = \frac{k}{4} \cdot \frac{\pi}{4} = \frac{k\pi}{16}$. $r = \sqrt{\frac{1}{\mathrm{M}} \mathrm{I}_0} = \sqrt{\frac{2}{k} \cdot \frac{k\pi}{16}} = \frac{1}{4}\sqrt{2\pi}$.

26. $\mathrm{M} = \int_0^{2\pi} \int_0^4 r \cdot r \, dr \, d\theta = \frac{64}{3} \int_0^{2\pi} d\theta = \frac{128}{3}\pi$. $\mathrm{I}_0 = \int_0^{2\pi} \int_0^4 r^2(r) r \, dr \, d\theta = \frac{1024}{5} \int_0^{2\pi} d\theta = \frac{2048}{5}\pi$.

$r = \sqrt{\frac{3}{128} \cdot \frac{2048}{5}} = \frac{4}{5}\sqrt{15}$

27. $\iint e^{x^2+y^2} \, dA = \int_0^{2\pi} \int_1^3 e^{r^2} \, r \, dr \, d\theta = \int_0^{2\pi} \frac{1}{2} e^{r^2} \Big]_1^3 d\theta = \frac{1}{2} \int_0^{2\pi} (e^9 - e) d\theta = (e^9 - e)\pi$

28. Evaluate by polar coordinates the double integral $\iint_R \frac{x}{\sqrt{x^2 + y^2}} dA$

where R is the region in the first quadrant bounded by the circle $x^2 + y^2 = 1$ and the coordinate axes.

- $\iint_R \frac{x}{\sqrt{x^2 + y^2}} dA = \int_0^{\pi/2} \int_0^1 \frac{r \cos \theta}{r} r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/2} \cos \theta \, d\theta = \frac{1}{2} \sin \theta \Big]_0^{\pi/2} = \frac{1}{2}$

Because $\left| x/\sqrt{x^2 + y^2} \right| \leq |x/x| = 1$ and the integrand is continuous except at the origin, the integral exists.

29. $x^2 + y^2 + z^2 = 4x$ and $y^2 + z^2 = x^2$ intersect in the circle $x = 2$, $y^2 + z^2 = 4$ which bisects the sphere $(x-2)^2 + y^2 + z^2 = 4$. The sphere has radius 2. Thus from Example 7 with $a = 2$, $\sigma = 2\pi(2)^2 = 8\pi$.

30. $x^2 + y^2 + z^2 = 36, x^2 + y^2 \leq 9. z = \sqrt{36 - x^2 - y^2}. z_x{}^2 + z_y{}^2 + 1 = \frac{x^2}{36 - x^2 - y^2} + \frac{y^2}{36 - x^2 - y^2} + 1 = \frac{36}{36 - x^2 - y^2}$

$\sigma = 2 \iint_R \frac{6}{\sqrt{36 - x^2 - y^2}} dA = 12 \int_0^{2\pi} \int_0^3 \frac{1}{\sqrt{36 - r^2}} r \, dr \, d\theta = 12 \int_0^{2\pi} -\sqrt{36 - r^2} \Big]_0^3 d\theta = 12 \int_0^{2\pi} (6 - 3\sqrt{3}) d\theta$

$= 72\pi(2 - \sqrt{3})$. Or, $z = f(r,\theta) = \sqrt{36 - r^2}$, $\sqrt{f_r{}^2 + r^{-2} f_\theta{}^2 + 1} = \sqrt{\left(\frac{-r}{\sqrt{36 - r^2}}\right)^2 + 0 + 1} = \frac{6}{\sqrt{36 - r^2}}$

31. $x^2 + y^2 + z^2 = 4z$ and $x^2 + y^2 = 3z$ intersect in the origin and the circle $z = 1$, $x^2 + y^2 = 3$. Because the larger part of the sphere is within the paraboloid, we find the area σ of the smaller part, which lies below the intersection, and subtract it from the area of of the whole sphere of radius 2 units. From Example 7 with $a = 2$, this area is $2 \cdot 2\pi(2)^2 = 16\pi$ square units.

$z = f(x,y) = 2 - \sqrt{4 - x^2 - y^2}$; $f_x(x,y) = \frac{-x}{\sqrt{4 - x^2 - y^2}}$; $f_y(x,y) = \frac{-y}{\sqrt{4 - x^2 - y^2}}$

$\sigma = \iint_R \sqrt{f_x{}^2(x,y) + f_y{}^2(x,y) + 1} \, dA = \iint_R \sqrt{\frac{x^2}{4 - x^4 - y^2} + \frac{y^2}{4 - x^2 - y^2} + 1} \, dA = \iint_R \frac{2}{\sqrt{4 - x^2 - y^2}} \, dA$

We use polar coordinates. If $x^2 + y^2 = r^2$ and $dA = r \, dr \, d\theta$, then

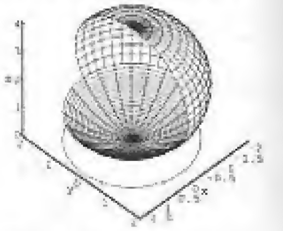
$\sigma = \int_0^{2\pi} \int_0^{\sqrt{3}} 2(4 - r^2)^{-1/2} r \, dr \, d\theta = \int_0^{2\pi} -2(4 - r^2)^{1/2} \Big]_0^{\sqrt{3}} d\theta = \int_0^{2\pi} (-2 + 4) d\theta = 2 \int_0^{2\pi} d\theta = 4\pi$

Alternatively, $z = f(r,\theta) = 2 - \sqrt{4 - r^2}$, $\sqrt{f_r{}^2 + r^{-2} f_\theta{}^2 + 1} = \sqrt{\left(\frac{r}{\sqrt{4 - r^2}}\right)^2 + 0 + 1} = \frac{2}{\sqrt{4 - r^2}}$

Hence the number of square units in the required area is $16\pi - 4\pi = 12\pi$.

32. Find the area of the portion of the surface of the paraboloid $x^2 + y^2 = 3z$ which lies within the sphere $x^2 + y^2 + z^2 = 4z$.

▷ We find the intersection of the paraboloid and the sphere. Eliminating x^2 and y^2 from the equations, we get $z^2 = z$. Therefore, the surfaces intersect in the planes $z = 0$ and $z = 1$. The curve of intersection in the surface $z = 1$ is the circle $x^2 + y^2 = 3$, $z = 1$. Completing the square on z in the equation of the sphere, we have
$x^2 + y^2 + (z-2)^2 = 4$



Thus, the sphere has center at $(0, 0, 2)$ and radius 2. The figure shows the back half of the sphere, and the surface, which lies over R, the circle $x^2 + y^2 = 3$ in the xy plane. We take
$f(x, y) = \frac{1}{3}(x^2 + y^2)$
Thus,
$f_x(x, y) = \frac{2}{3}x$ and $f_y(x, y) = \frac{2}{3}y$
We calculate the measure of the area and switch to polar coordinates.

$$\sigma = \iint_R \sqrt{f_x^{\,2}(x,y) + f_y^{\,2}(x,y) + 1}\,dA = \iint_R \sqrt{\tfrac{4}{9}(x^2+y^2) + 1}\,dA$$
$$= \int_0^{\sqrt{3}} \int_0^{2\pi} \sqrt{\tfrac{4}{9}r^2 + 1}\; r\,d\theta\,dr = 2\pi \int_0^{\sqrt{3}} \sqrt{\tfrac{4}{9}r^2 + 1}\; r\,dr = 2\pi(\tfrac{9}{8})\tfrac{2}{3}(\tfrac{4}{9}r^2 + 1)^{3/2}\Big]_0^{\sqrt{3}} = \tfrac{3}{2}\pi(\tfrac{7}{3}\sqrt{21} - 1)$$

Therefore, the area of the surface is $\frac{3}{2}\pi(\frac{7}{9}\sqrt{21} - 1) \approx 3.85\pi$ square units, while the area of the zone of the sphere from $z = 0$ to $z = 1$ is $2\pi rh = 2\pi(2)1 = 4\pi$ square units.

33. $az = xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$ is a hyperbolic paraboloid. $z = a^{-1}xy$. Inside $x^2 + y^2 = a^2$ in the first octant

$$\sigma = \iint_R \sqrt{z_x^{\,2} + z_y^{\,2} + 1}\,dA = \iint_R \sqrt{(a^{-1}y)^2 + (a^{-1}x)^2 + 1}\,dA = \iint_R a^{-1}\sqrt{x^2 + y^2 + a^2}\,dA$$
$$= \int_0^a \int_0^{\pi/2} a^{-1}\sqrt{r^2 + a^2}\,r\,dr\,d\theta = \tfrac{1}{2}\pi a^{-1}\int_0^a (r^2 + a^2)^{1/2} r\,dr = \tfrac{1}{3}\cdot\tfrac{1}{2}\pi a^{-1}(r^2 + a^2)^{3/2}\Big]_0^a$$
$$= \tfrac{1}{6}\pi a^{-1}[(2a^2)^{3/2} - (a^2)^{3/2}] = \tfrac{1}{6}\pi(2\sqrt{2} - 1)a^2$$

34. Find the area of the surface cut from the hyperbolic paraboloid $y^2 - x^2 = 6z$ by the cylinder $x^2 + y^2 = 36$.

▷ We take $f(x, y) = \frac{1}{6}(y^2 - x^2)$. Then $f_x(x, y) = -\frac{1}{3}x$ and $f_y(x, y) = \frac{1}{3}y$ and
$\sqrt{f_x^{\,2}(x,y) + f_y^{\,2}(x,y) + 1} = \sqrt{\frac{1}{9}(x^2 + y^2) + 1}$
The surface lies over (and under) the circle R in the xy plane that is the graph of $x^2 + y^2 = 36$. Therefore,

$$\sigma = \iint_R \sqrt{\tfrac{1}{9}(x^2 + y^2) + 1}\,dA = \int_0^6 \int_0^{2\pi} \sqrt{\tfrac{1}{9}r^2 + 1}\; r\,dr\,d\theta = 2\pi \int_0^6 \sqrt{\tfrac{1}{9}r^2 + 1}\; r\,dr$$
$$= 2\pi(\tfrac{9}{2})(\tfrac{2}{3})(\tfrac{1}{9}r^2 + 1)^{3/2}\Big]_0^6 = 6\pi(5\sqrt{5} - 1)$$

13.5 THE TRIPLE INTEGRAL

If f is a continuous function of x, y, and z on some region S in R^3, then the *triple integral* of f on S is given by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(u_i, v_i, w_i)\Delta_i V = \iiint_S f(x, y, z)\,dV$$

where S is divided into boxes by planes parallel to the coordinate planes and $\|\Delta\|$ is the length of the longest diagonal of a box.

We use the triple integral to find the total mass of a solid with variable density. If $\rho(x, y, z)$ is the measure of the volume density of a solid that occupies the region S, then the measure of the mass of the solid is given by

$$\mathrm{M} = \iiint_S \rho(x, y, z)\,dV$$

When $\rho(x, y, z) = 1$, the triple integral for mass gives the measure of the volume of the solid; earlier method are usually easier. We may use a thrice-iterated integral to calculate the value of a triple integral. If S is bounded by the planes $x = x_1$ and $x = x_2$, bounded by the cylinders $y = g_1(x)$ and $y = g_2(x)$ that are perpendicular to the xy plane, and bounded by the surfaces $z = h_1(x, y)$ and $z = h_2(x, y)$, where $x_1 < x_2$, $g_1(x) \le g_2(x)$, and $h_1(x, y) \le h_2(x, y)$ and the functions g_1, g_2, h_1, and h_2 are smooth (i.e. they have continuous derivatives or partial derivatives), then

$$\iiint_S f(x, y, z)\,dV = \int_{x_1}^{x_2} \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\,dz\,dy\,dx$$

The roles of x, y, and z may be interchanged, so that there are six possible orders of integration that may be used to evaluate a triple integral.

If the limits of integration are constants, so that R is a rectangular parallelpiped and f can be expressed as the product $f(x,y,z) = g(x)h(y)k(z)$, then the iterated integral can be expressed as a product

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dz\,dy\,dx = \int_{x_1}^{x_2} g(x)dx\int_{y_1}^{y_2} h(y)dy\int_{z_1}^{z_2} k(z)dz$$

Simpson's Rule In §7.6 we showed that Simpson's rule was exact for polynomials of degree ≤ 3. The following analog in 3-space can shorten many calculations. Let S be a tetrahedron with vertices A, B, C, D, centroid G and volume V and let $f(x,y,z)$ be a polynomial of degree ≤ 2 then

$$\iiint_S f(x,y,z)dV = \frac{V}{20}[f(A) + f(B) + f(C) + f(D) + 16f(G)].$$

Exercises 13.5

In Exercises 1–8, evaluate the iterated integral.

1. $\int_0^1\int_0^{1-x}\int_{2y}^{1+y^2} x\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\Big[xz\Big]_{2y}^{1+y^2} dy\,dx = \int_0^1\int_0^{1-x} x(1+y^2-2y)dy\,dx$

$= \int_0^1\int_0^{1-x} x(y-1)^2dy\,dx = \frac{1}{3}\int_0^1 x(y-1)^3\Big]_0^{1-x} dx = \frac{1}{3}\int_0^1 (x-x^4)dx = \frac{1}{3}\Big[\frac{1}{2}x^2 - \frac{1}{5}x^5\Big]_0^1 = \frac{1}{10}$

2. $\int_1^2\int_0^x\int_1^{x+xy} xy\,dz\,dy\,dx = \int_1^2\int_0^x xy(x+xy-1)dy\,dx = \int_1^2\int_0^x (x^2y + x^2y^2 - xy)dy\,dx$

$= \int_1^2\Big[\frac{1}{2}x^2y^2 + \frac{1}{3}x^2y^3 - \frac{1}{2}xy^2\Big]_0^x dx = \int_1^2(\frac{1}{2}x^4 + \frac{1}{3}x^5 - \frac{1}{2}x^3)dx = \Big[\frac{1}{10}x^5 + \frac{1}{18}x^6 - \frac{1}{8}x^4\Big]_1^2 = \frac{163}{40}$

3. $\int_0^1\int_0^x\int_0^{x+y}(x+y+z)dz\,dy\,dx = \int_0^1\int_0^x\Big[xz - yz + \frac{1}{2}z^2\Big]_0^{x+y} dy\,dx = \int_0^1\int_0^x\Big(\frac{3}{2}x^2 + 3xy + \frac{3}{2}y^2\Big)dy\,dx$

$= \int_0^1\Big[\frac{3}{2}x^2y + \frac{3}{2}xy^2 + \frac{1}{2}\Big]_0^x dx = \int_0^1\frac{7}{2}x^3dx = \Big[\frac{7}{8}x^4\Big]_0^1 = \frac{7}{8}$

4. $\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{2-y} z\,dx\,dz\,dy$

$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{2-y} z\,dx\,dz\,dy = \int_0^2\int_0^{\sqrt{4-y^2}}\Big]zx\Big]_{x=0}^{2-y} dz\,dy = \int_0^2\int_0^{\sqrt{4-y^2}}(2-y)z\,dz\,dy$

$= \int_0^2\Big](2-y)\frac{1}{2}z^2\Big]_{y=0}^{\sqrt{4-y^2}} = \frac{1}{2}\int_0^2(2-y)(4-y^2)dy$

$= \frac{1}{2}\int_0^2(8-4y-2y^2+y^3)dy = \frac{1}{2}\Big[8y - 2y^2 - \frac{2}{3}y^3 + \frac{1}{4}y^4\Big]_0^2 = \frac{10}{3}$

5. $\int_{-1}^0\int_e^{2e}\int_0^{\pi/3} y\ln z\tan x\,dx\,dz\,dy = \int_{-1}^0\int_e^{2e}\Big[y\ln z\ln|\sec x|\Big]_0^{\pi/3} dz\,dy = \int_{-1}^0\int_e^{2e} y\ln z\ln 2\,dz\,dy$

$= \int_{-1}^0\Big[y(\ln 2)z(\ln z - 1)\Big]_e^{2e} dy = \int_{-1}^0 2e(\ln 2)^2y\,dy = \Big[e(\ln 2)^2y^2\Big]_{-1}^0 = -e(\ln 2)^2$

6. $\int_1^2\int_y^{y^2}\int_0^{\ln x} ye^z dz\,dx\,dy = \int_1^2\int_y^{y^2} ye^z\Big]_0^{\ln x} dx\,dy = \int_1^2\int_y^{y^2} y(x-1)dx\,dy = \int_1^2 y\Big[\frac{1}{2}x^2 - x\Big]_y^{y^2} dy$

$= \int_1^2 y(\frac{1}{2}y^4 - \frac{1}{2}y^2 - y^2 + y)dy = \frac{1}{12}y^6 - \frac{3}{8}y^4 + \frac{1}{3}y^3\Big]_1^2 = \frac{47}{24}$

7. $\int_0^{\pi/2}\int_z^{\pi/2}\int_0^{xz}\cos\frac{y}{z}\,dy\,dx\,dz = \int_0^{\pi/2}\int_z^{\pi/2}(z\sin x)dx\,dz = \int_0^{\pi/2} z\cos z\,dz = \Big[z\sin z + \cos z\Big]_0^{\pi/2} = \frac{1}{2}\pi - 1$

8. $\int_0^2\int_0^y\int_0^{\sqrt{3}z}\frac{z}{x^2+z^2}dx\,dz\,dy$

$\int_0^2\int_0^y\int_0^{\sqrt{3}z}\frac{z}{x^2+z^2}dx\,dz\,dy = \int_0^2\int_0^y\Big[\tan^{-1}\Big(\frac{x}{z}\Big)\Big]_{x=0}^{\sqrt{3}z} = \int_0^2\int_0^y(\tan^{-1}\sqrt{3} - \tan^{-1}0)dz\,dy$

$= \int_0^2\int_0^y\frac{1}{3}\pi\,dz\,dy = \frac{1}{3}\pi\int_0^2 z\Big]_0^y dy = \frac{1}{3}\pi\int_0^2 y\,dy = \frac{1}{6}\pi y^2\Big]_0^2 = \frac{2}{3}\pi$

In Exercises 9–18, evaluate the triple integral.

In Exercises 9 and 10, S is bounded by the plane $12x + 20y + 15z = 60$ and the coordinate planes.

9. $\iiint_S y\,dV = \int_0^5\int_0^{4(1-x/5)}\int_0^{3(1-x/5-z/4)} y\,dy\,dz\,dx = \int_0^5\int_0^{4(1-x/5)}\frac{1}{2}y^2\Big]_0^{3(1-x/5-z/4)} dz\,dx$

$= \frac{9}{2}\int_0^5\int_0^{4(1-x/5)}\Big(1 - \frac{x}{5} - \frac{z}{4}\Big)dz\,dx = \frac{9}{2}\int_0^5 -\frac{4}{3}\Big(1 - \frac{x}{5} - \frac{z}{4}\Big)^3\Big]_0^{4(1-x/5)}dx = 6\int_0^5\Big(1 - \frac{x}{5}\Big)^3dx = 6\Big[-\frac{5}{2}\Big(1 - \frac{x}{2}\Big)^4\Big]_0^5 = \frac{15}{2}$

10. $\iiint_S (x^2+z^2)dV = \int_{y=0}^{3}\left[\int_{z=0}^{4(1-y/3)}\int_{x=0}^{5(1-y/3-z/4)} x^2 dx\, dz + \int_{x=0}^{5(1-y/3)}\int_{z=0}^{4(1-x/5-y/3)} z^2 dz\, dx\right]$

$= \int_0^3\left[\int_0^{4(1-y/3)} \tfrac{1}{3}x^3\Big]_0^{5(1-y/3-z/4)} dz + \int_0^{5(1-y/3)} \tfrac{1}{3}z^3\Big]_0^{4(1-x/5-y/3)} dx\right]dy$

$= \int_0^3\left[\int_0^{4(1-y/3)} \frac{125}{3}\left(1-\frac{y}{3}-\frac{z}{4}\right)^3 dz + \int_0^{5(1-y/3)} \frac{64}{3}\left(1-\frac{x}{5}-\frac{y}{3}\right)^3 dx\right]dy$

$= \int_0^3\left[-\frac{125}{3}\left(1-\frac{y}{3}\right)^4 - \frac{80}{3}\left(1-\frac{y}{3}\right)^4\right]dy = \frac{205}{3}\cdot\frac{3}{5}\left(1-\frac{y}{3}\right)^5\Big]_0^3 = 41$. Alternatively, because $f(x,y,z)$ is of degree 2, we may use Simpson's rule. The vertices are $(5,0,0), (0,3,0), (0,0,4)$, and $(0,0,0)$. $V = \frac{1}{6}\cdot 5\cdot 3\cdot 4 = 20$. $G = (\frac{5}{4},\frac{3}{4},1)$

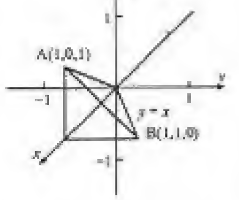
$I = \frac{V}{20}[f(5,0,0)+f(0,3,0)+f(0,0,4)+f(0,0,0)+16f(\frac{5}{4},\frac{3}{4},1)] = \frac{1}{2}(25+0+16+0+16\cdot\frac{41}{16}) = 41$

11. The vertices $(1,1,0)$, $(1,0,0)$, $(1,0,1)$ lie in the plane $z = x-y$; the 4th vertex is O.

$\iiint_S z\, dV = \int_0^1\int_0^x\int_0^{x-y} z\, dz\, dy\, dx = \int_0^1\int_0^x \frac{1}{2}(x-y)^2 dy\, dx = \int_0^1 -\frac{1}{6}(x-y)^3\Big]_0^x dx = \frac{1}{6}\int_0^1 x^3\, dx = \frac{1}{6}\left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{24}$

12. $\iiint_S yz\, dV$ is S is the region bounded by the tetrahedron having vertices $(0,0,0)$, $(1,1,0)$, $(1,0,0)$ and $(1,0,1)$.

▹ The figure shows the region S, which is bounded by the planes $x=0$ and $x=1$, the cylinders $y=0$ and $y=x$, and the surfaces $z=0$ and the plane of $(0,0,0)$, $(1,0,1)$ and $(1,1,0)$ which has equation $x-y-z=0$ or, equivalently, $z=x-y$. Therefore, the given triple integral may be expressed as an iterated integral as follows:



$$\iiint_S yz\, dV = \int_0^1\int_0^x\int_0^{x-y} yz\, dz\, dy\, dx = \tfrac{1}{2}\int_0^1\int_0^x y(x-y)^2 dy\, dx$$
$$= \tfrac{1}{2}\int_0^1\int_0^x (x^2y-2xy^2+y^3)dy\, dx = \tfrac{1}{2}\int_0^1\Big[\tfrac{1}{2}x^2y^2-\tfrac{2}{3}xy^3+\tfrac{1}{4}y^4\Big]_{y=0}^{x}$$
$$= \tfrac{1}{2}\int_0^1(\tfrac{1}{2}x^2-\tfrac{2}{3}x^4+\tfrac{1}{4}x^4)dx = \tfrac{1}{24}\int_0^1 x^4 dx = \tfrac{1}{120}$$

Also, because $f(x,y,z) = yz$ is of degree 2, we may use Simpson's rule. $G = (\frac{3}{4},\frac{1}{4},\frac{1}{4})$ and $V = \frac{1}{6}\cdot 1\cdot 1\cdot 1 = \frac{1}{6}$.

$\iiint_S yz dV = \frac{V}{20}[f(0,0,0)+f(1,1,0)+f(1,0,0)+f(1,0,1)+16f(\frac{3}{4},\frac{1}{4},\frac{1}{4})] = \frac{1}{120}[0+0+0+0+16\cdot\frac{1}{16}] = \frac{1}{120}$

13. S is bounded by the coordinate planes and the planes $x=2$, $y=3$, and $z=4$.

$\iiint_S xy\, dV = \int_0^2\int_0^3\int_0^4 xy\, dz\, dy\, dx = \int_0^2\int_0^3 4xy\, dy\, dx = \int_0^2 [2xy^2]_0^3\, dx = \int_0^2 18x\, dx = 9x^2\Big]_0^2 = 36$

14. S is the tetrahedron bounded by the planes $x+2y+3z=6$, $x=0$, $y=0$, $z=0$.

$\iiint_S x\, dV$. Method 1. Use the method of Exercise 9, integrating first with respect to x. Method 2. $f(x,y,z) = x$ is of degree 1. The vertices are $(6,0,0)$, $(0,3,0)$, $(0,0,2)$, $(0,0,0)$. $V = \frac{1}{6}\cdot 6\cdot 3\cdot 2 = 6$. $G = (\frac{3}{2},\frac{3}{4},\frac{1}{2})$.

$\iiint_S x\, dV = \frac{V}{20}[f(6,0,0)+f(0,3,0)+f(0,0,2)+f(0,0,0)+16f(\frac{3}{2},\frac{3}{4},\frac{1}{2})] = \frac{6}{20}(6+0+0+0+16\cdot\frac{3}{2}) = 9$

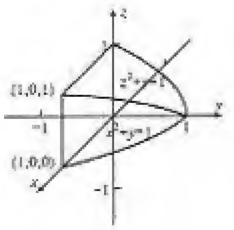
15. S is bounded by $z = x^2+y^2$, $z = 27-2x^2-2y^2$ intersecting in the circle R: $x^2+y^2=9$, $z=9$.

$\iiint_S dV = \iint_R\int_{x^2+y^2}^{27-2x^2-2y^2} dz\, dA = \iint_R (27-3x^2-3y^2)dA = \int_0^{2\pi}\int_0^3 (27-3r^2)r\, dr\, d\theta$

$= \int_0^{2\pi}\left[\frac{27}{2}r^2-\frac{3}{4}r^4\right]_0^3 d\theta = \int_0^{2\pi}\frac{243}{4}\, d\theta = \frac{243}{2}\pi$

16. $\iiint_S y^2 dV$ if S is the region bounded by the cylinders $x^2+y=1$ and $z^2+y=1$ and the plane $y=0$.

▹ The region S and integrand y^2 are symmetric with respect to the xy and yz planes. The figure shows S_1, the part of S in the first octant. We calculate the value of the triple integral over S_1 and multiply by 4. The region S_1 is bounded above by the cylinder $z = \sqrt{1-y}$ and below by the xy plane. Furthermore, the region S_1 is bounded by the cylinder $x = \sqrt{1-y}$ and the xz plane. The value of the given interval is thus



$$\iiint_S y^2 dV = 4\int_0^1\int_0^{\sqrt{1-y}}\int_0^{\sqrt{1-y}} y^2 dz\, dx\, dy = 4\int_0^1\int_0^{\sqrt{1-y}} y^2\sqrt{1-y}\, dx\, dy$$
$$= 4\int_0^1 y^2\sqrt{1-y}\sqrt{1-y}\, dy = 4\int_0^1 (y^2-y^3)dy = 4\left[\tfrac{1}{3}y^3-\tfrac{1}{4}y^4\right]_0^1 = \tfrac{1}{3}$$

17. S is bounded by the cylinder $x^2 + z^2 = 9$ and the planes $x + y = 3$, $z = 0$, and $y = 0$.

$$\iiint_S (xz + 3z)\,dV = \int_{-3}^{3}\int_0^{3-x}\int_0^{\sqrt{9-x^2}} z(x+3)\,dz\,dy\,dx = \frac{1}{2}\int_{-3}^{3}\int_0^{3-x}(9-x^2)(x+3)\,dy\,dx$$

$$= \frac{1}{2}\int_{-3}^{3}(9-x^2)(9-x^2)\,dx = \int_0^3 (81 - 18x^2 + x^4)\,dx = \left[81x - 6x^3 + \frac{1}{5}x^5\right]_0^3 = \frac{648}{5}$$

18. Take 8 times the part in the first octant. $V = 8\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}}\int_{z=0}^{\sqrt{4-x^2}} xyz\,dz\,dy\,dx$

$$= 4\int_0^2\int_0^{\sqrt{4-x^2}} xyz^2\Big]_{z=0}^{\sqrt{4-x^2}} dy\,dx = 44\int_0^2\int_0^{\sqrt{4-x^2}} x(4-x^2)y\,dy\,dx = 2\int_0^2 x(4-x^2)^2\,dx = -\tfrac{1}{3}(4-x^2)^3\Big]_0^2 = \frac{64}{3}$$

In Exercises 19–32, use triple integration.

▸ V cubic units is the volume of the region S and M kg is its mass.

19. S is bounded by the xy plane, the plane $z = y$, the cylinder $y^2 = x$ and the plane $x = 1$.

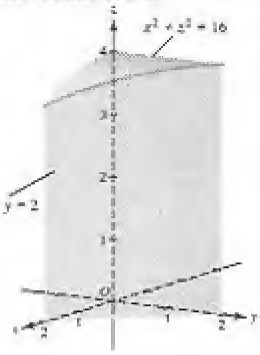
$$V = \iiint_S dV = \int_0^1\int_0^{\sqrt{x}}\int_0^y dz\,dy\,dx = \int_0^1\int_0^{\sqrt{x}} y\,dy\,dx = \int_0^1 \frac{1}{2}y^2\Big]_0^{\sqrt{x}} dx = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{4}x^2\Big]_0^1 = \frac{1}{4}$$

20. Find the volume of the solid in the first octant bounded by the cylinder $x^2 + z^2 = 16$, the plane $x + y = 2$, and the three coordinate planes.

▸ The figure shows the solid S, which is bounded by the planes $x = 0$ and $x = 2$, the cylinders $y = 0$ and $y = 2 - x$, and the surfaces $z = 0$ and $z = \sqrt{16 - x^2}$. Therefore, the measure of the volume of S is

$$V = \iiint_S dV = \int_0^2\int_0^{2-x}\int_0^{\sqrt{16-x^2}} dz\,dy\,dx = \int_0^2\int_0^{2-x}\sqrt{16-x^2}\,dy\,dx$$

$$= \int_0^2 (2-x)\sqrt{16-x^2}\,dx = 2\int_0^2\sqrt{16-x^2}\,dx - \int_0^2 x\sqrt{16-x^2}\,dx \qquad (1)$$



For the first integral in (1), let $x = 4\sin\theta$. Then

$$\int_0^2\sqrt{16-x^2}\,dx = \int_0^{\pi/6} 16\cos^2\theta\,d\theta = 8\int_0^{\pi/6}(1 + \cos 2\theta)\,d\theta$$

$$= 8\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/6} = 8\left(\tfrac{1}{6}\pi + \tfrac{1}{4}\sqrt{3}\right) = \tfrac{4}{3}\pi + 2\sqrt{3} \qquad (2)$$

For the second integral in (1) we have

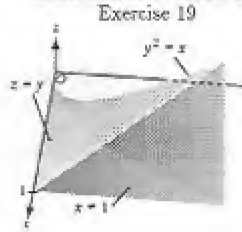
$$\int_0^2 x\sqrt{16-x^2}\,dx = -\tfrac{1}{3}(16-x^2)^{3/2}\Big]_0^2 = -\tfrac{1}{3}(24\sqrt{3} - 64) = \tfrac{64}{3} - 8\sqrt{3} \qquad (3)$$

Substituting from (2) and (3) into (1), we have

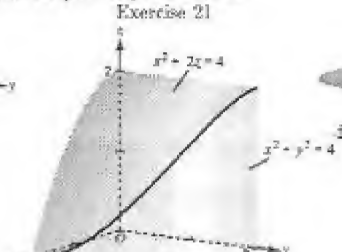
$$V = 2\left(\tfrac{4}{3}\pi + 2\sqrt{3}\right) - \left(\tfrac{64}{3} - 8\sqrt{3}\right) = \tfrac{8}{3}\pi + 12\sqrt{3} - \tfrac{64}{3}$$

Therefore, the volume is $\frac{4}{3}(2\pi + 9\sqrt{3} - 16)$ cubic units.

Exercise 19



Exercise 21



Exercise 22

21. S is in the first octant bounded by $x^2 + y^2 = 4$, $x^2 + 2z = 4$ and the coordinate planes.

$$V = \iiint_S dV = \int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{(4-x^2)/2} dz\,dy\,dx = \frac{1}{2}\int_0^2\int_0^{\sqrt{4-x^2}}(4-x^2)\,dy\,dx = \frac{1}{2}\int_0^2 (4-x^2)^{3/2}\,dx$$

Let $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$. Then

$$V = 8\int_0^{\pi/2}\cos^4\theta\,d\theta = 8\left[\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\int\cos^2\theta\,d\theta\right]_0^{\pi/2} = 8\left[0 + \frac{3}{4}\cdot\frac{1}{2}\cos\theta\sin\theta + \frac{3}{4}\cdot\frac{1}{2}\theta\right]_0^{\pi/2} = 8\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{2}\pi$$

22. The volume of a cone is $\frac{1}{3}$(base area)height. The base is the ellipse $4x^2 + 9y^2 = 36$, $x^2/9 + y^2/4 = 1$ of area $\pi(3)2 = 6\pi$. $V = \frac{1}{3}(6\pi)1 = 2\pi$

23. The projection of the intersection of $z = 3x^2 + y^2$ and $z = 4 - x^2$ is $4x^2 + y^2 = 4$.

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \Delta_i V = \iiint_S dV = 4\int_0^1 \int_0^{2\sqrt{1-x^2}} \int_{3x^2+y^2}^{4-x^2} dz\, dy\, dx = 4\int_0^1 \int_0^{2\sqrt{1-x^2}} (4 - 4x^2 - y^2) dy\, dx$$

Let $y = 2u$, $dy = 2\, du$. Then use $x^2 + u^2 = r^2$, $du\, dx = dA = r\, dr\, d\theta$.

$$V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} (4 - 4x^2 - 4u^2) du\, dx = 32\int_0^{\pi/2} \int_0^1 (1 - r^2) r\, dr\, d\theta = 32\int_0^{\pi/2} \left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 d\theta = 8\int_0^{\pi/2} d\theta = 4\pi$$

24. Find the volume of the solid enclosed by the sphere $x^2 + y^2 + z^2 = a^2$.

▸ If V cubic units is the volume of the solid,

$$V = \int_{-a}^{a} \left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\, dy \right] dx$$

The integral in brackets represents the area of a circle of radius $\sqrt{a^2 - x^2}$ and has the value $\pi(a^2 - x^2)$. Thus,

$$V = \int_{-a}^{a} \pi(a^2 - x^2) dx \tag{1}$$

$$= \pi\left[a^2x - \tfrac{1}{3}x^3\right]_{-a}^{a} = \tfrac{4}{3}\pi a^3$$

Therefore, the volume of the solid is $\frac{4}{3}\pi a^3$ cubic units. Note that (1) is equivalent to the method of slicing.

25. S is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

$$V = \iiint_S dV = 8\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\, dy\, dx = 8c\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx$$

Let $x = au$, $dx = a\, du$, $y = bv$, $dy = b\, dV$. Then let $u^2 + v^2 = r^2$, $du\, dv = dA = r\, dr\, d\theta$.

$$V = 8abc\int_0^1 \int_0^{\sqrt{1-u^2}} \sqrt{1 - u^2 - v^2}\, dV\, du = 8abc\int_0^{\pi/2} \int_0^1 (1 - r^2)^{1/2} r\, dr\, d\theta = 8abc\int_0^{\pi/2} -\tfrac{1}{3}(1 - r^2)^{3/2}\Big]_0^1 d\theta$$

$$= 8abc\int_0^{\pi/2} \frac{1}{3}\, d\theta = \frac{4}{3}\pi abc$$

26. S is bounded by the cylinders $z = 5x^2$ and $z = 3 - x^2$ and the planes $y + z = 4$, $y = 0$. $5x^2 = 3 - x^2$, $x^2 = \frac{1}{2}$

$$V = \iiint_S dV = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{5x^2}^{3-x^2} \int_0^{4-z} dy\, dz\, dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{5x^2}^{3-x^2} (4 - z) dz\, dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} (4z - \tfrac{1}{2}z^2)\Big]_{5x^2}^{3-x^2} dx$$

$$= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} [12 - 4x^2 - \tfrac{1}{2}(9 - 6x^2 + x^4) - 20x^2 + \tfrac{25}{2}x^4] dx = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} (\tfrac{15}{2} - 21x^2 + 12x^4) dx$$

$$= 2\left[\tfrac{15}{2}x - 7x^3 + \tfrac{12}{5}x^5\right]_0^{\sqrt{2}/2} = \tfrac{23}{5}\sqrt{2}$$

27. S is bounded by the cylinder $z = 4 - x^2$, $y = 5$ and the coordinate planes; $\rho(x, y, z) = k$.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k\, \Delta_i V = k\iiint_S dV = k\int_0^2 \int_0^5 \int_0^{4-x^2} dz\, dy\, dx = k\int_0^2 \int_0^5 z\Big]_0^{4-x^2} dy\, dx = k\int_0^2 \int_0^5 (4 - x^2) dy\, dx$$

$$= k\int_0^2 4y - x^2y\Big]_0^5 dx = k\int_0^2 (20 - 5x^2) dx = k\left[20x - \tfrac{5}{3}x^3\right]_0^2 = k\left(40 - \tfrac{40}{3}\right) = \tfrac{80}{3}k$$

28. Find the mass of the solid enclosed by the tetrahedron formed by the plane $100x + 25y + 16z = 400$ and the coordinate planes if the volume density varies as the distance from the yz plane. The volume density is measured in kg/m^3.

▸ Let $\rho(x, y, z) = kx$ be the measure of the volume density at (x, y, z). Then the measure of the mass of the solid S is given by

$$M = \iiint_S kx\, dV = \int_0^{25} \int_0^{(400-16z)/25} \int_0^{(400-25y-16z)/100} kx\, dx\, dy\, dz$$

$$= \int_0^{25} \int_0^{(400-16z)/25} \frac{k}{2}x^2\Big]_0^{(400-25y-16z)/100} dy\, dz = \int_0^{25} \int_0^{(400-16z)/25} \frac{k}{20{,}000}(400 - 25y - 16z)^2\, dy\, dz$$

$$= \int_0^{25} \frac{k}{20{,}000} \cdot \frac{1}{3(-25)}(400 - 25y - 16z)^3\Big]_0^{(400-16z)/25} dz = \int_0^{25} \frac{k}{1{,}500{,}000}(400 - 16z)^3\, dz$$

$$= \frac{k}{1{,}500{,}000} \cdot \frac{1}{4(-16)}(400 - 16z)^4\Big]_0^{25} = \frac{(400)^4 k}{1{,}500{,}000(64)} = \frac{800k}{3}$$

Thus, the mass is $\frac{800}{3}k$ kilograms.

Also, because $f(x, y, z) = kx$ is of degree 1, we may use Simpson's rule. The vertices are $(4, 0, 0)$, $(0, 16, 0)$, $(0, 0, 25)$, and $(0, 0, 0)$. The centroid is $G = (1, 4, \frac{25}{4})$ and $V = \frac{1}{6} \cdot 4 \cdot 16 \cdot 25 = \frac{800}{3}$. Therefore,

$$\iiint_S kx dV = \frac{V}{20}[f(4,0,0) + f(0,16,0) + f(0,0,25) + f(0,0,0) + 16f(1,4,\tfrac{25}{4})] = \tfrac{40}{3}[4k + 0 + 0 + 0 + 16k] = \tfrac{800}{3}k$$

29. S is bounded by $x = z^2$, $y = x^2$ and the planes $y = 0$ and $z = 0$. $\rho(x,y,z) = kxyz$ kg/m^3.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k u_i v_i w_i \Delta_i V = k \iiint_S xyz\, dV = k \int_0^1 \int_0^{x^2} \int_0^{\sqrt{x}} xyz\, dz\, dy\, dx = k \int_0^1 \int_0^{x^2} \frac{1}{2} xyz^2 \Big]_0^{\sqrt{x}} dy\, dx$$
$$= \frac{k}{2} \int_0^1 \int_0^{x^2} x^2 y\, dy\, dx = \frac{k}{2} \int_0^1 \frac{1}{2} x^2 y^2 \Big]_0^{x^2} dx = \frac{k}{4} \int_0^1 x^6 dx = \frac{1}{28} kx^7 \Big]_0^1 = \frac{1}{28} k$$

30. Take 4 times the part in the first quadrant. $m = 4 \int_{y=0}^{2} \int_{x=0}^{\sqrt{4-y^2}/2} \int_{z=0}^{4-4x^2-y^2} 3xz\, dz\, dx$

$$= 6 \int_0^2 \int_0^{\sqrt{4-y^2}/2} x(4 - 4x^2 - y^2)^2 dx\, dy = 6(-\tfrac{1}{24}) \int_0^2 (4 - 4x^2 - y^2)^3 \Big]_0^{\sqrt{4-y^2}/2} dy = \tfrac{1}{4} \int_0^2 (4 - y^2)^3 dy$$
$$= \tfrac{1}{4} \int_0^2 (64 - 48y^2 + 12y^4 - y^6) dy = \tfrac{1}{4}(64 \cdot 2 - 16 \cdot 8 + \tfrac{12}{5} \cdot 32 - \tfrac{1}{8} \cdot 128) = \tfrac{512}{35}$$

31. S is bounded by $z = xy$ and the planes $x = 1$, $y = 1$ and $z = 0$. $\rho(x,y,z) = 3\sqrt{x^2 + y^2}$ kg/m^3.

$$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 3\sqrt{u_i^2 + v_i^2}\, \Delta_i V = 3 \iiint_S \sqrt{x^2 + y^2}\, dV = 3 \int_0^1 \int_0^1 \int_0^{xy} \sqrt{x^2 + y^2}\, dz\, dy\, dx$$
$$= 3 \int_0^1 \int_0^1 xy(x^2 + y^2)^{1/2} dy\, dx = 3 \int_0^1 \tfrac{1}{3} x(x^2 + y^2)^{3/2} \Big]_0^1 dx = \int_0^1 [x(x^2 + 1)^{3/2} - x^4] dx = \tfrac{1}{5}(x^2 + 1)^{5/2} - \tfrac{1}{5} x^5 \Big]_0^1$$
$$= \tfrac{4}{5}\sqrt{2} - \tfrac{1}{5} - \tfrac{1}{5} = \tfrac{2}{5}(2\sqrt{2} - 1)$$

32. A solid has the shape of a right-circular cylinder of base radius r meters and height h meters. find the mass of the solid if the volume density varies as the distance from one of the bases. The volume density is measured in kg/m^3.

▸ Let one base of the solid be the region R enclosed by a circle in the xy plane with center at the origin and radius r meters. An equation of the circle is $x^2 + y^2 = r^2$. Let $\rho(x, y, z) = kz$. Then the number of units in the mass of the solid is

$$M = \iiint_S \rho(x, y, z) dV = \iint_R \int_0^h kz\, dz\, dA = \iint_R \tfrac{1}{2} kz^2 \Big]_0^h dA = \tfrac{1}{2} kh^2 \iint_R dA = \tfrac{1}{2} kh^2 (\pi r^2) = \tfrac{1}{2} \pi k h^2 r^2$$

The mass of the solid is $\frac{1}{2}\pi k h^2 r^2$ kilograms.

13.6 THE TRIPLE INTEGRAL IN CYLINDRICAL AND SPHERICAL COORDINATES

Cylindrical Coordinates Let f be a continuous function of r, θ, and z. Let S be a region in R^3 bounded by the planes $\theta = \theta_1$ and $\theta = \theta_2$, the cylinders $r = g_1(\theta)$ and $r = g_2(\theta)$ that are perpendicular to the polar plane, and by the surfaces $z = h_1(r, \theta)$ and $z = h_2(r, \theta)$, where $\theta_1 < \theta_2$, $g_1(\theta) \le g_2(\theta)$, and $h_1(r, \theta) \le h_2(r, \theta)$, and the functions g_1, g_2, h_1, and h_2 are smooth. Then the triple integral of f on S is equivalent to an iterated integral with

$$\iiint_S f(r, \theta, z) dV = \iiint_S f(r, \theta, z) r\, dz\, dr\, d\theta$$
$$= \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r, \theta, dz) r\, dz\, dr\, d\theta$$

Other orders of integration are also possible. Appropriate when there is symmetry with respect to an axis or a cylinder which contains an axis.

Spherical Coordinates Let f be a continuous function of ρ, θ, and ϕ. Let S be a region in R^3 bounded by the planes $\theta = \theta_1$ and $\theta = \theta_2$, the cones $\phi = g_1(\theta)$ and $\phi = g_2(\theta)$ and the surfaces $\rho = h_1(\phi, \theta)$ and $\rho = h_2(\phi, \theta)$ where $\theta_1 < \theta_2$, $g_1(\theta) \le g_2(\theta)$ and $h_1(\phi, \theta) \le h_2(\phi, \theta)$ and the functions g_1, g_2, h_1, and h_2 are smooth. Then the triple integral of f on S is equivalent to an iterated integral with

$$\iiint_S f(\rho, \theta, \phi) dV = \iiint_S f(\rho, \theta, \phi) \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$
$$= \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(\phi,\theta)}^{h_2(\phi,\theta)} f(\rho, \theta, \phi) \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Other orders of integration are also possible. Appropriate when a boundary is a cone, or a sphere centered at the origin or containing the origin.

Because the factors r and ρ^2, some integrals which are improper in rectangular coordinates become proper when we switch to cylindrical or spherical coordinates. See Ex. 32.

Exercises 13.6

In Exercises 1–6, evaluate the iterated integral.

1. $\int_0^{\pi/4}\int_0^{a}\int_0^{r\cos\theta} r\sec^3\theta\,dz\,dr\,d\theta = \int_0^{\pi/4}\int_0^{a} z\Big]_0^{r\cos\theta} r\sec^3\theta\,dr\,d\theta = \int_0^{\pi/4}\int_0^{a} r^2\sec^2\theta\,dr\,d\theta$
$= \int_0^{\pi/4} \frac{1}{3}r^3\Big]_0^{a}\sec^2\theta\,d\theta = \frac{1}{3}a^3\tan\theta\Big]_0^{\pi/4} = \frac{1}{3}a^3$

2. $\int_0^{\pi/4}\int_{2\sin\theta}^{2\cos\theta}\int_0^{r\sin\theta} r^2\cos\theta\,dz\,dr\,d\theta = \int_0^{\pi/4}\int_{2\sin\theta}^{2\cos\theta} r^3\sin\theta\cos\theta\,dr\,d\theta$
$= \int_0^{\pi/4}\frac{1}{4}[(2\cos\theta)^4 - (2\sin\theta)^4]\sin\theta\cos\theta\,d\theta = 4\int_0^{\pi/4}(1-2\sin^2\theta)\sin\theta\cos\theta\,d\theta = 4\Big[\frac{1}{2}\sin^2\theta - \frac{1}{2}\sin^4\theta\Big]_0^{\pi/4} = \frac{1}{2}$

3. $\int_0^{\pi}\int_0^{4}\int_0^{1} re^z\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{4} re^z\Big]_0^1 dr\,d\theta = (e-1)\int_0^{\pi}\int_0^{4} r\,dr\,d\theta = (e-1)\int_0^{\pi}\frac{1}{2}r^2\Big]_0^4 d\theta = (e-1)\int_0^{\pi} 8\,d\theta = 8\pi(e-1)$

4. $\int_0^{2\pi}\int_0^{\pi}\int_0^{2}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta$

▷ $\int_0^{2\pi}\int_0^{\pi}\int_0^{2}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\Big[\frac{1}{4}\rho^4\sin\phi\Big]_0^2 d\phi\,d\theta = 4\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = 4\int_0^{2\pi} -\cos\phi\Big]_0^{\pi} d\theta$
$= 4\int_0^{2\pi} 2\,d\theta = 16\pi$

5. $\int_0^{\pi/4}\int_0^{2a\cos\phi}\int_0^{2\pi}\rho^2\sin\phi\,d\theta\,d\rho\,d\phi = \int_0^{\pi/4}\int_0^{2a\cos\phi}\theta\Big]_0^{2\pi}\rho^2\sin\phi\,d\rho\,d\phi = 2\pi\int_0^{\pi/4}\int_0^{2a\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi$
$= 2\pi\int_0^{\pi/4}\frac{1}{3}\rho^3\Big]_0^{2a\cos\phi}\sin\phi\,d\phi = \frac{16}{3}\pi a^3\int_0^{\pi/4}\cos^3\phi\sin\phi\,d\phi = -\frac{16}{3}\pi a^3\Big[\frac{1}{4}\cos^4\phi\Big]_0^{\pi/4} = -\frac{4}{3}\pi a^3(\frac{1}{4}-1) = \pi a^3$

6. $\int_{\pi/4}^{\pi/2}\int_{\pi/4}^{\phi}\int_0^{a\csc\theta}\rho^3\sin^2\theta\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/4}^{\pi/2}\int_{\pi/4}^{\phi}\frac{1}{4}(a^4\csc^4\theta)\sin^2\theta\sin\phi\,d\theta\,d\phi = \frac{1}{4}a^4\int_{\pi/4}^{\pi/2}\int_{\pi/4}^{\phi}\csc^2\theta\sin\phi\,d\phi$
$= \frac{1}{4}a^4\int_{\pi/4}^{\pi/2}(1-\cot\phi)\sin\phi\,d\phi = \frac{1}{4}a^4\int_{\pi/4}^{\pi/2}(\sin\phi-\cos\phi)d\phi = \frac{1}{4}a^4\Big[-\cos\phi-\sin\phi\Big]_{\pi/4}^{\pi/2} = \frac{1}{4}a^4(\sqrt{2}-1)$

7. (a) $V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\Delta_i V = \iiint_S dV = 2\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = 2\int_0^{2\pi}\int_0^{a} rz\Big]_0^{\sqrt{a^2-r^2}} dr\,d\theta$
$= 2\int_0^{2\pi}\int_0^{a} r(a^2-r^2)^{1/2}dr\,d\theta = -\frac{2}{3}\int_0^{2\pi}(a^2-r^2)^{3/2}\Big]_0^{a} d\theta = \frac{2}{3}a^3\int_0^{2\pi} d\theta = \frac{4}{3}\pi a^3$

(b) $V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\Delta_i V = \iiint_S dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{a}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\int_0^{2\pi}\frac{1}{3}\rho^3\Big]_0^{a}\sin\phi\,d\theta\,d\phi$
$= \frac{1}{3}a^3\int_0^{\pi}\int_0^{2\pi}\sin\phi\,d\theta\,d\phi = \frac{1}{3}a^3\int_0^{\pi}\theta\Big]_0^{2\pi}\sin\phi\,d\phi = \frac{2}{3}\pi a^3\int_0^{\pi}\sin\phi\,d\phi = -\frac{2}{3}\pi a^3\cos\phi\Big]_0^{\pi} = \frac{4}{3}\pi a^3$

8. If S is the solid in the first octant bounded by the sphere $x^2+y^2+z^2=16$ and the coordinate planes, evaluate the triple integral $\iiint_S xyz\,dV$ by three methods: (a) using spherical coordinates; (b) using rectangular coordinates; (c) using cylindrical coordinates.

▷ (a) In spherical coordinates we have
$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi \quad dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$
Thus,
$$\iiint_S xyz\,dV = \iiint_S \rho^5\sin^3\phi\cos\phi\sin\theta\cos\theta\,d\rho\,d\theta\,d\phi$$
Because s is bounded by the point sphere $\rho=0$ and the sphere $\rho=4$, bounded by the cylinders $\theta=0$ and $\theta=\frac{1}{2}\pi$, and by the cones $\phi=0$ and $\phi=\frac{1}{2}\pi$, the triple integral may be replaced by an iterated integral. Thus,
$$\iiint_S xyz\,dV = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{4}\rho^5\sin^3\phi\cos\phi\sin\theta\cos\theta\,d\rho\,d\theta\,d\phi$$
$$= \int_0^4\rho^5 d\rho\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^{\pi/2}\sin^3\phi\cos\phi\,d\phi = \Big[\frac{1}{6}\rho\Big]_0^4\cdot\Big[\frac{1}{2}\sin^2\theta\Big]_0^{\pi/2}\cdot\Big[\frac{1}{4}\sin^4\phi\Big]_0^{\pi/2}$$
$$= \frac{1}{6}\cdot 4096\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{256}{3}$$

(b) In rectangular coordinates, s is bounded by the surface $z=0$ and $z=\sqrt{16-x^2-y^2}$, bounded by the cylinders $y=0$ and $y=\sqrt{16-x^2}$, and bounded by the planes $x=0$ and $x=4$. Thus,
$$\iiint_S xyz\,dV = \int_0^4\int_0^{\sqrt{16-x^2}}\int_0^{\sqrt{16-x^2-y^2}} xyz\,dz\,dy\,dx = \frac{1}{2}\int_0^4\int_0^{\sqrt{16-x^2}} xy(16-x^2-y^2)dy\,dx$$
$$= -\frac{1}{8}\int_0^4 x(16-x^2-y^2)^2\Big]_{y=0}^{\sqrt{16-x^2}} dx = \frac{1}{8}\int_0^4 x(16-x^2)^2 dx = -\frac{1}{48}(16-x^2)^3\Big]_0^4 = \frac{256}{3}$$

(c) In cylindrical coordinates, we have
$x = r\cos\theta \quad y = r\sin\theta \quad dV = r\,dz\,dr\,d\theta$
Thus,
$$\iiint_S xyz\,dV = \iiint_S r^3\sin\theta\cos\theta\, z\,dz\,dr\,d\theta$$
Because S is bounded by the surfaces $z=0$ and $z=\sqrt{16-r^2}$, by the cylinders $r=0$ and $r=4$, and by the planes $\theta=0$ and $\theta=\frac{1}{2}\pi$, the triple integral may be replaced by an iterated integral. Thus,
$$\iiint_S xyz\,dV = \int_0^{\pi/2}\int_0^4\int_0^{\sqrt{16-r^2}} r^3\sin\theta\cos\theta\, z\,dz\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^4 \sin\theta\cos\theta\, r^3(16-r^2)dr\,d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^4(16r^3-r^5)dr = \frac{1}{2}\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}\cdot\left[4r^4-\frac{1}{6}r^6\right]_0^4 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1024}{3} = \frac{256}{3}$$

In Exercises 9–16, use cylindrical coordinates.

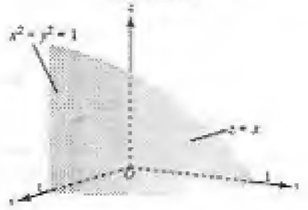
▸ V cubic units is the required volume, M kg is the required mass, ρ is the volume density.

9. $z = r\cos\theta$, $V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\Delta_i V = \iiint_S dV = \int_0^{\pi/2}\int_0^1\int_0^{r\cos\theta} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^2\cos\theta\,dr\,d\theta$
$= \frac{1}{3}\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{3}\sin\theta\Big]_0^{\pi/2} = \frac{1}{3}$

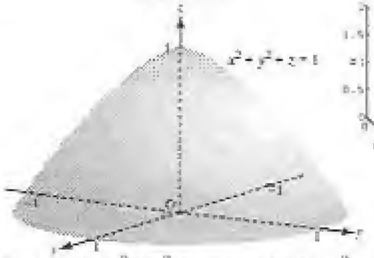
10. $z = 1-(x^2+y^2) = 1-r^2$, $V = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r(1-r^2)dr\,d\theta = \int_0^{2\pi} d\theta\left[\frac{1}{2}r^2-\frac{1}{4}r^4\right]_0^1 = \frac{1}{2}\pi$

11. $x^2+y^2 = r^2$, $V = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\Delta_i V = \iiint_S dV = \int_0^{2\pi}\int_0^2\int_8^{12-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (4-r^2)r\,dr\,d\theta$
$= \int_0^{2\pi} 2r^2-\frac{1}{4}r^4\Big]_0^2\,d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi$

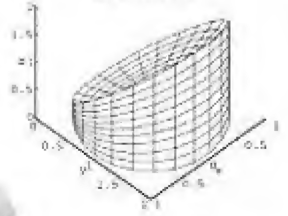
Exercise 9



Exercise 10



Exercise 12



12. Find the volume of the solid bounded by cylinder $x^2+y^2=2y$, paraboloid $x^2+y^2=2z$, and the xy plane.

▸ The figure shows the solid S, which is bounded above by the paraboloid whose cylindrical equation is $2z=r^2$, and below by the xy plane, and bounded by the cylinder whose cylindrical equation is $r = 2\sin\theta$, $0\le\theta\le\pi$. The volume of the solid is given by
$$V = \iiint_S dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2/2} r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{2\sin\theta}\frac{1}{2}r^3\,dr\,d\theta = \int_0^{\pi} 2\sin^4\theta\,d\theta$$
$$= 2\left[-\frac{1}{4}\sin^3\theta\cos\theta\Big]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\theta\,d\theta\right] = 2\left[0-\frac{3}{4}\cdot\frac{1}{2}\sin\theta\cos\theta+\frac{3}{4}\cdot\frac{1}{2}\theta\right]_0^{\pi} = \frac{3}{4}\pi$$

13. The solid is bounded by the sphere $r^2+z^2=a^2$. $\rho(r,\theta,z) = k(r^2+z^2)\text{kg/m}^3$.
$M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} k(r_i^2+z_i^2)\Delta_i V = k\iiint_S(r^2+z^2)dV = 2k\int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}}(r^2+z^2)r\,dz\,dr\,d\theta$
$= 2k\int_0^{2\pi}\int_0^a r^2z+\frac{1}{3}z^3\Big]_0^{\sqrt{a^2-r^2}} r\,dr\,d\theta = 2k\int_0^{2\pi}\int_0^a\left[r^2\sqrt{a^2-r^2}+\frac{1}{3}(a^2-r^2)^{3/2}\right]r\,dr\,d\theta$
Let $u^2 = a^2-r^2$, $2u\,du = -2r\,dr$. Then $M = 4\pi k\int_0^a\left[(a^2-u^2)u+\frac{1}{3}u^3\right]u\,du = 4\pi k\left[\frac{1}{3}a^2u^3-\frac{2}{15}u^5\right]_0^a = \frac{4}{5}a^5\pi k$

14. $x^2+y^2=4x$, $r^2 = 4r\cos\theta$, $r = 4\cos\theta$, $0\le\theta\le\frac{1}{2}\pi$. $x^2+y^2+z^2 = r^2+z^2 = 16$. $\rho = kz$.
$M = \int_0^{\pi/2}\int_0^{4\cos\theta}\int_0^{\sqrt{16-r^2}} kz\,r\,dz\,dr\,d\theta = k\int_0^{\pi/2}\int_0^{4\cos\theta}\frac{1}{2}(16-r^2)r\,dr\,d\theta = -\frac{1}{8}k\int_0^{\pi/2}(16-r^2)^2\Big]_0^{4\cos\theta}d\theta$
$= 32k\int_0^{\pi/2}(1-\sin^4\theta)d\theta = 16k\pi - 32k\left[-\frac{1}{4}\sin^3\theta\cos\theta\Big]_0^{\pi/2}+\frac{3}{4}\int_0^{\pi/2}\sin^2\theta\right]$
$= 16k\pi - 32k\left[0-\frac{3}{4}\cdot\frac{1}{2}\sin\theta\cos\theta\Big]_0^{\pi/2}+\frac{3}{4}\cdot\frac{1}{2}\theta\Big]_0^{\pi/2}\right] = 16k\pi - 6k\pi = 10k\pi$

15. S is bounded by $r = 5$, $z = r$, and the xy plane. $\rho = k$ slug/ft^3. I_z is in slug-ft^2.

$$I_z = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} r_i^2 k \, \Delta_i V = k \iiint_S r^2 dV = k \int_0^{2\pi} \int_0^5 \int_0^r r^2 (r \, dz \, dr \, d\theta) = k \int_0^{2\pi} \int_0^5 r^3 z \Big]_0^r dr \, d\theta = k \int_0^{2\pi} \int_0^5 r^4 dr \, d\theta$$
$$= k \int_0^{2\pi} \frac{1}{5} r^5 \Big]_0^5 d\theta = 625k \int_0^{2\pi} d\theta = 1250\pi k$$

16. Find the moment of inertia of the solid bounded by a right-circular cylinder of altitude h meters and radius a meters, with respect ot the axis of the cylinder. The volume density varies as the distance from the axis of the cylinder and is measured in kg/m^3.

▸ Take the z axis as the axis of the cylinder with the base of the cylinder in the polar plane. We are given that the measure of the volume density is $\rho(r, \theta, z) = kr$. If S is the region occupied by the solid, then the measure of its moment of inertia with respect to the axis of the cylinder is given by

$$I_z = \iiint_S r^2 (kr)(r \, dr \, d\theta \, dz) = k \left[\int_0^h dz \right] \cdot \left[\int_0^{2\pi} d\theta \right] \cdot \left[\int_0^a r^4 dr \right] = k \cdot h \cdot 2\pi \cdot \tfrac{1}{5} a^5 = \tfrac{2}{5} \pi k h a^5$$

Thus, the moment of inertia is $\frac{2}{5}\pi k h a^5$ kg-m^3.

17. $$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} k \bar{\rho}^2 \Delta_i = \iiint_S k\rho^2 dV = 8k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^2 \cdot \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$
$$= 8k \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{5} a^5 \sin \phi \, d\theta \, d\phi = \frac{8}{5} a^5 k \int_0^{\pi/2} \frac{\pi}{2} \cdot \sin \phi \, d\phi = \frac{4}{5} a^5 \pi k \Big[-\cos \phi \Big]_0^{\pi/2} = \frac{4}{5} a^5 \pi k$$

18. $\rho = kr = k\rho \sin \phi$. $$M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a k\rho \sin \phi \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = k \int_0^{2\pi} d\theta \int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 2\phi) d\phi \int_0^a \rho^3 d\rho$$
$$= k \cdot 2\pi \cdot \tfrac{1}{2} \Big[\phi - \tfrac{1}{2} \sin 2\phi \Big]_0^{\pi/2} \cdot \tfrac{1}{4} a^4 = \tfrac{1}{8} k a^4 \pi^2$$

In Exercises 19–22, use spherical coordinates.

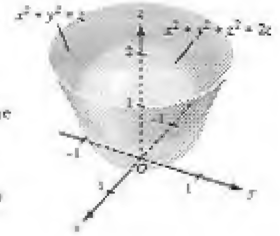
19. $x^2 + y^2 + z^2 = 4z$; $\rho^2 = 4\rho \cos \phi$; $\rho = 4 \cos \phi$ and $x^2 + y^2 + z^2 = 2z^2$; $\rho^2 = 2\rho^2 \cos^2 \phi$; $\cos^2 \phi = \frac{1}{2}$; $\phi = \frac{1}{4}\pi$

$$V = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \Delta_i V = \iiint_S dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{4 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3} \rho^3 \Big]_0^{4 \cos \phi} \sin \phi \, d\phi \, d\theta$$
$$= \frac{64}{3} \int_0^{2\pi} \int_0^{\pi/4} \cos^3 \phi \sin \phi \, d\phi \, d\theta = \frac{64}{3} \cdot 2\pi \Big[-\frac{1}{4} \cos^4 \phi \Big]_0^{\pi/4} = \frac{32}{3}\pi \Big(-\frac{1}{4} + 1 \Big) = 8\pi$$

20. Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 2z$ and above the paraboloid $x^2 + y^2 = z$.

▸ Eliminating x and y from the given equations, we obtain $z^2 = z$. Thus, the sphere and the paraboloid intersect in the plane $z = 1$, where $\phi = \frac{1}{4}\pi$, and in the origin, corresponding to $\phi = \frac{1}{2}\pi$. Because the sphere has the equation $x^2 + y^2 + (z-1)^2 = 1$, its center lies in the plane $z = 1$ and its radius is 1. The figure shows the solid. Because $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho \cos \phi$, a spherical equation of the sphere is

$$\rho^2 = 2\rho \cos \phi$$
$$\rho = 2 \cos \phi$$



Because $x^2 + y^2 = r^2 = (\rho \sin \phi)^2$, a spherical equation of the paraboloid is

$$\rho \cos \phi = \rho^2 \sin^2 \phi$$
$$\rho = \cot \phi \csc \phi$$

The measure of the volume of the entire sphere is $V_1 = \frac{4}{3}\pi$. The measure of the volume of the solid is $V_1 - V_2$, where V_2 is the measure of the volume of the sphere outside the paraboloid. Then

$$V_2 = \int_{\pi/4}^{\pi/2} \int_{\cot \phi \csc \phi}^{2 \cos \phi} \int_0^{2\pi} \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi = 2\pi \int_{\pi/4}^{\pi/2} \int_{\cot \phi \csc \phi}^{2 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi$$
$$= 2\pi \int_{\pi/4}^{\pi/2} \Big[\tfrac{1}{3} \rho^3 \sin \phi \Big]_{\cot \phi \csc \phi}^{2 \cos \phi} d\phi = \tfrac{2}{3}\pi \int_{\pi/4}^{\pi/2} (\cos^3 \phi \sin \phi - \cot^3 \phi \csc^2 \phi) d\phi$$
$$= \tfrac{2}{3}\pi \Big[-\tfrac{1}{4} \cos^4 \phi + \tfrac{1}{4} \cot^4 \phi \Big]_{\pi/4}^{\pi/2} = \tfrac{2}{3}\pi \cdot \tfrac{1}{4} = \tfrac{1}{6}\pi$$

Because $V_1 - V_2 = \frac{4}{3}\pi - \frac{1}{6}\pi = \frac{7}{6}\pi$, the volume of the solid is $\frac{7}{6}\pi$ cubic units.

21. $x^2+y^2=2x$; $r^2=2r\cos\theta$; $r=2\cos\theta$; $\rho=2\sin\theta\csc\phi$ and $x^2+y^2+z^2=2z^2$; $\rho^2=2\rho^2\cos^2\phi$; $\phi=\frac{1}{4}\pi$. If I_z kg-m^2 is the moment of inertia with respect to the z axis, then

$$I_z=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\bar{\rho}_i\sin\bar{\phi}_i)^2k\,\Delta_iV=\iiint_S k\rho^2\sin^2\phi\,dV=k\int_{-\pi/2}^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta\csc\phi}(\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=k\int_{-\pi/2}^{\pi/2}\int_{\pi/4}^{\pi/2}\tfrac{1}{5}\rho^5\Big]_0^{2\cos\theta\csc\phi}\sin^3\phi\,d\phi\,d\theta=\frac{32}{5}k\int_{-\pi/2}^{\pi/2}\int_{\pi/4}^{\pi/2}\cos^5\theta\csc^2\phi\,d\phi\,d\theta$$

$$=\frac{32}{5}k\int_{-\pi/2}^{\pi/2}-\cos^5\theta\cot\phi\Big]_{\pi/4}^{\pi/2}d\theta=\frac{32}{5}k\int_{-\pi/2}^{\pi/2}\cos^5\theta\,d\theta\overset{\text{even}}{=}\frac{64}{5}k\int_0^{\pi/2}(1-2\sin^2\theta+\sin^4\theta)\cos\theta\,d\theta$$

$$=\frac{64}{5}k\Big[\sin\theta-\tfrac{2}{3}\sin^3\theta+\tfrac{1}{5}\sin^5\theta\Big]_0^{\pi/2}=\frac{64}{5}k\Big(1-\tfrac{2}{3}+\tfrac{1}{5}\Big)=\tfrac{512}{75}k$$

22. $r^2=(\rho\sin\phi)^2$. $I_z=\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^2\rho^2\sin^2\phi\cdot k\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=k\int_0^{2\pi}d\theta\int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi\int_0^2\rho^4d\rho$

$$=2k\pi\Big[\tfrac{1}{3}\cos^3\phi-\cos\phi\Big]_0^{\pi}\Big[\tfrac{1}{5}\rho^5\Big]_0^2=2k\pi\cdot\tfrac{4}{3}\cdot\tfrac{32}{5}=\tfrac{256}{15}k\pi$$

In Exercises 23–28, use the coordinate system that you decide is best for the problem.

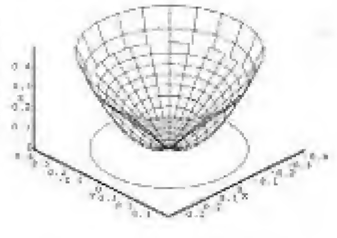
23. The solid is the upper half of the sphere $\rho=2$. $\rho(\rho,\theta,\phi)=k\rho$ kg/m^3.

$$M=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}k\bar{\rho}_i\Delta_iV=k\iiint_S\rho\,dV=4k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2\rho(r^2\sin\phi\,d\rho\,d\phi\,d\theta)=4k\int_0^{\pi/2}\int_0^{\pi/2}\tfrac{1}{4}\rho^4\Big]_0^2\sin\phi\,d\phi\,d\theta$$

$$=16k\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\,d\phi\,d\theta=16k\int_0^{\pi/2}(-\cos\phi)\Big]_0^{\pi/2}d\theta=16k\int_0^{\pi/2}d\theta=8k\pi$$

24. Find the mass of the homogeneous solid inside the paraboloid $3x^2+3y^2=z$ and outside the cone $x^2+y^2=z^2$ if the constant volume density is k kg/m^3.

▸ Cylindrical coordinates are best because a cylindrical equation of the paraboloid is $z=3r^2$ and a cylindrical equation of the cone is $z=r$. Eliminating z from the cylindrical equations, we obtain $r=3r^2$. Thus, $r=0$ or $r=\frac{1}{3}$. The figure shows the solid, which is bounded below by the paraboloid (shown cut open) above by the cone, and bounded by the cylinder $r=\frac{1}{3}$. If M is the measure of the mass of the solid, then



$$M=\int_0^{1/3}\int_{3r^2}^{r}\int_0^{2\pi}kr\,d\theta\,dz\,dr=2\pi k\int_0^{1/3}\int_{3r^2}^{r}r\,dz\,dr$$

$$=2\pi k\int_0^{1/3}r(3-3r^2)dr=2\pi k\Big[\tfrac{1}{3}r^3-\tfrac{3}{4}r^4\Big]_0^{1/3}=\tfrac{1}{162}\pi k$$

The mass is $\frac{1}{162}\pi k$ kg.

In Exercises 25 and 26, the solid is bounded by the spheres $\rho=a$ and $\rho=2a$. $\rho(\rho,\theta,\phi)=k\rho^{-2}$ slugs/ft^3.

25. If I_z slug-ft^2 is the moment of inertia of the solid about the z axis, then

$$I_z=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\bar{\rho}_i\sin\phi_i)^2(k\bar{\rho}_i^{-2})\Delta_iV=\iiint_S\rho^2\sin^2\phi(k\rho^{-2})dV=k\int_0^{\pi}\int_0^{2\pi}\int_a^{2a}\sin^2\phi(\rho^2\sin\phi\,d\rho\,d\theta\,d\phi)$$

$$=k\int_0^{\pi}\int_0^{2\pi}\tfrac{1}{3}\rho^3\Big]_a^{2a}\sin^3\phi\,d\theta\,d\phi=\tfrac{7}{3}a^3k\int_0^{\pi}\int_0^{2\pi}\sin^3\phi\,d\theta\,d\phi=\tfrac{7}{3}a^3k\int_0^{\pi}\theta\Big]_0^{2\pi}\sin^3\phi\,d\phi=\tfrac{14}{3}\pi a^3k\int_0^{\pi}\sin^3\phi\,d\phi$$

$$=\tfrac{14}{3}\pi a^3k\int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi=\tfrac{14}{3}\pi a^3k\Big[-\cos\phi+\tfrac{1}{3}\cos^3\phi\Big]_0^{\pi}=\tfrac{14}{3}\pi a^3k\Big(\tfrac{4}{3}\Big)=\tfrac{56}{9}\pi a^3k$$

26. $M=k\displaystyle\int_0^{\pi}\int_0^{2\pi}\int_a^{2a}k\rho^{-2}(\rho^2\sin\phi\,d\rho\,d\theta\,d\phi)=k\int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_a^{2a}d\rho=k\cdot 2\cdot 2\pi\cdot a=4\pi ka$

In Exercises 27 and 28, the solid is inside the paraboloid $x^2+y^2=z$ and outside the cone $x^2+y^2=z^2$. The constant volume density is k kg/m^3.

27. $x^2+y^2=z$; $z=r^2$ and $x^2+y^2=z^2$; $z=r$

$$M=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}k\,\Delta_iV=k\iiint_S dV=k\int_0^{2\pi}\int_0^1\int_{r^2}^{r}r\,dz\,dr\,d\theta=k\int_0^{2\pi}\int_0^1 rz\Big]_{r^2}^{r}dr\,d\theta=k\int_0^{2\pi}\int_0^1(r^2-r^3)dr\,d\theta$$

$$=k\int_0^{2\pi}\tfrac{1}{3}r^3-\tfrac{1}{4}r^4\Big]_0^1d\theta=\tfrac{1}{12}k\int_0^{2\pi}d\theta=\tfrac{1}{6}\pi k$$

If M_{xy} kg-m is the moment of mass with respect to the xy plane, then

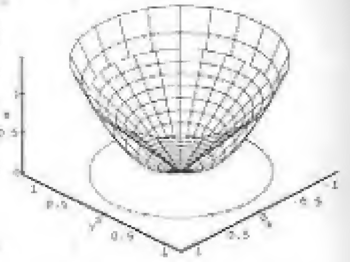
$$M_{xy}=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}z_i(k)\Delta_iV=k\iiint_S z\,dV=\int_0^{2\pi}\int_0^1\int_{r^2}^{r}zr\,dz\,dr\,d\theta=k\int_0^{2\pi}\int_0^1\tfrac{1}{2}rz^2\Big]_{r^2}^{r}dr\,d\theta$$

$$=\frac{k}{2}\int_0^{2\pi}\int_0^1(r^3-r^5)dr\,d\theta=\frac{k}{2}\int_0^{2\pi}\tfrac{1}{4}r^4-\tfrac{1}{6}r^6\Big]_0^1d\theta=\tfrac{1}{24}k\int_0^{2\pi}d\theta=\tfrac{1}{12}\pi k$$

Because of symmetry with respect to the z axis the center of mass is at $(0,0,\bar{z})$ where $\bar{z} = \frac{1}{M}M_{xy} = \frac{6}{\pi k} \cdot \frac{\pi k}{12} = \frac{1}{2}$.

28. Find the moment of inertia with respect to the z axis of the solid.

▸ Eliminating x and y from the given equations, we have $z^2 = z$. Thus, the given surfaces intersect in the planes $z = 0$ and $z = 1$. The figure shows the solid and its projection on the xy plane. We use a cylindrical coordinate system. Thus, the solid is bounded below by the paraboloid $z = r^2$ (shown cut open) and bounded above by the cone $z = r$, bounded by the cylinders $r = 0$ and $r = 1$, $0 \leq \theta \leq 2\pi$. Therefore, the measure of the moment of inertia with respect to the z axis is given by



$$I_z = \iiint_S kr^2 dV = k\int_0^{2\pi}\int_0^1\int_{r^2}^{r} r^2(r\,dz\,dr\,d\theta)$$

$$= k\left[\int_0^{2\pi} d\theta\right] \cdot \int_0^1 r^3(r - r^2)dr = 2\pi k\left[\tfrac{1}{5}r^5 - \tfrac{1}{6}r^6\right]_0^1 = 2\pi k \cdot \tfrac{1}{30} = \tfrac{1}{15}k\pi$$

The moment of inertia is $\frac{1}{15}\pi k$ kg m^3.

In Exercises 29–32, evaluate the iterated integral by using either cylindrical or spherical coordinates.

29. $\int_0^4\int_0^3\int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\,dy\,dx\,dz = \int_0^{\pi/2}\int_0^3\int_0^4 r(r\,dz\,dr\,d\theta) = \int_0^{\pi/2}\int_0^3 r^2 z\Big]_0^4 dr\,d\theta$

$= 4\int_0^{\pi/2}\int_0^3 r^2 dr\,d\theta = 4\int_0^{\pi/2} \frac{1}{3}r^3\Big]_0^3 d\theta = 36\int_0^{\pi/2} d\theta = 18\pi$

30. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{z}{\sqrt{x^2+y^2}} = \int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2}} \frac{z}{r}(r\,dz\,dr\,d\theta) = \int_0^{\pi/2}\int_0^1 \frac{1}{2}(1-r^2)dr\,d\theta = \frac{1}{2}\pi\left[\frac{1}{2}r - \frac{1}{6}r^3\right]_0^1 = \frac{1}{6}\pi$

31. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2 dz\,dx\,dy = \int_0^{\pi/2}\int_0^1\int_r^{\sqrt{2-r^2}} z^2(r\,dz\,dr\,d\theta) = \int_0^{\pi/2}\int_0^1 \frac{1}{3}rz^3\Big]_r^{\sqrt{2-r^2}}$

$= \frac{1}{3}\int_0^{\pi/2}\int_0^1 [r(2-r^2)^{3/2} - r^4]dr\,d\theta = \frac{1}{3}\int_0^{\pi/2}\left[-\frac{1}{5}(2-r^2)^{5/2} - \frac{1}{5}r^5\right]_0^1 = -\frac{1}{15}\int_0^{\pi/2}(2-4\sqrt{2})d\theta$

$= \frac{1}{15}(4\sqrt{2}-2)\theta\Big]_0^{\pi/2} = \frac{1}{15}\pi(2\sqrt{2}-1)$

32. $\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}} \frac{1}{x^2+y^2+z^2}dz\,dx\,dy$

▸ The given iterated integral is equivalent to a triple integral over a region S which is bounded by the surfaces $z = 0$ and $z = \sqrt{4-x^2-y^2}$, bounded by the cylinders $x = 0$ and $x = \sqrt{4-y^2}$, and bounded by the planes $y = 0$ and $y = 2$. Thus, S is the part of the spherical solid enclosed by $x^2+y^2+z^2$ that is in the first octant. We use spherical coordinates. We have

$x^2+y^2+z^2 = \rho^2$ and $dV = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

Therefore

$$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}} \frac{1}{x^2+y^2+z^2}dz\,dx\,dy = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \frac{1}{\rho^2}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \left[\int_0^{\pi/2}\sin\phi\,d\phi\right]\cdot\left[\int_0^{\pi/2}d\theta\right]\cdot\left[\int_0^2 d\rho\right] = 1\cdot\tfrac{1}{2}\pi\cdot 2 = \pi$$

Note that while the original integral is improper, the spherical integral is proper.

Miscellaneous Exercises for Chapter 13

1. $P_{\text{spherical}} = (3, \pi, \frac{1}{3}\pi)$. $r = \rho\sin\phi = 3\sin\frac{1}{3}\pi = 3(\frac{1}{2}\sqrt{3}) = \frac{3}{2}\sqrt{3}$. $\theta = \pi$, $z = 3\cos\frac{1}{3}\pi = 3(\frac{1}{2}) = \frac{3}{2}$.
Hence $P_{\text{cylindrical}} = (\frac{3}{2}\sqrt{3}, \pi, \frac{3}{2})$.

2. $P_{\text{cartesian}} = (-3, \sqrt{3}, 2)$. $\rho = \sqrt{9+3+4} = 4$, $\theta = \tan^{-1}\frac{\sqrt{3}}{-3} + \pi = \frac{5}{6}\pi$. $\phi = \cos^{-1}\frac{2}{4} = \frac{1}{3}\pi$. $P_{\text{spherical}} = (4, \frac{5}{6}\pi, \frac{1}{3}\pi)$

3. (a) $S_{\text{cartesian}}$ is $(x+y)^2 + 1 = z$. Because $x = r\cos\theta$ and $y = r\sin\theta$, $S_{\text{cylindrical}}$ is
$z = (r\cos\theta + r\sin\theta)^2 + 1$; $z = r^2\cos^2\theta + 2r^2\sin\theta\cos\theta + r^2\sin^2\theta + 1$; $z = r^2 + r^2\sin 2\theta + 1$
(b) $S_{\text{cartesian}}$ is $25x^2 + 4y^2 = 100$. Because $x = r\cos\theta$ and $y = r\sin\theta$, $S_{\text{cylindrical}}$ is
$25r^2\sin^2\theta + 4r^2\sin^2\theta = 100$; $r^2(25\cos^2\theta + 4\sin^2\theta) = 100$

4. Find an equation in spherical coordinates of the graph of each of the following equations:
(a) $x^2 + y^2 + 4z^2 = 4$ (b) $4x^2 - 4y^2 + 9z^2 = 36$

▷ (a) We have
$(x^2 + y^2 + z^2) + 3z^2 = 4$
Because $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho \cos \phi$, we obtain
$\rho^2 + 3\rho^2\cos^2\phi = 4$
$\rho^2(1 + 3\cos^2\phi) = 4$
(b) Substituting from Eqs. (III) of Section 15.9 into the given equation, we obtain

$$\begin{aligned} 4\rho^2\sin^2\phi\cos^2\theta - 4\rho^2\sin^2\phi\sin^2\theta + 9\rho^2\cos^2\phi &= 36 \\ 4\rho^2\sin^2\phi(\cos^2\theta - \sin^2\theta) + 9\rho^2\cos^2\phi &= 36 \\ 4\rho^2\sin^2\phi\cos 2\theta + 9\rho^2\cos^2\phi &= 36 \\ \rho^2(4\sin^2\phi\cos 2\theta + 9\cos^2\phi) &= 36 \end{aligned}$$

In Exercises 5–12, evaluate the iterated integral.

5. $\int_0^1 \int_x^{\sqrt{x}} x^2 y\, dy\, dx = \int_0^1 \frac{1}{2} y^2 \Big]_x^{\sqrt{x}} x^2\, dx = \frac{1}{2}\int_0^1 (x^3 - x^4)dx = \frac{1}{2}\Big[\frac{1}{4}x^4 - \frac{1}{5}x^5\Big]_0^1 = \frac{1}{2}\Big(\frac{1}{4} - \frac{1}{5}\Big) = \frac{1}{40}$

6. $\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xy\, dx\, dy = 0$ because R is symmetrical with respect to the axes and the integrand is odd.

7. $\int_0^{\pi/2} \int_0^{2\sin\theta} r\cos^2\theta\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{2} r^2 \Big]_0^{2\sin\theta} \cos^2\theta\, d\theta = 2\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\, d\theta$
$= \frac{1}{4}\int_0^{\pi/2} (1 - \cos 4\theta)d\theta = \frac{1}{4}\Big[\theta - \frac{1}{4}\sin 4\theta\Big]_0^{\pi/2} = \frac{1}{8}\pi$

8. $\int_0^{\pi} \int_0^{3(1+\cos\theta)} r^2 \sin\theta\, dr\, d\theta$

▷ $\int_0^{\pi} \int_0^{3(1+\cos\theta)} r^2\sin\theta\, dr\, d\theta = \int_0^{\pi} \sin\theta\Big[\frac{1}{3}r^3\Big]_0^{3(1+\cos\theta)} d\theta = 9\int_0^{\pi} (1 + \cos\theta)^3 \sin\theta\, d\theta = 9\Big[-\frac{1}{4}(1 + \cos\theta)^4\Big]_0^{\pi} = 36$

9. $\int_0^1 \int_0^z \int_0^{y+z} e^x e^y e^z dx\, dy\, dz = \int_0^1 \int_0^z e^x\Big]_0^{y+z} e^y e^z dy\, dz = \int_0^1 \int_0^z (e^{y+z} - 1)e^y e^z dy\, dz$
$= \int_0^1 \int_0^z (e^{2y}e^{2z} - e^y e^z)dy\, dz = \int_0^1 \Big[\frac{1}{2}e^{2y}e^{2z} - e^y e^z\Big]_0^z dz = \int_0^1 \Big(\frac{1}{2}e^{4z} - e^{2z} - \frac{1}{2}e^{2z} + e^z\Big)dz$
$= \int_0^1 \Big(\frac{1}{2}e^{4z} - \frac{3}{2}e^{2z} + e^z\Big)dz = \Big[\frac{1}{8}e^{4z} - \frac{3}{4}e^{2z} + e^z\Big]_0^1 = \frac{1}{8}e^4 - \frac{3}{4}e^2 + e - \frac{3}{8}$

10. $\int_1^2 \int_3^x \int_0^{\sqrt{3}y} \frac{y}{y^2 + z^2} dz\, dy\, dx = \int_1^2 \int_3^x \Big[\tan^{-1}\frac{z}{y}\Big]_0^{\sqrt{3}y} dy\, dx = \frac{1}{3}\pi \int_1^2 \int_3^x dy\, dx = \frac{1}{3}\pi \int_1^2 (x - 3)dx = \frac{1}{6}\pi(x - 3)^2\Big]_1^2 = -\frac{1}{2}\pi$

11. $\int_0^{\pi/2} \int_{\pi/6}^{\pi/2} \int_0^2 \rho^3 \sin\phi\cos\phi\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2} \int_{\pi/6}^{\pi/2} \frac{1}{4}\rho^4\Big]_0^2 \sin\phi\cos\phi\, d\phi\, d\theta = 4\int_0^{\pi/2} \int_{\pi/6}^{\pi/2} \sin\phi(\cos\phi)d\phi\, d\theta$
$= 4\int_0^{\pi/2} \frac{1}{2}\sin^2\phi\Big]_{\pi/6}^{\pi/2} d\theta = 2\int_0^{\pi/2} \Big(1 - \frac{1}{4}\Big)d\theta = \frac{3}{2}\int_0^{\pi/2} d\theta = \frac{3}{4}\pi$

12. $\int_0^a \int_0^{\pi/2} \int_0^{\sqrt{a^2-z^2}} zre^{-r^2} dr\, d\theta\, dz$

▷ $\int_0^a \int_0^{\pi/2} \int_0^{\sqrt{a^2-z^2}} zre^{-r^2} dr\, d\theta\, dz = \int_0^a \int_0^{\pi/2} z\Big[-\frac{1}{2}e^{-r^2}\Big]_0^{\sqrt{a^2-z^2}} d\theta\, dz = -\frac{1}{2}\int_0^a \int_0^{\pi/2} (ze^{z^2-a^2} - z)d\theta\, dz$
$= -\frac{1}{4}\pi \int_0^a (ze^{z^2-a^2} - z)dz = -\frac{1}{8}\pi\Big[e^{z^2-a^2} - z^2\Big]_0^a = \frac{1}{8}\pi(e^{-a^2} + a^2 - 1)$

In Exercises 13–16, evaluate the multiple integral.

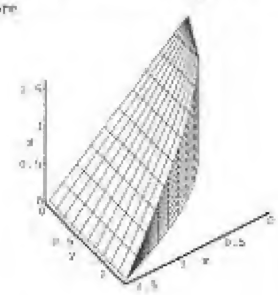
13. $\iint_R xy\, dA = \int_0^{\pi/2} \int_0^1 (r\cos\theta)(r\sin\theta)r\, dr\, d\theta = \int_0^{\pi/2} \int_0^1 r^3 \sin\theta\cos\theta\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{4}r^4\Big]_0^1 \sin\theta\cos\theta\, d\theta$
$= \frac{1}{4}\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \frac{1}{8}\sin^2\theta\Big]_0^{\pi/2} = \frac{1}{8}$

14. $\iint_R (x + y)dA \overset{x \text{ is odd}}{=} \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y\, dy\, dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos^2 x\, dx = \frac{1}{4}\int_{-\pi/2}^{\pi/2} (1 + \cos 2x)dx = \frac{1}{4}\pi$

15. $\iiint_S z^2 dV = 4\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{\sqrt{1-z}} z^2\, dy\, dx\, dz = 4\int_0^1 \int_0^{\sqrt{1-z}} y\Big]_0^{\sqrt{1-z}} z^2 dx\, dz$
$= 4\int_0^1 \int_0^{\sqrt{1-z}} z^2\sqrt{1 - z}\, dx = 4\int_0^1 x\Big]_0^{\sqrt{1-z}} z^2\sqrt{1 - z}\, dz = 4\int_0^1 (z^2 - z^3)dz = 4\Big[\frac{1}{3}z^3 - \frac{1}{4}z^4\Big]_0^1 = 4\Big(\frac{1}{3} - \frac{1}{4}\Big) = \frac{1}{3}$

16. $\iiint_S y\cos(x+z)dV$; S is the region bounded by the cylinder $x = y^2$ and the planes $x + z = \frac{1}{2}\pi$, $y = 0$, $z = 0$.

▷ The figure shows the solid S, which is bounded below by the plane $z = 0$, above by the plane $z = \frac{1}{2}\pi - x$, and bounded on the sides by the cylinders $y = 0$ and $y = \sqrt{x}$, $0 \le x \le \frac{1}{2}\pi$. Therefore



$$\iiint_S y\cos(x+z)dV = \int_0^{\pi/2}\int_0^{\sqrt{x}}\int_0^{\pi/2-x} y\cos(x+z)dz\,dy\,dx$$
$$= \int_0^{\pi/2}\int_0^{\sqrt{x}} y\Big[\sin(x+z)\Big]_0^{\pi/2-x} dy\,dx = \int_0^{\pi/2}\int_0^{\sqrt{x}} y(1-\sin x)dy\,dx$$
$$= \tfrac{1}{2}\int_0^{\pi/2} x(1-\sin x)dx = \tfrac{1}{2}\int_0^{\pi/2} x\,dx - \tfrac{1}{2}\int_0^{\pi/2} x\{\sin x\,dx\}$$
$$= \tfrac{1}{4}x^2\Big]_0^{\pi/2} - \tfrac{1}{2}\Big[-x\cos x + \sin x\Big]_0^{\pi/2} = \tfrac{1}{16}\pi^2 - \tfrac{1}{2}(1) = \tfrac{1}{16}(\pi^2 - 8)$$

17. $$\iint_R \frac{1}{x^2+y^2}dA = \int_0^{\pi/2}\int_1^2 \frac{1}{r^2}(r\,dr\,d\theta) = \int_0^{\pi/2}\int_1^2 \frac{1}{r}\,dr\,d\theta$$
$$= \int_0^{\pi/2} \ln|r|\Big]_1^2 d\theta = \ln 2\int_0^{\pi/2} d\theta = \tfrac{1}{2}\pi\ln 2$$

18. $$\int_0^1\int_{\sqrt{3}y}^{\sqrt{4-y^2}} \ln(x^2+y^2)dx\,dy = \int_0^{\pi/6}\int_0^2 \ln r\{2r\,dr\}d\theta = \tfrac{1}{6}\pi\Big[\lim_{a\to 0^+}\ln r\cdot r^2\Big]_a^2 - \int_0^2 r\,dr\Big] = \tfrac{1}{6}\pi(4\ln 2 - 2)$$
$$= \tfrac{1}{3}\pi(2\ln 2 - 1)$$

In Exercises 19 and 20, evaluate the iterated integral by reversing the order of integration.

19. $$\int_0^1\int_x^1 \sin y^2 dy\,dx = \int_0^1\int_0^y \sin y^2 dx\,dy = \int_0^1 x\Big]_0^y \sin y^2 dy = \int_0^1 y\sin y^2 dy = -\tfrac{1}{2}\cos y^2\Big]_0^1 = \tfrac{1}{2}(1-\cos 1)$$

20. $$\int_0^1\int_0^{\cos^{-1}y} e^{\sin x}dx\,dy$$

▷ The region R in the xy plane on which the double integral is taken is bounded by the curves $x = 0$ and $x = \cos x$ and the lines $x = 0$ and $x = \frac{1}{2}\pi$. Thus,
$$\int_0^1\int_0^{\cos^{-1}y} e^{\sin x}dx\,dy = \int_0^{\pi/2}\int_0^{\cos x} e^{\sin x}dy\,dx = \int_0^{\pi/2}\cos x\, e^{\sin x}dx = e^{\sin x}\Big]_0^{\pi/2} = e - 1$$

In Exercises 21 and 22, used double integrals to find the area of the region bounded by the curves. Sketch.

21. $$A = \iint_R dA = \int_{-1}^1\int_{x^4}^{x^2} dy\,dx = \int_0^1 y\Big]_{x^4}^{x^2} dx = \int_{-1}^1 (x^2 - x^4)dx = \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\Big]_{-1}^1 = (\tfrac{1}{3} - \tfrac{1}{5}) - (-\tfrac{1}{3} + \tfrac{1}{5}) = \tfrac{4}{15}$$

22. $$A = \int_0^1\int_{x^3}^{\sqrt{x}} dy\,dx = \int_0^1 (x^{1/2} - x^3)dx = \tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^4\Big]_0^1 = \tfrac{5}{12}$$

In Exercises 23 and 24, evaluate the iterated integral by changing to either cylindrical or spherical coordinates.

23. $$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^2 \sqrt{x^2+y^2}\,dz\,dy\,dx = \int_0^{\pi/2}\int_0^3\int_0^2 r(r\,dz\,dr\,d\theta) = \int_0^{\pi/2}\int_0^3 z\Big]_0^2 r^2 dr\,d\theta = 2\int_0^{\pi/2}\int_0^3 r^2 dr\,d\theta$$
$$= 2\int_0^{\pi/2} \tfrac{1}{3}r^3\Big]_0^3 d\theta = 18\int_0^{\pi/2} d\theta = 9\pi$$

24. $$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}} z\sqrt{4-x^2-y^2}\,dz\,dx\,dy$$

▷ The given iterated integral is equivalent to a triple integral over a region S which is bounded by the surfaces $z = 0$ and $z = \sqrt{4-x^2-y^2}$, bounded by the cylinders $x = 0$ and $x = \sqrt{4-y^2}$, and bounded by the planes $y = 0$ and $y = 2$. Thus, S is the part of the spherical solid enclosed by $x^2 + y^2 + z^2$ that is in the first octant. Because the integrand $z\sqrt{4-x^2-y^2}$ is more easily expressed in terms of cylindrical coordinates, we use cylindrical coordinates to evaluate the given integral. A cylindrical equation of the part of the sphere in the the first octant is $z = \sqrt{4-r^2}$, and an equation of its projection onto the xy plane is the circle $r = 2$, $0 \le \theta \le \frac{1}{2}\pi$. Therefore
$$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}} z\sqrt{4-x^2-y^2}\,dz\,dx\,dy = \int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$
$$= \int_0^{\pi/2}\int_0^2 \Big[\tfrac{1}{2}z^2 r\sqrt{4-r^2}\Big]_0^{\sqrt{4-r^2}} dr\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\int_0^2 r(4-r^2)^{3/2}dr\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\Big[-\tfrac{1}{5}(4-r^2)^{5/2}\Big]_0^2 d\theta$$
$$= -\tfrac{1}{10}\int_0^{\pi/2}(-4^{5/2})d\theta = \tfrac{8}{5}\pi$$

In Exercises 25 and 27, R is in the first quadrant bounded by the parabolas $x^2 = 4y$ and $x^2 = 8 - 4y$.

25. $A = \iint_R dx\,dy = \int_0^1 \int_0^{2\sqrt{y}} dx\,dy + \int_1^2 \int_0^{2\sqrt{2-y}} dx\,dy = \int_0^1 x\Big]_0^{2\sqrt{y}} dy + \int_1^2 x\Big]_0^{2\sqrt{2-y}} dy$

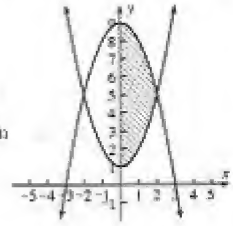
$= 2\int_0^1 y^{1/2} dy + 2\int_1^2 (2-y)^{1/2} dy = \frac{4}{3} y^{3/2}\Big]_0^1 - \frac{4}{3}(2-y)^{3/2}\Big]_1^2 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

27. $A = \iint_R dy\,dx = \int_0^2 \int_{x^2/4}^{(8-x^2)/4} dy\,dx = \int_0^2 y\Big]_{x^2/4}^{(8-x^2)/4} dx = \frac{1}{4}\int_0^2 (8 - 2x^2) dx = \frac{1}{4}\Big[8x - \frac{2}{3}x^3\Big]_0^2 = \frac{1}{4}(16 - \frac{16}{3}) = \frac{8}{3}$

In Exercises 26 and 28, use double integration to find the area of the region in the xy plane bounded by the parabolas $y = 9 - x^2$ and $y = x^2 + 1$.

26. Integrate first with respect to y.

$$A = 2\int_0^2 \int_{x^2+1}^{9-x^2} dy\,dx = 2\int_0^2 (8 - 2x^2) dx = 2\Big[8x - \frac{2}{3}x^3\Big]_0^2 = \frac{64}{3}$$



28. Integrate first with respect to x.

▷ The figure shows the region of which half is shaded. The parabolas intersect when

$$\begin{aligned} 9 - x^2 &= x^2 + 1 \\ 2x^2 &= 8 \\ x &= \pm 2 \\ y &= 5 \end{aligned}$$

Let

$$f(y) = \begin{cases} \sqrt{y-1} & \text{if } 1 \le y \le 5 \\ \sqrt{9-y} & \text{if } 5 \le y \le 9 \end{cases}$$

Then

$$A = \iint_R dx\,dy = 2\int_1^9 \int_0^{f(y)} dx\,dy = 2\Big[\int_1^5 \sqrt{y-1}\,dy + \int_5^9 \sqrt{9-y}\,dy\Big]$$

$$= 2 \cdot \frac{2}{3}(y-1)^{3/2}\Big]_1^5 + 2\Big(-\frac{2}{3}\Big)(9-y)^{3/2}\Big]_5^9 = \frac{4}{3} \cdot 8 + \frac{4}{3} \cdot 8 = \frac{64}{3}$$

Therefore, the area is $\frac{64}{3}$ square units.

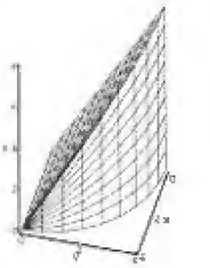
In Exercises 29 and 31, use double integration to find the the volume of the solid bounded by the planes $x = y$, $y = 0$, $z = 0$, $x = 1$, and $z = 1$.

29. $V = \iint_R 1\,dA = \int_0^1 \int_y^1 dx\,dy = \int_0^1 (1-y)dy = -\frac{1}{2}(1-y)^2\Big]_0^1 = \frac{1}{2}$

31. $V = \iint_R 1\,dA = \int_0^1 \int_0^x dy\,dx = \int_0^1 x\,dx = \frac{1}{2}x^2\Big]_0^1 = \frac{1}{2}$

In Exercises 30 and 32, use double integration to find the volume of the solid above the xy plane bounded by the cylinder $x^2 + y^2 = 16$ and the plane $z = 2y$.

30. $V = 2\int_0^4 \int_0^{\sqrt{16-y^2}} 2y\,dx\,dy = 4\int_0^4 \sqrt{16-y^2}\,y\,dy = -\frac{4}{3}(16-y^2)^{3/2}\Big]_0^4 = \frac{256}{3}$



32. Integrate first with respect to y.

▷ The figure shows the half of the solid that is in the first octant. We take R to be the first quadrant part of the circle $x^2 + y^2 = 16$. We regard R as being bounded by the lines $x = 0$ and $x = 4$ and the curves $y = 0$ and $y = \sqrt{16 - x^2}$. Thus, the measure of the volume is given by

$$V = 2\iint_R 2y\,dA = 2\int_0^4 \int_0^{\sqrt{16-x^2}} 2y\,dy\,dx = 2\int_0^4 (16 - x^2)dx$$

$$= 2\Big[16x - \frac{1}{3}x^3\Big]_0^4 = \frac{256}{3}$$

Therefore, the volume is $\frac{256}{3}$ cubic units.

33. $V = \iint_R (x^2 + y)dA = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} (x^2 + y)dy\,dx = \int_0^4 x^2 y + \frac{1}{2}y^2\Big]_{x^2/4}^{2\sqrt{x}} dx = \int_0^4 \Big(x^{5/2} + 2x - \frac{9}{32}x^4\Big)dx$

$= \frac{4}{7}x^{7/2} + x^2 - \frac{9}{160}x^5\Big]_0^4 = \frac{512}{7} + 16 - \frac{288}{5} = \frac{1104}{35}$

34. $x^2 = x + 1$, $x = -1, 2$. $M = \int_{-1}^2 \int_{x^2}^{x+2} x^2 y^2 dy\,dx = \frac{1}{3}\int_{-1}^2 x^2[(x+2)^3 - x^6]dx$

$= \frac{1}{3}\int_{-1}^2 [x^5 + 6x^4 + 12x^3 + 8x^2 - x^7]dx = \frac{1}{3}\Big[\frac{1}{6}x^6 + \frac{6}{5}x^5 + 3x^4 + \frac{8}{3}x^3 - \frac{1}{8}x^8\Big]_{-1}^2 = \frac{207}{10}$

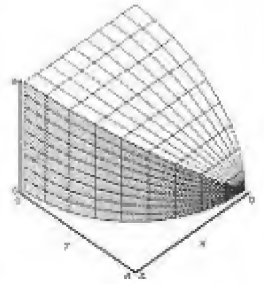
35. In the first octant $y \geq 0$ so $x^2 + y^2$ gives $y = \sqrt{9 - x^2}$. Then $\dfrac{\partial y}{\partial x} = -\dfrac{x}{\sqrt{9-x^2}}$ and $\dfrac{\partial y}{\partial z} = 0$.

$$\sigma = \int_0^3 \int_x^{3x} \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dz\, dx = \int_0^3 \int_0^{3x} \sqrt{\frac{x^2}{9-x^2} + 0 + 1}\, dz\, dx = \int_0^3 \frac{3z}{\sqrt{9-x^2}}\Big]_x^{3x} dx = 6\int_0^3 \frac{x}{\sqrt{9-x^2}}\, dx$$

$$= 6 \lim_{b \to 3^-} \int_0^b x(9-x^2)^{-1/2} dx = -6 \lim_{b\to 3^-} (9-x^2)^{1/2}\Big]_0^b = -6 \lim_{b \to 3^-} [(9-b^2)^{1/2} - 3] = 18$$

36. Find the area of the surface of the part of the cylinder $x^2 + y^2 = a^2$ that lies inside the cylinder $y^2 + z^2 = a^2$.

▸ The part of the surface that lies in the first octant is shown shaded in the figure. This is one-eighth of the entire surface. We take the yz plane as the horizontal plane. Solving the equation of the cylinder $x^2 + y^2 = a^2$ for x, we obtain $x = \pm\sqrt{a^2 - y^2}$. We take



$$f(y, z) = \sqrt{a^2 - y^2}$$

Thus,

$$f_y(y, z) = \frac{-y}{\sqrt{a^2 - y^2}} \quad \text{and } f_z(y, z) = 0$$

Hence,

$$\sqrt{f_y{}^2(y,z) + f_z{}^2(y,z) + 1} = \sqrt{\frac{y^2}{a^2 - y^2} + 1} = \frac{a}{\sqrt{a^2 - y^2}}$$

Let R be the region in the first quadrant of the yz plane bounded by the circle $y^2 + z^2 = a^2$. The measure of the surface area is given by

$$\sigma = 8 \iint_R \frac{a}{\sqrt{a^2 - y^2}} dA = \lim_{b \to a^-} 8a \int_0^b \int_0^{\sqrt{a^2 - y^2}} \frac{1}{\sqrt{a^2 - y^2}} dz\, dy = \lim_{b \to a^-} 8a \int_0^b dy = 8a^2$$

■ The surface area is $8a^2$ square units. Because the integrand is unbounded near $(0, a, 0)$, the integral is improper.

37. $$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^n r_i \Delta_i r\, \Delta_i \theta = \iint_R r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \int_{1/(1+\cos\theta)}^{1} r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2\Big]_{1/2\cos^2(\theta/2)}^{1} d\theta$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(1 - \frac{1}{4}\sec^4 \frac{\theta}{2}\right) d\theta = \int_0^{\pi/2} \left[1 - \frac{1}{4}\left(\tan^2 \frac{\theta}{2} + 1\right)\sec^2 \frac{\theta}{2}\right] d\theta = \left[\theta - \frac{1}{2}\left(\frac{1}{3}\tan^3 \frac{\theta}{2} + \tan \frac{\theta}{2}\right)\right]_0^{\pi/2} = \frac{\pi}{2} - \frac{2}{3}$$

38. The limaçon $r = 3 - \cos\theta$ meets the circle $r = 5\cos\theta$ when $3 - \cos\theta = 5\cos\theta$, $\cos\theta = \frac{1}{2}$, $\theta = \pm\frac{1}{3}\pi$.

$$M = 2\int_0^{\pi/3} \int_{3-\cos\theta}^{5\cos\theta} 2\sin\theta\, r\, dr\, d\theta = 2\int_0^{\pi/3} [25\cos^2\theta - (3 - \cos\theta)^2] \sin\theta\, d\theta = -\frac{2}{3}[25\cos^3\theta + (3-\cos\theta)^3]_0^{\pi/3} = \frac{19}{2}$$

39. $$M = \lim_{\|\Delta\| \to 0} \sum_{i=1}^n (u_i v_i{}^2) \Delta_i A = \iint_R xy^2 dA = \int_0^3 \int_0^2 xy^2 dy\, dx = \int_0^3 \frac{1}{3} xy^3 \Big]_0^2 dx = \frac{8}{3} \int_0^3 x\, dx = \frac{4}{3} x^2 \Big]_0^3 = 12$$

If M_x kg-m and M_y kg-m are the moments of mass with respect to the x and y axes,

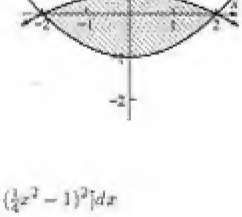
$$M_x = \lim_{\|\Delta\| \to 0} \sum_{i=1}^n v_i(u_i v_i{}^2)\Delta_i A = \iint_R xy^3 dA = \int_0^3 \int_0^2 xy^3 dy\, dx = \int_0^3 \frac{1}{4} xy^4 \Big]_0^2 dx = 4\int_0^3 x\, dx = 2x^2\Big]_0^3 = 18$$

$$M_y = \lim_{\|\Delta\| \to 0} \sum_{i=1}^n u_i(u_i v_i{}^2)\Delta_i A = \iint_R x^2 y^2 dA = \int_0^3 \int_0^2 x^2 y^2 dy\, dx = \int_0^3 \frac{1}{3} x^2 y^3 \Big]_0^2 dx = \frac{8}{3}\int_0^3 x^2\, dx = \frac{8}{9} x^3 \Big]_0^3 = 24$$

If $(\bar{x}, \bar{y})$ is the center of the mass, $\bar{x} = \dfrac{M_y}{M} = \dfrac{24}{12} = 2$ and $\bar{y} = \dfrac{M_x}{M} = \dfrac{18}{12} = \dfrac{3}{2}$.

40. Find the center of mass of the lamina in the shape of the region bounded by the parabolas $x^2 = 4 + 4y$ and $x^2 = 4 - 8y$ if the area density at any point is kx^2 kg/m^2.

▸ The figure shows the region R, which is bounded by the lines $x = -2$ and $x = 2$, bounded below by the curve $y = \frac{1}{4}x^2 - 1$, and above by the curve $y = -\frac{1}{8}x^2 + \frac{1}{2}$. Because the region is symmetrical with respect to the y axis, $\bar{x} = 0$. The measure of the total mass is given by



$$M = \iint_R kx^2 dA = k\int_{-2}^{2} \int_{x^2/4 - 1}^{-x^2/8 + 1/2} x^2 dy\, dx$$

$$= k\int_{-2}^{2} x^2[(-\tfrac{1}{8}x^2 + \tfrac{1}{2}) - (\tfrac{1}{4}x^2 - 1)]dx = 2k\int_0^2 (-\tfrac{3}{8}x^4 + \tfrac{3}{2}x^2)dx$$

$$= 2k\left[-\tfrac{3}{40}x^5 + \tfrac{1}{2}x^3\right]_0^2 = \tfrac{16}{5}k$$

Furthermore,

$$M_x = \iint_R kx^2 y\, dA = k\int_{-2}^{2} \int_{x^2/4-1}^{-x^2/8+1/2} x^2 y\, dy\, dx = \tfrac{1}{2}k \int_{-2}^{2} x^2[(-\tfrac{1}{8}x^2 + \tfrac{1}{2})^2 - (\tfrac{1}{4}x^2 - 1)^2]dx$$

$$= k\int_0^2 x^2[(-\tfrac{1}{8})^2(x^2-4)^2 - (\tfrac{1}{4})^2(x^2-4)^2]dx = -\tfrac{3}{64}\int_0^2 x^2(x^2-4)^2dx = -\tfrac{3}{64}\int_0^2 (x^6 - 8x^4 + 16x^2)dx$$

$$= -\tfrac{3}{64}\Big[\tfrac{1}{7}x^7 - \tfrac{8}{5}x^5 + \tfrac{16}{3}x^3\Big]_0^2 = -\tfrac{16}{35}k$$

Thus,

$$\bar{y} = \frac{1}{M}\cdot M_x = \frac{5}{16k}\Big(-\frac{16k}{35}\Big) = -\frac{1}{7}$$

Therefore, the center of mass is $(0, -\frac{1}{7})$.

41. $$M = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(\bar{r}_i\theta_i)(\bar{r}_i\Delta_i r\,\Delta_i\theta) = \iint_R r^2\theta\,dr\,d\theta = \int_0^{\pi/4}\int_0^{\cos 2\theta} r^2\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{3}r^3\theta\Big]_0^{\cos 2\theta}d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4}\theta\cos^3 2\theta\,d\theta = \frac{1}{3}\int_0^{\pi/4}\theta\{(1-\sin^2 2\theta)\cos 2\theta\,d\theta\} = \frac{1}{6}\Big[\theta\Big(\sin 2\theta - \frac{1}{3}\sin^3 2\theta\Big)\Big]_0^{\pi/4} - \frac{1}{6}\int_0^{\pi/4}\Big(\sin 2\theta - \frac{1}{3}\sin^3 2\theta\Big)d\theta$$

$$= \frac{1}{6}\cdot\frac{\pi}{4}\cdot\frac{2}{3} - \frac{1}{6}\int_0^{\pi/4}\Big(\frac{2}{3}+\frac{1}{3}\cos^2 2\theta\Big)\sin 2\theta\,d\theta = \frac{\pi}{36} + \frac{1}{12}\Big[\frac{2}{3}\cos 2\theta + \frac{1}{9}\cos^3 2\theta\Big]_0^{\pi/4} = \frac{\pi}{36} - \frac{1}{12}\cdot\frac{7}{9} = \frac{3\pi - 7}{108}$$

42. $$I_x = \iint_R y^2 k\sqrt{x^2+y^2}\,dA = k\int_0^a\int_0^{2\pi} r^2\sin^2\theta\cdot r\cdot r\,dr\,d\theta = k\int_0^a r^4dr\int_0^{2\pi}\tfrac{1}{2}(1-\cos 2\theta)d\theta = k\cdot\tfrac{1}{5}a^5\cdot\pi$$

43. $x^2 + y^2 = 4z$; $r^2 = 4z$ and $x^2 + y^2 = 4ay$; $r^2 = 4ar\sin\theta$; $r = 4a\sin\theta$.

$$V = \int_0^{\pi}\int_0^{4a\sin\theta}\int_0^{r^2/4} r\,dz\,d\theta = \frac{1}{4}\int_0^{\pi}\int_0^{4a\sin\theta} r^3dr\,d\theta = \frac{1}{16}\int_0^{\pi} r^4\Big]_0^{4a\sin\theta}d\theta = 16a^4\int_0^{\pi}\sin^4\theta\,d\theta$$

$$= 16a^4\Big[-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\Big(-\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\Big)\Big]_0^{\pi} = 16a^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\pi = 6\pi a^4$$

44. Use spherical coordinates to find the mass of a spherical solid of radius a meters if the volume density at each point is proportional to the distance of the point from the center of the sphere. The volume density is measured in kg/m^3.

▸ With the center of the sphere at the origin, a spherical equation of the sphere is $\rho = a$. We are given that the volume density is $k\rho$ kg/m^3. The measure of the mass of the solid is given by

$$M = \int_0^{\pi}\int_0^{2\pi}\int_0^a (k\rho)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = k\Big[\int_0^{\pi}\sin\phi\,d\phi\Big]\cdot\Big[\int_0^{2\pi}d\theta\Big]\cdot\Big[\int_0^a\rho^3\Big] = -k\cos\phi\Big]_0^{\pi}\cdot 2\pi\cdot\tfrac{1}{4}a^4 = \pi ka^4$$

The mass is πka^4 kg.

45. $$V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}}[r(4-r^2)^{1/2} - r]dr\,d\theta = \int_0^{2}\Big[-\frac{1}{3}(4-r^2)^{3/2} - \frac{1}{2}r^2\Big]_0^{\sqrt{3}}d\theta.$$

$$= 2\pi\Big[(-\tfrac{1}{3} - \tfrac{3}{2}) - (-\tfrac{8}{3})\Big] = \tfrac{5\pi}{3}$$

46. $$V = \int_0^2\int_0^{2x^2}\int_0^{8-y}dz\,dy\,dx = \int_0^2\int_0^{2x^2}(8-y)dy\,dx = \int_0^2\Big[8y - \tfrac{1}{2}y^2\Big]_0^{2x^2}dx = \int_0^2(16x^2 - 2x^4)dx$$

$$= \Big[\tfrac{16}{3}x^3 - \tfrac{2}{5}x^5\Big]_0^2 = \tfrac{448}{15}$$

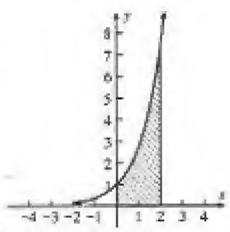
In Exercises 47 and 48, a lamina is the shape of the region bounded by the curve $y = e^x$, the line $x = 2$, and the coordinate axes. The area density at any point is xy kg/m^2.

47. If I_x kg-m^2 is the moment of inertia of the lamina about the x axis

$$I_x = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} v_i(u_i v_i)\Delta_i A = \iint_R xy^3dA = \int_0^2\int_0^{e^x} xy^3dy\,dx = \int_0^2 \frac{1}{4}xy^4\Big]_0^{e^x}dx = \frac{1}{4}\int_0^2 xe^{4x}dx$$

$$= \tfrac{1}{4}\Big[\tfrac{1}{4}xe^{4x} - \tfrac{1}{16}e^{4x}\Big]_0^2 = \tfrac{1}{64}[(8e^8 - e^8) + 1] = \tfrac{1}{64}(7e^8 + 1)$$

48. Find the moment of inertia about the y axis of the lamina.



▸ The figure shows the region. The measure of the moment of inertia about the y axis is given by

$$I_y = \iint_R x^3y\,dA = \int_0^2\int_0^{e^x} x^3y\,dy\,dx = \int_0^2 x^3\{\tfrac{1}{2}e^{2x}\,dx\}$$

We use integration by parts repeatedly with dv indicated with braces. Thus,

$$I_y = \Big[\tfrac{1}{4}x^3e^{2x} - \tfrac{3}{4}\int x^2\{e^{2x}dx\}\Big]_0^2 = \Big[\tfrac{1}{4}x^3e^{2x} - \tfrac{3}{8}x^2e^{2x} + \tfrac{3}{4}\int x\{e^{2x}dx\}\Big]_0^2$$

$$= \Big[\tfrac{1}{4}x^3e^{2x} - \tfrac{3}{8}x^2e^{2x} + \tfrac{3}{8}xe^{2x} - \tfrac{3}{8}\int e^{2x}dx\Big]_0^2 = \tfrac{1}{16}e^{2x}(4x^3 - 6x^2 + 6x - 3)\Big]_0^2 = \tfrac{1}{16}(17e^4 + 3)$$

Thus, the moment of inertia is $\frac{1}{16}(17e^4 + 3)$ kg·m^2.

In Exercises 49–51, lamina is the region bounded by the lemniscate $r^2 = 4\cos 2\theta$ and the area density is k kg/m^2.

49. If I_y kg-m^2 is the moment of inertia of the lamina about the y axis,

$$I_y = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (\bar{r}_i \cos \bar{\theta}_i)^2 (k)\Delta_i A = k \iint_R r^2\cos^2\theta\, r\, dr\, d\theta = 2k \int_{-\pi/4}^{\pi/4} \int_0^{2\sqrt{\cos 2\theta}} r^3 \cos^2\theta\, dr\, d\theta$$

$$= 2k \int_{-\pi/4}^{\pi/4} \tfrac{1}{4} r^4 \Big]_0^{2\sqrt{\cos 2\theta}} \cos^2\theta\, d\theta = 8k \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \cos^2\theta\, d\theta \overset{\text{sym}}{=} 16k \int_0^{\pi/4} \cos^2 2\theta \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 8k \int_0^{\pi/4} (\cos^2 2\theta + \cos^3 2\theta)\, d\theta = 8k \int_0^{\pi/4} \left[\tfrac{1}{2} + \tfrac{1}{2}\cos 4\theta + (1-\sin^2 2\theta)\cos 2\theta\right] d\theta$$

$$= 8k\left[\tfrac{1}{2}\theta + \tfrac{1}{8}\sin 4\theta + \tfrac{1}{2}\sin 2\theta - \tfrac{1}{6}\sin^3 2\theta\right]_0^{\pi/4} = 8k\left(\tfrac{1}{8}\pi + \tfrac{1}{2} - \tfrac{1}{6}\right) = k\left(\pi + \tfrac{8}{3}\right)$$

50. $M = 4k \int_0^{\pi/4} \int_0^{2\sqrt{\cos 2\theta}} r\, dr\, d\theta = 8k \int_0^{\pi/4} \cos 2\theta\, d\theta = 4k \sin 2\theta \Big]_0^{\pi/4} = 4k$. The mass is $4k$ kg.

51. If I_0 kg-m^2 is the polar moment of inertia of the lamina,

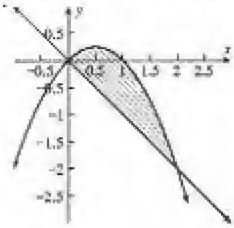
$$I_0 = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} r_i^2 (k)\Delta_i A = k \iint_R r^2 (r\, dr\, d\theta) = 2k \int_{-\pi/4}^{\pi/4} \int_0^{2\sqrt{\cos 2\theta}} r^3 dr\, d\theta = 2k \int_{-\pi/4}^{\pi/4} \tfrac{1}{4} r^4 \Big]_0^{2\sqrt{\cos 2\theta}} d\theta$$

$$= 8k \int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta = 8k \int_0^{\pi/4} (1+\cos 4\theta)\, d\theta = 8k\left[\theta + \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/4} = 2\pi k$$

From Exercises 50, we have $M = 4k$. If r meters is the radius of gyration, $r = \sqrt{\dfrac{I_0}{M}} = \sqrt{\dfrac{2\pi k}{4k}} = \tfrac{1}{2}\sqrt{2\pi}$.

52. Find the moment of inertia about the y axis of the lamina in the shape of the region bounded by the parabola $y = x - x^2$ and the line $x + y = 0$, if the area density at any point is $(x+y)$kg/m^2.

▹ The figure shows the region R. Let $\rho(x,y) = x + y$. The number of units in the moment of inertia about the y axis is



$$I_y = \iint_R \rho(x,y)x^2 dA = \int_0^2 \int_{-x}^{x-x^2} x^2(x+y)\, dy\, dx = \int_0^2 x^2 \tfrac{1}{2}(x+y)^2 \Big]_{-x}^{x-x^2} dx$$

$$= \tfrac{1}{2}\int_0^2 x^2(2x-x^2)^2 dx = \tfrac{1}{2}\int_0^2 (x^6 - 4x^5 + 4x^4)\, dx = \tfrac{1}{2}\left[\tfrac{1}{7}x^7 - \tfrac{2}{3}x^6 + \tfrac{4}{5}x^5\right]_0^2$$

$$= \frac{2^7}{14} - \frac{2^6}{3} + \frac{2^6}{5} = 2^6\left(\tfrac{1}{7} - \tfrac{1}{3} + \tfrac{1}{5}\right) = \tfrac{64}{105}$$

The moment of inertia is $\frac{64}{105}$ kg-m^2.

In Exercises 53 and 54, the solid is bounded by the spheres $\rho = 2$ and $\rho = 3$. $p = k\rho$ kg/m^2.

53. $M = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (k\rho_i)\Delta_i V = \iiint_S k\rho\, dV = \int_0^{\pi}\int_0^{2\pi}\int_2^3 k\rho \cdot \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = k\int_0^{\pi}\int_0^{2\pi} \tfrac{1}{4}\rho^4 \Big]_2^3$

$$= \frac{65}{4}k \int_0^{\pi}\int_0^{2\pi} \sin\phi\, d\theta\, d\phi = \frac{65}{4}k\int_0^{\pi} \theta\Big]_0^{2\pi} \sin\phi\, d\phi = \frac{65}{2}\pi k \int_0^{\pi} \sin\phi\, d\phi = \frac{65}{2}\pi k\Big[-\cos\phi\Big]_0^{\pi} = 65\pi k$$

54. $M_z = \int_0^{\pi}\int_0^{2\pi}\int_2^3 (\rho \sin\phi)^2 \cdot k\rho \cdot \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = 2\pi k \int_0^{\pi} (1-\cos^2\phi)\sin\phi\, d\phi \int_2^3 \rho^5 d\rho$

$$= 2\pi k\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi} \cdot \tfrac{665}{6} = \tfrac{2660}{9}\pi k$$

55. The solid is bounded by $z^2 = 4x^2 + 4y^2$; $z^2 = 4r^2$, $z = 0$ and $z = 4$. If I_z is in kg-m^2,

$$I_z = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \bar{r}_i^2 k\, \Delta_i V = \iiint_S kr^2 dV = k\int_0^{2\pi}\int_0^2\int_{2r}^4 r^2(r\, dz\, dr\, d\theta) = k\int_0^{2\pi}\int_0^2 r^3 z\Big]_{2r}^4 dr\, d\theta$$

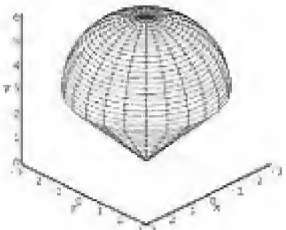
$$= 2k\int_0^{2\pi}\int_0^2 (4r^3 - 2r^4)\, dr\, d\theta = 2k\int_0^{2\pi}\left[\tfrac{1}{2}r^4 - \tfrac{1}{5}r^5\right]_0^2 d\theta = 2k\int_0^{2\pi} \tfrac{8}{5}\, d\theta = 2k\cdot\tfrac{8}{5}\cdot 2\pi = \tfrac{32}{5}\pi k$$

56. Find the center of mass of the solid bounded by the sphere $x^2 + y^2 + z^2 - 6z = 0$ and the cone $x^2 + y^2 = z^2$ and above the cone, if the volume density at any point is kz kg/m^3.

▹ Eliminating $x^2 + y^2$ from the given equations, we have $6z - z^2 = z^2$. $z(z-3) = 0$. Thus, the surfaces intersect in the planes $z = 0$ and $z = 3$. Furthermore, by completing the square we have for the equation of the sphere

$x^2 + y^2 + (z-3)^2 = 9$

The sphere has center $(0, 0, 3)$ and radius 3. The figure shows the solid S. Because the sphere and cone contain the origin, we use spherical coordinates. Because



$x^2 + y^2 + z^2 = \rho^2$ and $z = \rho \cos \phi$

the given equation of the sphere becomes

$$\rho^2 = 6\rho \cos \phi$$
$$\rho = 6 \cos \phi$$

in spherical coordinates. Because on the cone $z = r$, the equation of the cone is

$$\phi = \tfrac{1}{4}\pi$$

Thus, S is bounded by the surfaces $\rho = 0$ and $\rho = 6 \cos \phi$, bounded by the cones $\phi = 0$ and $\phi = \frac{1}{4}\pi$, for $0 \leq \theta \leq 2\pi$. Hence, the measure of the mass of the solid is

$$\begin{aligned} \mathrm{M} &= \iiint_S kz \, dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{6\cos\phi} (k\rho \cos \phi)\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \tfrac{1}{4}k \int_0^{\pi/4} \int_0^{2\pi} \sin \phi \cos \phi (6 \cos \phi)^4 d\theta \, d\phi \\ &= 648\pi k \int_0^{\pi/4} \sin \phi \cos^5\phi \, d\phi = 648\pi k\Big[-\tfrac{1}{6}\cos^6\phi\Big]_0^{\pi/4} = 648\pi k(1 - \tfrac{1}{8}) = \tfrac{189}{2}\pi k \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathrm{M}_{xy} &= \iiint_S kz^2 dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{6\cos\phi} k(\rho \cos \phi)^2 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= \tfrac{1}{5}k \int_0^{\pi/4} \int_0^{2\pi} \sin \phi \cos^2\phi (6 \cos \phi)^5 d\theta \, d\phi = \tfrac{15552}{5}\pi k \int_0^{\pi/4} \sin \phi \cos^7\phi \, d\phi = \tfrac{1944}{5}\pi k\Big[-\cos^8\phi\Big]_0^{\pi/4} \\ &= \tfrac{1944}{5}\pi k(1 - \tfrac{1}{16}) = \tfrac{729}{2}\pi k \end{aligned}$$

Then

$$\bar{z} = \frac{1}{\mathrm{M}}\mathrm{M}_{xy} = \frac{2}{189\pi k} \cdot \frac{729\pi k}{2} = \frac{27}{2}$$

Because S and the density function are symmetric with respect to the z axis, the center of mass lies on the z axis. Thus, the center of mass is $(0, 0, \frac{27}{7})$.

F O U R T E E N

INTRODUCTION TO THE CALCULUS OF VECTOR FIELDS

14.1 VECTOR FIELDS

14.1.1 Theorem Suppose that M and N are function of two variables x and y defined on an open disk $B((x_0, y_0); r)$ in R^2, and M_y and N_x are continuous on B. Then the vector
$M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$
is a gradient on B if and only if
$M_y(x, y) = N_x(x, y)$
at all points in B.

14.1.2 Theorem Let M, N, and R be functions of three variables x, y, and z defined on an open ball $B((x_0, y_0, z_0); r)$ in R^3, and M_y, M_z, N_x, N_z, R_x, and R_y are continuous on B. Then the vector
$M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$
is a gradient on B if and only if
$M_y(x, y, z) = N_x(x, y, z) \quad M_z(x, y, z) = R_x(x, y, z) \quad N_z(x, y, z) = R_y(x, y, z)$
at all points in B.

A *vector field* is a function that associates a vector with a point in space. A *scalar field* is a function that associates a scalar with a point in space.

Conservative If $\mathbf{F}$ is a vector field that is the gradient of some scalar field ϕ, that is, $\mathbf{F} = \nabla\phi$, then $\mathbf{F}$ is called a *conservative vector field* and ϕ is a *potential function* for $\mathbf{F}$. Also $d\phi = M\,dx + N\,dy + R\,dz$ is said to be an *exact differential.*

Let $\mathbf{F}$ be a vector field on some open ball in R^3 such that
$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$

Curl The *curl* of $\mathbf{F}$ is defined by
$$\text{curl } \mathbf{F}(x, y, z) = \left(\frac{\partial R}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$
if these partial derivatives exist.

In terms of vectors, Theorem 14.1.2 becomes

14.1.6 Theorem Let $\mathbf{F}$ be differentiable on an open ball B in R^3. Then $\mathbf{F} = \nabla f$ if and only if curl $\mathbf{F} \equiv \mathbf{0}$.
If curl $\mathbf{F} \equiv \mathbf{0}$, then $\mathbf{F}$ is said to be *irrotational.* In other words, $\mathbf{F}$ is irrotational if and only if it is conservative.

Divergence The divergence of $\mathbf{F}$, denoted by div $\mathbf{F}$, is defined by
$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial R}{\partial z}$$
if these partial derivatives exist.
If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl } \mathbf{F} + \text{curl } \mathbf{G}$ and $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div } \mathbf{F} + \text{div } \mathbf{G}$

14.1.5 Theorem Let $\mathbf{G}$ be differentiable on an open ball B. Then $\mathbf{G} = \text{curl } \mathbf{F}$ if and only if div $\mathbf{G} \equiv 0$. See Ex. 47.
If div $\mathbf{G} \equiv 0$, then $\mathbf{G}$ is said to be *solenoidal.*

If the del operator in three dimensions is defined by $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ then
$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & R \end{vmatrix} \quad \text{and} \quad \text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

The Laplacian of f is given by
$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

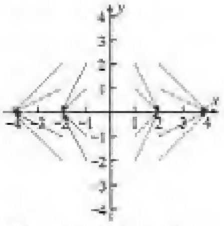
Two Dimensions If $\mathbf{F}$ is a vector field on some open disk in R^2, such that
$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$
then $\text{curl } \mathbf{F}(x, y) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$ and $\text{div } \mathbf{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

Exercises 14.1

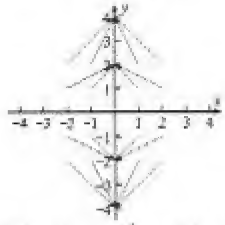
In Exercises 1–6, show on a figure the representations having initial point at (x, y) of the vectors in the vector field, where x is ± 1 or ± 2 and y is ± 1 or ± 2.

1. $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$

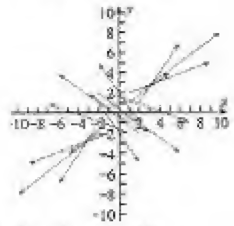
▷



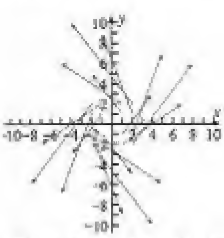
2. $\mathbf{F}(x, y) = -x\mathbf{i} + y\mathbf{j}$



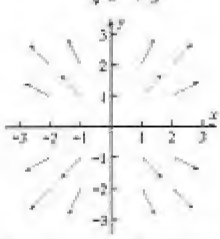
3. $\mathbf{F}(x, y) = 4y\mathbf{i} + 3x\mathbf{j}$



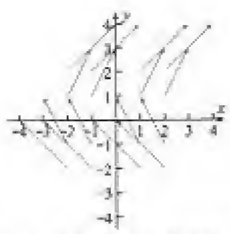
4. $\mathbf{F}(x, y) = -3y\mathbf{i} + 4x\mathbf{j}$



5. $\mathbf{F}(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}(x\mathbf{i} + y\mathbf{j})$



6. $\mathbf{F}(x, y) = y\mathbf{i} + 2\mathbf{j}$



▷ (Exercise 4) The table below lists the function value for $\mathbf{F}$ at each of the given points. For each point (x, y) in the table, we use (x, y) as the initial point, and sketch the geometric representation of the vector $\mathbf{F}(x, y)$. See the figure.

		y			
		-2	-1	1	2
	-2	$6\mathbf{i} - 8\mathbf{j}$	$3\mathbf{i} - 8\mathbf{j}$	$-3\mathbf{i} - 8\mathbf{j}$	$-6\mathbf{i} - 8\mathbf{j}$
x	-1	$6\mathbf{i} - 4\mathbf{j}$	$3\mathbf{i} - 4\mathbf{j}$	$-3\mathbf{i} - 4\mathbf{j}$	$-6\mathbf{i} - 4\mathbf{j}$
	1	$6\mathbf{i} + 4\mathbf{j}$	$3\mathbf{i} + 4\mathbf{j}$	$-3\mathbf{i} + 4\mathbf{j}$	$-6\mathbf{i} + 4\mathbf{j}$
	2	$6\mathbf{i} + 8\mathbf{j}$	$3\mathbf{i} + 8\mathbf{j}$	$-3\mathbf{i} + 8\mathbf{j}$	$-6\mathbf{i} + 8\mathbf{j}$

In Exercises 7–14, find a conservative vector field having the given potential function.

▷ The desired conservative vector field is $\mathbf{F} = \nabla f$.

7. $f(x, y) = 3x^2 + 2y^3$. $\mathbf{F}(x, y) = \nabla f(x, y) = 6x\mathbf{i} + 6y\mathbf{j}$.

8. $f(x, y) = 2x^4 - 5x^2y^2 + 4y^4$

▷ The required conservative vector field $\mathbf{F}$ is the gradient of f. Thus,
$\mathbf{F}(x, y) = \nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = (8x^3 - 10xy^2)\mathbf{i} + (-10x^2y + 16y^3)\mathbf{j}$

9. $f(x, y) = \tan^{-1} x^2 y$. $\mathbf{F}(x, y) = \nabla f(x, y) = \dfrac{2xy}{1 + x^4y^2}\mathbf{i} + \dfrac{x^2}{1 + x^4y^2}\mathbf{j}$.

10. $f(x, y) = ye^x - xe^y$. $\mathbf{F}(x, y) = \nabla f(x, y) = (ye^x - e^y)\mathbf{i} + (e^x - xe^y)\mathbf{j}$

11. $f(x, y, z) = 2x^3 - 3x^2y + xy^2 - 4y^3$. $\mathbf{F}(x, y, z) = \nabla f(x, y, z) = (6x^2 - 6xy + y^2)\mathbf{i} + (-3x^2 + 2xy - 12y^2)\mathbf{j}$.

12. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

▷ If $\mathbf{F}$ is the required vector field, then

$$\mathbf{F}(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k}$$

13. $f(x, y, z) = x^2ye^{-4z}$. $\mathbf{F}(x, y, z) = \nabla f(x, y, z) = 2xye^{-4z}\mathbf{i} + x^2e^{-4z}\mathbf{j} - 4x^2ye^{-4z}\mathbf{k}$.

14. $f(x, y, z) = z\sin(x^2 - y)$. $\mathbf{F}(x, y, z) = \nabla f(x, y, z) = 2xz\cos(x^2 - y)\mathbf{i} - z\cos(x^2 - y)\mathbf{j} + \sin(x^2 - y)\mathbf{k}$

In Exercises 15–20, determine if the vector field is conservative.

15. $\mathbf{F}(x,y) = (3x^2 - 2y^2)\mathbf{i} + (3 - 4xy)\mathbf{j} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$. $M_y(x,y) = -4y$, $N_x(x,y) = -4y$.
Because $M_y = N_x$, $\mathbf{F}$ is conservative.

16. $\mathbf{F}(x,y) = (e^x e^y + 6e^{2x})\mathbf{i} + (e^x e^y - 2e^y)\mathbf{j}$

▷ Let
$M(x,y) = e^x e^y + 6e^{2x}$ $\quad N(x,y) = e^x e^y - 2e^y$
$M_y(x,y) = e^x e^y$ $\quad N_x(x,y) = e^x e^y$
Because $M_y(x,y) = N_x(x,y)$, it follows that $\mathbf{F}$ is a gradient, and hence a conservative vector field.

17. $\mathbf{F}(x,y) = y\cos(x+y)\mathbf{i} - x\sin(x+y)\mathbf{j} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$.
$M_y(x,y) = \cos(x+y) - y\sin(x+y)$ $\quad N_x(x,y) = -\sin(x+y) - x\cos(x+y)$
Because $M_y \neq N_x$, $\mathbf{F}$ is not conservative.

18. $\mathbf{F}(x,y,z) = (3x^2 + 2yz)\mathbf{i} + (2xz + 6yz)\mathbf{j} + (2xy + 3y^2 - 2z)\mathbf{k} = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$
$M_y(x,y,z) = 2z$ $\quad N_x(x,y,z) = 2z$ $\quad R_x(x,y,z) = 2y$
$M_z(x,y,z) = 2y$ $\quad N_z(x,y,z) = 2x + 6y$ $\quad R_y(x,y,z) = 2x + 6y$
Because $M_y = N_x$, $M_z = R_x$, and $N_z = R_y$, $\mathbf{F}$ is conservative.

19. $\mathbf{F}(x,y,z) = (2ye^{2x} + e^z)\mathbf{i} + (3ze^{3y} + e^{2x})\mathbf{j} + (xe^z + e^{3y})\mathbf{k} = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$
$M_y(x,y,z) = 2e^{2x}$ $\quad N_x(x,y,z) = 2e^{2x}$ $\quad R_x(x,y,z) = e^z$
$M_z(x,y,z) = e^z$ $\quad N_z(x,y,z) = 3e^{3y}$ $\quad R_y(x,y,z) = 3e^{3y}$
Because $M_y = N_x$, $M_z = R_x$, and $N_z = R_y$, $\mathbf{F}$ is conservative.

20. $\mathbf{F}(x,y,z) = y\sec^2 x\mathbf{i} + (\tan x - z\sec^2 y)\mathbf{j} + x\sec z\tan z\mathbf{k}$

▷ We apply Theorem 14.1.2 to determine if $\mathbf{F}$ is a gradient. Let
$M(x,y,z) = y\sec^2 x$ $\quad N(x,y,z) = \tan x - z\sec^2 y$ $\quad R(x,y,z) = x\sec z\tan z$
$M_y(x,y,z) = \sec^2 x$ $\quad N_x(x,y,z) = \sec^2 x$ $\quad R_x(x,y,z) = \sec z\tan z$
$M_z(x,y,z) = 0$ $\quad N_z(x,y,z) = -\sec^2 y$ $\quad R_y(x,y,z) = 0$
We have $M_y(x,y,z) = N_x(x,y,z)$. However, because $M_z(x,y,z) \neq R_x(x,y,z)$, then $\mathbf{F}$ is not a gradient, and thus not conservative.

In Exercises 21–32, prove that the vector field is conservative and find a potential function.

▷ We may prove that $\mathbf{F}$ is conservative by exhibiting the potential function ϕ such that $\mathbf{F} = \nabla\phi$.

21. $\mathbf{F}(x,y) = y\mathbf{i} + x\mathbf{j} = \nabla\phi$ where $\phi(x,y) = xy + C$.

22. $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j} = \nabla\phi$ where $\phi(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$.

23. $\mathbf{F}(x,y) = e^x\sin y\mathbf{i} + e^x\cos y\mathbf{j} = \nabla\phi$ where $\phi(x,y) = e^x\sin y + C$.

24. $\mathbf{F}(x,y) = (\sin y\sinh x + \cos y\cosh x)\mathbf{i} + (\cos y\cosh x - \sin y\sinh x)\mathbf{j}$

▷ $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is the gradient of some function ϕ. We let
$M(x,y) = \sin y\sinh x + \cos y\cosh x$
$N(x,y) = \cos y\cosh x - \sin y\sinh x$
Then
$M_y(x,y) = \cos y\sinh x - \sin y\cosh x = N_x(x,y)$
Hence $\mathbf{F}$ is the gradient of some function ϕ. Furthermore, we have
$\phi_x(x,y) = \sin y\sinh x + \cos y\cosh x$ (1)
$\phi_y(x,y) = \cos y\cosh x - \sin y\sinh x$ (2)
Integrating with respect to x on both sides of (1), we have
$\phi(x,y) = \sin y\cosh x + \cos y\sinh x + c(y)$ (3)
Partial differentiating with respect to y on both sides of (3), we have
$\phi_x(x,y) = \cos y\cosh x - \sin y\sinh x + c'(y)$ (4)
Comparing (2) and (4) we find
$c'(y) = 0$
$c(y) = C$
Substituting for $c(y)$ into (3), we obtain the required potential function
$\phi(x,y) = \sin y\cosh x + \cos y\sinh x + C$

25. $\mathbf{F}(x,y) = (2xy^2 - y^3)\mathbf{i} + (2x^2y - 3xy^2 + 2)\mathbf{j} = \nabla\phi$ where $\phi(x,y) = x^2y^2 - xy^3 + 2y + C$.

26. $\mathbf{F}(x,y) = (3x^2+2y-y^2e^x)\mathbf{i} + (2x^2y - 3xy^2 + 2)\mathbf{j} = \nabla\phi$ where $\phi(x,y) = x^3 + 2xy - y^2e^x + C$.

27. $\mathbf{F}(x,y,z) = (x^2-y)\mathbf{i} - (x-3z)\mathbf{j} + (z+3y)\mathbf{k} = \nabla\phi$ where $\phi(x,y,z) = \frac{1}{3}x^3 - xy + 3yz + \frac{1}{2}z^2 + C$.

28. $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

▷ Because $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is a gradient, we let

$M(x,y,z) = yz \qquad N(x,y,z) = xz \qquad R(x,y,z) = xy$
$M_y(x,y,z) = z \qquad N_x(x,y,z) = z \qquad R_x(x,y,z) = y$
$M_z(x,y,z) = y \qquad N_z(x,y,z) = x \qquad R_y(x,y,z) = x$

Because
$M_y(x,y,z) = N_x(x,y,z) \quad M_z(x,y,z) = R_x(x,y,z) \quad N_z(x,y,z) = R_y(x,y,z)$
then $\mathbf{F}$ is the gradient of a potential function ϕ, such that
$\phi_x(x,y,z) = yz \quad \phi_y(x,y,z) = xz \quad \phi_z(x,y,z) = xy$ (1)
Because $\phi_x(x,y,z) = yz$, then by integrating with respect to x, we have
$\phi(x,y,z) = xyz + g(y,z)$
Partial-differentiating with respect to y on both sides, we get
$\phi_y(x,y,z) = xz + g_y(x,y)$
Because $\phi_y(x,y,z) = xz$ in (1), we conclude that $g_y(y,z) = 0$, and thus $g(y,z) = h(z)$, and
$\phi(x,y,z) = xyz + h(z)$
Partial-differentiating with respect to z, we have
$\phi_z(x,y,z) = xy + h'(z)$.
And because $\phi_z(x,y,z) = xy$ in (1), then $h'(z) = 0$, and $h(z) = C$. Thus, the potential function is given by
$\phi(x,y,z) = xyz + C$
It is perfectly legitimate to *guess* that $\phi(x,y,z) = xyz$ and merely verify Equations (1).

29. $\mathbf{F}(x,y,z) = (ze^x + e^y)\mathbf{i} + (xe^y - e^z)\mathbf{j} + (-ye^z + e^x)\mathbf{k} = \nabla\phi$ where $\phi(x,y,z) = ze^x + xe^y - ye^z + C$.

30. $\mathbf{F}(x,y,z) = (\tan y + 2xy\sec z)\mathbf{i} + (x\sec^2 y + x^2\sec z)\mathbf{j} + \sec z(x^2y\tan z - \sec z)\mathbf{k}$
$\phi(x,y,z) = x\tan y + x^2y\sec z - \tan z + C$

31. $\mathbf{F}(x,y,z) = (2x\cos y - 3)\mathbf{i} - (x^2\sin y + z^2)\mathbf{j} - (2yz-2)\mathbf{k} = \nabla\phi$ where $\phi = x^2\cos y - 3x - yz^2 + 2z + C$.

32. $\mathbf{F}(x,y,z) = (2y^3 - 8xz^2)\mathbf{i} + (6xy^2+1)\mathbf{j} - (8x^2z + 3z^2)\mathbf{k}$

▷ We have

$M(x,y,z) = 2y^3 - 8xz^2 \qquad N(x,y,z) = 6xy^2 + 1 \qquad R(x,y,z) = -(8x^2z + 3z^2)$
$M_y(x,y,z) = 6y^2 \qquad N_x(x,y,z) = 6y^2 \qquad R_x(x,y,z) = -16xz$
$M_z(x,y,z) = -16xz \qquad N_z(x,y,z) = 0 \qquad R_y(x,y,z) = 0$

Because
$M_y(x,y,z) = N_x(x,y,z) \quad M_z(x,y,z) = R_x(x,y,z) \quad N_z(x,y,z) = R_y(x,y,z)$
then $\mathbf{F}$ is the gradient of a potential function ϕ, such that
$\phi_x(x,y,z) = 2xy^3 - 8xz^2 \quad \phi_y(x,y,z) = 6xy^2 + 1 \quad \phi_z(x,y,z) = -8x^2z - 3z^2$ (1)
Integrating the first equation of (1) with respect to x, we get
$\phi(x,y,z) = 2xy^3 - 4x^2 + g(y,z)$ (2)
Partial-differentiating with respect to y on both sides, we get
$\phi_y(x,y,z) = 6xy^2 + g_y(y,z)$
Comparing with the second equation in (1), we get
$g_y(y,z) = 1$
$g(y,z) = y + h(z)$
Substituting the value for $g(y,z)$ into (2), we have
$\phi(x,y,z) = 2xy^3 - 4x^2z^2 + y + h(z)$ (3)
Partial-differentiating with respect to z, we have
$\phi_z(x,y,z) = = -8x^2z + h'(z)$
Comparing with the third equation in (1), we have
$h'(z) = -3z^2$
$h(z) = -z^3 + C$
Substituting into (3), we obtain the potential function
$\phi(x,y,z) = 2xy^3 - 4x^2z^2 + y - z^3 + C$

In Exercises 33–42, find curl **F** and div **F** for the vector field.

33. $\mathbf{F}(x,y) = 2x\mathbf{i} + 3y\mathbf{j}$. $M(x,y) = 2x$, $N(x,y) = 3y$. curl $\mathbf{F} = (N_x - M_y)\mathbf{k} = \mathbf{0}$; div $\mathbf{F} = M_x + N_y = 2 + 3 = 5$.

34. $\mathbf{F}(x,y) = \cos x\mathbf{i} - \sin y\mathbf{j}$. $M = \cos x$, $N = -\sin y$. curl $\mathbf{F} = (N_x - M_y)\mathbf{k} = \mathbf{0}$; div $\mathbf{F} = M_x + N_y = -\sin x - \cos y$

35. $\mathbf{F}(x,y) = e^x\cos y\mathbf{i} + e^x\sin y\mathbf{j}$. $M(x,y) = e^x\cos y$, $N(x,y) = e^x\sin y$. curl $\mathbf{F} = (N_x - M_y)\mathbf{k} =$ $(e^x\sin y + e^x\sin y)\mathbf{k} = 2e^x\sin y\mathbf{k}$. div $\mathbf{F} = M_x + N_y = e^x\cos y + e^x\cos y = 2e^x\cos y$.

36. $\mathbf{F}(x,y) = -\frac{y}{x}\mathbf{i} + \frac{1}{x}\mathbf{j}$

▹ Let $M(x,y) = -\frac{y}{x}$ $N(x,y) = \frac{1}{x}$

Then

$$\text{curl } \mathbf{F}(x,y) = [N_x(x,y) - M_y(x,y)]\mathbf{k} = \left(-\frac{1}{x^2} + \frac{1}{x}\right)\mathbf{k} = \frac{x-1}{x^2}\mathbf{k}$$

and

$$\text{div } \mathbf{F}(x,y) = M_x(x,y) + N_y(x,y) = \frac{y}{x^2}$$

37. $\mathbf{F}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$. $M(x,y,z) = x^2$, $N(x,y,z) = y^2$, $R(x,y,z) = z^2$.
curl $\mathbf{F} = (R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k} = \mathbf{0}$. div $\mathbf{F} = M_x + N_y + R_z = 2x + 2y + 2z$.

38. $\mathbf{F}(x,y,z) = xz^2\mathbf{i} + y^2\mathbf{j} + x^2z\mathbf{k}$. $M(x,y,z) = xz^2$, $N(x,y,z) = y^2$, $R(x,y,z) = x^2z$
curl $\mathbf{F} = (R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k} = \mathbf{0}$. div $\mathbf{F} = M_x + N_y + R_z = z^2 + 2y + x^2$.

39. $\mathbf{F}(x,y,z) = \cos y\mathbf{i} + \cos z\mathbf{j} + \cos x\mathbf{k}$. $M(x,y,z) = \cos y$, $N(x,y,z) = \cos z$, $R(x,y,z) = \cos x$.
curl $\mathbf{F} = (R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k} = \sin z\mathbf{i} + \sin x\mathbf{j} + \sin y\mathbf{k}$. div $\mathbf{F} = M_x + N_y + R_z = 0$.

40. $\mathbf{F}(x,y,z) = (y^2 + z^2)\mathbf{i} + xe^y\cos z\mathbf{j} - xe^y\cos z\mathbf{k}$

▹ curl $\mathbf{F}(x,y,z)$

$$= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 + z^2 & xe^y\cos z & -xe^y\cos z \end{vmatrix}$$

$$= [D_y(-xe^y\cos z) - D_z(xe^y\cos z)]\mathbf{i} - [D_x(-xe^y\cos z) - D_z(y^2 + z^2)]\mathbf{j} + [D_x(xe^y\cos z) - D_y(y^2 + z^2)]\mathbf{k}$$

$$= (-xe^y\cos z + xe^y\sin z)\mathbf{i} + (e^y\cos z)\mathbf{j} + (e^y\cos z - 2y)\mathbf{k}$$

$$\text{div } \mathbf{F}(x,y,z) = \nabla \cdot \mathbf{F} = D_x(y^2 + z^2) + D_y(xe^y\cos z) + D_z(-xe^y\cos z)$$

$$= 0 + xe^y\cos z + xe^y\sin z = xe^y(\cos z + \sin z)$$

41. $\mathbf{F}(x,y,z) = \sqrt{x^2 + y^2 + 1}\,\mathbf{i} + \sqrt{x^2 + y^2 + 1}\,\mathbf{j} + z^2\mathbf{k}$. $M = \sqrt{x^2 + y^2 + 1}$, $N = \sqrt{x^2 + y^2 + 1}$, $R = z^2$.
curl $\mathbf{F} = (R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k} = \dfrac{x - y}{\sqrt{x^2 + y^2 + 1}}\mathbf{k}$. div $\mathbf{F} = \dfrac{x + y}{\sqrt{x^2 + y^2 + 1}} + 2z$.

42. $\mathbf{F}(x,y,z) = \dfrac{x}{(x^2 + y^2)^{3/2}}\mathbf{i} + \dfrac{y}{(x^2 + y^2)^{3/2}}\mathbf{j} + \mathbf{k}$. $M = \dfrac{x}{(x^2 + y^2)^{3/2}}$, $N = \dfrac{y}{(x^2 + y^2)^{3/2}}$, $R = 1$.
curl $\mathbf{F} = (R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k} = \mathbf{0}$. div $\mathbf{F} = -\dfrac{1}{(x^2 + y^2)^{3/2}}$

In Exercises 43–46, prove that the scalar function is harmonic by showing that its Laplacian is zero.

43. $f(x,y) = e^y\sin x + e^x\cos y$. $\dfrac{\partial f}{\partial x} = e^y\cos x + e^x\cos y$, $\dfrac{\partial f}{\partial y} = e^y\sin x - e^x\sin y$.
$\dfrac{\partial^2 f}{\partial x^2} = -e^y\sin x + e^x\cos y$, $\dfrac{\partial^2 f}{\partial y^2} = e^y\sin x - e^x\cos y$. Because $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$, f is harmonic.

44. $f(x,y) = \ln\sqrt{x^2+y^2}$

▶ We have

$f(x,y) = \frac{1}{2}\ln(x^2+y^2)$

Then,

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2} \text{ and } \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Therefore, the Laplacian of f is

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial f^2}{\partial y^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

Because the Laplacian of f is zero, then f is harmonic.

45. $f(x,y,z) = 2x^2 + 3y^2 - 5z^2$, $\frac{\partial f}{\partial x} = 4x$, $\frac{\partial f}{\partial y} = 6y$, $\frac{\partial f}{\partial z} = -10z$.

$\frac{\partial^2 f}{\partial x^2} = 4$, $\frac{\partial^2 f}{\partial y^2} = 6$, $\frac{\partial f^2}{\partial z^2} = -10$. Because $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$, f is harmonic.

46. $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$, $f_x = -\frac{1}{2}x(x^2+y^2+z^2)^{-3/2}$.

$f_{xx} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} + \frac{3}{2}x^2(x^2+y^2+z^2)^{-5/2}$

$= \frac{1}{2}(x^2+y^2+z^2)^{-3/2}[-(x^2+y^2+z^2) + 3x^2] = \frac{1}{2}(x^2+y^2+z^2)(-2x^2+y^2+z^2)$. The sum of 3 such terms is 0.

47. Th 14.1.5. $\text{div}(\text{curl } \mathbf{F}) = \text{div}[(R_y - N_z)\mathbf{i} + (M_z - R_x)\mathbf{j} + (N_x - M_y)\mathbf{k}] = (R_{yx} - N_{zx}) + (M_{zy} - R_{xy}) + (N_{xz} - M_{yz})$
The mixed partial derivatives are equal if they are continuous (Th. 12.3.3) and so the sum is zero. Conversely, let $\mathbf{G} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ with $\text{div } \mathbf{G} = P_x + Q_y + R_z = 0$. We shall find $\mathbf{F} = S\mathbf{i} + T\mathbf{j}$ such that $\mathbf{G} = \text{curl } \mathbf{F}$, i.e. $P = -T_z$, $Q = S_z$, and $R = T_x - S_y$. For fixed x, y and any function $c(x,y)$, the first two are satisfied by

$T = -\int_{t=0}^{z} P(x,y,t)dt$ and $S = \int_{t=0}^{z} Q(x,y,t)dt + c(x,y)$. To satisfy the third we need

$0 = T_x - S_y - R = -\int_{t=0}^{z} (P_x(x,y,t) + Q_y(x,y,t))dt - c_y(x,y) - R(x,y,z)$

$= \int_{t=0}^{z} R_z(x,y,t)dt - c_y(x,y) - R(x,y,z)$ {hypothesis}

$= R(x,y,z) - R(x,y,0) - c_y(x,y) - R(x,y,z) = -R(x,y,0) - c_y(x,y)$

which is satisfied by $c(x,y) = -\int_{t=0}^{y} R(x,t,0)dt$. If $\mathbf{H}$ is any other solution then $\text{curl}(\mathbf{G} - \mathbf{H}) = \mathbf{G} - \mathbf{G} = \mathbf{0}$ so the difference is the gradient of a scalar function.

48. Prove Theorem 14.1.6.

▶ If f is a scalar field on an open ball in R^3, then the gradient of f is given by

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

The curl of the gradient of f is

$$\text{curl}(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}$$

Because the second partial derivatives of f are continuous on B, then by Theorem 12.3.3.

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Therefore, $\text{curl}(\nabla f) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

49. Because $M\mathbf{i} + N\mathbf{j} + R\mathbf{k}$ is a gradient, there exists a function f such that $f_x = M$, $f_y = N$, $f_z = R$. Because M_y, M_z, N_x, N_z, R_x, and R_y exist and are continuous on B, it follows from Theorem 12.3.3 that
$M_y = f_{xy} = f_{yx} = N_x$, $M_z = f_{xz} = f_{zx} = R_x$, and $N_z = f_{yz} = f_{zy} = R_y$.

14.2 LINE INTEGRALS

14.2.2 Definition Let C be a smooth curve lying in an open disk B in R^2 and having the vector equation

$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \quad a \le t \le b$

Let $\mathbf{F}$ be a vector field on B defined by

$\mathbf{F}(x,y) = \mathrm{M}(x,y)\mathbf{j} + \mathrm{N}(x,y)\mathbf{j}$

where M and N are continuous on B. Then, by using the differential form notation, the line integral of $\mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy$ over C is given by

$$\int_C \mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy = \int_a^b [\mathrm{M}(f(t),g(t))f'(t) + \mathrm{N}(f(t),g(t))g'(t)]dt$$

or, equivalently, by using vector notation, the line integral of $\mathbf{F}$ over C is given by

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_a^b \mathbf{F}(\mathbf{R}(t))\cdot \mathbf{R}'(t)dt \tag{I}$$

The line integral has the same value for any parametric representation of C. Furthermore, if C is closed curve, the value is the same for any initial point. The differential form notation seems to be easier to apply. See Exercise 4.

14.2.3 Definition Let the curve C consist of smooth arcs $\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_n$. Then the *line integral* of $\mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy$ over C is defined by the following equation:

$$\int_C \mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy = \sum_{i=1}^{n}\Big(\int_{C_i} \mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy\Big)$$

or, equivalently, by using vector notation, the line integral of $\mathbf{F}$ over C is defined by

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \sum_{i=1}^{n}\Big(\int_{a_i}^{b_i} \mathbf{F}(\mathbf{R}(t)\cdot \mathbf{R}'(t)dt\Big)$$

Line integrals are used to calculate the total work done by a variable force that causes a particle to move along a curve.

14.2.1 Definition Let C be a smooth curve lying in the open disk B in R^2 for which a vector equation is $\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, $a \le t \le b$. Furthermore, let a force field on B be defined by $\mathbf{F}(x,y) = \mathrm{M}(x,y)\mathbf{i} + \mathrm{N}(x,y)\mathbf{j}$, where M and N are continuous on B. If W is the measure of the work done by a force $\mathbf{F}$ moving a particle along C from $(f(a),g(a))$ to $(f(b),g(b))$, then

$$\mathrm{W} = \int_a^b [\mathrm{M}(f(t),g(t))f'(t) + \mathrm{N}(f(t),g(t))g'(t)]dt$$

or, equivalently, by using vector notation,

$$\mathrm{W} = \int_a^b \mathbf{F}(\mathbf{R}(t))\cdot \mathbf{R}'(t)dt$$

or, equivalently, using a line integral in differential form notation,

$$\mathrm{W} = \int_C \mathrm{M}(x,y)dx + \mathrm{N}(x,y)dy$$

The following definitions extend the concept of a line integral to functions of three variables.

Smooth, Closed The curve C having the vector equation $\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, $a \le t \le b$, is *smooth* on $[a,b]$ if f', g', and h' are continuous and never all 0 in (a,b). C is *closed* if $\mathbf{R}(a) = \mathbf{R}(b)$.

14.2.4 Definition Let C be a smooth curve lying in an open ball in R^3 and having the vector equation

$\mathbf{R}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \le t \le b$

Let $\mathbf{F}$ be a vector field on B defined by

$\mathbf{F}(x,y) = \mathrm{M}(x,y,z)\mathbf{j} + \mathrm{N}(x,y,z)\mathbf{j} + \mathrm{R}(x,y,z)\mathbf{k}$

where M, N, and R are continuous on B. Then the line integral of $\mathrm{M}(x,y,z)dx + \mathrm{N}(x,y,z)dy + \mathrm{R}(x,y,z)dz$ over C is given by

$$\int_C \mathrm{M}(x,y,z)dx + \mathrm{N}(x,y,z)dy + \mathrm{R}(x,y,z)dz =$$
$$\int_a^b [\mathrm{M}(f(t),g(t),h(t))f'(t) + \mathrm{N}(f(t),g(t),h(t))g'(t) + \mathrm{R}(f(t),g(t),h(t)]dt$$

or, equivalently, by using vector notation, the line integral of $\mathbf{F}$ over C is given by Eq. (I).

Exercises 14.2

In Exercises 1–22, evaluate the line integral over the curve.

1. $\mathbf{F}(x,y) = y\mathbf{i} + x\mathbf{j}$. C:$\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_C \mathbf{F}(\mathbf{R}(t))\cdot \mathbf{R}'(t)dt = \int_0^1 \langle t^2,t\rangle\cdot\langle 1,2t\rangle dt = \int_0^1 3t^2dt = t^3\Big]_0^1 = 1$$

2. $\mathbf{F}(x,y) = 2xy\mathbf{i} - 3x\mathbf{j}$; C: $\mathbf{R}(t) = 3t^2\mathbf{i} - t\mathbf{j}$, $0 \le t \le 1$.

$$\int_C \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t)dt = \int_0^1 \langle 2(3t^2)(-t), -3(3t^2)\rangle \cdot \langle 6t, -1\rangle dt = \int_0^1 (-36t^4 + 9t^2)dt = -\tfrac{36}{5}t^5 + 3t^3\Big]_0^1 = -\tfrac{21}{5}$$

3. $\mathbf{F}(x,y) = 2xy\mathbf{i} + (x - 2y)\mathbf{j}$. C: $\mathbf{R}(t) = \sin t\mathbf{i} - 2\cos t\mathbf{j}$, $0 \le t \le \pi$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^\pi \langle 2\sin t(-2\cos t), \sin t + 4\cos t\rangle \cdot \langle \cos t, 2\sin t\rangle dt$$

$$= \int_0^\pi (-4\sin t\cos^2 t + 2\sin^2 t + 8\sin t\cos t)dt = \Big[\tfrac{4}{3}\cos^3 t + \Big(t - \tfrac{1}{2}\cos 2t\Big) + 4\sin^2 t\Big]_0^\pi = \pi - \tfrac{8}{3}$$

4. $\int_C \mathbf{F} \cdot d\mathbf{R}$; $\mathbf{F}(x,y) = xy\mathbf{i} - y^2\mathbf{j}$; C: $\mathbf{R}(t) = t^2\mathbf{i} + t^3\mathbf{j}$, from the point $(1,1)$ to the point $(4,-8)$.

▶ We apply Definition 14.2.2. Because

$$\mathbf{F}(x,y) = \langle xy, -y^2\rangle \quad \text{and} \quad \mathbf{R}(t) = \langle t^2, t^3\rangle$$

then

$$\mathbf{F}(\mathbf{R}(t)) = \mathbf{F}(t^2, t^3) = \langle t^2t^3, -(t^3)^2\rangle = \langle t^5, -t^6\rangle \quad \text{and} \quad \mathbf{R}'(t) = \langle 2t, 3t^2\rangle$$

Furthermore, because $\mathbf{R}(1) = (1,1)$ and $\mathbf{R}(-2) = (4,-8)$, then $t = 1$ at the point $(1,1)$ and $t = -2$ at the point $(4,-8)$. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t)dt = \int_1^{-2} \langle t^5, -t^6\rangle \cdot \langle 2t, 3t^2\rangle dt = \int_1^{-2} (2t^6 - 3t^8)dt = \tfrac{2}{7}t^7 - \tfrac{1}{3}t^9\Big]_1^{-2} = \tfrac{939}{7}$$

We may also use the differential form notation. We have

$$x = t^2 \qquad y = t^3$$
$$dx = 2t\,dt \qquad dy = 3t^2\,dt$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C xy\,dx - y^2dy = \int_1^{-2} t^2(t^3)(2t\,dt) - (t^3)^2(3t^2dt) = \int_1^{-2} (2t^6 - 3t^8)dt = \tfrac{2}{7}t^7 - \tfrac{1}{3}t^9\Big]_1^{-2} = \tfrac{939}{7}$$

5. $\mathbf{F}(x,y) = (x - y)\mathbf{i} + (x + y)\mathbf{j}$. C: $\mathbf{R}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$, $0 \le t \le 2\pi$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^{2\pi} \langle 2\cos t - 2\sin t, 2\cos t + 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t\rangle dt$$

$$= \int_0^{2\pi} (-4\sin t\cos t + 4\sin^2 t + 4\cos^2 t + 4\sin t\cos t)dt = \int_0^{2\pi} 4\,dt = 8\pi$$

6. $x = 3\cos t$, $y = 2\sin t$. $\displaystyle\int_C (x - 2y)dx + xy\,dy = \int_0^{\pi/2} (3\cos t - 4\sin t)(-3\sin t\,dt) + 6\sin t\cos t(2\cos t\,dt)$

$$= \int_0^{\pi/2} [-9\cos t\sin t + 6(1 - \cos 2t) + 12\cos^2 t\sin t]dt = -\tfrac{9}{2}\sin^2 t + 6t - 3\sin 2t - 4\cos^3 t\Big]_0^{\pi/2} = 3\pi - \tfrac{1}{2}$$

7. $\mathbf{F}(x,y) = y\sin x\mathbf{i} - \cos x\mathbf{j}$. C: $\mathbf{R}(t) = \frac{1}{2}(t+1)\pi\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 \langle t\sin(t+1)\tfrac{\pi}{2}, -\cos(t+1)\tfrac{\pi}{2}\rangle \cdot \langle \tfrac{\pi}{2}, 1\rangle dt = \int_0^1 \Big[\tfrac{\pi}{2}t\sin(t+1)\tfrac{\pi}{2} - \cos(t+1)\tfrac{\pi}{2}\Big]dt = -t\cos(t+1)\tfrac{\pi}{2}\Big]_0^1 = 1$$

8. $\int_C \mathbf{F} \cdot d\mathbf{R}$; $\mathbf{F}(x,y) = 9x^2y\mathbf{i} + (5x^2 - y)\mathbf{j}$; C: the curve $y = x^3 + 1$ from $(1,2)$ to $(3,28)$.

▶ Because the equation of C is expressed in terms of x, we apply Definition 14.2.2 with parameter x. We replace y by $x^3 + 1$ and dy by $3x^2dx$ with $1 \le x \le 3$. We have

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C 9x^2y\,dx + (5x^2 - y)dy = \int_1^3 9x^2(x^3 + 1)dx + [5x^2 - (x^3 + 1)](3x^2\,dx)$$

$$= \int_1^3 (9x^5 + 9x^2 + 15x^4 - 3x^5 - 3x^2)dx = \int_1^3 (6x^5 + 15x^4 + 6x^2)dx = x^6 + 3x^5 + 2x^3\Big]_1^3 = 1{,}506$$

9. C: $y = x$ from $(0,0)$ to $(2,2)$. With x as parameter,

$$\int_C (x^2 + xy)dx + (y^2 - xy)dy = \int_0^2 (x^2 + x^2)dx + (x^2 - x^2)dx = \int_0^2 2x^2dx = \tfrac{2}{3}x^3\Big]_0^2 = \tfrac{16}{3}$$

10. C: $y = \frac{1}{2}x^2$ from $(0,0)$ to $(2,2)$. With x as parameter, $\displaystyle\int_C (x^2 + xy)dx + (y^2 - xy)dy$

$$= \int_0^2 (x^2 + \tfrac{1}{2}x^3)dx + (\tfrac{1}{4}x^4 - \tfrac{1}{2}x^3)(x\,dx) = \int_0^2 (x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^5 - \tfrac{1}{2}x^4)dx = \tfrac{1}{3}x^3 + \tfrac{1}{8}x^4 + \tfrac{1}{24}x^6 - \tfrac{1}{10}x^5\Big]_0^2 = \tfrac{62}{15}$$

11. On C_1, x is parameter, $0 \le x \le 2$, $y = 0$. On C_2, y is parameter, $0 \le y \le 2$, $x = 2$.

$$\int_C (x^2 + xy)dx + (y^2 - xy)dy = \int_0^2 x^2dx + \int_0^2 (y^2 - 2y)dy = \tfrac{1}{3}x^3\Big]_0^2 + \Big[\tfrac{1}{3}y^3 - y^2\Big]_0^2 = \tfrac{8}{3} + \Big(\tfrac{8}{3} - 4\Big) = \tfrac{4}{3}$$

12. $\int_C yx^2dx + (x + y)dy$, where C is the line $y = -x$ from the origin to the point $(1,-1)$.

▶ We let x be the parameter, and replace y by $-x$ and dy by $-dx$ with $0 \le x \le 1$. Thus,

$$\int_C yx^2dx + (x + y)dy = \int_0^1 -x^3dx + (x - x)(-dx) = -\int_0^1 x^3dx = -\tfrac{1}{4}x^4\Big]_0^1 = \tfrac{1}{4}$$

13. C: $y = -x^3$, $0 \le x \le 1$. With x as parameter

$$\int_C yx^2dx + (x+y)dy = \int_0^1 (-x^3)x^2dx + (x - x^3)(-3x^2)dx = \int_0^1 (2x^5 - 3x^3)dx = \left[\tfrac{1}{3}x^6 - \tfrac{3}{4}x^4\right]_0^1 = -\tfrac{5}{12}$$

14. C_1: $x = 0$, $y = 0$ to -1 and C_2: $y = -1$, $x = 0$ to 1.

$$\int_C yx^2dx + (x+y)dy = \int_0^{-1} y\,dy + \int_0^1 -x^2dx = \tfrac{1}{2}y^2\Big]_0^{-1} - \tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$$

15. With x as parameter, C_1: $y = 2x + 3$, $0 \le x \le 3$ and C_2: $y = x^2$, $3 \le x \le 5$.

$$\int_C 3xy\,dx + (4x^2 - 3y)dy = \int_0^3 3x(2x+3)dx + [4x^2 - 3(2x+3)]2\,dx + \int_0^5 3x(x^2)dx + (4x^2 - 3x^2)2x\,dx$$
$$= \int_0^3 (14x^2 - 3x - 18)dx + \int_0^5 5x^3dx = \left[\tfrac{14}{3}x^3 - \tfrac{3}{2}x^2 - 18x\right]_0^3 + \left[\tfrac{5}{2}x^4\right]_0^5 = 14 \cdot 9 - \tfrac{27}{2} - 54 + \tfrac{5}{4}(5^4 - 3^4) = \tfrac{1477}{2}$$

16. $\int_C (xy - z)dx + e^x dy + y\,dz$, where C is the line segment from $(1, 0, 0)$ to $(3, 4, 8)$.

▹ The segment has the direction of $\langle 1, 2, 4\rangle$. Thus, parametric equations of C are

$x = 1 + t \quad y = 2t \quad z = 4t \quad \text{for} \quad 0 \le t \le 2$

Because $dx = dt$, $dy = 2dt$, and $dz = 4dt$, we have

$$\int_C (xy - z)dx + e^x dy + y\,dz = \int_0^2 [(1+t)(2t) - 4t]dt + e^{1+t}(2dt) + (2t)(4dt)$$
$$= \int_0^2 (6t + 2t^2 + 2e^{t+1})dt = 3t^2 + \tfrac{2}{3}t^3 + 2e^{t+1}\Big]_0^2 = \tfrac{52}{3} + 2(e^3 - e)$$

17. C: $x = t$, $y = 2t$, $z = 4t$, $0 \le t \le 1$.

$$\int_C (x+y)dx + (y+z)dy + (x+z)dz = \int_0^1 (t + 2t)dt + (2t + 4t)2\,dt + (t + 4t)4\,dt = \int_0^1 35t\,dt = \tfrac{35}{2}$$

18. C: $x = t + 1$, $y = t^2$, $z = t^3$, $0 \le t \le 2$

$$\int_C (xy - z)dx + e^x dy + y\,dz = \int_0^2 (t^2 + 2te^{t+1} + 3t^4)dt = \tfrac{1}{3}t^3 + 2(t-1)e^{t+1} + \tfrac{3}{5}t^5\Big]_0^2 = 2e^3 + 2e + \tfrac{328}{15}$$

19. $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$; C: $\mathbf{R}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \langle t, a\cos t, a\sin t\rangle \cdot \langle -a\sin t, a\cos t, 1\rangle dt = \int_0^{2\pi} (-at\sin t + a^2\cos^2 t + a\sin t)dt$$
$$= a(t\cos t - \sin t) + \tfrac{1}{2}a^2 t + \tfrac{1}{4}a^2 \sin 2t - a\cos t\Big]_0^{2\pi} = (a \cdot 2\pi + a^2\pi - a) - (-a) = \pi(a^2 + 2a)$$

20. $\int_C \mathbf{F} \cdot d\mathbf{R}$; $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (6y^2 - xz)\mathbf{j} + 10z\mathbf{k}$; C: the twisted cubic $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$.

▹ We apply Definition 14.2.4. We have

$x = t \qquad y = t^2 \qquad z = t^3$

$dx = dt \qquad dy = 2t\,dt \qquad dz = 3t^2dt$

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C 2xy\,dx + (6y^2 - xz)dy + 10z\,dz$$
$$= \int_0^1 \{2t(t^2)dt + [6(t^2)^2 - t(t^3)](2t\,dt) + 10t^3(3t^2dt)\} = \int_0^1 (2t^3 + 40t^5)dt = \tfrac{1}{2}t^4 + \tfrac{20}{3}t^6\Big]_0^1 = \tfrac{43}{6}$$

21. $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (6y^2 - xz)\mathbf{j} + 10z\mathbf{k}$

C_1: $\mathbf{R}_1(t) = t\mathbf{k}$, $0 \le t \le 1$; C_2: $\mathbf{R}(t) = t\mathbf{j} + \mathbf{k}$, $0 \le t \le 1$; C_3: $\mathbf{R}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}$, $0 \le t \le 1$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_{C_1} \mathbf{F} \cdot d\mathbf{R}_1 + \int_{C_2} \mathbf{F} \cdot d\mathbf{R}_2 + \int_{C_3} \mathbf{F} \cdot d\mathbf{R}_3 = \int_0^1 10t\,dt + \int_0^1 6t^2\,dt + \int_0^1 2t\,dt = 5t^2\Big]_0^1 + 2t^3\Big]_0^1 + t^2\Big]_0^1 = 8$$

22. $x = y = z = t$, $0 \le t \le 1$. $\displaystyle\int_C 2xy\,dx + (6y^2 - xz)dy + 10z\,dz = \int_0^1 (2t^2 + 5t^2 + 10t)dt = \tfrac{7}{3}t^3 + 5t^2\Big]_0^1 = \tfrac{22}{3}$

In Exercises 23–36, find the total work done in moving an object along the given curve C if the motion is caused by the given force field. Assume that length is measured in meters and force is measured in newtons.

▹ W joules of work is done.

23. $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$; C: $\mathbf{R}(t) = t\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$.

$$W = \int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 \langle 2t^2, 2t^2\rangle \cdot \langle 1, 1\rangle dt = \int_0^1 4t^2dt = \tfrac{4}{3}t^3\Big]_0^1 = \tfrac{4}{3}$$

24. $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$; C is the arc of the parabola $y^2 = x$ from the origin to the point $(1, 1)$.

▹ We use Definition 14.2.1 in differential form notation with y as a parameter. We replace x by y^2, $0 \le y \le 1$, and dx by $2y\,dy$. Thus,

$$W = \int_C 2xy\,dx + (x^2 + y^2)dy = \int_0^1 2(y^2)y(2y\,dy) + [(y^2)^2 + y^2]dy = \int_0^1 (5y^4 + y^2)dy = \left[y^5 + \tfrac{1}{3}y^3\right]_0^1 = \tfrac{4}{3}$$

Thus, the total work done is $\frac{4}{3}$ joules.

25. $\mathbf{F}(x,y) = (y-x)\mathbf{i} + x^2y\mathbf{j}$; C: $\mathbf{R}(t)t\mathbf{i} + (3t-2)\mathbf{j}$, $1 \le t \le 2$.

$W = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^2 \langle 3t-2-t, t^2(3t-2)\rangle \cdot \langle 1,3\rangle dt = \int_0^2 (9t^3 - 6t^2 + 2t - 2)dt = \frac{9}{4}t^4 - 2t^3 + t^2 - 2t\Big]_0^2 = 20\frac{3}{4}$

26. C: $y = x^2$, $1 \le x \le 2$. $W = \int_C (y-x)dx + x^2y\,dy = \int_1^2 (x^2 - x)dx + x^4(2x\,dx) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x^6\Big]_1^2 = \frac{131}{6}$

27. $\mathbf{F}(x,y) = (y-x)\mathbf{i} + x^2y\mathbf{j}$; C_1: $\mathbf{R}_1(t) = t\mathbf{i} + t\mathbf{j}$, $1 \le t \le 2$; C_2: $\mathbf{R}_2(t) = 2\mathbf{i} + t\mathbf{j}$, $2 \le t \le 4$.

$W = \int_0^2 \langle t-t, t^2\cdot t\rangle\cdot\langle 1,1\rangle dt + \int_0^4 \langle t-2, 4t\rangle\cdot\langle 0,1\rangle dt = \int_0^2 t^3 dt + \int_0^4 4t\,dt = \frac{1}{4}t^4\Big]_0^2 + 2t^2\Big]_0^4 = 27\frac{3}{4}$

28. $\mathbf{F}(x,y) = -x^2y\mathbf{i} + 2y\mathbf{j}$; C is the line segment from $(a,0)$ to $(0,a)$.

▹ The segment has the direction of $\langle -1,1\rangle$. Thus, parametric equations of C are $x = a-t$ and $y = t$, $0 \le t \le a$.

$$W = \int_C -x^2y\,dx + 2y\,dy = \int_0^a -(a-t)^2t(-dt) + 2t\,dt = \int_0^a (t^3 - 2at^2 + a^2t + 2t)dt$$
$$= \frac{1}{4}t^4 - \frac{2}{3}at^3 + \frac{1}{2}a^2t^2 + t^2\Big]_0^a = \frac{1}{12}a^4 + a^2$$

Therefore, the total work done is $\frac{1}{12}a^4 + a^2$ joules.

29. $\mathbf{F}(x,y) = -x^2y\mathbf{i} + 2y\mathbf{j}$; C: $\mathbf{R}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $0 \le t \le \frac{1}{2}\pi$.

$W = \int_0^{\pi/2} \langle -a^2\cos^2 t\cdot a\sin t, 2\sin t\rangle\cdot\langle -a\sin t, a\cos t\rangle dt = \int_0^{\pi/2} (a^4\sin^2 t\cos^2 t + 2a^2\sin t\cos t)dt$

$= \int_0^{\pi/2}\left(\frac{1}{4}a^4\sin^2 2t + 2a^2\sin t\cos t\right)dt = \frac{1}{8}a^4\left(t - \frac{1}{4}\cos 4t\right) + a^2\sin^2 t\Big]_0^{\pi/2} = \frac{1}{16}a^4\pi + a^2$

30. C_1: $(a,0)$ to (a,a); C_2: (a,a) to $(0,a)$. $W = \int_{C_1} + \int_{C_2} -x^2y\,dx + 2y\,dy = \int_0^a 2y\,dy + \int_a^0 -ax^2dx = a^2 + \frac{1}{3}a^4$

31. $\mathbf{F}(x,y,z) = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$; C: $t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$.

$W = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \langle t+t, t+t, t+t\rangle\cdot\langle 1,1,1\rangle dt = \int_0^1 6t\,dt = 3t^2\Big]_0^1 = 3$

32. $\mathbf{F}(x,y,z) = z^2\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}$; C is the line segment from the origin to the point $(4,0,3)$.

▹ Parametric equation of C are

$x = 4t \quad y = 0 \quad z = 3t \quad 0 \le t \le 1$

Thus,

$$W = \int_C z^2dx + y^2dy + xz\,dz = \int_0^1 (3t)^2(4\,dt) + 0 + (4t)(3t)(3\,dt) = \int_0^1 72t^2dt = 24t^3\Big]_0^1 = 24$$

The total work done is 24 joules.

33. $\mathbf{F}(x,y,z) = e^x\mathbf{i} + e^y\mathbf{j} + e^z$; C: $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 2$.

$W = \int_0^2 \langle e^t, e^t, e^t\rangle\cdot\langle 1, t^2, t^3\rangle dt = \int_0^2 (e^t + 2te^t + 3t^2e^t)dt = e^t + e^t + e^t\Big]_0^2 = e^2 + e^4 + e^8 - 3$

34. $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 2$. $W = \int_C (xyz + x)dx + (x^2z + y)dy + (x^2y + z)dz =$

$\int_0^2 [(t^6 + t) + (t^5 + t^2)(2t) + (t^4 + t^3)(3t^2)]dt = \int_0^2 (t^6 + t + 2t^6 + 2t^3 + 3t^6 + 3t^5)dt = \frac{6}{7}t^7 + \frac{1}{2}t^6 + \frac{1}{2}t^4 + \frac{1}{2}t^2\Big]_0^2 = \frac{1062}{7}$

35. $\mathbf{F}(x,y,z) = (xyz + x)\mathbf{i} + (x^2z + y)\mathbf{j} + (x^2y + z)\mathbf{k}$

C_1: $\mathbf{R}_1(t) = t\mathbf{i}$, $0 \le t \le 1$; C_2: $\mathbf{R}_2(t) = \mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$; C_3: $\mathbf{R}_3(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$.

$W = \int_{C_1}\mathbf{F}\cdot d\mathbf{R}_1 + \int_{C_2}\mathbf{F}\cdot d\mathbf{R}_2 + \int_{C_3}\mathbf{F}\cdot d\mathbf{R}_3 = \int_0^1 t\,dt + \int_0^1 t\,dt + \int_0^1 (1+t)dt = \frac{1}{2}t^2\Big]_0^1 + \frac{1}{2}t^2\Big]_0^1 + t + \frac{1}{2}t^2\Big]_0^1 = 2\frac{1}{2}$

36. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + (yz - x)\mathbf{k}$; C is $\mathbf{R}(t) = 2t\mathbf{i} + t^2\mathbf{j} + 4t^3\mathbf{k}$, for $0 \le t \le 1$.

We have

$x = 2t \qquad y = t^2 \qquad z = 4t^3$

$dx = 2\,dt \qquad dy = 2t^2dt \qquad dz = 12t^2dt$

Thus,

$$W = \int_C x\,dx + y\,dy + (yz - x)dz = \int_0^1 (2t)(2\,dt) + (t^2)(2t^2dt) + [t^2(4t^3) - 2t](12t^2dt)$$
$$= \int_0^1 (48t^7 - 22t^3 + 4t)dt = 6t^8 - \frac{11}{2}t^4 + 2t^2\Big]_0^1 = \frac{5}{2}$$

The work is 2.5 joules.

14.3 LINE INTEGRALS INDEPENDENT OF THE PATH

14.3.1 Theorem Let C be any sectionally smooth curve lying in a open disk B in R^2 from the point (x_1, y_1) to the point (x_2, y_2). If **F** is a conservative vector field continuous on B and ϕ is a potential function for **F**, then the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{R}$$

is independent of the path C and

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(x_2, y_2) - \phi(x_1, y_1)$$

14.3.2 Theorem If C is any sectionally smooth closed curve lying in some open disk B in R^2 and **F** is a conservative vector field on B, then

$$\int_C \mathbf{F}\cdot d\mathbf{R} = 0$$

14.3.3 Theorem Let C be any sectionally smooth curve lying in a open ball B in R^3 from the point (x_1, y_1, z_1) to the point (x_2, y_2, z_2). If **F** is a conservative vector field continuous on B and ϕ is a potential function for **F**, then the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{R}$$

is independent of the path C and

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Exercises 14.3

In Exercises 1–12, use the result of the indicated Exercise in Exercises 14.1 to prove that the line integral is independent of the path. The evaluate the line integral by applying either Theorem 14.3.1 or 14.3.3 and using the potential function found in the indicated exercise. In each exercise C is any sectionally smooth curve from the point A to the point B.

1. $\mathbf{F} = \nabla\phi$ with $\phi(x,y) = xy$. $\int_C y\,dx + x\,dy = \phi(3,2) - \phi(1,4) = 6 - 4 = 2.$

2. $\mathbf{F} = \nabla\phi$ with $\phi(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. $\int_C x\,dx + y\,dy = \phi(1,3) - \phi(-5,2) = 5 - \frac{29}{2} = -\frac{19}{2}$

3. $\mathbf{F} = \nabla\phi$ with $\phi(x,y) = e^x\sin y$. $\int_C e^x\sin y\,dx + e^x\cos y\,dy = \phi(2,\frac{1}{2}\pi) - \phi(0,0) = e^2 - 0 = e^2$.

4. $\int_C (\sin y \sinh x + \cos y \cosh x)dx + (\cos y \cosh x - \sin y \sinh x)dy$; A is $(1,0)$ and B is $(2,\pi)$; Exercises 24.

▶ In Exercise 14.1.24 we proved that the vector field defined by
$\mathbf{F}(x,y) = (\sin y \sinh x + \cos y \cosh x)dx + (\cos y \cosh x - \sin y \sinh x)dy$
is conservative with a potential function given by
$\phi(x,y) = \sin y \cosh x + \cos y \sinh x$
Therefore, by Theorem 14.3.1, the given line integral is independent of the path C, and the value of the integral is

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(2,\pi) - \phi(1,0) = (\sin\pi\cosh 2 + \cos\pi\sinh 2) - (\sin 0\cosh 1 + \cos 0\sinh 1) = -\sinh 2 - \sinh 1$$

5. $\mathbf{F} = \nabla\phi$ with $\phi(x,y) = x^2y^2 - x^y3 + 2y$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,2) - \phi(-3,-1) = 0 - 4 = -4$.

6. $\mathbf{F} = \nabla\phi$ with $\phi(x,y) = x^3 + 2xy - y^2e^x$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,-3) - \phi(0,2) = (-5-9e) - (-4) = -1 - 9e$

7. $\mathbf{F} = \nabla\phi$ with $\phi(x,y,z) = \frac{1}{3}x^3 - xy + 3yz + \frac{1}{2}z^2$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(3,0,4) - \phi(-3,1,2) = 17 - 2 = 15$.

8. $\int_C yz\,dx + xz\,dy + xy\,dz$; A is $(0,-2,5)$ and B is $(4,1,-3)$; Exercise 28.

▶ In Exercise 14.1.28, we proved that the force field defined by
$\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
is conservative with a potential function given by
$\phi(x,y,z) = xyz$
Therefore, by Theorem 14.3.3, the given line integral is independent of the path C, and

$$\int_C yz\,dx + xz\,dy + xy\,dz = \phi(4,1,-3) - \phi(0,-2,5) = (4)(1)(-3) - (0)(-2)(5) = -12$$

9. $\mathbf{F} = \nabla\phi$ with $\phi(x,y,z) = ze^x + xe^y - ye^z$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,2,1) - \phi(1,0,2) = 1 - 2e - 2e - 1 = -4e$.

10. $\phi = x\tan y + x^2\sec z - \tan z$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(3, \frac{1}{4}\pi, \frac{1}{3}\pi) - \phi(2, \frac{1}{6}\pi, 0) = (3\cdot 1 + 9\cdot 2 - \sqrt{3}) - (\sqrt{3}+4) = 17 - 2\sqrt{3}$

11. $\mathbf{F} = \nabla\phi$ with $\phi(x,y,z) = x^2\cos y - 3z - yz^2 + 2z$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,\pi,0) - \phi(-1,0,3) = -4 - 10 = -14$.

12. $\int_C (2y^3 - 8xz^2)dx + (6xy^2+1)dy - (8x^2z + 3z^2)dz$; A is $(2,0,0)$ and B is $(3,2,1)$; Exercise 32.

▸ In Exercise 14.1.32, we proved that the vector field defined by
$\mathbf{F}(x,y,z) = (2y^3 - 8xz^2)\mathbf{i} + (6xy^2+1)\mathbf{j} - (8x^2 + 3z^2)\mathbf{k}$
is conservative with a potential function given by
$\phi(x,y,z) = 2xy^3 - 4x^2z^2 + y - z^3$
Therefore, by Theorem 14.3.3, the given line integral is independent of the path C, and

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(3,2,1) - \phi(2,0,0) = 13$$

In Exercises 13–20, show that the value of the line integral $\int_C \mathbf{F}\cdot d\mathbf{R}$ for the given function $\mathbf{F}$ and C is independent of the path, and evaluate the integral.

▸ In Exercises 13–19 we show the line integral is path-independent by exhibiting a potential.

13. $\mathbf{F}(x,y) = 2(x-y)\mathbf{i} + 2(3y-x)\mathbf{j} = \nabla\phi$ with $\phi(x,y) = x^2 - 2xy + 3y^2$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,3) - \phi(3,0) = 27 - 9 = 18$

14. $\mathbf{F}(x,y) = (3x^2 + 6xy - 2y^2)\mathbf{i} + (3x^2 - 4xy + 3y^2)\mathbf{j} = \nabla\phi$ with $\phi(x,y) = x^3 + 3x^2y - 2xy^2 + y^3$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,2) - \phi(3,0) = 8 - 27 = -19$$

15. $\mathbf{F}(x,y) = (4e^{2x} - 3e^xe^y)\mathbf{i} + (2e^{2y} - 3e^xe^y)\mathbf{j} = \nabla\phi$ with $\phi(x,y) = 2e^{2x} - 3e^{x+y} + e^{2y}$. The focus of the parabola $y^2 = 4x$ is at $(1,0)$ so the endpoint of the latus rectum is at $(1,2)$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(1,2) - \phi(0,0) = (2e^2 - 3e^1e^2 + e^4) - (2 - 3 + 1) = 2e^2 - 3e^3 + e^4$$

16. $\mathbf{F}(x,y) = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j}$; C is the segment of the line $3x + 4y = 12$ from the point where it intersects the y axis to the point where it intersects the x axis.

▸ By inspection, we see that
$\phi(x,y) = e^x\cos y$
is a potential function. In fact
$\nabla\phi(x,y) = D_x(e^x\cos y)\mathbf{i} + D_y(e^x\cos y)\mathbf{j} = e^x\cos y\mathbf{i} - e^x\sin y\mathbf{j}$
Thus, we may apply Theorem 14.3.1 to evaluate the integral.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(4,0) - \phi(0,3) = e^4 - \cos 3$$

17. $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 3y^2\mathbf{j} + \mathbf{k} = \nabla\phi$ with $\phi(x,y,z) = x^2 + y^3 + z$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,0,3) - \phi(\frac{3}{2},0,0) = 3 - \frac{9}{4} = \frac{3}{4}$

18. $\mathbf{F}(x,y,z) = (2xy + z^2)\mathbf{i} + (x^2 - 2yz)\mathbf{j} + (2xz - y^2)\mathbf{k} = \nabla\phi$ with $\phi(x,y,z) = x^2y + xz^2 - y^2z$

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(0,0,1) - \phi(0,1,0) = 0 - 0 = 0$$

19. $\mathbf{F}(x,y,z) = 2ye^{2x}\mathbf{i} + 2e^{2x}\mathbf{j} + 3z^2\mathbf{k} = \mathbf{F}\phi$ with $\phi(x,y,z) = ye^{2x} + z^3$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(\ln 2,2,2) - \phi(\ln 2,1,1) = 8 + 8 - 4 - 1 = 11$$

20. $\mathbf{F}(x,y,z) = \left(\frac{1}{z} - \frac{y}{x^2}\right)\mathbf{i} + \left(\frac{1}{x} + \frac{z}{y^2}\right)\mathbf{j} - \left(\frac{1}{y} + \frac{x}{z^2}\right)\mathbf{k}$;
C is any sectionally smooth curve from the point $(1,2,-1)$ to the point $(2,4,-2)$.

▸ Let

$$M(x,y,z) = \frac{1}{z} - \frac{y}{x^2} \qquad N(x,y,z) = \frac{1}{x} + \frac{z}{y^2} \qquad R(x,y,z) = \frac{-1}{y} - \frac{x}{z^2}$$

Then

$$M_y(x,y,z) = \frac{-1}{x^2} \qquad N_x(x,y,z) = \frac{-1}{x^2} \qquad R_x(x,y,z) = \frac{-1}{z^2}$$

$$M_z(x,y,z) = \frac{-1}{z^2} \qquad N_z(x,y,z) = \frac{1}{y^2} \qquad R_y(x,y,z) = \frac{1}{y^2}$$

Because $M_y(x,y,z) = N_x(x,y,z)$, $M_z(x,y,z) = R_x(x,y,z)$, and $N_z(x,y,z) = R_y(x,y,z)$, then the given vector field $\mathbf{F}$ is conservative and the line integral is independent of the path C. We let C be a line segment with direction $\langle 1,2,-1\rangle$. Parametric equation for C are

$$x = 1 + t \qquad y = 2 + 2t \qquad z = -(1 + t) \quad 0 \le t \le 1$$

Then

$$dx = dt \qquad dy = 2\,dt \qquad dz = -dt$$

and

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{R} &= \int_C \left(\frac{1}{z} - \frac{y}{x^2}\right)dx + \left(\frac{1}{x} + \frac{z}{y^2}\right)dy - \left(\frac{1}{y} + \frac{x}{z^2}\right)dz \\ &= \int_0^1 \left(\frac{-1}{1+t} - \frac{2+2t}{(1+t)^2}\right)dt + \left(\frac{1}{1+t} - \frac{1+t}{(2+2t)^2}\right)(2\,dt) - \left(\frac{1}{2+2t} + \frac{1+t}{(1+t)^2}\right)(-dt) \\ &= \int_0^1 (-3 + \tfrac{3}{2} + \tfrac{3}{2})\frac{dt}{1+t} = 0\end{aligned}$$

In Exercises 21–30, show that the line integral is independent of the path, and compute the value in any convenient manner. In Each exercise, C is any sectionally smooth curve from point A to point B.

▹ We may show the line integral is path-independent by exhibiting a potential for the vector function.

21. $\mathbf{F}(x, y) = (2y - x)\mathbf{i} + (y^2 + 2x)\mathbf{j} = \nabla\phi$ with $\phi(x, y) = 2xy - \frac{1}{2}x^2 + \frac{1}{3}y^3$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1, 2) - \phi(0, -1) = (4 - \tfrac{1}{2} + \tfrac{8}{3}) - (-\tfrac{1}{3}) = \tfrac{13}{2}$$

22. $\displaystyle\int_C (\ln x + 2y)dx + (e^y + 2x)dy = \int_C (\ln x\, dx + (2y\, dx + 2x\, dy) + e^y dy = x \ln x - x + 2xy + e^y \Big]_{(3,1)}^{(1,3)}$
$= -3 \ln 3 + 2 + e^3 - e$

23. $\mathbf{F}(x, y) = \tan y\mathbf{i} + x \sec^2 y\mathbf{j} = \nabla\phi$ with $\phi(x, y) = x \tan y$. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(4, \tfrac{1}{4}\pi) - \phi(-2, 0) = 4 - 0 = 4.$

24. $\int_C \sin y\, dx + (\sin y + x \cos y)dy$; A is $(-2, 0)$ and B is $(2, \frac{1}{6}\pi)$.

▹ Integrating $\sin y + x \cos y$ with respect to y, we conjecture that
$\phi = -\cos y + x \sin y$
is a potential for the integrand. Checking, we find
$\nabla(-\cos y + x \sin y) = \sin y\mathbf{i} + (\sin y + x \cos y)\mathbf{j}$
Thus, the integral is independent of the path and its value is

$$\phi(2, \tfrac{1}{6}\pi) - \phi(-2, 0) = (-\tfrac{1}{2}\sqrt{3} + 2 \cdot \tfrac{1}{2}) - (-1 - 2 \cdot 0) = 2 - \tfrac{1}{2}\sqrt{3}$$

25. $\mathbf{F}(x, y) = \dfrac{2y}{(xy + 1)^2}\mathbf{i} + \dfrac{2x}{(xy + 1)^2} = \nabla\phi$ with $\phi(x, y) = \dfrac{-2}{xy + 1}$. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1, 0) - \phi(0, 2) = -2 + 2 = 0$

26. $\displaystyle\int_C \frac{x\, dx + y\, dy + z\, dz}{x^2 + y^2 + z^2} = \int_{(1,0,0)}^{(1,2,3)} d[\tfrac{1}{2}\ln(x^2 + y^2 + z^2)] = \tfrac{1}{2}\ln(x^2 + y^2 + z^2)\Big]_{(1,0,0)}^{(1,2,3)} = \tfrac{1}{2}\ln 14$

27. $\mathbf{F}(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k} = \nabla\phi$ with $\phi(x, y, z) = zy + xy + yz$.

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1, 1, 1) - \phi(0, 0, 0) = 3 - 0 = 3$$

28. $\int_C (yz + x)dx + (xz + y)dy + (xy + z)dz$; A is $(0, 0, 0)$, and B is $(1, 1, 1)$.

▹ Let

$M(x, y, z) = yz + x \qquad N(x, y, z) = xz + y \qquad R(x, y, z) = xy + z$

Then

$M_y(x, y, z) = z \qquad N_x(x, y, z) = z \qquad R_x(x, y, z) = y$
$M_z(x, y, z) = y \qquad N_z(x, y, z) = x \qquad R_y(x, y, z) = x$

Because $M_y(x, y, z) = N_x(x, y, z)$, $M_z(x, y, z) = R_x(x, y, z)$, and $N_z(x, y, z) = R_y(x, y, z)$, then the given vector field $\mathbf{F}$ is conservative and the line integral is independent of the path C. We let C be a line segment from $(0, 0, 0)$ to $(1, 1, 1)$, with direction $\langle 1, 2, -1\rangle$. Parametric equations for C are
$x = t \quad y = t \quad z = t \quad 0 \le t \le 1$
Therefore,

$$\begin{aligned}\int_C (yz + x)dx + (xz + y)dy + (xy + z)dz &= \int_0^1 (t^2 + t)dt + (t^2 + t)dt + (t^2 + t)dt = \int_0^1 (3t^2 + 3t)dt \\ &= t^3 + \tfrac{3}{2}t^2\Big]_0^1 = \tfrac{5}{2}\end{aligned}$$

29. $\mathbf{F}(x, y, z) = (e^x \sin y + yz)\mathbf{i} + (e^x \cos y + z \sin y + xz)\mathbf{j} + (xy - \cos y)\mathbf{k} = \nabla\phi$ with
$\phi(x, y, z) = e^x \sin y + xyz - z \cos y$. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(0, \pi, 3) - \phi(2, 0, 1) = 3 + 1 = 4.$

30. $\displaystyle\int_C (2x \ln yz - 5ye^x)dx - (5e^x - x^2y^{-1})dy + (x^2z^{-1} + 2z)dz = x^2 \ln y + x^2 \ln z - 5ye^x + z^2\Big]_{(2,1,1)}^{(3,1,e)} = 8 + 6e^2 - 5e^3$

31. $\mathbf{F}(x,y,z) = (4x+2y-z)\mathbf{i} + (2x-2y+z)\mathbf{j} + (-x+y+2z)\mathbf{k} = \nabla\phi$ with
$\phi(x,y,z) = 2x^2 + 2xy - xz - y^2 + yz + z^2$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(-1,2,0) - \phi(4,-2,1) = -6 - 7 = -13$.

32. Prove Theorem 14.3.3.

▹ If C is a smooth curve, we may let parametric equations of C be
$x = f(t) \quad y = g(t) \quad z = h(t) \quad t_1 \le t \le t_2$
where
$(x_1, y_1, z_1) = (f(t_1), g(t_1), h(t_1))$ and $(x_2, y_2, z_2) = (f(t_2), g(t_2), h(t_2))$
Let
$\mathbf{F}(x,y,z) = \mathrm{M}(x,y,z)\mathbf{i} + \mathrm{N}(x,y,z)\mathbf{j} + \mathrm{R}(x,y,z)\mathbf{k}$
Because ϕ is a potential function for $\mathbf{F}$
$\phi_x(x,y,z) = \mathrm{M}(x,y,z) \quad \phi_y(x,y,z) = \mathrm{N}(x,y,z) \quad \phi_z(x,y,z) = \mathrm{R}(x,y,z)$
Then, with the help of the chain rule and the second fundamental theorem of the calculus,

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_C \mathrm{M}(x,y,z)dx + \mathrm{N}(x,y,z)dy + \mathrm{R}(x,y,z)dz \\
&= \int_C \phi_x(x,y,z)dx + \phi_y(x,y,z)dy + \phi_z(x,y,z)dz \\
&= \int_{t_1}^{t_2} \phi_x(f(t),g(t),h(t))f'(t)dt + \phi_y(f(t),g(t),h(t))g'(t)dt + \phi_z(f(t),g(t),h(t))h'(t)dt \\
&= \int_{t_1}^{t_2} d\phi(f(t),g(t),h(t)) \\
&= \phi(f(t),g(t),h(t)\Big]_{t_1}^{t_2} \\
&= \phi(f(t_2),g(t_2),h(t_2)) - \phi(f(t_1),g(t_1),h(t_1)) \\
&= \phi(x_2,y_2,z_2) - \phi(x_1,y_1,z_1)
\end{aligned}$$

If C is only sectionally smooth, then apply this result to each smooth piece; the values of ϕ at each intermediate point will cancel.

In Exercises 33-36, find the total work done in moving a particle along arc C if the motion is caused by the force field $\mathbf{F}$. Assume length is measured in meters and force is measured in newtons. (Hint: First show $\mathbf{F}$ is conservative.)

▹ W joules of work is done.

33. $\mathbf{F}(x,y) = 3(x+y)^2\mathbf{i} + 3(x+y)^2\mathbf{j} = \nabla\phi$ with $\phi(x,y) = (x+y)^3$. $\mathrm{W} = \phi(2,4) - \phi(0,0) = 216 - 0 = 216$.

34. $\mathrm{W} = \int_C (2xy - 5y + 2y^2)dx + (x^2 - 5x + 4xy)dy = x^2y - 5xy + 2xy^2\Big]_{(2,0)}^{(0,2)} = 0 - 0 = 0$

35. $\mathbf{F}(x,y,z) = 2y^2z^3\mathbf{i} + 4xyz^3\mathbf{j} + 6xy^2z^2\mathbf{k} = \nabla\phi$ with $\phi(x,y,z) = 2xy^2z^3$.
$\mathrm{W} = \phi(2,4,8) - \phi(1,1,1) = 32{,}768 - 2 = 32{,}766$

36. $\mathbf{F}(x,y,z) = 4y^2z\mathbf{i} + 8xyz\mathbf{j} + 4(3z^3 + xy^2)\mathbf{k}$; C: the arc of the circular helix $\mathbf{R}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + t\mathbf{k}$ from $t = 0$ to $t = \frac{1}{3}\pi$.

▹ Integrating the z component $12z^3 + 4xy^2$ with respect to z, we conjecture that
$\phi(x,y,z) = 3z^4 + 4xy^2z$
is a potential for the integrand. Checking, we find
$\nabla(3z^4 + 4xy^2z) = 4y^2z\mathbf{i} + 8xyz\mathbf{j} + 4(3z^3 + xy^2)\mathbf{k}$
Thus, the integral $\int_C$ is independent of the path and its value is
$\phi(\frac{3}{2}, \frac{3}{2}\sqrt{3}, \frac{1}{3}\pi) - \phi(3,0,0) = (\frac{27}{2}\pi + \frac{1}{27}\pi^4) - 0 = \frac{27}{2}\pi + \frac{1}{27}\pi^4$
The total work done is $(\frac{27}{2}\pi + \frac{1}{27}\pi^4)$ joules.

37. $\mathbf{F}(x,y,z) = \dfrac{k(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}}$.

(a) $\mathbf{F} = \nabla\phi$ with $\phi(x,y,z) = \dfrac{-k}{(x^2 + y^2 + z^2)^{1/2}}$. $\mathrm{W} = \phi(3,0,4) - \phi(3,0,0) = -\frac{k}{5} + \frac{k}{3} = \frac{2}{15}k$.

(b) With z as parameter, C: $x = 3$, $y = 4$, $0 \le z \le 4$.
$\mathrm{W} = k\int_0^4 (\frac{3}{(9+z^2)^{3/2}}, 0, \frac{z}{(9+z^2)^{3/2}})\cdot(0,0,1)dz = k\int_0^4 z(9+z^2)^{-3/2}dz = -k(9+z^2)^{-1/2}\Big]_0^4 = \frac{2}{15}k$

14.4 GREEN'S THEOREM

The work done in moving an object a round a sectionally smooth simple closed curve can be found by evaluating a line integral. Green's theorem gives us the option of using a double integral to evaluate the line integral. The line integral around C in the counterclockwise direction is denoted by $\oint_C$.

In the following three theorems, M and N are functions of two variables x and y which have continuous first partial derivatives on an open disk B in R^2, C is a sectionally smooth simple closed curve lying entirely in B and R is the region bounded by C.

14.4.1 Theorem (Green's Theorem)

$$\oint_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$$

14.4.3 Theorem (Gauss's Divergence Theorem in the Plane) If $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ and $\mathbf{n}(s)$ is the unit outward normal vector of C at P where s units is the length of arc measured from a particular point on C to P, then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \text{div}\,\mathbf{F}\,dA = \iint_R \nabla\cdot\mathbf{F}\,dA$$

The line integral $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$ is called the *flux* of $\mathbf{F}$ across C.

14.4.4 Theorem (Stokes's Theorem in the Plane) If $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ and $\mathbf{T}(s)$ is the unit tangent vector of C at P, where s units is the length of arc measured from a particular point on C to P, then

$$\oint_C \mathbf{F}\cdot\mathbf{T}ds = \iint_R \text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dA$$

The line integral $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$ is called the *circulation* of $\mathbf{F}$ around C.

When applying Theorems 14.4.3 and 14.4.4 we may use the equations

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C -N(x,y)dx + M(x,y)dy$$

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C M(x,y)dx + N(x,y)dy$$

The following theorem gives a method for using a line integral to find the area of a region bounded by a sectionally smooth simple curve.

14.4.2 Theorem If R is a region having as its boundary a sectionally smooth simple closed curve C, and A square units is the area of R, then

$$A = \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

The area of a hypocycloid of m cusps is $\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)a^2\pi$. The area of an astroid is $\frac{3}{8}a^2\pi$. See Exercise 25.

Exercises 14.4

In Exercises 1–8, evaluate the line integral by Green's theorem. Verify the result by the method of Section 14.2.

1. $\oint_C 4y\,dx + 3x\,dy = \iint_R \left[\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(4y)\right]dA = \iint_R -dA = -1$. Also C_1: $\mathbf{R}_1(t) = t\mathbf{i}$, $0 \le t \le 1$; C_2: $\mathbf{R}_2(t) = \mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$; C_3: $\mathbf{R}_3(t) = (1-t)\mathbf{i} + \mathbf{j}$, $0 \le t \le 1$; C_4: $\mathbf{R}_4(t) = (1-t)\mathbf{j}$, $0 \le t \le 1$. $\oint_C 4y\,dx + 3x\,dy = \int_0^1 0\,dt + \int_0^1 3\,dt + \int_0^1 (-4)dt + \int_0^1 0\,dt = 3 - 4 = -1$

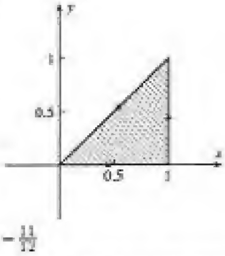
2. $\oint_C y^2dx + x^2dy = \iint_R [D_x(x^2) - D_y(y^2)]dA = \int_0^1\int_0^1 (2x-2y)dx\,dy = \int_0^1 x^2 - 2xy\Big]_0^1 dy = \int_0^1 (1-2y)dy$ $= y - y^2\Big]_0^1 = 0.$

3. $\oint_C 2xy\,dx - x^2y\,dy = \iint_R \left[\frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(2xy)\right]dA = \int_0^1 \int_0^{1-x} (-2xy - 2x)dy\,dx$
$= \int_0^1 [-xy^2 - 2xy]_0^{1-x}\,dx = \int_0^1 (-x^3 + 4x^2 - 3x)dx = -\frac{1}{4}x^4 + \frac{4}{3}x^3 - \frac{3}{2}x^2\Big]_0^1 = -\frac{1}{4} + 143 - \frac{3}{2} = -\frac{5}{12}$. Also

C_1: $\mathbf{R}_1(t) = t\mathbf{i}$, $0 \le t \le 1$; C_2: $\mathbf{R}_2(t) = (1-t)_{\mathbf{i}} + t\mathbf{j}$, $0 \le t \le 1$; C_3: $\mathbf{R}_3(t) = (1-t)\mathbf{j}$, $0 \le t \le 1$.

$\oint_C 2xy\,dx - x^2y\,dy = \int_0^1 0\,dt + \int_0^1 [2(1-t)t(-1) - (1-t)^2t]dt + \int_0^1 0\,dt = \int_0^1 (-t^3 + t^2 - 3t)dt = -\frac{5}{12}$

4. $\oint_C 2xy\,dx - x^2dy$, where C is the triangle with vertices at $(0,0)$, $(1,0)$, and $(1,1)$.



▷ Let
$M(x,y) = 2xy$ and $N(x,y) = -x^2y$
Then
$N_x(x,y) - M_y(x,y) = -2xy - 2x$
Because the region R that is enclosed by the curve C is bounded by the x axis, the line $x = 1$, and the line $y = x$ (see the figure), we have

$$\oint_C 2xy\,dx - x^2y\,dy = \iint_R [N_x(x,y) - M_y(x,y)]dA = \int_0^1 \int_0^x (-2xy - 2x)dy\,dx$$
$$= \int_0^1 -xy^2 - 2xy\Big]_0^x dx = \int_0^1 (-x^3 - 2x^2)dx = -\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3\Big]_0^1 = -\tfrac{11}{12}$$

We verify the result by the method of Section 14.2. Let C_1 be the segment from $(0,0)$ to $(1,0)$ (see the figure); let C_2 be the segment from $(1,0)$ to $(1,1)$; and let C_3 be the segment from $(1,1)$ to $(0,0)$. Then

$$\oint_C 2xy\,dx - x^2y\,dy = \oint_{C_1} 2xy\,dx - x^2y\,dy + \oint_{C_2} 2xy\,dx - x^2y\,dy + \oint_{C_3} 2xy\,dx - x^2y\,dy \qquad (1)$$

Parametric equations for C_1 are $x = t$, $y = 0$, $0 \le t \le 1$. Thus,

$$\oint_{C_1} 2xy\,dx - x^2y\,dy = \int_0^1 (2t)(0)dt - t^2(0)(0) = 0 \qquad (2)$$

Parametric equations for C_2 are $x = 1$, $y = t$, $0 \le t \le 1$. Therefore,

$$\oint_{C_2} 2xy\,dx - x^2y\,dy = \int_0^1 (2t)(0) - t\,dt = -\tfrac{1}{2}t^2\Big]_0^1 = -\tfrac{1}{2} \qquad (3)$$

Parametric equations for C_3 are $x = t$, $y = t$, from $t = 1$ to $t = 0$. Hence,

$$\oint_{C_3} 2xy\,dx - x^2y\,dy = \int_1^0 2t^2dt - t^3dt = \tfrac{2}{3}t^3 - \tfrac{1}{4}t^4\Big]_1^0 = 0 - (\tfrac{2}{3} - \tfrac{1}{4}) = -\tfrac{5}{12} \qquad (4)$$

Substituting from Eqs. (2), (3), and (4) into Eq. (1), we obtain

$$\oint_C 2xy\,dx - x^2y\,dy = 0 - \tfrac{1}{2} - \tfrac{5}{12} = -\tfrac{11}{12}$$

which agrees with the result obtained by using Green's theorem.

5. $\oint_C x^2y\,dx - y^2x\,dy = \iint_R \left[\frac{\partial}{\partial x}(-y^2x) - \frac{\partial}{\partial y}(x^2y)\right]dA = -\iint_R (x^2 + y^2)dA = -\int_0^{2\pi} \int_0^1 r^2(r\,dr\,d\theta)$
$= -\int_0^{2\pi} \frac{1}{4}\,d\theta = -\frac{\pi}{2}$. Also, C: $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le 2\pi$.

$\oint_C x^2y\,dx - y^2x\,dy = \int_0^{2\pi} [\cos^2 t \sin t(-\sin t) - \sin^2 t \cos t(\cos t)]dt = -2\int_0^{2\pi} \cos^2 t \sin^2 t\,dt$
$= -\frac{1}{2}\int_0^{2\pi} \sin^2 2t\,dt = -\frac{1}{2}\left[\frac{1}{2}t - \frac{1}{4}\cos 4t\right]_0^{2\pi} = -\frac{\pi}{2}$

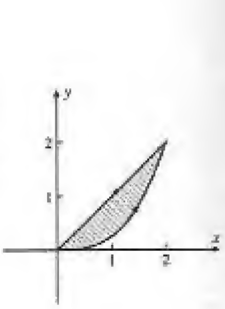
6. $\oint_C (x^2 - y^2)dx + 2xy\,dy = \iint_R [D_x(2xy) - D_y(x^2 - y^2)]dA = \iint_R (2y + 2y)dA = 0$ because y is odd. Also, let $x = \cos\theta$, $y = \sin\theta$. $\oint_C (x^2 - y^2)dx + 2xy\,dy = \int_0^{2\pi} (\cos^2\theta - \sin^2\theta)(-\sin\theta\,d\theta) + 2\sin\theta\cos\theta(\cos\theta\,d\theta) = 0$

7. $\oint_C x^2y\,dx - y^2x\,dy = \iint_R \left[\frac{\partial}{\partial x}(-y^2x) - \frac{\partial}{\partial y}(x^2y)\right]dA = -\int_0^2 \int_{x^3/4}^x (y^2 + x^2)dy\,dx$
$= -\int_0^2 \left[\frac{1}{3}y^3 + x^2y\right]_{x^3/4}^x dy = -\int_0^2 \left(\frac{4}{3} - \frac{1}{192}x^9 - \frac{1}{4}x^5\right)dx = -\left[\frac{1}{3}x^4 + \frac{1}{1920}x^{10} - \frac{1}{24}x^6\right]_0^2 = -\frac{32}{15}$. Also

C_1: $y = \frac{1}{4}x^3$, $0 \le x \le 2$; $-C_2$: $y = x$, $0 \le x \le 2$. $\oint_C x^2y\,dx - y^2x\,dy$
$= \int_0^2 \left[x^2\left(\frac{1}{4}x^3\right) - \left(\frac{1}{4}x^3\right)^2\left(\frac{3}{4}x^2\right)\right]dx - \int_0^2 (x^2x - x^2x)dx = \int_0^2 \left(\frac{1}{4}x^5 - \frac{3}{64}x^9\right)dx = \frac{1}{24}t^6 - \frac{3}{640}t^{10}\Big]_0^2 = -\frac{32}{15}$

8. $\oint_C (x^2 - y^2)dx + 2xy\,dy$, where C is the closed curve consisting of the arc of $4y = x^3$ from $(0,0)$ to $(2,2)$ and the line segment from $(2,2)$ to $(0,0)$.

▸ The figure shows the curve C and the region R that is enclosed by C. Applying Green's theorem, we have



$$\oint_C (x^2 - y^2)dx + 2xy\,dy = \iint_R [D_x(2xy) - D_y(x^2 - y^2)]dA = \iint_R (2y + 2y)dA$$
$$= \int_0^2 \int_{x^3/4}^{x} 4y\,dy\,dx = \int_0^2 2y^2\Big]_{x^3/4}^{x} dx$$
$$= 2\int_0^2 (x^2 - \tfrac{1}{16}x^6)dx = 2\Big[\tfrac{1}{3}x^3 - \tfrac{1}{112}x^7\Big]_0^2 = \tfrac{64}{21}$$

We verify the result by the method of Section 14.2. Let C_1 be the arc of the curve $4y = x^3$ from $(0,0)$ to $(2,2)$. Parametric equations of C_1 are

$x = t \quad y = \frac{1}{4}t^3 \quad 0 \le t \le 2$

Thus,

$$\oint_{C_1} (x^2 - y^2)dx + 2xy\,dy = \int_0^2 (t^2 - \tfrac{1}{16}t^6)dt + (\tfrac{1}{2}t^4)(\tfrac{3}{4}t^2dt) = \int_0^2 (t^2 + \tfrac{5}{16}t^6)dt = \tfrac{1}{3}t^3 + \tfrac{5}{112}t^7\Big]_0^2 = \tfrac{176}{21}$$

Let C_2 be the line segment from $(2,2)$ to $(0,0)$. Parametric equations of C_2 are:

$x = t \quad y = t$ from $t = 2$ to $t = 0$

Hence,

$$\oint_{C_2} (x^2 - y^2)dx + 2xy\,dy = \int_2^0 2t^2 dt = \tfrac{2}{3}t^3\Big]_2^0 = -\tfrac{16}{3}$$

Therefore,

$$\oint_C (x^2 - y^2)dx + 2xy\,dy = \oint_{C_1} (x^2 - y^2)dx + 2xy\,dy + \oint_{C_2} (x^2 - y^2)dx + 2xy\,dy = \tfrac{176}{21} - \tfrac{16}{3} = \tfrac{64}{21}$$

which agrees with the result obtained by using Green's theorem.

In Exercises 9–20, use Green's theorem to evaluate the line integral.

9. $\oint_C (x + y)dx + xy\,dy = \iint_R \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x + y)\right]dA = \int_0^2 \int_0^{x^3/4} (y - 1)dy\,x = \int_0^2 \left[\frac{1}{2}y^2 - y\right]_0^{x^3/4} dx$

$= \int_0^2 \left(\frac{1}{32}x^6 - \frac{1}{4}x^3\right)dx = \frac{1}{224}x^7 - \frac{1}{16}x^4\Big]_0^2 = \frac{4}{7} - 1 = -\frac{3}{7}$

10. $\oint_C y^2dx + x^2dy = \iint_R [D_x(x^2) - D_y(y^2)]dA = \int_0^1 \int_0^{x^2} (2x - 2y)dy\,dx = \int_0^1 (2xy - y^2)\Big]_0^{x^2} dx = \int_0^1 (2x^3 - x^4)dx$

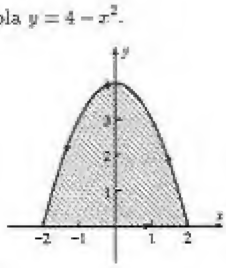
$= \frac{1}{2}x^4 - \frac{1}{5}x^5 = \frac{3}{10}$

11. $\oint_C (-x^2 + x)dy = \iint_R \left[\frac{\partial}{\partial x}(-x^2 + x) - \frac{\partial}{\partial y}(0)\right]dA = \int_0^1 \int_{2y}^{2y^2} (-2x + 1)dx\,dy = \int_0^1 -x^2 + x\Big]_{2y}^{2y^2} dy$

$= \int_0^1 (-4y^4 + 6y^2 - 2y)dy = -\frac{4}{5}y^5 + 2y^3 - y^2\Big]_0^1 = \frac{1}{5}$

12. $\oint_C (x^2 + y)dx$, where C is the closed curve determined by the x axis and the parabola $y = 4 - x^2$.

▸ The figure shows the curve C and the region R enclosed by C. We apply Green's theorem with $M(x,y) = x^2 + y$ and $N(x,y) = 0$



$$\oint_C (x^2 + y)dx = \iint_R (N_x - M_y)dA$$
$$= \iint_R (-1)dA$$
$$= -\int_{-2}^{2} \int_0^{4-x^2} dy\,dx$$
$$= -2\int_0^2 (4 - x^2)dx = -2\Big[4x - \tfrac{1}{3}x^3\Big]_0^2 = -\tfrac{32}{3}$$

13. $\oint_C \cos y\,dx + \cos x\,dy = \iint_R \left[\frac{\partial}{\partial x}(\cos x) - \frac{\partial}{\partial y}(\cos y)\right]dA = \int_0^{\pi/3} \int_0^{\pi/4} (-\sin x + \sin y)dy\,dx$

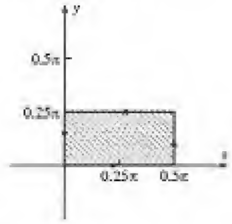
$= \int_0^{\pi/3} \Big[-y\sin x - \cos y\Big]_0^{\pi/4} dx = \int_0^{\pi/3} \left(-\frac{\pi}{4}\sin x - \frac{1}{2}\sqrt{2} + 1\right)dx = \left[\frac{\pi}{4}\cos x + \left(1 - \frac{1}{2}\sqrt{2}\right)x\right]_0^{\pi/3} = \frac{\pi}{24}(5 - 4\sqrt{2})$

14. $\oint_C e^{x+y}dx + e^{x+y}dy = \iint_R [D_x e^{x+y} - D_y e^{x+y}]dA = 0$

15. $\oint_C (\sin^4 x + e^{2x})dx + (\cos^3 y - e^y)dy = \iint_R \left[\frac{\partial}{\partial x}(\cos^3 y - e^y) - \frac{\partial}{\partial y}(\sin^4 x + e^{2x})\right]dA = \iint_R 0\ dA = 0$

16. $\oint_C x \sin y - y \cos x\ dy$, where C is the rectangle with vertices at $(0,0)$, $(\frac{1}{2}\pi, 0)$, $(\frac{1}{2}\pi, \frac{1}{4}\pi)$, and $(0, \frac{1}{4}\pi)$.

▹ The figure shows the rectangle C and the region R enclosed by C.



Applying Green's theorem, we have

$$\begin{aligned}\oint_C x \sin y - y \cos x\ dy &= \iint_R [D_x(-y\cos x) - D_y(x \sin y)]dA \\ &= \int_0^{\pi/2}\int_0^{\pi/4} (y \sin x - x \cos y)dy\ dx \\ &= \int_0^{\pi/2}\Big[\tfrac{1}{2}y^2 \sin x - x \sin y\Big]_0^{\pi/4} \\ &= \int_0^{\pi/2}(\tfrac{1}{32}\pi^2 \sin x - \tfrac{1}{2}\sqrt{2}x)dx \\ &= -\tfrac{1}{32}\pi^2 \cos x - \tfrac{1}{4}\sqrt{2}x^2\Big]_0^{\pi/2} = -\tfrac{1}{4}\sqrt{2}(\tfrac{1}{4}\pi^2) - (-\tfrac{1}{32}\pi^2 = \tfrac{1}{32}(1 - 2\sqrt{2})\pi^2\end{aligned}$$

17. The area of the ellipse $\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$ is $\pi(5)(2) = 10\pi$.

$$\oint_C \frac{x^2 y}{x^2+1}dx - \tan^{-1}x\,dy = \iint_R \left[\frac{\partial}{\partial x}(-\tan^{-1}x) + \frac{\partial}{\partial y}\left(\frac{x^2 y}{x^2+1}\right)\right]dA = \iint_R \left(\frac{-1}{1+x^2} - \frac{x^2}{x^2+1}\right)dA = -\iint_R dA = -10\pi$$

18. $\oint_C e^y \cos x\ dx + e^y \sin x\ dy = \iint_R [D_x(e^y \sin x) - D_y(e^y \cos x)]dA = \iint_R (e^y \cos x - e^y \cos x)dA = 0$

19. $\oint_C (e^x - x^2 y)dx + 3x^2 y\ dy = \iint_R \left[\frac{\partial}{\partial x}(3x^2 y) - \frac{\partial}{\partial y}(e^x - x^2 y)\right]dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (6xy + x^2)dy\ dx =$

$= \int_0^1 \Big[3xy^2 + x^2 y\Big]_{x^2}^{\sqrt{x}}\ dx = \int_0^1 (3x^3 + x^{5/2} - 3x^5 - x^4)dx = x^3 + \frac{2}{7}x^{7/2} - \frac{1}{2}x^6 - \frac{1}{5}x^5\Big]_0^1 = \frac{41}{70}$

20. $\oint_C \tan y + x \tan^2 y\ dy$, where C is the ellipse $x^2 + 4y^2 = 1$.

▹ Applying Green's theorem, we obtain

$$\begin{aligned}\oint_C \tan y + x \tan^2 y\ dy &= \iint_R [D_x(x \tan^2 y) - D_y(\tan y)]dA = \iint_R (\tan^2 y - \sec^2 y)dA \\ &= \iint_R (-1)dA = -(\text{area of R}) = -\tfrac{1}{2}\pi\end{aligned}$$

because the area of an ellipse of semiaxes 1 and $\frac{1}{2}$ is $\pi(1)(\frac{1}{2}) = \frac{1}{2}\pi$.

In Exercises 21–26, use Theorem 14.4.2 to find the area of the region.

▹ $A = \frac{1}{2}\oint_C x\ dy - y\ dx$

21. C_1: $\mathbf{R}_1(t) = t\mathbf{i}$, $0 \le t \le 4$; C_2: $\mathbf{R}_2(t) = (4-t)\mathbf{i} + 2t\mathbf{j}$, $0 \le t \le 1$;
C_3: $\mathbf{R}_3(t) = (3-2t)\mathbf{i} + (2-t)\mathbf{j}$, $0 \le t \le 1$; C_4: $\mathbf{R}_4(t) = (1-t)\mathbf{i} + (1-t)\mathbf{j}$, $0 \le t \le 1$.

$A = \frac{1}{2}\int_0^4 0\ dt + \frac{1}{2}\int_0^1 [2(4-t) + 2t]dt + \frac{1}{2}\int_0^1 [-(3-2t) + 2(2-t)]dt + \frac{1}{2}\int_0^1 [-(1-t) + (1-t)]dt$

$= 0 + \frac{1}{2}\int_0^1 8\ dt + \frac{1}{2}\int_0^1 1\ dt + 0 = \frac{9}{2}$

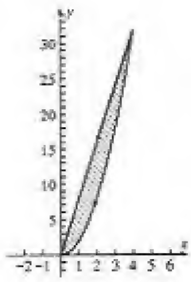
22. $x = a\cos\theta$, $y = a\sin\theta$. $A = \frac{1}{2}\int_0^{2\pi}(a\cos\theta)(a\cos\theta\ d\theta) - (a\sin\theta)(-a\sin\theta\ d\theta) = \frac{1}{2}a^2\int_0^{2\pi} d\theta = \pi a^2$

23. C_1: $\mathbf{R}_1 = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$; $-C_2$: $\mathbf{R}_2(t) = t\mathbf{i} + t^{1/2}\mathbf{j}$, $0 \le t \le 1$.

$A = \frac{1}{2}\int_0^1 (2t^2 - t^2)dt - \frac{1}{2}\int_0^1 \left(\frac{1}{2}t^{1/2} - t^{1/2}\right)dt = \frac{1}{2}\int_0^1 \left(t^2 + \frac{1}{2}t^{1/2}\right)dt = \frac{1}{2}\Big[\frac{1}{3}t^3 + \frac{1}{3}t^{3/2}\Big]_0^1 = \frac{1}{3}$

24. The region bounded by the parabola $y = 2x^2$ and the line $y = 8x$.

▸ The figure shows the region R, bounded by the curve C. Let C_1 be the arc of the parabola $y = 2x^2$ from $(0, 0)$ to $(4, 32)$, and let C_2 be the line segment $y = 8x$ from $(4, 32)$ to $(0, 0)$. By Theorem 14.4.2, the area of R is



$$A = \tfrac{1}{2}\oint_C x\,dy - y\,dx = \tfrac{1}{2}\oint_{C_1} x\,dy - y\,dx + \tfrac{1}{2}\oint_{C_2} x\,dy - y\,dx \qquad (1)$$

Parametric equations of C_1 are $x = t$ and $y = 2t^2$ for $0 \le t \le 4$. Thus,

$$\oint_{C_1} x\,dy - y\,dx = \int_0^4 t(4t\,dt) - 2t^2dt = \int_0^4 2t^2dt = \tfrac{2}{3}t^3\Big]_0^4 = \tfrac{128}{3} \qquad (2)$$

Parametric equations of C_2 are $x = t$ and $y = 8t$ from $t = 4$ to $t = 0$. Thus,

$$\oint_{C_2} x\,dy - y\,dx = \int_4^0 t(8\,dt) - 8t\,dt = 0 \qquad (3)$$

Substituting from Eqs. (2) and (3) into Eq. (1), we obtain

$A = \frac{1}{2}\cdot\frac{128}{3} + \frac{1}{2}\cdot 0 = \frac{64}{3}$

The area is $\frac{64}{3}$ square units.

25. Area of astroid $x = a\cos^3 t$, $y = a\sin^3 t$. $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} [a\cos^3(3a\sin^2 t\cos t) + a\sin^3 t(3a\cos^2 t\sin t)]dt$

$= \dfrac{3a^2}{2}\displaystyle\int_0^{2\pi}\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)dt = \dfrac{3a^2}{2}\displaystyle\int_0^{2\pi}\dfrac{1}{4}\sin^2 2t(1)dt = \dfrac{3a^2}{2}\cdot\dfrac{1}{8}\cdot 2\pi = \dfrac{3a^2\pi}{8}$

For the hypocyloid of $a/b = m$ cusps, let $k = m - 1$. Then $x = kb\cos t + b\cos kt$, $y = kb\sin t - b\sin kt$.

$A = \frac{1}{2}\displaystyle\int_0^{2\pi}[(kb\cos t + b\cos kt)(kb\cos t - kb\cos kt) - (kb\sin t - b\sin kt)(-kb\sin t - kb\sin kt)]dt$

$= \frac{1}{2}k(k-1)b^2\displaystyle\int_0^{2\pi}[1 + \cos(k-1)t]dt = k(k-1)b^2\pi = \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)a^2\pi.$

26. $x = t - \sin t$, $y = 1 - \cos t$. $A = \displaystyle\int_0^{2\pi} 0\,dt + \frac{1}{2}\int_{2\pi}^0 (t - \sin t)(\sin t\,dt) - (1 - \cos t)(1 - \cos t\,dt)$

$= \frac{1}{2}\displaystyle\int_{2\pi}^0 (t\sin t + 2\cos t - 2)dt = \frac{1}{2}\Big[-t\cos t + \sin t + 2\sin t - 2t\Big]_{2\pi}^0 = 3\pi$

In Exercises 27–30, verify Gauss's divergence theorem in the plane and Stokes's theorem in the plane for **F** and **R**.

▸ We must show (a) $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \iint \operatorname{div}\mathbf{F}\,dA$ and (b) $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dA$.

27. $\mathbf{F}(x, y) = 3x\mathbf{i} + 2y\mathbf{j}$, $\mathbf{R}(s) = \cos s\mathbf{i} + \sin s\mathbf{j}$, $0 \le s \le 2\pi$

(a) $\mathbf{N}(s) = \cos s\mathbf{i} + \sin s\mathbf{j}$. $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{N}ds = \int_0^{2\pi}(3\cos s\mathbf{i} + 2\sin s\mathbf{j})\cdot(\cos s\mathbf{i} + \sin s\mathbf{j})ds$

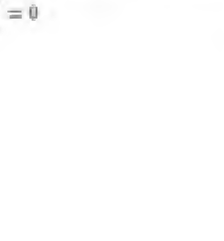
$= \displaystyle\int_0^{2\pi}(3\cos^2 s + 2\sin^2 s)ds = \frac{1}{2}\int_0^{2\pi}(\cos 2s + 5)ds = \frac{1}{2}(5)2\pi = 5\pi$ and $\displaystyle\iint_R \operatorname{div}\mathbf{F}\,dA = \iint_R (3 + 2)dA = 5\pi.$

(b) $\mathbf{T}(s) = -\sin s\mathbf{i} + \cos s\mathbf{j}$. $\displaystyle\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{2\pi}(3\cos s\mathbf{i} + 2\sin s\mathbf{j})\cdot(-\sin s\mathbf{i} + \cos s\mathbf{j})ds$

$= \displaystyle\int_0^{2\pi}(-3\cos s\sin s + 2\sin s\cos s)ds = -\frac{1}{2}\sin^2 s\Big]_0^{2\pi} = 0$ and $\displaystyle\iint_R \operatorname{curl}\mathbf{F}\cdot\mathbf{k}dA = \iint_R \mathbf{0}\cdot\mathbf{k}\,dA = 0.$

28. $\mathbf{F}(x, y) = 3y\mathbf{i} - 2x\mathbf{j}$ and $\mathbf{R}$ is the region bounded by $x^{2/3} + y^{2/3} = 1$.

▸ The figure shows the region R. The boundary curve C is an astroid.



Parametric equations of C are

$x = \cos^3 t \quad y = \sin^3 t \quad 0 \le t \le 2\pi$

Thus

$dx = -3\cos^2 t\sin t\,dt \qquad dy = 3\sin^2 t\cos t\,dt$

Because $\mathbf{F}(x, y) = 3y\mathbf{i} - 2x\mathbf{j}$, then

$M(x, y) = 3y = 3\sin^3 t \qquad N(x, y) = -2x = -2\cos^3 t$

The left side of Gauss's divergence theorem in the plane is

$$\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \oint_C -N(x,y)dx + M(x,y)dy = \int_0^{2\pi}[2\cos^3 t(-3\cos^2 t\sin t) + 3\sin^3 t(3\sin^2 t\cos t)]\,dt$$

$$= \int_0^{2\pi}(9\sin^5 t\cos t - 6\cos^5 t\sin t)\,dt = \tfrac{3}{2}\sin^6 t + \cos^6 t\Big]_0^{2\pi} = 0$$

The right side of Gauss's divergence theorem in the plane is

$$\iint_R \operatorname{div} \mathbf{F}\, dA = \iint_R [M_x(x,y) + N_y(x,y)]dA = \iint_R [D_x(3y) + D_y(-2x)]dA = \iint_R 0\, dA = 0$$

and we have verified Gauss's divergence theorem in the plane for the given function and region.
The left side of Stokes's theorem in the plane is

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{T}\, ds &= \oint_C M(x,y)\, dx + N(x,y)\, dy = \int_0^{2\pi} [3 \sin^3 t)(-3 \cos^2 t \sin t) + (-2 \cos^3 t)(3 \sin^2 t \cos t)]\, dt \\
&= \int_0^{2\pi} -3 \sin^2 t \cos^2 t(3 \sin^2 t + 2 \cos^2 t)\, dt = \int_0^{2\pi} -3\left(\frac{\sin 2t}{2}\right)^2\left[3\,\frac{1-\cos 2t}{2} + 2\,\frac{1+\cos 2t}{2}\right] dt \\
&= -\tfrac{3}{8}\int_0^{2\pi} \sin^2 2t(5 - \cos 2t)\, dt = -\tfrac{3}{8}\int_0^{2\pi} [\tfrac{5}{2}(1 - \cos 4t) - \sin^2 2t \cos 2t]\, dt \\
&= -\tfrac{3}{8}\left[\tfrac{5}{2}t - \tfrac{5}{8}\sin 4t - \tfrac{1}{6}\sin^3 2t\right]_0^{2\pi} = -\tfrac{3}{8}(5\pi) = -\tfrac{15}{8}\pi
\end{aligned}$$

The right side of Stokes's theorem in the plane is

$$\begin{aligned}
\iint_R \operatorname{curl} \mathbf{F}\cdot\mathbf{k}\, dA &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R \left[\frac{\partial}{\partial x}(-2x) - \frac{\partial}{\partial y}(3y)\right] dA = \iint_R -5\, dA \\
&= -5\int_{-1}^{1}\int_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} dy\, dx = -10\int_{-1}^{1} (1 - x^{2/3})^{3/2}\, dx
\end{aligned}$$

Let $x = \cos^3\theta$ and $dx = -3\cos^2\theta \sin\theta\, d\theta$. Thus

$$\begin{aligned}
\iint_R \operatorname{curl} \mathbf{F}\cdot\mathbf{k}\, dA &= -10\int_{-\pi}^{0} (1 - \cos^2\theta)^{3/2}(-3\cos^2\theta \sin\theta)\, d\theta = -30\int_0^{\pi} (\sin^2\theta)^{3/2}\cos^2\theta \sin\theta\, d\theta \\
&= -30\int_0^{\pi} (\sin^2\theta \cos^2\theta)\sin^2\theta\, d\theta = -30\int_0^{\pi}\left(\frac{\sin 2\theta}{2}\right)^2 \frac{1-\cos 2\theta}{2}\, d\theta \\
&= -\tfrac{15}{4}\int_0^{\pi} \sin^2 2\theta(1 - \cos 2\theta)\, d\theta = -\tfrac{15}{4}\int_0^{\pi} [\tfrac{1}{2}(1 - \cos 4\theta) - \sin^2 2\theta \cos 2\theta]\, d\theta \\
&= -\tfrac{15}{4}\left[\tfrac{1}{2}\theta - \tfrac{1}{8}\sin 4\theta - \tfrac{1}{6}\sin^3 2\theta\right]_0^{\pi} = -\tfrac{15}{4}(\tfrac{1}{2}\pi) = -\tfrac{15}{8}\pi
\end{aligned}$$

and we have verified Stokes's theorem in the plane for the given vector field and region.

29. $\mathbf{F}(x,y) = x^2\mathbf{i} + y^2\mathbf{j}$. $\mathbf{R}(t) = 5\cos t\mathbf{i} + 2\sin t\mathbf{j}$, $0 \le t \le 2\pi$.

(a) $\oint_C \mathbf{F}\cdot\mathbf{N}\, ds = \oint_C -y^2dx + x^2dy = \int_0^{2\pi}[-(4\sin^2 t)(-5\sin t) + (25\cos^2 t)(2\cos t)]dt$

$= \int_0^{2\pi}(20\sin^3 t + 50\cos^3 t)dt = 0$ and $\iint_R \operatorname{div}(x^2\mathbf{i} + y^2\mathbf{j})dA = \iint_R (2x + 2y)dA = 0$ (odd powers).

(b) $\oint_C \mathbf{F}\cdot\mathbf{T}ds = \oint_C x^2dx + y^2dy = \int_0^{2\pi}[25\cos^2 t\frac{d}{dt}(\cos t) + 4\sin^2 t\frac{d}{dt}(\sin t)]dt = \frac{25}{3}\cos^3 t + \frac{4}{3}\sin^3 t\Big]_0^{2\pi} = 0$

and $\iint_R \operatorname{curl} \mathbf{F}\cdot\mathbf{k}\, dA = \iint_R \mathbf{0}\cdot\mathbf{k}\, dA = 0$

30. $\mathbf{F}(x,y) = y^2\mathbf{i} + x^2\mathbf{j}$. R: $x = 5\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$

(a) $\oint_C \mathbf{F}\cdot\mathbf{N}\, ds = \oint_C -x^2dx + y^2dx = \oint_C d(-\frac{1}{3}x^3 + \frac{1}{3}y^3) = 0$ and $\iint_R \operatorname{div}(y^2\mathbf{i} + x^2\mathbf{j})dA = \iint_R 0\, dA = 0$

(b) $\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C y^2dx + x^2dy = \int_0^{2\pi} 4\sin^2 t(-5\sin t\, dt) + 25\cos^2 t(2\cos t\, dt) = 0$ and

$\iint_R \operatorname{curl} \mathbf{F}\cdot\mathbf{k}\, dA = \iint_R (2x - 2y)dA = 0$ because each term is odd

In Exercises 31–34, use Green's theorem to find the total work done in moving an object in the counterclockwise direction once around curve C if the motion is caused by the force field $\mathbf{F}(x,y)$. Length is measured in meters and force is measured in newtons.

▹ In Exercises 31–33 W joules of work is done.

31. The ellipse $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$ has area $\pi(4)(2) = 8\pi$ cm^2.

$W = \oint_C (3x + y)dx + (4x - 5y)dy = \iint_R \left[\frac{\partial}{\partial x}(4x - 5y) - \frac{\partial}{\partial y}(3x + y)\right]dA = \iint_R 3\, dA = 3\cdot 8\pi = 24\pi$

32. C is the circle $x^2 + y^2 = 25$; $\mathbf{F}(x, y) = (e^x + y^2)\mathbf{i} + (x^2y + \cos y)\mathbf{j}$

▶ Let R be the region enclosed by the circle. The number of joules in the work is given by

$$W = \oint_C (e^x + y^2)dx + (x^2y + \cos y)dy = \iint_R [D_x(x^2y + \cos y) - D_y(e^x + y^2)]dA = \iint_R (2xy - 2y)dA = 0$$

because the integrand is an odd function of y and R is symmetric with respect to the x axis.

33. $W = \oint_C (e^x + y^2)dx + (e^y + x^2)dy = \iint_R \left[\frac{\partial}{\partial x}(e^y + x^2) - \frac{\partial}{\partial y}(e^x + y^2)\right]dA = \int_0^2 \int_0^{2-x} (2x - 2y)dy\, dx$

$= \int_0^2 \left[2xy - y^2\right]_0^{2-x} dx = \int_0^2 [2x(2-x) - (2-x)^2]dx = \int_0^2 (-3x^2 + 8x - 4)dx = -x^3 + 4x^2 - 4x\Big]_0^2 = 0$

34. $W = \oint_C (xy + y^2)dx + xy\, dy = \iint_R [D_x(xy) - D_y(xy + y^2)]dA = \iint_R (-y - x)dA$

$= -\int_{-2}^{2} \int_0^{3\sqrt{4-x^2}/2} y\, dy\, dx = -\int_0^2 \frac{9}{4}(4 - x^2)dx = -\frac{9}{4}\left[4x - \frac{1}{3}x^3\right]_0^2 = -12$

In Exercises 35–38, find the rate of flow of the fluid out of the region R bounded by the curve C if **F** is the velocity field of the fluid. Velocity is measured in centimeters per second; area is measured in square centimeters.

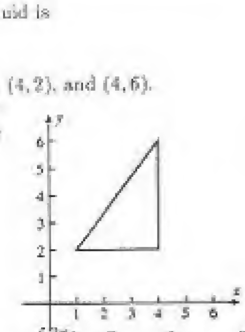
35. $\mathbf{F}(x, y) = (y^2 + 6x)\mathbf{i} + (2y - x^2)\mathbf{j}$. The ellipse $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$ has area $\pi(2)(1) = 2\pi$ cm^2.

From Gauss' divergence theorem the number of cm^2/sec in the outflow of the fluid is

$$\text{flux} = \oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \text{div } \mathbf{F}\, dA = \iint_R (6 + 2)dA = 8 \cdot 2\pi = 16\pi$$

36. $\mathbf{F}(x, y) = (5x - y^2)\mathbf{i} + (3x - 2y)\mathbf{j}$; C is the right triangle having vertices at (1, 2), (4, 2), and (4, 6).

▶ The rate of flow of the fluid is the flux. We apply Gauss's theorem in the plane to evaluate the line integral. The figure shows the the region R. The area of the right triangle is 6 cm^2. Thus,



$$\text{flux} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \text{div } \mathbf{F}\, dA$$
$$= \iint_R (D_x(5x - y^2) + D_y(3x - 2y)]dA = \iint_R 3\, dA = 3(6) = 18$$

The rate of flow of the fluid out of the region R is 18 cm^2/sec.

37. $\mathbf{F}(x, y) = x^3\mathbf{i} + y^3\mathbf{j}$. The number of cm^2/sec in the outflow of the fluid is

$$\text{flux} = \oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \text{div } \mathbf{F}\, dA = \iint_R (3x^2 + 3y^2)dA = \int_0^{2\pi} \int_0^1 3r^2(r\, dr\, d\theta) = \int_0^{2\pi} \frac{3}{4} r^4\Big]_0^1 d\theta = \frac{3}{4} \cdot 2\pi = \frac{3}{2}\pi$$

38. $\text{flux} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \text{div}(xy^2\mathbf{i} + yx^2\mathbf{j})dA = \iint_R (y^2 + x^2)dA = \int_0^{2\pi} \int_0^3 r^2(r\, dr\, d\theta) = 2\pi \cdot \frac{1}{4}r^4\Big]_0^3 = \frac{81}{2}\pi$

In Exercises 39–42, **F** is the velocity field of a fluid around a closed curve C. Use Stokes's theorem in the plane to compute $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, and from the result determine which of the following applies: (i) the circulation of the fluid is counterclockwise; (ii) the circulation of the fluid is clockwise; (iii) **F** is irrotational; (iv) the circulation is 0 but **F** is not irrotational.

39. $\mathbf{F}(x, y) = 4y\mathbf{i} + 6x\mathbf{j}$. curl $\mathbf{F} = (6 - 4)\mathbf{k} = 2\mathbf{k}$. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \text{curl } \mathbf{F} \cdot \mathbf{k}\, dA = \iint_R 2\, dA > 0$. Hence (i).

40. $\mathbf{F}(x, y) = 8y\mathbf{i} + 3x\mathbf{j}$; C is the ellipse $4x^2 + 9y^2 = 1$.

▶ The curl of **F** is given by

$$\text{curl } \mathbf{F} = [D_x(3x) - D_y(8y)]\mathbf{k} = -5\mathbf{k}$$

The area of the region R bounded by the ellipse is $\pi ab = \pi(\frac{1}{2})(\frac{1}{3}) = \frac{1}{6}\pi$. Therefore, applying Stokes's theorem in the plane, we have

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \text{curl } \mathbf{F} \cdot \mathbf{k}\, dA = \iint_R (-5\mathbf{k}) \cdot \mathbf{k}\, dA = -5\iint_R dA = -5(\tfrac{1}{6}\pi) = -\tfrac{5}{6}\pi$$

Because $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds < 0$, the circulation of the fluid is clockwise.

41. $\mathbf{F}(x, y) = \sin^2 x\mathbf{i} + \cos^2 y\mathbf{j}$. curl $\mathbf{F} = \mathbf{0}$. Hence (iii).

42. $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \text{curl}(y^3\mathbf{i} + x^3\mathbf{j})dA = \iint_R (3x^2 - 3y^2)dA = 0$ Hence (iv).

43. $\oint_C N(x, y)dy = \iint_R \frac{\partial N}{\partial x}\, dA$ is Green's theorem with $M(x, y) = 0$.

14.5 SURFACE INTEGRALS

Let S be the surface that is the graph of the equation $z = f(x, y)$ were f and its first partial derivatives are continuous on a region D in the xy plane, and let G be a function of x, y, z which is continuous on S. Then the surface integral of G over S can be expressed as a double integral over the region D and is given by

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x{}^2(x,y) + f_y{}^2(x,y) + 1}\; dA$$

In polar coordinates, if the surface is given by $z = f(r, \theta)$, then

$$\iint_S G(r,\theta,z)d\sigma = \iint_R G(r,\theta,f(r,\theta))\sqrt{f_r{}^2(r,\theta) + r^{-2}f_\theta{}^2(r,\theta) + 1}\; r\, dr\, d\theta$$

If the measure of the area density at the point (x, y, z) on a surface S is $\rho(x, y, z)$, and if M is the measure of the mass of S, then

$$M = \iint_S \rho(x,y,z)d\sigma$$

Let $\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be the velocity field for a fluid that flows through a surface S. If s has the equation $z = f(x, y)$, and S lies over a region D in the xy plane, where f and its first partial derivatives are continuous on D, and $\mathbf{n}$ is a unit upper normal vector for the surface, then the flux of $\mathbf{F}$ across S is the surface integral of $\mathbf{F}\cdot\mathbf{n}$ over S, and this surface integral may be evaluated by a double integral over D. The flux of $\mathbf{F}$ across S is given by

$$\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma = \iint_D (-Mf_x - Nf_y + R)dA \tag{9}$$

where z is replaced by $f(x, y)$.

Exercises 14.5

In Exercises 1-14, evaluate the surface integral $\iint_S G(x,y,z)d\sigma$ for G and S.

1. $f(x,y) = z = \sqrt{4 - x^2 - y^2}$, $f_x(x,y) = \dfrac{-x}{\sqrt{4-x^2-y^2}}$, $f_y(x,y) = \dfrac{-y}{\sqrt{4-x^2-y^2}}$. Circle D has area 4π. $\iint_S z\, d\sigma$

$$= \iint_D \sqrt{4-x^2-y^2}\sqrt{\frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2} + 1}\; dA = \iint_D \sqrt{4-x^2-y^2}\sqrt{\frac{4}{4-x^2-y^2}}\; dA = 2\iint_D dA = 8\pi$$

2. $f(x,y) = z = 1 - x - y$, $f_x(x,y) = -1$, $f_y(x,y) = -1$.

$$\iint_S x\, d\sigma = \iint_D x\sqrt{1+1+1}dA = \int_0^1\int_0^{1-x}\sqrt{3}x\, dy\, dx = \sqrt{3}\int_0^1 (1-x)x\, dx = \sqrt{3}\Big[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{6}\sqrt{3}$$

3. $f(x,y) = z = 2 - x - y$, $f_x(x,y) = -1$, $f_y(x,y) = -1$.

$$\iint_S (x + 2y - z)d\sigma = \iint_D (2x + 3y - 2)\sqrt{(-1)^2 + (-1)^2 + 1}\; dA = \sqrt{3}\int_0^2\int_0^{2-x}(2x+3y-2)dy\, dx$$

$$= \sqrt{3}\int_0^2 \Big[(2x-2)y + \tfrac{3}{2}y^2\Big]_0^{2-x} dx = \sqrt{3}\int_0^2 \Big[(2x-2)(2-x) + \tfrac{3}{2}(2-x)^2\Big]dx$$

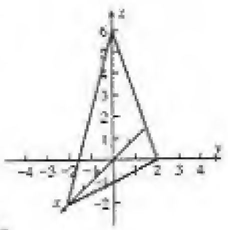
$$= \sqrt{3}\int_0^2 \Big[-2x^2 + 6x - 4 + \tfrac{3}{2}(2-x)^2\Big]dx = \sqrt{3}\Big[-\tfrac{2}{3}x^3 + 3x^2 - 4x - \tfrac{1}{2}(2-x)^3\Big]_0^2 = \tfrac{8}{3}\sqrt{3}$$

4. $G(x,y,z) = z$; S is the portion of the plane $2x + 3y + z = 6$ in the first octant.

▷ The figure shows the surface S. The projection of S onto the xy plane is the triangular region D, bounded by the x and y axes and the line $2x + 3y = 6$. Because

$$z = f(x,y) = 6 - 2x - 3y$$

then the surface integral of G over S is given by



$$\iint_S z\, d\sigma = \iint_D (6 - 2x - 3y)\sqrt{f_x{}^2(x,y) + f_y{}^2(x,y)}\; dA$$

$$= \int_0^3\int_0^{(6-2x)/3}(6-2x-3y)\sqrt{(-2)^2 + (-3)^2 + 1}\; dy\, dx$$

$$= \sqrt{14}\int_0^3\int_0^{(6-2x)/3}(6-2x-3y)dy\, dx = \int_0^3 (6-2x)y - \tfrac{3}{2}y^2\Big]_0^{(6-2x)/3} dx$$

$$= \sqrt{14}\int_0^3 \tfrac{1}{6}(6-2x)^2\, dx = \sqrt{14}(-\tfrac{1}{36})(6-2x)^3\Big]_0^3 = 6\sqrt{14}$$

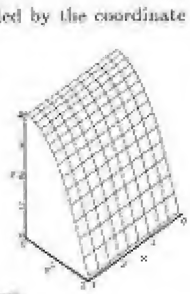
5. $f(x,y) = z = 6 - 2x - 3y$. $f_x(x,y) = -2$, $f_y(x,y) = -3$. $\iint_S xyz\, d\sigma = \iint_D xy(6-2x-3y)\sqrt{4+9+1}\, dA$
$= \sqrt{14}\int_0^3\int_0^{(6-2x)/3} x[(6-2x)y - 3y^2]dy\, dx = \sqrt{14}\int_0^3 x\left[\frac12(6-2x)y^2 - y^3\right]_0^{(6-2x)/3} dx$
$= \frac{1}{54}\sqrt{14}\int_0^3 x(6-2x)^3 dx = \frac{1}{54}\sqrt{14}\left[-\frac18 x(6-2x)^4 - \frac{1}{80}(6-2x)^5\right]_0^3 = \frac{1}{54}\sqrt{14}\frac{6^5}{80} = \frac92\sqrt{14}$

6. $f(x,z) = y = \sqrt{1-x^2}$. $f_x(x,z) = \frac{-x}{\sqrt{1-x^2}}$, $f_z(x,z) = 0$. $\iint_S x^2\sqrt{\frac{x^2}{1-x^2} + 0 + 1}dA = \int_0^1\int_0^1 \frac{x^2}{\sqrt{1-x^2}}dz\, dx$
$= \int_0^1 \frac{x^2}{\sqrt{1-x^2}}dx = \int_0^{\pi/2}\frac{\sin^2\theta}{\cos\theta}\cos\theta\, d\theta = \int_0^{\pi/2}\frac12(1-\cos 2\theta)d\theta = \frac14\pi$

7. $f(x,y) = z = x^2$. $f_x(x,y) = 2x$, $f_y(x,y) = 0$. $\iint_S x\, d\sigma = \iint_D x\sqrt{4x^2+1}\, dA = \int_0^1\int_0^2 x\sqrt{4x^2+2}\, dy\, dx$
$= \frac14\int_0^1 8x\sqrt{4x^2+1}\, dx = \frac14\cdot\frac23(4x^2+1)^{3/2}\Big]_0^1 = \frac12(5\sqrt5 - 1)$

8. $G(x,y,z) = y$; S is the portion of the cylinder $z = 4 - y^2$ in the first octant bounded by the coordinate planes and the plane $x = 3$.

▷ The figure shows the surface S. The projection of S onto the xy plane is the rectangular region D, bounded by the x and y axes and the lines $x = 3$ and $y = 2$. Because
$z = f(x,y) = 4 - y^2$
then the surface integral of G over S is given by



$$\iint_S y\, d\sigma = \iint_D y\sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}dA = \int_0^2\int_0^3 y\sqrt{(-2y)^2+1}dx\, dy$$
$$= \int_0^2 3y\sqrt{4y^2+1}dy = 3(\tfrac18)(\tfrac23)(4y^2+1)^{3/2}\Big]_0^2 = \tfrac14(17\sqrt{17} - 1)$$

In Exercises 9 and 10. $f(x,y) = z = \sqrt{x^2+y^2}$. $f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, $f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$

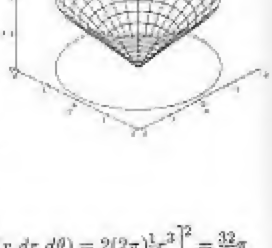
9. $\iint_S z^2 d\sigma = \iint_D (x^2+y^2)\sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}\, dA = \sqrt2\iint_D (x^2+y^2)dA = \sqrt2\int_0^{2\pi}\int_0^2 r^2(r\, dr\, d\theta)$
$= \sqrt2\int_0^{2\pi}\frac14 r^4\Big]_0^2 = \frac14\sqrt2\int_0^{2\pi} 15\, d\theta = \frac{15}{2}\sqrt2\pi$

10. $\iint_S xyz\, d\sigma = \sqrt2\iint_D xy\sqrt{x^2+y^2}dA = 0$ because integrand is odd

11. $f(x,y) = z = 2 - \frac23 x - \frac12 y$. $f_x(x,y) = -\frac23$, $f_y(x,y) = -\frac12$.
$\iint_S (x+y)d\sigma = \iint_D (x+y)\sqrt{\frac49 + \frac14 + 1}\, dA = \frac16\sqrt{61}\int_0^3\int_0^{4-4x/3}(x+y)dy\, dx = \frac16\sqrt{61}\int_0^3 \frac12(x+y)^2\Big]_0^{4-4x/3}$
$= \frac{1}{12}\sqrt{61}\int_0^3\left[\left(4 - \frac{x}{3}\right)^2 - x^2\right]dx = \frac{1}{12}\sqrt{61}\left[-\left(4 - \frac{x}{3}\right)^3 - \frac13 x^3\right]_0^3 = \frac{1}{12}\sqrt{61}(-27 - 9 + 64) = \frac73\sqrt{61}$

12. $G(x,y,z) = \sqrt{x^2+y^2+z^2}$; S is the portion of the cone $x^2+y^2 = z^2$ between the xy plane and the plane $z = 2$.

▷ The figure shows the surface S. The projection of S onto the xy plane is the region D bounded by the circle $x^2+y^2 = 4$. Solving the equation of the surface for z, we obtain
$z = f(x,y) = \sqrt{x^2+y^2}$
Then
$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ $f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$



We evaluate the surface integral of G over S with the help of polar coordinates
$$\iint_S \sqrt{x^2+y^2+z^2}d\sigma = \iint_D \sqrt{x^2+y^2+(x^2+y^2)}\sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}dA$$
$$= \iint_D \sqrt{2(x^2+y^2)}\sqrt2 dA = 2\iint_D \sqrt{x^2+y^2}dA = 2\int_0^{2\pi}\int_0^2 r(r\, dr\, d\theta) = 2(2\pi)\tfrac13 r^3\Big]_0^2 = \tfrac{32}{3}\pi$$

13. $f(x,y) = z = \sqrt{4-x^2}$. $f_x(x,y) = \dfrac{-x}{\sqrt{4-x^2}}$, $f_y(x,y) = 0$

$$\iint_S xyz\, d\sigma = \iint_D xy\sqrt{4-x^2}\sqrt{\frac{x^2}{4-x^2}+1}\, dA = \int_1^3\int_{-1}^2 2xy\, dx\, dy = \int_1^3 x^2y\Big]_{-1}^2 dy = 0$$

14. Let T be the entire sphere. $\iint_S x^2 d\sigma = \frac{1}{2}\iint_T x^2 d\sigma = \frac{1}{2}\iint_T y^2 d\sigma = \frac{1}{2}\iint_T z^2 d\sigma = \frac{1}{6}\iint_T (x^2+y^2+z^2) d\sigma$

$$= \tfrac{1}{6}\cdot 81\iint_T d\sigma = \tfrac{1}{6}\cdot 81\cdot 4\pi = 54\pi$$

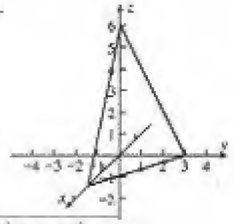
In Exercises 15–20, find the mass of the surface S if the area density at any point (x,y,z) on S is $\rho(x,y,z)$ kg/m^2.

15. $f(x,y) = z = \sqrt{4-x^2-y^2}$. $f_x(x,y) = \dfrac{-x}{\sqrt{4-x^2-y^2}}$, $f_y(x,y) = \dfrac{-y}{\sqrt{4-x^2-y^2}}$. If M kg is the mass

$$M = \iint_S k\sqrt{x^2+y^2+z^2}\, d\sigma = \iint_D 2k\sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}\, dA = 4k\iint_D \frac{1}{\sqrt{4-x^2-y^2}}\, dA$$
$$= 4k\int_0^{2\pi}\int_0^1 \frac{1}{r}(r\, dr\, d\theta) = 4k\int_0^{2\pi} 1\, d\theta = 4k\cdot 2\pi = 8\pi k$$

16. S is the portion of the plane $3x+2y+z=6$ in the first octant; $\rho(x,y,z) = y+2z$.

▹ The figure shows the surface S. The mass of the surface is M kg, where M is given by the surface integral



$$M = \iint_S (y+2z) d\sigma$$

The projection of S onto the xy plane is the region bounded by the axes and the line $3x+2y=6$. solving the equation of the surface for z, we obtain
$z = f(x,y) = 6-3x-2y$
Then, substituting for z,

$$M = \iint_D (y+2z)\sqrt{f_x{}^2(x,y)+f_y{}^2(x,y)+1}\, dA = \iint_D (12-6x-3y)\sqrt{(-3)^2+(-2)^2+1}\, dA$$
$$= 3\sqrt{14}\int_0^2\int_0^{3(2-x)/2}(4-2x-y)dy\, dx = 3\sqrt{14}\int_0^2 2(2-x)y - \tfrac{1}{2}y^2\Big]_0^{3(2-x)/2} dx$$
$$= 3\sqrt{14}\int_0^2 [3(2-x)^2 - \tfrac{9}{8}(2-x)^2]dx = \tfrac{45}{8}\sqrt{14}\int_0^2 (2-x)^2 dx = -\tfrac{15}{8}\sqrt{14}(2-x)^3\Big]_0^2 = 15\sqrt{14}$$

17. $f(x,y) = z = 9-x^2-y^2$. $f_x(x,y) = -2x$, $f_y(x,y) = -2y$. $d\sigma = \sqrt{4x^2+4y^2+1}\, dA$. D is a circle of radius 3.
If M kg is the mass, $M = \iint_S d\sigma/\sqrt{4x^2+4y^2+1} = \iint_D dA = 9\pi$.

18. Let T be the entire sphere.

$$M = \iint_S (x^2+y^2)d\sigma = \tfrac{1}{2}\iint_T (x^2+y^2)d\sigma = \tfrac{1}{2}\cdot\tfrac{2}{3}\iint_T (x^2+y^2+z^2)d\sigma = \tfrac{1}{3}\iint_T 1 d\sigma = \tfrac{1}{3}\cdot 4\pi = \tfrac{4}{3}\pi$$

19. $f(x,y) = z = \sqrt{x^2+y^2}$. $f_x(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$, $f_y(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$. If M kg is the mass

$$M = \iint_S y^2z^2 d\sigma = \iint_D y^2(x^2+y^2)\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}\, dA = \sqrt{2}\iint_D y^2(x^2+y^2)dA$$
$$= \sqrt{2}\int_0^{2\pi}\int_0^3 r^2\sin^2\theta(r^2)r\, dr\, d\theta = \sqrt{2}\int_0^{2\pi}\frac{1}{6}r^6\Big]_0^3 \sin^2\theta d\theta = \frac{665}{6}\sqrt{2}\int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)d\theta = \frac{665}{6}\sqrt{2}\pi$$

20. S is the portion of the sphere $x^2+y^2+z^2=16$ in the first octant; $\rho(x,y,z) = kz^2$, where k is a constant.

▹ The mass of the surface is M kg, where M is given by the surface integral

$$M = \iint_S kz^2\, d\sigma \qquad (1)$$

By symmetry we also have

$$M = \iint_S kx^2\, d\sigma \quad (2) \quad \text{and} \qquad M = \iint_S ky^2\, d\sigma \quad (3)$$

Adding (1), (2) and (3) and dividing by 3, we get

$$M = \tfrac{1}{3}\iint_S k(x^2+y^2+z^2)\, d\sigma = \tfrac{16}{3}k\iint_S d\sigma = \tfrac{16}{3}k(\tfrac{1}{8}\text{ area of sphere of radius } 4) = \tfrac{16}{3}\cdot\tfrac{1}{8}\cdot 4\pi(4)^2 = \tfrac{128}{3}k\pi$$

In Exercises 21–24, find the flux of **F** across the surface S where $\mathbf{F}(x, y, z)$ gives the velocity field of the fluid.

21. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. $f(x, y) = z = 6 - 3x - 2y$. $f_x = -3$, $f_y = -2$. D is a triangle of area 3.

By equation (9), flux $= \iint_S \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R [-x(-3) - y(-2) + (6 - 3x - 2y)]dA = \iint_D 6\, dA = 18$.

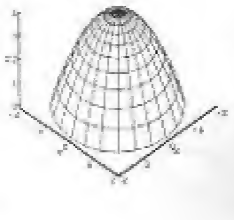
22. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has direction of $\mathbf{n}$ and so $\mathbf{F} \cdot \mathbf{n} = \sqrt{x^2 + y^2 + z^2} = 1$ and $\iint_S \mathbf{F} \cdot \mathbf{n} ds = \iint_S ds = 2\pi$

23. $\mathbf{F} = -2y\mathbf{i} + 2x\mathbf{j} + 5\mathbf{k}$. $f(x, y) = z = \sqrt{16 - x^2 - y^2}$. $f_x = \frac{-x}{\sqrt{16 - x^2 - y^2}} = -\frac{x}{z}$. $f_y = \frac{-y}{\sqrt{16 - x^2 - y^2}} = -\frac{y}{z}$

By equation (9), flux $= \iint_S \mathbf{F} \cdot \mathbf{N}\, d\sigma = \iint_D \left[(2y)\left(-\frac{x}{z}\right) + (-2x)\left(-\frac{y}{z}\right) + 5\right]dA = 5\iint_D dA = 5\pi(3)^2 = 45\pi$.

24. $\mathbf{F}(x, y, z) = 3x\mathbf{i} + 3y\mathbf{j} + 6z\mathbf{k}$: S is the portion of the paraboloid $z = 4 - x^2 - y^2$ above the xy plane.

▸ The figure shows the surface S, which lies above the region D in the xy plane.
D is bounded by the circle $x^2 + y^2 = 4$, whose area is 4π. Let
$z = f(x, y) = 4 - x^2 - y^2$
and
$M(x, y, z) = 3x \quad N(x, y, z) = 3y \quad R(x, y, z) = 6z$
If $\mathbf{n}$ is a unit upper normal vector for S, then the flux of **F** across S is given by



$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_D (-Mf_x - Nf_y + R)dA$$
$$= \iint_D [(-3x)(-2x) - (3y)(-2y) + 6(4 - x^2 - y^2)]dA = 24\iint_D dA$$
$$= 24(4\pi) = 96\pi$$

25. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + 2z\mathbf{k}$. Take the sides of the cube in the order $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, d\sigma = \int_0^1\int_0^1 (2z\mathbf{k}) \cdot (-\mathbf{i})dy\, dz + \int_0^1\int_0^1 (\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}) \cdot \mathbf{i}\, dy\, dz + \int_0^1\int_0^1 (x^2\mathbf{i} + 2z\mathbf{k}) \cdot (-\mathbf{j})dx\, dz$$
$$+ \int_0^1\int_0^1 (x^2\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}) \cdot \mathbf{j}\, dx\, dz + \int_0^1\int_0^1 (x^2\mathbf{i} + xy\mathbf{j}) \cdot (-\mathbf{k})dx\, dy + \int_0^1\int_0^1 (x^2\mathbf{i} + xy\mathbf{j} + 2\mathbf{k}) \cdot \mathbf{k}\, dx\, dy$$
$$= 0 + \int_0^1\int_0^1 dy\, dz + 0 + \int_0^1\int_0^1 x\, dx\, dz + 0 + \int_0^1\int_0^1 2\, dx\, dy = 1 + \tfrac{1}{2} + 2 = \tfrac{7}{2}$$

26. $\mathbf{F}(x, y, z) = 3x\mathbf{i} + y^2\mathbf{j} + yz\mathbf{k}$. Take the sides of the cube in the order $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, d\sigma = \int_0^1\int_0^1 (y^2\mathbf{j} + yz\mathbf{k}) \cdot (-\mathbf{i})dy\, dz + \int_0^1\int_0^1 (3\mathbf{i} + y^2\mathbf{j} + yz\mathbf{k}) \cdot \mathbf{i}\, dy\, dz + \int_0^1\int_0^1 (3x\mathbf{i}) \cdot (-\mathbf{j})dx\, dz$$
$$+ \int_0^1\int_0^1 (3x\mathbf{i} + \mathbf{j} + z\mathbf{k}) \cdot \mathbf{j}\, dx\, dz + \int_0^1\int_0^1 (3x\mathbf{i} + y^2\mathbf{j}) \cdot (-\mathbf{k})dx\, dy + \int_0^1\int_0^1 (3x\mathbf{i} + y^2\mathbf{j} + y\mathbf{k}) \cdot \mathbf{k}\, dx\, dy$$
$$= 0 + \int_0^1\int_0^1 3dy\, dz + 0 + \int_0^1\int_0^1 dx\, dz + 0 + \int_0^1\int_0^1 y\, dx\, dy = 3 + 1 + \tfrac{1}{2} = \tfrac{9}{2}$$

14.6 GAUSS'S DIVERGENCE THEOREM AND STOKES'S THEOREM

In the following two theorems, M, N and R are functions of three variables x, y, and z, having continuous first partial derivatives on an open ball in R^3 and we let
$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$.

14.6.1 Theorem (Gauss's Divergence Theorem) Let S be a sectionally smooth closed surface lying in B, let E be the region bounded by S, and let $\mathbf{n}$ be a unit outward normal vector of S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

14.6.2 Theorem (Stokes's Theorem) Let S be a sectionally smooth two-sided surface with one side called "up" lying in B and let C be a sectionally smooth simple curve that is the boundary of S. $\mathbf{n}$ is a unit upward normal and $\mathbf{T}$ is a unit tangent vector to C, where s units is the length of arc measured from a particular point on C to P in the counterclockwise direction when viewed from above. Then

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

Stokes's theorem does not apply to a Mobius band: it is one-sided.

Exercises 14.6

In Exercises 1–4, verify Gauss's divergence theorem for **F** and S.

1. $\mathbf{F} = 2z\mathbf{k}$. div $\mathbf{F} = 2$. E is the sphere $x^2+y^2+z^2=1$. $\iiint_E \text{div } \mathbf{F}\, dV = 2\iiint_E dV = 2\left(\frac{4}{3}\pi\right) = \frac{8}{3}\pi$.

   On the upper hemisphere, by equation 14.5.9, $\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_D (0+0+2z)dA$

   $= 2\iint_D (1-x^2-y^2)^{1/2} dA = 2\int_0^{2\pi}\int_0^1 (1-r^2)^{1/2} r\, dr\, d\theta = 2\int_0^{2\pi} -\frac{1}{3}(1-r^2)^{3/2}\Big]_0^1 d\theta = \frac{2}{3}\int_0^{2\pi} d\theta = \frac{4}{3}\pi$

   On the lower hemisphere the signs of z and **N** are reversed so the integral is the same.
   Hence the total flux is $\frac{8}{3}\pi$ and Gauss' divergence theorem is verified.

2. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. E is the sphere $x^2+y^2+z^2=9$. $\iiint_E \text{div } \mathbf{F}\, dV = 3\iiint_E dV = 3(\frac{4}{3}\pi 27) = 108\pi$

   $\mathbf{F}\cdot\mathbf{n} = \sqrt{x^2+y^2+z^2} = 3$ so $\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = 3\cdot 4\pi(3^2) = 108\pi$

3. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j}$. div $\mathbf{F} = y+0+y = 2y$. E is the cube $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$.

   $\iiint_E \text{div } \mathbf{F}\, dV = \int_0^1\int_0^1\int_0^1 2y\, dy\, dx\, dz = \int_0^1 2y\, dy = y^2\Big]_0^1 = 1$. Alternatively,

   $\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = \int_0^1\int_0^1 (yz\mathbf{j})\cdot(-\mathbf{i})dy\, dz + \int_0^1\int_0^1 (y\mathbf{i}+yz\mathbf{j})\cdot\mathbf{i}\, dy\, dz + 0 + \int_0^1\int_0^1 (x\mathbf{i}+z\mathbf{j})\cdot\mathbf{j}\, dx\, dz$

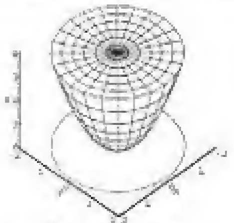
   $+ \int_0^1\int_0^1 (xy\mathbf{i})\cdot(-\mathbf{k})dx\, dy + \int_0^1\int_0^1 (xy\mathbf{i}+y\mathbf{j})\cdot\mathbf{k}\, dx\, dy$

   $= 0 + \int_0^1\int_0^1 y\, dy + 0 + \int_0^1\int_0^1 z\, dx\, dz + 0 + 0 = \frac{1}{2}y^2\Big]_0^1 + \frac{1}{2}z^2\Big]_0^1 = \frac{1}{2}+\frac{1}{2} = 1$

4. $\mathbf{F}(x,y,z) = 4x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$; S is the boundary of the region enclosed by the paraboloid $z = x^2+y^2$ and the plane $z = 4$.

▸ The figure shows the surface S. S_1 is the part of the paraboloid below the plane and S_2 is the part of S in the plane. The projection of S on the xy plane is the region D bounded by the circle $x^2+y^2=4$. We have

$f(x,y) = x^2+y^2 \quad M(x,y,z) = 4x \quad N(x,y,z) = -2y \quad R(x,y,z) = z = x^2+y^2$



The left side of Gauss's divergence theorem is

$$\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_{S_1} \mathbf{F}\cdot\mathbf{N}\, d\sigma + \iint_{S_2} \mathbf{F}\cdot\mathbf{N}\, d\sigma \qquad (1)$$

Because the outward unit normal to S_1 is downward, we have

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_D (Mf_x + Nf_y - R)dA = \iint_D [4x(2x) - 2y(2y) - (x^2+y^2)]dA = \iint_D (7x^2 - 5y^2)\, dA$$

We switch to polar coordinates with $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\, dr\, d\theta$.

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{N}\, d\sigma = \int_0^2\int_0^{2\pi} (7r^2\cos^2\theta - 5r^2\sin^2\theta)r\, d\theta\, dr = \int_0^2\int_0^{2\pi}\left[7\left(\frac{1+\cos 2\theta}{2}\right) - 5\left(\frac{1-\cos 2\theta}{2}\right)\right]r^3\, d\theta\, dr$$

$$= \int_0^2\int_0^{2\pi} (1+6\cos 2\theta)r^3\, d\theta\, dr = \int_0^2 2\pi r^3\, dr = \frac{1}{2}\pi r^4\Big]_0^2 = 8\pi \qquad (2)$$

On S^2, $\mathbf{N} = \mathbf{k}$, and $\mathbf{F}\cdot\mathbf{N} = z = 4$. Thus

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_{S_2} 4\, d\sigma = 4(\text{area of circle of radius } 2) = 4\pi(2)^2 = 16\pi \qquad (3)$$

Substituting from (2) and (3) into (1), we have

$$\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = 8\pi + 16\pi = 24\pi$$

The right side of Gauss's divergence theorem is

$$\iiint_E \text{div } \mathbf{F}\, dV = \iiint_E \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(z)\right] dV = 3\iiint_E dV$$

We compute the volume of E by the method of slicing. A section perpendicular to the z axis is a circle of radius $\sqrt{z}$ and area πz. Thus

$$\iiint_E \text{div } \mathbf{F}\, dV = 3\int_0^4 \pi z\, dz = 3\cdot\frac{1}{2}\pi z^2\Big]_0^4 = 24\pi$$

Thus we have verified Gauss's divergence theorem for the given **F** and S.

In Exercises 5–8, for the **F** and S of the exercise in Ex. 14.5 find the flux of **F** across S by Gauss's divergence theorem.

5. Ex. 21. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. div $\mathbf{F} = 1 + 1 + 1 = 3$. E is a tetrahedron of volume $\frac{1}{6}\cdot 2\cdot 3\cdot 6 = 6$.

   Flux $= \iiint_E 3\, dV = 3\cdot 6 = 18$ and the flux over the faces parallel to the coordinate planes are 0.

6. Ex. 22. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. On the xy plane, $\mathbf{F}\cdot\mathbf{n} = (x\mathbf{i} + y\mathbf{j})\cdot\mathbf{k} = 0$. Let T be the closed hemisphere $x^2 + y^2 + z^2 = 1$ and E the region it bounds, of volume $\frac{2}{3}\pi$.

   flux $= \iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint_T \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint_E \text{div}\,\mathbf{F}\, dA = \iiint_E 3\, dA = 2\pi$

7. $\text{div}(x^2\mathbf{i} + xy\mathbf{j} + 2z\mathbf{k}) = 2x + x + 2 = 3x + 2$. Flux $= \int_0^1\int_0^1\int_0^1 (3x+2)dx\, dy\, dz = \int_0^1\int_0^1 \frac{7}{2}\, dy\, dz = \frac{7}{2}$.

8. Exercise 14.5.26. $\mathbf{F}(x,y,z) = 3x\mathbf{i} + y^2\mathbf{j} + yz\mathbf{k}$; S is the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$.

▸ We have

$$\begin{aligned}\text{div}\,\mathbf{F}(x,y,z) &= D_x(3x) + D_y(y^2) + D_z(yz)\\ &= 3 + 2y + y = 3 + 3y\end{aligned}$$

Applying the divergence theorem, we have

$$\text{flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint_E \text{div}\,\mathbf{F}\, dV = \int_0^1\int_0^1\int_0^1 (3+3y)dx\, dz\, dy = \int_0^1 (3+3y)dy = 3y + \tfrac{3}{2}y^2\Big]_0^1 = \tfrac{9}{2}$$

In Exercises 9–16, use Gauss's divergence theorem to evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma$ for **F** and S.

9. $\mathbf{F}(x,y,z) = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$. div $\mathbf{F} = 2xyz + 2xyz + 2xyz = 6xyz$. E is the rectangular parallelpiped $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$.

   Flux $= \iiint_E 6xyz\, dV = 6\int_0^1 x\, dx\int_0^2 y\, dy\int_0^3 z\, dz = 6\cdot\frac{1}{2}x^2\Big]_0^1\cdot\frac{1}{2}y^2\Big]_0^2\cdot\frac{1}{2}z^2\Big]_0^3 = 6\cdot\frac{1}{2}\cdot\frac{4}{2}\cdot\frac{9}{2} = 27$

10. $\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint_E \text{div}(6x\mathbf{i} + 3y\mathbf{j} + 2z\mathbf{k})dV = \iiint_E 11\, dV = 11\cdot\frac{1}{6}(3)(1)(2) = 11$

11. $\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iiint_E \text{div}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})dV = \iiint_E 3\, dV = 3\cdot\frac{4}{3}\pi(2)^2 = 32\pi$

12. $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; S is the boundary of the region enclosed on the side by the cylinder $x^2 + y^2 = 9$, below by the xy plane, and above by the plane $z = 4$.

▸ The volume of the region E enclosed by the surface S is

$V = \pi r^2 h = \pi(3^2)(4) = 36\pi$

Because

div $\mathbf{F} = D_x(x) + D_y(y) + D_z(z) = 1 + 1 + 1 = 3$

then by Gauss's divergence theorem we have

$$\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iiint_E \text{div}\,\mathbf{F}\, dV = \iiint_E 3\, dv = 3(36\pi) = 108\pi$$

13. $\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iiint_E \text{div}(x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})dV = \iiint_E 2(x+y+z)dV$

$$\begin{aligned}&= 2\int_0^3\int_0^4\int_0^{2\pi} (r\cos\theta + r\sin\theta + z)r\, d\theta\, dz\, dr = 2\int_0^3\int_0^4 \Big[r\sin\theta - r\cos\theta + z\theta\Big]_0^{2\pi} r\, dz\, dr\\ &= 4\pi\int_0^3\int_0^4 zr\, dz\, dr = 4\pi\int_0^3 8r\, dr = 4\pi(8)\tfrac{9}{2} = 144\pi\end{aligned}$$

14. $\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint_E \text{div}(x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})dV = \iiint_E (2x + 2y + 2z)dV = 0$ (odd)

15. $\iint_S \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iiint_E \text{div}(2x\mathbf{i} + 2yz\mathbf{j} + 3z\mathbf{k})dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} (5+2z)dz\, dy\, dx$

$$\begin{aligned}&= \int_0^1\int_0^{1-x}\Big[5z + z^2\Big]_0^{1-x-y} = \int_0^1\int_0^1 [5(1-x-y) + (1-x-y)^2]dx = \int_0^1\Big[-\tfrac{5}{2}(1-x-y)^2 - \tfrac{1}{3}(1-x-y)^3\Big]_0^{1-x} dx\\ &= \int_0^1\Big[\tfrac{5}{2}(1-x)^2 + \tfrac{1}{3}(1-x)^3\Big]dx = -\tfrac{5}{6}(1-x)^3 - \tfrac{1}{12}(1-x)^4\Big]_0^1 = \tfrac{5}{6} + \tfrac{1}{12} = \tfrac{11}{12}\end{aligned}$$

16. $\mathbf{F}(x,y,z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2+y^2+z^2}$

▹ Let

$M = \dfrac{x}{x^2+y^2+z^2} \quad N = \dfrac{y}{x^2+y^2+z^2} \quad R = \dfrac{z}{x^2+y^2+z^2}$

Then

$$\frac{\partial M}{\partial x} = \frac{1(x^2+y^2+z^2) - x(2x)}{x^2+y^2+z^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{1(x^2+y^2+z^2) - y(2y)}{x^2+y^2+z^2} = \frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial R}{\partial x} = \frac{1(x^2+y^2+z^2) - z(2x)}{x^2+y^2+z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

Thus,

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial R}{\partial z} = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

Applying Gauss's divergence theorem and using spherical coordinates to evaluate the triple integral, we obtain

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iiint_E \text{div } \mathbf{F}\, dV = \iiint_E \frac{1}{x^2+y^2+z^2} dV = \int_0^{\pi}\int_0^{2\pi}\int_1^2 \frac{1}{\rho^2}\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$
$$= \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_1^2 d\rho = -\cos\phi\Big]_0^{\pi}(2\pi)1 = 4\pi$$

In Exercises 17–22, verify Stokes's theorem for **F** and S.

▹ We must show that $\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \iint_S \text{curl } \mathbf{F}\cdot\mathbf{N}\, d\sigma$.

17. $\mathbf{F}(x,y,z) = y^2\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k}$, S is $x^2+y^2+z^2 = 1$, $z \geq 0$. $\mathbf{R} = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \leq t \leq 2\pi$.

$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C y^2dx + x^2dy + z^2dz = \int_0^{2\pi}[(\sin^2 t)(-\sin t) + (\cos^2 t)(\cos t) + 0]dt = 0 \text{ \{odd\}}$$

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_S (2x-2y)\mathbf{k}\cdot\mathbf{N}\, d\sigma = \iint_S 2x\,\mathbf{k}\cdot\mathbf{N}\, d\sigma - \iint_S 2y\,\mathbf{k}\cdot\mathbf{N}\, d\sigma = 0 \text{ \{symmetry\}}$$

18. Same as 17, with direction of circle and **n** reversed.

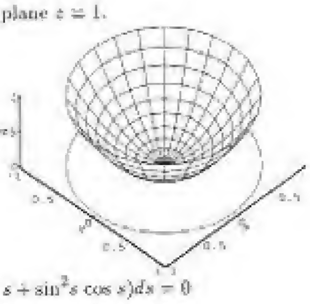
19. $\mathbf{F}(x,y,z) = y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$. $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \leq t \leq 2\pi$.

$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C y^2dx + x\, dy + z^2dz = \int_0^{2\pi} [\sin^2 t(-\sin t) + \cos t(\cos t)]dt = \int_0^{2\pi} (-\sin^3 t + \cos^2 t)dt$$
$$= \int_0^{2\pi} (-\sin^3 t + \tfrac{1}{2} + \tfrac{1}{2}\cos 2t)dt = 0 + \pi + 0 = \pi. \text{ Also, from Equation 14.5.9}$$
$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_S (1-2y)\mathbf{k}\cdot\mathbf{N}\, d\sigma = \iint_D (-2y)dA = \int_0^1\int_0^{2\pi} (1-2r\sin\theta)r\, d\theta\, dr = \int_0^1 (2\pi)r\, dr = \pi$$

20. $\mathbf{F}(x,y,z) = xy\mathbf{i} + y^2\mathbf{j} + 2\mathbf{k}$; S is the portion of the paraboloid $z = x^2+y^2$ below the plane $z = 1$.

▹ The figure shows the surface S and its projection D onto the xy plane. D is bounded by the circle $x^2+y^2 = 1$. The boundary of S is the circle C with center at the point $(0, 0, 1)$ and radius 1. A vector equation of C with arc length s as parameter is given by

$\mathbf{R}(s) = \cos s\mathbf{i} + \sin s\mathbf{j} + \mathbf{k} \quad 0 \leq s \leq 2\pi$



The unit tangent vector to the curve C is

$\mathbf{T}(s) = D_s\mathbf{R}(s) = -\sin s\mathbf{i} + \cos s\mathbf{j}$

Because

$\mathbf{F}(\mathbf{R}(s)) = \mathbf{F}(\cos s, \sin s, 1) = \cos s \sin s\mathbf{i} + \sin^2 s\mathbf{j} + 2\mathbf{k}$

then the left side of Stokes's theorem for the given **F** and S is

$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \int_0^{2\pi}(\cos s\sin s\mathbf{i} + \sin^2 s\mathbf{j} + 2\mathbf{k})\cdot(-\sin s\mathbf{i} + \cos s\mathbf{j})ds = \int_0^{2\pi}(-\sin^2 s\cos s + \sin^2 s\cos s)ds = 0$$

We evaluate the right side of Stokes's theorem for the given **F** and S.

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & y^2 & 2 \end{vmatrix} = -x\mathbf{k}$$

Thus,

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint_S -x\mathbf{k}\cdot\mathbf{n} = 0$$
because $-x$ is an odd function and S is symmetric with respect to the yz plane.
Because both sides of Stokes's theorem are zero, we have verified Stokes's theorem for the given $\mathbf{F}$ and S.

21. $\mathbf{F}(x,y,z) = -3y\mathbf{i} + 3x\mathbf{j} + 2\mathbf{k}$. $\mathbf{R}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + \mathbf{k}$.
$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C -3y\, dx + 3x\, dy + 2\, dz = \int_0^{2\pi}[-3(3\sin t)(-3\sin t) + 3(3\cos t)(3\cos t) + 2\cdot 0]dt$$
$$= \int_0^{2\pi} 27\, dt = 54\pi. \text{ Also } \iint_S \text{curl } \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_S 6\mathbf{k}\cdot\mathbf{k}\, d\sigma = 6\iint_S d\sigma = 6(9\pi) = 54\pi$$

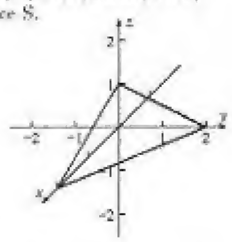
22. $\mathbf{F}(x,y,z) = 2z\mathbf{i} + 3x\mathbf{j} + 4z\mathbf{k}$. $\mathbf{R}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$.
$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C 2z\, dx + 3x\, dy + 4z\, dz = \int_0^{2\pi}[0 + 3(2\cos t)(2\cos t) + 0]dt = 6\int_0^{2\pi}(1+\cos 2t)dt = 12\pi$$
$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint_S (2\mathbf{j} + 3\mathbf{k})\cdot\mathbf{k}\, d\sigma = 3\iint_S d\sigma = 3\cdot\pi(2^2) = 12\pi$$

In Exercises 23–28, use Stokes's theorem to evaluate the line integral $\oint_C \mathbf{F}\cdot\mathbf{T}\, ds$ for $\mathbf{F}$ and C.

23. $\text{curl}(4y\mathbf{i} - 3z\mathbf{j} + x\mathbf{k}) = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$. $\mathbf{N} = \frac{1}{3}\sqrt{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$. The equilateral triangle has area $\frac{1}{2}\sqrt{3}$.
$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \iint_S \text{curl } \mathbf{F}\cdot\mathbf{N}\, d\sigma = \tfrac{1}{3}\sqrt{3}\iint_S (3-1-4)d\sigma = \tfrac{1}{3}\sqrt{3}(-2)\left(\tfrac{1}{2}\sqrt{3}\right) = -1$$

24. $\mathbf{F}(x,y,z) = (y-x)\mathbf{i} + (x-z)\mathbf{j} + (x-y)\mathbf{k}$; C is the triangle having vertices at $(2,0,0)$, $(0,2,0)$, and $(0,0,1)$.

▸ The figure shows the triangle. We calculate the curl of $\mathbf{F}$ before choosing the surface S.



$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-x & x-z & x-y \end{vmatrix} = -\mathbf{j}$$
To simplify the calculation, we choose S to consist of the three triangles S_x with normal $\mathbf{i}$, S_y with normal $\mathbf{j}$, and S_z with normal $\mathbf{k}$ cut off the coordinate planes by the given triangle. Applying Stokes's theorem, we have
$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint_{S_x} -\mathbf{j}\cdot\mathbf{i}\, d\sigma + \iint_{S_y} -\mathbf{j}\cdot\mathbf{j}\, d\sigma + \iint_{S_z} -\mathbf{j}\cdot\mathbf{k}\, d\sigma$$
$$= 0 - (\text{area of } S_x) + 0 = -1$$

25. $\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \iint_S \text{curl}(-y\mathbf{i} + x\mathbf{j} + z\mathbf{k})\cdot\mathbf{N}\, d\sigma = \iint_S 2\mathbf{k}\cdot\mathbf{k}\, d\sigma = 2\iint_S d\sigma = 2(4\pi) = 8\pi$

26. $\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \iint_S \text{curl}(yz\mathbf{i} + xy\mathbf{j} + xz\mathbf{k})\cdot\mathbf{n}\, d\sigma = \iint_S [(y-z)\mathbf{j} + (y-z)\mathbf{k}]\cdot\mathbf{k}\, d\sigma = \int_0^2\int_0^2 y\, dy\, dx = 4$

27. $P = (1,0,0)$, $Q = (0,1,0)$, $R = (0,0,2)$. C is triangle PQR.
$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \iint_S \text{curl}[(2y + \sin^{-1}x)\mathbf{i} + e^y\mathbf{j} + (x + \ln(z^2+4))\mathbf{k}]\cdot\mathbf{N}\, d\sigma = \iint_S (-\mathbf{j} - 2\mathbf{k})\cdot\mathbf{N}\, d\sigma$$
$$= \iint_{OPR} -1\, dA + \iint_{OPQ} -2\, dA = (-1)1 + (-2)\tfrac{1}{2} = -2$$

28. $\mathbf{F}(x,y,z) = (2z - e^x)\mathbf{i} + (x^3 + \sin y)\mathbf{j} + (y^2 - \tan z)\mathbf{k}$; C has vector equation $\mathbf{R}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}, 0 \le t \le 2\pi$.

▸
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z - e^x & x^3 + \sin y & y^2 - \tan z \end{vmatrix} = 2y\mathbf{i} + 2\mathbf{j} + 3x^2\mathbf{k}$$
Because C is in a plane parallel to the xy plane, $\mathbf{N} = \mathbf{k}$. Applying Stokes's theorem, we have
$$\oint_C \mathbf{F}\cdot\mathbf{T} ds = \iint_S \text{curl } \mathbf{F}\cdot\mathbf{N}\, d\sigma = \iint_S 3x^2\, dA$$
We switch to polar coordinates with $x = r\cos\theta$, $dA = r\, dr\, d\theta$. Thus
$$\oint_C \mathbf{F}\cdot\mathbf{T} ds = \int_0^{2\pi}\int_0^1 3(r\cos\theta)^2 r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 3r^3\cos^2\theta\, dr\, d\theta = \int_0^{2\pi}\tfrac{3}{4}\cos^2\theta\, d\theta$$
$$= \int_0^{2\pi}\tfrac{3}{8}(1+\cos 2\theta)\, d\theta = \tfrac{3}{8}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = \tfrac{3}{4}\pi$$

Miscellaneous Exercises for Chapter 14

In Exercises 1–4, determine if the vector is a gradient and if it is, find a function having the gradient.

1. We apply Theorem 14.1.1 to the vector $2xe^{x^2}\ln y\mathbf{i} + \frac{e^{x^2}}{y}\mathbf{j} = \mathrm{M}(x,y)\mathbf{i} + \mathrm{N}(x,y)\mathbf{j}$. Then
   $\mathrm{M}_y(x,y) = \frac{2x}{y}e^{x^2}$ and $\mathrm{N}_x(x,y) = \frac{2x}{y}e^{x^2}$. $\mathrm{M}_y = \mathrm{N}_x$, so the given vector is a gradient $\nabla f(x,y)$.
   $f_x(x,y) = 2xe^{x^2}\ln y$ and $f_y = \frac{e^{x^2}}{y}$. $f(x,y) = e^{x^2}\ln y + g(y)$; $f_y(x,y) = \frac{e^{x^2}}{y} + g'(y) = \frac{e^{x^2}}{y}$.
   $g'(y) = 0$; $g(y) = \mathrm{C}$. Therefore $f(x,y) = e^{x^2}\ln y + \mathrm{C}$.

2. $\mathrm{M} = e^x\tan y - \sec y$, $\mathrm{N} = e^x\sec^2 y - x\sec y\tan y$. $\mathrm{M}_y = e^x\sec^2 y - \sec y\tan y = \mathrm{N}_x$
   $f_x(x,y) = e^x\tan y - \sec y$, $f(x,y) = e^x\tan y - x\sec y + g(y)$; $f_y(x,y) = e^x\sec^2 y - x\sec y\tan y + g'(y) = \mathrm{N}$
   $g'(y) = 0$, $g(y) = \mathrm{C}$. $f(x,y) = e^x\tan y - x\sec y + \mathrm{C}$

3. We apply Theorem 14.1.2 to the vector $\left[\frac{-y}{(x+z)^2} + \frac{1}{x^2}\right]\mathbf{i} + \frac{1}{x+z}\mathbf{j} + \left[\frac{-y}{(x+z)^2} + \frac{2}{z^2}\right]\mathbf{k}$.
   Let $\mathrm{M}(x,y,z) = \frac{-y}{(x+z)^2} + \frac{1}{x^2}$, $\mathrm{N}(x,y,z) = \frac{1}{x+z}$, and $\mathrm{R}(x,y,z) = \frac{-y}{(x+z)^2} + \frac{2}{z^2}$.
   $\mathrm{M}_y(x,y,z) = \frac{-1}{(x+z)^2}$ $\qquad$ $\mathrm{N}_x(x,y,z) = \frac{-1}{(x+z)^2}$ $\qquad$ $\mathrm{R}_x(x,y,z) = \frac{2y}{(x+z)^3}$
   $\mathrm{M}_z(x,y,z) = \frac{2y}{(x+z)^3}$ $\qquad$ $\mathrm{N}_z(x,y,z) = \frac{-1}{(x+z)^2}$ $\qquad$ $\mathrm{R}_y(x,y,z) = \frac{-1}{(x+z)^2}$
   Because $\mathrm{M}_y = \mathrm{N}_x$, $\mathrm{M}_z = \mathrm{R}_x$, and $\mathrm{N}_z = \mathrm{R}_y$, the given vector is a gradient $\nabla f(x,y,z)$ where
   $f_x(x,y,z) = \frac{-y}{(x+z)^2} + \frac{1}{x^2}$ $\qquad$ $f_y(x,y,z) = \frac{1}{x+z}$ $\qquad$ $f_z(x,y,z) = \frac{-y}{(x+z)^2} + \frac{2}{z^2}$
   From f_x, $f(x,y,z) = \frac{y}{x+z} - \frac{1}{x} + g(y,z)$; $f_y(x,y,z) = \frac{1}{x+z} + g_y(y,z) = \frac{1}{x+z}$. $g_y(y,z) = 0$;
   $g(y,z) = \mathrm{h}(z)$. Hence $f(x,y,z) = \frac{y}{x+z} - \frac{1}{x} + \mathrm{h}(z)$. $f_z(x,y,z) = \frac{-y}{(x+z)^2} + h'(z) = \frac{-y}{(x+z)^2} + \frac{2}{z^2}$;
   $h'(z) = \frac{2}{z^2}$; $\mathrm{h}(z) = -\frac{2}{z} + \mathrm{C}$. Therefore $f(x,y,z) = \frac{y}{x+z} - \frac{1}{x} - \frac{2}{z} + \mathrm{C}$.

4. $y(\cos x - z\sin x)\mathbf{i} + z(\cos x + \sin y)\mathbf{j} - (\cos y - y\cos x)\mathbf{k}$

▷ Let
   $\mathrm{M}(x,y,z) = y\cos x - yz\sin x$ $\quad$ $\mathrm{N}(x,y,z) = z\cos x + z\sin y$ $\quad$ $\mathrm{R}(x,y,z) = -\cos y + y\cos x$
   We have
   $\mathrm{M}_y(x,y,z) = \cos x - z\sin x$ and $\mathrm{N}_x(x,y,z) = -z\sin x$
   Because $\mathrm{M}_y(x,y,z) \neq \mathrm{N}_x(x,y,z)$, then the vector is not a gradient.

In Exercises 5 and 6, find a conservative vector field having the potential function f.

5. (a) $\mathbf{F}(x,y) = \nabla f = \nabla(2x^2y + 3xy^3) = (4xy + 3y^3)\mathbf{i} + (2x^2 + 9xy^2)\mathbf{j}$
   (b) $\mathbf{F}(x,y,z) = \nabla f = \nabla(xe^y - yze^y) = e^y\mathbf{i} + (xe^y - yze^y - ze^y)\mathbf{j} - ye^y\mathbf{k}$

6. (a) $\mathbf{F}(x,y) = \nabla f = \nabla(e^x\cos y + x\sin y) = (e^x\cos y + \sin y)\mathbf{i} + (-e^x\sin y + x\cos y)\mathbf{j}$
   (b) $\mathbf{F}(x,y,z) = \nabla f = \nabla(x^2 + y^2 + z^2)^{-1} = -2(x^2 + y^2 + z^2)^{-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

In Exercises 7–10, prove that the vector field $\mathbf{F}$ is conservative, and find a potential function.

7. $\mathbf{F}(x,y) = \frac{2y^2}{1+4x^2y^4}\mathbf{i} + \frac{4xy}{1+4x^2y^4}\mathbf{j}$ is conservative because $\mathbf{F} = \nabla\phi$ with $\phi(x,y) = \tan^{-1}2xy^2 + \mathrm{C}$.

8. $\mathbf{F}(x,y,z) = (6x - 4y)\mathbf{i} + (z - 4x)\mathbf{j} + (y - 8z)\mathbf{k}$

▷ Let
   $\mathrm{M}(x,y,z) = 6x - 4y$ $\qquad$ $\mathrm{N}(x,y,z) = z - 4x$ $\qquad$ $\mathrm{R}(x,y,z) = y - 8z$
   Then
   $\mathrm{M}_y(x,y,z) = -4$ $\qquad$ $\mathrm{N}_x(x,y,z) = -4$ $\qquad$ $\mathrm{R}_x(x,y,z) = 0$
   $\mathrm{M}_z(x,y,z) = 0$ $\qquad$ $\mathrm{N}_z(x,y,z) = 1$ $\qquad$ $\mathrm{R}_y(x,y,z) = 1$
   Because $\mathrm{M}_y = \mathrm{N}_x$, $\mathrm{M}_z = \mathrm{R}_x$, and $\mathrm{N}_z = \mathrm{R}_y$, then $\mathbf{F}$ is the gradient of some scalar field, and thus $\mathbf{F}$ is conservative. If ϕ is a potential function, then
   $\phi_x(x,y,z) = \mathrm{M}(x,y,z) = 6x - 4y$
   Integrating with respect to x, we get
   $\phi(x,y,z) = 3x^2 - 4xy + g(y,z)$

Partial-differentiating with respect to y, we have

$\phi_y(x,y,z) = -4x + g_y(y,z)$

Because $\phi_y(x,y,z) = \mathrm{N}(x,y,z)$, then

$$\begin{aligned} -4x + g_y(y,z) &= z - 4x \\ g_y(y,z) &= z \\ g(y,z) &= yz + h(z) \end{aligned}$$

and

$\phi(x,y,z) = 3x^2 - 4xy + yz + h(z)$

Partial-differentiating with respect to z, we obtain

$\phi_z(x,y,z) = y + h'(z)$

Because $\phi_z(x,y,z) = \mathrm{R}(x,y,z)$, then

$$\begin{aligned} y + h'(z) &= y - 8z \\ h'(z) &= -8z \\ h(z) &= -4z^2 + C \end{aligned}$$

and a potential function is

$\phi(x,y,z) = 3x^2 - 4xy + yz - 4z^2 + C$

9. $\mathbf{F}(x,y,z) = z^2\sec^2 x\mathbf{i} + 2ye^{3z}\mathbf{j} + (3y^2 + 2z\tan x)\mathbf{k}$ is conservative: $\mathbf{F} = \nabla\phi$ with $\phi(x,y,z) = z^2\tan x + y^2e^{3z} + C$.

10. $\mathbf{F}(x,y) = (y\sin x - \sin y)\mathbf{i} - (x\cos y + \cos x)\mathbf{j} = \nabla(-y\cos x - x\sin y)$

In Exercises 11–14, find curl F and div F.

11. $\mathbf{F}(x,y,z) = e^{yz}\mathbf{i} + e^{xz}\mathbf{j} + e^{xy}\mathbf{k}$. $\text{div } \mathbf{F} = \frac{\partial}{\partial x}(e^{yz}) + \frac{\partial}{\partial y}(e^{xz}) + \frac{\partial}{\partial z}(e^{xy}) = 0$.

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{yz} & e^{xz} & e^{xy} \end{vmatrix} = (xe^{xy} - xe^{xz})\mathbf{i} + (ye^{yz} - ye^{xy})\mathbf{j} + (ze^{xz} - ze^{yz})\mathbf{k}$$

12. $\mathbf{F}(x,y) = \sin y\mathbf{i} + \sin x\mathbf{k}$

▷ $$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin y & 0 & \sin x \end{vmatrix} = -\cos x\mathbf{j} - \cos y\mathbf{k}$$

and

$$\text{div } \mathbf{F} = D_x(\sin y) + D_y(0) + D_z(\sin x) = 0$$

13. $\mathbf{F}(x,y) = \frac{1}{y}\mathbf{i} - \frac{2x}{y}\mathbf{j}$. $\text{curl } \mathbf{F} = \left[\frac{\partial}{\partial x}\left(-\frac{2x}{y}\right) - \frac{\partial}{\partial y}\left(\frac{1}{y}\right)\right]\mathbf{k} = \left(-\frac{2}{y} + \frac{1}{y^2}\right)\mathbf{k}$. $\text{div } \mathbf{F} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right) + \frac{\partial}{\partial y}\left(-\frac{2x}{y}\right) = \frac{2x}{y^2}$

14. $\mathbf{F}(x,y,z) = \frac{x}{y}\mathbf{i} + \frac{y}{z}\mathbf{j} + \frac{z}{x}\mathbf{k}$. $\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/y & y/z & z/x \end{vmatrix} = \frac{y}{z^2}\mathbf{i} + \frac{z}{x^2}\mathbf{j} + \frac{x}{y^2}\mathbf{k}$. $\text{div } \mathbf{F} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{\partial}{\partial y}\left(\frac{y}{z}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x}\right) = \frac{1}{y} + \frac{1}{z} + \frac{1}{x}$

In Exercises 15–22, evaluate the line integral over curve C.

15. $\mathbf{F}(x,y) = 3y\mathbf{i} - 4x\mathbf{j}$; $\mathbf{R}(t) = 2t^2\mathbf{i} - t\mathbf{j}$. $\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \langle 3(-t), -4(2t^2)\rangle\cdot\langle 4t, -1\rangle dt = \int_0^1 -4t^2 dt = -\frac{4}{3}$.

16. $\int_C \mathbf{F}\cdot d\mathbf{R}$: $\mathbf{F}(x,y) = (x+y)\mathbf{i} + (y-x)\mathbf{j}$; C: $\mathbf{R}(t) = t^3\mathbf{i} + t^2\mathbf{j}$ from the point $(8,4)$ to the point $(1,1)$.

▷ Because $\mathbf{R}(2) = 8\mathbf{i} + 4\mathbf{j}$, then $t = 2$ at the point $(8,4)$. Because $\mathbf{R}(1) = \mathbf{i} + \mathbf{j}$, then $t = 1$ at the point $(1,1)$. Furthermore,

$\mathbf{F}(\mathbf{R}(t)) = \mathbf{F}(t^3, t^2) = (t^3 + t^2)\mathbf{i} + (t^2 - t^3)\mathbf{j}$

and

$\mathbf{R}'(t) = 3t^2\mathbf{i} + 2t\mathbf{j}$

Therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_2^1 [(t^3 + t^2)\mathbf{i} + (t^2 - t^3)\mathbf{j}]\cdot(3t^2\mathbf{i} + 2t\mathbf{j})dt = \int_2^1 (3t^5 + 3t^4 + 2t^3 - 2t^4)dt = \tfrac{1}{2}t^6 + \tfrac{1}{5}t^5 + \tfrac{1}{2}t^4\Big]_2^1 = -\tfrac{226}{5}$$

17. $\mathbf{R}(t) = 4\sin t\mathbf{i} - \cos t\mathbf{j}$, $0 \le t \le \frac{1}{2}\pi$. $\int_C (2x + 3y)dx + xy\,dy$

$$\begin{aligned} &= \int_0^{\pi/2} [2(4\sin t) + 3(-\cos t)]4\cos t\,dt + (4\sin t)(-\cos t)(\sin t\,dt) \\ &= \int_0^{\pi/2} (32\sin t\cos t - 12\cos^2 t - 4\sin^2 t\cos t)dt = 16\sin^2 t - 12\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) - \tfrac{4}{3}\sin^3 t\Big]_0^{\pi/2} \\ &= 16 - 3\pi - \tfrac{4}{3} = \tfrac{44}{3} - 3\pi \end{aligned}$$

18. $x = 3\sin t,\ y = 3\cos t.\ \int_C (2x+y)dx + (x-2y)dy = \int_0^{2\pi} [(6\sin t + 3\cos t)3\cos t + (3\sin t - 6\cos t)(-3\sin t)]dt$

$= \int_0^{2\pi} (36 \sin t \cos t + 9\cos^2 t - 9\sin^2 t)dt = 0$

19. $\mathbf{R}(t) = (t-1)\mathbf{i} + (t+1)\mathbf{j} + t^2\mathbf{k},\ 0 \le t \le 1.\ \int_C y^2 dx + z^2 dy + x^2 dz$

$= \int_0^1 [(t+1)^2 + (t^2)^2 + (t-1)^2 \cdot 2t]dt = \int_0^1 (t^4 + 2t^3 - 3t^2 + 4t + 1)dt = \frac{1}{5}t^5 + \frac{1}{2}t^4 - t^3 + 2t^2 + t\Big]_0^1 = \frac{27}{10}$

20. $\int_C xe^y dx - xe^z dy + e^z dz$; C is $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, for $0 \le t \le 1$.

▹ Parametric equations of C are

$x = t \quad y = t^2 \quad z = t^3 \quad \text{for } 0 \le t \le 1$

Therefore,

$$\int_C xe^y dx - xe^z dy + e^z dz = \int_0^1 [te^{t^2}dt - te^{t^3}(2t\,dt) + e^{t^3}(3t^2 dt)] = \tfrac{1}{2}e^{t^2} - \tfrac{2}{3}e^{t^3} + e^{t^3}\Big]_0^1$$
$$= \tfrac{1}{2}(e-1) - \tfrac{2}{3}(e-1) + (e-1) = \tfrac{5}{6}(e-1)$$

21. $\mathbf{F}(x,y,z) = 3xy\mathbf{i} + (4y^2 - xz)\mathbf{j} + 6z\mathbf{k}$; $\mathbf{R}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},\ 0 \le t \le 1$.

$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 \langle 3t(t^2), 4t^4 - t^4, 6t^3\rangle \cdot \langle 1, 2t, 3t^2\rangle dt = \int_0^1 (3t^3 + 24t^5)dt = \frac{3}{4}t^4 + 4t^6\Big]_0^1 = \frac{19}{4}$

22. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + z\mathbf{k}.\ \int_C \mathbf{F} \cdot d\mathbf{R} = \int_{(2,0,0)}^{(2,0,2\pi)} 2x\,dx + 3y\,dy + z\,dz = x^2 + \frac{3}{2}y^2 + \frac{1}{2}z^2\Big]_{(2,0,0)}^{(2,0,2\pi)} = 2\pi^2$

In Exercises 23–30, prove that the value of the line integral is independent of the path, and compute the value in any convenient manner. In each exercise C is any sectionally smooth curve from point A to point B.

23. $\mathbf{F}(x,y) = 2xe^y\mathbf{i} + x^2e^y\mathbf{j} = \nabla\phi$ with $\phi(x,y) = x^2e^y$. $\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(3,2) - \phi(1,0) = 9e^2 - 1$.

24. $\int_C \left(\frac{1}{y} - y\right)dx + \left(-\frac{x}{y^2} - x\right)dy$; A is $(0,1)$ and B is $(6,3)$.

▹ Let

$$M(x,y) = \frac{1}{y} - y \qquad N(x,y) = -\frac{x}{y^2} - x$$
$$M_y(x,y) = -\frac{1}{y^2} - 1 \qquad N_x(x,y) = -\frac{1}{y^2} - 1$$

Because $M_y(x,y) = N_x(x,y)$, then the line integral is independent of the path. We take a path from A to B consisting of two parts. Let C_1 be the line segment from $A(0,1)$ to $D(0,3)$, and let C_2 be the line segment from $D(0,3)$ to $B(6,3)$. Because $x = 0$ at every point on C_1,

$$\int_{C_1} \left(\frac{1}{y} - y\right)dx + \left(-\frac{x}{y^2} - x\right)dy = 0$$

At every point on C_2 we have $y = 3$ and $dy = 0$ and $0 \le x \le 6$. Thus,

$$\int_{C_1} \left(\frac{1}{y} - y\right)dx + \left(-\frac{x}{y^2 - x}\right)dy = \int_0^6 (\tfrac{1}{3} - 3)dx = -16$$

By adding the two parts of the line integral, we have

$$\int_C \left(\frac{1}{y} - y\right)dx + \left(-\frac{x}{y^2} - x\right)dy = 0 + (-16) = -16$$

Alternatively, by inspection

$$\int_C \left(\frac{1}{y} - y\right)dx + \left(-\frac{x}{y^2} - x\right)dy = \int_{(0,1)}^{(6,3)} d\left(\frac{x}{y} - xy\right) = \frac{x}{y} - xy\Big]_{(0,1)}^{(6,3)} = [\tfrac{6}{3} - 6(3)] - 0 = -16$$

25. $\mathbf{F}(x,y) = (\cos y - y\cos x)\mathbf{i} - (\sin x + x\sin y)\mathbf{j} = \nabla\phi$ with $\phi(x,y) = x\cos y - y\sin x$.

$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(\pi,0) - \phi(0,\frac{1}{2}\pi) = \pi$

26. $\int_C [(2xy - 2y)\mathbf{i} + (x^2 - 2x + 3y^2)\mathbf{j}] \cdot d\mathbf{R} = x^2y - 2xy + y^3\Big]_{(2,-1)}^{(3,2)} = (18 - 12 + 8) - (-4 + 4 - 1) = 15$

27. $\mathbf{F}(x,y,z) = 3y\mathbf{i} + (3x + 4y)dy\mathbf{j} - 2z\mathbf{k} = \nabla\phi$ with $\phi(x,y,z) = 3xy + 2y^2 - z^2$.

$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1,2,0) - \phi(0,1,-1) = 14 - 1 = 13$

28. $\int_C z \sin y\,dx + xz\cos y\,dy + x\sin y\,dz$; A is $(0,0,0)$ and B is $(3,3,\frac{1}{2}\pi)$.

▸ Because integrating $z \sin y$ with respect to x, integrating $xz \cos y$ with respect to y, and integrating $x \sin y$ with respect to z all give $xz \sin y$, we have

$$\int_C z\sin y\,dx + xz\cos y\,dy + x\sin y\,dz = \int_{(0,0,0)}^{(3,3,\pi/2)} d(xz\sin y) = xz\sin y\Big]_{(0,0,0)}^{(2,3,\pi/2)}$$
$$= 2\cdot\tfrac{1}{2}\pi\sin 3 - 0 = \pi\sin 3$$

29. $\mathbf{F}(x,y,z) = \left(\frac{1}{y} - \frac{2z}{x^2}\right)\mathbf{i} - \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathbf{j} + \left(\frac{2}{x} + \frac{y}{z^2}\right)\mathbf{k} = \nabla\phi$ with $\phi(x,y,z) = \frac{x}{y} + \frac{2z}{x} - \frac{y}{z}$.

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \phi(4,2,-2) - \phi(2,-1,1) = (2 - 1 + 1) - (-2 + 1 - 1) = 4$$

30. $\int_C [(2xy + 3yz)\mathbf{i} + (x^2 - 4yz + 3xz)\mathbf{j} + (3xy - 2y^2)\mathbf{k}]\cdot d\mathbf{R} = x^2y - 2y^2z + 3xyz\Big]_{(0,2,1)}^{(1,-1,4)} = -13$

In Exercises 31–34, use Green's theorem to evaluate the line integral.

31. The ellipse $x^2/3^2 + y^2/4^2 = 1$ has area $\pi(3)(4) = 12\pi$.

$$\oint_C (3x + 2y)dx + (3x + y^2)dy = \iint_R \left[\frac{\partial}{\partial x}(3x + y^2) - \frac{\partial}{\partial y}(3x + 2y)\right]dA = \iint_R dA = 12\pi$$

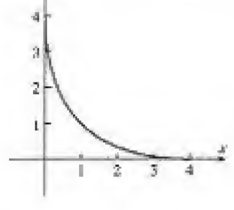
32. $\oint_C \ln(y+1)dx - \frac{xy}{y+1}dy$, where C is the closed curve determined by the curve $\sqrt{x} + \sqrt{y} = 2$ and the intervals $[0,4]$ on the x and y axes.

▸ The figure shows the region R, bounded below by the x axis and bounded above by the parabola (A.10.35) $y = (2 - \sqrt{x})^2$, $0 \le x \le 4$. Because

$\mathrm{M}(x,y) = \ln(y+1)$ and $\mathrm{N}(x,y) = -\frac{xy}{y+1}$

then

$\mathrm{N}_x(x,y) - \mathrm{M}_y(x,y) = \frac{-y}{y+1} - \frac{1}{y+1} = -1$



Applying Green's theorem (14.4.1), we have

$$\oint_C \ln(y+1)dx - \frac{xy}{y+1}dy = \iint_R [\mathrm{N}_x(x,y) - \mathrm{M}_y(x,y)]dA = \iint_R -dA$$
$$= -\int_0^4\int_0^{(2-\sqrt{x})^2} dy\,dx = -\int_0^4 (4 - 4\sqrt{x} + x)dx = -\left[4x - \tfrac{8}{3}x^{3/2} + \tfrac{1}{2}x^2\right]_0^4 = -\tfrac{8}{3}$$

33. $\oint_C e^x\sin y\,dx + e^x\cos y\,dy = \iint_R \left[\frac{\partial}{\partial x}(e^x\cos y) - \frac{\partial}{\partial y}(e^x\sin y)\right]dA = \iint_R (e^x\cos y - e^x\cos y)dA = 0$

34. $\oint_C (x^2 - y^3)dx + (y^2 + x^3)dy = \iint_R [\mathrm{D}_x(y^2 + x^3) - \mathrm{D}_y(x^2 - y^3)]dA = \iint_R (3x^2 + 3y^2)dA$

$$= 3\int_0^{2\pi}\int_0^1 r^2 r\,dr\,d\theta = 6\pi\cdot\tfrac{1}{4}r^4\Big]_0^1 = \tfrac{3}{2}\pi$$

In Exercises 35 and 36, use Theorem 14.4.2 to find the area of the region.

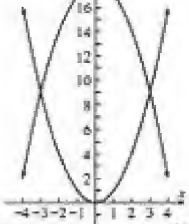
35. $x^2 = x + 2$ gives $x = -1, 2$. C_1: $y = x^2$, $-\mathrm{C}_2$: $y = x + 2$, $-1 \le x \le 2$. By Theorem 14.4.2,

$$\mathrm{A} = \tfrac{1}{2}\oint_C x\,dy - y\,dx = \tfrac{1}{2}\int_{-1}^{2} x(2x\,dx) - x^2dx - \tfrac{1}{2}\int_{-1}^{2} x\,dx - (x+2)dx = \tfrac{1}{2}\int (x^2 + 2)dx = \tfrac{1}{2}\left[\tfrac{1}{2}x^3 + 2x\right]_{-1}^{2} = \tfrac{9}{2}$$

36. The region enclosed by the two parabolas $y = x^2$ and $x^2 = 18 - y$.

▸ The figure shows the region. The boundary C is the union of C_1 and C_2, where C_1 is the parabola $y = x^2$ from $(-3,9)$ to $(3,9)$, and C_2 is parabola $y = 18 - x^2$ from $(3,9)$ to $(-3,9)$. By Theorem 14.4.2 the area of the region enclosed by C is

$$\mathrm{A} = \tfrac{1}{2}\oint_C x\,dy - y\,dx = \tfrac{1}{2}\left[\oint_{C_1} x\,dy - y\,dx + \oint_{C_2} x\,dy - y\,dx\right] \quad (1)$$



Because $y = x^2$ on C_1 and $-3 \le x \le 3$, then $dy = 2x$, and

$$\oint_{C_1} x\,dy - y\,dx = \int_{-3}^{3}[x(2x\,dx) - x^2dx] = \tfrac{1}{3}x^3\Big]_{-3}^{3} = 18 \quad (2)$$

Because $y = 18 - x^2$ for C_2 and C_2 is from $(3,9)$ to $(-3,9)$, then

$$\oint_{C_2} x\,dy - y\,dx = \int_{3}^{-3}[x(-2x)dx - (18 - x^2)dx] = \int_{-3}^{3}(x^2 + 18)dx = \tfrac{1}{3}x^3 + 18x\Big]_{-3}^{3} = 126 \quad (3)$$

Substituting from (2) and (3) into (1), we get

$$\mathrm{A} = \tfrac{1}{2}(18 + 126) = 72$$

The area of the region is 72 square units.

In Exercises 37–40, find the total work done in moving an object along C if the motion is caused by the force field. Length is measured in meters and force is measured in newtons.

37. $\mathbf{F}(x,y) = 2x^2y\mathbf{i} + (x^2+3y)\mathbf{j}$; $y = 3x^2+2x+4$, $0 \le x \le 1$. If W joules work is done

$$W = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \langle 2x^2(3x^2+2x+4), x^2+3(3x^2+2x+4)\rangle \cdot \langle 1, 6x+2\rangle dx$$
$$= \int_0^1 (6x^4+64x^3+64x^2+84x+24)dx = \tfrac{6}{5}t^5 + 16t^4 + \tfrac{64}{3}t^3 + 42t^2 + 24t\Big]_0^1 = \tfrac{1568}{15}$$

38. $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le \frac{1}{2}\pi$. $W = \int_C xy^2dx - x^2y\,dy = \int_0^{\pi/2} 16[\cos t\sin^2 t(-\sin t) - \cos^2 t\sin t(\cos t)]dt$

$$= -16\int_0^{\pi/2}(\sin^3 t\cos t + \cos^3 t\sin t)dt = -4\Big[\sin^4 t - \cos^4 t\Big]_0^{\pi/2} = -8 \text{ joules}$$

39. $\mathbf{F}(x,y,z) = (xy-z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. $\mathbf{R}(t) = 4t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$. If W joules work is done

$$W = \int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^1 \langle 4t^2-2t, t, 2t\rangle\cdot\langle 4,1,2\rangle dt = \int_0^1 (16t^2-3t)dt = \tfrac{16}{3}t^3 - \tfrac{3}{2}t^2\Big]_0^1 = \tfrac{23}{6}$$

40. $\mathbf{F}(x,y,z) = xyz\mathbf{i} + e^y\mathbf{j} + (x+z)\mathbf{k}$; C: $\mathbf{R}(t) = 3t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$; $0 \le t \le 3$

▹ If W joules is the work, then

$$W = \int_0^3 \mathbf{F}(3t,t^2,2t)\cdot d\mathbf{R}(t) = \int_0^3 (6t^4\mathbf{i} + e^{t^2}\mathbf{j} + 5t\mathbf{k})\cdot(3\mathbf{i}+2t\mathbf{j}+2\mathbf{k})dt = \int_0^3 (18t^4 + 2te^{t^2} + 10t)dt$$
$$= \tfrac{18}{5}t^5 + e^{t^2} + 5t^2\Big]_0^3 = [\tfrac{18}{5}(3^5) + e^9 + 5(3^2)] - e^0 = \tfrac{4594}{5} + e^9$$

The work is approximately 9022 joules.

In Exercises 41 and 42, verify Gauss's divergence theorem and Stokes's theorem in the plane for **F** and R.

41. $\mathbf{F}(x,y) = 4y\mathbf{i} + 3x\mathbf{j}$. C: $\mathbf{R}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j}$.

(a) $\oint_C \mathbf{F}\cdot\mathbf{N}\,ds = \oint_C -3x\,dx + 4y\,dy = \int_0^{2\pi}[-3\cos^3 t\,\frac{d}{dt}(\cos^3 t) + \sin^3 t\,\frac{d}{dt}(\sin^3 t)]dt$

$= -\frac{3}{2}(\cos^3 t)^2 + 2(\sin^3 t)^2\Big]_0^{2\pi} = 0$ and $\iint_R \operatorname{div}\mathbf{F}\,dA = \iint_R 0\,dA = 0$

(b) $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C 4y\,dx + 3x\,dy = \int_0^{2\pi}[4\sin^3 t(-3\cos^2 t\sin t) + 3\cos^3 t(3\sin^2 t\cos t)]dt$

$= \int_0^{2\pi} 3\sin^2 t\cos^2 t(3\cos^2 t - 4\sin^2 t)dt = \frac{3}{8}\int_0^{2\pi}\sin^2 2t(7\cos 2t - 1)dt$

$= \frac{21}{8}\int_0^{2\pi}\sin^2 2t\cos 2t\,dt - \frac{3}{16}\int_0^{2\pi}(1-\cos 4t)dt = 0 - \frac{3}{16}\cdot 2\pi = -\frac{3}{8}\pi$. Also, by Exercise 14.4.25 with $a = 1$,

R has area $\frac{3}{8}\pi$. $\iint_R \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dA = \iint_R(-\mathbf{k})\cdot\mathbf{k}\,dA = -\iint_R dA = -\frac{3}{8}\pi$.

42. $\mathbf{F}(x,y) = 3x^2\mathbf{i} + 4y^2\mathbf{j}$. $x = 4\cos t$, $y = 3\sin t$.

(a) $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C -4y^2dx + 3x^2dy = \int_0^{2\pi}[-36\sin^2 t(-4\sin t) + 48\cos^2 t(3\cos t)]dt = 0$ (odd power)

$\iint_R \operatorname{div}(3x^2\mathbf{i} + 4y^2\mathbf{j})dA = \iint_R (6x+8y)dA = 0$

(b) $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C 3x^2dx + 4y^2dy = \oint_C d(x^3 + \frac{4}{3}y^3) = 0$. $\iint_R \operatorname{curl}(3x^2\mathbf{i} + 4y^2\mathbf{j})dA = \iint_R 0\,dA = 0$

In Exercises 43 and 44, use Green's theorem to find the total work done in moving an object in the counterclockwise direction once around C if the motion is caused by the force field $\mathbf{F}(x,y)$. Length is measured in meters and force is measured in newtons.

43. $W = \oint_C \mathbf{F}\cdot d\mathbf{R} = \iint_R\left[\frac{\partial}{\partial x}(x^2+e^y) - \frac{\partial}{\partial y}(xy^2+\cos x)\right]dA = \iint_R(2x-2xy)dA = 0$ (odd function of x)

44. C is the ellipse $9x^2 + y^2 = 9$; $\mathbf{F}(x,y) = (2x-3y)\mathbf{i} + (x+2y)\mathbf{j}$.

▹ The area of the the region R enclosed by the ellipse is $\pi ab = \pi(1)(3) = 3\pi$. Let

$M(x,y) = 2x - 3y \quad N(x,y) = x + 2y$

If W joules is the total work done, then by Green's theorem

$$W = \oint_C M(x,y)dx + N(x,y)dy = \iint_R[N_x(x,y) - M_y(x,y)]dA = \iint_R[1-(-3)]dA = 4\iint_R dA$$
$$= 4(3\pi) = 12\pi$$

The total work is 12 joules.

45. $\mathbf{F} = (4x - 3y)\mathbf{i} + (5y - 4x^2)\mathbf{j}$. The triangle $(0,1)$, $(0,4)$, $(4,4)$ has area $\frac{1}{2} \cdot 3 \cdot 4 = 6$. By the divergence theorem the number of cm^2/sec in the flow is $\oint_C \mathbf{F} \cdot \mathbf{N}\, d\sigma = \iint_R \text{div}\, \mathbf{F}\, dA = \iint_R (4+5)dA = 9 \cdot 6 = 54$

46. C: $\frac{x^2}{16} + \frac{y^2}{4} = 1$. $\oint_C \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R \text{div}[(y^2 + 12x)\mathbf{i} + (4y - x^2)\mathbf{j}]dA = \iint_R (12+4)dA = 16 \cdot \pi(4)(2) = 128\pi$

47. $\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{y}{x^2+y^2}\mathbf{j} = \nabla\phi$ with $\phi(x,y) = -\tan^{-1}\frac{y}{x}$ ($\phi(x,y) = \tan^{-1}\frac{x}{y}$ is not defined on the x axis).

$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(-\sqrt{2}, \sqrt{2}) - \phi(\sqrt{2}, \sqrt{2}) = \frac{1}{4}\pi - (-\frac{1}{4}\pi) = \frac{1}{2}\pi$.

48. Apply Green's theorem to compute the area of the quadrilateral having vertices $(0,0), (3,2), (1,5)$, and $(-2,1)$.

▷ Let C be the boundary of the given quadrilateral region R. We choose functions M and N so that $N_x(x,y) - M_y(x,y) = 1$. Then the right side of Green's theorem will give the measure of the area of R, and we may calculate the area by evaluating the integral on the left side of Green's theorem. There are many possible choices for the functions M and N. We let $M(x,y) = 0$ and $N(x,y) = x$. The area of the region R is given by

$$A = \oint_C M(x,y)dx + N(x,y)dy = \oint_C x\, dy$$

The closed curve C consists of four pieces.

We evaluate $\int_C x\, dy$ where C is the segment from (x_1, y_1) to (x_2, y_2). If m is the slope of the segment, then $dy/dx = m$ and $dy = m\, dx$. Thus

$$\int_C x\, dy = \int_{x_1}^{x_2} \frac{y_2 - y_1}{x_2 - x_1}\, x\, dx = \frac{1}{2} \cdot \frac{y_2 - y_1}{x_2 - x_1} x^2 \Big]_{x_1}^{x_2} = \frac{1}{2} \cdot \frac{y_2 - y_1}{x_2 - x_1}({x_2}^2 - {x_1}^2) = \tfrac{1}{2}(y_2 - y_1)(x_2 + x_1) \qquad (1)$$

Applying (1) to the segment from $(0,0)$ to $(3,2)$, we get

$$\int_{C_1} x\, dy = \tfrac{1}{2}(2-0)(3+0) = 3$$

Applying (1) to the segment from $(3,2)$ to $(1,5)$, we get

$$\int_{C_2} x\, dy = \tfrac{1}{2}(5-2)(1+3) = 6$$

Applying (1) to the segment from $(1,5)$ to $(-2,1)$, we get

$$\int_{C_3} x\, dy = \tfrac{1}{2}(1-5)(-2+1) = 2$$

Applying (1) to the segment from $(-2,1)$ to $(0,0)$, we get

$$\int_{C_4} x\, dy = \tfrac{1}{2}(0-1)(0-2) = 1$$

Therefore,

$$A = \oint_C x\, dy = \oint_{C_1} x\, dy + \oint_{C_2} x\, dy + \oint_{C_3} x\, dy + \oint_{C_4} x\, dy = 3 + 6 + 2 + 1 = 12$$

The area of the quadrilateral is 12 square units.

49. $f(x,y) = z = 3x + 2y$. $f_x(x,y) = 3$, $f_y(x,y) = 2$. Triangle D has side $3x + 2y = 6$.

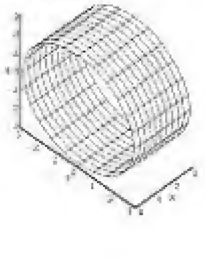
$\iint_S xy\, d\sigma = \iint_D xy\sqrt{2^2 + 3^2 + 1}\, dA = \sqrt{14}\int_0^2\int_0^{(6-3x)/2} xy\, dy\, dx = \sqrt{14}\int_0^2 \frac{1}{2}xy^2\Big]_0^{(6-3x)/2} dx$

$= \frac{1}{8}\sqrt{14}\int_0^2 x(6-3x)^2 dx = \frac{1}{8}\sqrt{14}\Big[-\frac{1}{9}x(6-3x)^3 - \frac{1}{108}(6-3x)^4\Big]_0^2 = \frac{1}{8}\sqrt{14} \cdot \frac{6^4}{108} = \frac{3}{2}\sqrt{14}$

50. See Exercise 14.5.6

51. $f(x,y) = z = 9 - x^2$. $f_x(x,y) = -2x$, $f_y(x,y) = 0$. $\iint_S x\, d\sigma = \iint_D x\sqrt{4x^2+1}\, dA = \frac{1}{8}\int_0^2\int_0^3 8x(4x^2+1)^{1/2} dx\, dy$

$= \frac{1}{8}\int_0^2 \frac{2}{3}(4x^2+1)^{3/2}\Big]_0^3 dy = \frac{1}{12}\int_0^2 [(37)^{3/2} - 1]dy = \frac{1}{6}[(37)^{3/2} - 1]$



52. Evaluate the surface integral $\iint_S xyz\, d\sigma$, where S is the portion of the cylinder $y^2 + z^2 = 9$ between the planes $x = 1$ and $x = 4$.

▷ The surface is shown in the figure. Because the surface is symmetric with respect to the xz plane $y = 0$, and the integrand xyz is an odd function of y, then the value of the integral is 0.

53. $\rho(x,y,z) = kz^2$, $f(x,y) = z = \sqrt{4-x^2-y^2}$, $f_x = \dfrac{-x}{\sqrt{4-x^2-y^2}}$, $f_y = \dfrac{-y}{\sqrt{4-x^2-y^2}}$. The mass is M kg.

$$M = \iint_S \rho(x,y,z)d\sigma = \iint_D k(4-x^2-y^2)\sqrt{\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}+1}\,dA = 2k\iint_D \sqrt{4-x^2-y^2}\,dA$$
$$= 2k\int_0^{\pi/2}\int_0^2 r\cdot r\,dr\,d\theta = 2k\int_0^{\pi/2}\frac{8}{3}\,d\theta = 2k\cdot\frac{\pi}{2}\cdot\frac{8}{3} = \frac{8}{3}k\pi$$

54. $z = \sqrt{4-x^2-y^2}$. $M = \iint_S (4-z)d\sigma = 4\iint_S d\sigma - \iint_D \sqrt{4-x^2-y^2}\sqrt{\dfrac{x^2}{4-x^2-y^2}+\dfrac{y^2}{4-x^2-y^2}+1}\,dA$

$$= 4\iint_S d\sigma - 2\iint_D dA = 4\cdot 2\pi(2^2) - 2\cdot\pi(2)^2 = 24\pi$$

55. $f(x,y) = z = \sqrt{x^2+y^2}$. $f_x(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$, $f_y(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$. $d\sigma = \sqrt{\dfrac{x^2}{x^2+y^2}+\dfrac{y^2}{x^2+y^2}+1}\,dA = \sqrt{2}\,dA$.

D is an annulus of radii 1 and 4. $\rho(x,y,z) = (10-z)\mathrm{kg/m^2}$. M kg is the funnel's mass.

$$M = \iint_S (10-z)d\sigma = \iint_D [10-\sqrt{x^2+y^2}]\sqrt{2}\,dA = \int_0^{2\pi}\int_0^4 (10-r)r\,dr\,d\theta = \sqrt{2}\theta\Big]_0^{2\pi}\Big[5r^2-\frac{1}{3}r^3\Big]_0^4 = 108\sqrt{2}\pi$$

56. S is that part of the sphere $x^2+y^2+z^2 = 9$ that is above the region D in the xy plane enclosed by the circle $x^2+y^2 = 1$. If the velocity field of a fluid is given by $\mathbf{F}(x,y,z) = -y\mathbf{i}+x\mathbf{j}+3\mathbf{k}$, find the flux of **F** across S.

▷ We have

$M(x,y,z) = -y \quad N(x,y,z) = x \quad R(x,y,z) = 3$

solving the equation of the sphere for z, we obtain

$z = f(x,y) = \sqrt{9-x^2-y^2}$

Thus

$$f_x(x,y) = \frac{-x}{\sqrt{9-x^2-y^2}} \quad f_y(x,y) = \frac{-y}{\sqrt{9-x^2-y^2}}$$

If **n** is a unit upper normal vector of S, the flux of **F** across S is given by

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_D (-Mf_x - Nf_y + R)dA = \iint_D \Big(\frac{-xy}{\sqrt{9-x^2-y^2}}+\frac{xy}{\sqrt{9-x^2-y^2}}+3\Big)dA = 2\iint_D dA$$

Because the area of the region D is $\pi r^2 = \pi$, then the flux of **F** across S is 3π cubic units per unit of time.

57. $\mathbf{F}(x,y,z) = 2x\mathbf{i}+2y\mathbf{j}+3z\mathbf{k}$. $f(x,y) = z = 4-x^2-y^2$. $f_x(x,y) = -2x$, $f_y(x,y) = -2y$. D is $x^2+y^2 \le 4$

$$\text{flux} = \iint_S \mathbf{F}\cdot\mathbf{N}\,d\sigma = \iint_D [(-2x)(-2x)+(-2y)(-2y)+3(4-x^2-y^2)]dA = \iint_D (x^2+y^2+12)dA$$
$$= \int_0^{2\pi}\int_0^2 (r^2+12)r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_0^2 (r^3+12r)dr = \theta\Big]_0^{2\pi}\Big[\frac{1}{4}r^4+6r^2\Big]_0^2 = 2\pi\cdot 28 = 56\pi$$

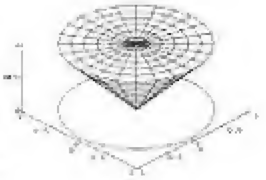
58. $\mathbf{F}(x,y,z) = \frac{1}{2}z\mathbf{i}$, S: $x^2+y^2+z^2 = 4$.

$\iint_S \frac{1}{2}z\mathbf{i}\cdot\mathbf{n}\,d\sigma = 0$ because $\mathbf{i}\cdot\mathbf{n}$ is even, z is odd with respect to xy plane. $\iiint_E \mathrm{div}(\frac{1}{2}z\mathbf{i})dV = \iiint_E 0\,dV = 0$.

In Exercises 59 and 60, Use Gauss's divergence theorem to evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{N}\,d\sigma$ for **F** and S.

59. $\mathbf{F}(x,y,z) = 2x\mathbf{i}+y\mathbf{j}+2z\mathbf{k}$. $\iint_S \mathbf{F}\cdot\mathbf{N}\,d\sigma = \iiint_E \mathrm{div}\,\mathbf{F}\,dV = \iiint_E 5\,dV = 5(16\pi)(2) = 160\pi$.

60. $\mathbf{F}(x,y,z) = x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$; S is the boundary of the region enclosed by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 1$.



▷ Refer to the figure. By Gauss's divergence theorem

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,d\sigma = \iiint_E \mathrm{div}\,\mathbf{F}\,dV = \iiint_E (2x+2y+2z)\,dV$$

Because the region is symmetric with respect to the planes $x = 0$ and $y = 0$, the integrals of the first two terms are 0. To evaluate the third, we switch to cylindrical coordinates with $dV = r\,dr\,d\theta\,dz$. Thus

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,d\sigma = \int_0^1\int_0^{2\pi}\int_0^z 2zr\,dr\,d\theta\,dz = \int_0^1\int_0^{2\pi} r^2z\Big]_0^z d\theta\,dz$$
$$= \int_0^1\int_0^{2\pi} z^3\,d\theta\,dz = \int_0^1 2\pi z^3\,dz = \frac{1}{2}\pi z^4\Big]_0^1 = \frac{1}{2}\pi$$

In Exercises 61 and 62, verify Stokes's theorem for $\mathbf{F}$ and S.

61. C: $\mathbf{R}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$, $0 \le t \le 2\pi$. $\mathbf{F} = z\mathbf{i} + 4x\mathbf{j} + 2z\mathbf{k} = 12\cos t\mathbf{j}$. By equation 14.4.11

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{2\pi} [0 + (12\cos t)(3\cos t)]dt = 36\int_0^{2\pi} \cos^2 t\,dt = 36\cdot\tfrac{1}{2}\cdot 2\pi = 36\pi.$$ Also

curl $\mathbf{F} = 1\mathbf{j} + 4\mathbf{k}$. $f(x,y) = z = 9 - x^2 - y^2$. From equation 19.5.9,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{N}\,d\sigma = \iint_D (0 - 1\cdot 2y + 4)dA = \iint_D 4\,dA = 4(9\pi) = 36\pi$$

62. C: $\mathbf{R}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$, $0 \le t \le 2\pi$. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$. $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C xy\,dx = 0$ because y is odd.

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S (-y\mathbf{i} - z\mathbf{j} - x\mathbf{k})\cdot\tfrac{1}{4}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})d\sigma = -\tfrac{1}{4}\iint_S (xy + yz + xz)d\sigma = 0 \text{ (odd in } x \text{ and } y)$$

In Exercises 63 and 64, use Stokes's theorem to evaluate the line integral $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$ for $\mathbf{F}$ and C.

63. S is $x^2 + y^2 \le 1$, $z = 1$. $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \operatorname{curl}[(z + \ln(x^2+1))\mathbf{i} + (\cos y - x^2)\mathbf{j} + (3y^2 - e^z)\mathbf{k}]\cdot\mathbf{N}\,d\sigma$

$$= \iint_S (6y\mathbf{i} - \mathbf{j} - 2x\mathbf{k})\cdot\mathbf{k}\,dA = \iint_S -2x\,dA = 0 \text{ because } x \text{ is odd.}$$

64. $\mathbf{F}(x,y,z) = -2y\mathbf{i} + 3x\mathbf{j} + z\mathbf{k}$; C is the circle $x^2 + y^2 = 1$ in the xy plane.

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2y & 3x & z \end{vmatrix} = 5\mathbf{k}$$

Applying Stokes's theorem, we have

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S 5\mathbf{k}\cdot\mathbf{n}\,d\sigma$$

Because S is any sectionally smooth surface whose boundary is C, we may take S to be that portion of the xy plane enclosed by the given circle C. Because $\mathbf{n}$ is a unit upper normal vector of S, then $\mathbf{n} = \mathbf{k}$ and $\mathbf{k}\cdot\mathbf{n} = 1$. The surface area of S is $\pi(1)^2 = \pi$. We conclude that

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = 5\iint_S d\sigma = 5\pi$$

In Exercises 65 and 66, prove the identity if f is a real-valued function and $\mathbf{v}$ is a vector-valued function.

Let $\mathbf{v} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$.

65.
$$\begin{aligned}\nabla\cdot(f\mathbf{v}) &= D_x(fp) + D_y(fq) + D_z(fr) = f_x p + f p_x + f_y q + f q_y + f_z r + f r_z \\ &= (f_x p + f_y q + f_z r) + f(p_x + q_y + r_z) = (\nabla f)\cdot\mathbf{v} + f(\nabla\cdot\mathbf{v})\end{aligned}$$

66. $\nabla\times(f\mathbf{v}) = \nabla\times(fp\mathbf{i} + fq\mathbf{j} + fr\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ fp & fq & fr \end{vmatrix} = \left[\frac{\partial}{\partial y}(fr) - \frac{\partial}{\partial z}(fq)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(fr) - \frac{\partial}{\partial z}(fp)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(fq) - \frac{\partial}{\partial y}(fp)\right]\mathbf{k}$$

$$= [f_y r - f_z q]\mathbf{i} - [f_x r - f_z p]\mathbf{j} + [f_x q - f_y p]\mathbf{k} + [f r_y - f q_z]\mathbf{i} - [f r_x - f p_z]\mathbf{j} + [f q_x - f p_y]\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ p & q & r \end{vmatrix} + f\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ p & q & r \end{vmatrix} = \nabla f\times\mathbf{v} + f(\nabla\times\mathbf{v})$$